Metrics admitting projective and c-projective vector fields

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Gianni Manno, Jan Schumm, and Andreas Vollmer

Abstract. Vector fields on a manifold with an affine connection are called *projective* (resp. *c-projective*) if their local flows preserve geodesics (resp. J-planar curves). In this expository paper, metrics with such vector fields are discussed. Emphasis is put on Lie's classical 1882 problem of finding a local description of surfaces with projective vector fields, known results on this problem, as well as on various extensions in the literature.

Corresponding author: Gianni Manno

In memory of A. M. Vinogradov, whose Mathematics was and will be of great impact.

1. Introduction

Let M be a smooth manifold of dimension n equipped with a connection ∇. In the present paper, all connections are torsion-free and affine. A curve $\gamma : I \subseteq \mathbb{R} \mapsto \gamma(t) \in M$ such that

$$\nabla_{\dot\gamma}\dot\gamma = \alpha(t)\dot\gamma \tag{1.1}$$

for some function $\alpha \in C^\infty(I)$, is called an *unparametrized geodesic* of ∇.

Definition 1.1. Two connections on the same manifold M are *projectively equivalent* if they share the same unparametrized geodesics. The collection of all connections projectively equivalent to a given connection ∇ is called the *projective class* of ∇, denoted by $[\nabla]_{\mathfrak{p}}$, and it is a *projective structure* on M.

Definition 1.2. A *projective transformation* is a (local) diffeomorphism of M that sends unparametrized geodesics into unparametrized geodesics. A vector field on M is *projective* if its (local) flow acts by projective transformations.

The projective vector fields of a given connection ∇ form a Lie algebra [**26**], called the *projective Lie algebra*, which we denote by $\mathfrak{p}(\nabla)$.

The above definitions can be given also in the case that M is equipped with a *metric* g, i.e., a symmetric, non-degenerate $(0, 2)$-tensor field of arbitrary signature, by considering its Levi-Civita connection. In this case, the projective class of g will be denoted by $[g]_{\mathfrak{p}}$ and its Lie algebra of projective vector fields by $\mathfrak{p}(g)$.

If g is positive definite, it is called a *Riemannian* metric. A metric is of *constant curvature* if its sectional curvatures coincide and are constant. In particular, if M is 2-dimensional, this means that the Gaussian curvature is constant.

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Definition 1.3. A projective vector field v is *homothetic* (for the metric g) if the Lie derivative of g along v satisfies $\mathcal{L}_v g = \lambda g$ for some constant $\lambda \in \mathbb{R}$. If v is not homothetic, it is called an *essential* projective vector field.

In the 1880s, Sophus Lie posed the following problem for 2-dimensional surfaces [**25**, **26**]:[1]

> **Lie's 1882 problem:** It is being required to determine the form of the arc element of any surface whose geodesic curves admit an infinitesimal transformation.

In fact, Lie distinguishes two distinct problems, dubbed Lie's First Problem and Second Problem, respectively. The first is the one stated above, and asks to describe metrics whose geodesic curves admit (at least) one infinitesimal transformation. The other asks to describe metrics whose geodesic curves admit *at least* two independent infinitesimal transformations[2]. For details we refer to Section 3 below. The distinction made by Lie turns out to be much deeper than it might initially seem. Indeed we shall see that the solution techniques found in the literature are fundamentally different in either case.

Lie's formulation of the problem appears ambiguous on various levels, and we would like to mention at least the following aspects here. Indeed, Lie merely asks us to *determine* ("bestimmen") the metrics. This might be understood as the task to find a list of candidates (which is not required to be sharp, i.e. might contain metrics not satisfying the criteria). However, a more thorough answer would provide a sharp list of metrics (such that any entry has the desired properties). Lastly, one might even ask for a proper classification, e.g. up to projective or isometric transformations.

Another ambiguity arises as the projective Lie algebra action could show *singularities*, i.e. it can have orbits of different dimensions. While Lie's classical problem is well understood today around generic points, such singularities are much less well understood (some partial results are obtained in [**30**]).

A last, and less deep, ambiguity is that Lie explicitly asks us only for the metrics as such. However, one would naturally ask, as well, for the full and explicit projective vector fields that these metrics admit. For simplicity of exposition, we focus on the following interpretation of Lie's problem.

> **Lie's classical problem:** Determine a sharp list of local forms of 2-dimensional metrics admitting a projective Lie algebra of dimension greater than zero, along with the admitted projective vector fields.

The idea of Lie's problem is, of course, not restricted to the 2-dimensional case. The analogous problem for any dimension $n \geq 2$ can be stated and studied as well:

> **Extended Lie problem:** Determine a sharp list of local forms of n-dimensional metrics ($n \geq 3$) admitting a projective Lie algebra of dimension greater than zero, along with the admitted projective vector fields.

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[1] German original: "Es wird verlangt, die Form des Bogenelementes einer jeden Fläche zu bestimmen, deren geodätische Curven eine infinitesimale Transformation gestatten".

[2] Lie's Second Problem (original German version in [**25**]): "Man soll die Form des Bogenelementes einer jeden Fläche bestimmen, deren geodätische Curven mehrere infinitesimale Transformationen gestatten." A rough translation is: "One should determine the form of the arc element of any surface whose geodesic curves admit several infinitesimal transformations."

We are going to come back to this problem in Section 4. Before proceeding to a detailed exposition of the established solution of Lie's classical problem, we mention that Lie's problem also has a natural analog in the realm of Kähler geometry. In this context one needs to study infinitesimal transformations which preserve the so-called J-planar curves, a generalization of geodesics for complex manifolds. Detailed definitions and a synopsis of some established results are given in Section 5.

2. Projective structures, their metrizability and degree of mobility

In the present section the Einstein summation convention is applied unless otherwise specified. Condition (1.1) reads, in a system of coordinates $(y^1, \ldots, y^n)$ of M,

$$\ddot{y}^k + \Gamma^k_{ij}\dot{y}^i\dot{y}^j = \alpha(t)\dot{y}^k \,.$$

By eliminating the function $\alpha(t)$, a system of $n-1$ ordinary differential equations (ODEs) describing *unparametrized geodesics* is obtained:

$$y^k_{xx} + \Gamma^k_{11} + \sum_{i=2}^n (2\Gamma^k_{1i} - \delta^k_i\Gamma^1_{11})y^i_x + \sum_{i,j=2}^n (\Gamma^k_{ij} - \delta^k_i\Gamma^1_{1j} - \delta^k_j\Gamma^1_{1i})y^i_x y^j_x - \sum_{i,j=2}^n \Gamma^1_{ij}\, y^i_x y^j_x y^k_x \,, \tag{2.1}$$

where $k \in \{2, \ldots, n\}$ and

$$(y^1, y^2, \ldots, y^n) = (x, y^2, \ldots, y^n) \,.$$

Since the equivalence class $[\nabla]_{\mathfrak{p}}$ of connections that are projectively equivalent to ∇ is represented by (2.1), we call (2.1) the *projective connection associated to* ∇.

Note that the System (2.1) is of special type, i.e. of the form

$$y^k_{xx} = f^k_{11} + \sum_{i=2}^n f^k_{1i}y^i_x + \sum_{i,j=2}^n f^k_{ij}y^i_x y^j_x + \sum_{i,j=2}^n f_{ij}\, y^i_x y^j_x y^k_x \,, \tag{2.2}$$

where $k \in \{2, \ldots, n\}$ and where f^k_{ab} and f_{ij} are functions of $(x, y^2, \ldots, y^n)$.

Definition 2.1. A *projective connection* on a smooth manifold M is a system of ODEs of type (2.2).

Remark 2.2. Essentially, a projective connection on M realizes a projective structure on M in terms of a system of ODEs. A straightforward computation shows that the class of ODEs (2.2) is closed under *point transformations*, i.e., transformations

$$(x, y^2, \ldots, y^n) \to (u, v^2, \ldots, v^n) \tag{2.3}$$

with $u, v^2, \ldots, v^n$ functions of $(x, y^2, \ldots, y^n)$.

Definition 2.3. Two systems of type (2.2) are called *(point-)equivalent* if there exists a (point-)transformation (2.3) mapping one into the other.

One might now ask oneself under which conditions a projective structure is *metrizable*, by which we mean the following:

Metrizability problem: Given a connection ∇, decide whether there exists a Levi-Civita connection ∇^g (for a metric g) such that $\nabla^g \in [\nabla]_{\mathfrak{p}}$.

From an analytic viewpoint this is equivalent, due to (2.1) and (2.2), to proving the existence of functions $g_{ij} = g_{ij}(y^1, \dots, y^n)$, i.e. of a metric tensor g, satisfying the following system:

$$
\left\{\begin{array}{ll}
\Gamma^k_{11} = -f^k_{11}\,, & k \in \{2, \dots, n\} \\
2\Gamma^k_{1i} - \delta^k_i \Gamma^1_{11} = -f^k_{1i}\,, & k \in \{2, \dots, n\} \\
\Gamma^k_{ij} - \delta^k_i \Gamma^1_{1j} - \delta^k_j \Gamma^1_{1i} = -f^k_{ij}\,, & k \in \{2, \dots, n\} \\
\Gamma^1_{ij} = f_{ij} &
\end{array}\right. \tag{2.4}
$$

where $i, j \in \{2, \dots, n\}$ and

$$
\Gamma^k_{ij} = \frac{1}{2} g^{kh} \left(\frac{\partial g_{jh}}{\partial y^i} + \frac{\partial g_{ih}}{\partial y^j} - \frac{\partial g_{ij}}{\partial y^h} \right) , \quad i, j, k \in \{1, \dots, n\}\,,
$$

are the *Christoffel symbols*. System (2.4) is a non-linear system of partial differential equations (PDEs), but it can be linearized thanks to the following proposition.

PROPOSITION 2.4 ([**19**]). A metric g on an n-dimensional manifold lies in the projective class of a given connection ∇ if and only if σ^{ij} defined by

$$
\sigma^{ij} = |\det(g)|^{\frac{1}{n+1}} g^{ij} \tag{2.5}
$$

is a solution of the linear system

$$
\nabla_a \sigma^{bc} - \frac{1}{n+1} (\delta^c_a \nabla_i \sigma^{ib} + \delta^b_a \nabla_i \sigma^{ic}) = 0 \tag{2.6}
$$

with

$$
\nabla_a \sigma^{bc} = \sigma^{bc}_{,a} + \Gamma^b_{ad} \sigma^{dc} + \Gamma^c_{ad} \sigma^{db} - \frac{2}{n+1} \Gamma^d_{da} \sigma^{bc}\,,
$$

where Γ^k_{ij} are the Christoffel symbols.

REMARK 2.5. The tensor σ whose components are (2.5) is a *weighted tensor field* on M and as such an element of $S^2(M) \otimes (\Lambda^n(M))^{\frac{2}{n+1}}$, where $S^2(M)$ denotes the space of symmetric $(2,0)$-tensor fields on M. In dimension 2, specifically, linearization can also be achieved using the weighted tensor field $a = |\det(g)|^{-\frac{2}{3}} g$ instead of $\sigma = |\det(g)|^{\frac{1}{3}} g^{-1}$. Note that the components of a and σ are related by the classical adjugate. Indeed, in the papers treating the 2-dimensional case, i.e. [**13, 33, 39**], the weighted tensor field a is frequently used rather than σ.

DEFINITION 2.6. The vector space of the solutions to System (2.6) is going to be denoted by Σ. The *degree of mobility* of a metric g is the dimension of Σ.

Important consequences of Proposition 2.4 are the following: firstly, System (2.6) is projectively invariant as it depends on the projective class $[\nabla]_{\mathfrak{p}}$ only. Thus, if v is a projective vector field, the Lie derivative $\mathcal{L}_v$ sends solutions of (2.6) to solutions of (2.6), i.e., it is a linear map on Σ:

$$
\mathcal{L}_v : \Sigma \to \Sigma
$$

with

$$
\mathcal{L}_v \sigma^{ij} = |\det(g)|^{\frac{1}{n+1}} \left(\frac{g^{mn} \mathcal{L}_v g_{mn}}{n+1} g^{ij} + \mathcal{L}_v g^{ij} \right) . \tag{2.7}
$$

Secondly, if σ_1 and σ_2 are solutions to (2.6), also $c_1\sigma_1 + c_2\sigma_2$ is a solution for any $c_1, c_2 \in \mathbb{R}$. Therefore, if g_1 and g_2 are projectively equivalent metrics on an n-dimensional manifold M, then

$$\frac{\left(c_1|\det(g_1)|^{\frac{1}{n+1}}g_1^{-1} + c_2|\det(g_2)|^{\frac{1}{n+1}}g_2^{-1}\right)^{-1}}{\det\left(c_1|\det(g_1)|^{\frac{1}{n+1}}g_1^{-1} + c_2|\det(g_2)|^{\frac{1}{n+1}}g_2^{-1}\right)} \tag{2.8}$$

is a projectively equivalent metric to g_1 (and g_2) for any $c_1, c_2 \in \mathbb{R}$, whenever it is defined.

Important papers concerning the metrizability of projective structures are [**12**, **18**]. The paper [**12**] finds differential invariants responsible for the metrizability of a given 2-dimensional projective structure and then explicit obstructions to the existence of a Levi-Civita connection within a given projective class. Indeed the authors studied the overdetermined system (2.6) by employing the prolongation method. Analogous results are contained in [**18**], where the authors face the problem of metrizability of 3-dimensional projective structures.

We conclude this section by introducing, for any pair of projectively equivalent metrics $g, \hat{g}$ on M, the $(1,1)$-tensor field

$$L = L(g,\hat{g}) = \left|\frac{\det(\hat{g})}{\det(g)}\right|^{\frac{1}{n+1}} \hat{g}^{-1}g = \hat{\sigma}\sigma^{-1}\,, \tag{2.9}$$

which has been an important tool to solve both Lie's classical as well as Lie's extended problem

3. Classical Lie's problems

Now we review Lie's second problem (Section 3.1) and Lie's first problem (Section 3.2), which have been solved using quite different techniques. To start with, some clarifications about Lie's problem are in order. As we already pointed out, the set of projective vector fields form a finite dimensional Lie algebra: in what follows we assume that the action of the projective Lie algebra has orbits of constant dimension. Then one can use the results by S. Lie. He found all possible Lie algebras that can arise from projective vector fields of a 2-dimensional projective connection [**25**]. Moreover, Lie obtained the possible realizations in terms of vector fields on (open neighborhoods in) a 2-dimensional manifold [**26**], see also [**27**, **46**]. In Section 3.1 and 3.2 below we shall explain how to obtain a *local* description of metrics admitting at least one projective vector field. Global aspects of Lie's theorem, in the case of Riemannian metrics, can be found in [**34**–**37**].

We start treating Lie's second problem that, from a chronological viewpoint, was the first to be solved [**13**]. Historically, it was believed that A.V. Aminova had solved Lie's second problem in [**1**, **2**]. Some years later a gap in her proof was pointed out and filled by [**13**]. This gap will be addressed below. Indeed, Aminova obtained an exhaustive (but not sharp) list of 2-dimensional metrics admitting nonzero projective vector fields, some of which are however only described via ODEs.

3.1. Lie's second problem: 2-dimensional metrics g with $\dim \mathfrak{p}(g) \geq 2$.

We now discuss the aforementioned gap pointed out in [**13**]. Briefly speaking, it occurred due to a misinterpretation of a statement by G. Kœnigs in the 1890s, see [**16**, p. 374]. This statement is summarized in the following theorem.

THEOREM 3.1. *A 2-dimensional metric whose geodesic flow admits three independent quadratic integrals (in momenta) is a metric of a surface of revolution.*[3]

From Theorem 3.1 it follows that the metric admits a Killing vector field or, equivalently, a linear integral.

The issue here is how the number of independent integrals is counted. Indeed, Kœnigs does not include the Hamiltonian obtained from the metric into the count of the integrals, essentially counting integrals of motion up to the addition of multiples of the Hamiltonian. Instead, Aminova applies the following interpretation:

NON-THEOREM 3.1. *If the geodesic flow of a 2-dimensional metric admits three independent quadratic integrals in momenta (where in the tally the Hamiltonian is included), then the metric is a metric of a surface of revolution.*

The following counter-example shows that this interpretation is not viable.

EXAMPLE 3.2. All quadratic first integrals in the momenta p, q of the metric $g = (4x^2 + y^2 + 1)(dx^2 + dy^2)$ are linear combinations of

$$\begin{cases} H = \frac{p^2+q^2}{4x^2+y^2+1} \\ F_1 = \frac{y^2p^2-(4x^2+1)q^2}{4x^2+y^2+1} \\ F_2 = \frac{-xy^2p^2+y(4x^2+y^2+1)pq-x(4x^2+2y^2+1)q^2}{4x^2+y^2+1} \end{cases}$$

It is not difficult to confirm that this metric admits no non-zero Killing vector field.

Kœnigs' theorem, in an enriched form, therefore would be as follows:

THEOREM 3.3. *The geodesic flow of a* 2*-dimensional metric admits* 1*,* 2*,* 3*,* 4 *or* 6 *linearly independent quadratic first integrals in momenta (including the Hamiltonian). If the metric admits* 4 *linearly independent quadratic first integrals, then it admits a non-zero linear first integral (or, equivalently, a non-zero Killing vector field). If the metric admits* 6 *linearly independent quadratic first integrals, then it is of constant curvature.*

The following theorem provides a link between quadratic integrals of the geodesic flow of a metric g and its projective class $[g]_{\mathfrak{p}}$. In particular, in dimension 2, the degree of mobility (see Definition 2.6) coincides with the dimension of the vector space of quadratic integrals. The insight goes back, at least, to the end of the 19th century and is mentioned by Levi-Civita in [**24**, page 276], who attributes it to Painlevé [**45**, Ch. 2] (see also [**48**]).

THEOREM 3.4. *Let g and $\hat{g}$ be metrics on an n-dimensional manifold M. If they are projectively equivalent, then the function*

$$K : v \in TM \to K(v, v) := \left(\frac{\det(g)}{\det(\hat{g})}\right)^{\frac{2}{n+1}} g(v, v)$$

is a quadratic first integral in velocities of g. If $n = 2$ the above implication is an equivalence.

[3]French original: "Tout ds^2 dont les géodésiques admettent trois intégrales quadratiques indépendantes convient à une surface de révolution".

The statement of Non-Theorem 3.1 appears crucial in the proof in [**1**, **2**]. Indeed, one can construct a quadratic first integral starting from a projective vector field (see Proposition 5 of [**31**] and also [**21**, **43**]). Then, if a metric admits two independent projective vector fields, it would admit, in view of Non-Theorem 3.1 of the above reasoning, also a Killing vector field. This, of course, would be a huge simplification of Lie's second problem as the components of the metric tensor to be found would depend on one variable only. The aforementioned gap was observed and remedied by R. L. Bryant and coworkers in [**13**]. Indeed, the implication

$$\begin{gathered} g \text{ is a 2-dimensional metric with } \dim\mathfrak{p}(g)\geq 2\\ \Longrightarrow \text{existence of a Killing vector field of } g \end{gathered}$$

is actually true, but so far only an easy a posteriori confirmation exists using the complete list of Theorem 3.5 below. Therefore, the list of metrics obtained by Aminova in [**1**, **2**] is correct and obtained from that in Theorem 3.5 below after suitable changes of the coordinates. However, the list of metrics of [**1**, **2**] contains some functions given in terms of certain differential equations, contrary to that of Theorem 3.5, proved in Sections 3.1.1–3.1.4.

THEOREM 3.5 ([**13**]). *Let g be a 2-dimensional metric on a manifold M and assume $\dim\mathfrak{p}(g)\geq 2$. Then, almost every point of M has a neighborhood U such that the restriction of g to U has constant curvature, or such that there exists a coordinate system (x,y) on U in which the metric g takes one of the following forms:*

(1) *Metrics with $\dim\mathfrak{p}(g)=2$.*
  (a) $\varepsilon_1 e^{(b+2)\,x}\,dx^2+\varepsilon_2 e^{b\,x}\,dy^2$, *where $b\in\mathbb{R}\setminus\{-2,0,1\}$ and $\varepsilon_i\in\{-1,1\}$ are constants,*
  (b) $a\left(\frac{e^{(b+2)\,x}dx^2}{(e^{b\,x}+\varepsilon_2)^2}+\varepsilon_1\frac{e^{b\,x}dy^2}{e^{b\,x}+\varepsilon_2}\right)$, *where $a\in\mathbb{R}\setminus\{0\}$, $b\in\mathbb{R}\setminus\{-2,0,1\}$, and $\varepsilon_i\in\{-1,1\}$ are constants, and*
  (c) $a\left(\frac{e^{2\,x}dx^2}{x^2}+\varepsilon\frac{dy^2}{x}\right)$, *where $a\in\mathbb{R}\setminus\{0\}$, and $\varepsilon\in\{1,-1\}$ are constants.*

(2) *Metrics with $\dim\mathfrak{p}(g)=3$.*
  (a) $\varepsilon_1 e^{3x}dx^2+\varepsilon_2 e^{x}dy^2$, *where $\varepsilon_i\in\{-1,1\}$ are constants,*
  (b) $a\left(\frac{e^{3x}dx^2}{(e^x+\varepsilon_2)^2}+\varepsilon_1\frac{e^x dy^2}{(e^x+\varepsilon_2)}\right)$, *where $a\in\mathbb{R}\setminus\{0\}$, $\varepsilon_i\in\{-1,1\}$ are constants, and*
  (c) $a\left(\frac{dx^2}{(2x^2+cx+\varepsilon_2)^2x}+\varepsilon_1\frac{x dy^2}{(2x^2+cx+\varepsilon_2)}\right)$, *where $a>0$, $\varepsilon_i\in\{-1,1\}$, $c\in\mathbb{R}$ are constants.*

No two distinct metrics from this list are isometric.

3.1.1. *Preliminary results and ideas for the solution of Lie's second problem.* In the 2-dimensional case, Systems (2.1) and (2.2) read, respectively,

$$y_{xx}=-\Gamma^2_{11}+(\Gamma^1_{11}-2\Gamma^2_{12})y_x-(\Gamma^2_{22}-2\Gamma^1_{12})y_x^2-\Gamma^1_{22}y_x^3$$

and

$$y_{xx}=A_0(x,y)+A_1(x,y)y_x+A_2(x,y)y_x^2+A_3(x,y)y_x^3\,. \tag{3.1}$$

S. Lie proved [**25**] that the algebras that are the projective Lie algebras of 2-dimensional metrics, or, equivalently, the symmetry Lie algebras of the ODEs (3.1), can only be the following ones:

$$0\,,\quad \mathbb{R}\,,\quad \mathfrak{s}:[X,Y]=X\,,\quad \mathfrak{sl}(2,\mathbb{R})\,,\quad \mathfrak{sl}(3,\mathbb{R})\,, \tag{3.2}$$

where $\mathfrak{s}$ is the 2-dimensional non-commutative Lie algebra. Note that $\mathfrak{s}$ is a subalgebra of both $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(3,\mathbb{R})$. Since in this section we are interested in metrics admitting at least a 2-dimensional Lie algebra of projective vector fields, we shall concentrate on the last three ones. Lie also showed that an ODE of type (3.1) admitting $\mathfrak{sl}(3,\mathbb{R})$ as symmetry Lie algebra is equivalent (see Definition 2.3) to

$$y_{xx} = 0\,, \tag{3.3}$$

i.e., to the ODE whose solutions are the lines of $\mathbb{R}^2$. In other words, all 2-dimensional metrics admitting an 8-dimensional Lie algebra of projective vector fields are of constant curvature. Liouville proved [**29**] that

$$(L_1 dx + L_2 dy) \otimes (dx \wedge dy) \tag{3.4}$$

where

$$\begin{aligned} L_1 &= 2A_{1xy} - A_{2xx} - 3A_{0yy} - 6A_0A_{3x} \\ &\quad -3A_3A_{0x} + 3A_0A_{2y} + 3A_2A_{0y} + A_1A_{2x} - 2A_1A_{1y} \\ L_2 &= 2A_{2xy} - A_{1yy} - 3A_{3xx} + 6A_3A_{0y} \\ &\quad +3A_0A_{3y} - 3A_3A_{1x} - 3A_1A_{3x} - A_2A_{1y} + 2A_2A_{2x} \end{aligned}$$

is a differential invariant of the projective connection (3.1): the quantity (3.4) vanishes if and only if (3.1) is equivalent to (3.3) (see also [**15**, **49**]). Note also that in the list (3.2) there is a dimensional "gap", since $\mathfrak{sl}(2,\mathbb{R})$ is 3-dimensional and $\mathfrak{sl}(3,\mathbb{R})$ 8-dimensional. This means that an ODE (3.1) admitting more than three independent projective vector fields automatically admits $\mathfrak{sl}(3,\mathbb{R})$ as symmetry Lie algebra, implying that such an ODE is equivalent to (3.3).

It remains to investigate 2-dimensional metrics admitting $\mathfrak{s}$ and $\mathfrak{sl}(2,\mathbb{R})$ (cf. list (3.2)) as projective Lie algebras. To this end, a result by Lie [**26**–**28**] concerning all possible realizations of the algebra $\mathfrak{s}$ as an algebra of vector fields on $\mathbb{R}^2$ is very helpful. He showed that, in a neighborhood of almost every point[4], there exist coordinates (x,y) such that vector fields X, Y satisfying $[X,Y] = X$ are, up to the trivial case $X = 0$, are described by

- *Non-Transitive case:* $X = e^y\partial_y$, $Y = -\partial_y$;
- *Transitive case:* $X = \partial_y$, $Y = \partial_x + y\partial_y$.

3.1.2. *The non-transitive case.* By a direct computation one confirms that the most general projective connection (3.1) admitting $e^y\partial_y$ and ∂_y as infinitesimal symmetries is

$$y_{xx} = A_1(x)y_x + y_x^2\,. \tag{3.5}$$

Since the Liouville invariant (3.4) of (3.5) vanishes, (3.5) admits a symmetry Lie algebra isomorphic to $\mathfrak{sl}(3,\mathbb{R})$, i.e., (3.5) is equivalent to (3.3). We conclude that 2-dimensional metrics g such that $\dim \mathfrak{p}(g) = 2$ with $\mathfrak{p}(g)$ acting non-transitively are of constant curvature.

[4]We recall the assumption that the orbits of the projective Lie algebra action are of constant dimension.

3.1.3. *The transitive case with* $\dim\mathfrak{p}(g)=2$. By a direct computation one confirms that the most general projective connection (3.1) admitting ∂_y and $\partial_x+y\partial_y$ as infinitesimal symmetries is

$$y_{xx}=K_0e^x+K_1y_x+K_2e^{-x}y_x^2+K_3e^{-2x}y_x^3\,,\quad K_i\in\mathbb{R}\,. \tag{3.6}$$

In this case, the linear system (2.6) reads as follows:

$$\begin{cases} \sigma^{22}_{,x}-\frac{2}{3}K_1\,\sigma^{22}-2K_0e^x\,\sigma^{12}=0 \\ \sigma^{22}_{,y}-2\sigma^{12}_{,x}-\frac{4}{3}K_2e^{-x}\,\sigma^{22}-\frac{2}{3}K_1\,\sigma^{12}+2\,K_0e^x\,\sigma^{11}=0 \\ -2\sigma^{12}_{,y}+\sigma^{11}_{,x}-2\,K_3e^{-2x}\,\sigma^{22}+\frac{2}{3}K_2e^{-x}\,\sigma^{12}+\frac{4}{3}K_1\,\sigma^{11}=0 \\ \sigma^{11}_{,y}+2\,K_3e^{-2x}\,\sigma^{12}+\frac{2}{3}K_2e^{-x}\,\sigma^{11}=0 \end{cases} \tag{3.7}$$

Since we are supposing $\dim\mathfrak{p}(g)=2$, we can further assume (see page 454 of [**13**])

$$K_2\neq 0 \text{ and } K_1=K_3=0\,,\quad \text{or}$$
$$K_2=0, K_3\neq 0 \text{ and if } K_0=0 \text{ then } K_1\notin\{\tfrac{1}{2},2\}$$

As the coefficients of the above linear system do not depend on y, if $\sigma=\sigma^{ij}$ is a solution to (3.7), also $\frac{\partial^k\sigma}{\partial y^k}$ is a solution ($\forall\,k\in\mathbb{N}$), implying that any linear combination $\sum_{k=0}c_k\frac{\partial^k\sigma}{\partial y^k}$ is a solution ($c_k\in\mathbb{R}$). We stress that, according to Theorems 3.3 and 3.4, the dimension of the solution space to (3.7) can be equal to $0,1,2,3,4,6$ (but 0 would imply non-metrizability). There is another restriction:

Proposition 3.6 ([**13**, p. 453]). A 2-dimensional metric g with $\dim\mathfrak{p}(g)=2$ has degree of mobility less than four.

Due to Proposition 3.6, four independent solutions to System (3.7) cannot exist, so there must exist constants c_i, with $(c_0,c_1,c_2,c_3)\neq(0,0,0,0)$ such that

$$c_0\sigma+c_1\frac{\partial\sigma}{\partial y}+c_2\frac{\partial^2\sigma}{\partial y^2}+c_3\frac{\partial^3\sigma}{\partial y^3}=0\,. \tag{3.8}$$

One finds solutions to (3.8), depending on the multiplicity of the roots of its characteristic polynomial, and then one substitutes them into System (3.5). A long and tedious computation (see [**13**, pages 453-455]) shows that $\frac{\partial\sigma}{\partial y}=0$, i.e., in view of formula (2.5), the metrics we are looking for admit ∂_x as a Killing vector field. This leads to obtain the list of metrics of Theorem 3.5 with $\dim\mathfrak{p}(g)=2$.

3.1.4. *The transitive case with* $\dim\mathfrak{p}(g)=3$. In this case, one can find a basis X,Y,Z of projective vector fields satisfying the following commutation relations:

$$[X,Y]=X\,,\quad [X,Z]=Y\,,\quad [Y,Z]=Z\,. \tag{3.9}$$

We can then exploit the realization of X and Y, in terms of vector fields, given in Section 3.1.1 for the transitive case, $X=\partial_y$ and $Y=\partial_x+y\partial_y$, in order to find also a coordinate description of the vector field Z. Indeed, conditions (3.9) translate into a system of PDEs, with the components of Z as unknown functions, that can be easily integrated:

$$Z=(y+C_1e^x)\partial_x+\left(\frac{y^2}{2}+C_2e^{2x}\right)\partial_y\,. \tag{3.10}$$

The only possibility for the projective connection (3.6) to admit an infinitesimal symmetry of type (3.10) and, in the same time, to admit *exactly* a 3-dimensional symmetry Lie algebra, is that $K_0 = K_2 = 0$, $K_1 = \frac{1}{2}$, $K_3 \neq 0$, $C_1 = C_2 = 0$. To sum up, we have to consider the projective connection

$$y_{xx} = \frac{1}{2} y_x + K_3 e^{-2x} y_x^3 \,, \quad K_3 \neq 0 \tag{3.11}$$

with the following vector fields as a basis of the projective Lie algebra:

$$X = \partial_y \,, \quad Y = \partial_x + y\partial_y \,, \quad Z = y\partial_x + \frac{y^2}{2}\partial_y \,. \tag{3.12}$$

The following proposition ([**13**, page 456]) is crucial for obtaining the list of metrics of Theorem 3.5 in the case $\dim \mathfrak{p}(g) = 3$.

PROPOSITION 3.7. A 2-dimensional metric g such that $\dim \mathfrak{p}(g) \geq 3$ admits a non-zero Killing vector field.

Without loss of generality, we may suppose that X or Y or Z of (3.12) is a Killing vector field for the metric g, thus obtaining three distinct cases. We discuss only the case in which $X = \partial_y$ is a Killing vector field for the metric g, as the other two cases can be treated similarly. In the case under consideration, the metric coefficients g_{ij} depend only on x, so that also σ^{ij} (see (2.5)) depend only on x. Substituting $\sigma^{ij} = \sigma^{ij}(x)$ and $A_0 = A_2 = 0$, $A_1 = \frac{1}{2}$, $A_3 = K_3 e^{-2x}$ (cf. (3.11)) into System (3.7), one obtains a system that yields

$$g = e^{3x} C_2 (-C_1 e^x + 2C_2 K_3)^2 \, dx^2 - \frac{e^x}{C_2^2(-C_1 e^x + 2C_2K_3)} \, dy^2 \,, \quad C_i \in \mathbb{R} \,.$$

By virtue of an appropriate change of coordinate, the metric g is transformed into either a metric 2a or 2b in Theorem 3.5. The metrics 2c in Theorem 3.5 are obtained by a similar consideration of the remaining cases.

3.2. Lie's first problem. We now consider 2-dimensional metrics with a projective Lie algebra of dimension exactly 1, i.e. we require that $\dim \mathfrak{p}(g) = 1$. Thereby we follow the chronological order: historically Lie's second problem has been solved before Lie's first problem. The techniques employed in these solutions are also quite different. Indeed one could follow the same strategy as in [**13**], outlined above, but in the case under consideration the projective connection (3.1) would depend on univariate functions rather than constants, contrary to (3.6), making the general solution to System (3.7) harder to obtain. As mentioned in the beginning of Section 3, Aminova [**1**, **2**] had already found a non-sharp list of normal forms for 2-dimensional metrics g with $\dim \mathfrak{p}(g) = 1$ (involving unsolved ODEs).

The hypothesis $\dim \mathfrak{p}(g) = 1$ implies that, if v is a non-zero projective vector field, then any other projective vector field will be a scalar multiple of v. We recall that, due to Theorems 3.3 and 3.4, the degree of mobility of an arbitrary 2-dimensional metric satisfies

$$\dim(\Sigma) \in \{1, 2, 3, 4, 6\} \,.$$

The now discuss each case individually, beginning with $\dim(\Sigma) = 6$. Due to Theorems 3.3 and 3.4, the metric is then of constant curvature. It therefore admits an 8-dimensional Lie algebra of projective vector fields and the projective Lie algebra is isomorphic to $\mathfrak{sl}(3, \mathbb{R})$.

The next possible case is $\dim(\Sigma) = 4$. By virtue of Theorems 3.3 and 3.4, a Killing vector field exists. In a neighborhood of a point p such that $v_p \neq 0$, there exist coordinates (x, y) such that $v = \partial_x$: in this system of coordinates

$$g = E(y)dx^2 + 2F(y)dxdy + G(y)dy^2 .$$

Of course, this is a non-sharp description as, for instance, in the case of constant E, F and G this metric is of constant curvature and hence does not meet the hypothesis.

We now turn to the case when $\dim(\Sigma) = 1$, in which all metrics projectively equivalent to the metric g are proportional to g by a non-zero factor, and therefore v necessarily is homothetic for any such multiple. In suitable local coordinates we achieve, around points where v does not vanish, $v = \partial_x$ and

$$g = e^{\lambda x} \left(E(y)\, dx^2 + 2\, F(y)\, dxdy + G(y)\, dy^2\right) ,$$

which is a non-sharp description (for instance, again, if E, F and G are constants then g admits another projective vector field, namely ∂_y).

We are thus left with two cases, $\dim(\Sigma) \in \{2, 3\}$. As pointed out in Section 2, the Lie derivative $\mathcal{L}_v$ along v is a linear operator on Σ, see (2.7). Furthermore, the operator $\mathcal{L}_v$ is non-degenerate [**39**, Lemma 3].

– If $\dim(\Sigma) = 2$, then since $\dim \mathfrak{p}(g) = 1$, we can rescale v by a non-zero constant factor and choose an appropriate basis of Σ such that $\mathcal{L}_v$ assumes one of the following Jordan forms (for a constant λ):

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} , \quad \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} , \quad \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} . \tag{3.13}$$

– If $\dim(\Sigma) = 3$, there similarly exists an appropriate basis of Σ and a rescaling of v such that $\mathcal{L}_v$ assumes ones of the following Jordan forms:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} , \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{pmatrix} , \quad \begin{pmatrix} \lambda & -1 & 0 \\ 1 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix} , \quad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{pmatrix} \tag{3.14}$$

for constants λ, μ. We observe that the non-degenerate operator $\mathcal{L}_v$ always possesses a 2-dimensional invariant subspace $\mathcal{V}$. Therefore, up to a non-zero constant factor, the restriction of $\mathcal{L}_v$ to $\mathcal{V}$ is described by one of the matrices (3.13).

The previous discussion reduces Lie's first problem, at least partially, to the following: find $\mathcal{L}_v$-invariant subspaces $\Sigma' \subseteq \Sigma$ of dimension 2 such that $\dim \mathfrak{p}(g) = 1$, then try to extend them to the full space Σ. Two pre-existing results make this approach feasible: first, we can assume $\mathcal{L}_v$ to have Jordan form as in (3.13). Second, the basis of Σ', in suitable local coordinates on the manifold, can be taken in canonical form according to the following proposition.

PROPOSITION 3.8 ([**5, 10, 17**]). *Let g_1 and g_2 be projectively equivalent, non-proportional 2-dimensional metrics. Then, in a neighborhood of almost every point, there are coordinates (x, y) such that the metrics assume one of the following three normal forms:*

	Liouville	Complex-Liouville	Jordan-Block
g_1	$(f_1 - f_2)(dx^2 \pm dy^2)$	$(\overline{h(z)} - h(z))\,(d\bar{z}^2 - dz^2)$	$(1 + xf_2')dxdy$
g_2	$\left(\frac{1}{f_1} - \frac{1}{f_2}\right)\left(\frac{dx^2}{f_1} \pm \frac{dy^2}{f_2}\right)$	$\left(\frac{1}{\overline{h(z)}} - \frac{1}{h(z)}\right)\left(\frac{d\bar{z}^2}{\overline{h(z)}} - \frac{dz^2}{h(z)}\right)$	$\frac{1+xf_2'}{f_2^4}\,(-2f_2\, dxdy$ $+(1 + xf_2')\, dy^2)$

Here, $f_1 = f_1(x)$ and $f_2 = f_2(y)$ are functions of one variable only, and in the complex Liouville case we use coordinates $z = x + iy$, $\overline{z} = x - iy$.

The strategy for the solution of Lie's first problem pursued in [**39**] is as follows. Let us assume $\dim(\Sigma) = 2$. Let $\{\sigma_1, \sigma_2\}$ be a basis of Σ. The equations

$$\mathcal{L}_v \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = A \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix}$$

where A is one of the matrices in (3.13), translate into a system of PDEs with the metric coefficients of g_1, g_2 and the components of v as unknown functions, because of (2.5) and (2.7). By substituting g_1 and g_2 with those contained in the table of Proposition 3.8, we obtain (for each considered matrix (3.13)) three distinct cases, each of which is a system of 6 PDEs that can be integrated. Once g_1 and g_2 are obtained from the above computation, Formula (2.8) gives all metrics in the projective class $[g_1]_{\mathfrak{p}}$ of g_1 (and, of course, g_2).

If $\dim \Sigma = 3$, a similar reasoning allows one to find a suitable basis of Σ, such that $\mathcal{L}_v$ is described by one of the matrices (3.14), starting from a basis of the invariant subspace $\mathcal{V}$ that is inherited from the previous case.

The next question is how to obtain a list of pairwise non-isometric metrics lying in the same projective class. This question was answered in [**33**]. The idea is the following. Since v is a projective vector field, its local flow preserves the equivalence class $[g_1]_{\mathfrak{p}}$: two metrics in $[g_1]_{\mathfrak{p}}$ linked by the local flow of v are thus isometric. An additional tool for distinguishing non-isometric metrics in the same projective class is to study invariants of tensor (2.9).

The full proof is rather tedious and technical. We therefore refer the interested reader to the literature [**2, 32, 39**]. The result can be summarized as follows: if a metric g satisfies $\dim \mathfrak{p}(g) = 1$, then locally it is either of the form

$$g = e^{\mu x}(E(y)dx^2 + 2F(y)\, dxdy + G(y)\, dy^2)\,, \qquad \mu \in \mathbb{R}\,,$$

and the projective vector fields are multiples of ∂_x, or it can be transformed into exactly one metric from an (extensive) list of normal forms, see [**32**]. We also refer the reader to the concise table in Appendix A of [**31**].

4. Higher dimensional extensions

Having discussed Lie's classical problem of metrics with infinitesimal symmetries of their geodesics, we shall now turn our attention to extensions of this problem. The most natural extension is to ask for metrics in higher dimension $n \geq 3$ that have projective vector fields.

While Lie's classical problem (see page 1) is solved, the same cannot be said about higher dimensions. Various partial solutions have been obtained, however. Typically, these works impose additional restrictions on the type of metrics. In particular, Levi-Civita's metrics have received some attention, which were obtained by Levi-Civita when studying pairs of metrics that share the same unparametrized geodesics; a detailed introduction to these metrics can be found in the next subsection. We present two major results from the literature.

Firstly, *in arbitrary dimension $n \geq 3$, Levi-Civita metrics of the most basic type* have been investigated by Aleksandr Solodovnikov [**47**]. He obtained a description in terms of ordinary differential equations which we review in Section 4.2. For a given metric, these equations determine its projective vector fields. On the other hand, they also allow one to find metrics with projective vector fields.

Secondly, the latter point has been executed for *Levi-Civita metrics of arbitrary type in dimension* $n = 3$ [**33**]. Two of the authors thus obtained a sharp description (see Section 4.1), not only for Levi-Civita metrics of basic type, but for all Levi-Civita metrics in dimension 3. In particular, this suffices to solve Lie's problem for Riemannian metrics in dimension $n = 3$ (see Section 4.4). For Levi-Civita metrics of non-basic type this also reveals a splitting-gluing phenomenon effectively reducing the problem to Lie's classical, 2-dimensional problem.

These situations will be elaborated upon in the dedicated sections 4.2 as well as 4.3 and 4.4, respectively. We stress that the Einstein convention is not used. The next section is dedicated to Levi-Civita metrics in general.

4.1. Levi-Civita metrics. In his 1896 work [**24**], Levi-Civita considers pairs of non-proportional, Riemannian metrics $g, \hat{g}$ that share the same unparametrized geodesics. His aim is to find local canonical forms for such pairs. He relies on previous work by Appel [**4**], Painlevé [**45**] and Liouville [**29**]. For the purposes here, the most important result of Levi-Civita is that in a sufficiently small neighborhood, there exist local coordinates to put the metrics g and $\hat{g}$ into a canonical form:

Proposition 4.1 ([**7, 8, 38**]). Let g be a Riemannian, and $\hat{g}$ a metric projectively equivalent and non-proportional to g. Then (almost everywhere, in a neighborhood) local coordinates exist such that $g, \hat{g}$ assume the form

$$g = \sum_{i=1}^{r} P_i (dx_i^1)^2 + \sum_{i=r+1}^{m} \left[P_i \sum_{\alpha_i,\beta_i}^{k_i} (h_i(x_i))_{\alpha_i,\beta_i} dx_i^{\alpha_i} dx_i^{\beta_i} \right] \tag{4.1a}$$

$$\hat{g} = \sum_{i=1}^{r} P_i \rho_i (dx_i^1)^2 + \sum_{i=r+1}^{m} \left[P_i \rho_i \sum_{\alpha_i,\beta_i}^{k_i} (h_i(x_i))_{\alpha_i,\beta_i} dx_i^{\alpha_i} dx_i^{\beta_i} \right] \tag{4.1b}$$

where the functions $(h_i)_{\alpha\beta}$ depend on a subset (x_i) of the coordinates only, denoted by $(x_i^1, \ldots, x_i^{k_i})$, and where

$$P_i = \pm \prod (f_i - f_j) \qquad \text{and} \qquad \rho_i = \frac{1}{f_i \prod_\alpha f_\alpha^{k_\alpha}}\,,$$

with f_i denoting the eigenvalue of $L = L(g, \hat{g})$ (see (2.9)) for the eigendistribution $\mathcal{D}_i$. The numbers k_i are larger or equal to 2, and the dimension of the manifold is

$$n = r + \sum_{i=r+1}^{m} k_i.$$

The description is sharp in the sense that the metrics g, $\hat{g}$ of this form are indeed projectively equivalent. Note that these normal forms include three major "building components": Univariate functions $f_i(x_i)$, constants ρ_μ and lower-dimensional (but otherwise arbitrary) metrics h_μ. The building components f_i and ρ_μ of g, $\hat{g}$ take on a specific meaning: they are eigenvalues of L. We remark that the corresponding eigenspaces yield integrable distributions [**7**]. The metrics h_μ correspond to constant eigenvalues ρ_μ of the endomorphism L.

Definition 4.2. We say that a Levi-Civita metric is of *basic type* if L has only 1-dimensional eigenspaces.

While projectively equivalent, the metrics g, $\hat{g}$ of Levi-Civita type do not necessarily admit projective vector fields. However, they are natural candidates for such metrics: let us now consider a metric g that admits a projective vector field v. If v is not homothetic, i.e. $\mathcal{L}_v g$ is not proportional to g, the construction introduced in Section 2 allows us to write down, using (2.5),

$$\hat{\sigma} = \mathcal{L}_v \sigma = \mathcal{L}_v(|\det(g)|^{\frac{1}{n+1}} g^{-1})$$

which solves (2.6). If $\hat{\sigma}$ is non-degenerate, then

$$\hat{g} = |\det(\hat{\sigma})|^{-1} \hat{\sigma}^{-1}$$

is a metric, such that $(g, \hat{g})$ is a pair of metrics satisfying Levi-Civita's hypothesis. If $\hat{\sigma}$ is degenerate, we replace it by a suitable linear combination of $\hat{\sigma}$ and σ to obtain such a metric. This confirms: Riemannian metrics with a projective vector field that is not homothetic are of Levi-Civita's type.

We conclude the section by mentioning that Levi-Civita's metrics are not confined to Riemannian geometry. Indeed, using the canonical form (4.1) we obtain many examples $g, \hat{g}$ of projectively equivalent metrics that are of mixed signature.

4.2. Solodovnikov's solution for basic Levi-Civita metrics. We now present a result obtained by A. Solodovnikov in the 1950s, which characterizes the Riemannian Levi-Civita metrics of basic type with homothetic or non-homothetic projective vector fields. As the signature turns out not to be crucial for the reasoning, we shall consider metrics of the more general form

$$g = \sum_{i=1}^{n} \prod_{j=1}^{n} (f_j(x_j) - f_i(x_i))\, dx_i^2 \qquad (n \geq 3)\,, \tag{4.2}$$

with $\hat{g}$ having the corresponding form via (4.1b). As explained in detail in [**7**, **8**], the projective vector fields for such metrics are of the form

$$v = \sum_i v_i(x_i) \partial_{x_i}\,.$$

Briefly speaking the reason for this form of v is that the distributions associated to eigenspaces of L are integrable, cf. (2.9), see also [**7**]. The ordinary differential equations contained in the following theorem were obtained by A. Solodovnikov.

Theorem 4.3. *Let g be a metric (4.2) with non-zero projective vector field v.*

(i) If v is homothetic, then there are constants $c, a \in \mathbb{R}$ such that

$$\begin{aligned} v_i(x_i) f_i'(x_i) &= f_i(x_i)^2 + c f_i(x_i) + a \\ 2v_i'(x_i) &= (3-n) f_i(x_i) - c(n-1) \end{aligned}$$

(ii) If v is not homothetic, then there are constants $c, \ell, \lambda \in \mathbb{R}$ such that

$$\begin{aligned} v_i(x_i) f_i'(x_i) &= c f_i(x_i) + \ell \\ 2v_i'(x_i) &= -c(n-1) + \lambda \end{aligned}$$

Proof. One can show, that in the adapted coordinates we are using, each component v_i of the projective vector field depends on the variable x_i only (in other words, the eigendistributions of L are integrable). This implies $\nabla_i v_j = 0$ if

$i \neq j$. Solodovnikov then shows that if v is not homothetic, then without loss of generality we have

$$\nabla_i v_i = (f_i + \sum_j f_j) \prod_{j\neq i} |f_i - f_j|$$

(achieving this form might require a common translation $f_i \to f_i + \mu$, $\mu \in \mathbb{R}$, which does not modify the metric g). He obtains the following solution:

$$2\frac{\partial v_i}{\partial x_i} = -\sum_{j\neq i} \frac{\frac{\partial f_j}{\partial x_j} v_j - \frac{\partial f_i}{\partial x_i} v_i}{f_j - f_i} + 2f_i + \sum_{j\neq i} f_j \,.$$

Since the left hand side depends on x_i only, this must hold also for the expression on the right hand side. In particular, for $j \neq i$,

$$-\frac{\frac{\partial f_j}{\partial x_j} v_j - \frac{\partial f_i}{\partial x_i} v_i}{f_j - f_i} + 2f_i = F_i^j(x_i)\,,$$

should be univariate. It necessarily follows that $c_{ij} = f_i - F_i^j$ is a constant and that $c_{ij} = c_{ji}$. Writing this out, we have that

$$\tfrac{\partial f_i}{\partial x_i} - f_i^2 - c_{ij} f_i = \tfrac{\partial f_j}{\partial x_j} - f_j^2 - c_{ji} f_j =: a$$

is a constant, and, indeed, a does not depend on the choice of i, j. Resubstituting into the formula for $\frac{\partial v_i}{\partial x_i}$, the claim is obtained for the case of a non-homothetic vector field. The equations for homothetic vector fields can be developed similarly. □

The theorem establishes a characterization of metrics (4.2) with projective vector fields. For given functions $f_i(x_i)$, employing it directly allows one to check if a projective vector field exists. More importantly, however, we view these equations as ODEs for both v and f (ignoring labels as the equations are identical for all blocks). This allows us to find in explicit terms all the metrics with homothetic and non-homothetic vector fields, along with the vector fields themselves. As a metric can admit both kinds of projective vector fields simultaneously, the solutions should be expected to overlap. We follow this train of thought in the next section where we discuss 3-dimensional Levi-Civita metrics.

4.3. Levi-Civita metrics in dimension $n = 3$. The current section reviews the work by two of the authors [**33**], addressing the case of Levi-Civita metrics in dimension $n = 3$. Writing out formula (4.1a), we obtain the following two cases of Levi-Civita metrics in dimension 3:

Type 111 $\quad g = (X - Y)(X - Z)dx^2 + (Y - X)(Y - Z)dy^2 + (Z - X)(Z - Y)dz^2$

Type 21 $\quad g = (X - \rho)dx^2 + (\rho - X)h$

where $X = X(x)$, $Y = Y(y)$, $Z = Z(z)$ and where

$$-h = h_{11}(y,z)dy^2 + 2h_{12}(y,z)dydz + h_{22}(y,z)dz^2\,.$$

Redefining $\zeta(x) = X(x) - \rho$, we write the type-21 metric in more compact form as

$$g = \zeta(x)(dx^2 + h)\,.$$

The task is now to find the metrics among these that admit projective vector fields. This is particularly easy for the first case, as we simply have to solve Solodovnikov's equations. Note that these simplify in the case $n = 3$. As the equations are of Riccati type, the solutions are readily obtained. For example, one solution is

$$\begin{aligned} g = {} & k_1(\tanh(x) - \tanh(y))(\tanh(x) - \tanh(z))dx^2 \\ & + k_2(\tanh(y) - \tanh(x))(\tanh(y) - \tanh(z))dy^2 \\ & + k_3(\tanh(z) - \tanh(x))(\tanh(z) - \tanh(y))dz^2 \,. \end{aligned}$$

The full list of solutions can be found in [**33**, Lemmas 10 and 11].

Now let us turn to the 21-type. One can show, again using that eigenspaces correspond to integrable distributions, that if v is projective for a 21-type metric, $v = (\alpha(x), u^2(y,z), u^3(y,z))$.

Proposition 4.4. If $v = \alpha(x)\partial_x + u$ is a projective vector field of a 21-type Levi-Civita metric, then u is homothetic for h. Conversely, if u is homothetic for h with $\mathcal{L}_u h = -Ch$, then v is homothetic if and only if

$$\begin{aligned} \zeta\,(\alpha\zeta'' - \alpha'\zeta' - C\zeta') &= \alpha\zeta'^2 \\ \zeta\,(-2\alpha''\zeta - \alpha\zeta'' + \alpha'\zeta') &= \alpha\zeta'^2 \end{aligned}$$

This reduces the problem effectively to one dimension lower, as it permits us to use the solution of Lie's classical 2-dimensional problem, and then construct all 3-dimensional 21-type metrics from these by solving the above system of differential equations. The next section gives a qualitative impression of the results obtained.

4.4. A sharp description of Riemannian metrics in $n = 3$. We note the following easy lemma.

Lemma 4.5. Let g be a Riemannian metric of dimension $n = 3$ that admits a non-vanishing projective vector field v. Then:

- If v is homothetic, then locally, around almost every point of the manifold, there are local coordinates such that $v = \partial_x$ and $g = e^{\mu x}h(y,z)$;
- If v is not homothetic, then g is of constant curvature, or a Levi-Civita metric of either type 111 or type 21.

Proof. The first part of the claim is obvious by the rectification theorem. For the second part, assume v is not homothetic. Let $\sigma = |\det(g)|^{\frac{1}{n+1}}g^{-1}$ and then let $\hat\sigma = \mathcal{L}_v\sigma$. If $\hat\sigma$ is non-degenerate, then $\hat g = |\det\hat\sigma|^{-1}\hat\sigma^{-1}$ is a metric such that $(g,\hat g)$ is a Levi-Civita pair. If $\hat\sigma$ is degenerate, we replace $\hat\sigma$ by a linear combination of $\hat\sigma$ and σ and then obtain $\hat g$ in the obvious, analogous way. $\square$

The lemma immediately gives us a sharp description of 3-dimensional Riemannian metrics with projective vector fields.

Theorem 4.6. *Let g be a Riemannian metric of $n = 3$ with a non-vanishing projective vector field. Then, locally, in a small neighborhood of almost any point, it falls into exactly one of the following cases:*

(1) *The metric g is of constant curvature and after a local change of coordinates its Lie algebra of projective vector fields is isomorphic to $\mathfrak{sl}(n+1,\mathbb{R})$.*
(2) *The metric g has non-constant curvature and admits a homothetic vector field and, after a local change of coordinates,*

$$g = e^{\mu x}h(y,z)$$

(recall: we exclude cases of constant curvature as these have already been covered).

(3) *The metric g has non-constant curvature and admits no homothetic vector field. Then, after a local change of coordinates,*

$$g = k_1(f(x) - f(y))(f(x) - f(z))e^{bx}dx^2 + k_2(f(y) - f(x))(f(y) - f(z))e^{by}dy^2 + k_3(f(z) - f(x))(f(z) - f(y))e^{bz}dz^2 ,$$

where $(b \in \mathbb{R})$

$$(i)\ f : t \to \tanh(t) , \quad (ii)\ f : t \to \tan(t) , \quad (iii)\ f : t \to \frac{1}{t} , b = 2 .$$

The projective vector field is then a multiple of $\partial_x + \partial_y + \partial_z$.

(4) *The metric g has non-constant curvature and admits a non-homothetic vector field and, after a local change of coordinates,*

$$g = k\, f'(x)(dx^2 + h) ,$$

where $k \in \mathbb{R} \setminus \{0\}$ and where h does not admit any non-zero homothetic vector field. Moreover,

$$(i)\ f(x) = \frac{1}{x} \qquad (ii)\ f(x) = \tan(x) \qquad (iii)\ f(x) = \tanh(x) ,$$

and the respective projective vector field is a multiple of $f(x)\partial_x$.

PROOF. The first two cases are, of course, classical. As outlined in the previous section, if g has no homothetic vector fields, it is necessarily of Levi-Civita type 111 or 21, see (4.1). Therefore there is an endomorphism L with at least one non-constant eigenvalue of multiplicity one. In the 111 case, we integrate Solodovnikov's ODE system. In the case of 21-type metrics one proceeds analogously. □

We conclude the section by commenting on the case when both non-homothetic and homothetic vector fields exist. We proceed along the degree of mobility, and in the case of 21-type Levi-Civita metrics also the dimension of the homothetic algebra of h.

Due to a theorem by Matveev-Kiosak [**22**], the degree of mobility is either $D = 1$ or $D = 2$, if the metric is of non-constant curvature. If $D = 1$, then we are in the first case. So assume $D = 2$, which guarantees that g is Levi-Civita. Let us begin with the 111-type. One finds that L has at least one constant eigenvalue, and thus that the projective Lie algebra of g is generically 1-dimensional. However, there are also cases with an algebra of dimension 2 or 3, and the details can be found in [**33**].

Finally, let us assume $D = 2$ and that the metric is of 21-type. Then h has to have non-constant Gaussian curvature (as one can show that otherwise g would have constant curvature already). The projective vector field of g is necessarily of the form $v = \alpha(x)\partial_x + u$ where u is a homothetic vector field of h. Therefore we arrive at the following cases:

- $\zeta = const$. Then the dimension $\dim \mathfrak{p}(g) = \dim \mathfrak{p}(h) + 2$ is at least 2 and at most 5.
- h has constant curvature. Then $\dim \mathfrak{p}(g) \in \{3, 4\}$.
- $\dim \mathfrak{p}(h) = 1$. Then $\dim \mathfrak{p}(g) = 2$.

- $\dim \mathfrak{p}(h) = 1$. Then $\dim \mathfrak{p}(g) \in \{2, 3\}$.

Here, we do not have space to list the metrics realizing the cases in (2) of Theorem 4.6. But these can be found in terms of local canonical forms in [**33**], see Propositions 9, 10 and 11 of the reference (111-case), as well as in Corollary 3 and Theorem 3 (21-case).

THEOREM 4.7. *Let g be a local Riemannian metric of $n = 3$, not of constant curvature, with a non-vanishing, homothetic vector field v. Then, locally, in a small neighborhood of almost any point, the projective Lie algebra of g is of dimension*

$$1 \leq \dim \mathfrak{p}(g) \leq 5\,,$$

where each number is realized. In particular, if $\dim \mathfrak{p}(g) = 1$, all projective vector fields are multiples of v. In the other extreme, $\dim \mathfrak{p}(g) = 5$, the metric h is of constant curvature and $g = k(h + dz^2)$ ($k \neq 0$). Then $v = u + (k_1 z + k_0)\partial_z$ with constants $k_0, k_1 \in \mathbb{R}$.

5. Complex Lie problem

The complex analog of projective geometry is called *c-projective geometry.* This field studies what is now known as *J-planar curves* which were treated in 1954 by Otsuki and Tashiro [**44**] after S. Bochner [**6**, Theorem 2] had shown in 1947 that two metrics which are Kähler w.r.t. the same complex structure J and are geodesically equivalent must be affinely equivalent (i.e. their Levi-Civita connections coincide). A comprehensive overview of the topic can be found in [**14**].

Let M be an n-dimensional complex manifold with complex structure J (here n is the complex dimension, thus the real dimension of M is $2n$) equipped with a *complex connection* ∇, i.e., a connection such that $\nabla J = 0$. A *J-planar curve* is a curve $\gamma : I \subseteq \mathbb{R} \to M$ such that

$$\nabla_{\dot\gamma}\dot\gamma = \alpha(t)\dot\gamma + \beta(t)J(\dot\gamma) \tag{5.1}$$

for some smooth functions $\alpha, \beta \in C^\infty(I)$. It is a natural generalization of unparametrized geodesics. Note that Equation (5.1) is equivalent to $\nabla_{\dot\gamma}\dot\gamma \wedge \dot\gamma \wedge J(\dot\gamma) = 0$ whereas unparametrized geodesics are characterized by $\nabla_{\dot\gamma}\dot\gamma \wedge \dot\gamma = 0$.

DEFINITION 5.1. Two complex connections on the same complex manifold (M, J) are *c-projectively equivalent* if they share the same J-planar curves. The set of all connections c-projectively equivalent to a given connection ∇ is called the *c-projective class* of ∇.

DEFINITION 5.2. A *c-projective transformation* is a local diffeomorphism of M that sends J-planar curves into J-planar curves. A *c-projective vector field* is a vector field on M whose local flow acts by c-projective transformations.

The c-projective vector fields of a complex connection form a Lie algebra.

The above definitions can also be given in the case that M is equipped with a *Kähler* metric g by considering its Levi-Civita connection. A metric is said to be of *constant holomorphic curvature* if the sectional curvature, restricted to J-invariant planes, is constant on M. We can then state the Lie problem in the complex case:

Complex Lie problem: Determine a sharp list of local forms of $2n$-dimensional Kähler metrics admitting a non-zero c-projective vector field.

Already in the case $2n = 4$ many different cases can occur. Assume that g and $\hat{g}$ are c-projectively equivalent and consider the $(1,1)$-tensor (this is the c-projective analog of (2.9))

$$A = \left|\frac{\det(\hat{g})}{\det(g)}\right|^{\frac{1}{2(n+1)}} \hat{g}^{-1} g\,.$$

Then for any $t \in \mathbb{R}$ the vector field

$$J \operatorname{grad} \sqrt{\det(t\mathbb{1} - A)}$$

is Killing for g and $\hat{g}$; these *canonical Killing vector fields* [**11**, **14**, **23**] are the examples of c-projective vector fields that are easiest to find. A list of local normal forms for c-projectively equivalent metrics was published in [**11**].

Example 5.3 (c-projectively equivalent metrics). Let g and $\hat{g}$ be c-projectively equivalent metrics of dimension $2n = 4$ and let A have two real non-constant eigenvalues ρ, σ. Then, in a neighbourhood of almost any point, there exist coordinates (x, y, s, t) such that $\rho = \rho(x)$, $\sigma = \sigma(y)$ and the metric g, the Kähler form $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ and the endomorphism A take the forms

$$\begin{aligned} g &= (\rho - \sigma)(dx^2 + \epsilon dy^2) + \frac{1}{\rho - \sigma}\left[(\rho')^2(ds + \sigma dt)^2 + \epsilon(\sigma')^2(ds + \rho dt)^2\right] \\ \omega &= d((\rho + \sigma)ds + \rho\sigma dt) \\ A &= \rho\partial_x \otimes dx + \sigma\partial_y \otimes dy + (\rho + \sigma)\partial_s \otimes ds + \rho\sigma\partial_s \otimes dt - \partial_t \otimes ds \end{aligned}$$

where $\epsilon = 1$ in the case of positive signature and $\epsilon = -1$ in the case of split signature. J and $\hat{g}$ can be computed from the given data via the formulae for ω and A mentioned above. This is the c-projective analog of the Liouville case of Proposition 3.8.

For the full c-projective analog of Proposition 3.8 see [**9**, Theorem 3.1] and for the generalization to all dimensions [**11**, Theorem 1.6]. The classification of the metrics in [**11**, Theorem 1.6] w.r.t. the existence of canonical Killing vector fields is evident but some metrics may admit non-canonical Killing vector fields as well as other c-projective vector fields. Due to its significance regarding other problems, in particular the Yano-Obata conjecture [**20**, **40**], existing research has been focused on the following problem.

Complex Lie problem for non-affine c-projective vector fields: Determine a sharp list of local forms of $2n$-dimensional Kähler metrics admitting a non-affine c-projective vector field.

In the case of dimension $2n = 4$ there is a characterization in terms of the degree of mobility:

Proposition 5.4. Let (M, g, J) be a connected Kähler manifold of real dimension 4 (of arbitrary signature) and of non-constant holomorphic sectional curvature. Then the degree of mobility is at most 2 [**3**, **20**].

If the degree of mobility is one, then only homothetic and Killing vector fields are possible. The case of degree of mobility three cannot occur and if the degree of mobility attains the maximal value four, then the metric is of constant holomorphic sectional curvature [**41**]. In the case that the degree of mobility is two, a list of metrics admitting a non-affine c-projective vector field is given in [**9**, Theorems 1.2 and 1.5].

Example 5.5 ([**9**, Theorem 1.2, case L1]). The Kähler structure defined by

$$g = (x-y)(c_1^2 dx^2 + \epsilon c_2^2 dy^2) + \frac{1}{x-y}\left[\frac{1}{c_1^2}(ds + y dt)^2 + \epsilon \frac{1}{c_2^2}(ds + x dt)^2\right]$$
$$\omega = d\left[(x+y)\, ds + xy\, dt\right]$$

admits the non-affine c-projective vector field $v = \partial_x + \partial_y - t\partial_s$.

The list in [**9**, Theorems 1.2 and 1.5] is complete in the sense that for any metric admitting a non-affine c-projective vector field in a neighborhood of almost any point there exist coordinates such that the metric takes one of the given forms. But the list is not sharp in the sense that some of the metrics contained are diffeomorphic to others in the list, and some of them have constant holomorphic sectional curvature.

In real dimension $2n \geq 6$ Proposition 5.4 is true only under the additional assumption of closedness [**20**], but fails without the additional assumption [**41**]. The results in [**42**, §2.2] allow the construction of (local) metrics with degree of mobility ≥ 3 and non-constant holomorphic sectional curvature. For Riemannian Kähler metrics that admit a c-projectively equivalent metric that is not affinely equivalent the possible degrees of mobility and the possible dimensions of the space of non-affine c-projective vector fields have been determined in [**41**].

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Politecnico di Torino, via Duca degli Abruzzi 24, 10129 Torino (Italy)
Email address: giovanni.manno@polito.it

Politecnico di Torino, via Duca degli Abruzzi 24, 10129 Torino (Italy)
Email address: jsmaths@gmx.net

Politecnico di Torino, via Duca degli Abruzzi 24, 10129 Torino (Italy)
Email address: andreas.vollmer@polito.it