On the exponents in the factorizations of r consecutive numbers

ON THE EXPONENTS IN THE FACTORIZATIONS OF r CONSECUTIVE NUMBERS

CARLO SANNA†

Abstract. Let $f(n)$ be the number of distinct exponents in the prime factorization of the natural number n. For every r-tuple of positive integers $\boldsymbol{k} = (k_1, \dots, k_r)$ and for all $x > 1$, let $\mathcal{N}_{\boldsymbol{k}}(x)$ be the set of natural numbers $n \leq x$ such that $f(n + i - 1) = k_i$ for $i = 1, \dots, r$.

We prove that

$$\#\mathcal{N}_{\boldsymbol{k}}(x) = A_{\boldsymbol{k}} x + O_r\big(x^{\alpha_r}\big),$$

where $A_{\boldsymbol{k}} \geq 0$ depends only on $\boldsymbol{k}$ and $\alpha_r \in (0, 1)$ depends only on r. Moreover, we provide a characterization of the $\boldsymbol{k}$'s such that $A_{\boldsymbol{k}} > 0$. This extends a previous result of the author, who considered the case $r = 1$.

1. Introduction

Let $n > 1$ be an integer with prime factorization $n = p_1^{a_1} \cdots p_s^{a_s}$, where $p_1 < \cdots < p_s$ are prime numbers and $a_1, \dots, a_s$ are positive integers. Several arithmetic functions of n defined in terms of the exponents $a_1, \dots, a_s$ have been studied, including: the product [12], the arithmetic mean [2, 3, 4, 5], and the maximum and minimum [7, 8, 11, 13] of the exponents.

Let f be the arithmetic function defined by $f(1) := 0$ and $f(n) := \#\{a_1, \dots, a_s\}$, that is, $f(n)$ is the number of distinct exponents in the prime factorization of n. For every natural number k, let $\mathcal{N}_k$ be the set of positive integers n such that $f(n) = k$. In a previous paper, the author proved the following result [10, Corollary 1.1].

Theorem 1.1. *For each positive integer k, there exists $A_k > 0$ such that*

$$\#\mathcal{N}_k(x) = A_k x + O_\varepsilon\big(x^{1/2+\varepsilon}\big),$$

for all $x > 1$ and $\varepsilon > 0$.

We extend Theorem 1.1 by considering the values of f on r-tuples of consecutive integers. For $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{Z}_{>0}^r$, let $\mathcal{N}_{\boldsymbol{k}}$ be the set of positive integers n such that $f(n+i-1) = k_i$ for $i = 1, \dots, r$. Our first result is the following.

Theorem 1.2. *For each $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{Z}_{>0}^r$, there exists $A_{\boldsymbol{k}} \geq 0$ such that*

$$\#\mathcal{N}_{\boldsymbol{k}}(x) = A_{\boldsymbol{k}} x + O_r\big(x^{\alpha_r}\big), \tag{1}$$

for all $x > 1$, where $\alpha_r \in (0, 1)$ depends only on r. In particular, for $r \geq 2$ we can take $\alpha_r = 1 - 1/(45r + 3)$.

We did not try to minimize the exponent of x in the error term of (1), and some improvements are surely possible. However, our method cannot provide an exponent below $2/3$.

Note that, contrary to Theorem 1.1, the constant in the asymptotic formula of Theorem 1.2 may be equal to zero. The next result characterizes when this is the case. Recall that a natural number n is said to be *powerful* if $p \mid n$ implies $p^2 \mid n$, for every prime number p.

Theorem 1.3. *For $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{Z}_{>0}^r$, we have that $A_{\boldsymbol{k}} > 0$ if and only if there exists a positive integer n_0 such that $n_0 + i - 1$ is not powerful and $f(n_0 + i - 1) = k_i$ for $i = 1, \dots, r$.*

2010 *Mathematics Subject Classification.* Primary: 11N25; Secondary: 11N37, 11N64.

Key words and phrases. Prime factorization; exponents; powerful number.

† C. Sanna is a member of the INdAM group GNSAGA.

Note that Theorem 1.3 is not an effective criterion, since no upper bound for n_0 is provided. It seems likely that the only obstruction to the existence of n_0 comes from congruences modulo prime numbers not exceeding $r^{1/2}$. For example, it is easy to check that n_0 does not exist if $\boldsymbol{k} = (1,1,1,1)$. Indeed, for every four consecutive integers at least one of them, say m, is divisible by 4 and consequently if m is not powerful then $f(m) > 1$. However, we did not find a simple way to generalize these kind of reasonings. We leave such problem as an open question for the interested reader.

Theorem 1.2 is a result about consecutive natural numbers each of which has a small (prescribed) number of distinct exponents in its prime factorization. We remark that, on the opposite side, Aktaş and Ram Murty [1], motivated by a question of Recamán Santos [9], considered tuples of consecutive natural numbers each of which has all distinct exponents in its prime factorization, and showed that only finitely many (and effectively computable) such 23-tuples exist, assuming the explicit abc-conjecture.

2. Notation

We employ the Landau–Bachmann "Big Oh" notation O, as well as the associated Vinogradov symbols $\ll$ and $\gg$, with their usual meanings. Any dependence of the implied constants is explicitly stated or indicated with subscripts. For every set of positive integers $\mathcal{S}$, we put $\mathcal{S}(x) := \mathcal{S} \cap [1, x]$ for all $x > 1$. Throughout, the letter p is reserved for prime numbers. We write $(n_1, \dots, n_s)$ and $[n_1, \dots, n_s]$ to denote the greatest common divisor and least common multiple of the integers $n_1, \dots, n_s$, respectively. The first notation should not be mistaken for the s-tuple notation $(n_1, \dots, n_s)$, which we also use.

3. Preliminaries

We need some basic results on powerful numbers. Let $\mathcal{P}$ be the set of powerful numbers.

Lemma 3.1. *We have $\#\mathcal{P}(x) \ll x^{1/2}$, for all $x > 1$.*

Proof. See [6]. □

Lemma 3.2. *We have*

$$\sum_{\substack{\ell \in \mathcal{P} \\ \ell > y}} \frac{1}{\ell} \ll \frac{1}{y^{1/2}},$$

for all $y > 1$.

Proof. The claim follows easily from Lemma 3.1 by partial summation. □

For every natural number n, let $\lambda(n)$ be the *powerful part* of n, that is, the greatest powerful number that divides n.

Lemma 3.3. *For each positive integer r and for all $x, y > 1$, the number of positive integers $n \leq x$ such that $\lambda(n + i - 1) > y$, for some $i \in \{1, \dots, r\}$, is at most $O_r\big(x/y^{1/2}\big)$.*

Proof. By Lemma 3.2, the quantity at issue is at most

$$\sum_{i=1}^{r} \sum_{\substack{\ell \in \mathcal{P} \\ \ell > y}} \frac{x + i - 1}{\ell} \ll_r \frac{x}{y^{1/2}},$$

as claimed. □

For $\boldsymbol{a} = (a_1, \dots, a_r) \in \mathbb{Z}_{>0}^r$, $\boldsymbol{b} = (b_1, \dots, b_r) \in \mathbb{Z}_{\geq 0}^r$, and $x > 1$, let $\mathcal{Q}_{\boldsymbol{a},\boldsymbol{b}}(x)$ be the set of positive integers $n \leq x$ such that $a_1 n + b_1, \dots, a_r n + b_r$ are all squarefree. Furthermore, put

$$Q_{\boldsymbol{a},\boldsymbol{b}} := \prod_p \left(1 - \frac{\varrho_{\boldsymbol{a},\boldsymbol{b}}(p)}{p^2}\right),$$

where $\varrho_{\boldsymbol{a},\boldsymbol{b}}(p)$ is the number of $n \in \{1, \dots, p^2\}$ such that at least one of $a_1 n + b_1, \dots, a_r n + b_r$ is divisible by p^2. We need the following asymptotic formula for $\#\mathcal{Q}_{\boldsymbol{a},\boldsymbol{b}}(x)$.

Theorem 3.4. *Let* $\boldsymbol{a} = (a_1, \dots, a_r) \in \mathbb{Z}_{>0}^r$ *and* $\boldsymbol{b} = (b_1, \dots, b_r) \in \mathbb{Z}_{\geq 0}^r$*. We have*

$$\#\mathcal{Q}_{\boldsymbol{a},\boldsymbol{b}}(x) = Q_{\boldsymbol{a},\boldsymbol{b}}\, x + O\big([a_1, \dots, a_r]^3 x^{2/3}\big), \tag{2}$$

for all $x \geq \max b_i / \max a_i$ *with* $\log x / \log\log x \geq 25r$.

Proof. In the special case in which $a_i = 1$ for $i = 1, \dots, r$, Tsang proved an asymptotic formula for $\#\mathcal{Q}_{\boldsymbol{a},\boldsymbol{b}}(x)$ using the Buchstab–Rosser sieve [14, Theorem 1]. In the general case, the proof of (2) proceeds exactly as Tsang's proof, with only a few changes. First, [14, Eq. (10)] (which in our notation is $\varrho_{\boldsymbol{a},\boldsymbol{b}}(p) \leq r$ and is false if, for example, $p > r$ and $p^2 \mid (a_i, b_i)$ for all i) has to be replaced with the upper bound

$$\varrho_{\boldsymbol{a},\boldsymbol{b}}(p) \leq \begin{cases} r & \text{if } p \nmid [a_1, \dots, a_r], \\ p^2 & \text{otherwise,} \end{cases}$$

which in turn accounts for an extra factor $[a_1, \dots, a_r]^2$ on the right-hand sides of [14, Eqs. (7) and (8)] and their subsequent. Second, the ninth equation on [14, p. 269] becomes

$$\max_{\substack{n \leq x \\ 1 \leq i \leq r}} |a_i n + b_i| \leq (L+1)x,$$

which holds picking L so that $(L+1)/2 = \max a_i$ and assuming $x \geq \max b_i / \max a_i$. After these changes, Tsang's proof yields

$$\#\mathcal{Q}_{\boldsymbol{a},\boldsymbol{b}}(x) = Q_{\boldsymbol{a},\boldsymbol{b}}\, x + E(x),$$

where

$$-C\left(\frac{r}{\log x}(Lx)^{1/2} + x^{3/5}\frac{r^{12/5}}{(\log x)^{8/5}}\right) \leq \frac{E(x)}{[a_1, \dots, a_r]^2} \leq Cx^{2/3}\left(\frac{r}{\log x}\right)^{4/3}$$

and $C > 0$ is an absolute constant. Since $r \ll \log x$ (the condition $\log x / \log\log x \geq 25r$ is required in Tsang's proof) and $L \ll \max a_i \leq [a_1, \dots, a_r]$, we have that (2) follows easily after some simplifications. $\square$

We need the following easy lemma about a least common multiple of "almost" pairwise coprime integers.

Lemma 3.5. *Let* $\ell_1, \dots, \ell_r$ *and* D *be positive integers such that* $(\ell_i, \ell_j) \mid D$ *for* $i, j = 1, \dots, r$ *with* $i \neq j$*. Then* $[\ell_1, \dots, \ell_r] \gg_{D,r} \ell_1 \cdots \ell_r$.

Proof. Clearly, $R := \ell_1 \cdots \ell_r / [\ell_1, \dots, \ell_r]$ is an integer. We have to bound R in terms of D, r. Let p be a prime number. If $p \nmid D$, then from $(\ell_i, \ell_j) \mid D$ it follows that there exists at most one i such that $p \mid \ell_i$. Consequently, $p \nmid R$. If $p \mid D$ then, without loss of generality, we can assume that $\nu_p(\ell_1) \leq \cdots \leq \nu_p(\ell_r)$, where ν_p denotes the p-adic valuation. Hence, for every positive integer $i < r$ we have

$$\nu_p(\ell_i) = \min\{\nu_p(\ell_i), \nu_p(\ell_r)\} = \nu_p((\ell_i, \ell_r)) \leq \nu_p(D),$$

so that

$$\nu_p(R) = \nu_p(\ell_1) + \cdots + \nu_p(\ell_{r-1}) \leq (r-1)\nu_p(D).$$

We have thus proved that $R \mid D^{r-1}$ and the claim follows. $\square$

4. Proof of Theorem 1.2

Let us fix $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{Z}_{>0}^r$. We assume that $x > 1$ is sufficiently large so that $\log x / \log\log x \geq 25r$. In what follows, implied constants may depend on r. Let $y = y(x) > 1$ be a function of x to be chosen later. Let $\mathcal{N}_{\boldsymbol{k}}'$ be the set of $n \in \mathcal{N}_{\boldsymbol{k}}$ such that $n + i - 1$ is not powerful and $\lambda(n + i - 1) \leq y$ for $i = 1, \dots, r$. By Lemma 3.1 and Lemma 3.3, we have

$$\#\mathcal{N}_{\boldsymbol{k}}(x) = \#\mathcal{N}_{\boldsymbol{k}}'(x) + O\left(x^{1/2} + \frac{x}{y^{1/2}}\right). \tag{3}$$

Pick $n \in \mathcal{N}'_{\boldsymbol{k}}(x)$ and put, for a moment, $\ell_i := \lambda(n+i-1)$ for $i = 1, \ldots, r$. Since $f(n+i-1) = k_i$ and $n+i-1$ is not powerful, it follows that $f(\ell_i) = k_i - 1$. Moreover, since $\ell_i \mid n+i-1$, we get that $(\ell_i, \ell_j) \mid i-j$ for $i, j = 1, \ldots, r$.

In light of that, let $\mathcal{L}_{\boldsymbol{k}}$ be the set of r-tuples $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_r)$ of powerful numbers such that $f(\ell_i) = k_i - 1$ and $(\ell_i, \ell_j) \mid i-j$ for $i, j = 1, \ldots, r$. Also, let $\mathcal{L}_{\boldsymbol{k}}(y)$ be the subset of $\mathcal{L}_{\boldsymbol{k}}$ consisting of the $\boldsymbol{\ell}$'s such that $\ell_i \leq y$ for $i = 1, \ldots, r$, and put $\mathcal{L}'_{\boldsymbol{k}}(y) := \mathcal{L}_{\boldsymbol{k}} \setminus \mathcal{L}_{\boldsymbol{k}}(y)$. Furthermore, for each $\boldsymbol{\ell} \in \mathcal{L}_{\boldsymbol{k}}(y)$, let $\mathcal{N}'_{\boldsymbol{k},\boldsymbol{\ell}}(x)$ be the set of $n \in \mathcal{N}'_{\boldsymbol{k}}(x)$ such that $\lambda(n+i-1) = \ell_i$ for $i = 1, \ldots, r$. Clearly, the sets $\mathcal{N}'_{\boldsymbol{k},\boldsymbol{\ell}}(x)$, with $\boldsymbol{\ell} \in \mathcal{L}_{\boldsymbol{k}}(y)$, constitute a partition of $\mathcal{N}'_{\boldsymbol{k}}(x)$.

Pick $\boldsymbol{\ell} \in \mathcal{L}_{\boldsymbol{k}}(y)$. Let us count the number of $n \in \mathcal{N}'_{\boldsymbol{k},\boldsymbol{\ell}}(x)$. The condition $\lambda(n+i-1) = \ell_i$ is equivalent to $\ell_i \mid n+i-1$, $(\ell_i, m_i) = 1$, and m_i is squarefree, where $m_i := (n+i-1)/\ell_i$ for $i = 1, \ldots, r$. In turn, $\ell_i \mid n+i-1$ and $(\ell_i, m_i) = 1$ are equivalent to $n \equiv \ell_i s_i - i + 1 \pmod{\ell_i^2}$, for some $s_i \in \{1, \ldots, \ell_i\}$ with $(s_i, \ell_i) = 1$. Letting $L := [\ell_1, \ldots, \ell_r]$ and putting these r congruences moduli $\ell_1^2, \ldots, \ell_r^2$ together using the Chinese Remainder Theorem, we get that n modulo L^2 must belong to a set $\Omega_{\boldsymbol{\ell}} \subseteq \{1, \ldots, L^2\}$ completely determined by $\boldsymbol{\ell}$. In particular,

$$\#\Omega_{\boldsymbol{\ell}} \leq \varphi(\ell_1) \cdots \varphi(\ell_r) \leq \ell_1 \cdots \ell_r \leq y^r, \tag{4}$$

since there are at most $\varphi(\ell_i)$ choices for each s_i. Thus $n = L^2 u + v$, for some integers u, v satisfying $0 \leq u \leq (x-v)/L^2$ and $v \in \Omega_{\boldsymbol{l}}$. Finally, $m_i = a_i u + b_i$, where $\boldsymbol{a}(\boldsymbol{\ell}) = (a_1, \ldots, a_r)$ and $\boldsymbol{b}(\boldsymbol{\ell}, v) = (b_1, \ldots, b_r)$ are integer vectors defined by $a_i = L^2/\ell_i$ and $b_i = (v+i-1)/\ell_i$ for $i = 1, \ldots, r$. Note that

$$[a_1, \ldots, a_r] \leq L^2 \leq (\ell_1 \cdots \ell_r)^2 \leq y^{2r}. \tag{5}$$

Hence, we are asking that $a_1 u + b_1, \ldots, a_r u + b_r$ are all squarefree and greater than 1 (this latter condition to ensure that each $n+i-1$ is not powerful). Therefore, setting

$$S_{\boldsymbol{\ell}} := \sum_{v \,\in\, \Omega_{\boldsymbol{l}}} Q_{\boldsymbol{a}(\boldsymbol{\ell}), \boldsymbol{b}(\boldsymbol{\ell}, v)},$$

we obtain

$$\begin{aligned} \#\mathcal{N}'_{\boldsymbol{k},\boldsymbol{\ell}}(x) &= \sum_{v \,\in\, \Omega_{\boldsymbol{l}}} \left(\#\mathcal{Q}_{\boldsymbol{a}(\boldsymbol{\ell}), \boldsymbol{b}(\boldsymbol{\ell}, v)}\left(\frac{x - v}{[\ell_1, \ldots, \ell_r]^2} \right) + O(1) \right) \\ &= \sum_{v \,\in\, \Omega_{\boldsymbol{l}}} \left(Q_{\boldsymbol{a}(\boldsymbol{\ell}), \boldsymbol{b}(\boldsymbol{\ell}, v)} \frac{x}{[\ell_1, \ldots, \ell_r]^2} + O\big(x^{2/3} y^{6r}\big) \right) \\ &= \frac{S_{\boldsymbol{\ell}}}{[\ell_1, \ldots, \ell_r]^2}\, x + O\big(x^{2/3} y^{7r}\big), \end{aligned} \tag{6}$$

where we employed Theorem 3.4, (5), and (4).

Noting that $S_{\boldsymbol{\ell}} \leq \#\Omega_{\boldsymbol{\ell}} \leq \ell_1 \cdots \ell_r$ and using Lemma 3.5 and Lemma 3.2, we get

$$\begin{aligned} \sum_{\boldsymbol{\ell} \,\in\, \mathcal{L}'_{\boldsymbol{k}}(y)} \frac{S_{\boldsymbol{\ell}}}{[\ell_1, \ldots, \ell_r]^2} &\leq \sum_{\boldsymbol{\ell} \,\in\, \mathcal{L}'_{\boldsymbol{k}}(y)} \frac{\ell_1 \cdots \ell_r}{[\ell_1, \ldots, \ell_r]^2} \ll \sum_{\boldsymbol{\ell} \,\in\, \mathcal{L}'_{\boldsymbol{k}}(y)} \frac{1}{\ell_1 \cdots \ell_r} \\ &\ll \left(\sum_{\ell \,\in\, \mathcal{P}} \frac{1}{\ell} \right)^{r-1} \sum_{\substack{\ell \,\in\, \mathcal{P} \\ \ell \,>\, y}} \frac{1}{\ell} \ll \frac{1}{y^{1/2}}. \end{aligned} \tag{7}$$

In particular, the series

$$A_{\boldsymbol{k}} := \sum_{\boldsymbol{\ell} \,\in\, \mathcal{L}_{\boldsymbol{k}}} \frac{S_{\boldsymbol{\ell}}}{[\ell_1, \ldots, \ell_r]^2} \tag{8}$$

converges.

Hence, summing (6) over all the $\boldsymbol{\ell} \in \mathcal{L}_{\boldsymbol{k}}(y)$, using $\#\mathcal{L}_{\boldsymbol{k}}(y) \ll y^{r/2}$ (by Lemma 3.1) and (7), we get

$$\begin{aligned}
\#\mathcal{N}'_{\boldsymbol{k}}(x) &= \sum_{\boldsymbol{\ell} \in \mathcal{L}_{\boldsymbol{k}}(y)} \#\mathcal{N}'_{\boldsymbol{k},\boldsymbol{\ell}}(x) = \sum_{\boldsymbol{\ell} \in \mathcal{L}_{\boldsymbol{k}}(y)} \left(\frac{S_{\boldsymbol{\ell}}}{[\ell_1, \dots, \ell_r]^2}\, x + O\big(x^{2/3} y^{7r}\big) \right) \\
&= A_{\boldsymbol{k}} x - \sum_{\boldsymbol{\ell} \in \mathcal{L}'_{\boldsymbol{k}}(y)} \frac{S_{\boldsymbol{\ell}}}{[\ell_1, \dots, \ell_r]^2}\, x + O\big(x^{2/3} y^{15r/2}\big) \\
&= A_{\boldsymbol{k}} x + O\left(\frac{x}{y^{1/2}} + x^{2/3} y^{15r/2} \right),
\end{aligned}$$

which together with (3) yields

$$\#\mathcal{N}_{\boldsymbol{k}}(x) = A_{\boldsymbol{k}} x + O\left(x^{1/2} + \frac{x}{y^{1/2}} + x^{2/3} y^{15r/2} \right).$$

At this point, picking $y = x^{2/(45r+3)}$ gives the desired asymptotic formula.

5. Proof of Theorem 1.3

Let us fix $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r_{>0}$. Suppose that there exists a positive integer n_0 such that $n_0 + i - 1$ is not powerful and $f(n_0 + i - 1) = k_i$ for $i = 1, \dots, r$. We shall prove that $A_{\boldsymbol{k}} > 0$. Put $\ell_i := \lambda(n_0 + i - 1)$ for $i = 1, \dots, r$. Also, let $L := [\ell_1, \dots, \ell_r]$ and define $\boldsymbol{a} = (a_1, \dots, a_r) \in \mathbb{Z}^r_{>0}$ and $\boldsymbol{b} = (b_1, \dots, b_r) \in \mathbb{Z}^r_{\geq 0}$ by $a_i := L^2/\ell_i$ and $b_i := (n_0 + i - 1)/\ell_i$ for $i = 1, \dots, r$. Note that, by construction, $\ell_i \mid a_i$, $(b_i, \ell_i) = 1$, and b_i is squarefree. Since $n_0 + i - 1$ is not powerful and $f(n_0 + i - 1) = k_i$, we have that $f(\ell_i) = k_i - 1$. Let u be a positive integer and set $n := L^2 u + n_0$. We have $n + i - 1 = \ell_i m_i$, where $m_i := a_i u + b_i$, for $i = 1, \dots, r$. In particular, $(\ell_i, m_i) = (\ell_i, b_i) = 1$. Thus $n \in \mathcal{N}_{\boldsymbol{k}}$ whenever $a_1 u + b_1, \dots, a_r u + b_r$ are all squarefree. By Theorem 3.4, this happens for a set of u with natural density $Q_{\boldsymbol{a},\boldsymbol{b}}$. Recalling that $b_1, \dots, b_r$ are all squarefree, for every prime number p we have that none of $a_1 p^2 + b_1, \dots, a_r p^2 + b_r$ is divisible by p^2 and, consequently, $\varrho_{\boldsymbol{a},\boldsymbol{b}}(p) < p^2$. Hence, $Q_{\boldsymbol{a},\boldsymbol{b}} > 0$ and $A_{\boldsymbol{k}} > Q_{\boldsymbol{a},\boldsymbol{b}}/L^2 > 0$, as desired.

Now suppose that $A_{\boldsymbol{k}} > 0$. We shall prove that there exists a positive integer n_0 such that $n_0 + i - 1$ is not powerful and $f(n_0 + i - 1) = k_i$ for $i = 1, \dots, r$. Thanks to Theorem 1.2 and Lemma 3.1, we have

$$\begin{aligned}
&\#\big(\mathcal{N}_{\boldsymbol{k}}(x) \setminus \{n \leq x : n + i - 1 \in \mathcal{P} \text{ for some } i \in \{1, \dots, r\}\}\big) \\
&\qquad \geq A_{\boldsymbol{k}} x + O_r\big(x^{\alpha_r}\big) - O_r\big(x^{1/2}\big) \gg_{\boldsymbol{k}} x > 0,
\end{aligned}$$

for sufficiently large x, and the existence of n_0 follows.

6. Acknowledgments

The author thanks Daniele Mastrostefano (University of Warwick) for suggestions that improved the paper.

References

1. K. Aktaş and M. Ram Murty, *On the number of special numbers*, Proc. Indian Acad. Sci. Math. Sci. **127** (2017), no. 3, 423–430.
2. Hui Zhong Cao, *On the average of exponents*, Northeast. Math. J. **10** (1994), no. 3, 291–296.
3. J.-M. De Koninck, *Sums of quotients of additive functions*, Proc. Amer. Math. Soc. **44** (1974), 35–38.
4. J.-M. De Koninck and A. Ivić, *Sums of reciprocals of certain additive functions*, Manuscripta Math. **30** (1979/80), no. 4, 329–341.
5. R. L. Duncan, *On the factorization of integers*, Proc. Amer. Math. Soc. **25** (1970), 191–192.
6. S. W. Golomb, *Powerful numbers*, Amer. Math. Monthly **77** (1970), 848–855.
7. I. Kátai and M. V. Subbarao, *On the maximal and minimal exponent of the prime power divisors of integers*, Publ. Math. Debrecen **68** (2006), no. 3-4, 477–488.
8. I. Niven, *Averages of exponents in factoring integers*, Proc. Amer. Math. Soc. **22** (1969), 356–360.

9. B. Recamán Santos, *Consecutive numbers with mutually distinct exponents in their canonical prime factorization*, MathOverflow (2015), https://mathoverflow.net/questions/201489.
10. C. Sanna, *On the number of distinct exponents in the prime factorization of an integer*, Proc. Indian Acad. Sci. Math. Sci. **130** (2020), no. 1, Paper No. 27.
11. K. Sinha, *Average orders of certain arithmetical functions*, J. Ramanujan Math. Soc. **21** (2006), no. 3, 267–277.
12. D. Suryanarayana and R. Sitaramachandra Rao, *The number of square-full divisors of an integer*, Proc. Amer. Math. Soc. **34** (1972), 79–80.
13. ———, *On the maximum and minimum exponents in factoring integers*, Arch. Math. (Basel) **28** (1977), no. 3, 261–269.
14. K. M. Tsang, *The distribution of r-tuples of squarefree numbers*, Mathematika **32** (1985), no. 2, 265–275 (1986).

Politecnico di Torino, Department of Mathematical Sciences
Corso Duca degli Abruzzi 24, 10129 Torino, Italy
Email address: carlo.sanna.dev@gmail.com