A formulation of volumetric growth as a mechanical problem subjected to non-holonomic and rheonomic constraint

Alfio Grillo[1] and Salvatore Di Stefano[2]

[1] *Dip. di Scienze Matematiche "G. L. Lagrange", Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129, Torino, Italia. (alfio.grillo@polito.it)*[*]

[2] *Dip. di Ingegneria Civile, Ambientale, del Territorio, Edile e di Chimica, Politecnico di Bari, Via E. Orabona 4, 70125, Bari, Italia. (salvatore.distefano@poliba.it)*

Abstract

Starting from a reformulation of the mass balance law based on the Bilby-Kröner-Lee (BKL-) decomposition of the deformation gradient tensor, we study some peculiar mechanical aspects of growth in a monophasic continuum by regarding the reformulated mass balance equation as a non-holonomic and rheonomic constraint. Such constraint restricts the admissible rates of the growth tensor, i.e., one of the two factors of the BKL-decomposition, to comply with a growth law provided phenomenologically. For our purposes, we put the constraint in Pfaffian form, and treat time as a fictitious, additional Lagrangian parameter, subjected to the condition that its rate must be unitary. Then, by taking some suggestions from the literature, we assume the existence of generalized forces conjugated with the virtual variations of the growth tensor, and we write a constrained version of the Principle of Virtual Work (PVW) that leads to a mixed boundary value problem whose unknowns are the motion, the growth tensor, and the Lagrange multipliers of the considered theory. This allows to extrapolate a physical interpretation of the role that the growth-conjugated forces play on the components of the growth tensor, especially on the distortional ones, i.e., those that are not directly related to the variation of mass of the body. The core message of our work is conceptual: we show that the growth laws usually encountered in the literature, which are prescribed phenomenologically, but may be difficult to justify theoretically, can be put in the framework of the Principle of Virtual Work by regarding them as constraints. Moreover, we retrieve more particularized frameworks of growth available in the literature, while being able to switch to a theory of growth of grade one, such as a Cahn-Hilliard model of growth.

Keywords

Growth mechanics; Bilby-Kröner-Lee multiplicative decomposition; Non-holonomic constraints; Lagrange multipliers; Principle of Virtual Work; Dissipation; Cahn-Hilliard model.

1 Introduction

In Mechanics, the Principle of Virtual Work (PVW)[1] constitutes a well-established paradigmatic method for determining the conditions of (dynamic) equilibrium for systems with finite number of

---

[*]Corresponding author

[1]In this work, we refer to the Principle of Virtual Work (PVW) in a generalized sense, thereby including also D'Alembert's Principle. In Continuum Mechanics, this can be done by counting inertial forces among the body forces

degrees of freedom (see e.g. [87, 14, 17, 47]). Even in the presence of holonomic constraints, be they rheonomic (i.e., explicitly time-dependent) or not, or of non-holonomic and scleronomic (i.e., not explicitly dependent on time) constraints, the *Lagrange multiplier method* makes the constrained version of the PVW a straightforward generalization of the case in which such constraints are absent [87]. However, to the best of our knowledge, the formulation of the PVW becomes less obvious when the considered constraints are non-holonomic *and* rheonomic, although constraints of this type are the focus of several studies [106, 92, 121, 95, 104, 91], especially after the formulation of the so-called *vakonomic dynamics* due to Kozlov [82, 83, 84, 85].

In Continuum Mechanics, the PVW has a rather long history, too (see e.g. [52]), and, as reported by Germain [53], it has been employed in an increasingly consistent manner for determining the balance of forces at the basis of the continuum theories developed through the years. By virtue of the intrinsic elegance of its formulation, which has its origin in the concept of duality between kinematics and dynamics, the PVW puts naturally Mechanics in the framework of Differential Geometry, as suggested by Epstein and Segev [42], Marsden and Hughes [98], and many other authors.

In addition to the problems involving "non-classical" continua, such as beams and plates [33] modeled after Cosserat media [29], materials with microstructure [53, 103, 119, 21], micromorphic and micropolar media [60, 43], or multipolar media [61], and generalized continua [34], the PVW has been used extensively also in the context of the mechanics of inelastic phenomena, such as standard and strain gradient plasticity [32, 75], where it served as a point of departure for re-interpreting the theories proposed, for example, by Aifantis [3, 4], or for developing new theories [71, 72, 73], and performing studies based on such theories [13, 69].

In some of the above referenced works (see e.g. [71, 73]), the theory is accompanied by constraints imposed on the tensor field that describes the plastic distortions, which are indeed assumed to be isochoric, and associated with null plastic spin. In this respect, the elastoplasticity developed by Gurtin and Anand [71, 73] is only one example of continuum theories with internal constraints, i.e., constraints that, rather than being expressed through contact conditions between bodies, or through Dirichlet boundary conditions on one or more kinematic descriptors of a given body, restrict the admissibility of such descriptors in the internal points of the body itself. In fact, many other examples of internal constraints may be cited, which can be either holonomic or non-holonomic, and some of those have been reviewed by Capriz and Podio Guidugli [22] for *"oriented materials"*, and by Batra [18] and Carlson *et al.* [23] for thermo-elasticity and hyperelasticity, respectively.

In our opinion, it is important to emphasize that, in those theories mentioned above, in which the PVW is used to study materials with microstructure, the PVW is formulated by extending the kinematics of the "classical" continua[2] so as to include the structural degrees of freedom of

per unit volume [75].

[2]By *"classical" continua*, we mean continua that do not possess an *active* microstructure. For a continuum of this type, the microstructure, if at all considered, evolves *passively* under the action of either kinematic or dynamic entities, such as the deformation gradient tensor or Cauchy stress tensor, respectively, that are inherent to changes of configuration of the continuum under study. However, no specific micro-structural descriptor is introduced, be it viewed as an internal variable or as a representation of a structural degree of freedom. For example, fiber-reinforced materials can be studied as "classical" continua when the evolution of their microstructure, consisting in a reorientation of the fibers, is assumed to be a passive consequence of their deformation (see e.g. [44, 118] and the references therein). On the other hand, they can also be modeled as non-classical materials, when their microstructure is assumed to be active. In this case, the reorientation of the fibers is a dynamic process, coupled with the deformation, but virtually independent of it, that represents the manifestation of one or more micro-structural degrees of freedom [107]. One of these can be, for instance, the mean angle of the probability density distribution

the materials under study. Indeed, whereas the kinematics of a classical continuum is limited to describe its motion in the three-dimensional Euclidean space, the extended kinematics describes also the evolution of the microstructure of the continuum itself. In fact, this is achieved through the introduction, for a given body, of suitable kinematic descriptors, which, along with their variations, are virtually independent of the changes of shape of the body, and only have to be compatible with the internal constraints that are possibly present.

The concept of extended kinematics is at the basis also of the theories of plasticity proposed by Cermelli *et al.* [24], and Gurtin and Anand [71, 73], in which the Bilby-Kröner-Lee (BKL) decomposition of the deformation gradient tensor is introduced, and the factor of such decomposition termed *tensor of plastic distortions* is taken as the descriptor of the structural changes that, in a body, are brought about by plasticity. In particular, Cermelli *et al.* [24] define stress-like generalized forces, which they call *"couple densities"*[3], and study their balance under the constraint of isochoric plastic distortions. Moreover, in the investigation of the dissipation inequality, they compute the power that the *"couple densities"* produce on the rate of the tensor of plastic distortions. In these respects, our approach has some similarities with the works by Cermelli *et al.* [24], and Fried and Sellers [50], although, as discussed in detail below, our results are found within the context of growth, and following a procedure explicitly based on a constrained version of the PVW for the case of non-holonomic and rheonomic constraints, and on the use of the Lagrange multiplier technique.

Within a line of thought similar to the one followed by Cermelli *et al.* [24], the idea of the extended kinematics summarized above was adopted by DiCarlo and Quiligotti [38] in the context of Biomechanics for addressing growth and remodeling. These processes are both anelastic, and consist of the variation of mass and change of material properties of biological tissues [117] or cellular complexes [49, 48, 110, 57, 35], respectively. In fact, in several biologically relevant situations, both growth and remodeling are described by having recourse to the BKL decomposition [115], or to similar decompositions (see e.g. [39]), and the factor of the decomposition employed that accounts for the anelastic distortions accompanying growth or remodeling is sometimes referred to as *growth tensor* or *remodeling tensor*. This tensor, thus, replaces the tensor of plastic distortions encountered in elastoplasticity. Yet, a fundamental difference exists between plasticity and remodeling, on the one side, and growth, on the other side. This difference is due to the fact that, whereas in elastoplasticity and remodeling the tensor of inelastic distortions is often assumed to be isochoric [116, 111], the growth tensor is required to comply with the mass balance law in the following sense: the trace of a suitably defined rate of the growth tensor can be set equal to the normalized source/sink of mass describing growth (see e.g. [41, 8, 97, 96]). The relation obtained this way between the source/sink of mass and the growth tensor can be interpreted in different ways. In particular, if the source/sink of mass is supplied from the outset, e.g. phenomenologically [8, 10, 100, 58], the relation in question amounts to translating the mass balance law into a *non-holonomic and rheonomic constraint* on the growth tensor.

The just given interpretation of the mass balance constitutes the core of our present work. To expand this idea, we first have to review some crucial points of the derivation outlined by DiCarlo

[16, 64, 67, 63, 31] that describes the orientation of the fibers in composite materials with statistical fiber distributions (see e.g. [88, 45, 93, 78]and the references therein.)

[3]A list of works that introduce generalized forces similar to the *"couple densities"* [24] and study the balance laws associated with them can be found in the work by Cermelli *et al.* [24], where these laws are called *"ancillary"*, and in the work by Fried and Sellers [50].

and Quiligotti [38], who base their approach to the mechanics of growth on the PVW. For their purposes, indeed, they regard the growth tensor as the basic descriptor of the growth kinematics, introduce the generalized virtual velocity associated with it, define a set of generalized forces dual to the virtual velocity of the growth tensor, and obtain the balance of these forces as a consequence of the localization of the integral equation expressing the PVW. We remark, however, that, although DiCarlo and Quiligotti [38] speak of growth (and remodeling) in their paper, they do not mention the mass balance law, nor do they discuss any *a priori* condition that the growth tensor should fulfill, at least not explicitly. A review of their approach and its connection with the one presented hereafter is the subject of a forthcoming work of ours.

Compared with the formulation summarized above, and with others that have come afterwards (see e.g. [107]), we believe that the approach that we are proposing is novel because it treats the mass balance law as a *non-holonomic and rheonomic constraint* on the growth tensor, and provides a *constrained version* of the PVW relying on the Lagrange multiplier technique. More specifically, by mimicking the PVW employed in computational mechanics for systems subjected to internal constraints, as is the case, e.g., for incompressibility [79, 20], and adapting the procedure to the non-holonomic and rheonomic case, we append the constraint on the growth tensor to the "standard version" of the PVW [38] in order to determine the full set of equations that govern the dynamics of the growing body under investigation. To the best of our knowledge, this procedure is not standard for the case of non-holonomic and rheonomic constraints and, indeed, it has been obtained by adapting some results put forward by Nadile [106] and Llibre *et al.* [95] in completely different frameworks.

Although being conceived for the mechanics of volumetric growth, our results are meant to apply to all those situations in which the kinematic variables describing the structural changes of a body must satisfy one or more *a priori* conditions, dictated, for example, by the phenomenology under study.

In our work, we also retrieve some results obtained by Gurtin [74], who provides a rational derivation of the Cahn-Hilliard model for mass transport, and we reinterpret them in light of the constrained version of the PVW within the context of growth mechanics. Our purpose, in this case, is to show that, framed as we do in our approach, the formulation developed by Gurtin [74] can be regarded as a "precursor" of a growth problem (although his paper, in fact, was published two years later than the paper by Rodriguez *et al.* [115]). In this respect, our study aims to build connections with other formulations of growth (see e.g. [41]) and to highlight both the similarities and the conceptual differences among the considered approaches.

In order to give prominence to the theoretical results of our study, we prefer to show no numerical simulations here, and to dedicate another work to the numerical aspects of our approach.

In our opinion, our analysis may contribute to construct a unified formulation of inelastic processes, based on the paradigmatic procedure of the PVW, which is made compliant, when necessary, with phenomenological laws treated as constraints.

2 General Notation

In this section, we briefly give the notation used throughout the rest of our work. To this end, we introduce the three-dimensional Euclidean space $\mathscr{S}$, the reference placement of the body under study, i.e., $\mathscr{B} \subset \mathscr{S}$, the time line $\mathscr{I}$, the time interval $[t_{\rm in}, t_{\rm fin}] \subset \mathscr{I}$, and the map $\chi(\,\cdot\,, t) : \mathscr{B} \to \mathscr{S}$, which, for every time $t \in [t_{\rm in}, t_{\rm fin}] \subset \mathscr{I}$, transforms univocally each point X of $\mathscr{B}$ into the point

$x = \chi(X,t) \in \mathscr{S}$, so that $\chi(\mathscr{B},t) =: \mathscr{B}_t \subset \mathscr{S}$ represents the change of shape of the body from its reference placement to the placement $\mathscr{B}_t$ attained at time t. Note that, with a slight abuse of terminology, we shall refer to this map simply as "motion" in the sequel. In addition, we define the auxiliary maps

$$\mathcal{X} : \mathscr{B} \times \mathscr{I} \to \mathscr{B}, \qquad \mathcal{X}(X,t) = X, \tag{1a}$$

$$\mathcal{T} : \mathscr{B} \times \mathscr{I} \to \mathscr{I}, \qquad \mathcal{T}(X,t) = t \tag{1b}$$

(see e.g [46] and the references therein), which enjoy the properties

$$T\mathcal{X}(X,t) = \boldsymbol{I}(X,t), \qquad \dot{\mathcal{X}}(X,t) = \boldsymbol{0}, \qquad \forall\ (X,t) \in \mathscr{B} \times \mathscr{I}, \tag{2a}$$

$$\mathrm{Grad}\mathcal{T}(X,t) = \boldsymbol{0}, \qquad \dot{\mathcal{T}}(X,t) = 1, \qquad \forall\ (X,t) \in \mathscr{B} \times \mathscr{I}, \tag{2b}$$

where $T\mathcal{X}(X,t)$ is the tangent map of $\mathcal{X}(\,\cdot\,,t)$ at $X \in \mathscr{B}$, $\boldsymbol{I}(X,t) : T_X\mathscr{B} \to T_X\mathscr{B}$ is the identity tensor, and $T_X\mathscr{B}$ is the tangent space of $\mathscr{B}$ attached at $X \in \mathscr{B}$ (see e.g. [98]). For completeness, we also define the transpose of the identity tensor, i.e., $\boldsymbol{I}^{\mathrm{T}}(X,t) : T_X^*\mathscr{B} \to T_X^*\mathscr{B}$, with $T_X^*\mathscr{B}$ being the dual space of $T_X\mathscr{B}$, as well as the metric tensor associated with $\mathscr{B}$, i.e., $\boldsymbol{G}(X,t)$, for which it identically holds that $\dot{\boldsymbol{G}}(X,t) = \boldsymbol{0}$, for all $(X,t) \in \mathscr{B} \times \mathscr{I}$. By virtue of the maps χ and $\mathcal{T}$, any function $f : \mathscr{B}_t \times \mathscr{I} \to \mathbb{K}$, with $\mathbb{K}$ representing the set of real numbers or any vector or tensor space, can be expressed as a function of the points of $\mathscr{B}$ and time by means of the composition with the pair $(\chi,\mathcal{T})$, i.e., $f \circ (\chi,\mathcal{T}) : \mathscr{B} \times \mathscr{I} \to \mathbb{K}$, provided the composition makes sense.

Granted the usual differentiability properties of $\chi(\,\cdot\,,t)$, we introduce the deformation gradient tensor $\boldsymbol{F}(X,t) := T\chi(X,t) : T_X\mathscr{B} \to T_x\mathscr{S}$, where $T\chi(X,t)$ is the *tangent map* of $\chi(\,\cdot\,,t)$ at $X \in \mathscr{B}$, with $t \in [t_{\mathrm{in}}, t_{\mathrm{fin}}]$, while $T_x\mathscr{S}$ is the tangent space of $\mathscr{S}$ attached at $x \equiv \chi(X,t) \in \mathscr{S}$ (see e.g. [98]). We also introduce the right Cauchy-Green deformation tensor $\boldsymbol{C}(X,t) := \boldsymbol{F}^{\mathrm{T}}(x,t)\boldsymbol{g}(x,t)\boldsymbol{F}(X,t)$, with $x \equiv \chi(X,t)$, and where $\boldsymbol{g}(x,t)$ is the metric tensor at $x \in \mathscr{S}$. It is understood that $\partial_t\boldsymbol{g}(x,t) = \boldsymbol{0}$, for all $x \in \mathscr{S}$ and $t \in \mathscr{I}$. We remark that $\boldsymbol{F}^{\mathrm{T}}(x,t)$ is defined as $\boldsymbol{F}^{\mathrm{T}}(x,t) : T_x^*\mathscr{S} \to T_X^*\mathscr{B}$, where $T_x^*\mathscr{S}$ is the dual space of $T_x\mathscr{S}$, and the notation $\boldsymbol{F}^{\mathrm{T}} \circ (\chi,\mathcal{T})$ should be used, when it is necessary to rephrase $\boldsymbol{F}^{\mathrm{T}}$ as a function of the points of $\mathscr{B}$ and time. However, when there it no room for confusion, to reduce the notational burden, we omit the composition with $(\chi,\mathcal{T})$, and tacitly redefine $\boldsymbol{F}^{\mathrm{T}}$ as a function of the points of $\mathscr{B}$ and time.

We recall the BKL decomposition, $\boldsymbol{F}(X,t) = \boldsymbol{F}_{\mathrm{e}}(X,t)\boldsymbol{K}(X,t)$, where $\boldsymbol{F}_{\mathrm{e}}(X,t)$ and $\boldsymbol{K}(X,t)$ are referred to as *tensor of elastic distortions* and *growth tensor*, respectively. The latter one, indeed, describes the anelastic distortions induced in the body by the variation of mass due to growth. For future use, we set $J := \det \boldsymbol{F}$, $J_{\mathrm{e}} := \det \boldsymbol{F}_{\mathrm{e}}$, and $J_{\boldsymbol{K}} := \det \boldsymbol{K}$. Here, we do not fuss over the physical meanings attributed to the BKL decomposition, since a huge literature is available on the topic (see e.g. [115, 117, 8, 38, 80, 51, 96, 6, 64, 19, 81, 86, 112, 65, 62, 114, 36, 5, 31]). However, we mention that, although $\boldsymbol{K}$ is in principle a two-point tensor, in the sequel we consider it a mixed tensor from $T_X\mathscr{B}$ into a "relaxed" copy of this vector space (see e.g. [28, 36]).

Finally, the maps $\mathcal{X}$ and $\mathcal{T}$ defined in Equations (1a) and (1b) are useful to express the explicit dependence of physical quantities on the points of $\mathscr{B}$ and time. For instance, if h a is physical quantity that can be written as $h(X,t) = \hat{h}(\boldsymbol{F}(X,t), \boldsymbol{K}(X,t), X, t)$, where $\hat{h}$ is the constitutive representation of h, then the notation $h = \hat{h} \circ (\boldsymbol{F}, \boldsymbol{K}, \mathcal{X}, \mathcal{T})$ applies.

3 Mass balance as a constraint on the growth tensor

In this section, we briefly review the mass balance law of the body under study in the context of the theory of volumetric growth based on the BKL decomposition. To this end, we recall that such decomposition permits to rewrite the balance of mass of the considered body as a relation between the trace of the rate $\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}$ and the normalized source/sink of mass R_γ, i.e. [41, 97, 9, 8],

$$\mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}) \equiv \boldsymbol{K}^{-\mathrm{T}} : \dot{\boldsymbol{K}} = R_\gamma, \qquad \text{in } \mathscr{B} \times [t_{\mathrm{in}}, t_{\mathrm{fin}}], \tag{3}$$

where R_γ is defined as $R_\gamma := J r_\gamma / \varrho_{\mathrm{R}}$, r_γ is the "true" source/sink of mass, and ϱ_{R} is the mass density per unit volume of the body's reference placement. Equation (3) is obtained from the mass balance law, expressed in local form and with respect to the reference placement of the body, i.e., $\dot{\varrho}_{\mathrm{R}} = J r_\gamma$, by exploiting the relation $\varrho_{\mathrm{R}} = J_{\boldsymbol{K}} \varrho_\nu$, where ϱ_ν is the mass density per unit volume of the body's natural state, and using the identity $\dot{J}_{\boldsymbol{K}} = J_{\boldsymbol{K}} \mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}})$, under the hypothesis that ϱ_ν is constant in time.

We assume that R_γ is prescribed phenomenologically through a *growth law* of the type [100, 99]

$$R_\gamma \equiv R_{\gamma(\mathrm{ph})} := \bar{R}_{\gamma(\mathrm{ph})} \circ (\wp, \omega) = \hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega), \qquad \text{in } \mathscr{B} \times [t_{\mathrm{in}}, t_{\mathrm{fin}}], \tag{4}$$

where ω is the mass fraction of the nutrient substances (e.g. glucose or oxygen) that promote the accretion of the tissue's mass, and $\wp := -\frac{1}{3}\mathrm{tr}\boldsymbol{\sigma}$ is the mechanical pressure in the tissue [100, 99], which, under suitable constitutive assumptions, can be expressed as $\wp = \hat{\wp} \circ (\boldsymbol{F}, \boldsymbol{K})$. As reported in [100, 99], the growth law (4) is calibrated in such a way that R_γ can be positive (mass accretion), null, or negative (mass resorption), depending on whether ω exceeds, equals, or goes below a certain threshold mass fraction, ω_{cr}. Finally, the dependence on $\wp$ is introduced since it is believed that pressure, when it is positive, has the capability of slowing down the rate of mass accretion [25], whereas it has no relevant influence on $R_{\gamma(\mathrm{ph})}$, when it is negative.

To complete the description of $R_{\gamma(\mathrm{ph})}$, the evolution of the nutrients' mass fraction has to be described. This is done by taking into account the mass balance law of the nutrients. In this work, we assume that they are free to move within the body, and that such a motion can be modeled in terms of Fickean diffusion. In particular, it can be shown that, within the monophasic framework[4], and written with respect to the reference placement of the body, the mass balance law of the nutrients becomes the diffusion-reaction equation, defined in $\mathscr{B} \times [t_{\mathrm{in}}, t_{\mathrm{fin}}]$, given by

$$J_{\boldsymbol{K}} \varrho_\nu \dot{\omega} - \mathrm{Div}\big(J_{\boldsymbol{K}} \varrho_\nu \boldsymbol{D}\, \mathrm{Grad}\omega\big) = -J_{\boldsymbol{K}} \varrho_\nu r_{\mathrm{n}} \omega - J_{\boldsymbol{K}} \varrho_\nu [\hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)]\omega, \tag{5}$$

where r_{n} is the rate at which the nutrients are absorbed by the tumor, and $\boldsymbol{D} := \hat{\boldsymbol{D}} \circ (\boldsymbol{F}, \boldsymbol{K})$ is the diffusivity tensor (see e.g. [15, 35] for possible constitutive expressions of $\boldsymbol{D}$). Moreover, following [8, 99, 36, 113], we can equip Equation (5) with boundary and initial conditions of the type

$$\omega = \omega_{\mathrm{b}}, \qquad \text{on } \partial^\omega_{\mathrm{D}}\mathscr{B}, \tag{6a}$$

$$[-J_{\boldsymbol{K}} \varrho_\nu \boldsymbol{D}\, \mathrm{Grad}\omega]\boldsymbol{N} = \jmath_{\mathrm{b}}, \qquad \text{on } \partial^\omega_{\mathrm{N}}\mathscr{B}, \tag{6b}$$

$$\omega(X, t_{\mathrm{in}}) = \omega_{\mathrm{in}}(X), \qquad \text{in } \mathscr{B}, \tag{6c}$$

[4] For biological tissues and tumors, the monophasic framework is clearly much less descriptive than the biphasic, or the multiphasic, one [36, 69, 113]. However, it is sufficient for conveying the message contained in our work.

where ω_{b} and $\jmath_{\mathrm{b}}$ are the nutrients' mass fraction and flux imposed on the Dirichlet portion $\partial^{\omega}_{\mathrm{D}}\mathscr{B}$ and on the Neumann portion $\partial^{\omega}_{\mathrm{N}}\mathscr{B}$ of $\partial\mathscr{B}$ associated with ω, respectively.

Once $R_{\gamma(\mathrm{ph})}$ is assigned from the outset, we regard Equation (3) as an *a priori* restriction on the rate $\dot{\boldsymbol{K}}$, for given $\boldsymbol{F}$, $\boldsymbol{K}$, and ω, and, accordingly, we rewrite it as

$$\hat{\mathcal{C}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \omega) := \boldsymbol{K}^{-\mathrm{T}} : \dot{\boldsymbol{K}} - \hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega) = 0. \tag{7}$$

Equation (7) defines a *non-holonomic* and *rheonomic constraint* on $\boldsymbol{K}$ (see e.g. [87] for a classification of these constraints). The constraint is non-holonomic because it cannot be integrated, and, indeed, there exists no scalar function $f := \hat{f} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)$ whose time derivative coincides with $\hat{\mathcal{C}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \omega)$; it is rheonomic because $\hat{\mathcal{C}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \omega)$ depends on time not only through $\boldsymbol{F}$, $\boldsymbol{K}$ and $\dot{\boldsymbol{K}}$, which are kinematic variables of the model, but also through ω, as prescribed by $\hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)$.

Remark 3.1 (More general form of the constraint $\hat{\mathcal{C}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \omega) = 0$)
The expression of the growth law and of the corresponding constraint can be made more general than the ones in Equations (4) *and* (7) *by introducing a new function $\check{R}_{\gamma(\mathrm{ph})}$, such that $R_{\gamma(\mathrm{ph})} = \check{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \mathcal{X}, \mathcal{T})$, and the new function $\check{\mathcal{C}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \mathcal{X}, \mathcal{T})$, such that*

$$\check{\mathcal{C}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \mathcal{X}, \mathcal{T}) := \boldsymbol{K}^{-\mathrm{T}} : \dot{\boldsymbol{K}} - \check{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \mathcal{X}, \mathcal{T}) = 0, \tag{8}$$

where the composition with the maps $\mathcal{X}$ and $\mathcal{T}$ is meant to account for the explicit dependence of the growth law on material points and time virtually in all possible ways. A dependence of this type, for example, should be considered when an explicit expression of Equation (4) *features material parameters that are functions of the material points and time, rather than being constants, as assumed later. According to Equation* (8)*, the non-integrability of the constraint may be rephrased by saying that there exists no scalar function $f = \check{f} \circ (\boldsymbol{F}, \boldsymbol{K}, \mathcal{X}, \mathcal{T})$, whose time derivative coincides with $\check{\mathcal{C}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \mathcal{X}, \mathcal{T})$. This is imputable to the functional form of the growth law $\check{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \mathcal{X}, \mathcal{T})$. We also mention that, within the framework of bone mechanics, there exist mathematical models of growth (see e.g. [90]) in which the growth law $R_{\gamma(\mathrm{ph})}$ is given as a function of a biomechanical stimulus expressed through the convolution integral of the strain energy density of the material with a suitably defined kernel [90, 55].*

Remark 3.2 (The limit case of holonomic, rheonomic constraint)
The constraint (8) *turns out to be integrable if the growth law takes on the simple form $R_{\gamma} := [R_{\gamma\mathrm{p}} \circ \mathcal{X}][R_{\gamma\mathrm{t}} \circ \mathcal{T}]$, where $R_{\gamma\mathrm{p}}$ is a function of material points, and $R_{\gamma\mathrm{t}}$ is a function of time that admits primitives in $[t_{\mathrm{in}}, t_{\mathrm{fin}}]$. Indeed, in this case, the constraint reads*

$$\check{\mathcal{C}}_{\boldsymbol{K}} \circ (\boldsymbol{K}, \dot{\boldsymbol{K}}, \mathcal{X}, \mathcal{T}) := \boldsymbol{K}^{-\mathrm{T}} : \dot{\boldsymbol{K}} - [R_{\gamma\mathrm{p}} \circ \mathcal{X}][R_{\gamma\mathrm{t}} \circ \mathcal{T}] = 0, \tag{9}$$

and it can be obtained by requiring the vanishing of the total time derivative of the function

$$f := \check{f} \circ (\boldsymbol{K}, \mathcal{X}, \mathcal{T}) = \log\det \boldsymbol{K} - [R_{\gamma\mathrm{p}} \circ \mathcal{X}][S_{\gamma\mathrm{t}} \circ \mathcal{T}] + f_0 \circ \mathcal{X}, \tag{10}$$

where $S_{\gamma\mathrm{t}}$ is one primitive of $R_{\gamma\mathrm{t}}$ over $[t_{\mathrm{in}}, t_{\mathrm{fin}}]$, i.e., $\dot{S}_{\gamma\mathrm{t}}(t) = R_{\gamma\mathrm{t}}(t)$ for $t \in [t_{\mathrm{in}}, t_{\mathrm{fin}}]$, and f_0 is an arbitrary function of material points, only. Accordingly, Equation (10) *can be rephrased as a holonomic constraint that prescribes $\check{f} \circ (\boldsymbol{K}, \mathcal{X}, \mathcal{T})$ to remain constant in time over $[t_{\mathrm{in}}, t_{\mathrm{fin}}]$.*

Moreover, since, without loss of generality, the constant value of $\check{f} \circ (\boldsymbol{K}, \mathcal{X}, \mathcal{T})$ *can be assumed to be zero, the constraint becomes*

$$\check{f} \circ (\boldsymbol{K}, \mathcal{X}, \mathcal{T}) = 0 \quad \Rightarrow \quad \log\det \boldsymbol{K} - [R_{\gamma\mathrm{p}} \circ \mathcal{X}][S_{\gamma\mathrm{t}} \circ \mathcal{T}] = -f_0 \circ \mathcal{X}. \tag{11}$$

Finally, if $S_{\gamma\mathrm{t}}$ *is chosen as* $S_{\gamma\mathrm{t}}(t) := \int_{t_{\mathrm{in}}}^{t} R_{\gamma\mathrm{t}}(s)\mathrm{d}s$*, and the initial condition* $\det \boldsymbol{K}(X, t_{\mathrm{in}}) = 1$ *is imposed, then, we achieve the identification* $f_0(X) \equiv 0$*, which yields*

$$\det \boldsymbol{K}(X,t) \equiv J_{\boldsymbol{K}}(X,t) = \exp\left(R_{\gamma\mathrm{p}}(X) \int_{t_{\mathrm{in}}}^{t} R_{\gamma\mathrm{t}}(s)\mathrm{d}s\right), \qquad \forall\ (X,t) \in \mathscr{B} \times [t_{\mathrm{in}}, t_{\mathrm{fin}}]. \tag{12}$$

Hence, the growth problem is reformulated as a problem subjected to an a priori *condition on* $\det \boldsymbol{K}$*. This, in turn, could be understood as a "prescribed dilatation, or volumetric contraction, due to growth", and features some similarities with the theory of swelling [120]. Moreover, if the growth law were switched off, the condition* $\det \boldsymbol{K}(X,t) = 1$ *would be obtained, thereby recovering isochoric inelastic distortions.*

Before closing this section, we notice that, regardless of whether the constraint under study is expressed as in Equation (7) or as in Equation (8), a direct consequence of the introduction of the map $\mathcal{T}$ (see Equations (1b) and (2b) for its properties) is that the constraint can be rewritten as a *Pfaffian form* [95], i.e., in terms of new functions that formally depend on the rate $\dot{\mathcal{T}}$ as follows

$$\hat{\mathcal{V}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \dot{\mathcal{T}}, \omega) := \boldsymbol{K}^{-\mathrm{T}} : \dot{\boldsymbol{K}} - [\hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)]\dot{\mathcal{T}} = 0, \tag{13a}$$

$$\check{\mathcal{V}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \dot{\mathcal{T}}, \mathcal{X}, \mathcal{T}) := \boldsymbol{K}^{-\mathrm{T}} : \dot{\boldsymbol{K}} - [\check{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \mathcal{X}, \mathcal{T})]\dot{\mathcal{T}} = 0, \tag{13b}$$

where we have exploited Equation (2b).

4 Time as a constrained, fictitious Lagrangian parameter

To our knowledge, the non-holonomic and rheonomic nature of the constraint (7) presents some technical difficulties in the formulation of the Principle of Virtual Work (see e.g. [87]). Specifically, the main issue is that, when the method of Lagrange multipliers is invoked, the term $\hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)$ in Equation (7), or $\check{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \mathcal{X}, \mathcal{T})$ in Equation (8), cannot be combined, as it stands, with the virtual works expended on the virtual variations of χ and $\boldsymbol{K}$. This difficulty, however, can be circumvented by having recourse to an alternative formulation of the constraint, in which time is viewed as a *fictitious, additional Lagrangian parameter* of the considered problem. Before entering the details, we remark that this way of proceeding is not new *per se* (see e.g. [106, 95], and [87] for the case in which time is treated as an *"ignorable variable"*), and it can be put in our context on the basis of the rationale exposed in Appendix A1. Here, for the sake of conciseness, we say that the main reason for undertaking this path is to study the constraint expressed by Equation (7), or (8), within the setting of the Principle of Virtual Work. Indeed, regarding time as a Lagrangian parameter allows to introduce its virtual variations, along with a system of generalized, fictitious forces, dual to such variations and satisfying their own balance law. These forces produce virtual work against the virtual variations of time, and this virtual work can be combined with the work done on the same virtual variations by the term $\hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)$, or

$\check{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \mathcal{X}, \mathcal{T})$, multiplied by a suitable Lagrange multiplier. A more detailed discussion on this topic is done below in this section as well as in the next one, and in Appendix A1.

For the purposes outlined above, in addition to $\mathcal{T}$ in Equation (1b), we introduce

$$\mathfrak{T} : \mathscr{B} \times \mathscr{I} \to \mathscr{I}, \qquad \mathfrak{T}(X,t) = t_X. \tag{14}$$

In principle, the auxiliary map $\mathfrak{T}$ differs from $\mathcal{T}$ in that the re-mapped time $t_X = \mathfrak{T}(X,t)$ is not *a priori* required to be equal to $t = \mathcal{T}(X,t)$, whereas the latter equality is true by definition.

To clarify the introduction of $\mathfrak{T}$, and its relation with $\mathcal{T}$, let us notice that, formally, $\mathfrak{T}$ has the same "dignity" as χ and $\boldsymbol{K}$, and has the property of returning a unique instant of time $t_X = \mathfrak{T}(X,t) \in \mathscr{I}$, for each pair $(X,t) \in \mathscr{B} \times \mathscr{I}$, whereas $\chi(X,t) = x$ defines a unique position in space, and $\boldsymbol{K}(X,t)$ describes how the body elements of $T_X\mathscr{B}$ are relaxed at time $t \in \mathscr{I}$. Hence, while χ is a *space-like Lagrangian parameter*, and $\boldsymbol{K}$ is a *structural Lagrangian parameter*, $\mathfrak{T}$ could be termed *time-like Lagrangian parameter*, and, as a Lagrangian parameter of the theory, it can be associated with a dynamic equation [106]. Yet, $\mathfrak{T}$ is fictitious, because its evolution is known *a priori* on physical grounds. Indeed, for consistency with the Galileian laws of composition of velocities and accelerations, $\mathfrak{T}$ is restricted to produce, at most, the time translation

$$\mathfrak{T}(X,t) = \mathfrak{T}_0(X) + t = \mathfrak{T}_0(X) + \mathcal{T}(X,t) =: t_X \in \mathscr{I}, \quad \forall\ (X,t) \in \mathscr{B} \times \mathscr{I}, \tag{15}$$

where $\mathfrak{T}_0(X)$ is an arbitrary point-dependent time shift. In particular, Equation (15) guarantees the equality $\dot{\mathfrak{T}}(X,t) = 1$, for all $(X,t) \in \mathscr{B} \times \mathscr{I}$, which means that the "velocity of time" is equal to unity for all body points and for all times. Note that Equation (15) is a direct consequence of the fact that, in Galileian mechanics, *time is absolute*, since it is postulated to flow at the same rate for all observers. In fact, the equality $\dot{\mathfrak{T}}(X,t) = 1$ recasts Equation (15) in differential form, and can be interpreted as a *constraint* on $\mathfrak{T}$, or, better, on $\dot{\mathfrak{T}}$, which can be written as

$$\hat{\mathcal{V}}_{\mathfrak{T}}(\dot{\mathfrak{T}}(X,t), \dot{\mathcal{T}}(X,t)) := \dot{\mathfrak{T}}(X,t) - \dot{\mathcal{T}}(X,t) = \dot{\mathfrak{T}}(X,t) - 1 = 0, \tag{16}$$

where $\hat{\mathcal{V}}_{\mathfrak{T}} : \mathbb{R}^2 \to \mathbb{R}$ is defined by $\hat{\mathcal{V}}_{\mathfrak{T}}(a_1, a_2) = a_1 - a_2 = 0$, for all $(a_1, a_2) \in \mathbb{R}^2$. Finally, by having recourse to the composition of maps, we obtain

$$\hat{\mathcal{V}}_{\mathfrak{T}} \circ (\dot{\mathfrak{T}}, \dot{\mathcal{T}}) := \dot{\mathfrak{T}} - \dot{\mathcal{T}} = 0. \tag{17}$$

We denote by $T_t\mathscr{I}$ the one-dimensional tangent space of $\mathscr{I}$ at t, and by $T\mathscr{I} := \sqcup_{t \in \mathscr{I}} T_t\mathscr{I}$ the tangent bundle of $\mathscr{I}$. Moreover, we define $\delta\mathcal{T} : \mathscr{B} \times \mathscr{I} \to T\mathscr{I}$ and $\delta\mathfrak{T} : \mathscr{B} \times \mathscr{I} \to T\mathscr{I}$ such that, for each $(X,t) \in \mathscr{B} \times \mathscr{I}$, $\delta\mathcal{T}(X,t) \in T_t\mathscr{I}$ and $\delta\mathfrak{T}(X,t) \in T_t\mathscr{I}$ represent a *virtual time translation* and the *virtual displacement associated with* $\mathfrak{T}(X,t)$, respectively. We notice that, since $\delta\mathfrak{T}(X,t)$ is defined as a virtual displacement, it has to be compatible with the imposed constraints, and, thus, it must satisfy Equations (16) and (17) in the form [95],

$$\hat{\mathcal{V}}_{\mathfrak{T}}(\delta\mathfrak{T}(X,t), \delta\mathcal{T}(X,t)) = \delta\mathfrak{T}(X,t) - \delta\mathcal{T}(X,t) = 0, \quad \forall\ (X,t) \in \mathscr{B} \times \mathscr{I}, \tag{18a}$$

$$\hat{\mathcal{V}}_{\mathfrak{T}} \circ (\delta\mathfrak{T}, \delta\mathcal{T}) = \delta\mathfrak{T} - \delta\mathcal{T} = 0. \tag{18b}$$

Before going further, we notice that the constraint in Equation (13a), and, equivalently in Equation (13b), is linear in the rates $\dot{\boldsymbol{K}}$ and $\dot{\mathcal{T}}$. Thus, by recalling the definition of $\delta\mathcal{T}$, and introducing the *virtual variation of the growth tensor* $\delta\boldsymbol{K} : \mathscr{B} \times \mathscr{I} \to [T\mathscr{B}]^1{}_1$, where $[T\mathscr{B}]^1{}_1$ is the space of tensors mapping vectors of $T_X\mathscr{B}$ into vectors of $T_X\mathscr{B}$, we can rephrase Equation (13a) in the Lagrange-Chetaev form [95][5]. This is obtained by replacing $\dot{\boldsymbol{K}}$ and $\dot{\mathcal{T}}$ with the "virtual displacements" $\delta\boldsymbol{K}$ and $\delta\mathcal{T}$, respectively, i.e.,

$$\hat{\mathcal{V}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \delta\boldsymbol{K}, \delta\mathcal{T}, \omega) = \boldsymbol{K}^{-\mathrm{T}} : \delta\boldsymbol{K} - [\hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)]\delta\mathcal{T} = 0. \tag{19}$$

Moreover, Equations (17) and (18b) permit to rephrase the expressions of the constraint (13a) and (19) by substituting $\dot{\mathcal{T}}$ with $\dot{\mathfrak{T}}$ and $\delta\mathcal{T}$ with $\delta\mathfrak{T}$, thereby obtaining

$$\hat{\mathcal{V}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \dot{\mathfrak{T}}, \omega) := \boldsymbol{K}^{-\mathrm{T}} : \dot{\boldsymbol{K}} - [\hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)]\dot{\mathfrak{T}} = 0, \tag{20a}$$

$$\hat{\mathcal{V}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \delta\boldsymbol{K}, \delta\mathfrak{T}, \omega) = \boldsymbol{K}^{-\mathrm{T}} : \delta\boldsymbol{K} - [\hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)]\delta\mathfrak{T} = 0. \tag{20b}$$

Clearly, while the constraint expressed in Equation (20a) has physical dimensions of the reciprocal of time, the one rewritten in Equation (19) is non-dimensional. However, these two versions can be made dimensionally coherent with each other by exploiting the fact that $\hat{\mathcal{V}}_{\boldsymbol{K}}$ is homogeneous of degree 1 in its third and fourth argument. Indeed, given a strictly positive constant $t_{\mathrm{c}} > 0$, which may represent a characteristic time scale associated with the accretion or resorption of mass, it holds true that

$$\hat{\mathcal{V}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, t_{\mathrm{c}}\dot{\boldsymbol{K}}, t_{\mathrm{c}}\dot{\mathfrak{T}}, \omega) = t_{\mathrm{c}}[\hat{\mathcal{V}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \dot{\mathfrak{T}}, \omega)] = 0, \tag{21}$$

which is equivalent to Equation (20a), and non-dimensional.

Similarly, although the constraint in Equation (18b) has physical dimensions of time, whereas that in Equation (17) is non-dimensional, we can write

$$\hat{\mathcal{V}}_{\mathfrak{T}} \circ (t_{\mathrm{c}}\dot{\mathfrak{T}}, t_{\mathrm{c}}\dot{\mathcal{T}}) = t_{\mathrm{c}}[\hat{\mathcal{V}}_{\mathfrak{T}} \circ (\dot{\mathfrak{T}}, \dot{\mathcal{T}})] = 0, \tag{22}$$

thereby obtaining a constraint with the same physical dimensions as those in Equation (18b). We shall use this result in the constrained formulation of the Principle of Virtual Work.

Remark 4.1 (The maps $\mathfrak{T}$ and $\mathcal{T}$)
A direct integration of Equation (17) *with respect to time brings us back to Equation* (15)*, in which* $\mathfrak{T}_0(X)$ *takes on the meaning of point-dependent integration constant. This can be particularized, for example, by requiring* $\mathrm{Grad}\,\mathfrak{T}(X,t) = \mathbf{0}$*, for all* $(X,t) \in \mathscr{B} \times \mathscr{I}$*, so that the further condition* $\mathrm{Grad}\,\mathfrak{T}_0(X) = \mathbf{0}$ *applies for all* $X \in \mathscr{B}$*. Thus, we can set* $\mathfrak{T}(X,t) = t_X = t_0 + t$*, with* $t_0 := \mathfrak{T}_0(X)$ *being an arbitrary constant for all* $X \in \mathscr{B}$*, and, if we finally choose* $t_0 = 0$*, we obtain the unique solution* $\mathfrak{T}(X,t) = t \equiv \mathcal{T}(X,t)$*. However, in spite of this result, we find it convenient for the forthcoming discussion to maintain a conceptual distinction between* $\mathfrak{T}$ *and* $\mathcal{T}$*. Indeed, whereas* $\mathfrak{T}$ *is a (fictitious) Lagrangian parameter of the theory, constrained by Equation* (17) *to have unitary generalized velocity,* $\mathcal{T}$ *is an auxiliary function that, through the composition of maps, is often*

[5] We say that a constraint is expressed in the "Lagrange-Chetaev form" if it can be written in a form in which the generalized virtual velocities involved in the constraint are replaced with the corresponding generalized displacements. In fact, this is possible, granted that the constraint complies with the Chetaev —or Lagrange-Chetaev— condition [95], in which case the constraint is also referred to as *"ideal"* [95]. By adapting the terminology used by Llibre et al. [95] to our context, the constraint $\hat{\mathcal{V}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}, \dot{\mathfrak{T}}, \omega) = 0$ is ideal, since it is linear in the generalized velocities $\dot{\boldsymbol{K}}$ and $\dot{\mathfrak{T}}$, and, thus, it fulfills the Lagrange-Chetaev conditions, which reads $[\partial_{\dot{\boldsymbol{K}}}\hat{\mathcal{V}} \circ (\ldots)] : \delta\boldsymbol{K} + [\partial_{\dot{\mathfrak{T}}}\hat{\mathcal{V}} \circ (\ldots)]\delta\mathfrak{T} = \boldsymbol{K}^{-\mathrm{T}} : \delta\boldsymbol{K} - [\hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)]\delta\mathfrak{T} = 0$.

useful to express the explicit time dependence of some physical quantities in a formally correct way. Above all, the main reason for distinguishing between $\mathfrak{T}$ and $\mathcal{T}$ is the one that has been anticipated at the beginning of this section: since $\mathfrak{T}$ is declared as a Lagrangian parameter, its virtual variation $\delta\mathfrak{T}$ admits the introduction of fictitious forces, dual to $\delta\mathfrak{T}$, that produce virtual work on $\delta\mathfrak{T}$, and, because of the identity $\delta\mathfrak{T} = \delta\mathcal{T}$, this virtual work can be added to the one done on $\delta\mathcal{T}$ by the quantity $\mu_{\boldsymbol{K}}[\hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)]$ or $\mu_{\boldsymbol{K}}[\check{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \mathcal{X}, \mathcal{T})]$, where $\mu_{\boldsymbol{K}}$ is the Lagrange multiplier associated with the constraint (20a)*, put in the form* (19) *(see Equation* (24a) *below).*

5 Principle of Virtual Work revisited

To account for the fact that the Principle of Virtual Work has to be written for arbitrary generalized virtual displacements that are in harmony with the imposed constraints, we rephrase the PVW formulated by DiCarlo and Quiligotti [38] for growth mechanics as explained below. First, we recall that the kinematic descriptors of the present theory, which is of grade one in χ, and of grade zero in $\boldsymbol{K}$ [38] and $\mathfrak{T}$, are given by

$$(\chi, \boldsymbol{F}, \boldsymbol{K}, \mathfrak{T}, \delta\chi, \mathrm{Grad}\delta\chi, \delta\boldsymbol{K}, \delta\mathfrak{T}). \tag{23}$$

Then, since we are going to append the constraints, both in the rescaled forms (21) and (22) and in the Lagrange-Chetaev forms [95] (19) and (18b), to the expression of the PVW that one would have in the absence of constraints, we introduce the Lagrange multipliers $\mu_{\boldsymbol{K}}$ and $\mu_{\mathfrak{T}}$, along with their virtual variations $\delta\mu_{\boldsymbol{K}}$ and $\delta\mu_{\mathfrak{T}}$, so that the following duality pairings apply

$$\mu_{\boldsymbol{K}} \div [\hat{\mathcal{V}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \delta\boldsymbol{K}, \delta\mathcal{T}, \omega)], \qquad \mu_{\mathfrak{T}} \div [\hat{\mathcal{V}}_{\mathfrak{T}} \circ (\delta\mathfrak{T}, \delta\mathcal{T})], \tag{24a}$$

$$\delta\mu_{\boldsymbol{K}} \div [\hat{\mathcal{V}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, t_{\mathrm{c}}\dot{\boldsymbol{K}}, t_{\mathrm{c}}\dot{\mathfrak{T}}, \omega)], \qquad \delta\mu_{\mathfrak{T}} \div [\hat{\mathcal{V}}_{\mathfrak{T}} \circ (t_{\mathrm{c}}\dot{\mathfrak{T}}, t_{\mathrm{c}}\dot{\mathcal{T}})], \tag{24b}$$

where the symbol "$\div$" indicates the conjugation induced by duality.

By invoking duality again, we introduce the *internal generalized forces* that expend virtual work on the virtual variations of the associated kinematic descriptors, i.e.,

$$\boldsymbol{P} \div \mathrm{Grad}\delta\chi, \quad \boldsymbol{Y}_{\mathrm{u}} \div \boldsymbol{K}^{-1}\delta\boldsymbol{K}, \quad \mathcal{Y}_{\mathrm{u}} \div \delta\mathfrak{T}, \tag{25}$$

where $\boldsymbol{P}$ is the "classical" first Piola-Kirchhoff stress tensor; $\boldsymbol{Y}_{\mathrm{u}}$ is referred to as *internal growth-conjugated stress* in the sequel, since it is a stress-like quantity dual to $\boldsymbol{K}^{-1}\delta\boldsymbol{K}$; and $\mathcal{Y}_{\mathrm{u}}$ is termed *internal time-conjugated force*, since it is dual to $\delta\mathfrak{T}$. The subscript "u" in $\boldsymbol{Y}_{\mathrm{u}}$ and $\mathcal{Y}_{\mathrm{u}}$ indicates that these forces are "unconstrained", in the sense that, because of the presence of the Lagrangian multipliers $\mu_{\boldsymbol{K}}$ and $\mu_{\mathfrak{T}}$, they are associated with arbitrary (and, thus, "unconstrained") variations $\delta\boldsymbol{K}$ and $\delta\mathfrak{T}$, respectively.

Finally, we consider the external generalized forces

$$\boldsymbol{f}, \boldsymbol{\tau} \div \delta\chi, \quad \boldsymbol{Z} \div \boldsymbol{K}^{-1}\delta\boldsymbol{K}, \quad \mathcal{Z} \div \delta\mathfrak{T}, \tag{26}$$

where $\boldsymbol{f}$ and $\boldsymbol{\tau}$ are the body forces per unit volume and the boundary contact forces per unit area of "classical" Continuum Mechanics, respectively, while, from here on, $\boldsymbol{Z}$ and $\mathcal{Z}$ are said to be *external growth-conjugated stress-like force*, and *external time-conjugated force*, respectively.

Remark 5.1 (The external time-conjugated force $\mathcal{Z}$)
The external time-conjugated force $\mathcal{Z}$ is introduced by analogy with the external growth-conjugated stress-like force $\boldsymbol{Z}$. Indeed, as for $\boldsymbol{Z}$, whose origin has been discussed in [38, 37] for the case of growth, and that can be found also in [74, 24] for different problems, the rationale behind the introduction of $\mathcal{Z}$ in our model is the one that has been anticipated at the beginning of the previous section as well as in Remark 4.1, and it can be summarized as follows. We admit that the definition of $\mathfrak{T}$ as a Lagrangian parameter of the theory, and the definition of its virtual variation, i.e., $\delta\mathfrak{T}$, give room to the existence of forces dual to $\delta\mathfrak{T}$, which can be either internal or external, depending on the type of interaction that they model.

With the premises outlined above, the constrained version of the PVW can be put in the form

$$\begin{aligned}
&\int_{\mathscr{B}} \boldsymbol{P} : \mathrm{Grad}\delta\chi + \int_{\mathscr{B}} \boldsymbol{Y}_{\mathrm{u}} : \boldsymbol{K}^{-1}\delta\boldsymbol{K} + \int_{\mathscr{B}} \mathcal{Y}_{\mathrm{u}}\delta\mathfrak{T} \\
&+ \int_{\mathscr{B}} \mu_{\boldsymbol{K}}[\hat{\mathcal{V}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \delta\boldsymbol{K}, \delta\mathcal{T}, \omega)] + \int_{\mathscr{B}} \mu_{\mathfrak{T}}[\hat{\mathcal{V}}_{\mathfrak{T}} \circ (\delta\mathfrak{T}, \delta\mathcal{T})] \\
&+ \int_{\mathscr{B}} \delta\mu_{\boldsymbol{K}}[\hat{\mathcal{V}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, t_{\mathrm{c}}\dot{\boldsymbol{K}}, t_{\mathrm{c}}\dot{\mathfrak{T}}, \omega)] + \int_{\mathscr{B}} \delta\mu_{\mathfrak{T}}[\hat{\mathcal{V}}_{\mathfrak{T}} \circ (t_{\mathrm{c}}\dot{\mathfrak{T}}, t_{\mathrm{c}}\dot{\mathcal{T}})] \\
&= \int_{\mathscr{B}} \boldsymbol{f}\delta\chi + \int_{\partial^{\chi}_{\mathrm{N}}\mathscr{B}} \boldsymbol{\tau}\delta\chi + \int_{\mathscr{B}} \boldsymbol{Z} : \boldsymbol{K}^{-1}\delta\boldsymbol{K} + \int_{\mathscr{B}} \mathcal{Z}\delta\mathfrak{T},
\end{aligned} \tag{27}$$

where $\partial\mathscr{B} = \partial^{\chi}_{\mathrm{N}}\mathscr{B} \sqcup \partial^{\chi}_{\mathrm{D}}\mathscr{B}$ is the boundary of $\mathscr{B}$, while $\partial^{\chi}_{\mathrm{N}}\mathscr{B}$ and $\partial^{\chi}_{\mathrm{D}}\mathscr{B}$ represent the Neumann and the Dirichlet portions of $\partial\mathscr{B}$, respectively. Clearly, since the four integrals featuring the constraints are identically zero, the PVW expressed in Equation (27) is only formally different from that of the unconstrained theory of growth put forward by DiCarlo and Quiligotti [38].

Before proceeding, it is worth mentioning that an approach similar to ours can be found in the context of the bone remodeling formulated under the assumption of *"optimal response"* [89]. We also notice that, in the case of bone remodeling, the role that in our theory is played by the structural descriptor $\boldsymbol{K}$, is sometimes assigned to a scalar variable, termed *"microdeformation"* [55, 54], which is related to purely dissipative effects [30].

5.1 Dynamic equations in local form

By performing standard calculations, and writing explicitly $\hat{\mathcal{V}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, \delta\boldsymbol{K}, \delta\mathcal{T}, \omega)$ and $\hat{\mathcal{V}}_{\mathfrak{T}} \circ (\delta\mathfrak{T}, \delta\mathcal{T})$, Equation (27) can be recast in the form

$$\begin{aligned}
&\int_{\partial^{\chi}_{\mathrm{N}}\mathscr{B}} \{\boldsymbol{\tau} - \boldsymbol{P}\boldsymbol{N}\}\delta\chi + \int_{\mathscr{B}} \{\mathrm{Div}\boldsymbol{P} + \boldsymbol{f}\}\delta\chi \\
&+ \int_{\mathscr{B}} \left\{\boldsymbol{Z} - \mu_{\boldsymbol{K}}\boldsymbol{I}^{\mathrm{T}} - \boldsymbol{Y}_{\mathrm{u}}\right\} : \boldsymbol{K}^{-1}\delta\boldsymbol{K} + \int_{\mathscr{B}} \{\mathcal{Z} - \mathcal{Y}_{\mathrm{u}} - \mu_{\mathfrak{T}}\}\delta\mathfrak{T} \\
&+ \int_{\mathscr{B}} \left\{\mu_{\boldsymbol{K}}[\hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)] + \mu_{\mathfrak{T}}\right\}\delta\mathcal{T} \\
&- \int_{\mathscr{B}} \delta\mu_{\boldsymbol{K}}[\hat{\mathcal{V}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, t_{\mathrm{c}}\dot{\boldsymbol{K}}, t_{\mathrm{c}}\dot{\mathfrak{T}}, \omega)] - \int_{\mathscr{B}} \delta\mu_{\mathfrak{T}}[\hat{\mathcal{V}}_{\mathfrak{T}} \circ (t_{\mathrm{c}}\dot{\mathfrak{T}}, t_{\mathrm{c}}\dot{\mathcal{T}})] = 0,
\end{aligned} \tag{28}$$

which has to hold true for arbitrary $\delta\chi$ vanishing on $\partial^{\chi}_{\mathrm{D}}\mathscr{B}$, and for arbitrary $\delta\boldsymbol{K}$, $\delta\mathfrak{T}$, $\delta\mathcal{T}$, $\delta\mu_{\boldsymbol{K}}$, and $\delta\mu_{\mathfrak{T}}$. Moreover, localizing Equation (28), and appending the Dirichlet condition for χ on $\partial^{\chi}_{\mathrm{D}}\mathscr{B}$ lead to the *mixed formulation*

$$\mathrm{Div}\boldsymbol{P} + \boldsymbol{f} = \boldsymbol{0}, \qquad \text{in } \mathscr{B}, \tag{29a}$$

$$\chi = \chi_{\mathrm{b}}, \qquad \text{on } \partial^{\chi}_{\mathrm{D}}\mathscr{B}, \tag{29b}$$

$$\boldsymbol{P}\boldsymbol{N} = \boldsymbol{\tau}, \qquad \text{on } \partial^{\chi}_{\mathrm{N}}\mathscr{B}, \tag{29c}$$

$$\left(\boldsymbol{Y}_{\mathrm{u}} + \mu_{\boldsymbol{K}}\boldsymbol{I}^{\mathrm{T}}\right) - \boldsymbol{Z} = \boldsymbol{0}, \qquad \text{in } \mathscr{B}, \tag{29d}$$

$$(\mathcal{Y}_{\mathrm{u}} + \mu_{\mathfrak{T}}) - \mathcal{Z} = 0, \qquad \text{in } \mathscr{B}, \tag{29e}$$

$$\mu_{\boldsymbol{K}}[\hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)] + \mu_{\mathfrak{T}} = 0, \qquad \text{in } \mathscr{B}, \tag{29f}$$

$$\hat{\mathcal{V}}_{\boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}, t_{\mathrm{c}}\dot{\boldsymbol{K}}, t_{\mathrm{c}}\dot{\mathfrak{T}}, \omega) = 0, \qquad \text{in } \mathscr{B}, \tag{29g}$$

$$\hat{\mathcal{V}}_{\mathfrak{T}} \circ (t_{\mathrm{c}}\dot{\mathfrak{T}}, t_{\mathrm{c}}\dot{\mathcal{T}}) = 0, \qquad \text{in } \mathscr{B}. \tag{29h}$$

Equation (29a) is the local form of the balance of linear momentum of "classical" Continuum Mechanics, equipped with the Dirichlet boundary condition (29b) and the Neumann boundary condition (29c), where χ_{b} is the motion prescribed on $\partial^{\chi}_{\mathrm{D}}\mathscr{B}$, and $\boldsymbol{N}$ is the field of co-normals defined over $\partial\mathscr{B}$.

Equation (29d) is equivalent to the balance of the growth-conjugated stress-like generalized forces that was obtained by DiCarlo and Quiligotti [38] in their picture of growth mechanics, provided the identification $\boldsymbol{Y} \equiv \boldsymbol{Y}_{\mathrm{u}} + \mu_{\boldsymbol{K}}\boldsymbol{I}^{\mathrm{T}}$ is made, where $\boldsymbol{Y}$ denotes the *overall, internal, growth-conjugated stress* dual to $\boldsymbol{K}^{-1}\delta\boldsymbol{K}$ (see also [24] for a similar force balance obtained in the context of plasticity), and corresponds to what DiCarlo and Quiligotti [38] indicate with "$-\mathbb{C}$" and call *"remodelling self-couple"*. Clearly, our external force $\boldsymbol{Z}$ corresponds to the external *"remodelling couple"* denoted by "$\mathbb{B}$" by DiCarlo and Quiligotti [38]. In this respect, we notice that, apart from having re-defined $\boldsymbol{Y}$ as the sum of its unconstrained part, i.e., $\boldsymbol{Y}_{\mathrm{u}}$, and the part given by the Lagrange multiplier, i.e., $\mu_{\boldsymbol{K}}\boldsymbol{I}^{\mathrm{T}}$, Equation (29d) is not new and, in fact, it is rather well-established in several papers on inelastic processes (see e.g. [76, 24, 38, 77, 72, 37, 7, 107, 6, 12, 64, 105, 66, 31, 5, 69, 27, 26]). However, a novelty of our approach is that we are viewing growth as a constrained problem, as testified by the presence of the Lagrange multiplier $\mu_{\boldsymbol{K}}$ in $\boldsymbol{Y} \equiv \boldsymbol{Y}_{\mathrm{u}} + \mu_{\boldsymbol{K}}\boldsymbol{I}^{\mathrm{T}}$. To this end, a constitutive law relating $\boldsymbol{Y}_{\mathrm{u}}$ to $\dot{\boldsymbol{K}}$ will be sought for and, consequently, Equation (29d) will be turned into an ordinary differential equation in $\boldsymbol{K}$, and solved with respect to this tensorial variable.

Equation (29e) defines the balance of the generalized forces dual to $\delta\mathfrak{T}$, for which, in analogy with Equation (29d), one can identify $\mathcal{Y} \equiv \mathcal{Y}_{\mathrm{u}} + \mu_{\mathfrak{T}}$. Equation (29f), instead, defines a balance between the Lagrange multipliers of the theory. We emphasize that, in spite of the fact that Equation (29e) has the same structure as Equation (29d), it has a different meaning. Indeed, since the evolution of the fictitious Lagrangian parameter $\mathfrak{T}$ is entirely described by the constraint (29h), which yields $\mathfrak{T}(X, t) = t$, for all $(X, t) \in \mathscr{B} \times \mathscr{I}$ (see also the discussion in Remark 4.1), Equation (29e) determines the difference $\mathcal{Z} - \mathcal{Y}_{\mathrm{u}}$ [106]. In this respect, we highlight that Equations (29e) and (29f) are new in the theory of volumetric growth, at least to the best of our knowledge. However, similar equations were obtained by Nadile [106] in a completely different context. Finally, Equations (29g) and (29h) return the constraints.

After the constitutive framework is established, Equations (29a) and (29d)–(29h) constitute a set of 16 scalar equations in the unknowns χ, $\boldsymbol{K}$, $\mu_{\boldsymbol{K}}$, $\mu_{\mathfrak{T}}$, $\mathcal{Z} - \mathcal{Y}_{\mathrm{u}}$, and $\mathfrak{T}$, which amount to 16 scalar unknowns. Hence, the problem is closed. In particular, Equation (29g) allows to determine the Lagrange multiplier $\mu_{\boldsymbol{K}}$ (although, given the specific hypotheses adopted in this work, a different procedure will be used in the sequel), while Equation (29h) is used to obtain $\mathfrak{T}(X,t) = t$, so that the Lagrange multiplier $\mu_{\mathfrak{T}}$ is determined by means of Equation (29f). In our opinion, the constraint on time constitutes a novelty in our approach, since it has not been considered in the previous formulations of growth which we are aware of.

5.2 Preparation of the initial and boundary value problem (IBVP)

After substituting the explicit expressions of the constraints (17) and (20a) into Equations (29h) and (29g), respectively, and dropping the strictly positive constant $t_{\rm c}$ featuring in these equations, we proceed with the solution of the system (29a)–(29h).

First, we notice that Equations (29e) and (29f) can be decoupled from the other ones and rewritten as

$$\mathcal{Y}_{\rm u} + \mu_{\mathfrak{T}} = \mathcal{Z}, \qquad \text{in } \mathscr{B}, \tag{30a}$$

$$\mu_{\mathfrak{T}} = -\mu_{\boldsymbol{K}}[\hat{R}_{\gamma(\rm ph)} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)], \qquad \text{in } \mathscr{B}, \tag{30b}$$

so that the Lagrange multiplier $\mu_{\mathfrak{T}}$ can be determined by the right-hand-side of Equation (30b), once $\mu_{\boldsymbol{K}}$ is known. Accordingly, Equation (30a) becomes

$$\mathcal{Z} - \mathcal{Y}_{\rm u} = \mu_{\mathfrak{T}} = -\mu_{\boldsymbol{K}}[\hat{R}_{\gamma(\rm ph)} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)], \qquad \text{in } \mathscr{B}. \tag{31}$$

Second, Equations (29d) and (29g) can be studied separately from the other ones, so that one obtains

$$\boldsymbol{Y}_{\rm u} + \mu_{\boldsymbol{K}} \boldsymbol{I}^{\rm T} - \boldsymbol{Z} = \boldsymbol{0}, \qquad \text{in } \mathscr{B}, \tag{32a}$$

$$\boldsymbol{K}^{-\rm T} : \dot{\boldsymbol{K}} - \hat{R}_{\gamma(\rm ph)} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega) = 0, \qquad \text{in } \mathscr{B}. \tag{32b}$$

Consequently, $\mu_{\boldsymbol{K}}$ can be computed by separating the spherical part of Equation (32a) from its deviatoric counterpart, so that Equations (32a) and (32b) become

$$\mathrm{dev}\boldsymbol{Y}_{\rm u} = \mathrm{dev}\boldsymbol{Z}, \qquad \text{in } \mathscr{B}, \tag{33a}$$

$$\boldsymbol{K}^{-\rm T} : \dot{\boldsymbol{K}} = \hat{R}_{\gamma(\rm ph)} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega), \qquad \text{in } \mathscr{B}, \tag{33b}$$

$$\mu_{\boldsymbol{K}} = \tfrac{1}{3}\mathrm{tr}\boldsymbol{Z} - \tfrac{1}{3}\mathrm{tr}\boldsymbol{Y}_{\rm u}, \qquad \text{in } \mathscr{B}. \tag{33c}$$

Now, if $\boldsymbol{Z}$ is supplied through a function independent of $\dot{\boldsymbol{K}}$ and other time derivatives of $\boldsymbol{K}$ of order higher than the first, and if $\boldsymbol{Y}_{\rm u}$ is expressed constitutively as a function of $\boldsymbol{K}$ and $\dot{\boldsymbol{K}}$, then Equations (33a) and (33b) become a set of first-order ordinary differential equations. Thus, once $\boldsymbol{Z}$ is assigned and $\boldsymbol{Y}_{\rm u}$ is provided constitutively, the Lagrangian multiplier $\mu_{\boldsymbol{K}}$ is determined by the right-hand-side of Equation (33c), while Equations (33a) and (33b) are sufficient to determine the 9 independent components of $\boldsymbol{K}$. Therefore, the boundary value problem that has to be solved according to the formulation presented in this work is given by

$$
\begin{aligned}
&\mathrm{Div}\boldsymbol{P} + \boldsymbol{f} = \boldsymbol{0}, && \text{in } \mathscr{B}, && \text{(34a)}\\
&\chi = \chi_{\mathrm{b}}, && \text{on } \partial^{\chi}_{\mathrm{D}}\mathscr{B}, && \text{(34b)}\\
&\boldsymbol{P}\boldsymbol{N} = \boldsymbol{\tau}, && \text{on } \partial^{\chi}_{\mathrm{N}}\mathscr{B}, && \text{(34c)}\\
&\mathrm{dev}\boldsymbol{Y}_{\mathrm{u}} = \mathrm{dev}\boldsymbol{Z}, && \text{in } \mathscr{B}, && \text{(34d)}\\
&\boldsymbol{K}^{-\mathrm{T}} : \dot{\boldsymbol{K}} = \hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega), && \text{in } \mathscr{B}, && \text{(34e)}
\end{aligned}
$$

which, apart from the boundary conditions (34b) and (34c), involves 12 independent equations in 12 unknowns (3 equations for χ and 9 equations for $\boldsymbol{K}$), while $\mu_{\boldsymbol{K}}$, $\mu_{\mathfrak{T}}$, and $\mathcal{Z} - \mathcal{Y}_{\mathrm{u}}$ can be computed *a posteriori* by means of Equations (33c), (30b), and (31). To do that, it is necessary to supply $\boldsymbol{P}$ and $\boldsymbol{Y}_{\mathrm{u}}$ constitutively.

6 Constitutive laws and final form of the initial and boundary value problem

Given for granted that the constitutive expressions for $\boldsymbol{P}$ and $\boldsymbol{Y}_{\mathrm{u}}$ comply with all the axioms of the theory of constitutive laws, we focus here on their thermodynamical admissibility. To this end, we adhere to the framework presented by Gurtin [74], which we slightly adapt to our purposes, and, in the following study of the dissipation inequality, we consider the limit case in which $\mathcal{Z}$ is assumed to vanish from the outset. However, in Appendix A2, we study the opposite point of view, in which $\mathcal{Z}$ *is not* assumed to be zero, and is rather regarded as an unknown of the model. Hence, by taking inspiration from [24, 74, 75] for the general structure of the dissipation associated with a fixed region $\mathscr{R} \subset \mathscr{B}$, we write here

$$
\int_{\mathscr{R}} \mathcal{D}_{\mathrm{R}} = -\overset{\cdot}{\overline{\int_{\mathscr{R}} \Psi_{\mathrm{R}}}} + \underbrace{\int_{\mathscr{R}} \boldsymbol{f}\boldsymbol{v} + \int_{\partial\mathscr{R}} (\boldsymbol{P}\boldsymbol{N})\boldsymbol{v}}_{\mathcal{P}^{(\mathrm{net},\chi)}_{\mathrm{ext}}} + \underbrace{\int_{\mathscr{R}} \boldsymbol{Z} : \boldsymbol{K}^{-1}\dot{\boldsymbol{K}}}_{\mathcal{P}^{(\mathrm{net},\boldsymbol{K})}_{\mathrm{ext}}} + \underbrace{\int_{\mathscr{R}} \mu_{\mathrm{ch}}[\hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)]}_{\mathcal{M}_{\mathrm{ext}}} \geq 0, \tag{35}
$$

and refer to Gurtin [74] for an explanation of the fifth term on the right-hand-side of Equation (35) (see Remark 6.1). We emphasize that, within the theoretical framework presented in this section, one cannot expect to obtain $\mathcal{Y}_{\mathrm{u}}$ through the study of Equation (35). Indeed, the force $\mathcal{Y}_{\mathrm{u}}$ determined below follows directly from the force balance (31) under the simplifying assumption of vanishing $\mathcal{Z}$. A different approach, in which $\mathcal{Y}_{\mathrm{u}}$ is determined constitutively and $\mathcal{Z}$ solves the force balance (31) is shown in Appendix A2.

In Equation (35), $\mathcal{D}_{\mathrm{R}}$ and Ψ_{R} are the dissipation density and Helmholtz free energy density per unit volume of the reference placement, respectively, $\mathcal{P}^{(\mathrm{net},\chi)}_{\mathrm{ext}}$ is the external net power [24] associated with the Lagrangian parameter χ through the velocity $\boldsymbol{v} := \dot{\chi}$; $\mathcal{P}^{(\mathrm{net},\boldsymbol{K})}_{\mathrm{ext}}$ denotes the external net power conjugated with $\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}$, and μ_{ch} is identified with a generalized *chemical potential*, whose product with the growth law $R_{\gamma(\mathrm{ph})} \equiv \hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)$ defines $\mathcal{M}_{\mathrm{ext}}$, i.e., in a modified version of Gurtin's words [74], the power that is added to or subtracted from the system by means of the addition (when $R_{\gamma(\mathrm{ph})} > 0$) or depletion of mass (when $R_{\gamma(\mathrm{ph})} < 0$).

Remark 6.1 (On the presence of $\mathcal{M}_{\text{ext}}$ in Equation (35))
In his expression of the dissipation inequality, Gurtin [74] introduces the term that we have denoted here by $\mathcal{M}_{\text{ext}}$ because, in his model, the mass source is declared as an entity that operates on the system that he considers from the world outside it [74]. Indeed, in analogy with the external powers $\mathcal{P}_{\text{ext}}^{(\text{net},\chi)}$ and $\mathcal{P}_{\text{ext}}^{(\text{net},\boldsymbol{K})}$, the term $\mathcal{M}_{\text{ext}}$ must feature explicitly in the dissipation inequality, since it constitutes the external power due to the transport of mass. As such, and as anticipated above, it is represented by the action of the generalized force dual to the variation of mass, i.e., the chemical potential μ_{ch}, on the mass source/sink, which is the conjugated generalized rate. Within our approach, the mass source/sink is given phenomenologically from the outset, and is thus regarded as external, thereby making our formulation similar to the one presented by Gurtin [74]. This concept is explained also by Fried and Sellers [50], although they investigate a different situation. Indeed, also other approaches are possible. In fact, Fried and Sellers [50] elaborate a model in which their source/sink of mass is introduced as a supply/loss of mass operating from the inside of the system under study. Consistently with this point of view, their source/sink of mass cannot feature explicitly in the definition of the dissipation inequality, although it can be made to appear in the subsequent expression of the dissipation obtained by exploiting the mass balance law. This difference between the approach proposed by Gurtin [74], and slightly modified in our work, and the approach proposed by Fried and Sellers [50] is, in fact, essential. Indeed, since Fried and Sellers [50] treat the mass source/sink as an internal constitutive variable, they have to determine a constitutive law for it.

6.1 Local dissipation

We proceed with the localization of the dissipation inequality (35) under the assumption that $\mathscr{R} \subset \mathscr{B}$ is independent of time [24]. To this end, we apply Gauss' Theorem, and enforce the dynamic equations (29a) and (29d), thereby obtaining

$$\mathcal{D}_{\text{R}} = -\dot{\Psi}_{\text{R}} + \boldsymbol{P} : \dot{\boldsymbol{F}} + (\boldsymbol{Y}_{\text{u}} + \mu_{\boldsymbol{K}} \boldsymbol{I}^{\text{T}}) : \boldsymbol{K}^{-1}\dot{\boldsymbol{K}} + \mu_{\text{ch}}[\hat{R}_{\gamma(\text{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)] \geq 0. \tag{36}$$

By recalling Equation (7), which implies $\hat{R}_{\gamma(\text{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega) = \boldsymbol{I}^{\text{T}} : \boldsymbol{K}^{-1}\dot{\boldsymbol{K}}$, Equation (36) features the term $(\mu_{\boldsymbol{K}} + \mu_{\text{ch}})[\boldsymbol{I}^{\text{T}} : \boldsymbol{K}^{-1}\dot{\boldsymbol{K}}]$, which can be eliminated by setting $\mu_{\text{ch}} = -\mu_{\boldsymbol{K}}$, thereby obtaining

$$\mathcal{D}_{\text{R}} = -\dot{\Psi}_{\text{R}} + \boldsymbol{P} : \dot{\boldsymbol{F}} + \boldsymbol{Y}_{\text{u}} : \boldsymbol{K}^{-1}\dot{\boldsymbol{K}} \geq 0. \tag{37}$$

To study Equation (37), we write Ψ_{R} as $\Psi_{\text{R}} = J_{\boldsymbol{K}}\Psi_{\nu}$, where Ψ_{ν} is the body's Helmholtz free energy density per unit volume of the natural state, and, under the hypothesis of hyperelastic material, we express Ψ_{ν} constitutively as $\Psi_{\nu} = \hat{\Psi}_{\nu} \circ (\boldsymbol{F}\boldsymbol{K}^{-1})$, so that Ψ_{R} can be re-defined as $\Psi_{\text{R}} = \hat{\Psi}_{\text{R}} \circ (\boldsymbol{F}, \boldsymbol{K}) = J_{\boldsymbol{K}}[\hat{\Psi}_{\nu} \circ (\boldsymbol{F}\boldsymbol{K}^{-1})]$. Then, by computing the time derivative of Ψ_{R}, substituting it into Equation (37), and making the identifications

$$\boldsymbol{P} = \hat{\boldsymbol{P}} \circ (\boldsymbol{F}, \boldsymbol{K}) = \frac{\partial \hat{\Psi}_{\text{R}}}{\partial \boldsymbol{F}} \circ (\boldsymbol{F}, \boldsymbol{K}) = (\det \boldsymbol{K}) \left(\frac{\partial \hat{\Psi}_{\nu}}{\partial \boldsymbol{F}\boldsymbol{K}^{-1}} \circ (\boldsymbol{F}\boldsymbol{K}^{-1}) \right) \boldsymbol{K}^{-\text{T}}, \tag{38a}$$

$$\boldsymbol{H} = \hat{\boldsymbol{H}} \circ (\boldsymbol{F}, \boldsymbol{K}) = \boldsymbol{K}^{\text{T}} \left(\frac{\partial \hat{\Psi}_{\text{R}}}{\partial \boldsymbol{K}} \circ (\boldsymbol{F}, \boldsymbol{K}) \right) = [\hat{\Psi}_{\text{R}} \circ (\boldsymbol{F}, \boldsymbol{K})]\boldsymbol{I}^{\text{T}} - \boldsymbol{F}^{\text{T}}[\hat{\boldsymbol{P}} \circ (\boldsymbol{F}, \boldsymbol{K})], \tag{38b}$$

$$\boldsymbol{Y}_{\text{u,d}} := \boldsymbol{Y}_{\text{u}} - [\hat{\boldsymbol{H}} \circ (\boldsymbol{F}, \boldsymbol{K})], \tag{38c}$$

where $\boldsymbol{H}$ is Eshelby stress tensor, and $\boldsymbol{Y}_{\mathrm{u,d}}$ is the *dissipative part* of $\boldsymbol{Y}_{\mathrm{u}}$ (see also [24, 38, 37]), the dissipation inequality reduces to

$$\mathcal{D}_{\mathrm{R}} = \boldsymbol{Y}_{\mathrm{u,d}} : \boldsymbol{K}^{-1}\dot{\boldsymbol{K}} \geq 0. \tag{39}$$

6.2 Constitutive laws

By looking at Equation (39), and restricting our study to the linear theory and to the isotropic case, we express $\boldsymbol{Y}_{\mathrm{u,d}}$ by using a decomposition of fourth-order tensors [108] that yields (see also [70, 64, 94])

$$\begin{aligned}\boldsymbol{Y}_{\mathrm{u,d}} := \hat{\boldsymbol{Y}}_{\mathrm{u,d}} \circ (\boldsymbol{F}, \boldsymbol{K}, \dot{\boldsymbol{K}}) =& \tfrac{1}{3} J_{\boldsymbol{K}} \mathfrak{a}_\nu \mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}) \boldsymbol{I}^{\mathrm{T}} + J_{\boldsymbol{K}} \mathfrak{b}_\nu \{\boldsymbol{C}\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}\boldsymbol{C}^{-1} + (\boldsymbol{K}^{-1}\dot{\boldsymbol{K}})^{\mathrm{T}}\} \\ &+ J_{\boldsymbol{K}} \mathfrak{c}_\nu \{\boldsymbol{C}\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}\boldsymbol{C}^{-1} - (\boldsymbol{K}^{-1}\dot{\boldsymbol{K}})^{\mathrm{T}}\} \\ =& [\hat{\mathbb{T}} \circ (\boldsymbol{F}, \boldsymbol{K})] : (\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}),\end{aligned} \tag{40}$$

where $\mathfrak{a}_\nu$, $\mathfrak{b}_\nu$, and $\mathfrak{c}_\nu$ are constant material parameters such that $\mathfrak{a}_\nu + 2\mathfrak{b}_\nu \geq 0$, $\mathfrak{b}_\nu \geq 0$, and $\mathfrak{c}_\nu \geq 0$, while $\hat{\mathbb{T}} \circ (\boldsymbol{F}, \boldsymbol{K})$ is the constitutive function, mapped in the space of fourth-order tensors, given by

$$\hat{\mathbb{T}} \circ (\boldsymbol{F}, \boldsymbol{K}) := \tfrac{1}{3} J_{\boldsymbol{K}} \mathfrak{a}_\nu \boldsymbol{I}^{\mathrm{T}} \otimes \boldsymbol{I}^{\mathrm{T}} + J_{\boldsymbol{K}} \mathfrak{b}_\nu \big\{\boldsymbol{C} \underline{\otimes} \boldsymbol{C}^{-1} + \boldsymbol{I}^{\mathrm{T}} \overline{\otimes} \boldsymbol{I}\big\} + J_{\boldsymbol{K}} \mathfrak{c}_\nu \big\{\boldsymbol{C} \underline{\otimes} \boldsymbol{C}^{-1} - \boldsymbol{I}^{\mathrm{T}} \overline{\otimes} \boldsymbol{I}\big\}. \tag{41}$$

Next, we discuss the external forces $\boldsymbol{f}$ and $\boldsymbol{Z}$. Whereas $\boldsymbol{f}$ is often assumed to be negligible in the mechanics of tumor growth, $\boldsymbol{Z}$ may be important in the evolution of a tumor [37]. To us, this generalized force may be the expression of genetic and/or epigenetic and/or chemical information, appropriately "translated" into mechanical interactions (cf. the interpretation given by [37]). In our opinion, providing accurate models of such interactions is not straightforward and further research in this direction is therefore necessary. In an ongoing work of ours, we propose a possible expression for $\boldsymbol{Z}$ on the basis of several discussions with colleagues and former co-workers[6]. However, since providing an explicit expression for $\boldsymbol{Z}$ is not the focus of the present work, we simply assume here that it can be assigned as $\boldsymbol{Z} := \hat{\boldsymbol{Z}} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega, \mathrm{Grad}\,\omega)$. In particular, we choose $\boldsymbol{Z}$ such that $\boldsymbol{C}^{-1}\boldsymbol{Z}$, and, thus, also $\boldsymbol{C}^{-1}\mathrm{dev}\boldsymbol{Z}$, are symmetric second-order tensor fields, so that, according to Equation (34d), $\boldsymbol{C}^{-1}\mathrm{dev}\boldsymbol{Y}_{\mathrm{u}}$ is symmetric, too. Consequently, the term multiplied by the generalized viscosity $\mathfrak{c}_\nu$ in Equation (40) has to be zero.

6.3 Final form of the IBVP

In summary, the set of equations describing the considered growing medium consists of Equation (34a), i.e., the linear momentum balance law, which determines the medium's deformation, χ; Equations (34d), i.e., the balance of forces dual to the unconstrained part of $\dot{\boldsymbol{K}}$; Equation (34e), i.e., the constraint describing the medium's growth; Equation (5), which represents the diffusion-reaction equation for the nutrients' mass fraction, ω. The above mentioned list of equations is equipped with the boundary conditions (34b) and (34c), assessing prescribed deformations and tractions, and with the boundary conditions (6a) and (6b) assigned on ω. Analogously, we prescribe

[6] We acknowledge, in particular, several discussions done with Ms. Francesca Ballatore and especially with Ms. Valentina Licari at the time of her Master of Science thesis [94].

an initial condition of the type $\boldsymbol{K}(X, t_{\rm in}) = \boldsymbol{K}_{\rm in}(X)$ for the growth tensor and the initial condition for ω expressed in Equation (6c). Finally, we obtain the IBVP

$$
\begin{aligned}
&\mathrm{Div}\boldsymbol{P} = -\boldsymbol{f}, && \text{in } \mathscr{B}, && \text{(42a)}\\
&\chi = \chi_{\rm b}, && \text{on } \partial^{\chi}_{\rm D}\mathscr{B}, && \text{(42b)}\\
&\boldsymbol{P}\boldsymbol{N} = \boldsymbol{\tau}, && \text{on } \partial^{\chi}_{\rm N}\mathscr{B}, && \text{(42c)}\\
&2J_{\boldsymbol{K}}\mathfrak{b}_{\nu}\mathrm{dev}_{\boldsymbol{C}}\mathrm{sym}[(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}})\boldsymbol{C}^{-1}] = -\boldsymbol{C}^{-1}\mathrm{dev}\boldsymbol{H} + \boldsymbol{C}^{-1}\mathrm{dev}\boldsymbol{Z}, && \text{in } \mathscr{B}, && \text{(42d)}\\
&2J_{\boldsymbol{K}}\mathfrak{c}_{\nu}\mathrm{skew}[(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}})\boldsymbol{C}^{-1}] = \boldsymbol{0}, && \text{in } \mathscr{B}, && \text{(42e)}\\
&\boldsymbol{K}^{-\rm T} : \dot{\boldsymbol{K}} = R_{\gamma(\rm ph)}, && \text{in } \mathscr{B}, && \text{(42f)}\\
&\boldsymbol{K}(X, t_{\rm in}) = \boldsymbol{K}_{\rm in}(X), && \text{in } \mathscr{B}, && \text{(42g)}\\
&J_{\boldsymbol{K}}\varrho_{\nu}\dot{\omega} - \mathrm{Div}\big(J_{\boldsymbol{K}}\varrho_{\nu}\boldsymbol{D}\,\mathrm{Grad}\omega\big) = -J_{\boldsymbol{K}}\varrho_{\nu}r_{\rm n}\omega - J_{\boldsymbol{K}}\varrho_{\nu}R_{\gamma(\rm ph)}\omega, && \text{in } \mathscr{B}, && \text{(42h)}\\
&\omega = \omega_{\rm b}, && \text{on } \partial^{\omega}_{\rm D}\mathscr{B}, && \text{(42i)}\\
&[-J_{\boldsymbol{K}}\varrho_{\nu}\boldsymbol{D}\,\mathrm{Grad}\omega]\boldsymbol{N} = \jmath_{\rm b}, && \text{on } \partial^{\omega}_{\rm N}\mathscr{B}, && \text{(42j)}\\
&\omega(X, t_{\rm in}) = \omega_{\rm in}(X), && \text{in } \mathscr{B}, && \text{(42k)}
\end{aligned}
$$

where the operator $\mathrm{dev}_{\boldsymbol{C}}$ is defined by $\mathrm{dev}_{\boldsymbol{C}}\boldsymbol{T} := \boldsymbol{T} - \frac{1}{3}\mathrm{tr}(\boldsymbol{C}\boldsymbol{T})\boldsymbol{C}^{-1}$, for all second-order, contravariant tensors $\boldsymbol{T}$. Note that Equations (42d) and (42e) are obtained by left-multiplying Equation (34d) by $\boldsymbol{C}^{-1}$, employing Equation (40) for the constitutive representation of $\boldsymbol{Y}_{\rm u}$, and extracting once the symmetric part and once the skew-symmetric part of the resulting expression. Moreover, we remark that Equation (42d) is equivalent to

$$
J_{\boldsymbol{K}}\mathfrak{b}_{\nu}\,\mathrm{dev}\{\boldsymbol{C}\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}\boldsymbol{C}^{-1} + (\boldsymbol{K}^{-1}\dot{\boldsymbol{K}})^{\rm T}\} = -\mathrm{dev}\boldsymbol{H} + \mathrm{dev}\boldsymbol{Z}, \qquad \text{in } \mathscr{B}. \tag{43}
$$

Once the IBVP (42a)–(42k) is solved, the Lagrange multipliers $\mu_{\boldsymbol{K}}$ and $\mu_{\mathfrak{T}}$ can be computed *a posteriori* as prescribed by Equations (33c) and (30b), respectively, i.e.,

$$
\mu_{\boldsymbol{K}} = \tfrac{1}{3}\mathrm{tr}\boldsymbol{Z} - \tfrac{1}{3}\mathrm{tr}\boldsymbol{H} - \tfrac{1}{3}J_{\boldsymbol{K}}[\mathfrak{a}_{\nu} + 2\mathfrak{b}_{\nu}]\mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}) = \tfrac{1}{3}\mathrm{tr}\boldsymbol{Z} - \tfrac{1}{3}\mathrm{tr}\boldsymbol{H} - \tfrac{1}{3}J_{\boldsymbol{K}}[\mathfrak{a}_{\nu} + 2\mathfrak{b}_{\nu}]R_{\gamma(\rm ph)}, \quad \text{in } \mathscr{B}, \tag{44a}
$$

$$
\mu_{\mathfrak{T}} = -\mu_{\boldsymbol{K}}R_{\gamma(\rm ph)} = -\big\{\tfrac{1}{3}\mathrm{tr}\boldsymbol{Z} - \tfrac{1}{3}\mathrm{tr}\boldsymbol{H} - \tfrac{1}{3}J_{\boldsymbol{K}}[\mathfrak{a}_{\nu} + 2\mathfrak{b}_{\nu}]R_{\gamma(\rm ph)}\big\}R_{\gamma(\rm ph)}, \qquad \text{in } \mathscr{B}. \tag{44b}
$$

Also the internal generalized force conjugated with $\dot{\mathfrak{T}}$, i.e., $\mathcal{Y}_{\rm u}$, can be determined *a posteriori* by means of Equation (31), with $\mathcal{Z} = 0$, as

$$
\mathcal{Y}_{\rm u} = -\mu_{\mathfrak{T}} = \mu_{\boldsymbol{K}}R_{\gamma(\rm ph)} = \big\{\tfrac{1}{3}\mathrm{tr}\boldsymbol{Z} - \tfrac{1}{3}\mathrm{tr}\boldsymbol{H} - \tfrac{1}{3}J_{\boldsymbol{K}}[\mathfrak{a}_{\nu} + 2\mathfrak{b}_{\nu}]R_{\gamma(\rm ph)}\big\}R_{\gamma(\rm ph)}. \tag{45}
$$

In our opinion, Equations (44a), (44b), and (45) deserve specific comments, which we summarize in Remarks 8.1 and 8.2 of section 8.

6.4 The limit case of spherical growth tensor

In several problems addressing the growth of a tumor in the so-called *"avascular stage"* [25], the growth tensor is often assumed to be spherical from the outset [9, 11, 100, 59, 58]. By considering the decomposition $\boldsymbol{K} = J_{\boldsymbol{K}}^{1/3}\tilde{\boldsymbol{K}}$ (see e.g. [20] in which such decomposition is used for $\boldsymbol{F}$ to study incompressibility in finite deformations), which implies $\det \boldsymbol{K} = J_{\boldsymbol{K}}$ and, thus, necessarily $\det \tilde{\boldsymbol{K}} = 1$, the hypothesis of spherical growth tensor amounts to set $\tilde{\boldsymbol{K}} = \boldsymbol{I}$ and, thus, $\boldsymbol{K} = {J_{\boldsymbol{K}}}^{1/3}\boldsymbol{I}$. Consequently, $\boldsymbol{K}$ features one free component only, i.e., $J_{\boldsymbol{K}}$, which, in turn, is restricted by Equation (3) to fulfill the constraint

$$\dot{J}_{\boldsymbol{K}} = R_{\gamma(\mathrm{ph})} J_{\boldsymbol{K}}, \tag{46}$$

as can be seen by employing the identity $\mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}) = \dot{J}_{\boldsymbol{K}}/J_{\boldsymbol{K}}$.

Since, in the present case, it holds true that $\boldsymbol{K}^{-1}\dot{\boldsymbol{K}} = \frac{1}{3}(\dot{J}_{\boldsymbol{K}}/J_{\boldsymbol{K}})\boldsymbol{I}$, and, accordingly, $\boldsymbol{K}^{-1}\delta\boldsymbol{K} = \frac{1}{3}(\delta J_{\boldsymbol{K}}/J_{\boldsymbol{K}})\boldsymbol{I}$, the non-spherical parts of the growth-conjugated, stress-like generalized forces $\boldsymbol{Y}_{\mathrm{u}}$ and $\boldsymbol{Z}$ are filtered out from the PVW stated in Equations (27) and (28), so that Equation (29d), or, equivalently, Equation (32a), reduces to

$$(y_{\mathrm{u}} + \mu_{\boldsymbol{K}}) - z = 0, \tag{47}$$

with $y_{\mathrm{u}} := \frac{1}{3}\mathrm{tr}\boldsymbol{Y}_{\mathrm{u}}$ and $z := \frac{1}{3}\mathrm{tr}\boldsymbol{Z}$ being the scalar coefficients of the spherical parts of $\boldsymbol{Y}_{\mathrm{u}}$ and $\boldsymbol{Z}$, respectively (cf. Equation (33c)). Note that all the other equations of the boundary value problem (29a)–(29h) remain unchanged, and that Equation (29g), or its explicit form (32b), simply returns Equation (46). A direct consequence of these facts is that Equation (33a) is eliminated from the model and, consistently with what is usually done in some works (see e.g. [8, 11, 100, 99, 58] and the references therein), $J_{\boldsymbol{K}}$ is entirely defined by Equation (46), which, however, is here understood as a constraint. As such, it requires the Lagrange multiplier $\mu_{\boldsymbol{K}}$, which is computed by the force balance (47), i.e.,

$$\mu_{\boldsymbol{K}} = z - y_{\mathrm{u}}, \tag{48}$$

similarly to Equation (33c). There remains to determine y_{u}. To do this, we go through the dissipation inequality, for example in the form of Equation (37), which we reformulate for the case at hand by assuming $\Psi_{\mathrm{R}} = \hat{\Psi}_{\mathrm{R}} \circ (\boldsymbol{F}, J_{\boldsymbol{K}})$. Hence, after some calculations, we obtain

$$y_{\mathrm{u}} = y_{\mathrm{u,d}} + J_{\boldsymbol{K}} \left[\frac{\partial \hat{\Psi}_{\mathrm{R}}}{\partial J_{\boldsymbol{K}}} \circ (\boldsymbol{F}, J_{\boldsymbol{K}}) \right], \tag{49}$$

where the second summand on the right-hand-side of Equation (49) is the scalar coefficient of the spherical part of Eshelby stress tensor and $y_{\mathrm{u,d}}$ is the dissipative part of y_{u}, and must comply with Equation (39), which now reads

$$\mathcal{D}_{\mathrm{R}} = y_{\mathrm{u,d}}(\dot{J}_{\boldsymbol{K}}/J_{\boldsymbol{K}}) \geq 0. \tag{50}$$

Within the linear theory, we can take $y_{\mathrm{u,d}} = \kappa_\nu \dot{J}_{\boldsymbol{K}}$, with $\kappa_\nu > 0$, so that Equation (48) becomes

$$\mu_{\boldsymbol{K}} = z - J_{\boldsymbol{K}} \left[\frac{\partial \hat{\Psi}_{\mathrm{R}}}{\partial J_{\boldsymbol{K}}} \circ (\boldsymbol{F}, J_{\boldsymbol{K}}) \right] - \kappa_\nu \dot{J}_{\boldsymbol{K}} = z - J_{\boldsymbol{K}} \left[\frac{\partial \hat{\Psi}_{\mathrm{R}}}{\partial J_{\boldsymbol{K}}} \circ (\boldsymbol{F}, J_{\boldsymbol{K}}) \right] - J_{\boldsymbol{K}} \kappa_\nu R_{\gamma(\mathrm{ph})}. \tag{51}$$

For completeness, we recall that $\mathcal{Y}_{\mathrm{u}}$ and $\mu_{\mathfrak{Z}}$ are determined by adapting Equation (45) to the case under study. In conclusion, a theory of the type just described permits to compute χ and $J_{\boldsymbol{K}}$ by solving Equations (42a)–(42c) and (46), whereas the Lagrange multiplier $\mu_{\boldsymbol{K}}$ can be determined *a posteriori*, since it is decoupled from the rest of the model equations.

7 Comparison with Gurtin's derivation of Cahn-Hilliard equation

A rather different situation from that depicted in the previous section arises when the mass balance law accounts for diffusion. We consider this circumstance by adhering to the framework developed by Gurtin [74], in which the evolution of the mass density of a body is studied under the assumption that such mass density describes an order parameter[7]. We notice that, in the context of tumor growth, Cahn-Hilliard based models have been proposed, e.g., in [1, 2], whereas a model involving the curvature induced by the growth tensor was proposed in [36].

To model diffusion, we rewrite the mass balance law as $\dot{\varrho}_{\mathrm{R}} = -\mathrm{Div}\mathfrak{J}_{\mathrm{R}} + J r_\gamma$, where $\mathfrak{J}_{\mathrm{R}}$ is a diffusive mass flux vector, and the mass density ϱ_{R} is now regarded as a body's order parameter [74]. In fact, by using the relation $\varrho_{\mathrm{R}} = J_{\boldsymbol{K}}\varrho_\nu$, and assuming ϱ_ν to be constant, one may regard $J_{\boldsymbol{K}}$ as the "effective" order parameter of the theory and rewrite the mass balance law as

$$\frac{\dot{J}_{\boldsymbol{K}}}{J_{\boldsymbol{K}}} = -\frac{1}{J_{\boldsymbol{K}}}\mathrm{Div}\mathfrak{v} + R_{\gamma(\mathrm{ph})}, \tag{52}$$

with $\mathfrak{v} := \varrho_\nu^{-1}\mathfrak{J}_{\mathrm{R}}$ being a normalized mass flux vector having physical dimensions of velocity.

Equation (52) is the new form of the constraint on $J_{\boldsymbol{K}}$, and is a partial differential equation, in which $\mathfrak{v}$ has to be determined constitutively. To accomplish this task, we briefly review Gurtin's approach [74], and slightly modify it in order to match our growth problem. To start with, we recall that Gurtin's fundamental hypothesis is that the mass balance law described by Equation (52) has to be studied in conjunction with an additional balance of forces [74], dual to the variations of the order parameter. In this respect, the force balance in Equation (47) has to be re-interpreted accordingly, and, by introducing the virtual variations of the mass density ϱ_{R} and of $J_{\boldsymbol{K}}$, i.e., $\delta\varrho_{\mathrm{R}}$ and $\delta J_{\boldsymbol{K}}/J_{\boldsymbol{K}}$, we write it in the following two equivalent forms

$$-\mathrm{Div}\boldsymbol{\xi} + \pi = \gamma, \qquad \text{work-conjugate with } \delta\varrho_{\mathrm{R}} \text{ (see [74])}, \tag{53a}$$

$$-\mathrm{Div}\mathfrak{f} + \left(\mathfrak{f}\frac{\mathrm{Grad}J_{\boldsymbol{K}}}{J_{\boldsymbol{K}}} + q_{\mathrm{u}} + \mu_{\boldsymbol{K}}\right) = z, \qquad \text{work-conjugate with } \frac{\delta J_{\boldsymbol{K}}}{J_{\boldsymbol{K}}}. \tag{53b}$$

Thus, a comparison of Equation (53b) with Equation (47) allows to identify the internal generalized force y_{u} with the combination

$$y_{\mathrm{u}} \equiv -\mathrm{Div}\mathfrak{f} + \left(\mathfrak{f}\frac{\mathrm{Grad}J_{\boldsymbol{K}}}{J_{\boldsymbol{K}}} + q_{\mathrm{u}}\right) = -\mathrm{Div}\mathfrak{f} + q_{\mathrm{u,eff}}, \tag{54a}$$

$$q_{\mathrm{u,eff}} := \mathfrak{f}\frac{\mathrm{Grad}J_{\boldsymbol{K}}}{J_{\boldsymbol{K}}} + q_{\mathrm{u}}, \tag{54b}$$

where q_{u} is the generalized internal force associated with growth, while $q_{\mathrm{u,eff}}$ describes an *effective internal force*, in which the term $\mathfrak{f}J_{\boldsymbol{K}}^{-1}\mathrm{Grad}J_{\boldsymbol{K}}$ accounts for the inhomogeneity of $J_{\boldsymbol{K}}$, and is indeed remnant of the *"inhomogeneity force"* introduced by Epstein and Maugin [41] in the context of growth mechanics (see also [101] for a more general context). We remark that Equations (53a) and (53b) can be found by adapting the PVW in Equation (27) to a theory of grade one in $J_{\boldsymbol{K}}$. This, in general, also requires prescribing $\mathfrak{f}\boldsymbol{N}$ on the portions of $\partial\mathscr{B}$ on which contact forces dual to $\delta J_{\boldsymbol{K}}/J_{\boldsymbol{K}}$, or to $\delta\varrho_{\mathrm{R}}$, are assigned. However, by assuming for simplicity no contact forces of this type, the

[7] The content of this section has been partially inspired by Fried and Sellers [50].

adapted expression of the PVW is obtained by adding the internal virtual work $\int_{\mathscr{B}}(\mathfrak{f}/J_{\boldsymbol{K}})\mathrm{Grad}\delta J_{\boldsymbol{K}}$ to the left-hand side of Equation (27), replacing the second, fourth and sixth integrand on the same side with $q_{\mathrm{u}}\delta J_{\boldsymbol{K}}/J_{\boldsymbol{K}}$, $\mu_{\boldsymbol{K}}\{\delta J_{\boldsymbol{K}}/J_{\boldsymbol{K}} + [J_{\boldsymbol{K}}^{-1}\mathrm{Div}\boldsymbol{\mathfrak{v}} - R_{\gamma(\mathrm{ph})}]\delta\mathcal{T}\}$, and $\delta\mu_{\boldsymbol{K}}t_{\mathrm{c}}\{\dot{J}_{\boldsymbol{K}}/J_{\boldsymbol{K}} + [J_{\boldsymbol{K}}^{-1}\mathrm{Div}\boldsymbol{\mathfrak{v}} - R_{\gamma(\mathrm{ph})}]\}$, respectively, and substituting the third integrand on the right-hand side with $z\delta J_{\boldsymbol{K}}/J_{\boldsymbol{K}}$. This way, the Lagrange multiplier $\mu_{\mathfrak{T}}$ is equal to the product of $\mu_{\boldsymbol{K}}$ with the negative of the right-hand side of Equation (52).

Note that, up to the sign convention, we used Gurtin's notation in Equation (53a) for the vectorial generalized force $\boldsymbol{\xi}$ as well as for the scalar-valued, internal generalized force π and external generalized force γ [74]. Moreover, the forces $\mathfrak{f}$, q_{u}, and z, which we introduced in Equation (53b) for our problem, are connected with $\boldsymbol{\xi}$, π, and γ, respectively, through the conversion formulae

$$\mathfrak{f} \equiv \varrho_{\mathrm{R}}\boldsymbol{\xi}, \tag{55a}$$

$$q_{\mathrm{u}} + \mu_{\boldsymbol{K}} \equiv \varrho_{\mathrm{R}}\pi, \tag{55b}$$

$$z \equiv \varrho_{\mathrm{R}}\gamma. \tag{55c}$$

We emphasize that the Lagrange multiplier $\mu_{\boldsymbol{K}}$ does not feature explicitly in the derivation of Equation (53a) done by Gurtin [74]. Rather, another multiplier, with different sign and different physical dimensions, is introduced when the dissipation inequality is investigated. In fact, Gurtin's Lagrange multiplier is a rescaled version of μ_{ch} featuring in Equation (35).

Before going further, we deem appropriate to recall that Gurtin [74] considered a body that can be described as a *"lattice"*, or as a *"network"*, whose sites are free to experience relative motions with respect to their underlying lattice structure [74]. A physical interpretation of $\boldsymbol{\xi}$, π, and γ, which are said to be *"microforces"* [74], is provided also by Podio Guidugli [109]. Here, by slightly reformulating Gurtin and Podio Guidugli's words [74, 109], we say that $\boldsymbol{\xi}$ models contact interactions that a given region $\mathscr{R} \subset \mathscr{B}$ of the body exchanges with the neighboring regions through its boundary $\partial\mathscr{R}$, π describes the interactions exchanged between the lattice and the particles occupying the lattice sites within $\mathscr{R}$ (given that the particles and the lattice are subsystems of the system realized by the complex made of particles and lattice, the force π is internal to the latter system), while γ accounts for non-contact interactions between $\mathscr{R}$ and its environment.

To motivate the employment of the above outlined framework for a problem of growth, and especially of tumor growth, we remark that, as anticipated above, the external force denoted by γ or z may represent, for instance, genetic or epigenetic interactions that are capable of modifying the tumor mass through changes of its density ϱ_{R}, or, equivalently, of the descriptor $J_{\boldsymbol{K}}$, while the complex consisting of lattice and particles (cf. [109]) may be taken as a representation of the system comprising the cells (which play the role of the particles) and the network of collage filaments (i.e., the "lattice").

Next, we turn to the dissipation inequality. Hence, we modify Equation (35) to account for the powers associated with the mass flux $\boldsymbol{\mathfrak{v}}$ and the force $\mathfrak{f}$, thereby obtaining

$$\int_{\mathscr{R}} \mathcal{D}_{\mathrm{R,new}} = \int_{\mathscr{R}} \mathcal{D}_{\mathrm{R,old}} + \int_{\partial\mathscr{R}} \mathfrak{f}[\dot{J}_{\boldsymbol{K}}/J_{\boldsymbol{K}}]\boldsymbol{N} - \int_{\partial\mathscr{R}} \mu_{\mathrm{ch}} J_{\boldsymbol{K}}^{-1}\boldsymbol{\mathfrak{v}}\boldsymbol{N} \geq 0, \tag{56}$$

where $\int_{\mathscr{R}} \mathcal{D}_{\mathrm{R,old}}$ coincides with the right-hand-side of Equation (35) in which, however, the identification $\boldsymbol{Z} : \boldsymbol{K}^{-1}\dot{\boldsymbol{K}} = zJ_{\boldsymbol{K}}^{-1}\dot{J}_{\boldsymbol{K}}$ is made. By localizing this result, setting $\mu_{\mathrm{ch}} = -\mu_{\boldsymbol{K}}$, and using the force balance (53b) and the mass balance (52), we find

$$\mathcal{D}_{\mathrm{R,new}} = -\dot{\Psi}_{\mathrm{R}} + \boldsymbol{P} : \dot{\boldsymbol{F}} + q_{\mathrm{u}}J_{\boldsymbol{K}}^{-1}\dot{J}_{\boldsymbol{K}} + \mathfrak{f}J_{\boldsymbol{K}}^{-1}\mathrm{Grad}\dot{J}_{\boldsymbol{K}} + \boldsymbol{\mathfrak{v}}\,\mathrm{Grad}(J_{\boldsymbol{K}}^{-1}\mu_{\boldsymbol{K}}) \geq 0. \tag{57}$$

To complete the study of the dissipation inequality (57), we choose $\boldsymbol{F}$, $J_{\boldsymbol{K}}$, $\mathrm{Grad}J_{\boldsymbol{K}}$, $\dot{J}_{\boldsymbol{K}}$, $J_{\boldsymbol{K}}^{-1}\mu_{\boldsymbol{K}}$, and $\mathrm{Grad}(J_{\boldsymbol{K}}^{-1}\mu_{\boldsymbol{K}})$ as independent variables, and we express Ψ_{R}, $\boldsymbol{P}$, q_{u}, $\mathfrak{f}$, and $\mathfrak{v}$ as functions of these variables. Then, by following the Coleman-Noll procedure, and exploiting the fact that the constitutive expressions of Ψ_{R}, $\boldsymbol{P}$, and $\mathfrak{f}$ can be proven to be independent of $\dot{J}_{\boldsymbol{K}}$, $J_{\boldsymbol{K}}^{-1}\mu_{\boldsymbol{K}}$, and $\mathrm{Grad}(J_{\boldsymbol{K}}^{-1}\mu_{\boldsymbol{K}})$, we obtain

$$\boldsymbol{P} \equiv \hat{\boldsymbol{P}} \circ (\boldsymbol{F}, J_{\boldsymbol{K}}, \mathrm{Grad}J_{\boldsymbol{K}}) := \frac{\partial \hat{\Psi}_{\mathrm{R}}}{\partial \boldsymbol{F}} \circ (\boldsymbol{F}, J_{\boldsymbol{K}}, \mathrm{Grad}J_{\boldsymbol{K}}), \tag{58a}$$

$$q_{\mathrm{u}} := q_{\mathrm{u,d}} + J_{\boldsymbol{K}}\left[\frac{\partial \hat{\Psi}_{\mathrm{R}}}{\partial J_{\boldsymbol{K}}} \circ (\boldsymbol{F}, J_{\boldsymbol{K}}, \mathrm{Grad}J_{\boldsymbol{K}})\right], \tag{58b}$$

$$\mathfrak{f} \equiv \hat{\mathfrak{f}} \circ (\boldsymbol{F}, J_{\boldsymbol{K}}, \mathrm{Grad}J_{\boldsymbol{K}}) := J_{\boldsymbol{K}}\left[\frac{\partial \hat{\Psi}_{\mathrm{R}}}{\partial \mathrm{Grad}J_{\boldsymbol{K}}} \circ (\boldsymbol{F}, J_{\boldsymbol{K}}, \mathrm{Grad}J_{\boldsymbol{K}})\right], \tag{58c}$$

where $q_{\mathrm{u,d}}$ is referred to as the dissipative part of q_{u}, in analogy with the definition of the force $y_{\mathrm{u,d}}$ in Equation (49). Thus, we are left with the residual dissipation

$$\mathcal{D}_{\mathrm{R,new}} = q_{\mathrm{u,d}} J_{\boldsymbol{K}}^{-1}\dot{J}_{\boldsymbol{K}} + \mathfrak{v}\,\mathrm{Grad}(J_{\boldsymbol{K}}^{-1}\mu_{\boldsymbol{K}}) \geq 0, \tag{59}$$

which allows to take

$$q_{\mathrm{u,d}} \equiv \hat{q}_{\mathrm{u,d}} \circ (\boldsymbol{F}, J_{\boldsymbol{K}}, \mathrm{Grad}J_{\boldsymbol{K}}, \dot{J}_{\boldsymbol{K}}, J_{\boldsymbol{K}}^{-1}\mu_{\boldsymbol{K}}, \mathrm{Grad}(J_{\boldsymbol{K}}^{-1}\mu_{\boldsymbol{K}})) := \kappa_{\nu}\dot{J}_{\boldsymbol{K}}, \tag{60a}$$

$$\mathfrak{v} \equiv \hat{\mathfrak{v}} \circ (\boldsymbol{F}, J_{\boldsymbol{K}}, \mathrm{Grad}J_{\boldsymbol{K}}, \dot{J}_{\boldsymbol{K}}, J_{\boldsymbol{K}}^{-1}\mu_{\boldsymbol{K}}, \mathrm{Grad}(J_{\boldsymbol{K}}^{-1}\mu_{\boldsymbol{K}})) := \mathfrak{M}\,\mathrm{Grad}\left(\frac{\mu_{\boldsymbol{K}}}{J_{\boldsymbol{K}}}\right), \tag{60b}$$

where $\kappa_{\nu} > 0$ can be understood as a strictly positive, bulk generalized viscosity, and the positive semi-definite, second-order tensor field $\mathfrak{M}$ is said to be the medium's *mobility tensor* [74].

Now, the growth law $R_{\gamma(\mathrm{ph})}$ is prescribed phenomenologically, the normalized mass flux $\mathfrak{v}$ is given by Equation (60b), and the Lagrange multiplier $\mu_{\boldsymbol{K}}$ can be expressed as a combination of the other forces featuring in the force balance (53b), i.e.,

$$\mu_{\boldsymbol{K}} = z - q_{\mathrm{u}} - \mathfrak{f}\frac{\mathrm{Grad}J_{\boldsymbol{K}}}{J_{\boldsymbol{K}}} + \mathrm{Div}\mathfrak{f} = z - \kappa_{\nu}\dot{J}_{\boldsymbol{K}} - J_{\boldsymbol{K}}\,\mathcal{E}_{J_{\boldsymbol{K}}}\Psi_{\mathrm{R}}, \tag{61}$$

where we introduced the notation

$$\mathcal{E}_{J_{\boldsymbol{K}}}\Psi_{\mathrm{R}} := \frac{\partial \hat{\Psi}_{\mathrm{R}}}{\partial J_{\boldsymbol{K}}} \circ (\boldsymbol{F}, J_{\boldsymbol{K}}, \mathrm{Grad}J_{\boldsymbol{K}}) - \mathrm{Div}\left[\frac{\partial \hat{\Psi}_{\mathrm{R}}}{\partial \mathrm{Grad}J_{\boldsymbol{K}}} \circ (\boldsymbol{F}, J_{\boldsymbol{K}}, \mathrm{Grad}J_{\boldsymbol{K}})\right]. \tag{62}$$

Thus, we have enough information to determine $J_{\boldsymbol{K}}$ and $\mu_{\boldsymbol{K}}$, which, indeed, are obtained by solving the system of equations

$$\frac{\dot{J}_{\boldsymbol{K}}}{J_{\boldsymbol{K}}} = -\frac{1}{J_{\boldsymbol{K}}}\mathrm{Div}\left[\mathfrak{M}\mathrm{Grad}\left(\frac{\mu_{\boldsymbol{K}}}{J_{\boldsymbol{K}}}\right)\right] + R_{\gamma(\mathrm{ph})}, \tag{63a}$$

$$\frac{\mu_{\boldsymbol{K}}}{J_{\boldsymbol{K}}} = \frac{z}{J_{\boldsymbol{K}}} - \kappa_{\nu}\frac{\dot{J}_{\boldsymbol{K}}}{J_{\boldsymbol{K}}} - \mathcal{E}_{J_{\boldsymbol{K}}}\Psi_{\mathrm{R}}. \tag{63b}$$

8 Conclusions

In this work, we have proposed a formulation of the mechanics of bulk growth in which the rate of variation of the mass of a body is assigned from the outset through a growth law prescribed phenomenologically. To better capture the implications of our approach, we summarize our results as follows.

8.1 Main results within the theory of null grade in the growth tensor

In Section 3, and, in particular, by means of Equations (3), (4), (7), and (8), we have shown the phenomenological assignment of the growth law and the rephrasing of the mass balance law in light of the BKL decomposition, which allows to interpret the mass balance law itself as a non-holonomic and rheonomic constraint on the growth tensor. This constraint has then been put in Pfaffian form [95] in Equation (20a), thereby establishing the basis for introducing the virtual variations $\delta \boldsymbol{K}$, $\delta \mathcal{T}$, and $\delta \mathfrak{T}$, where $\mathfrak{T}$ is the fictitious Lagrangian parameter representing time [106]. The main result of this formulation is given by Equations (19) and (20b), in which the constraint on the growth tensor is expressed in terms of the generalized virtual displacements $\delta \boldsymbol{K}$, $\delta \mathcal{T}$, and $\delta \mathfrak{T}$, and is attached to the "constrained version" of the PVW. This version of the PVW constitutes the crux of our work, and is presented in detail in Section 5, where we revise the PVW, and obtain the boundary value problem (34a)–(34e) of interest for the study at hand.

In Section "Constitutive laws and final form of the initial and boundary value problem", after presenting the constitutive framework, studying the dissipation inequality, and showing the final form of the initial and boundary value problem in Equations (42a)–(42k), we obtain the first results concerning the generalized internal forces $\mathcal{Y}_{\rm u}$ and $\boldsymbol{Y}_{\rm u,d}$ as well as the Lagrange multipliers $\mu_{\mathfrak{T}}$ and $\mu_{\boldsymbol{K}}$. These results serve as comments to Equations (44a), (44b), and (45), and can be summarized in the following remarks:

Remark 8.1 (The case of no mass variation, i.e., $R_{\gamma(\rm ph)} = 0$)
If we set $R_{\gamma(\rm ph)} = 0$, thereby switching off the variation of mass, we find $\mathcal{Y}_{\rm u} = -\mu_{\mathfrak{T}} = 0$. This result, which trivially follows from Equation (45), is consistent with the fact that, for $R_{\gamma(\rm ph)} = 0$, the constraint (7) becomes holonomic, and amounts to requiring that the growth-induced distortions are isochoric, i.e., $\dot{J}_{\boldsymbol{K}} = 0$ (see also Equation (34e)). In this case, the fictitious Lagrangian parameter $\mathfrak{T}$ need not be introduced at all, and, accordingly, Equations (29e) and (30a) disappear from the model, while the Lagrange multiplier $\mu_{\boldsymbol{K}}$ reduces to $\mu_{\boldsymbol{K}} = \frac{1}{3}{\rm tr}\boldsymbol{Z} - \frac{1}{3}{\rm tr}\boldsymbol{H}$, with $-\frac{1}{3}{\rm tr}\boldsymbol{H}$ acquiring the meaning of a generalized, "configurational" pressure [64], and $\boldsymbol{H}$ being evaluated for admissible tensors $\boldsymbol{K}$. On the other hand, the evolution of $\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}$, which is a deviatoric tensor, is governed by Equation (34d). Thus, in general, even for vanishing ${\rm dev}\boldsymbol{Z}$, the configurational force $-{\rm dev}\boldsymbol{H}$ may trigger the evolution of non-trivial, plastic-like distortions, described by the isochoric tensor field $\boldsymbol{K}$, as shown in Equation (42d). In this respect, some linear models of the biomechanical process known as "remodeling" are recovered [7, 64, 105, 68]. Finally, we notice that, since it holds true that ${\rm tr}\boldsymbol{Y}_{\rm u,d} = J_{\boldsymbol{K}}[\mathfrak{a}_\nu + 2\mathfrak{b}_\nu]{\rm tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}})$ and ${\rm tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}}) = R_{\gamma(\rm ph)}$, the vanishing of $R_{\gamma(\rm ph)}$ also implies the vanishing of the volumetric part of the dissipative generalized force $\boldsymbol{Y}_{\rm u,d}$. Still, the converse is not true, as highlighted by Remark 8.2.

Remark 8.2 (The case of vanishing ${\rm tr}\boldsymbol{Y}_{\rm u,d}$)
Since it descends from Equation (40) that ${\rm tr}\boldsymbol{Y}_{\rm u,d} = J_{\boldsymbol{K}}[\mathfrak{a}_\nu + 2\mathfrak{b}_\nu]{\rm tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}})$, one may consider the

case $\mathfrak{a}_\nu + 2\mathfrak{b}_\nu = 0$*, thereby characterizing the situation in which the spherical component of the dissipative generalized force* $\boldsymbol{Y}_{\mathrm{u,d}}$ *vanishes identically, regardless of the values taken by* $\mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}})$*. Also in this situation, the Lagrange multiplier* $\mu_{\boldsymbol{K}}$ *reduces to the configurational pressure* $\mu_{\boldsymbol{K}} = \frac{1}{3}\mathrm{tr}\boldsymbol{Z} - \frac{1}{3}\mathrm{tr}\boldsymbol{H}$*, but, as long as the condition* $R_{\gamma(\mathrm{ph})} \neq 0$ *is fulfilled, growth persists and the evolution of* $\boldsymbol{K}$ *is fully determined by Equations* (34d) *and* (34e)*. Growth, thus, continues to be a dissipative process.*

In Section 6.4, we showed how our approach can be used to recover systematically the situation, often encountered in the modeling of tumor growth, in which the growth tensor is assumed to be spherical from the outset [8, 11, 59, 56]. Also for this situation, which can be described in terms of the determinant of the growth tensor, i.e., $J_{\boldsymbol{K}}$, we discussed separately the limit cases of no growth (cf. Remark 8.1) and of vanishing dissipative force $y_{\mathrm{u,d}}$ (cf. Remark 8.2). These can be summarized as follows:

- The case of no growth (see Remark 8.1) trivially restricts $J_{\boldsymbol{K}}$ to remain equal to its initial distribution, $J_{\boldsymbol{K}}^{\mathrm{in}}$, while $\mu_{\boldsymbol{K}}$ becomes $\mu_{\boldsymbol{K}} = z - J_{\boldsymbol{K}}^{\mathrm{in}}[\partial_{J_{\boldsymbol{K}}}\hat{\Psi}_{\mathrm{R}} \circ (\boldsymbol{F}, J_{\boldsymbol{K}}^{\mathrm{in}})]$, with z being possibly null. The quantity $J_{\boldsymbol{K}}^{\mathrm{in}}$, in fact, need not be unitary, in general, because the "initial" state of the body may coincide with a state in which growth has occurred and has then come to a stop (see e.g. [36]). Clearly, also in this case, no growth implies the vanishing of $y_{\mathrm{u,d}}$, whereas the vice versa, in general, does not apply.
- The discussion done in Remark 8.2 on the vanishing of $y_{\mathrm{u,d}}$ is recovered for $\kappa_\nu = 0$, thereby yielding $y_{\mathrm{u}} = J_{\boldsymbol{K}}[\partial_{J_{\boldsymbol{K}}}\hat{\Psi}_{\mathrm{R}} \circ (\boldsymbol{F}, J_{\boldsymbol{K}})]$, and $\mu_{\boldsymbol{K}} = z - J_{\boldsymbol{K}}[\partial_{J_{\boldsymbol{K}}}\hat{\Psi}_{\mathrm{R}} \circ (\boldsymbol{F}, J_{\boldsymbol{K}})]$, while the evolution of $J_{\boldsymbol{K}}$ remains prescribed by Equation (46).

8.2 Main results within the Cahn-Hilliard theory

A result that we deem particularly relevant for our formulation is the reframing of the Cahn-Hilliard model provided by Gurtin [74] within the context of the mechanics of bulk growth. This has been discussed in Section 7. In particular, we remark that Equations (63a) and (63b), which are obtained by adapting Gurtin's procedure [74] to our problem, boil down to Equations (46) and (51), respectively, if the body's mobility tensor $\mathfrak{M}$ is assumed to be null and if the dependence on the strain energy density $\hat{\Psi}_{\mathrm{R}}$ on $\mathrm{Grad}J_{\boldsymbol{K}}$ is suppressed. Furthermore, we think that, in order to highlight how our work is connected with that of others through a strongly similar physics, it could be interesting to evaluate again the case of vanishing growth and the case of vanishing dissipative force $q_{\mathrm{u,d}}$ within the framework of the Cahn-Hilliard model. This can be summarized as follows:

Remark 8.3 (Vanishing $R_{\gamma(\mathrm{ph})}$ and vanishing $q_{\mathrm{u,d}}$ within the Cahn-Hilliard approach) *Equations* (63a) *and* (63b) *show that, quite differently from what has been said in Remark 8.1, the condition* $R_{\gamma(\mathrm{ph})} = 0$ *does not imply, in this case,* $\dot{J}_{\boldsymbol{K}} = 0$*. Rather, it prescribes that* $J_{\boldsymbol{K}}$ *evolves according to the Cahn-Hilliard equation* (63a)*, with* $R_{\gamma(\mathrm{ph})} = 0$*, and that the Lagrange multiplier* $\mu_{\boldsymbol{K}}$ *is determined by Equation* (63b)*, possibly augmented with the supplementary condition* $z = 0$*, if required. Thus, even in the absence of "true" growth, the movement of mass within the body, described by the re-distribution of* $J_{\boldsymbol{K}}$*, is driven by diffusion. This result, in fact, recalls what has been obtained obtained by Epstein [40] in a work in which he hypothesized a sort of diffusion equation for a tensor-valued field representing the* "material inhomogeneities" *of a body [101] (see* *also [36] for the case in which Epstein's framework was extended to biphasic media for studying tumor growth). Indeed, within this scenario, a theory of growth based on the assumption of purely volumetric growth tensor* $\boldsymbol{K} = J_{\boldsymbol{K}}^{1/3}\boldsymbol{I}$ *boils down to a Cahn-Hilliard model of diffusion for* $J_{\boldsymbol{K}}$*, with the dissipative term* $-\kappa_\nu \dot{J}_{\boldsymbol{K}}$*, i.e.,*

$$\dot{J}_{\boldsymbol{K}} = -\mathrm{Div}\left[\mathfrak{M}\mathrm{Grad}\left(\frac{\mu_{\boldsymbol{K}}}{J_{\boldsymbol{K}}}\right)\right], \tag{64a}$$

$$\frac{\mu_{\boldsymbol{K}}}{J_{\boldsymbol{K}}} = -\kappa_\nu \frac{\dot{J}_{\boldsymbol{K}}}{J_{\boldsymbol{K}}} - \mathcal{E}_{J_{\boldsymbol{K}}}\Psi_{\mathrm{R}}. \tag{64b}$$

There is, however, another nuance concealed in the Cahn-Hilliard approach. Indeed, whereas in the model discussed in Section "The limit case of spherical growth tensor" *the case of no growth* ($R_{\gamma(\mathrm{ph})} = 0$) *also yields the vanishing of* $y_{\mathrm{u,d}}$*, because it implies* $\dot{J}_{\boldsymbol{K}} = 0$*, this implication does not hold true in the present framework, since the condition* $R_{\gamma(\mathrm{ph})} = 0$ *does not require* $\dot{J}_{\boldsymbol{K}} = 0$*, and, thus, it does not lead to* $q_{\mathrm{u,d}} = 0$*. In fact, one can compell* $q_{\mathrm{u,d}}$ *to be null by choosing* $\kappa_\nu = 0$*, thereby re-obtaining the standard version of the Cahn-Hilliard model.*

8.3 Connections with other theories of growth

The formulation of the mechanics of bulk growth proposed in this work may be regarded as a "bridge" between the perspective supplied by Epstein and Maugin [41] and the ones developed by DiCarlo and Quiligotti [38] and, later, by DiCarlo [37]. To explain this, let us briefly recall the most important results of these two approaches.

8.3.1 Growth viewed as a "flow rule".

Epstein and Maugin [41] write the mass balance law in a way similar to our Equation (3)[8], but, in their case, the source/sink of mass *is not* given phenomenologically. Rather, after showing how to determine what they call *"transplant operator"* [41], i.e., *formally* the inverse of the growth tensor used in our work, Epstein and Maugin [41] compute the source/sink of mass *a posteriori* as $R_\gamma = \mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}})$ (in our notation). For a comparison, the Reader is referred to Equation (9.21) of [41], in which our R_γ is written as "Π/ϱ_0", and our $\mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}})$ replaces their "$-\mathrm{tr}\boldsymbol{L}_{\boldsymbol{K}}$". To show how to obtain $\boldsymbol{K}$, Epstein and Maugin [41] determine an evolution law for it, in which a suitable rate of $\boldsymbol{K}$ is expressed as a function of its power-conjugated generalized force, i.e., Eshelby stress tensor. This result, in a sense, may be understood as a flow rule for a viscoplastic medium, as recognized by DiCarlo [37], although its range of validity within a growth theory may need further investigations. Indeed, in our opinion, one ought to make sure that $\mathrm{tr}(\boldsymbol{K}^{-1}\dot{\boldsymbol{K}})$ does not vanish identically for vanishing Eshelby stress, since this would imply that stress is the only activator, or deactivator, of growth.

8.3.2 The "Eshelbian coupling" pointed out by DiCarlo and Quiligotti [38].

As anticipated in the Introduction, DiCarlo and Quiligotti [38] formulate a model of growth whose core is the balance of the stress-like forces dual to the virtual variations of $\boldsymbol{K}$. In fact, this

[8] In fact, this is true up to the presence of a mass flux vector, which we neglect in the first part of our work and consider only afterwards, when we compare our approach with the one by Gurtin [74].

force balance is obtained without any *a priori* constraint on the growth tensor or its rate, and, after establishing the constitutive framework and studying the system's dissipation inequality, it is written as $\boldsymbol{Y}_{\mathrm{d}}+\boldsymbol{H}=\boldsymbol{Z}$ (in our notation), where $\boldsymbol{Y}_{\mathrm{d}}$ is the dissipative part of the overall generalized internal force $\boldsymbol{Y}$ (compare with our Equation (29d)). This way, DiCarlo and Quiligotti [38] highlight the interaction between Eshelby stress tensor $\boldsymbol{H}$ and the stress-like generalized force $\boldsymbol{Y}$ ("$-\mathbb{C}$" in their notation). This interaction, referred to as *"Eshelbian coupling"* by DiCarlo and Quiligotti [38], is mentioned also by DiCarlo [37] in conjunction with the role attributed to the generalized force $\boldsymbol{Z}$ ("$\mathbb{B}$" in his notation). Indeed, by rephrasing DiCarlo's words [37], one may say that $\boldsymbol{Z}$ resolves the biochemical interactions that, possibly occurring at different scales, promote or hinder the growth of a biological system, and models their influence on the mechanics of the system's structural evolution at the continuum scale. Thus, if we correctly interpret DiCarlo's thoughts [37], it is hard to construct a biologically consistent mechanical theory of growth without $\boldsymbol{Z}$, since Eshelby stress tensor alone is unable to capture the necessary biological information that guides growth. This, in turn, contrasts with the model of growth provided by Epstein and Maugin [41].

Even though we agree on the fact that the picture proposed by Epstein and Maugin [41] may be too restrictive in "real" biological situations, since we believe that it is the force unbalance $\boldsymbol{Z}-\boldsymbol{H}$, rather than $\boldsymbol{H}$, that should be considered in biological growth, our interpretation of the role of Eshelby stress tensor is different from that given by DiCarlo and Quiligotti [38]. Indeed, in our opinion, $\boldsymbol{H}$ is the *"driving force"* [41] of the contribution to the overall variation of mass of a body that is ascribable to the development and redistribution of *"material inhomogeneities"* [41, 101]. On the other hand, we think that also $\boldsymbol{Z}-\boldsymbol{H}$ may fail to describe some growth laws supported by experiments. It is exactly this observation that suggested us to reformulate growth as a constrained problem. This way, indeed, one is free to assign from the outset the growth law that best fits a given phenomenology by just paying the price of introducing the Lagrange multiplier $\mu_{\boldsymbol{K}}$. In this respect, this part of our approach seems to comply with the biochemical interactions discussed in [37]. Furthermore, the Lagrange multiplier, although being by definition the dual force of the variation of mass, need not feature explicitly in the mass balance, i.e., the constraint of the theory, unless one resorts, for instance, to diffusion models, like the Cahn-Hilliard one discussed by Gurtin [74], in which, besides R_{γ} (in our notation), the transport of mass is considered and associated with the gradient of $\mu_{\boldsymbol{K}}$. In addition, in our approach (here limited to the case of isotropic material), the deviatoric tensor $\mathrm{dev}\boldsymbol{Z}-\mathrm{dev}\boldsymbol{H}$ is the *"driving force"*, as predicted by Epstein and Maugin [41], of the isochoric distortions associated with growth, but not directly related to the variation of mass. These distortions, indeed, make the growth tensor generally non-spherical, thereby allowing for models even more general than those usually encountered in the description of tumor growth. In this respect, we have in mind also those growth models formulated for bone, skin, arteries or heart mechanics, in which the growth tensor is assumed to be symmetric, but non-spherical, and with principal (anisotropy) directions assigned from the outset (see [6, 5] for a review). According to our model, instead, also in all these cases, $\boldsymbol{K}$ has to be computed by solving Equations (42a)–(42k), and it is a suitably modeled external force $\boldsymbol{Z}$ that determines, through its interaction with Eshelby stress tensor, i.e., through $\mathrm{dev}\boldsymbol{Z}-\mathrm{dev}\boldsymbol{H}$, how much $\boldsymbol{K}$ deviates from a spherical tensor and which symmetries it may possess (cf. Equation (42d)).

8.3.3 Towards a unified approach to inelastic processes.

As a final remark, let us notice that, since our approach is based on a growth law given *a priori*, the variation of mass considered in our work need not be correlated, in principle, with any measure of stress, although we do let $R_{\gamma(\mathrm{ph})}$ depend on $\wp$, as reported in Equation (4). This fact emphasizes that the constrained approach, although being constrained, guarantees a certain freedom in the choice of the growth law; a freedom that is balanced by the restrictions placed on tensor $\boldsymbol{K}$. In this respect, however, we think that our approach may be used also in physical situations deeply different from growth, in which it is anyway necessary, or preferable, to assign the evolution of $\boldsymbol{K}$ *a priori*. Indeed, as an outlook for future research, we have in mind to reformulate in the context of growth and/or remodeling some models of the inelastic phenomena taken from the literature, like, for instance, Gurtin's constrained plasticity [71, 72, 73], or the plastic flow rules suggested by Mićunović [102], and obtained experimentally for the case of non-associative plasticity. Furthermore, a natural extension of a theory of growth of grade one in the inelastic variable $\boldsymbol{K}$ would consist in switching to constitutive relations of grade two in the deformation. Although such approaches, in fact, have been proposed for the case of bone remodeling by adopting linear energy densities (see e.g. [54]), the framework of growth might call for the generalization of these energies to the nonlinear case.

Conflict of Interests

The Authors declare that they have no conflict of interests.

Acknowledgments

The Authors wish to thank Prof. Luigi Preziosi for enlightening discussions on the phenomenological growth law used in this work, Ms. Francesca Ballatore for fruitful discussions about the growth-conjugated, generalized external force, and Ms. Valentina Licari for very precious discussions about the preliminary stages of this work. A. Grillo is indebted with Prof. Eliot Fried for providing both published and unpublished literature, and for sharing his view on growth mechanics. A. Grillo is partially funded by the *Dipartimento di Eccellenza*, Politecnico di Torino (Italy), Project No. E11G18000350001. S. Di Stefano acknowledges *Regione Puglia* in the context of the REFIN research project *"Riciclo di materiali e sostenibilità: modelli di delaminazione per dispositivi laminati"* and INdAM (National Institute of High Mathematics) in the context of *"Progetto Giovani GNFM 2020-2022"*. This work is partially supported by MIUR (Italian Ministry of Education, Universities and Research) through the PRIN project n. 2020F3NCPX on *"Mathematics for industry 4.0 (Math4I4)"* and through the PRIN project n. 2017KL4EF3 on *"Mathematics of active materials: From mechanobiology to smart devices."*

A1: Time as a fictitious, additional Lagrangian parameter

To explain the reasons for regarding time as a fictitious, additional Lagrangian parameter of the problem under investigation, let us briefly review the approach of Analytical Mechanics to a generic discrete mechanical system [17] subjected to non-holonomic and rheonomic constraints [87]. To this end, let us consider a system of this type, described by $n \geq 1$, $n \in \mathbb{N}$, free generalized coordinates, which, as is customary in Analytical Mechanics, are denoted by $q^1, \dots, q^n$. For each $k = 1, \dots, n$, let $q^k : [t_{\text{in}}, t_{\text{fin}}] \to \mathbb{R}$ be a function of time fulfilling all the differentiability hypotheses that are necessary for the forthcoming discussion. Let us also consider $m \in \mathbb{N}$, $m \leq n$, linearly independent, non-holonomic and rheonomic constraints, i.e., restrictions on the generalized velocities $\dot{q}^1, \dots, \dot{q}^n$ that, under the hypotheses of linearity in $\dot{q}^1, \dots, \dot{q}^n$, can be expressed as

$$\check{\mathcal{C}}^i(q(t), \dot{q}(t), t) := \sum_{k=1}^{n} [a^i{}_k(q(t), t)]\dot{q}^k(t) + b^i(q(t), t) = 0, \quad i = 1, \dots, m, \tag{65}$$

where q denotes the array $q := (q^1, \dots, q^n)$, while the coefficients $a^i{}_k(q(t), t)$ and $b^i(q(t), t)$ are given functions of the generalized coordinates and time. We remark that the constraints defined in Equation (65) are analogous, for discrete systems, to those introduced in Equations (7) and (8).

Since the matrix constructed with the functions $a^i{}_k(q(t), t)$ has maximal rank for all t and $q(t)$, only $n - m$ generalized velocities can be taken as linearly independent in Equation (65). Accordingly, if the constraints are employed explicitly to select *a priori* the admissible motions of the system, the remaining m generalized velocities are to be understood as functions of the linearly independent ones as well as of the coefficients $b^i(q(t), t)$ and $a^i{}_k(q(t), t)$. The relations obtained this way must be respected also by the virtual velocities of the considered mechanical system, since, by definition, they must be instantaneously in harmony with the imposed constraints. In this respect, it can be noticed that, even when the linearly independent velocities are assumed to vanish, the coefficients $b^i(q(t), t)$, when they are nonzero, render the m dependent velocities (be they virtual or real) nonzero, too.

On the other hand, if the constraints (65) are accounted for through the method of Lagrange multipliers, framed within the context of the Principle of Virtual Work, the coefficients $b^i(q(t), t)$ necessitate a dedicated study. Indeed, since they are not multiplied by any virtual displacement, they spoil the standard procedure on which the Principle of Virtual Work is based. To see this, let us define the virtual displacements $\delta q^1, \dots, \delta q^n$, and let us recall that, at each fixed time $t \in [t_{\text{in}}, t_{\text{fin}}]$, and for each $k = 1, \dots, n$, the symbol $\delta q^k(t)$ represents a virtual variation of the value taken by q^k at time t, i.e., $q^k(t)$. Hence, the collection $\delta q(t) := (\delta q^1(t), \dots, \delta q^n(t))$ represents a virtual variation of the system's global configuration at time t, i.e., $q(t)$, and the corresponding virtual work can be written as $\sum_{k=1}^{n} \mathcal{Q}_k(t)\delta q^k(t)$, where $\mathcal{Q}_k(t)$ denotes the Lagrange generalized force dual to $\delta q^k(t)$[9].

Granted this background, as suggested by Lanczos [87], the constraints (65) can be reformulated as

$$\hat{\mathcal{C}}^i(q(t), \delta q(t), \delta t(t), t) = \sum_{k=1}^{n} [a^i{}_k(q(t), t)]\delta q^k(t) + b^i(q(t), t)\delta t(t) = 0, \quad i = 1, \dots, m, \tag{66}$$

where $\delta t(t)$ ("δt", in Lanczos' original notation [87]) is a translation of time attached at the instant of time t.

By introducing m unknown, time-dependent Lagrange multipliers $\mu_1, \dots, \mu_m$, the quantity $\sum_{i=1}^{m} \sum_{k=1}^{n} \mu_i(t)\hat{\mathcal{C}}^i(q(t), \delta q(t), \delta t(t), t)$ produces the term $\sum_{i=1}^{m} \mu_i(t)[b^i(q(t), t)]\delta t(t)$, which, however, cannot be combined with any of the summands of the virtual work $\sum_{k=1}^{n} \mathcal{Q}_k(t)\delta q^k(t)$, since

[9] It is out of the scopes of this discussion to provide a thorough analysis of the constitutive expressions of Lagrange generalized forces.

none of those features the variation $\delta t(t)$. To solve this problem, we proceed in two steps. First, we introduce the fictitious Lagrangian parameter $\mathfrak{T} : \mathscr{I} \to \mathscr{I}$, such that $\mathfrak{T}(t) = t_0 + t$, where $t_0 \in \mathscr{I}$ is a given constant. We say that the map $\mathfrak{T}$ is a "fictitious Lagrangian parameter" because its evolution is already prescribed and is consistent with the transformation of time of Galileian mechanics, thereby yielding the condition $\dot{\mathfrak{T}}(t) = 1$, for all $t \in \mathscr{I}$. This condition, in fact, can be regarded as an additional constraint, and is satisfied as $\delta\mathfrak{T}(t) = \delta t(t)$, when it is written in terms of the virtual variation of $\mathfrak{T}$ at t, denoted by $\delta\mathfrak{T}(t)$.

The second step consists of giving room to a generalized force dual to $\delta\mathfrak{T}(t)$ [106], hereafter called $\mathcal{Q}_{\mathfrak{T}}(t)$, so that the Principle of Virtual Work, augmented by the method of Lagrange multipliers, and accounting for all the constraints, yields

$$
\begin{aligned}
&\sum_{k=1}^{n} \mathcal{Q}_k(t)\delta q^k(t) + \mathcal{Q}_{\mathfrak{T}}(t)\delta\mathfrak{T}(t) \\
&+ \sum_{i=1}^{m} \mu_i(t)\left\{ \sum_{k=1}^{n} [a^i{}_k(q(t),t)]\delta q^k(t) + b^i(q(t),t)\delta t(t) \right\} + \mu_{\mathfrak{T}}(t)\left\{ \delta\mathfrak{T}(t) - \delta t(t) \right\} = 0,
\end{aligned}
\tag{67}
$$

where $\mu_{\mathfrak{T}}(t)$ is the Lagrange multiplier associated with the constraint $\delta\mathfrak{T}(t) - \delta t(t) = 0$. Hence, by putting together all the terms multiplied by the same virtual variation, Equation (67) can be rewritten as

$$
\begin{aligned}
&\sum_{k=1}^{n} \left\{ \mathcal{Q}_k(t) + \sum_{i=1}^{m} \mu_i(t)[a^i{}_k(q(t),t)] \right\} \delta q^k(t) \\
&+ \left\{ \mathcal{Q}_{\mathfrak{T}}(t) + \mu_{\mathfrak{T}}(t) \right\} \delta\mathfrak{T}(t) + \left\{ \sum_{i=1}^{m} \mu_i(t)[b^i(q(t),t)] - \mu_{\mathfrak{T}}(t) \right\} \delta t(t) = 0,
\end{aligned}
\tag{68}
$$

and leads to the system of equations

$$
\mathcal{Q}_k(t) + \sum_{i=1}^{m} \mu_i[a^i{}_k(q(t),t)] = 0, \qquad k = 1, \dots, n, \tag{69a}
$$

$$
\mathcal{Q}_{\mathfrak{T}}(t) + \mu_{\mathfrak{T}}(t) = 0, \qquad \mathcal{Q}_{\mathfrak{T}}(t) = -\mu_{\mathfrak{T}}(t), \tag{69b}
$$

$$
\sum_{i=1}^{m} \mu_i(t)[b^i(q(t),t)] - \mu_{\mathfrak{T}}(t) = 0, \qquad \mu_{\mathfrak{T}}(t) = \sum_{i=1}^{m} \mu_i(t)[b^i(q(t),t)], \tag{69c}
$$

which, in conjunction with Equation (65), allow to determine the n Lagrangian parameters $q^1, \dots, q^n$, the m Lagrange multipliers $\mu_1, \dots, \mu_m$, as well as $\mu_{\mathfrak{T}}$ and $\mathcal{Q}_{\mathfrak{T}}$. Note that, for brevity, in equations (67) and (68), we have omitted the terms $\sum_{i=1}^{m} \delta\mu_i(t) t_{\mathrm{c}} \check{\mathcal{C}}^i(q(t), \dot{q}(t), t)$ and $\delta\mu_{\mathfrak{T}} t_{\mathrm{c}}[\dot{\mathfrak{T}}(t) - 1]$, with $t_{\mathrm{c}} > 0$ being a characteristic time. These terms, however, are identically zero.

A2: The case of non-vanishing external time-conjugated force

In this section, we sketch the main changes that take place in the procedure shown in section 6, if the hypothesis concerning the vanishing of $\mathcal{Z}$ is relaxed and, rather, $\mathcal{Z}$ is regarded as an unknown of the problem. In this case, following Gurtin's approach [74], we postulate that the dissipation inequality reads

$$\int_{\mathscr{R}} \mathcal{D}_{\mathrm{R}} = \int_{\mathscr{R}} \mathcal{D}_{\mathrm{R,old}} + \int_{\mathscr{R}} \mathcal{Z}\dot{\mathfrak{T}} - \int_{\mathscr{R}} \mu_{\mathfrak{T}}\dot{\mathcal{T}} \geq 0, \tag{70}$$

where $\int_{\mathscr{R}} \mathcal{D}_{\mathrm{R,old}}$ coincides with the right-hand side of Equation (35), with $\mu_{\mathrm{ch}} \equiv -\mu_{\boldsymbol{K}}$, the term $\mathcal{Z}\dot{\mathfrak{T}}$ is the external power done by $\mathcal{Z}$ on $\dot{\mathfrak{T}}$, and the term $-\mu_{\mathfrak{T}}\dot{\mathcal{T}}$ is introduced in analogy with the last summand on the right-hand side of Equation (35) to account for the fact that $\dot{\mathfrak{T}}$ is constrained to be equal to $\dot{\mathcal{T}}$ from the outset, thereby allowing to identify $\dot{\mathcal{T}}$ as a "source" for $\dot{\mathfrak{T}}$. By performing the same localization procedure that has led to Equation (37) from Equation (35), recalling the force balance $\mathcal{Y}_{\mathrm{u}} + \mu_{\mathfrak{T}} = \mathcal{Z}$ of Equation (29e), and enforcing the constraint $\dot{\mathfrak{T}} = \dot{\mathcal{T}}$, we obtain now

$$\mathcal{D}_{\mathrm{R}} = -\dot{\Psi}_{\mathrm{R}} + \boldsymbol{P} : \dot{\boldsymbol{F}} + \boldsymbol{Y}_{\mathrm{u}} : \boldsymbol{K}^{-1}\dot{\boldsymbol{K}} + \mathcal{Y}_{\mathrm{u}}\dot{\mathfrak{T}} \geq 0. \tag{71}$$

Thus, under the constitutive hypotheses presented in section 6.1, which declare $\hat{\Psi}_{\mathrm{R}}$ as independent of $\mathfrak{T}$, and by assuming that $\mathcal{Y}_{\mathrm{u}}\dot{\mathfrak{T}} = \mathcal{Y}_{\mathrm{u}}\dot{\mathcal{T}} = \mathcal{Y}_{\mathrm{u}}$ is not dissipative (recall that the last equality descends from the identity $\dot{\mathcal{T}}(X,t) = 1$), we conclude that the condition $\mathcal{Y}_{\mathrm{u}} = 0$ must hold, and, thus, that Equation (71) yields Equation (39), i.e., $\mathcal{D}_{\mathrm{R}} = \boldsymbol{Y}_{\mathrm{u,d}} : \boldsymbol{K}^{-1}\dot{\boldsymbol{K}} \geq 0$. Hence, the study of the residual dissipation inequality, the solution of the IBVP (42a)–(42k), and the *a posteriori* determination of $\mu_{\boldsymbol{K}}$ and $\mu_{\mathfrak{T}}$ as shown in Equations (44a) and (44b) can proceed as shown in the main body of our work. However, the difference with respect to the model presented above is that Equation (29e) now determines $\mathcal{Z}$, because $\mathcal{Y}_{\mathrm{u}}$ vanishes for constitutive reasons, so that Equations (31) and (45) now become

$$\mathcal{Z} = \mu_{\mathfrak{T}} = -\mu_{\boldsymbol{K}}[\hat{R}_{\gamma(\mathrm{ph})} \circ (\boldsymbol{F}, \boldsymbol{K}, \omega)] = -\left\{\tfrac{1}{3}\mathrm{tr}\boldsymbol{Z} - \tfrac{1}{3}\mathrm{tr}\boldsymbol{H} - \tfrac{1}{3}J_{\boldsymbol{K}}[\mathfrak{a}_\nu + 2\mathfrak{b}_\nu]R_{\gamma(\mathrm{ph})}\right\}R_{\gamma(\mathrm{ph})}. \tag{72}$$

Therefore, also the conclusion reported in Remark 8.1 must be rephrased accordingly, by saying that, for $R_{\gamma(\mathrm{ph})} = 0$, the condition $\mathcal{Z} = \mu_{\mathfrak{T}} = 0$ complies with the fact that the constraint (7) turns into a holonomic constraint.

References

[1] A. Agosti, C. Cattaneo, C. Giverso, D. Ambrosi, and P. Ciarletta. A computational framework for the personalized clinical treatment of glioblastoma multiforme. *Zeitschrift für Angewandte Mathematik und Mechanik - ZAMM*, 98:2307–2327, 2018.

[2] A. Agosti, C. Giverso, E. Faggiano, A. Stamm, and P. Ciarletta. A personalized mathematical tool for neuro-oncology: A clinical case study. *International Journal of Non-Linear Mechanics*, 107:170–181, 2018.

[3] E.C. Aifantis. On the microstructural origin of certain inelastic models. *Transactions on ASME Journal of Engineering Materials and Technology*, 106:326–330, 1984.

[4] E.C. Aifantis. The physics of plastic deformation. *International Journal of Plasticity*, 3:211–247, 1987.

[5] D. Ambrosi, M. B. Amar, C. J. Cyron, A. DeSimone, A. Goriely, J. D. Humphrey, and E. Kuhl. Growth and remodelling of living tissues: perspectives, challenges and opportunities. *J. R. Soc. Interface*, 16:20190233, 2019.

[6] D. Ambrosi, G.A. Ateshian, E.M. Arruda, S.C. Cowin, J. Dumais, A. Goriely, G.A. Holzapfel, J.D. Humphrey, R. Kemkemer, E. Kuhl, J.E. Olberding, L.A. Taber, and K. Garikipati. Perspectives on biological growth and remodeling. *Journal of the Mechanics and Physics of Solids*, 59(4):863–883, apr 2011.

[7] D. Ambrosi and F. Guana. Stress-modulated growth. *Math. Mech. Solids*, 12:319–342, 2007.

[8] D. Ambrosi and F. Mollica. On the mechanics of a growing tumor. *International Journal of Engineering Science*, 40(12):1297–1316, jul 2002.

[9] D. Ambrosi and L. Preziosi. On the closure of mass balance models for tumor growth. *Mathematical Models and Methods in Applied Sciences*, 12(05):737–754, may 2002.

[10] D. Ambrosi and L. Preziosi. Cell adhesion mechanisms and stress relaxation in the mechanics of tumours. *Biomechanics and Modeling in Mechanobiology*, 8:397–413, 2009.

[11] D. Ambrosi and L. Preziosi. Cell adhesion mechanisms and stress relaxation in the mechanics of tumours. *Biomech Model Mechanobiol*, 8:397–413, 2009.

[12] D. Ambrosi, L. Preziosi, and G. Vitale. The interplay between stress and growth in solid tumors. *Mech. Res. Commun.*, 42:87–91, 2012.

[13] L. Anand, O. Aslan, and A. Chester. A large-deformation gradient theory for elastic-plastic materials: Strain softening and regularization of shear bands. *International Journal of Plasticity*, 30–31:116–143, 2012.

[14] V. I. Arnold. *Mathematical Methods of Classical Mechanics, 2nd Edition*. Springer Verlag, New York, 1989.

[15] G.A. Ateshian and J.A. Weiss. Anisotropic hydraulic permeability under finite deformation. *J. Biomech. Engng.*, 132:111004–1–111004–7, 2010.

[16] F. Baaijens, C. Bouten, and N. Driessen. Modeling cartilage remodeling. *J. Biomech.*, 43:166–175, 2010.

[17] P. Ballard. The dynamics of discrete mechanical systems with perfect unilateral constraints. *Archive for Rational Mechanics and Analysis*, 154:199–274, 2000.

[18] G. Batra. On hamilton's principle for thermo-elastic fluids and solids, and internal constraints in thermo-elasticity. *Archive for Rational Mechanics and Analysis*, 99:37–59, 1987.

[19] M. Böl and A. B. Albero. On a new model for inhomogeneous volume growth of elastic bodies. *J. Mech. Beh. Biom. Mat.*, 29:582–593, 2014.

[20] J. Bonet and R.D. Wood. *Nonlinear Continuum Mechanics for Finite Element Analysis*. Cambridge University Press, New York, 2008.

[21] G. Capriz. *Continua with Microstructure. Springer Tracts in Natural Philosophy, vol. 35.* Springer, New York, 1989.

[22] G. Capriz and P. Podio Guidugli. Formal structure and classification of theories of oriented materials. *Annali di Matematica Pura e Applicata*, 4(2):281–292, 1977.

[23] D. E. Carlson, E. Fried, and D. A. Tortorelli. Geometrically-based consequences of internal constraints. *Journal of Elasticity*, 70:101–109, 2003.

[24] P. Cermelli, E. Fried, and S. Sellers. Configurational stress, yield and flow in rate-independent plasticity. *Proc. R. Soc. Lond. A*, 457:1447–1467, 2001.

[25] M.A.J. Chaplain, L. Graziano, and L. Preziosi. Mathematical modelling of the loss of tissue compression responsiveness and its role in solid tumour development. *Mathematical Medicine and Biology: A Journal of the IMA*, 23(3):197–229, sep 2006.

[26] J. Ciambella, G. Lucci, P. Nardinocchi, and L. Preziosi. Passive and active fiber reorientation in anisotropic materials. *International Journal of Engineering Science*, 176(103688):1–10, 2022.

[27] J. Ciambella and P. Nardinocchi. Torque-induced reorientation in active fibre-reinforced materials. *Soft Matter*, 15(9):2081–2091, 2019.

[28] V. Ciancio, M. Dolfin, M. Francaviglia, and S. Preston. Uniform materials and the multiplicative decomposition of the deformation gradient in finite elasto-plasticity. *J. Non-Equilib. Thermodyn.*, 33(3):199–234, 2008.

[29] E. Cosserat and F. Cosserat. *Théorie des corps déformable.* Hermann, Paris, 1909.

[30] S. C. Cowin and J. W. Nunziato. Linear elastic materials with voids. *Journal of Elasticity*, 13(2):125–147, 1983.

[31] E. Crevacore, S. Di Stefano, and A. Grillo. Coupling among deformation, fluid flow, structural reorganisation and fibre reorientation in fibre-reinforced, transversely isotropic biological tissues. *International Journal of Nonlinear Mechanics*, 111:1–13, 2019.

[32] G. Del Piero. On the method of virtual power in continuum mechanics. *Journal of Mechanics of Materials and Structures*, 4(2):281–292, 2009.

[33] G. Del Piero. A rational approach to cosserat continua, with application to plate and beam theories. *Mechanics Research Communications*, 58:97–104, 2014.

[34] G. Del Piero. Continuum mechanics — an axiomatic framework for the mechanics of generalized continua. *Rend. Lincei Mat. Appl.*, 29:31–61, 2018.

[35] S. Di Stefano, A. Giammarini, C. Giverso, and A. Grillo. An elasto-plastic biphasic model of the compression of multicellular aggregates: the influence of fluid on stress and deformation. *Z. Angew. Math. Phys.*, 73:79–118, 2022.

[36] Salvatore Di Stefano, Ariel Ramírez-Torres, Raimondo Penta, and Alfio Grillo. Self-influenced growth through evolving material inhomogeneities. *International Journal of Non-Linear Mechanics*, 106:174–187, 2018.

[37] A. DiCarlo. Surface and bulk growth unified. In Maugin G.A. Steinmann P., editor, *Mechanics of Material Forces. Advances in Mechanics and Mathematics*, volume 11, pages 53–64. Springer, Boston, MA, 2005.

[38] A. DiCarlo and S. Quiligotti. Growth and balance. *Mechanics Research Communications*, 29(6):449–456, nov 2002.

[39] W. Ehlers, M. M. Rehm, P. Schröder, D. Stöhr, and A. Wagner. Multiphasic modelling and computation of metastatic lung-cancer cell proliferation and atrophy in brain tissue based on experimental data. *Biomech Model Mechanobiol*, 21(1):277–315, 2022.

[40] M. Epstein. Self-driven continuous dislocations and growth. In Maugin G.A. Steinmann P., editor, *Mechanics of Material Forces. Advances in Mechanics and Mathematics*, volume 11, pages 129–139. Springer, Boston, MA, 2005.

[41] M. Epstein and G.A. Maugin. Thermomechanics of volumetric growth in uniform bodies. *International Journal of Plasticity*, 16(7-8):951–978, jun 2000.

[42] M. Epstein and R. Segev. Differentiable manifolds and the principle of virtual work in continuum mechanics. *J. Math. Phys.*, 21(5):1243–1245, 1980.

[43] A. C. Eringen. Mechanics of micromorphic continua. In E. Kröner, editor, *Mechanics of Generalized Continua, IUTAM Symposium*, pages 18–35. Springer-Verlag, Berlin, 1968.

[44] S. Federico and T.C. Gasser. Non-linear elasticity of biological tissues with statistical fibre orientation. *J. R. Soc. Interface*, 7:955–966, 2010.

[45] S. Federico and W. Herzog. On the anisotropy and inhomogeneity of permeability in articular cartilage. *Biomech. Model. Mechanobiol.*, 7:367–378, 2008.

[46] Salvatore Federico, Mawafag F. Alhasadi, and Alfio Grillo. Eshelby's inclusion theory in light of Noether's theorem. *Mathematics and Mechanics of Complex Systems*, 7(3):247–285, 2019.

[47] John Michael Finn. *Classical Mechanics.* Infinity Science Press LLC, Hingham, Massachusetts, New Delhi, 2008.

[48] G. Forgacs, R.A. Foty, Y. Shafrir, and M.S. Steinberg. Viscoelastic properties of living embryonic tissues: a quantitative study. *Biophysical Journal*, 74:2227–2234, 1998.

[49] R.A. Foty, C.M. Pfleger, G. Forgacs, and M.S. Steinberg. Surface tensions of embryonic tissues predict their mutual envelopment behavior. *Development*, 122:1611–1620, 1996.

[50] E. Fried and S. Sellers. The mechanics of volumetric growth: A plasticity-like approach. *Private Communication*, pages 1–24, 2010.

[51] K. Garikipati, E.M. Arruda, K. Grosh, H. Narayanan, and S. Calve. A continuum treatment of growth in biological tissue: the coupling of mass transport and mechanics. *J. Mech. Phys. Solids*, 52:1595–1625, 2004.

[52] P. Germain. Sur l'application de la méthode des puissances virtuelles en mécanique des milieux continus. *C. R. Acad. Sci. Paris Sér. A*, 274:1051–1055, 1972.

[53] P. Germain. The method of virtual power in continuum mechanics. part 2: Microstructure. *SIAM J. Appl. Math.*, 25(3):556–575, 1973.

[54] I. Giorgio, U. Andreaus, F. dell'Isola, and T. Lekszycki. Viscous second gradient porous materials for bones reconstructed with bio-resorbable grafts. *Extreme Mechanics Letters*, 13:141–147, 2017.

[55] I. Giorgio, U. Andreaus, D. Scerrato, and F. dell'Isola. A visco-poroelastic model of functional adaptation in bones reconstructed with bio-resorbable materials. *Biomech Model Mechanobiol*, 15:1325–1343, 2016.

[56] C. Giverso, S. Di Stefano, A. Grillo, and L. Preziosi. A three dimensional model of multicellular aggregate compression. *Soft Matter*, 15(48):10005–10019, 2019.

[57] C. Giverso and L. Preziosi. Modelling the compression and reorganization of cell aggregates. *Mathematical Medicine and Biology*, 29:181–204, 2012.

[58] C. Giverso and L. Preziosi. Influence of the mechanical properties of the necrotic core on the growth and remodelling of tumour spheroids. *International Journal of Non-Linear Mechanics*, 108:20–32, 2019.

[59] C. Giverso, M. Scianna, and A. Grillo. Growing avascular tumours as elasto-plastic bodies by the theory of evolving natural configurations. *Mech. Res. Commun.*, 68:31–39, 2015.

[60] A. E. Green. Micro-materials and multipolar continuum mechanics. *International Journal of Engineering Science*, 3:533–537, 1965.

[61] A. E. Green and R. S. Rivlin. Multipolar continuum mechanics: functional theory. *Proc. Roy. Soc. Ser. A*, 284:303–324, 1965.

[62] A. Grillo, M. Carfagna, and S. Federico. Non-Darcian flow in fibre-reinforced biological tissues. *Meccanica*, 52:3299–3320, 2017.

[63] A. Grillo, M. Carfagna, and S. Federico. An allen-cahn approach to the remodelling of fibre-reinforced anisotropic materials. *J. Eng. Math.*, 109(1):139–172, 2018.

[64] A. Grillo, S. Federico, and G. Wittum. Growth, mass transfer, and remodeling in fiber-reinforced, multi-constituent materials. *Int. J. Nonlinear Mech.*, 47:388–401, 2012.

[65] A. Grillo, R. Prohl, and G. Wittum. A poroplastic model of structural reorganisation in porous media of biomechanical interest. *Continuum Mech. Therm.*, 28:579–601, 2016.

[66] A. Grillo, R. Prohl, and G. Wittum. A generalised algorithm for anelastic processes in elastoplasticity and biomechanics. *Math. Mech. Solids*, 22(3):502–527, 2017.

[67] A. Grillo, G. Wittum, A. Tomic, and S. Federico. Remodelling in statistically oriented fibre-reinforced materials and biological tissues. *Math. Mech. Solids*, 20(9):1107–1129, 2015.

[68] Alfio Grillo, Salvatore Di Stefano, and Salvatore Federico. Growth and remodelling from the perspective of noether's theorem. *Mechanics Research Communications*, 97:89–95, 2019.

[69] Alfio Grillo, Salvatore Di Stefano, Ariel Ramírez-Torres, and Michele Loverre. A study of growth and remodeling in isotropic tissues, based on the anand-aslan-chester theory of strain-gradient plasticity. *GAMM-Mitteilungen*, 42(4), may 2019.

[70] Alfio Grillo and Gabriel Wittum. Growth and mass transfer in multi-constituent biological materials. *American Institute of Physics Conference Proceedings*, 1281:355–358, 2010.

[71] M. E. Gurtin and L. Anand. The decomposition $\mathbf{F} = \mathbf{F}^{\mathrm{e}}\mathbf{F}^{\mathrm{p}}$, material symmetry, and plastic irrotationality for solids that are isotropic-viscoplastic or amorphous. *International Journal of Plasticity*, 21:1686–1719, 2005.

[72] M. E. Gurtin and L. Anand. A theory of strain-gradient plasticity for isotropic, plastically irrotational materials. Part I: Small deformations. *Journal of the Mechanics and Physics of solids*, 53:1624–1649, 2005.

[73] M. E. Gurtin and L. Anand. A theory of strain-gradient plasticity for isotropic, plastically irrotational materials. Part II: Finite deformations. *International Journal of Plasticity*, 21:2297–2318, 2005.

[74] M.E. Gurtin. Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a microforce balance. *Physica D*, 92:178–192, 1996.

[75] M.E. Gurtin, E. Fried, and L. Anand. *The Mechanics and Thermodynamics of Continua.* Cambridge University Press, 2010.

[76] Morton E. Gurtin. On the plasticity of single crystals: free energy, microforces, plastic-strain gradients. *Journal of the Mechanics and Physics of Solids*, 48(5):989–1036, may 2000.

[77] Morton E. Gurtin. On a framework for small-deformation viscoplasticity: free energy, microforces, strain gradients. *International Journal of Plasticity*, 19:47–90, 2003.

[78] G.A. Holzapfel, R.W. Ogden, and S. Sherifova. On fibre dispersion modelling of soft biological tissues: a review. *Proc. R. Soc. A*, 475(2224):1–22, 2019.

[79] T. J. R. Hughes. *The Finite Element Method – Linear Static and Dynamic Finite Element Analysis.* Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1987.

[80] J. D. Humphrey and K. R. Rajagopal. A constrained mixture model for growth and remodeling of soft tissues. *Mathematical Models and Methods in Applied Sciences*, 12(03):407–430, mar 2002.

[81] R.K. Jain, J.D. Martin, and T. Stylianopoulos. The role of mechanical forces in tumor growth and therapy. *Annual Review of Biomedical Engineering*, 16:321–346, 2014.

[82] V. V. Kozlov. Dynamics of systems with non-integrable restrictions i. *Vestn. Mosk. Univ., Ser. I Mat. Mekh.*, 3:92–100, 1982.

[83] V. V. Kozlov. Dynamics of systems with non-integrable restrictions ii. *Vestn. Mosk. Univ., Ser. I Mat. Mekh.*, 3:70–76, 1982.

[84] V. V. Kozlov. Dynamics of systems with non-integrable restrictions iii. *Vestn. Mosk. Univ., Ser. I Mat. Mekh.*, 3:102–111, 1983.

[85] V. V. Kozlov. The problem of realizing constraints in dynamics. *J. Appl. Math. Mech.*, 56:594–600, 1992.

[86] Ellen Kuhl. Growing matter: A review of growth in living systems. *J. Mech. Behav. Biomed. Mater.*, 29:529–543, jan 2014.

[87] Cornelius Lanczos. *The Variational Principles of Mechanics.* Dover Publications, Inc. New York, 1970.

[88] Y. Lanir. Constitutive equations for fibrous connective tissues. *J. Biomech.*, 16:1–12, 1983.

[89] T. Lekszycki. Modelling of bone adaptation based on an optimal response hypothesis. *Meccanica*, 37:343–354, 2002.

[90] T. Lekszycki and F. dell'Isola. A mixture model with evolving mass densities for describing synthesis and resorption phenomena in bones reconstructed with bio-resorbable materials. *ZAMM - Z. Angew. Math. Mech.*, 92(6):426–444, 2012.

[91] N. A. Lemos. Complete inequivalence of nonholonomic and vakonomic mechanics. *Acta Mech.*, 233:47–56, 2022.

[92] A. D. Lewis and R. M. Murray. Variational principles for constrained systems: Theory and experiment. *International Journal of Non-Linear Mechanics*, 30(6):793–815, 1995.

[93] K. Li, R. W. Ogden, and G. A. Holzapfel. Computational method for excluding fibers under compression in modeling soft fibrous solids. *Eur. J. Mech. A/Solids*, 57:178–193, 2016.

[94] Valentina Licari. Considerazioni sulla possibilità di formulare alcune leggi evolutive della crescita volumetrica di aggregati cellulari come equazioni dinamiche di teorie meccaniche dei processi anelastici. Master's thesis, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129, Torino, Italy, 11 2021.

[95] Jaume Llibre, Rafael Ramírez, and Natalia Sadovskaia. Nonlinear dynamics. *A new approach to the vakonomic mechanics*, 78:2219–2247, 2014.

[96] B. Loret and F.M.F. Simões. A framework for deformation, generalized diffusion, mass transfer and growth in multi-species multi-phase biological tissues. *Eur. J. Mech. A*, 24:757–781, 2005.

[97] V.A. Lubarda and A. Hoger. On the mechanics of solids with a growing mass. *International Journal of the Mechanics and Physics of Solids*, 39:4627–4664, 2002.

[98] J.E. Marsden and T.J.R. Hughes. *Mathematical Foundations of Elasticity.* Dover Publications, Inc., Mineola, New York, 1983.

[99] P. Mascheroni, M. Carfagna, A. Grillo, D.P. Boso, and B.A. Schrefler. An avascular tumor growth model based on porous media mechanics and evolving natural states. *Mathematics and Mechanics of Solids*, 23(4):686–712, jun 2018.

[100] P. Mascheroni, C. Stigliano, M. Carfagna, D.P. Boso, L. Preziosi, P. Decuzzi, and B.A. Schrefler. Predicting the growth of glioblastoma multiforme spheroids using a multiphase porous media model. *Biomech. Model. Mechanobiol.*, 15(5):1215–1228, jan 2016.

[101] G. A. Maugin. *Material Inhomogeneities in Elasticity.* Chapman and Hall, London, 1993.

[102] M.V. Mićunović. *Thermomechanics of Viscoplasticity.* Springer New York, 2009.

[103] R. D. Mindlin. Microstructure in linear elasticity. *Arch. Rational Mech. Anal.*, 16:51–78, 1964.

[104] E. Minguzzi. A unifying mechanical equation with applications to non-holonomic constraints and dissipative phenomena. *Journal of Geometric Mechanics*, 7(4):473–482, 2015.

[105] Manuela Minozzi, Paola Nardinocchi, Luciano Teresi, and Valerio Varano. Growth-induced compatible strains. *Math. Mech. Solids*, 22(1):62–71, aug 2016.

[106] A. Nadile. Sull'esistenza per i sistemi anolonomi soggetti a vincoli reonomi di un integrale analogo a quello dell'energia. *Bollettino Unione Matematica Italiana*, 5:297–301, 1950.

[107] T. Olsson and A. Klarbring. Residual stresses in soft tissue as a consequence of growth and remodeling: application to an arterial geometry. *Eur. J. Mech. A*, 27(6):959–974, 2008.

[108] K. A. Pericak-Spector and S. J. Spector. On the representation theorem for linear isotropic tensor functions. *Journal of Elasticity*, 39:181–185, 1995.

[109] P. Podio Guidugli. Models of phase segregation and diffusion of atomic species on a lattice. *Ricerche di Matematica*, 55:105–118, 2006.

[110] L. Preziosi, D. Ambrosi, and C. Verdier. An elasto-visco-plastic model of cell aggregates. *J. Theor. Biol.*, 262(1):35–47, 2010.

[111] L. Preziosi and G. Vitale. A multiphase model of tumor and tissue growth including cell adhesion and plastic reorganization. *Math. Models Methods Appl. Sci.*, 21(09):1901–1932, sep 2011.

[112] A. Ramírez-Torres, R. Rodríguez-Ramos, J. Merodio, J. Bravo-Castillero, R. Guinovart-Díaz, and J.C.L. Alfonso. Action of body forces in tumor growth. *International Journal of Engineering Science*, 89:18–34, apr 2015.

[113] Ariel Ramírez-Torres, Salvatore Di Stefano, and Alfio Grillo. Influence of non-local diffusion in avascular tumour growth. *Mathematics and Mechanics of Solids*, 26(9):1264–1293, 2021.

[114] Ariel Ramírez-Torres, Reinaldo Rodríguez-Ramos, José Merodio, Raimondo Penta, Julián Bravo-Castillero, Raúl Guinovart-Díaz, Federico J. Sabina, Catherine García-Reimbert, Igor Sevostianov, and Aura Conci. The influence of anisotropic growth and geometry on the stress of solid tumors. *International Journal of Engineering Science*, 119:40–49, oct 2017.

[115] E.K. Rodriguez, A. Hoger, and A.D. McCulloch. Stress-dependent finite growth in soft elastic tissues. *J. Biomech.*, 27:455–467, 1994.

[116] J.C. Simo and T.J.R. Hughes. *Computational Inelasticity.* Springer, New York, 1998.

[117] L.A. Taber. Biomechanics of growth, remodeling, and morphogenesis. *Applied Mechanics Reviews*, 48(8):487, 1995.

[118] A. Tomic, A. Grillo, and S. Federico. Poroelastic materials reinforced by statistically oriented fibres — numerical implementation and application to articular cartilage. *IMA J. Appl. Math.*, 79:1027–1059, 2014.

[119] R. A. Toupin. Theory of elasticity with couple-stress. *Arch. Rational Mech. Anal.*, 17:85–112, 1964.

[120] V. Zamani, H. Demirkoparan, and T. J. Pence. Material swelling with partial confinement in the internally balanced generalization of hyperelasticity. *Mathematics and Mechanics of Solids*, In Press, 2022.

[121] G. Zampieri. Nonholonomic versus vakonomic dynamics. *Journal of Differential Equations*, 163:335–347, 2000.