Continuity of the non-convex play operator in the space of rectifiable curves

CONTINUITY OF THE NON-CONVEX PLAY OPERATOR IN THE SPACE OF RECTIFIABLE CURVES

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Abstract. In this paper we prove that the vector play operator with a uniformly prox-regular characteristic set of constraints is continuous with respect to the BV-norm and to the BV-strict metric in the space of rectifiable curves, i.e. in the space of continuous functions of bounded variation. We do not assume any further regularity of the characteristic set. We also prove that the non-convex play operator is rate independent.

Keywords: Evolution variational inequalities, Play operator, Sweeping processes, Functions of bounded variation, Prox-regular sets

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1. Introduction

Several phenomena in elasto-plasticity, ferromagnetism, and phase transitions are modeled by the following evolution variational inequality in a real Hilbert space $\mathcal{H}$ with the inner product $\langle \cdot, \cdot \rangle$:

$$\langle z - u(t) + y(t), y'(t) \rangle \le 0 \qquad \forall z \in \mathcal{Z}, \quad t \in [0, T]\,, \tag{1.1}$$

$$u(t) - y(t) \in \mathcal{Z} \qquad \forall t \in [0, T]\,. \tag{1.2}$$

Here $u : [0, T] \longrightarrow \mathcal{H}$ is a given "input" function, $T > 0$ being the final time of evolution, and $y : [0, T] \longrightarrow \mathcal{H}$ is the unknown function, y' being its derivative. It is assumed that the set $\mathcal{Z}$ in the constraint (1.2) is a closed convex subset of $\mathcal{H}$, and it is usually called the *characteristic set*. We refer to the monographs [20, 27, 37, 5, 22, 28] for surveys on these physical models. It is well-known (see, e.g., [22]), that if u is

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absolutely continuous, then there exists a unique absolutely continuous solution y of (1.1)-(1.2) together with the given initial condition

$$u(0) - y(0) = z_0 \in \mathcal{Z}. \tag{1.3}$$

If we set $\mathsf{P}(u, z_0) := y$ we have defined a solution operator $\mathsf{P} : W^{1,1}([0,T]\,;\mathcal{H}) \times \mathcal{Z} \longrightarrow W^{1,1}([0,T]\,;\mathcal{H})$ which is called the *play operator*. Here $W^{1,1}([0,T]\,;\mathcal{H})$ denotes the space of $\mathcal{H}$-valued Lipschitz continuous functions defined on $[0,T]$ (precise definitions will be given in Sections 2 and 3). An important feature of P is its *rate independence*, i.e.

$$\mathsf{P}(u \circ \phi, z_0) = \mathsf{P}(u, z_0) \circ \phi, \tag{1.4}$$

whenever $\phi : [0,T] \longrightarrow [0,T]$ is an increasing surjective Lipschitz continuous reparametrization of time. The play operator can be extended to the space of rectifiable curves in $\mathcal{H}$, i.e. to the space of continuous $\mathcal{H}$-valued functions of bounded variation $C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$ ([22]). This can be done by reformulating (1.1) as an integral variation inequality:

$$\int_0^T \langle z(t) - u(t) + y(t), \mathrm{d}\, y(t)\rangle \le 0, \qquad \forall z \in BV([0,T]\,;\mathcal{Z}), \tag{1.5}$$

where the integral can be interpreted as a Riemann-Stieltjes integral (see e.g. [22]), but also as a Lebesgue integral with respect to the differential measure $\mathrm{D}y$, the distributional derivative of y (see [32] for the equivalence of the two formulations). By [22] for every $u \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$ there exists a unique $y \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$ such that (1.5), (1.2), (1.3) hold. Therefore the play operator can be extended to the operator $\mathsf{P} : C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H}) \times \mathcal{Z} \to C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$. Its domain of definition is naturally endowed with the strong BV-norm defined by

$$\|u\|_{BV} := \|u\|_\infty + \mathrm{V}(u, [0,T]), \qquad u \in BV([0,T]\,;\mathcal{H}), \tag{1.6}$$

where $\|u\|_\infty$ is the supremum norm of u and $\mathrm{V}(u, [0,T])$ is the total variation of u. For absolutely continuous inputs the BV-norm is exactly the standard $W^{1,1}$-norm, and the continuity of P on $W^{1,1}(0,T;\mathcal{H})$ in this special case was proved in [21] for finite dimensional $\mathcal{H}$ and in [22] for separable Hilbert spaces. For such spaces $\mathcal{H}$, assuming $\mathcal{Z}$ having a smooth boundary, the BV-norm continuity of P on $BV([0,T]\,;\mathcal{H}) \cap C([0,T]\,;\mathcal{H})$ (respectively on $BV([0,T]\,;\mathcal{H})$) was proved in [6] (respectively in [25]). Under this additional regularity of $\mathcal{Z}$, in [6, 25] it is also shown that P is locally Lipschitz continuous. In [19] we were able to drop the regularity of $\mathcal{Z}$ and we proved that P is BV-norm continuous on $BV([0,T]\,;\mathcal{H})$ for an arbitrary characteristic set $\mathcal{Z}$.

Another relevant topology in BV is the one induced by the so-called *strict metric*, which is defined by

$$d_s(u,v) := \|u-v\|_\infty + |\,\mathrm{V}(u,[0,T]) - \mathrm{V}(v,[0,T])|, \qquad u,v \in BV([0,T]\,;\mathcal{H}). \tag{1.7}$$

Indeed every $u \in BV([0,T]\,;\mathcal{H})$ can be approximated by a sequence $u_n \in AC([0,T]\,;\mathcal{H})$ converging to u in the strict metric. In [22] it is proved that P is continuous on $C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$ with respect to the strict metric (shortly, "strictly continuous"), provided $\mathcal{Z}$ has a smooth boundary. In [32] this regularity assumption is dropped and it is proved that P is continuous on $C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$ with respect to the strict metric for every characteristic convex set $\mathcal{Z}$. In [32] it is also proved that in general P is not strictly continuous on the whole $BV([0,T]\,;\mathcal{H})$. For other results on the continuity properties of P we refer to [31, 33, 18].

Previous results are concerned with the case of a convex set $\mathcal{Z}$, but the characteristic set of constraints can be non-convex in some applications, e.g. in problems of crowd motion modeling (see [38]).

In the following we will restrict ourselves to uniform prox-regular sets - these are closed sets having a neighborhood where the projection exists and is unique. For the notion of prox-regularity we refer the reader to [16, 39, 8, 29, 12]. Following e.g. [11, 24], we see that the proper formulation of (1.5) in the case of a prox-regular set $\mathcal{Z}$ reads

$$\int_0^T \langle z(t)-u(t)+y(t), \mathrm{d}y(t)\rangle \le \frac{1}{2r}\int_0^T \|z(t)-u(t)+y(t)\|^2 \,\mathrm{d}V_y(t) \quad \forall z \in BV([0,T]\,;\mathcal{Z}), \tag{1.8}$$

where $V_y(t) = \mathrm{V}(y,[0,t])$ for $t \in [0,T]$ and $\|\cdot\|$ is the norm in $\mathcal{H}$. It is well-known (cf., e.g., [15] or [24]) that for every $u \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$ there exists a unique $y = \mathsf{P}(u,z_0) \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$ which satifies (1.8), (1.2), (1.3). Thus also in the non-convex case the solution operator

$$\mathsf{P} : C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H}) \times \mathcal{Z} \to C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H}),$$

of problem (1.8), (1.2), (1.3) can be defined, which we will call *non-convex play operator*. In [23] it is proved that in $W^{1,1}([0,T]\,;\mathcal{H})$ the operator P is continuous (and also local Lipschitz continuous) with respect to the strong BV-norm under the assumption that $\mathcal{Z}$ satisfies a suitable regularity assumption, to be more precise it is required that $\mathcal{Z}$ is the sublevel set of a Lipschitz continuous function. In the present paper we prove that P is BV-norm continuous on the larger space $C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$ and for every characteristic prox-regular set $\mathcal{Z}$. We also prove that it is continuous with respect to the strict metric on the space of continuous functions of bounded variation. The technique of our proof is obtained via a reparametrization method by the arc length. In order to perform this reparametrization we use the rate independence of P, which, to the best of our knowledge is proved here for the first time for the non-convex case. The question of the BV-norm continuity on the whole space $BV([0,T]\,;\mathcal{H})$ will be addressed in a future paper: in that case the presence of jumps makes the problem considerably more difficult and the reparametrization method studied in [34, 35] is needed.

The plan of the paper is the following: In Section 2 we recall the preliminaries needed to prove our results, which are stated in Section 3. In Section 4 we perform all the proofs.

2. Preliminaries

The set of integers greater or equal to 1 will be denoted by $\mathbb{N}$.

2.1. **Prox-regular sets.** Throughout this paper we assume that

$$\begin{cases} \mathcal{H} \text{ is a real Hilbert space with the inner product } \langle x, y\rangle, \\ \mathcal{H} \neq \{0\}, \\ \|x\| := \langle x, x\rangle^{1/2} \qquad \text{for } x \in \mathcal{H}. \end{cases} \tag{2.1}$$

If $\mathcal{S} \subseteq \mathcal{H}$ and $x \in \mathcal{H}$ we set $d_{\mathcal{S}}(x) := \inf\{\|x - s\| \ : \ s \in \mathcal{S}\}$.

Definition 2.1. *If $\mathcal{K}$ is a closed subset of $\mathcal{H}$, $\mathcal{K} \neq \varnothing$, and $y \in \mathcal{H}$, we define the* set of projections of y onto $\mathcal{K}$ *by setting*

$$\operatorname{Proj}_{\mathcal{K}}(y) := \left\{x \in \mathcal{K} \ : \ \|x - y\| = \inf_{z \in \mathcal{K}} \|z - y\|\right\} \tag{2.2}$$

and the (exterior) normal cone of $\mathcal{K}$ at x *by*

$$N_{\mathcal{K}}(x) := \{\lambda(y - x) \ : \ x \in \operatorname{Proj}_{\mathcal{K}}(y),\ y \in \mathcal{H},\ \lambda \geq 0\}. \tag{2.3}$$

We recall the notion of prox-regularity (see [8, Theorem 4.1-(d)]) which can also be called "mild non-convexity".

Definition 2.2. *If $\mathcal{K}$ is a closed subset of $\mathcal{H}$ and if $r \in \,]0, \infty[$, we say that $\mathcal{K}$ is* r-prox-regular *if for every $y \in \{v \in \mathcal{H} \ : \ 0 < d_{\mathcal{K}}(v) < r\}$ we have that $\operatorname{Proj}_{\mathcal{K}}(y) \neq \varnothing$ and*

$$x \in \operatorname{Proj}_{\mathcal{K}}\left(x + r\frac{y - x}{\|y - x\|}\right), \qquad \forall x \in \operatorname{Proj}_{\mathcal{K}}(y).$$

It is well-known and easy to prove that if $x \in \mathrm{Proj}_{\mathcal{K}}(y_0)$ for some $y_0 \in \mathcal{H}$, then $\mathrm{Proj}_{\mathcal{K}}(y) = \{x\}$ for every y lying in the segment with endpoints y_0 and x. Thus it follows that if $\mathcal{K}$ is r-prox-regular for some $r > 0$, then $\mathrm{Proj}_{\mathcal{K}}(y)$ is a singleton for every $y \in \{v \in \mathcal{H} \ : \ 0 < d_{\mathcal{K}}(v) < r\}$.

Prox-regularity can be characterized by means of a variational inequality, indeed in [29, Theorem 4.1] and in [12, Theorem 16] one can find the proof of the following:

Theorem 2.1. *Let $\mathcal{K}$ be a closed subset of $\mathcal{H}$ and let $r \in \left]0,\infty\right[$. Then $\mathcal{K}$ is r-prox-regular if and only if for every $x \in \mathcal{K}$ and $n \in N_{\mathcal{K}}(x)$ we have*

$$\langle n, z - x\rangle \leq \frac{\|n\|}{2r}\|z - x\|^2, \qquad \forall z \in \mathcal{K}.$$

2.2. **Functions of bounded variation.** Let I be an interval of $\mathbb{R}$. The set of $\mathcal{H}$-valued continuous functions defined on I is denoted by $C(I;\mathcal{H})$. For a function $f : I \longrightarrow \mathcal{H}$ and for $S \subseteq I$ we write $\mathrm{Lip}(f,S) := \sup\{\|f(t) - f(s)\|/|t - s| \ : \ s,t \in S,\ s \neq t\}$, $\mathrm{Lip}(f) := \mathrm{Lip}(f,I)$, the Lipschitz constant of f, and $Lip(I;X) := \{f : I \longrightarrow \mathcal{H} \ : \ \mathrm{Lip}(f) < \infty\}$, the set of $\mathcal{H}$-valued Lipschitz continuous functions on I.

Definition 2.3. *Given an interval $I \subseteq \mathbb{R}$, a function $f : I \longrightarrow \mathcal{H}$, and a subinterval $J \subseteq I$, the* variation of f on J *is defined by*

$$\mathrm{V}(f,J) := \sup\left\{\sum_{j=1}^{m}\|(f(t_{j-1}) - f(t_j)\| \ : \ m \in \mathbb{N},\ t_j \in J\ \forall j,\ t_0 < \cdots < t_m\right\}.$$

If $\mathrm{V}(f,I) < \infty$ *we say that* f is of bounded variation on I *and we set*

$$BV(I;\mathcal{H}) := \{f \in I :\longrightarrow \mathcal{H} \ : \ \mathrm{V}(f,I) < \infty\}.$$

It is well known that the completeness of $\mathcal{H}$ implies that every $f \in BV(I;\mathcal{H})$ admits one sided limits $f(t-), f(t+)$ at every point $t \in I$, with the convention that $f(\inf I-) := f(\inf I)$ if $\inf I \in I$, and that $f(\sup I+) := f(\sup I)$ if $\sup I \in I$. If I is bounded we have $Lip(I;\mathcal{H}) \subseteq BV(I;\mathcal{H})$.

2.3. **Differential measures.** Given an interval I of the real line $\mathbb{R}$, the family of Borel sets in I is denoted by $\mathscr{B}(I)$. If $\mu : \mathscr{B}(I) \longrightarrow [0,\infty]$ is a measure, $p \in [1,\infty]$, then the space of $\mathcal{H}$-valued functions which are p-integrable with respect to μ will be denoted by $L^p(I,\mu;\mathcal{H})$ or simply by $L^p(\mu;\mathcal{H})$. For the theory of integration of vector valued functions we refer, e.g., to [26, Chapter VI]. When $\mu = \mathcal{L}^1$, where $\mathcal{L}^1$ is the one dimensional Lebesgue measure, we write $L^p(I;\mathcal{H}) := L^p(I,\mu;\mathcal{H})$.

We recall that a *$\mathcal{H}$-valued measure on I* is the map $\nu : \mathscr{B}(I) \longrightarrow \mathcal{H}$ such that $\nu(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty}\nu(B_n)$ for every sequence (B_n) of mutually disjoint sets in

$\mathscr{B}(I)$. The *total variation of* ν is the positive measure $|\nu| : \mathscr{B}(I) \longrightarrow [0,\infty]$ defined by

$$|\nu|(B) := \sup\left\{\sum_{n=1}^{\infty}\|\nu(B_n)\| \ : \ B = \bigcup_{n=1}^{\infty} B_n,\ B_n \in \mathscr{B}(I),\ B_h \cap B_k = \varnothing \text{ if } h \neq k\right\}.$$

The vector measure ν is said to be *with bounded variation* if $|\nu|(I) < \infty$. In this case the equality $\|\nu\| := |\nu|(I)$ defines a complete norm on the space of measures with bounded variation (see, e.g. [14, Chapter I, Section 3]).

If $\mu : \mathscr{B}(I) \longrightarrow [0,\infty]$ is a positive bounded Borel measure and if $g \in L^1(I,\mu;\mathcal{H})$, then $g\mu : \mathscr{B}(I) \longrightarrow \mathcal{H}$ denotes the vector measure defined by

$$g\mu(B) := \int_B g \,\mathrm{d}\mu, \qquad B \in \mathscr{B}(I).$$

In this case we have that

$$|g\mu|(B) = \int_B \|g(t)\| \,\mathrm{d}\mu \qquad \forall B \in \mathscr{B}(I) \tag{2.4}$$

(see [14, Proposition 10, p. 174]).

Assume that $\nu : \mathscr{B}(I) \longrightarrow \mathcal{H}$ is a vector measure with bounded variation and $f : I \longrightarrow \mathcal{H}$ and $\phi : I \longrightarrow \mathbb{R}$ are two *step maps with respect to* ν, i.e. there exist $f_1,\ldots,f_m \in \mathcal{H}$, $\phi_1,\ldots,\phi_m \in \mathcal{H}$ and $A_1,\ldots,A_m \in \mathscr{B}(I)$ mutually disjoint such that $|\nu|(A_j) < \infty$ for every j and $f = \sum_{j=1}^m \mathbb{1}_{A_j} f_j$, $\phi = \sum_{j=1}^m \mathbb{1}_{A_j}\phi_j$. Here $\mathbb{1}_S$ is the characteristic function of a set S, i.e. $\mathbb{1}_S(x) := 1$ if $x \in S$ and $\mathbb{1}_S(x) := 0$ if $x \notin S$. For such step maps we define $\int_I \langle f, \mathrm{d}\nu\rangle := \sum_{j=1}^m \langle f_j, \nu(A_j)\rangle \in \mathbb{R}$ and $\int_I \phi \,\mathrm{d}\nu := \sum_{j=1}^m \phi_j \nu(A_j) \in \mathcal{H}$.

If $St(|\nu|;\mathcal{H})$ (resp. $St(|\nu|)$) is the set of $\mathcal{H}$-valued (resp. real valued) step maps with respect to ν, then the maps $St(|\nu|;\mathcal{H}) \longrightarrow \mathcal{H} : f \longmapsto \int_I \langle f, \mathrm{d}\nu\rangle$ and $St(|\nu|) \longrightarrow \mathcal{H} : \phi \longmapsto \int_I \phi \,\mathrm{d}\nu$ are linear and continuous when $St(|\nu|;\mathcal{H})$ and $St(|\nu|)$ are endowed with the L^1-seminorms $\|f\|_{L^1(|\nu|;\mathcal{H})} := \int_I \|f\| \,\mathrm{d}|\nu|$ and $\|\phi\|_{L^1(|\nu|)} := \int_I |\phi| \,\mathrm{d}|\nu|$. Therefore they admit unique continuous extensions $\mathsf{I}_\nu : L^1(|\nu|;\mathcal{H}) \longrightarrow \mathbb{R}$ and $\mathsf{J}_\nu : L^1(|\nu|) \longrightarrow \mathcal{H}$, and we set

$$\int_I \langle f, \mathrm{d}\nu\rangle := \mathsf{I}_\nu(f), \quad \int_I \phi \ \mathrm{d}\nu := \mathsf{J}_\nu(\phi), \qquad f \in L^1(|\nu|;\mathcal{H}), \quad \phi \in L^1(|\nu|).$$

If μ is a bounded positive measure and $g \in L^1(\mu;\mathcal{H})$, arguing first on step functions, and then taking limits, it is easy to check that

$$\int_I \langle f, \mathrm{d}\,(g\mu)\rangle = \int_I \langle f, g\rangle \,\mathrm{d}\mu, \qquad \forall f \in L^\infty(\mu;\mathcal{H}).$$

The following results (cf., e.g., [14, Section III.17.2-3, p. 358-362]) provide the connection between functions with bounded variation and vector measures which will be implicitly used in this paper.

Theorem 2.2. *For every $f \in BV(I;\mathcal{H})$ there exists a unique vector measure of bounded variation $\nu_f : \mathscr{B}(I) \longrightarrow \mathcal{H}$ such that*

$$\nu_f(]c,d[) = f(d-) - f(c+), \qquad \nu_f([c,d]) = f(d+) - f(c-),$$
$$\nu_f([c,d[) = f(d-) - f(c-), \qquad \nu_f(]c,d]) = f(d+) - f(c+).$$

whenever $\inf I \le c < d \le \sup I$ *and the left hand side of each equality makes sense. Conversely, if $\nu : \mathscr{B}(I) \longrightarrow \mathcal{H}$ is a vector measure with bounded variation, and if $f_\nu : I \longrightarrow \mathcal{H}$ is defined by $f_\nu(t) := \nu([\inf I, t[\cap I)$, then $f_\nu \in BV(I;\mathcal{H})$ and $\nu_{f_\nu} = \nu$.*

Proposition 2.1. *Let $f \in BV(I;\mathcal{H})$, let $g : I \longrightarrow \mathcal{H}$ be defined by $g(t) := f(t-)$, for $t \in \operatorname{int}(I)$, and by $g(t) := f(t)$, if $t \in \partial I$, and let $V_g : I \longrightarrow \mathbb{R}$ be defined by $V_g(t) := \mathrm{V}(g, [\inf I, t] \cap I)$. Then $\nu_g = \nu_f$ and $|\nu_f|(I) = \nu_{V_g}(I) = \mathrm{V}(g, I)$.*

The measure ν_f is called the *Lebesgue-Stieltjes measure* or *differential measure* of f. Let us see the connection between the differential measure and the distributional derivative. If $f \in BV(I;\mathcal{H})$ and if $\overline{f} : \mathbb{R} \longrightarrow \mathcal{H}$ is defined by

$$\overline{f}(t) := \begin{cases} f(t) & \text{if } t \in I, \\ f(\inf I) & \text{if } \inf I \in \mathbb{R},\ t \notin I,\ t \le \inf I, \\ f(\sup I) & \text{if } \sup I \in \mathbb{R},\ t \notin I,\ t \ge \sup I, \end{cases} \tag{2.5}$$

then, as in the scalar case, it turns out (cf. [32, Section 2]) that $\nu_f(B) = \mathrm{D}\overline{f}(B)$ for every $B \in \mathscr{B}(\mathbb{R})$, where $\mathrm{D}\overline{f}$ is the distributional derivative of $\overline{f}$, i.e.

$$-\int_{\mathbb{R}} \varphi'(t)\overline{f}(t)\,\mathrm{d}t = \int_{\mathbb{R}} \varphi\,\mathrm{d}\mathrm{D}\overline{f}, \qquad \forall \varphi \in C_c^1(\mathbb{R};\mathbb{R}),$$

where $C_c^1(\mathbb{R};\mathbb{R})$ is the space of continuously differentiable functions on $\mathbb{R}$ with compact support. Observe that $\mathrm{D}\overline{f}$ is concentrated on I: $\mathrm{D}\overline{f}(B) = \nu_f(B \cap I)$ for every $B \in \mathscr{B}(I)$, hence in the remainder of the paper, if $f \in BV(I,\mathcal{H})$ then we will simply write

$$\mathrm{D}f := \mathrm{D}\overline{f} = \nu_f, \qquad f \in BV(I;\mathcal{H}), \tag{2.6}$$

and from the previous discussion it follows that

$$\|\mathrm{D}f\| = |\mathrm{D}f|(I) = \|\nu_f\| = \mathrm{V}(f, I), \qquad \forall f \in BV(I;\mathcal{H}). \tag{2.7}$$

If I is bounded and $p \in [1, \infty]$, then the classical Sobolev space $W^{1,p}(I;\mathcal{H})$ consists of those functions $f \in C(I;\mathcal{H})$ for which $\mathrm{D}f = g\mathcal{L}^1$ for some $g \in L^p(I;\mathcal{H})$ and $W^{1,\infty}(I;\mathcal{H}) = Lip(I;\mathcal{H})$. Let us also recall that if $f \in W^{1,1}(I;\mathcal{H})$ then the derivative $f'(t)$ exists $\mathcal{L}^1$-a.e. in $t \in I$, $\mathrm{D}f = f'\mathcal{L}^1$, and $\mathrm{V}(f, I) = \int_I \|f'(t)\| \,\mathrm{d}t$ (see e.g. [4, Appendix]).

3. Main results

From now on we will assume that

$$\text{(3.1)} \qquad \mathcal{Z} \text{ is a } r\text{-prox-regular subset of } \mathcal{H} \text{ for some } r > 0, \qquad T > 0.$$

We will consider on $BV([0,T]\,;\mathcal{H})$ the classical complete BV-norm defined by (1.6), where

$$\|f\|_\infty := \sup\{\|f(t)\| \ : \ t \in [0,T]\}.$$

The norm (1.6) is equivalent to the norm defined by

$$|||f|||_{BV} := \|f(0)\| + \mathrm{V}(f, [0,T]), \qquad f \in BV([0,T]\,;\mathcal{H}).$$

From (2.7) it also follows that

$$\|f\|_{BV} = \|f\|_\infty + \|\,\mathrm{D}f\| = \|f\|_\infty + |\mathrm{D}f|\,([0,T]), \qquad \forall f \in BV([0,T]\,;\mathcal{H}).$$

where $\mathrm{D}f$ is the differential measure of f and $|\mathrm{D}f|$ is the total variation measure of $\mathrm{D}f$. We also have

$$\|f\|_{BV} = \|f\|_\infty + \int_0^T \|f'(t)\| \,\mathrm{d}t \qquad \forall f \in W^{1,1}([0,T]\,;\mathcal{H}).$$

On $BV([0,T]\,;\mathcal{H})$ we will consider also the so-called *strict metric* defined by (1.7). We say that $f_n \to f$ *strictly on* $[0,T]$ if $d_s(f_n, f) \to 0$ as $n \to \infty$. Let us recall that d_s is not complete and the topology induced by d_s is not linear. We now define the so-called "non-convex play operator".

Definition 3.1. *Assume that* (2.1) *and* (3.1) *hold. We call* (non-convex) play operator *the mapping*

$$\mathsf{P} : C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H}) \times \mathcal{Z} \longrightarrow C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$$

associating with every $(u, z_0) \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H}) \times \mathcal{Z}$ *the unique function* $y = \mathsf{P}(u, z_0) \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$ *such that*

$$(3.2) \quad u(t) - y(t) \in \mathcal{Z} \quad \forall t \in [0,T]\,,$$

$$(3.3) \quad \int_{[0,T]} \langle z(t) - u(t) + y(t), \mathrm{d}\mathrm{D}y(t) \rangle \le \frac{1}{2r} \int_{[0,T]} \|z(t) - u(t) + y(t)\|^2 \,\mathrm{d}\,|\mathrm{D}y|\,(t) \qquad \forall z \in BV([0,T]\,;\mathcal{H}), z([0,T]) \subseteq \mathcal{Z},$$

$$(3.4) \quad u(0) - y(0) = z_0.$$

The existence and uniqueness of such a function $y = \mathsf{P}(u, z_0)$ *is well-known and is guaranteed by Theorem 3.1 below.*

The integrals in (3.3) are Lebesgue integrals with respect to the measures $\mathrm{D}y$ and $|\mathrm{D}y|$. The inequality can be equivalently written using Riemann-Stieltjes integrals, by virtue of [32, Lemma A.9] and the discussion in Section 2.3.

Here is the existence and uniqueness theorem mentioned in Definition 3.1.

Theorem 3.1. *Assume that* (2.1) *and* (3.1) *hold,* $u \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$ *and* $z_0 \in \mathcal{Z}$. *Then there exists a unique function* $y \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$ *such that* (3.2)–(3.4) *hold, in other words the non-convex play operator is well defined in* $C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$.

As we pointed out in the previous definition the existence and uniqueness of solution to problem (3.2)–(3.4) is well-known. The reader can refer for instance to [24], where the problem is dealt with exclusively within the framework of the integral formulation. But the result could also be inferred by a careful comparison of [11, Proposition 3.1] and of [15, Corollary 3.1]. However, since the literature contains different formulations, and the equivalence of those is not always explicitly proved, for the sake of completeness we show here how the existence and uniqueness of a solution to (3.2)–(3.4) can be derived from [15], which to the best of our knowledge contains the first proof of the existence of solution to the non-convex problem (3.5)–(3.8) (see also [7, 10, 1, 2]). We will need the following auxiliary result showing that (3.3) can be equivalently stated as a differential inclusion. We will prove it in the next section.

Proposition 3.1. *Assume that* (2.1) *and* (3.1) *hold and that* $u \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$ *and* $z_0 \in \mathcal{Z}$. *Then a function* $y \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$ *satisfies* (3.2)–(3.4) *if and only if there exists a measure* $\mu : \mathscr{B}([0,T]) \longrightarrow [0,\infty[$ *and a* *function* $v \in L^1(\mu, \mathcal{H})$ *such that*

$$\mathrm{D}y = v\mu, \tag{3.5}$$

$$u(t) - y(t) \in \mathcal{Z} \qquad \forall t \in [0,T] \tag{3.6}$$

$$-\, v(t) \in N_{u(t)-\mathcal{Z}}(y(t)) \qquad \text{for } \mu\text{-a.e. } t \in [0,T] \tag{3.7}$$

$$u(0) - y(0) = z_0. \tag{3.8}$$

Now let observe that thanks to [15, Corollary 3.1] we have that under the assumptions of Proposition 3.1 there exists a unique solution to (3.5)–(3.8). Thus by virtue of Proposition 3.1 we infer Theorem 3.1.

When the "input" function u of the play operator is more regular, we have the following characterization of solutions (see, e.g., [24, Corollary 6.3]).

Proposition 3.2. *Assume that* (2.1) *and* (3.1) *hold. If* $u \in W^{1,p}([0,T]\,;\mathcal{H})$, $z_0 \in \mathcal{Z}$, *and if* $y = \mathsf{P}(u, z_0)$ *satisfies* (3.2)–(3.4), *then* $y \in W^{1,p}([0,T]\,;\mathcal{H})$ *and*

$$u(t) - y(t) \in \mathcal{Z} \quad \forall t \in [0,T]\,, \tag{3.9}$$

$$\langle z - u(t) + y(t), y'(t)\rangle \le \frac{\|y'(t)\|}{2r}\|z(t) - u(t) + y(t)\|^2 \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,T],\ \forall z \in \mathcal{Z}, \tag{3.10}$$

$$u(0) - y(0) = z_0. \tag{3.11}$$

Moreover y *is the unique function in* $W^{1,p}([0,T]\,;\mathcal{H})$ *such that* (3.10)–(3.11) *hold.*

Now we can state our main theorems. The first result states that P is continuous with respect to the BV-norm on $C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$.

Theorem 3.2. *Assume that* (2.1) *and* (3.1) *hold. The play operator* $\mathsf{P} : C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H}) \times \mathcal{Z} \longrightarrow C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$ *is continuous with respect to the* BV*-norm* (1.6), *i.e. if*

$$\|u - u_n\|_{BV} \to 0, \quad \|z_0 - z_{0n}\| \to 0 \qquad \text{as } n \to \infty,$$

then

$$\|\mathsf{P}(u, z_0) - \mathsf{P}(u_n, z_{0n})\|_{BV} \to 0 \qquad \text{as } n \to \infty$$

whenever $u, u_n \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$ *and* $z_0, z_{0,n} \in \mathcal{Z}$ *for every* $n \in \mathbb{N}$.

We will also prove that the play operator is continuous with respect to the strict metric.

Theorem 3.3. *Assume that* (2.1) *and* (3.1) *hold. The play operator* $\mathsf{P} : C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H}) \times \mathcal{Z} \longrightarrow C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$ *is continuous with respect to the strict metric* d_s *defined by* (1.7)*, i.e. if*

$$d_s(u,u_n) \to 0, \quad \|z_0 - z_{0n}\| \to 0 \qquad \text{as } n \to \infty,$$

then

$$d_s(\mathsf{P}(u,z_0), \mathsf{P}(u_n,z_{0n})) \to 0 \qquad \text{as } n \to \infty$$

whenever $u, u_n \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$ *and* $z_0, z_{0,n} \in \mathcal{Z}$ *for every* $n \in \mathbb{N}$.

The proofs of our main theorems are strongly based on the fact that the play operator is rate independent, which is the property (3.12) of P proved in the following theorem.

Theorem 3.4. *Assume that* (2.1) *and* (3.1) *hold,* $u \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$*, and* $z_0 \in \mathcal{Z}$*. If* $\phi : [0,T] \longrightarrow [0,T]$ *is a continuous function such that* $(\phi(t)-\phi(s))(t-s) \geq 0$ *and* $\phi([0,T]) = [0,T]$*, and if* $y := \mathsf{P}(u,z_0)$ *satisfies* (3.2)–(3.4)*, then*

$$\mathsf{P}(u \circ \phi, z_0) = \mathsf{P}(u,z_0) \circ \phi. \tag{3.12}$$

We will prove Theorems 3.2, 3.3, and 3.4 in Section 4.

4. Proofs

Let us start with an integral characterization of the differential inclusion (3.7).

Lemma 4.1. *Assume that* $r > 0$, $T > 0$, $\mu : \mathscr{B}([0,T]) \longrightarrow [0,\infty[$ *is a measure. If* $u \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$, $v \in L^1(\mu;\mathcal{H})$, $y \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$*, and* $u(t) - y(t) \in \mathcal{Z}$ *for every* $t \in [0,T]$*, then the following two conditions are equivalent.*

(i) $-v(t) \in N_{u(t)-\mathcal{Z}}(y(t))$ *for* μ-a.e. $t \in [0,T]$.

(ii) *For every* $z \in BV([0,T]\,;\mathcal{H})$ *such that* $z([0,T]) \subseteq \mathcal{Z}$ *one has*

$$\int_{[0,T]} \langle z(t) - u(t) + y(t), v(t)\rangle \,\mathrm{d}\mu(t) \leq \frac{1}{2r}\int_{[0,T]} \|v(t)\|\,\|z(t) - u(t) + y(t)\|^2 \,\mathrm{d}\mu(t).$$

Proof. Assume first that (i) holds and let $z \in BV([0,T]\,;\mathcal{H})$ be such that $z(t) \in \mathcal{Z}$ for every $t \in [0,T]$. Then it follows that

$$\langle z(t) - u(t) + y(t), v(t)\rangle \leq \frac{\|v(t)\|}{2r}\|z(t) - u(t) + y(t)\|^2 \qquad \text{for } \mu\text{-a.e. } t \in [0,T],$$

and after integrating with respect to μ over $[0,T]$ we infer the condition (ii).
Now assume that (ii) is satisfied. Let L be the set of μ-Lebesgue points of v, according

to the definition given in Theorem 5.2 of the Appendix. If we fix $t \in L$, and choose $\zeta \in \mathcal{Z}$ and $f \in C([0,T]\,;\mathcal{H})$ arbitrarily, it is trivially seen that t is a μ-Lebesgue point of f, and we have

$$\begin{aligned}(4.1)\qquad &\int_{[t-h,t+h]\cap[0,T]} |\langle f(\tau), v(\tau)\rangle - \langle f(t), v(t)\rangle| \,\mathrm{d}\mu(\tau)\\ &\le \int_{[t-h,t+h]\cap[0,T]} (\|f\|_\infty \|v(\tau)-v(t)\| + \|v(t)\|\,\|f(\tau)-f(t)\|)\,\mathrm{d}\mu(\tau),\end{aligned}$$

therefore

$$(4.2)\qquad \lim_{h\searrow 0} \frac{1}{\mu([t-h,t+h]\cap[0,T])}\int_{[t-h,t+h]\cap[0,T]} \langle f(\tau), v(\tau)\rangle\,\mathrm{d}\mu(\tau) = \langle f(t), v(t)\rangle.$$

An analogous argument also shows that t is a μ-Lebesgue point of the real function $\tau \longmapsto \|v(\tau)\|\,\|f(\tau)\|^2$, indeed

$$\begin{aligned}&\big|\|v(\tau)\|\,\|f(\tau)\|^2 - \|v(t)\|\,\|f(t)\|^2\big|\\ &\le \|v(\tau)-v(t)\|\,\|f\|_\infty^2 + \|v(t)\|\,\big|\|f(\tau)\|^2 - \|f(t)\|^2\big|\end{aligned}$$

and $\tau \longmapsto \|f(\tau)\|^2$ is continuous. Therefore

$$\lim_{h\searrow 0} \frac{1}{\mu([t-h,t+h]\cap[0,T])}\int_{[t-h,t+h]\cap[0,T]} \|v(\tau)\|\,\|f(\tau)\|^2\,\mathrm{d}\tau = \|v(t)\|\,\|f(t)\|^2.$$

For any $h > 0$ we define the function $z : [0,T] \longrightarrow \mathcal{H}$ by

$$z(\tau) := \mathbb{1}_{[0,T]\cap[t-h,t+h]}(\tau)\zeta + \mathbb{1}_{[0,T]\smallsetminus[t-h,t+h]}(\tau)(u(\tau)-y(\tau)), \qquad \tau \in [0,T]\,.$$

We have that z is of bounded variation and that $z(\tau) \in \mathcal{Z}$ for every $\tau \in [0,T]$, thus we can take such a z in the condition (ii) and we get

$$\begin{aligned}&\int_{[t-h,t+h]\cap[0,T]} \langle \zeta - u(\tau) + y(\tau), v(\tau)\rangle\,\mathrm{d}\mu(\tau)\\ &\le \frac{1}{2r}\int_{[t-h,t+h]\cap[0,T]} \|v(\tau)\|\,\|\zeta - u(\tau) + y(\tau)\|^2\,\mathrm{d}\mu(\tau).\end{aligned}$$

Dividing this inequality by $\mu([t-h,t+h]\cap[0,T])$ and taking the limit as $h \searrow 0$, and taking the continuous function $f(t) = \zeta - u(t) + y(t)$, by the previous discussion we get $\langle \zeta - u(t) + y(t), v(t)\rangle \le \|v(t)\|\,\|\zeta - u(t) + y(t)\|^2/(2r)$. Therefore, as $\mu(L) = 0$, we have proved that

$$\langle \zeta - u(t) + y(t), v(t)\rangle \le \frac{\|v(t)\|}{2r}\|\zeta - u(t) + y(t)\|^2 \quad \forall \zeta \in \mathcal{Z}, \quad \text{for } \mu\text{-a.e. } t \in [0,T],$$

i.e. the condition (i) holds using Theorem 2.1. $\square$

Now we can prove Proposition 3.1.

Proof of Proposition 3.1. Let us first assume that (3.5)–(3.8) hold for some measure $\mu : \mathscr{B}([0,T]) \longrightarrow [0,\infty[$ and some function $v \in L^1(\mu, \mathcal{H})$. In particular it follows that $\lVert \mathrm{D}y \rVert = \|v\|\mu$, hence $\lVert \mathrm{D}y \rVert$ is μ-absolutely continuous, thus by the Radon-Nicodym theorem there exists $h \in L^1(\mu; \mathbb{R})$ such that $|h(t)| \le 1$ for μ-a.e. $t \in [0,T]$, and $\lVert \mathrm{D}y \rVert = h\mu$. On the other hand, thanks to Theorem 5.3 from the Appendix there exists $g \in L^1(\lVert \mathrm{D}y \rVert; \mathcal{H})$ such that $\|g(t)\| = 1$ for μ-a.e. $t \in [0,T]$, and $\mathrm{D}y = g \lVert \mathrm{D}y \rVert$, therefore $\mathrm{D}y = hg\mu$. In particular it follows (see (2.4)) that $\lVert \mathrm{D}y \rVert = |h|\|g\|\mu$ and $v(t) = h(t)g(t)$ for μ-a.e. $t \in [0,T]$. Hence, applying also Lemma 4.1, we obtain that

$$
\begin{aligned}
\int_{[0,T]} \langle z(t) - u(t) + y(t), \mathrm{d}\mathrm{D}y(t) \rangle &= \int_{[0,T]} \langle z(t) - u(t) + y(t), v(t) \rangle \, \mathrm{d}\mu(t) \\
&\le \frac{1}{2r} \int_{[0,T]} \|v(t)\| \|z(t) - u(t) + y(t)\|^2 \, \mathrm{d}\mu(t) \\
&= \frac{1}{2r} \int_{[0,T]} \|h(t)g(t)\| \|z(t) - u(t) + y(t)\|^2 \, \mathrm{d}\mu(t) \\
&= \frac{1}{2r} \int_{[0,T]} \|z(t) - u(t) + y(t)\|^2 |h(t)| \|g(t)\| \, \mathrm{d}\mu(t) \\
&= \frac{1}{2r} \int_{[0,T]} \|z(t) - u(t) + y(t)\|^2 \, \mathrm{d} \lVert \mathrm{D}y \rVert (t)
\end{aligned}
$$

and (3.3) is proved. Vice-versa let us assume that (3.2)–(3.4) hold. Then the condition (ii) of Lemma 4.1 is obtained by taking $\mu = \lVert \mathrm{D}y \rVert$ and v equal to the density of $\mathrm{D}y$ with respect to $\lVert \mathrm{D}y \rVert$, and we are done. $\square$

Now we prove that P is rate independent.

Proof of Theorem 3.4. Set $y := \mathsf{P}(u, z_0)$, and recall that $V_y(t) = \mathrm{V}(y, [0,t])$ for every $t \in [0,T]$. Hence $\lVert \mathrm{D}y \rVert = \mathrm{D}V_y$, and by the vectorial Radon-Nikodym theorem ([26, Corollary VII.4.2]) there exists $v \in L^1(\lVert \mathrm{D}y \rVert; \mathcal{H})$ such that $\mathrm{D}y = v\,\mathrm{D}V_y$. Let us fix $z \in BV([0,T]; \mathcal{H})$ such that $z([0,T]) \subseteq [0,T]$ and recall the following well-known formula holding for any measure $\mu : \mathscr{B}([0,T]) \longrightarrow [0,\infty[$, $g \in L^1(\mu; \mathcal{H})$, and $A \in \mathscr{B}([0,T])$:

$$
\int_{\phi^{-1}(A)} g(\phi(t)) \, \mathrm{d}\mu(t) = \int_A g(\tau) \, \mathrm{d}(\phi_* \mu)(\tau),
$$

where $\phi_*\mu : \mathscr{B}([0,T]) \longrightarrow [0,\infty[$ is the measure defined by $\phi_*\mu(B) := \mu(\phi^{-1}(B))$ for $B \in \mathscr{B}([0,T])$ (this formula can be proved by approximating g by a sequence of step functions and then taking the limit). If $0 \le \alpha \le \beta \le T$ we have

$$
\phi_* \,\mathrm{D}(V_y \circ \phi)([\alpha,\beta]) = \mathrm{D}\,(V_y \circ \phi)(\phi^{-1}([\alpha,\beta])) = \mathrm{D}V_y([\alpha,\beta]),
$$

hence

$$\phi_* \, \mathrm{D}(V_y \circ \phi) = \mathrm{D}V_y,$$

and for $0 \le a \le b \le T$ we find that

$$\begin{aligned}\mathrm{D}(y \circ \phi)([a,b]) &= y(\phi(b)) - y(\phi(a)) = \mathrm{D}y([\phi(a), \phi(b)]) \\ &= \int_{[\phi(a),\phi(b)]} v(\tau) \, \mathrm{d}\mathrm{D}V_y(\tau) = \int_{[\phi(a),\phi(b)]} v(\tau) \, \mathrm{d}\phi_*(\mathrm{D}(V_y \circ \phi))(\tau) \\ &= \int_{[a,b]} v(\phi(t)) \, \mathrm{d}\mathrm{D}(V_y \circ \phi)(t) = (v \circ \phi) \, \mathrm{D}(V_y \circ \phi)([a,b]),\end{aligned}$$

so that

$$\mathrm{D}(y \circ \phi) = (v \circ \phi) \, \mathrm{D}(V_y \circ \phi), \qquad \big|\mathrm{D}(y \circ \phi)\big| = \|v \circ \phi\| \, \mathrm{D}(V_y \circ \phi).$$

If $\psi(\tau) := \inf \phi^{-1}(\tau)$, then ψ is increasing and $\tau = \phi(\psi(\tau))$. Therefore, since $\mathrm{D}(V_y \circ \phi) = 0$ on every interval where ϕ is constant, we find that for every $h \in C(\mathbb{R}^2)$ we have

$$\int_{[0,T]} h(z(t), \phi(t)) \, \mathrm{d}\mathrm{D}(V_y \circ \phi)(t) = \int_{[0,T]} h(z(\psi(\phi(t)), \phi(t)) \, \mathrm{d}\mathrm{D}(V_y \circ \phi)(t).$$

Hence

$$\begin{aligned} &\int_{[0,T]} \langle z(t) - u(\phi(t)) + y(\phi(t)), \mathrm{d}\mathrm{D}(y \circ \phi)(t)\rangle \\ &= \int_{[0,T]} \langle z(t) - u(\phi(t)) + y(\phi(t)), v(\phi(t))\rangle \, \mathrm{d}\mathrm{D}(V_y \circ \phi)(t) \\ &= \int_{[0,T]} \langle z(\psi(\phi(t))) - u(\phi(t)) + y(\phi(t)), v(\phi(t))\rangle \, \mathrm{d}\mathrm{D}(V_y \circ \phi)(t) \\ &= \int_{[0,T]} \langle z(\psi(\tau)) - u(\tau) + y(\tau), v(\tau)\rangle \mathrm{d}\mathrm{D}V_y(\tau) \\ &= \int_{[0,T]} \langle z(\psi(\tau)) - u(\tau) + y(\tau), \mathrm{d}\mathrm{D}y(\tau)\rangle, \end{aligned} \tag{4.3}$$

and

$$
\begin{aligned}
(4.4) \qquad & \int_{[0,T]} \|z(t) - u(\phi(t)) + y(\phi(t))\|^2 \, \mathrm{d} \left| \mathrm{D}(y \circ \phi) \right| (t) \\
&= \int_{[0,T]} \|z(t) - u(\phi(t)) + y(\phi(t))\|^2 \|v(\phi(t))\| \, \mathrm{d}\mathrm{D}(V_y \circ \phi)(t) \\
&= \int_{[0,T]} \|z(\psi(\phi(t)) - u(\phi(t)) + y(\phi(t))\|^2 \|v(\phi(t))\| \, \mathrm{d}\mathrm{D}(V_y \circ \phi)(t) \\
&= \int_{[0,T]} \|z(\psi(\tau)) - u(\tau) + y(\tau)\|^2 \|v(\tau)\| \, \mathrm{d}\mathrm{D}V_y(\tau) \\
&= \int_{[0,T]} \|z(\psi(\tau)) - u(\tau) + y(\tau)\|^2 \, \mathrm{d} \left| \mathrm{D}y \right| (\tau).
\end{aligned}
$$

Since $y = \mathsf{P}(u, z_0)$ we have that the right hand side of (4.3) is less or equal to the right hand side of (4.4) times $\frac{1}{2r}$ and this implies that

$$
\begin{aligned}
(4.5) \qquad & \int_{[0,T]} \langle z(t) - u(\phi(t)) + y(\phi(t)), \mathrm{d}\mathrm{D}(y \circ \phi)(t) \rangle \\
& \leq \frac{1}{2r} \int_{[0,T]} \|z(t) - u(\phi(t)) + y(\phi(t))\|^2 \, \mathrm{d} \left| \mathrm{D}(y \circ \phi) \right| (t),
\end{aligned}
$$

which is what we wanted to prove. $\square$

In the next result, we prove a normality rule for the non-convex play operator, thereby we generalize to the non-convex case the result in [22, Proposition 3.9]. The idea of the proof is analogous to the one of [22, Proposition 3.9].

Proposition 4.1. *Assume that* (2.1) *and* (3.1) *hold,* $u \in Lip([0,T]\,; \mathcal{H})$, $z_0 \in \mathcal{Z}$, *and that* $y = \mathsf{P}(u, z_0)$. *Let* $x = \mathsf{S}(u, z_0) : [0,T] \longrightarrow \mathcal{H}$ *and* $w = \mathsf{Q}(u, z_0) : [0,T] \longrightarrow \mathcal{H}$ *be defined by*

$$
(4.6) \qquad x(t) := \mathsf{S}(u, z_0)(t) := u(t) - y(t), \qquad t \in [0,T],
$$

$$
(4.7) \qquad w(t) := \mathsf{Q}(u, z_0)(t) := y(t) - x(t), \qquad t \in [0,T].
$$

Then $w = \mathsf{Q}(u, z_0) \in Lip([0,T]\,; \mathcal{H})$, $x = \mathsf{S}(u, z_0) \in Lip([0,T]\,; \mathcal{H})$, $x(t) \in Z$ *for every* $t \in [0,T]$, *and*

$$
(4.8) \qquad \langle y'(t), x'(t) \rangle = 0 \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,T],
$$

and

$$
(4.9) \qquad \|w'(t)\| = \|u'(t)\| \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0,T].
$$

Proof. Let $t \in [0,T]$ be a point where x is differentiable. Taking $z(t) = x(t+h) \in \mathcal{Z}$ for every $h > 0$ sufficiently small, we have

$$\frac{1}{h}\langle y'(t), x(t) - x(t+h)\rangle \geq -\frac{\|y'(t)\|}{2rh}\|x(t) - x(t+h)\|^2$$

therefore letting $h \searrow 0$ we get

$$\langle y'(t), -x'(t)\rangle \geq 0. \tag{4.10}$$

Taking $z(t) = x(t-h)$ for every $h > 0$ we also have

$$\frac{1}{h}\langle y'(t), x(t) - x(t-h)\rangle \geq -\frac{\|y'(t)\|}{2rh}\|x(t) - x(t-h)\|^2$$

therefore letting $h \searrow 0$ we get

$$\langle y'(t), x'(t)\rangle \geq 0,$$

which together with (4.10) yields (4.8). This formula implies that

$$\|w'(t)\|^2 = \|y'(t) - x'(t)\|^2 = \langle y'(t) - x'(t), y'(t) - x'(t)\rangle = \|y'(t)\|^2 + \|x'(t)\|^2, \tag{4.11}$$

and

$$\|u'(t)\|^2 = \|y'(t) + x'(t)\|^2 = \langle y'(t) + x'(t), y'(t) + x'(t)\rangle = \|y'(t)\|^2 + \|x'(t)\|^2, \tag{4.12}$$

therefore (4.9) follows. □

Let us observe that, in the previous proposition, the geometrical meaning of (4.11)-(4.12), is that $w'(t)$ and $u'(t)$ are the diagonals of the rectangle with sides $x'(t)$ and $y'(t)$, so that we have (4.9).

In order to prove the BV-norm continuity of P on $C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$ we need the following two auxiliary results. The first is the following:

Proposition 4.2. *Assume that* (2.1) *holds. For every* $f \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$, *let* $\ell_f : [0,T] \longrightarrow [0,T]$ *be defined by*

$$\ell_f(t) = \begin{cases} \dfrac{T}{\mathrm{V}(f,[0,T])}\,\mathrm{V}(f,[0,t]) & \textit{if } \mathrm{V}(f,[0,T]) \neq 0, \\ \\ 0 & \textit{if } \mathrm{V}(f,[0,T]) = 0, \end{cases} \tag{4.13}$$

which we call normalized arc-length of f. *Then there exists* $\widetilde{f} \in Lip([0,T]\,;\mathcal{H})$, *the* reparametrization of f by the normalized arc-length, *such that*

$$f = \widetilde{f} \circ \ell_f. \tag{4.14}$$

Moreover there exists a $\mathcal{L}^1$*-representative* $\widetilde{f}'$ *of the distributional derivative of* $\widetilde{f}$ *such that*

$$\|\widetilde{f}'(\sigma)\| = \frac{V(f,[0,T])}{T}, \qquad \forall \sigma \in [0,T]\,. \tag{4.15}$$

Proof. The existence of a function $\widetilde{f} \in Lip([0,T]\,;\mathcal{H})$ satisfying (4.14) is easy to prove (see e.g. [30, Proposition 3.1]). Moreover we know from [32, Lemma 4.3] that if g is a $\mathcal{L}^1$-representative of the distributional derivative of f, then $\|g(\sigma)\| = V(f,[0,T])/T$ for every $\sigma \in F$, for some $F \subseteq [0,T]$ with full measure in $[0,T]$. Thus (4.15) follows if we define the following Lebesgue representative of the derivative of $\widetilde{f}$:

$$\widetilde{f}'(\sigma) := \begin{cases} g(\sigma) & \text{if } \sigma \in F, \\ \\ \dfrac{(V(f,[0,T])}{T} e_0 & \text{if } \sigma \notin F, \end{cases}$$

where $e_0 \in \mathcal{H}$ is chosen so that $\|e_0\| = 1$. $\square$

Then, as for the Lipschitz case, we need to introduce the operator Q defined by $\mathsf{Q}(v) = 2\mathsf{P}(v) - v$ for $v \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$.

Lemma 4.2. *Assume that* $v \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$, $z_0 \in \mathcal{Z}$, *and let* $\mathsf{Q} : C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H}) \times \mathcal{Z} \longrightarrow C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$ *be defined by*

$$\mathsf{Q}(v,z_0) := 2\mathsf{P}(v,z_0) - v, \qquad v \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H}). \tag{4.16}$$

Then Q *is rate independent, i.e.*

$$\mathsf{Q}(v \circ \phi, z_0) = \mathsf{Q}(v,z_0) \circ \phi \qquad \forall v \in C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H}) \tag{4.17}$$

for every continuous function $\phi : [0,T] \longrightarrow [0,T]$ *such that* $(\phi(t) - \phi(s))(t-s) \geq 0$ *and* $\phi([0,T]) = [0,T]$. *Moreover if* ℓ_v *is the arc-length defined in* (4.13), *then*

$$\mathrm{D}\mathsf{Q}(v,z_0) = ((\mathsf{Q}(\widetilde{v},z_0))' \circ \ell_v)\, \mathrm{D}\ell_v, \tag{4.18}$$

i.e.

$$\mathrm{D}\mathsf{Q}(v,z_0)(B) = \int_B (\mathsf{Q}(\widetilde{v},z_0))'(\ell_v(t))\, \mathrm{d}\mathrm{D}\ell_v(t), \qquad \forall B \in \mathscr{B}([0,T]), \tag{4.19}$$

where formulas (4.18)–(4.19) *hold with any* $\mathcal{L}^1$*-representative* $(\mathsf{Q}(\widetilde{v}, z_0))'$ *of the distributional derivative of* $\mathsf{Q}(\widetilde{v}, z_0)$*. Finally we can take such an* $\mathcal{L}^1$*-representative so that*

$$\|(\mathsf{Q}(\widetilde{v}, z_0))'(\sigma)\| = \frac{\mathrm{V}(v, [0,T])}{T}, \qquad \forall \sigma \in [0,T]\,. \tag{4.20}$$

Proof. From Theorem 3.4 it follows that

$$\mathsf{Q}(v \circ \phi, z_0) = 2\mathsf{P}(v \circ \phi, z_0) - v \circ \phi = 2\mathsf{P}(v, z_0) \circ \phi - v \circ \phi = \mathsf{Q}(v, z_0) \circ \phi,$$

which is (4.17). Moreover, since $\widetilde{v}$ is Lipschitz continuous, we have that $\mathsf{Q}(\widetilde{v}, z_0) \in Lip([0,T]\,;\mathcal{H})$, therefore by [32, Theorem A.7] we infer that, if $\mathsf{Q}(\widetilde{v}, z_0)'$ is any $\mathcal{L}^1$-representative of the distributional derivative of $\mathsf{Q}(\widetilde{v}, z_0)$, then the bounded measurable function $\mathsf{Q}(\widetilde{v}, z_0)' \circ \ell_v$ is a density of $\mathsf{Q}(v, z_0)$ with respect to the measure $\mathrm{D}\ell_v$, i.e. (4.18) holds. Finally (4.20) follows from (4.9) of Proposition 4.1 and from (4.15) of Proposition 4.2. $\square$

Now we can prove our first main result.

Proof of Theorem 3.2. Let us consider $u \in BV([0,T]\,;\mathcal{H})$ and $u_n \in BV([0,T]\,;\mathcal{H})$ for every $n \in \mathbb{N}$, and assume that $\|u_n - u\|_{BV([0,T];\mathcal{H})} \to 0$ as $n \to \infty$. Then let $\ell := \ell_u$ and $\ell_n := \ell_{u_n}$ be the normalized arc-length functions defined in (4.13), so we have

$$u = \widetilde{u} \circ \ell, \quad u_n = \widetilde{u}_n \circ \ell_n \qquad \forall n \in \mathbb{N}.$$

Let us also set

$$w := \mathsf{Q}(u, z_0), \quad w_n := \mathsf{Q}(u_n, z_{0,n}), \qquad n \in \mathbb{N}, \tag{4.21}$$

where the operator Q is defined in Lemma 4.2. By the proof of [24, Theorem 5.5] we have that $\mathsf{P}(u_n, z_{0n}) \to \mathsf{P}(u, z_0)$ uniformly on $[0,T]$, because $\|u_n - u\|_\infty \to 0$ as $n \to \infty$. Therefore from formula (4.16) it follows that

$$w_n \to w \qquad \text{uniformly on } [0,T]. \tag{4.22}$$

Let us observe that $\mathsf{Q}(\widetilde{u}, z_0)$ and $\mathsf{Q}(\widetilde{u}_n, z_0)$ are Lipschitz continuous for every $n \in \mathbb{N}$ and let us define the bounded measurable functions $h : [0,T] \longrightarrow \mathcal{H}$ and $h_n : [0,T] \longrightarrow \mathcal{H}$ by

$$h(t) := (\mathsf{Q}(\widetilde{u}, z_0))'(\ell_u(t)), \quad h_n(t) := (\mathsf{Q}(\widetilde{u}_n, z_{0n}))'(\ell_n(t)), \qquad t \in [0,T]\,, \tag{4.23}$$

where, by Lemma 4.2, formula (4.20), we have that the $\mathscr{L}^1$-representatives of the distributional derivatives of $\mathsf{Q}(\widetilde{u}, z_0)$ and $\mathsf{Q}(\widetilde{u}_n, z_0)$ can be chosen in such a way that

$$\|(\mathsf{Q}(\widetilde{u}, z_0))'(\sigma)\| = \frac{\mathrm{V}(u, [0,T])}{T}, \quad \|(\mathsf{Q}(\widetilde{u}_n, z_0))'(\sigma)\| = \frac{\mathrm{V}(u_n, [0,T])}{T}, \qquad \forall \sigma \in [0,T]\,.$$

Therefore

$$\|h(t)\| = \frac{\mathrm{V}(u, [0,T])}{T}, \quad \|h_n(t)\| = \frac{\mathrm{V}(u_n, [0,T])}{T}, \qquad \forall t \in [0,T]\,,\ \forall n \in \mathbb{N}. \tag{4.24}$$

Since $u_n \to u$ in $BV([0,T]\,;\mathcal{H})$, from the inequality

$$|\,\mathrm{V}(u, [a,b]) - \mathrm{V}(u_n, [a,b])| \le \mathrm{V}(u - u_n, [a,b]), \tag{4.25}$$

holding for $0 \le a \le b \le T$, we infer that $\mathrm{V}(u_n, [0,T]) \to \mathrm{V}(u, [0,T])$ as $n \to \infty$, hence the sequence $\{\mathrm{V}(u_n, [0,T])\}$ is bounded. Therefore from (4.24) we infer that there exists $C > 0$ such that

$$\sup\{\|h_n(t)\|\ :\ t \in [0,T]\} \le C \qquad \forall n \in \mathbb{N} \tag{4.26}$$

and

$$\lim_{n\to\infty} \|h_n(t)\| = \lim_{n\to\infty} \frac{\mathrm{V}(u_n, [0,T])}{T} = \frac{\mathrm{V}(u, [0,T])}{T} = \|h(t)\|, \qquad \forall t \in [0,T]\,. \tag{4.27}$$

It follows that

$$\begin{aligned}
\lim_{n\to\infty} \int_{[0,T]} \|h_n(t)\|^2 \,\mathrm{d}\mathrm{D}\ell(t) &= \lim_{n\to\infty} \int_{[0,T]} \left(\frac{\mathrm{V}(u_n, [0,T])}{T}\right)^2 \mathrm{d}\mathrm{D}\ell(t) \\
&= \int_{[0,T]} \left(\frac{\mathrm{V}(u, [0,T])}{T}\right)^2 \mathrm{d}\mathrm{D}\ell(t) \\
&= \int_{[0,T]} \|h(t)\|^2 \,\mathrm{d}\mathrm{D}\ell(t),
\end{aligned}$$

hence

$$\lim_{n\to\infty} \|h_n\|^2_{L^2(\mathrm{D}\ell;\mathcal{H})} = \|h\|^2_{L^2(\mathrm{D}\ell;\mathcal{H})}. \tag{4.28}$$

Now let us observe that from Lemma 4.2 and formulas (4.19), (4.21) and (4.23), we have that

$$\mathrm{D}w = h\,\mathrm{D}\ell, \qquad \mathrm{D}w_n = h_n\,\mathrm{D}\ell_n. \tag{4.29}$$

Let us also recall that the vector space of (vector) measures $\nu : \mathscr{B}([0,T]) \longrightarrow \mathcal{H}$ can be endowed with the complete norm $\|\nu\| := |\nu|([0,T])$, where $|\nu|$ is the total variation measure of ν. Moreover from the definition of variation, inequality (4.25), and the triangle inequality, we infer that

$$\|\mathrm{D}\ell - \mathrm{D}\ell_n\| = \left|\mathrm{D}(\ell - \ell_n)\right|([0,T]) = \mathrm{V}(\ell - \ell_n, [0,T]) \to 0 \qquad \text{as } n \to \infty. \tag{4.30}$$

From (4.26) it follows that

$$|\mathrm{D}w_n|(B) = \int_B \|h_n(t)\| \,\mathrm{d}\mathrm{D}\ell_n(t) \leq C\,|\mathrm{D}\ell_n|(B), \qquad \forall B \in \mathscr{B}([0,T]), \tag{4.31}$$

therefore, since $\mathrm{D}\ell_n \to \mathrm{D}\ell$ in the space of real measures, we infer that for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|\mathrm{D}\ell|(B) < \delta \implies \sup_{n \in \mathbb{N}} |\mathrm{D}w_n|(B) < \varepsilon$$

for every $B \in \mathscr{B}([0,T])$. This allows us to apply the weak sequential compactness Dunford-Pettis theorem for vector measures (cf. Theorem 5.1 of the Appendix) and we deduce that, at least for a subsequence, $\mathrm{D}w_n$ is weakly convergent to some measure $\nu : \mathscr{B}([0,T]) \longrightarrow \mathcal{H}$. On the other hand, by (4.22), we have that $w_n \to w$ uniformly, therefore invoking Lemma 5.1 of the Appendix we can identify the weak limit ν with $\mathrm{D}w$ and we infer that

$$\mathrm{D}w_n \ \text{ converges weakly to } \ \mathrm{D}w. \tag{4.32}$$

In particular for every bounded Borel function $\varphi : [0,T] \longrightarrow \mathcal{H}$, the functional $\nu \longmapsto \int_{[0,T]} \langle \varphi(t), \mathrm{d}\nu(t) \rangle$ is linear and continuous on the space of measures with bounded variation and

$$\lim_{n\to\infty} \int_{[0,T]} \langle \varphi(t), \mathrm{d}\mathrm{D}w_n(t) \rangle = \int_{[0,T]} \langle \varphi(t), \mathrm{d}\mathrm{D}w(t) \rangle,$$

that is

$$\lim_{n\to\infty} \int_{[0,T]} \langle \varphi(t), h_n(t) \rangle \,\mathrm{d}\mathrm{D}\ell_n(t) = \int_{[0,T]} \langle \varphi(t), h(t) \rangle \,\mathrm{d}\mathrm{D}\ell(t). \tag{4.33}$$

On the other hand, by (4.26) there exists $\eta \in L^2(\mathrm{D}\ell; \mathcal{H})$ such that h_n is weakly convergent to η in $L^2(\mathrm{D}\ell; \mathcal{H})$, therefore if we set $\psi_n(t) := \langle \varphi(t), h_n(t) \rangle$ and $\psi(t) :=$

$\langle\varphi(t),\eta(t)\rangle$ for $t\in[0,T]$, ψ_n is weakly convergent to ψ in $L^2(\mathrm{D}\ell;\mathbb{R})$, and

$$\begin{aligned}
&\left|\int_{[0,T]}\psi_n(t)\,\mathrm{d}\mathrm{D}\ell_n(t)-\int_{[0,T]}\psi(t)\,\mathrm{d}\mathrm{D}\ell(t)\right| \\
&\le\int_{[0,T]}|\psi_n(t)|\,\mathrm{d}\,\left\|\mathrm{D}(\ell_n-\ell)\right\|(t)+\left|\int_{[0,T]}(\psi_n(t)-\psi(t))\,\mathrm{d}\mathrm{D}\ell(t)\right| \\
&\le\|\varphi\|_\infty\|h_n\|_\infty\left\|\mathrm{D}(\ell_n-\ell)\right\|([0,T])+\left|\int_{[0,T]}(\psi_n(t)-\psi(t))\,\mathrm{d}\mathrm{D}\ell(t)\right|\to 0
\end{aligned}\tag{4.34}$$

as $n\to\infty$, because (4.26) and (4.30) hold, and ψ_n is weakly convergent to ψ in $L^2(\mathrm{D}\ell;\mathbb{R})$. Therefore we have found that

$$\lim_{n\to\infty}\int_{[0,T]}\langle\varphi(t),h_n(t)\rangle\,\mathrm{d}\mathrm{D}\ell_n(t)\ =\int_{[0,T]}\langle\varphi(t),\eta(t)\rangle\,\mathrm{d}\mathrm{D}\ell(t),$$

hence, by (4.33),

$$\int_{[0,T]}\langle\varphi(t),\mathrm{d}(h\,\mathrm{D}\ell)(t)\rangle=\int_{[0,T]}\langle\varphi(t),\mathrm{d}(\eta\,\mathrm{D}\ell)(t)\rangle.\tag{4.35}$$

The arbitrariness of φ and (4.35) implies that $\eta\,\mathrm{D}\ell=h\,\mathrm{D}\ell$ (cf. [14, Proposition 35, p. 326]), hence $\eta(t)=h(t)$ for $\mathrm{D}\ell$-a.e. $t\in[0,T]$ and we have found that

$$h_n\rightharpoonup h\qquad\text{in } L^2(\mathrm{D}\ell;\mathcal{H}).\tag{4.36}$$

Since $L^2(\mathrm{D}\ell;\mathcal{H})$ is a Hilbert space, from (4.28) and (4.36) we deduce that

$$h_n\to h\qquad\text{in } L^2(\mathrm{D}\ell;\mathcal{H}),\tag{4.37}$$

and, since $\mathrm{D}\ell([0,T])$ is finite,

$$h_n\to h\qquad\text{in } L^1(\mathrm{D}\ell;\mathcal{H}).\tag{4.38}$$

Hence, at least for a subsequence which we do not relabel, $h_n(t)\to h(t)$ for $\mathrm{D}\ell$-a.e. $t\in[0,T]$, thus

$$\begin{aligned}
V(w_n-w,[0,T])&=\|\,\mathrm{D}(w_n-w)\|=\|\,\mathrm{D}w_n-\mathrm{D}w\|=\|h_n\,\mathrm{D}\ell_n-h\,\mathrm{D}\ell\| \\
&\le\|h_n\,\mathrm{D}(\ell_n-\ell)\|+\|(h_n-h)\,\mathrm{D}\ell\| \\
&\le C\|\,\mathrm{D}(\ell_n-\ell)\|+\int_{[0,T]}\|h_n(t)-h(t)\|\,\mathrm{d}\mathrm{D}\ell(t)\to 0
\end{aligned}$$

as $n\to\infty$, which proves that $\|w-w_n\|_{BV}\to 0$ as $n\to\infty$. We can conclude recalling (4.21) and that $\mathsf{Q}(v)=2\mathsf{P}(v)-v$ for every $v\in C([0,T]\,;\mathcal{H})\cap BV([0,T]\,;\mathcal{H})$. $\square$

We can finally infer the strict continuity of the play operator on $C([0,T]\,;\mathcal{H}) \cap BV([0,T]\,;\mathcal{H})$.

Proof of Theorem 3.3. The proof of Theorem 3.3 is now a consequence of Theorem 3.2 and [32, Theorem 3.4]. $\square$

5. Appendix

In this appendix we collect some results on vector measures which are needed in some proofs of the paper. As we pointed out in Section 2.3, if $I \subseteq \mathbb{R}$, then the vector space of $\mathcal{H}$-valued measures $\nu : \mathscr{B}(I) \longrightarrow \mathcal{H}$ with bounded variation is a real Banach space when endowed with the norm $\|\nu\| := |\nu|\,(I)$. Therefore we can define on it the notion of weak convergence.

Definition 5.1. *Assume that* (2.1) *hold and that* $I \subseteq \mathbb{R}$ *is an interval. Let* $\mathfrak{M}(I;\mathcal{H})$ *denote the real Banach space of* $\mathcal{H}$*-valued measures on* $\mathscr{B}(I)$ *having bounded variation according to Section 2.3. If* $\nu, \nu_n \in \mathfrak{M}(I;\mathcal{H})$ *for every* $n \in \mathbb{N}$*, then we say that* ν_n is weakly convergent to ν *if* $\lim_{n\to\infty}\langle T, \nu_n\rangle = \langle T, \nu\rangle$ *for every linear continuous function* T *belonging to the topological dual space of* $\mathfrak{M}(I;\mathcal{H})$*.*

For the reader's convenience we restate the Dunford-Pettis weak compactness theorem for measures [13, Theorem 5, p. 105] in a form which is suitable to our purposes.

Theorem 5.1. *Assume that* (2.1) *hold and that* $I \subseteq \mathbb{R}$ *is an interval and let* $\mathfrak{B}$ *be a bounded subset of* $\mathfrak{M}(I;\mathcal{H})$*. Then* $\mathfrak{B}$ *is weakly sequentially precompact if and only if there exists a bounded positive measure* $\nu : \mathscr{B}(I) \longrightarrow [0,\infty[$ *such that for every* $\varepsilon > 0$ *there is a* $\delta > 0$ *which satisfies the implication*

$$\forall \varepsilon > 0\ \exists \delta > 0 \quad : \quad \left(B \in \mathscr{B}(I),\ \nu(B) < \delta \implies \sup_{\mu \in \mathfrak{B}} |\mu|\,(B) < \varepsilon \right). \tag{5.1}$$

Theorem 5.1 is stated in [13, Theorem 5, p. 105] as a topological precompactness result. An inspection of the proof easily shows that this is actually a sequential precompatness theorem, since an isometric isomorphism reduces it to the well-known Dunford-Pettis weak sequential precompactness theorem in $L^1(\nu;\mathcal{H})$ (see, e.g., [13, Theorem 1, p. 101]).
The following lemma is a vector measure counterpart of a well-known weak derivative argument and is proved in [19, Lemma 7.1].

Lemma 5.1. *Assume that* (2.1) *hold and that* $I \subseteq \mathbb{R}$ *is an interval. Let* $w, w_n \in BV(I;\mathcal{H})$ *for every* $n \in \mathbb{N}$ *and let* $\nu : \mathscr{B}(I) \longrightarrow \mathcal{H}$ *be a measure with bounded variation. If* $w_n \to w$ *uniformly on* I *and* $\mathrm{D}w_n \rightharpoonup \nu$*, then* $\mathrm{D}w = \mu$*.*

Now are going to state the theorem concerning Lebesgue point of vector value functions with respect to a Borel measure.

Theorem 5.2. *Assume that* (2.1) *hold, that* $I \subseteq \mathbb{R}$ *is an interval, and let* $\mu : \mathscr{B}(I) \longrightarrow [0, \infty[$ *be a finite Borel measure on* I*. If* $f \in L^1(\mu; \mathcal{H})$ *then there exists* $L \in \mathscr{B}(I)$ *such that* $\mu(I \smallsetminus L) = 0$, $\mu([t-h, t+h] \cap I) > 0$ *for every* $h > 0$, *and*

$$\lim_{h \searrow 0} \frac{1}{\mu([t-h,t+h] \cap I)} \int_{[t-h,t+h]\cap I} \|f(\tau) - f(t)\| \, \mathrm{d}\mu(\tau) = 0 \qquad \forall t \in L. \tag{5.2}$$

The points t *satisfying* (5.2) *are called* μ-Lebesgue points of f.

A proof of this theorem can be found in [17] in a much more general framework. In order to help the reader we show how derive it. The family $\mathcal{V} := \{(t, [t-h, t+h] \cap I) \ : \ t \in I, h > 0\}$ satisfies the definition of *Vitali relation* given in [17, Section 2.8.16, p. 151]. In [17, Section 2.9.1, p. 153] the left hand side of (5.2) is called $\mathcal{V}$*-derivative* of $\psi : \tau \longmapsto \|f(\tau) - f(t)\|$ with respect to μ at t. Since there exists a μ-zero measure set Z such that $f([0,T] \smallsetminus Z)$ is separable (see, e.g., [26, Property M11, p. 124]), we can repeat "mutatis-mutandi" the proof of [17, Corollary 2.9.9., p. 156] (where it is formally assumed that $\mathcal{H}$ is separable), and we infer the result of Theorem 5.2. We conclude with the following result which is proved e.g. in [26, Section VII, Theorem 4.1].

Theorem 5.3. *Assume that* (2.1) *hold, that* $I \subseteq \mathbb{R}$ *is an interval and let* $\nu : \mathscr{B}(I) \longrightarrow \mathcal{H}$ *is a Borel measure with bounded variation. Then there exists* $g \in L^1(\|\nu\|; \mathcal{H})$ *such that* $\|g(t)\| = 1$ *for* μ*-a.e.* $t \in [0,T]$, *and* $\nu = g \|\nu\|$ *(cf.* (2.4)*).*

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