Enhanced Quadratic Programming via Pseudo-Transient Continuation: An Application to Model Predictive Control

Enhanced Quadratic Programming via Pseudo-Transient Continuation: An Application to Model Predictive Control

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***Abstract*—In this letter, we present a novel fast solver for convex quadratic programs (QPs) based on pseudo-transient continuation (PTC). Tailored for real-time applications with strict computational requirements, our solver offers high execution speed and guaranteed global convergence to the optimal solution. PTC is a numerical technique that transforms multivariate nonlinear equations into autonomous systems that converge to the solution sought. In our approach, we recast the general QP Karush-Kuhn-Tucker (KKT) conditions into a system of equations and employ PTC to solve the latter to attain the optimal solution. Importantly, we provide theoretical guarantees demonstrating the global convergence of our PTC-based solver to the optimal solution of any given QP. To showcase the effectiveness of PTC, we employ it within the domain of Model Predictive Control (MPC). Specifically, numerical simulations are carried out on the MPC control of a quadrotor – a demanding dynamical system – highlighting excellent results in accurately executing the control task and ensuring lower computational times compared to conventional QP solvers.**

***Index Terms*—Pseudo-transient continuation, quadratic programming, optimal control**

I. Introduction

QUADRATIC programming (QP) solvers that join effectiveness with a simple implementation are becoming essential in the field of optimal control, specifically when dealing with real-time applications with strict timing constraints and limited computational resources [1]. To address this need, in this letter we present a novel high-performance QP solution method based on pseudo-transient continuation (PTC). PTC is a technique that reformulates fixed-point iteration and Newton's methods, by constructing an autonomous system that converges to the solution of a given multivariate nonlinear equation [2], [3]. In our setting, starting from a general convex QP problem, we recast the associated Karush-Kuhn-Tucker (KKT) conditions into a system of equations, where the QP global optimum serves as the solution. Employing PTC to solve the latter equations ensures achieving the globally optimal solution, with a notably reduced computational load and a straightforward practical implementation. Additionally, our approach offers theoretical guarantees for the PTC-based solver to globally converge, in Lyapunov sense, to the optimum of any given QP problem. Such global convergence is indeed an aspect that has been often overlooked, since PTC has been mainly investigated in terms of its local and numerical convergence (see, e.g., [4]–[6]). However, in optimal control scenarios, securing the globally optimal solution is utterly important, as it ensures enhanced system performance and stability compared to approximate or suboptimal solutions [7]. An alternative approach for solving KKT-based systems of equations, based on a piecewise smooth Newton method, was proposed in [8]. Yet, PTC, being a gradient-free approach, offers unquestionable advantages, compared to Newton-like methods, when dealing with ill-conditioned and possibly non-smooth problems; also, PTC constructs an autonomous system from KKT conditions, allowing for a convergence analysis based on well-assessed Lyapunov arguments. The effectiveness of the proposed methodology is tested by employing the PTC-based solver in the context of Model Predictive Control (MPC), which stands as the predominant scenario for employing fast real-time optimization in control applications. Specifically, the case study of the MPC control of an Euler-Lagrange quadrotor model is considered [9]. The quadrotor is tasked with performing swift maneuvers in a constrained environment; such task yields a potentially large-scale MPC problem for which real-time requirements may still pose a challenge [10]. The problem is tackled with PTC and, for comparison purposes, with several other solvers, based on the most employed conventional QP methods (i.e., active-set [1], interior-point [11], [12], operator splitting [13], and augmented Lagrangian [14]). Numerical simulations confirm the excellent performance of PTC in accurately executing the control tasks. Most notably, PTC delivers significantly lower computational

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times compared to the conventional methods.

Outline: The remainder of this letter is structured as follows. Section II contains our main contribution: we introduce the application of PTC to efficiently solve general QPs, we recast KKT conditions in light of the PTC framework, and we prove PTC global convergence conditions. Section III presents the MPC framework employed to showcase the effectiveness of PTC. Section IV validates our PTC-based solver on the MPC quadrotor control case study, through extensive simulations and comparison with conventional algorithms. Our conclusions are drawn in Section V, along with perspectives for future research avenues.

II. Pseudo-Transient Continuation for Quadratic Programming

Consider a multivariate nonlinear equation in the form

$$F(x) = 0,\ F : \mathbb{R}^n \to \mathbb{R}^n, \tag{1}$$

where F is continuous and has a set of solutions $S = \{x \in \mathbb{R}^n : F(x) = 0\}$. Pseudo-transient continuation (PTC) seeks a functional $f(F(x)) : \mathbb{R}^n \to \mathbb{R}^n$ such that the autonomous system

$$\dot{x} = f(F(x)) \tag{2}$$

has a set of equilibrium points $S^* \subseteq S$ and converges (at least locally) to one of such equilibria, i.e.,

$$f(F(x(t))) \to 0,\ x(t) \to x^* \in S^* \ \text{ for } \ t \to +\infty. \tag{3}$$

In this work, we leverage PTC to efficiently solve general convex quadratic programs (QPs) in the following form:

$$\min_y \frac{1}{2} y^\top H y + c^\top y \quad \text{s.t.} \quad Cy = p, \quad Dy \le q, \tag{4}$$

where $y \in \mathbb{R}^n$ is the vector of optimization variables; $H \in \mathbb{R}^{n,n}$, $H = H^\top \succ 0$; $C \in \mathbb{R}^{N_E,n}$ and $p \in \mathbb{R}^{N_E}$ represent the N_E equality constraints, C has full row rank; $D \in \mathbb{R}^{N_I,n}$ and $q \in \mathbb{R}^{N_I}$ represent the N_I inequality constraints. Since the cost and the inequality constraints of (4) are convex functions, it admits a unique global minimum y^* [15]. PTC allows to solve (4) for its global optimum with high computational performance. In this perspective, two fundamental steps have to be performed:

1) by manipulating the Karush-Kuhn-Tucker (KKT) conditions associated with (4), recast them as a system of equations like (1), having as unique solution the global optimum of (4);
2) derive sufficient conditions characterizing F such that, for given functionals f, the global asymptotic convergence of (2) to its unique equilibrium is guaranteed.

These two steps are assessed in Sections II-A and II-B, respectively.

A. Conversion of QP KKT Conditions into a System of Equations

Let us consider the Lagrangian of (4), i.e.,

$$\mathcal{L}(y, \mu, \lambda) = \frac{1}{2} y^\top H y + c^\top y - \mu^\top (Cy - p) - \lambda^\top (Dy - q), \tag{5}$$

where $\mu \in \mathbb{R}^{N_E}$ and $\lambda \in \mathbb{R}^{N_I}$ are the Lagrange multipliers. It is well known that, if a triple (y^*, μ^*, λ^*) satisfies the KKT conditions, i.e.,

$$\nabla_y \mathcal{L}(y, \mu, \lambda) = Hy + c - C^\top \mu - D^\top \lambda = 0, \tag{6a}$$
$$Cy = p, \tag{6b}$$
$$Dy \le q, \quad \lambda \le 0, \quad \lambda^\top (Dy - q) = 0, \tag{6c}$$

then y^* is also the global minimum of (4) [15]. According to [16], conditions (6c) can be equivalently rewritten as

$$D_{i,\cdot}\, y = q_i \ \text{ if } \lambda_i \le 0, \quad D_{i,\cdot}\, y < q_i \ \text{ if } \lambda_i = 0, \tag{7}$$

where $D_{i,\cdot}$ denotes the i-th row of D. (7) is in the form of a mixed complementarity problem, which is then equivalent to the following variational inequality [17]:

$$\lambda^\top (z - Dy) \ge 0,\ \forall z \in Z,\ Z = \{z \in \mathbb{R}^{N_I} : z \le q\}. \tag{8}$$

Additionally, (7) is also equivalent to the following system of piecewise affine equations [16]:

$$Dy = \phi_{[-\infty,q]}(Dy - \alpha\lambda), \tag{9}$$

where $\alpha \in \mathbb{R}_{>0}$ and $\phi_{[a,b]}(z)$ denotes the lower-upper truncation of the vector $z \in \mathbb{R}^{N_I}$ by $[a, b]$, which can be defined component-wise as

$$\phi_{[a,b]}(z) \equiv [\phi_{[a_i,b_i]}(z_i)]_{i=1}^{N_I},$$
$$\phi_{[a_i,b_i]}(z_i) = \begin{cases} a_i & \text{if } z_i < a_i, \\ z_i & \text{if } a_i \le z_i \le b_i, \\ b_i & \text{if } z_i > b_i. \end{cases} \tag{10}$$

Hereafter, for notation clarity, we denote $\phi_{[-\infty,q]}$ as ϕ.

The KKT conditions (6) can be then rewritten as a system of equations as follows:

$$Hy + c - C^\top \mu - D^\top \lambda = 0, \tag{11a}$$
$$Cy - p = 0, \tag{11b}$$
$$Dy - \phi(Dy - \alpha\lambda) = 0. \tag{11c}$$

The system of equations (11) can be simplified as

$$y = G' D^\top \lambda + h', \tag{12a}$$
$$\mu = (CH^{-1}C^\top)^{-1}(p - CH^{-1}(D^\top \lambda - c)), \tag{12b}$$
$$G\lambda + h - \phi((G - \alpha I)\lambda + h) = 0, \tag{12c}$$

where

$$G' = H^{-1} - H^{-1}C^\top (CH^{-1}C^\top)^{-1} CH^{-1},$$
$$h' = H^{-1}(C^\top (CH^{-1}C^\top)^{-1}(CH^{-1}c + p) - c),$$
$$G = DG'D^\top, \quad h = Dh'. \tag{13}$$

We clearly see that only (12c) has to be solved for λ, since y and μ are function of λ only. Also, since (12) admits a unique solution (y^*, μ^*, λ^*) [15], then λ^* is the unique solution of (12c).

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B. Global Convergence of PTC for QPs Solution

To prove that (12) can be solved through PTC and exhibits global convergence to its optimum, we introduce the following Theorem, reporting a sufficient condition for the PTC autonomous system (2) to be globally asymptotically stable in the sense of Lyapunov.

Assumption 1: Assume that the multivariate nonlinear equation (1), i.e., $F(x) = 0$, has a unique solution x^*.

Assumption 2: Assume that there exists a symmetric and positive definite matrix $M \in \mathbb{R}^{n,n}$ such that

$$(x - x^*)^\top M F(x) > 0, \quad \forall x \neq x^*. \tag{14}$$

Assumption 2 has a straightforward interpretation when considering a single-variable function $F : \mathbb{R} \to \mathbb{R}$. Let $x(t) = x^* + vt$ be a parametrization over the direction v, where $t \in \mathbb{R}$ and $\|v\| = 1$. In the single-variable case, v = 1 and $M \in \mathbb{R}_{>0}$; substituting $x(t)$ in (14) yields

$$\begin{aligned} &(x - x^*)MF(x) = tMF(x(t)) > 0, \ \forall t \neq 0 \\ &\Rightarrow \begin{cases} F(x(t)) > 0 & \text{if } t > 0, \\ F(x(t)) < 0 & \text{if } t < 0 \end{cases} \Rightarrow \begin{cases} F(x) > 0 & \text{if } x > x^*, \\ F(x) < 0 & \text{if } x < x^*. \end{cases} \end{aligned} \tag{15}$$

Single-variable functions satisfying (15) also fulfill Assumption 2. Thus, (14) is a generalization of (15) for multivariate functions $F : \mathbb{R}^n \to \mathbb{R}^n$.

Theorem 1: Let Assumptions 1-2 hold. Then, the system

$$\dot{x} = -\beta F(x), \quad \beta \in \mathbb{R}_{>0} \tag{16}$$

has a unique equilibrium point in x^* and such equilibrium is globally asymptotically stable (GAS).

Proof: We start by proving that (16) has a unique equilibrium point in x^*. An equilibrium point satisfies the equation $-\beta F(x) = 0 \Rightarrow F(x) = 0$, which, by assumption, has a unique solution in x^*, which is then also the unique equilibrium point of (16).

Now, we prove the global asymptotic stability of x^* through Lyapunov stability theory. Without any loss of generality, let's consider $x^* = 0$. We then construct a Lyapunov candidate function $V(x)$ as follows:

$$V(x) = \frac{1}{2} x^\top M x, \tag{17}$$

where M is from Assumption 2. $V(x)$ is positive definite (PD) and radially unbounded. Let's now compute $\dot{V}(x)$, i.e.,

$$\begin{aligned} \dot{V}(x) &= \frac{d}{dt}\left(\frac{1}{2}x^\top M x\right) = x^\top M \dot{x} = x^\top M(-\beta F(x)) = \\ &= -\beta x^\top M F(x) < 0, \ \forall x \neq x^*. \end{aligned} \tag{18}$$

Thus, by Assumption 2 with $x^* = 0$, $\dot{V}(x)$ is negative definite (ND). According to Lyapunov stability theory, since the candidate function $V(x)$ is PD and radially unbounded and $\dot{V}(x)$ is ND, then the equilibrium x^* of (16) is GAS. ■

Now, in the following Proposition, we prove that the KKT system of equations (12) satisfies the assumptions of Theorem 1 and, thus, can be effectively solved via PTC. First, we introduce a technical lemma:

Lemma 1: Let $Z = \{z \in \mathbb{R}^n : z \leq q\}$ and consider $\phi_{[-\infty,q]} \equiv \phi$, as defined in (10). Then, it holds

$$(z - \phi(a))^\top (\phi(a) - a) \geq 0, \ \ \forall a \in \mathbb{R}^n, \ \ \forall z \in Z. \tag{19}$$

Proof: By definition (10), $\phi(a)$ is the global optimum of the function $g(z) = \|z - a\|^2$ over the convex set Z, i.e., $\phi(a) = \arg\min_{z\in Z} g(z)$. Recalling the first order optimality conditions for constrained optimization problems [15], if z^* is an optimum of $g(z)$, then it holds

$$(z - z^*)^\top \nabla g(z^*) \geq 0, \ \ \forall z \in Z. \tag{20}$$

Being $\phi(a)$ an optimum of g and $\nabla g(z) = 2(z - a)$, then

$$(z - \phi(a))^\top (\phi(a) - a) \geq 0, \ \ \forall a \in \mathbb{R}^n, \ \ \forall z \in Z. \tag{21}$$

■

Proposition 1: Let us consider (12c) and denote it as $F(\lambda) = 0$. Then, the system

$$\dot{\lambda} = -\beta F(\lambda), \quad \beta \in \mathbb{R}_{>0} \tag{22}$$

has a unique equilibrium point λ^*, coinciding with the solution of (12c), and such equilibrium is GAS.

Proof: Since $F(\lambda) = 0$ admits a unique solution λ^*, it satisfies Assumption 1 of Theorem 1.

From (8), it holds

$$(z - Dy^*)^\top \lambda^* \geq 0, \ \forall z \in Z = \{z \in \mathbb{R}^{N_I} : z \leq q\}, \tag{23}$$

where (y^*, λ^*) is the solution of (12). By replacing $z = \phi(G\lambda + h - \alpha\lambda) \in Z$ and recalling from (12a) that $Dy^* = G\lambda^* + h$, we have

$$\alpha \left(\phi(G\lambda + h - \alpha\lambda) - (G\lambda^* + h)\right)^\top \lambda^* \geq 0, \tag{24}$$

where $\alpha > 0$ was multiplied to the left-hand side of the inequality in preparation for the next calculations.

Through Lemma 1, considering $a = G\lambda + h - \alpha\lambda$ and $z = G\lambda^* + h = Dy^* \in Z$, we also have that

$$\begin{aligned} &\left(G\lambda^* + h - \phi(G\lambda + h - \alpha\lambda)\right)^\top \cdot \\ &\left(\phi(G\lambda + h - \alpha\lambda) - (G\lambda + h - \alpha\lambda)\right) \geq 0. \end{aligned} \tag{25}$$

By adding together both sides of the inequalities (24), (25) and recalling that $F(\lambda) = G\lambda + h - \phi(G\lambda + h - \alpha\lambda)$, we obtain

$$\begin{aligned} &\left(G\lambda^* + h - \phi(G\lambda + h - \alpha\lambda)\right)^\top \cdot \\ &\left(\phi(G\lambda + h - \alpha\lambda) - (G\lambda + h - \alpha\lambda) - \alpha\lambda^*\right) \geq 0 \\ \Rightarrow &\left(F(\lambda) - G(\lambda - \lambda^*)\right)^\top \left(F(\lambda) - \alpha(\lambda - \lambda^*)\right) \leq 0. \end{aligned} \tag{26}$$

By expanding (26),

$$\begin{aligned} &-\alpha F(\lambda)^\top (\lambda - \lambda^*) - (\lambda - \lambda^*)^\top G F(\lambda) \leq \\ &\leq -F(\lambda)^\top F(\lambda) - \alpha(\lambda - \lambda^*)^\top G(\lambda - \lambda^*) < 0, \ \forall \lambda \neq \lambda^* \\ \Rightarrow &(\lambda - \lambda^*)^\top (\alpha I + G) F(\lambda) > 0, \ \forall \lambda \neq \lambda^*. \end{aligned} \tag{27}$$

From (13), the matrix $\alpha I + G$ results symmetric and PD; therefore, by (27), $F(\lambda)$ satisfies also Assumption 2 of Theorem 1. Then, by the latter Theorem, we conclude that system (22) has a unique and GAS equilibrium in λ^*. ■

With Proposition 1, we have shown that any QP problem in the form (4), when rewritten as a system of equations (12), satisfies the global convergence conditions for PTC expressed by Theorem 1. Therefore, system (22) globally converges to the exact global minimum of (4). This result allows to effectively employ PTC for quadratic programming, enabling its application in optimization-based scenarios, among which optimal control.

III. Application of PTC to Model Predictive Control

Among the possible QP applications, Model Predictive Control (MPC) is the most relevant scenario that joins optimization with control. MPC problems require a fast and reliable solution to enable real-time implementation on hardwares with limited computational capability. This makes MPC a good benchmark to assess the performance of PTC.

In this letter, we confine our application to a standard linear MPC scheme based on an affine time-variant (ATV) model [18]. Given a discrete-time (DT) ATV plant

$$x_{k+1} = A_k x_k + B_k u_k + b_k, \quad k \geq 0, \tag{28}$$

where $x_k \in \mathbb{R}^{n_x}$ and $u_k \in \mathbb{R}^{n_u}$, the MPC optimal control problem, at each time k, is defined as follows:

$$\min_{\hat{u},\hat{x}} J_k(\hat{u},\hat{x}) = \min_{\hat{u},\hat{x}} \sum_{i=0}^{N_p-1} \left(\|\hat{x}_{i|k} - x_{r,k+i}\|_Q^2 + \|\hat{u}_{i|k}\|_R^2 \right) + \sum_{i=1}^{N_p-1} \left(\|\hat{u}_{i|k} - \hat{u}_{i-1|k}\|_{R_\Delta}^2 \right) + \|\hat{x}_{N_p|k} - x_{r,k+N_p}\|_P^2 \tag{29a}$$

s.t. $\forall i \in \{0, 1, \ldots, N_p - 1\}$

$$\hat{x}_{0|k} = x_k, \quad \hat{x}_{i+1|k} = A_k \hat{x}_{i|k} + B_k \hat{u}_{i|k} + b_k, \tag{29b}$$

$$\hat{u}_{i|k} \in \mathcal{U}, \quad \hat{x}_{i|k} \in \mathcal{X}, \tag{29c}$$

where $\hat{u}_k = [\hat{u}_{i|k}]_{i=0}^{N_p-1}$, $\hat{x}_k = [\hat{x}_{i|k}]_{i=0}^{N_p}$ are the inputs and states predicted i steps ahead at time k, respectively; x_r is the state reference trajectory; (29a) is the MPC cost function, with $Q = Q^\top \succeq 0$, $P = P^\top \succeq 0$, $R_\Delta = R_\Delta^\top \succeq 0$, and $R = R^\top \succ 0$ ($\|x\|_M^2 \equiv \frac{1}{2}x^\top M x$) [19]; (29b) are the prediction model constraints; (29c) are inputs and states constraints, with $\mathcal{U} = \{u \in \mathbb{R}^{n_u} : \underline{u} \leq u \leq \overline{u}\}$ and $\mathcal{X} = \{x \in \mathbb{R}^{n_x} : \underline{x} \leq x \leq \overline{x}\}$.

The MPC optimal control problem (29) can be rewritten to match the QP formulation (4) [19], [20], i.e.,

$$\min_{y_k} \frac{1}{2} y_k^\top H y_k + c_k^\top y_k \quad \text{s.t.} \quad C_k y_k = p_k, \quad D y_k \leq q, \tag{30}$$

where $y_k = \left[\hat{x}_k^\top, \hat{u}_k^\top\right]^\top \in \mathbb{R}^{n_x(N_p+1)+n_u N_p}$ are the optimization variables at time k. (30) can be fast solved for its global optimum by PTC with global convergence guarantees, in view of the results presented in Section II.

IV. Simulations and Results

In this section, we assess the performance of PTC to solve the MPC optimal control problem (30) and compare it with several alternate solvers. As nonlinear plant to control, we select the following Euler-Lagrange quadrotor model [9]:

$$\begin{aligned}
\ddot{x} &= \tfrac{1}{m}(c_\phi s_\theta c_\phi + s_\phi s_\psi) f - \tfrac{\beta_x}{m}\dot{x}, \\
\ddot{y} &= \tfrac{1}{m}(c_\phi s_\theta s_\phi - s_\phi c_\psi) f - \tfrac{\beta_y}{m}\dot{y}, \\
\ddot{z} &= -g + \tfrac{1}{m} c_\phi c_\theta f - \tfrac{\beta_z}{m}\dot{z}, \\
\ddot{\phi} &= \tfrac{I_y - I_z}{I_x}\dot{\theta}\dot{\psi} + \tfrac{1}{I_x}\tau_x, \\
\ddot{\theta} &= \tfrac{I_z - I_x}{I_y}\dot{\phi}\dot{\psi} + \tfrac{1}{I_y}\tau_y, \\
\ddot{\psi} &= \tfrac{I_x - I_y}{I_z}\dot{\phi}\dot{\theta} + \tfrac{1}{I_z}\tau_z,
\end{aligned} \tag{31}$$

where $c_x \equiv \cos(x)$, $s_x \equiv \sin(x)$. In (31), the system states are the quadrotor position $p = [x, y, z]^\top$ and orientation $\alpha = [\phi, \theta, \psi]^\top$, along with their time derivatives, and the inputs are the thrust force f and torques $\tau = [\tau_x, \tau_y, \tau_z]^\top$ (i.e., $x = [p^\top, \dot{p}^\top, \alpha^\top, \dot{\alpha}^\top]^\top \in \mathbb{R}^{12}$ and $u = [f, \tau^\top]^\top \in \mathbb{R}^4$). The control task is to track a lemniscate reference trajectory $p_{r,k} = [x_{r,k}, y_{r,k}, z_{r,k}]^\top$, partially crossing an infeasible region of space (to mimic a vertical obstacle, see Fig. 1a). States and inputs are also subject to bound constraints (see Fig. 1b, 1c). To deploy the linear MPC (29) for the sake of controlling the nonlinear continuous-time (CT) plant (31), we perform the preparation steps adopted in the sequential quadratic programming (SQP) approach with real-time iteration (RTI) [18]: the prediction model (29b) is obtained by first discretizing the CT plant with time step T_s and then linearizing it, at each time k, around a guess point computed from the optimal states and inputs at the previous time instant [18]. The CT plant (31) is controlled through the 1-step receding horizon policy, i.e., over each discrete time interval $[kT_s, (k+1)T_s]$, $k \geq 0$, the plant is fed with the first sample $\hat{u}_{0|k}^*$ of the optimal inputs $\hat{u}_k^*$, that are the solution of (30).

PTC is compared with the following conventional QP solvers: the active-set solvers quadprog and DAQP [1]; the interior-point solvers Gurobi, MOSEK, qpSWIFT [11], and PIQP [12]; the operator splitting solver OSQP [13]; the augmented Lagrangian solver QPALM [14]. DAQP, qpSWIFT, PIQP, OSPQ, and QPALM are open-source solvers for embedded applications, coded in C, while quadprog, Gurobi, and MOSEK are well-known commercial solvers. Comparison encompasses the following aspects, with reference to (30):

1) successful attainment of the control task;
2) deviation between the obtained closed-loop trajectories and the true globally optimal ones;
3) value of optimality conditions at each time k, i.e., primal residual $r_{p,k}$ and dual resisual $r_{d,k}$,

$$r_{p,k} = \max\{\|C_k y_k^* - p_k\|_\infty, \|\max\{D y_k^* - q, 0\}\|_\infty\}, \tag{32a}$$

$$r_{d,k} = \|H y_k^* + c_k - C_k^\top \mu_k^* - D^\top \lambda_k^*\|_\infty; \tag{32b}$$

4) execution time at each control step.

A. Implementation Details

Simulations are performed with MATLAB® 2023b on a 13th Gen Intel® Core™ i7 CPU at 1.7 GHz. The PTC-based solver is implemented in MATLAB. The full source code is available online[1].

The PTC autonomous system (22) is numerically integrated using the explicit Runge-Kutta 2(3) method with adaptive step size (RK23). The integration of (22) starts from the initial

[1] github.com/lorenzocalogero/QP_PTC.

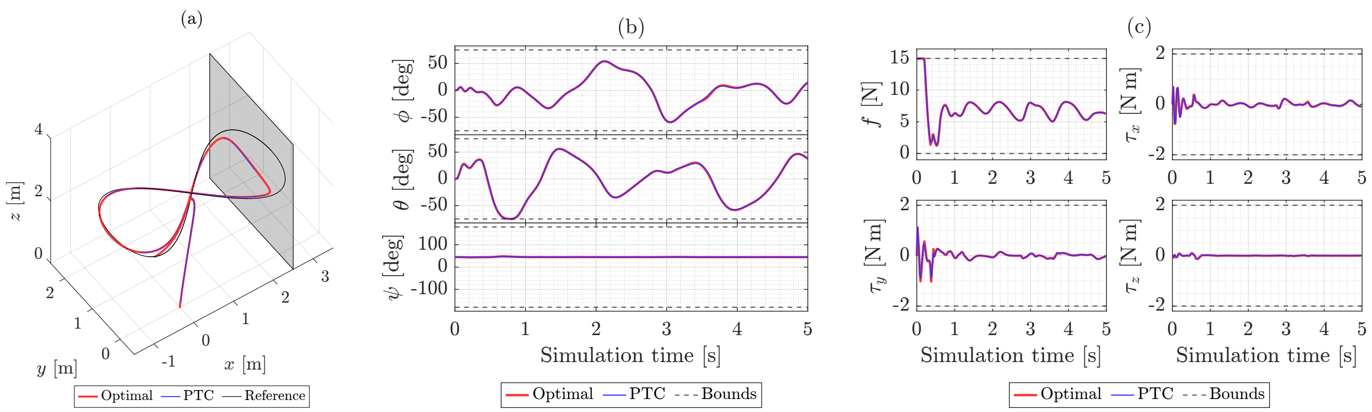

Fig. 1. MPC quadrotor control: comparison of PTC closed-loop trajectories with the globally optimal ones. (a) Position; (b) Orientation; (c) Control inputs.

value $\lambda = 0$ and is stopped when either of the following termination criteria is met:

1) the primal and dual residuals are small enough, i.e.,

$$r_{p,k} \leq \epsilon_{\text{abs},p} + \epsilon_{\text{rel},p} \max\{\|C_k y_k^*\|_\infty, \|p_k\|_\infty, \|D y_k^*\|_\infty, \|q\|_\infty\}, \quad (33a)$$

$$r_{d,k} \leq \epsilon_{\text{abs},d} + \epsilon_{\text{rel},d} \max\{\|H y_k^*\|_\infty, \|C_k^\top \mu_k^*\|_\infty, \|D^\top \lambda_k^*\|_\infty\}, \quad (33b)$$

where ϵ_{abs} and ϵ_{rel} are absolute and relative optimality tolerance parameters;

2) a maximum integration time $T_{\max}$ is reached.

The optimality tolerances are set to $\epsilon_{\text{abs}} = \epsilon_{\text{rel}} = 1 \times 10^{-6}$ for all solvers under comparison. For additional data and parameters, we refer the reader to the source code mentioned above.

B. Simulation and Comparison Results

1) Attainment of the control task and deviation from the global optimum: Figure 1 reports the quadrotor closed-loop trajectories obtained by solving the MPC problem (30) with PTC. An estimate of the globally optimal trajectories is obtained with Gurobi by setting very low optimality tolerances ($\epsilon_{\text{abs}} = \epsilon_{\text{rel}} = 1 \times 10^{-9}$). Since these trajectories are rather coincident, we are confident that PTC has successfully attained the control task while adhering to the given state and input constraints.

2) Optimality conditions: To assess the goodness of the obtained MPC solutions, we evaluate the primal and dual residuals (32) at each time instant and for every solver, as reported in Figure 3. We see that PTC consistently achieves low residuals that stay below the thresholds defined by the optimality criteria (33). On the contrary, some of the other solvers fail to deliver an acceptable dual residual.

Furthermore, Figure 2 reports the overlay of all PTC primal and dual residual curves, collected at each time instant, with respect to the RK23 iterations (which is the algorithm used to numerically integrate the PTC autonomous system (22), see Sec. IV-A). The iteration axis is normalized to accommodate curves with varying total iteration counts. PTC exhibits a good and consistent convergence of the primal residual, while the dual residual is always close to 0, since the dual optimality

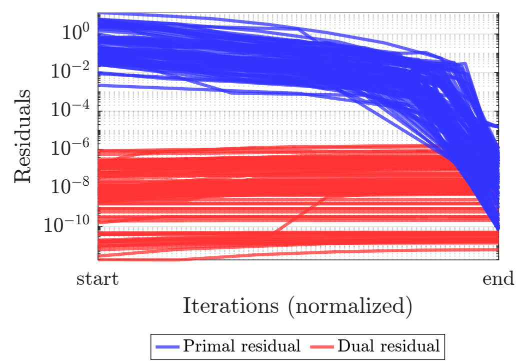

Fig. 2. Convergence of primal and dual residuals of PTC at each iteration of the RK23 integration algorithm.

TABLE I
EXECUTION TIME AND ITERATIONS (SINGLE MPC CONTROL STEP).

Solver	Exec. time [ms]		Iterations	
	Max	Mean	Max	Mean
PTC	4.19	0.85	20	3
quadprog	98.04	45.02	90	7
Gurobi	14.23	10.96	24	16
MOSEK	43.20	20.17	49	24
OSQP	53.64	6.17	2875	329
qpSWIFT	6.33	4.73	18	14
DAQP	8.74	1.43	23	3
PIQP	4.98	4.28	13	11
QPALM	13.29	6.82	49	28

condition (11a) is directly employed in (12b) to compute the dual variable μ from λ.

3) Execution time: Figure 4 compares, for each solver, the execution time of each MPC control step. Such comparison is summarized in Table I, including the number of iterations required by each solver. We observe that PTC outperforms all other solvers in terms of both maximum and average computational time. A more robust comparison of the execution time is performed by means of a Monte Carlo simulation, employing 50 randomly selected initial states x_0 from the feasible set $\mathcal{X}$. Figure 5 illustrates the results, indicating that PTC consistently achieves a lower average execution time and exhibits a tighter Monte Carlo envelope compared to other solvers. We observe that PTC execution time increases as the system states and

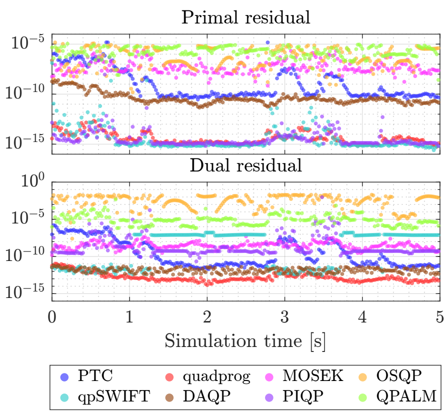

Fig. 3. Primal and dual residuals on the MPC solution obtained by each solver.

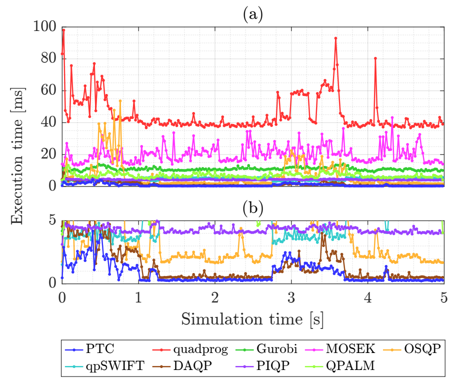

Fig. 4. Execution time of each MPC control step. (a) Execution time; (b) Detail of the range $[0, 5]$ ms.

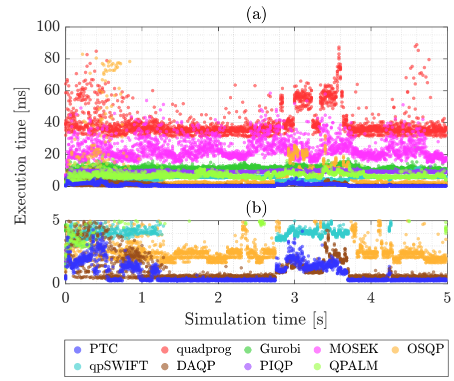

Fig. 5. Monte Carlo simulation for the execution time of each MPC control step. (a) Execution time; (b) Detail of the range $[0, 5]$ ms.

inputs approach the infeasible regions $\mathbb{R}^{n_x} \setminus \mathcal{X}$ and $\mathbb{R}^{n_u} \setminus \mathcal{U}$; this happens since, when x_k and u_k are far from the infeasible regions, many state and input constraints stay inactive; for such constraints, $\lambda^* = 0$, coinciding with the initial value of (22) and, hence, resulting in a much faster computation. The same behaviour is also shared by active-set solvers.

V. Conclusions

In this letter, the application of pseudo-transient continuation (PTC) to solve convex quadratic programs with high computational performance has been presented. PTC allows to solve multivariate nonlinear equations by transforming them into autonomous systems that converge to the solutions sought. To this aim, the Karush-Kuhn-Tucker (KKT) conditions of a general QP problem have been recast into a system of equations, to which PTC has been applied. The global convergence of PTC has then been proved to guarantee the attainment of the global optimum for any given QP.

The effectiveness of PTC has been showcased within the context of Model Predictive Control (MPC). Specifically, the case study of the MPC control of a quadrotor – a dynamically rich system that requires swift control actions to perform fast and aggressive maneuvers – has been considered. Extensive numerical simulations have demonstrated the excellent capability of PTC to carry out the control task, consistently outperforming conventional QP solvers in terms of computational time. These outcomes, joined with the ease of implementation, underscore the practical feasibility of PTC in real-time optimal control scenarios.

Many extensions of this effort are envisaged. Future research avenues will concern a detailed study of the computational complexity of the PTC-based solver and its extension to generic nonlinear MPC problems, by employing it as a sequential quadratic programming (SQP) solver.

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