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An Orbifold Geometry in Holography,

or:

How the spindle fits into supergravity and supersymmetric quantum field theory

By

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I would like to dedicate this thesis to my wife

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Abstract

In this thesis, I summarize my work on supergravity and supersymmetric quantum field theories in the context of holography [1–3]. From the gravity side of the correspondence, I present a solution of $D=4$, $\mathcal{N}=4$ gauged supergravity, firstly found in [1], conjectured to be a near-horizon solution of an extremal dyonically charged, rotating, and accelerating supersymmetric black hole in AdS4. The solution is distinguished by the presence of an orbifold geometry: the shape of the event horizon is $\Sigma = \mathbb{WCP}^1[n_+, n_-]$. Following the steps of [4], I provide an exhaustive thermodynamic analysis of the solution, motivating the setup of the dual supersymmetric field theory. I then consider a three-dimensional $\mathcal{N}=2$ supersymmetric field theory defined on a general complex-valued background, capable of accommodating the spindle. Finally, I present the novel "spindle index," introduced in [2], and its derivation by a localization computation involving an application of the equivariant orbifold index theorem [3].

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Chapter 1

Introductions

1.1 Holographic Principle and AdS/CFT Correspondence

The holographic principle posits that the physical information contained within a volume is in perfect equivalence with the information that can be extracted from the volume's boundary. In the realm of quantum gravity, more specifically in the context of string theory, this concept was first introduced in the early 1990s by 't Hooft, Thorn, and Susskind [5–7]. Starting with Maldacena's groundbreaking work in 1997 [8], the scientific community was able to concretely realize the holographic principle within the frameworks of string theory, M-theory, and supergravity theories. In Maldacena's work, he presented his conjecture, now widely recognized as the AdS/CFT correspondence (Anti-de Sitter and Conformal Field Theory). This conjecture asserts the equivalence of two seemingly unrelated theories. In its initial application, these theories were the type IIB string theory on an AdS_5 x S^5 background and the $\mathscr{N}=4$ super Yang-Mills in $d=4$ on a flat background. An in-depth understanding of this duality relies on string theories, given their inherent association with "branes," extended objects of intricate nature. These branes act as intermediaries between the two theories, with their world-volume defining the background spacetime of the dual field theory and the configurations of string on them determining the dual fields.

This thesis primarily centers on supergravity theories in $D=4$, representing consistent truncations of $D=11$ supergravity. This supergravity, which first appeared in the 1978 paper by Cremmer et al. [9], is now often presented as a classical, low-energy limit of M-theory, implying that solutions in $D = 4$ have direct counterparts as solutions involving M2-branes within the framework of M-theory. Consequently, the associated dual field theories manifest in three dimensions. For a more comprehensive understanding of the process of uplifting four-dimensional gravity solutions, we refer to Appendix B.

Within the realm of theoretical physics, the holographic principle holds a multitude of potential applications. However, I will specifically focus on its relevance in the field of black hole physics. Theoretical research in black hole physics is of paramount importance because it provides a unique avenue for investigating the quantum nature of gravity and potentially arriving at a coherent theory of quantum gravity that aligns with our existing field theories. In $D = 4$, a productive approach, initially introduced in references [10, 11], has been developed to replicate the Bekenstein-Hawking entropy of supersymmetric asymptotically locally AdS_4 black holes. This approach involves a meticulous analysis of specific supersymmetric statistical ensembles within the corresponding dual $d = 3$ SCFT, which can be precisely computed using localization techniques. This achievement stands as one of the most significant results in the recent theoretical research on black hole physics.

This thesis is inserted in an ongoing and prolific research endeavor that began with the 2020 publication [12], where the central focus is the complex weighted projective space $\mathbb{WCP}^1_{[n_N,n_S]}$, commonly known as the spindle. In these four years, many works related to spindles have been published, bearing witness to the vitality of this research topic. Here, we provide a partial summary [1–3, 12–27]. In this thesis the spindle will embody both the structure of a black hole's horizon in supergravity and the curved geometric background where the dual SQFT is defined. A crucial characteristic of the spindle is its role as an extension of the sphere. In essence, through a carefully controlled limit, the spindle can be reduced to the sphere. Consequently, every paper involving the spindle can be seen as a generalization of prior works involving spheres. This broader perspective enables researchers to unveil essential general features that may have eluded previous examination due to the inherent limitations of spheres. The second, and undeniably more striking, aspect of the spindle geometry lies in its capacity to accommodate supersymmetry in a more intricate manner when compared to spheres, which is called anti-twist. Moreover there exist instances of spindle solutions, as evidenced in [15], where supersymmetry can manifest through both the conventional topological twist and the novel anti-twist. This anti-twist can be regarded as an extension of the concept of a no-twist, which is typically associated with spheres. This intriguing possibility has led to the emergence of a novel index, elegantly unifying both the topological twisted index and the superconformal index. This unification is exemplified in [2] and is comprehensively elucidated in chapter 4.

1.2 Supergravity in Four Dimensions

AdS/CFT correspondence has proven to be a powerful tool in recent years for gaining insights into the microscopic degrees of freedom of various classes of black holes. In the context of $D=4$ dimensions, two main classes of black holes have been discussed using this approach. One class consists of static black holes, for which the dual field theory is typically defined on $S^1 \times \Sigma_g$, where Σ_g is a genus g Riemann surface equipped with a constant curvature metric, and in order to preserve supersymmetry one performs the so-called topological twist [28]. A second class consists of rotating Kerr-Newman-AdS black holes with spherical horizons, for which the dual field theory is defined on a "spinning" $S^1 \times S^2$ [29].

In [13] a different class of asymptotically locally AdS_4 black holes has been considered in the context of holography. These are a family of solutions to Einstein-Maxwell theory with a cosmological constant, or equivalently minimal $D=4$, $\mathcal{N}=2$ gauged supergravity, originally constructed by Plebański and Demiański [30, 31]. The Plebański-Demiański solutions describe the most general dyonic, rotating and accelerating black holes in minimal gauged supergravity, and have a number of striking features. The term "accelerating" refers to the fact that the black hole curvature singularity can be shown to have a uniform proper acceleration, and more generally in a natural frame any world-line with constant space-like coordinates also has this property – see, *e.g.* section III of [31]. It is well-known that the acceleration is associated with conical deficit angles, which may be interpreted as being sourced by strings in the black hole geometry [32][1]. The conical deficits manifest themselves as orbifold singularities on the horizon, which becomes a spindle $\mathbb{\Sigma}$. An important property of these Plebański-Demiański solutions is that, in the context of minimal gauged supergravity, they admit a supersymmetric and extremal sub-family of dyonic

[1] It is also well-known that such accelerating black holes emit gravitational radiation [33], but the resulting energy loss is balanced by the force exerted by the strings, which keeps the acceleration constant.

accelerating and rotating black holes [34], whose near horizon geometry is a spinning $\mathrm{AdS}_2 \times \Sigma$ solution [13].

In chapter 2 of this thesis, we will discuss extensions of the above solutions to non-minimal supergravities, focusing on an $\mathcal{N} = 4$ supergravity model that arises as a consistent truncation of $D = 11$ supergravity. Alternatively, this can be regarded as minimal $\mathcal{N} = 2$ gauged supergravity coupled to one vector multiplet with a particular prepotential. As such, solutions of this model can be uplifted to solutions of $D = 11$ supergravity, and therefore interpreted holographically as dual to $\mathcal{N} = 4$ [35] (or $\mathcal{N} = 2$), $d = 3$ SCFTs arising on M2-branes.

Several solutions to this supergravity theory are known in the literature, and we have summarized those relevant for holography in the diagram in Figure 1.2 below. In particular, there exist two notable classes of solutions. A solution

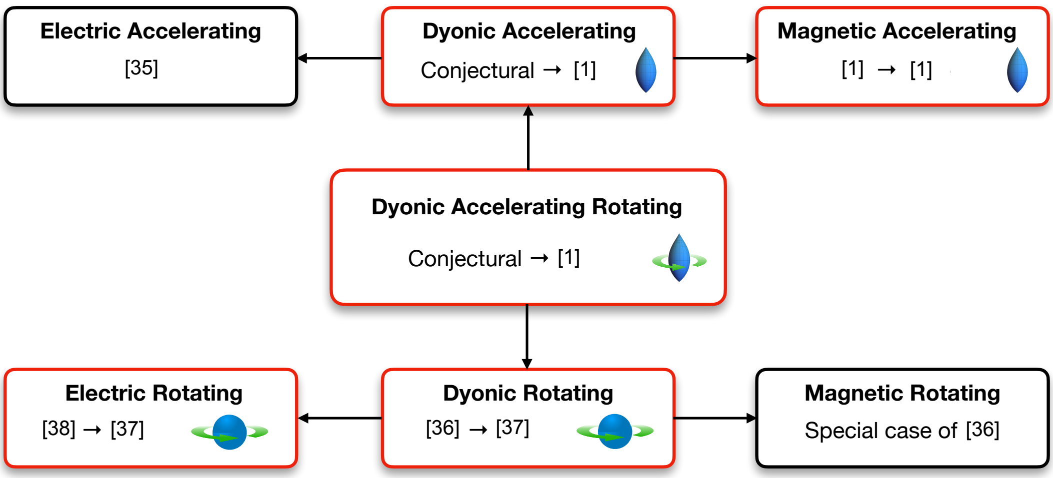

Fig. 1.1 Summary of AdS_4 black holes with either spherical or spindle horizons in $D = 4$, $\mathcal{N} = 4$ gauged supergravity. The solutions in the red frames admit a supersymmetric and extremal limit and their near horizon $\mathrm{AdS}_2 \times \Sigma$ geometries are represented pictorially. From bottom-left to top-right: a spinning sphere, a spinning spindle, and a non-spinning spindle. In all cases the reference on the left refers to the non-extremal black holes, and that on the right refers to the near horizon solution in the supersymmetric limit.

describing electrically charged, non-rotating, accelerating black holes was presented in [36]. This is a multi-charge generalization of the AdS C-metric in the Einstein-Maxwell theory, which is a member of the solutions in [30], and does not admit a supersymmetric limit. Reference [37] constructed a dyonic, rotating, but non-accelerating, family of black hole solutions. Imposing supersymmetry on this family leads to a dyonic rotating solution, recovered in [38], that also discusses the $\mathrm{AdS}_2 \times S^2$ near horizon solution. In turn, switching off the magnetic charge, the solution reduces to a multi-electric charge version of the Kerr-Newman black hole of Maxwell-Einstein theory, that was previously discovered in [39]. The supersymmetric limit of this was discussed in [40], and is the multi-electric charge counterpart of the extremal Kerr-Newman black hole of Maxwell-Einstein theory [41]. Based on these and on the family of black hole solutions of the minimal theory [30, 31], my colleagues and I have conjectured in [1] that there should exist a family of multi-charge, dyonic, accelerating and rotating AdS_4 black holes, from which all the other solutions should arise as special cases. Unfortunately, the construction of this general family of black holes has remained a challenge. Nevertheless, we have constructed a family of supersymmetric multi-charge spinning spindle solutions, namely rotating $\mathrm{AdS}_2 \times \mathbb{\Sigma}$ solutions. We expect this family to arise as the near horizon limit of the corresponding family of supersymmetric and extremal black holes conjectured above. Interestingly, the Bekenstein-Hawking entropy of the black holes as a function of the physical charges, as well as the spindle deficit angles, can be obtained purely from the near horizon solutions.

In AdS/CFT one identifies the holographically renormalized on-shell action in gravity with minus the logarithm of the dual field theory partition function in a grand canonical ensemble. The latter is a function of the associated chemical potentials, and the two ensembles are related by a Legendre transform. While the black hole entropy can be computed from the near horizon solution, in order to compute the on-shell action we in principle need the full (non-extremal) black hole solutions. Nevertheless, in [1], we have presented a conjectural formula for the euclidean on-shell action associated to the full black hole. In the special case where the conjectural black hole reduces to the solution of the minimal theory, in [4] it has been shown that the entropy can be derived from a Legendre transform of the Euclidean on-shell action, similarly to [42, 43], thus setting the stage for a direct analysis of the dual $d = 3$ SCFT, defined on $S^1 \times \mathbb{\Sigma}$.

1.3 Supersymmetric Field Theory in Three Dimensions

In order to assess the validity of the AdS/CFT correspondence, it becomes imperative to calculate quantities within supersymmetric quantum field theories (SQFTs) defined on curved manifolds. Starting from the foundational work presented in [44], a systematic framework has emerged for delineating the necessary conditions that background fields, including the metric, must meet to enable the realization of a supersymmetric theory on the given background. The background fields that allow the realization of supersymmetry are identified with the rigid limit of the gravity multiplet fields of an off-shell formulation of supergravity.

In the context of [44], the focus is directed toward four-dimensional theories. Similar insights have been extended to three-dimensional theories, as elaborated in [45], where the off-shell realization of supergravity pertains to new minimal supergravity. It is in the framework of the rigid limit of new minimal supergravity in $d = 3$ that, in our paper [2], we introduced a comprehensive category of rigid supersymmetric backgrounds conducive to accommodating the spindle. This particular background preserves two distinct Killing spinors $(\zeta, \widetilde{\zeta})$ with R-charges ± 1, respectively and it represents the most general background invariant under two real Killing vectors. Furthermore, it exhibits a metric characterized by complex values in order to accomodate the intrinsically complex geometry of [4], which is discussed in Chapter 3.

Once we have an SQFT on a curved background, the computation of quantities becomes feasible through localization techniques. The roots of these techniques can be traced back to [46], but their modern wide application gained momentum after [47], where the author successfully calculated the partition function of different theories on an S^4 background. Localization techniques can be viewed as a stationary phase approximation of the path integral of the theory. Thanks to supersymmetry, it turns out that this approximation yields an exact result. In a supersymmetric theory, there exists a fermionic symmetry generated by a supersymmetric charge $\mathscr{Q}$. This $\mathscr{Q}$ can either be a nilpotent operator or can square to a bosonic symmetry. In both cases, it is possible to deform the action of the theory with a $\mathscr{Q}$-exact term without changing the path integral: quantum physical states of the theory reside on the cohomology (or equivariant cohomology) group generated by the operator $\mathscr{Q}$.

Introducing a deformation parameter $t \in \mathbb{R}$, we express the partition function as

$$Z(t) = \int e^{-S+t\mathscr{Q}X} \,. \tag{1.3.1}$$

Taking the limit $t \to \infty$ and making a judicious choice of the deformation action X, the path integral localizes to the saddle points:

$$Z(t=\infty) = \sum_{\text{saddle points}} e^{-S_{\text{classical}}} \frac{\det \text{fermions}}{\det \text{bosons}} \,. \tag{1.3.2}$$

At this point, the computation involves two contributions: one from the values of the field on the saddle points, termed the "classical contribution," and a second from small oscillations around these stable configurations, termed the "one-loop determinant."

In the contest of three-dimensional SQFTs, the supersymmetric localization was employed to compute the topologically twisted index in [28] and the superconformal index [48, 49]. The topologically twisted index is defined as the partition function of a supersymmetric theory defined on a background $\Sigma \times S^1$ where a topological twist is performed on Σ, otherwise if there is no flux through Σ we refer to the partition function as superconformal index. As discussed in the previous section, the large-N limit of these quantities provides a microscopic interpretation of the entropy of magnetically charged supersymmetric black holes in AdS_4 with an event horizon shaped like Σ [10].

In [2], it was demonstrated that performing the new anti-twist through a spindle $\mathbb{\Sigma}$ leads to an associated index, generalizing the superconformal index. Moreover, the unification of this index with the topologically twisted index resulted in a comprehensive newly defined index termed "the spindle index." This nomenclature emphasizes the fundamental role of the orbifold nature of the spindle in achieving this result. Moreover, in [3], it was shown that the same procedure can be applied to other orbifolds. In Chapter 4 of this thesis, we provide specific details regarding the new background capable of accommodating the spindle, as well as the localization computation that gives rise to the spindle index.

Chapter 2

Supergravity Side/Spindle Black Holes

2.1 The supergravity model

In this chapter, we present the supersymmetric $\mathrm{AdS}_2 \times \Sigma$ solutions of $D=4$, $\mathcal{N}=4$ gauged supergravity, firstly found in [1]. This theory can also be described, in the language of $D=4$, $\mathcal{N}=2$ supergravity [50], as a theory with no hypermultiplets and one vector multiplet, with prepotential $F=-\mathrm{i}X^0X^1$ and electric Fayet-Iliopoulos gauging. Yet another viewpoint is that it is a truncation of the STU model [51], where the four Abelian gauge fields are set pairwise equal and two of the complex scalars are identified. Introducing the axio-dilaton

$$z=\frac{X^1}{X^0}=\mathrm{e}^{-\xi}+\mathrm{i}\,\chi\,, \tag{2.1.1}$$

we can write the bosonic action of the theory as

$$\begin{aligned}S=\frac{1}{16\pi G_{(4)}}\int\Big[&(R-g^2\mathcal{V})\star 1-\frac{1}{2}\mathrm{d}\xi\wedge\star\mathrm{d}\xi-\frac{1}{2}\mathrm{e}^{2\xi}\,\mathrm{d}\chi\wedge\star\mathrm{d}\chi-\mathrm{e}^{-\xi}F_2\wedge\star F_2\\&+\chi\,F_2\wedge F_2-\frac{1}{1+\chi^2\mathrm{e}^{2\xi}}\left(\mathrm{e}^{\xi}F_1\wedge\star F_1+\chi\,\mathrm{e}^{2\xi}F_1\wedge F_1\right)\Big]\,,\end{aligned} \tag{2.1.2}$$

where $F_i = \mathrm{d}A_i$, $i = 1,2$, and the scalar potential $\mathscr{V}$ is given by

$$\mathscr{V} = -\left(4 + 2\cosh\xi + \mathrm{e}^{\xi}\chi^2\right). \tag{2.1.3}$$

We shall henceforth set $g = 1$, so that in the AdS$_4$ vacuum of the theory there is an effective cosmological constant $\Lambda = -3$. We also remark that minimal $D = 4$, $\mathscr{N} = 2$ gauged supergravity is obtained via the consistent truncation $\xi = 0 = \chi$, $A_1 = A_2 = A$.

Although we shall not consider the fermionic completion of the action (2.1.2), it will be important to consider the supersymmetry variations of the gravitini and gaugini of this theory, which must vanish for bosonic backgrounds that preserve some amount of supersymmetry. While it is customary to formulate $D = 4$, $\mathscr{N} = 2$ supergravity in terms of Weyl fermions, we follow [52] and combine them into complex Dirac fermions: a gravitino ψ_μ, a dilatino λ and a supersymmetry parameter ε. In terms of these, the Killing spinor equations (KSEs) can be written as

$$\begin{aligned}
\delta\psi_\mu &= \left[\nabla_\mu - \frac{\mathrm{i}}{2}(A_1 + A_2)_\mu + \frac{\mathrm{i}}{4}\mathrm{e}^{\xi}\,\partial_\mu\chi\,\gamma_5 + \frac{1}{4}\left(\mathrm{e}^{\xi/2} + \mathrm{e}^{-\xi/2}\right)\gamma_\mu + \frac{\mathrm{i}}{4}\chi\,\mathrm{e}^{\xi/2}\,\gamma_\mu\,\gamma_5\right. \\
&\quad \left. + \frac{\mathrm{i}}{8}\left(\frac{\mathrm{e}^{\xi/2}}{1+\chi^2\mathrm{e}^{2\xi}}\not{F}_1 + \mathrm{e}^{-\xi/2}\not{F}_2\right)\gamma_\mu - \frac{1}{8}\frac{\chi\,\mathrm{e}^{3\xi/2}}{1+\chi^2\mathrm{e}^{2\xi}}\not{F}_1\,\gamma_\mu\,\gamma_5\right]\varepsilon = 0\,, \\
\delta\lambda &= \left[\mathrm{i}\,\mathrm{e}^{-\xi}\,\not{\partial}\xi - \frac{\mathrm{e}^{-\xi}}{2}\left(\frac{\mathrm{e}^{\xi/2}}{1+\chi^2\mathrm{e}^{2\xi}}\not{F}_1 - \mathrm{e}^{-\xi/2}\not{F}_2\right) - \mathrm{i}\,\mathrm{e}^{-\xi}\left(\mathrm{e}^{\xi/2} - \mathrm{e}^{-\xi/2}\right)\right. \\
&\quad \left. + \left(\not{\partial}\chi + \frac{\mathrm{i}}{2}\frac{\chi\,\mathrm{e}^{\xi/2}}{1+\chi^2\mathrm{e}^{2\xi}}\not{F}_1 + \chi\,\mathrm{e}^{-\xi/2}\right)\gamma_5\right]\varepsilon = 0\,.
\end{aligned} \tag{2.1.4}$$

Being a truncation of the maximal $D = 4$, $\mathscr{N} = 8$ gauged supergravity, all supersymmetric solutions of this theory can be uplifted on S^7 to supersymmetric solutions of $D = 11$ supergravity. The details of the uplift for this specific truncation can be found in [53]. We shall discuss uplifting of the metric in appendix B, where global regularity of the $D = 11$ solutions will require quantization of the magnetic charges of the $D = 4$ solutions.

2.2 Local AdS_2 solutions

In this section we present a class of rotating, dyonically charged $\mathrm{AdS}_2 \times \Sigma$ solutions of the $D = 4$, $\mathcal{N} = 4$ supergravity model introduced in the previous section. We conjecture these solutions to arise as the near horizon limit of accelerating, rotating and dyonic black holes, that are also extremal and supersymmetric.

The local form of the solutions is given by

$$\begin{aligned} \mathrm{d}s_4^2 &= \frac{1}{4}\lambda(y)\left(-\rho^2\,\mathrm{d}\tau^2 + \frac{\mathrm{d}\rho^2}{\rho^2}\right) + \frac{\lambda(y)}{q(y)}\,\mathrm{d}y^2 + \frac{q(y)}{4\,\lambda(y)}\,(\mathrm{d}z + \mathtt{j}\,\rho\,\mathrm{d}\tau)^2\,, \\ A_i &= \frac{h_i(y)}{\lambda(y)}\,(\mathrm{d}z + \mathtt{j}\,\rho\,\mathrm{d}\tau)\,, \quad \mathrm{e}^{\xi} = \frac{g_1(y)}{\lambda(y)}\,, \quad \chi = \frac{g_2(y)}{g_1(y)}\,, \end{aligned} \tag{2.2.1}$$

where all the functions that we introduced are polynomials in y, given by

$$\begin{aligned} \lambda(y) &= y^2 + \mathtt{j}^2 - 2c_2\,, \\ q(y) &= (y^2+\mathtt{j}^2)^2 - 4(1-\mathtt{j}^2+c_2)\,y^2 + 4c_1\sqrt{1-\mathtt{j}^2}\,y - c_1^2 + 4c_2\,(c_2-\mathtt{j}^2)\,, \\ h_1(y) &= \frac{\sqrt{1-\mathtt{j}^2}}{2}\,(1-c_3)\,\lambda(y) - \frac{1}{2}\left(c_1 + 2\sqrt{1-\mathtt{j}^2}\,\sqrt{2c_2-c_3^2\,\mathtt{j}^2}\right)y \\ &\quad + (2c_2-\mathtt{j}^2)\sqrt{1-\mathtt{j}^2} + \frac{1}{2}c_1\sqrt{2c_2-c_3^2\,\mathtt{j}^2}\,, \\ h_2(y) &= \frac{\sqrt{1-\mathtt{j}^2}}{2}\,(1+c_3)\,\lambda(y) - \frac{1}{2}\left(c_1 - 2\sqrt{1-\mathtt{j}^2}\,\sqrt{2c_2-c_3^2\,\mathtt{j}^2}\right)y \\ &\quad + (2c_2-\mathtt{j}^2)\sqrt{1-\mathtt{j}^2} - \frac{1}{2}c_1\sqrt{2c_2-c_3^2\,\mathtt{j}^2}\,, \\ g_1(y) &= y^2 + 2\sqrt{2c_2-c_3^2\,\mathtt{j}^2}\,y + 2c_2 + (1-2c_3)\,\mathtt{j}^2\,, \\ g_2(y) &= 2c_3\,\mathtt{j}\,y + 2\,\mathtt{j}\sqrt{2c_2-c_3^2\,\mathtt{j}^2}\,. \end{aligned} \tag{2.2.2}$$

Note that the solution depends on the four parameters $\mathtt{j}$, c_i ($i = 1,2,3$), where $\mathtt{j}$ has the interpretation of a rotation parameter. We can interpret the number of independent parameters in terms of our conjecture that this arises as the near horizon limit of a supersymmetric and extremal accelerating black hole. One can imagine a full black hole metric with seven parameters, representing mass, acceleration, angular momentum and two pairs of dyonic charges. We would then expect two constraints on the parameters to come from the supersymmetry conditions, and one from the requirement of extremality, resulting in a four-parameter solution, as for that described above. One can then think of the four parameters as representing the two pairs of dyonic charges, with mass, acceleration and angular momentum related to them by supersymmetry and extremality.

2.3 Killing spinors

Let us now justify our claim that the solution (2.2.1) is supersymmetric, by showing explicitly the associated Killing spinors.

We choose the orthonormal frame

$$\begin{aligned} e^0 &= \frac{1}{2}\sqrt{\lambda(y)}\,\rho\,\mathrm{d}\tau\,, \qquad & e^1 &= \frac{1}{2}\sqrt{\lambda(y)}\,\frac{\mathrm{d}\rho}{\rho}\,, \\ e^2 &= \sqrt{\frac{\lambda(y)}{q(y)}}\,\mathrm{d}y\,, \qquad & e^3 &= \sqrt{\frac{q(y)}{4\lambda(y)}}\,(\mathrm{d}z + \mathrm{j}\,\rho\,\mathrm{d}\tau)\,. \end{aligned} \tag{2.3.1}$$

The four-dimensional gamma matrices are then taken to be

$$\gamma_a = \beta_a \otimes 1_2\,, \qquad a = 0,1 \tag{2.3.2}$$

$$\gamma_2 = \beta_3 \otimes \sigma^1\,, \qquad \gamma_3 = \beta_3 \otimes \sigma^2\,, \tag{2.3.3}$$

with the two-dimensional gamma matrices β_a are defined by

$$\beta_0 = \mathrm{i}\,\sigma^2\,, \qquad \beta_1 = \sigma^1\,, \qquad \beta_3 \equiv \beta_0\beta_1 = \sigma^3\,, \tag{2.3.4}$$

where σ^i are the Pauli matrices.

We consider the following Killing spinor equation (KSE) for AdS_2:

$$\nabla_a \theta = \frac{\mathrm{i}}{2} n\,\beta_a \beta_3\,\theta\,, \tag{2.3.5}$$

with $n = \pm 1$. This is solved by Majorana spinors that can be decomposed as $\theta_{1,2} = \theta_{1,2}^{(+)} + \theta_{1,2}^{(-)}$, with the Majorana-Weyl spinors $\theta_{1,2}^{(\pm)}$ of chirality $\beta_3\,\theta_{1,2}^{(\pm)} = \pm\theta_{1,2}^{(\pm)}$, given by

$$\theta_1^{(+)} = \begin{pmatrix} \sqrt{\rho} \\ 0 \end{pmatrix}, \qquad \theta_1^{(-)} = \begin{pmatrix} 0 \\ \mathrm{i}\, n\sqrt{\rho} \end{pmatrix}, \tag{2.3.6}$$

$$\theta_2^{(+)} = \begin{pmatrix} \sqrt{\rho}\,\tau - \frac{1}{\sqrt{\rho}} \\ 0 \end{pmatrix}, \quad \theta_2^{(-)} = \begin{pmatrix} 0 \\ \mathrm{i}\, n\left(\sqrt{\rho}\,\tau + \frac{1}{\sqrt{\rho}}\right) \end{pmatrix}. \tag{2.3.7}$$

We are finally ready to discuss the explicit Killing spinors, which solve both equations in (2.1.4), and can be written as

$$\varepsilon_1 = \theta_1^{(+)} \otimes \eta_1 + \theta_1^{(-)} \otimes \eta_2\,, \tag{2.3.8}$$

$$\varepsilon_2 = \theta_2^{(+)} \otimes \eta_1 + \theta_2^{(-)} \otimes \eta_2\,, \tag{2.3.9}$$

where $\eta_{1,2}$ are two two-dimensional spinors, given by

$$\begin{aligned} \eta_1 &= \begin{pmatrix} \mathrm{e}^{-\frac{\mathrm{i}}{2}\arctan\left(\frac{g_1'(y)}{2\mathrm{j}\,(1-c_3)}\right)} \frac{q_+(y)^{1/2}}{\lambda(y)^{1/4}} \\ \mathrm{i}\,\mathrm{e}^{\frac{\mathrm{i}}{2}\arctan\left(\frac{g_1'(y)}{2\mathrm{j}\,(1-c_3)}\right)} \frac{q_-(y)^{1/2}}{\lambda(y)^{1/4}} \end{pmatrix}, \\ \eta_2 &= n\,\mathrm{e}^{\mathrm{i}\arccos \mathrm{j}} \begin{pmatrix} \mathrm{i}\,\mathrm{e}^{\frac{\mathrm{i}}{2}\arctan\left(\frac{g_1'(y)}{2\mathrm{j}\,(1-c_3)}\right)} \frac{q_+(y)^{1/2}}{\lambda(y)^{1/4}} \\ -\mathrm{e}^{-\frac{\mathrm{i}}{2}\arctan\left(\frac{g_1'(y)}{2\mathrm{j}\,(1-c_3)}\right)} \frac{q_-(y)^{1/2}}{\lambda(y)^{1/4}} \end{pmatrix}. \end{aligned} \tag{2.3.10}$$

We have also defined

$$q_\pm(y) \equiv \lambda(y) \pm \left(c_1 - 2\sqrt{1-\mathrm{j}^2}\,y\right), \tag{2.3.11}$$

which satisfy $q(y) = q_+(y)\,q_-(y)$.

Finally, we conclude with some comments about the counting of supercharges. As we have just discussed, we have a solution to $D=4$ supergravity which admits two independent Dirac Killing spinors, given by (2.3). This is equivalent to four Majorana, or four Weyl spinors, hence the solution can be described as $\frac{1}{2}$−BPS from the point of view of $D=4$, $\mathcal{N}=2$ supergravity, or $\frac{1}{4}$−BPS from the point of view of $D=4$, $\mathcal{N}=4$ supergravity. In the dual $d=1$ superconformal quantum mechanics

(SCQM), the complex spinor ε_1 gives two real Poincaré supercharges, while ε_2 gives two real conformal supercharges. Thus, the SCQM has $\mathscr{N}=2$ supersymmetry in one dimension, since in the field theory counting one usually includes only Poincaré supercharges, with superalgebra $\mathfrak{su}(1,1|1)$.

2.4 Global analysis

We would now like to determine conditions on the parameters $\mathtt{j}$ and c_i ($i=1,2,3$) such that the two-dimensional metric

$$\mathrm{d}s^2_{\Sigma} = \frac{\lambda(y)}{q(y)}\,\mathrm{d}y^2 + \frac{q(y)}{4\lambda(y)}\,\mathrm{d}z^2\,, \tag{2.4.1}$$

obtained from (2.2.1) on slices of constant τ and ρ, is a smooth orbifold metric on a spindle. Clearly, we want $\lambda(y)>0$ and $q(y)\geq 0$, which is also enough to guarantee the correct signature of the metric (2.2.1). For (2.4.1) to be a metric on a compact space, we also want to take $y\in[y_a,y_b]$, with $y_a<y_b$ two roots of $q(y)=0$, such that $q(y)>0$ for $y\in(y_a,y_b)$. Since the coefficient of y^4 in $q(y)$ is positive, this is only possible if there are four single[1] real roots, and $y_{a,b}$ are taken to be the middle two roots.

A sufficient condition for $\lambda(y)$ to be positive is that it has no real roots, which is the case for $c_2<\frac{\mathtt{j}^2}{2}$. As for the roots of $q(y)$, they admit a simple expression as

$$\begin{aligned}
y_1 &= -\sqrt{1-\mathtt{j}^2}-\sqrt{1+c_1+2c_2-2\mathtt{j}^2}\,,\\
y_2 &= -\sqrt{1-\mathtt{j}^2}+\sqrt{1+c_1+2c_2-2\mathtt{j}^2}\,,\\
y_3 &= +\sqrt{1-\mathtt{j}^2}-\sqrt{1-c_1+2c_2-2\mathtt{j}^2}\,,\\
y_4 &= +\sqrt{1-\mathtt{j}^2}+\sqrt{1-c_1+2c_2-2\mathtt{j}^2}\,,
\end{aligned} \tag{2.4.2}$$

and note that for at least two of the roots to be real we need $\mathtt{j}\in[-1,1]$. Since the sign of $\mathtt{j}$ can be reabsorbed with a change of the sign of τ, we are actually free to set $\mathtt{j}\in[0,1]$. We further note that $y_{1,2}$ are real and distinct for $c_1>-f(\mathtt{j},c_2)$, while $y_{3,4}$ are real and distinct for $c_1<f(\mathtt{j},c_2)$, with $f(\mathtt{j},c_2)=1+2c_2-2\mathtt{j}^2$. Thus, a

[1] If there is a double root, that is necessarily either y_a or y_b, but then (2.4.1) would not yield a complete metric on a compact space.

necessary condition to have four distinct real roots is that $f(\mathrm{j}, c_2) > 0$, which leads to the constraint $c_2 > \mathrm{j}^2 - \frac{1}{2}$. Note that in this case we also have $y_1 < y_2 < y_3 < y_4$, so we must set $a = 2$, $b = 3$ and take $y \in [y_2, y_3]$. We also note that the dilaton e^{ξ} should be positive for ξ to be real. Its denominator $\lambda(y)$ is positive in the ranges discussed above, while the numerator $g_1(y)$ is a polynomial in y of degree two which is always positive since it has a negative discriminant, given by $-4(1-c_3)^2\,\mathrm{j}^2$. Finally, we should also take c_2 such that the square root $\sqrt{2c_2 - c_3^2\,\mathrm{j}^2}$ appearing in (2.2.2) is real, which requires $2c_2 \geq c_3^2\,\mathrm{j}^2$. This gives a non-empty intersection with the other conditions (in particular $c_2 < \frac{1}{2}\mathrm{j}^2$) only if $|c_3| < 1$.

To summarize, we have shown that when[2]

$$0 \leq \mathrm{j} \leq 1\,, \quad \max\left(\mathrm{j}^2 - \tfrac{1}{2}, \tfrac{1}{2}c_3^2\,\mathrm{j}^2\right) < c_2 < \tfrac{1}{2}\mathrm{j}^2\,, \quad |c_1| < 1 + 2c_2 - 2\mathrm{j}^2\,, \quad |c_3| < 1\,, \tag{2.4.3}$$

we can take $y \in [y_2, y_3]$, with $q(y) \geq 0$ and $\lambda(y) > 0$ in that interval. We shall from now on assume that these conditions hold, and study the global regularity of (2.4.1) under this assumption.

Let us then consider the behaviour of the metric (2.4.1) near the poles $y_{a,b}$. For any y_i such that $q(y_i) = 0$, setting $y = \frac{r^2}{4} + y_i$ we find

$$\mathrm{d}s^2_{\Sigma} \simeq \frac{\lambda(y_i)}{q'(y_i)}\left(\mathrm{d}r^2 + r^2\frac{q'(y_i)^2}{16\lambda(y_i)^2}\,\mathrm{d}z^2\right)\,. \tag{2.4.4}$$

Then, (2.4.1) is a smooth metric[3] on $\mathbb{WCP}^1_{[n_-,n_+]}$ if

$$\frac{q'(y_2)}{4\lambda(y_2)}\Delta z = \frac{2\pi}{n_+}\,, \qquad -\frac{q'(y_3)}{4\lambda(y_3)}\Delta z = \frac{2\pi}{n_-}\,, \tag{2.4.5}$$

[2] Note that from (2.4.3) it seems that taking $\mathrm{j} = 0$ also forces $c_2 = 0$. This is, however, not the case, since the correct way to turn off the rotation parameter is that of taking a limit $c_3 \to \infty$ and $\mathrm{j} \to 0$, with constant product $c_3\,\mathrm{j}$.

[3] In the orbifold sense: the metric is regular everywhere except for the poles $y = y_{2,3}$, where there are conical deficit angles $2\pi\left(1 - \frac{1}{n_\pm}\right)$.

with $n_\pm$ coprime positive integers. Notice here that $\lambda > 0$, while $q'(y_2) > 0$ and $q'(y_3) < 0$, which determines the signs in (2.4.5). These equations are solved by

$$c_1 = \frac{(n_-^2 - n_+^2)\,(1+2c_2-2\mathrm{j}^2)}{n_-^2+n_+^2}\,, \qquad \Delta z = \frac{\sqrt{2}\sqrt{n_-^2+n_+^2}}{n_-\,n_+\,\sqrt{1+2c_2-2\mathrm{j}^2}}\,\pi\,. \tag{2.4.6}$$

Using these conditions, and using the expression

$$\sqrt{g_\Sigma}\,R_\Sigma = \frac{\mathrm{d}}{\mathrm{d}y}\frac{q(y)\,\lambda'(y) - q'(y)\,\lambda(y)}{2\lambda(y)^2}\,, \tag{2.4.7}$$

for the Ricci scalar of the metric (2.4.1), we can also check that the orbifold Euler number

$$\chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma R_\Sigma\,\mathrm{vol}_\Sigma = \frac{n_-+n_+}{n_-\,n_+}\,, \tag{2.4.8}$$

takes the correct value for the spindle. Note that the last condition in (2.4.3) is trivially satisfied for all values of $n_\pm$ due to the constraint (2.4.6).

Chapter 3

Black-Hole Thermodynamics

Since the seminal works of Bekenstein[54, 55] and Hawking[56, 57] in the early 1970s, a remarkable trajectory of research has unfolded, enriching the field of black hole physics through the lens of thermodynamics. The laws of black hole thermodynamics can be considered a semiclassical limit of a comprehensive quantum gravity theory, offering fundamental insights that guide our quest for such a theory. This becomes apparent when examining the Bekenstein-Hawking formula for black hole entropy, once constants of nature are reinstated:

$$S_{BH} = \frac{Ak_B c^3}{4G\hbar} \tag{3.0.1}$$

The presence of the fundamental constants $\hbar$ and G indicates the quantum nature of gravity. Furthermore, within this formula lie the foundations of the holographic nature of quantum gravity, as the entropy is proportional to the area of the event horizon rather than the volume of the black hole.

In this chapter we will work with two solution. The conjectural near horizon solution presented in chapter 2 and the black hole solution of the minimal theory worked out in [13]. This solution can be thought as a special case of the conjectural full black hole of chapter 2. Looking at the solution (2.2.1) and the functions (2.2.2), one can see that the choice $c_2 = c_3 = 0$ sets to zero the scalars and gives $A_1 = A_2$. It is then straightforward to see that our multi-charge spindle solution reproduces in this limit the AdS_2 solutions discussed in [13], if one sets $c_1 = \mathtt{a}$, with all the other coordinates and parameters unchanged. We mention this special solution because in this case we have the full black hole solution and it is possible to provide a complete thermodynamic analysis, as done in [4].

3.1 Near Horizon Analysis

It is possible to compute the conserved charges associated with the conjectural black hole of which (2.2.1) represents the near horizon limit, as well as its entropy just from the near-horizon solution. In the full black hole solution, these conserved charges would usually be defined as integrals over a constant time surface Σ_∞ at infinity, the integrand being constructed from an appropriately conserved current. This approach will be followed in Chapter 4. However, at least for the electric and magnetic charges and angular momentum, using Stokes' Theorem we may equivalently evaluate these quantities as integrals over the horizon Σ, which may then be computed in the near horizon solution, following [13].

First, we define the magnetic charges to be

$$P_i \equiv \frac{1}{2\pi}\int_{\Sigma} F_i\,. \tag{3.1.1}$$

Since $\mathrm{d}F_i = 0$, these charges will be equal to $\frac{1}{2\pi}\int_{\Sigma_\infty} F_i$ for any solution in which the horizon Σ is homologous to a spacelike surface Σ_∞ at infinity. After a computation we find

$$\begin{aligned} P_1 + P_2 &= \frac{n_- - n_+}{n_- n_+} \equiv 4Q_m\,, \\ P_1 - P_2 &= -2\sqrt{2c_2 - c_3^2\,\mathrm{j}^2}\,\frac{\Delta z}{2\pi}\,. \end{aligned} \tag{3.1.2}$$

The first equation in (3.1.2) gives the anti-topological twist; this nomenclature was introduced in [17], due to the relative minus sign $(n_- - n_+)/n_- n_+$ in this expression for the total flux. This may be contrasted with the orbifoldEuler number of the spindle $\chi(\Sigma)$ given by (2.4.8). The latter would be the total magnetic flux $P_1 + P_2$ if supersymmetry was realized by a topological twist, appropriately identifying the spin connection on Σ with the R-symmetry gauge fields, so that the Killing spinor is constant. However, for the anti-topological twist here the spinors are sections of non-trivial bundles over Σ, and so certainly not constant.

To define the electric charges, we notice that while in general $\mathrm{d}\star F_i \neq 0$, the two-forms

$$\mathscr{F}_1 \equiv \frac{\mathrm{e}^{\xi}}{1+\chi^2\,\mathrm{e}^{2\xi}}\left(\star F_1 + \chi\,\mathrm{e}^{\xi}\,F_1\right), \qquad \mathscr{F}_2 \equiv \mathrm{e}^{-\xi}\star F_2 - \chi\,F_2\,, \tag{3.1.3}$$

are closed by virtue of the equations of motion. We thus define

$$Q_i \equiv -\frac{1}{2\pi}\int_{\Sigma}\mathscr{F}_i\,, \tag{3.1.4}$$

which by a similar comment to that above will be equal to the corresponding integrals evaluated on Σ_∞. We find

$$\begin{aligned} Q_1 + Q_2 &= 2\mathrm{j}\,\frac{\Delta z}{2\pi} \equiv 4Q_e\,,\\ Q_1 - Q_2 &= -c_3\,(Q_1+Q_2)\,. \end{aligned} \tag{3.1.5}$$

The first equation provides the definition of the total electric charge Q_e.

Even without knowing the full black hole metric of which (2.2.1) is the near horizon limit, we can still compute its entropy using the Bekenstein-Hawking formula

$$\begin{aligned} S_{BH} &= \frac{\text{Area}}{4} = \frac{1}{4}\frac{y_3-y_2}{2}\,\Delta z\\ &= \frac{\pi}{4}\left(\frac{\sqrt{2(n_-^2+n_+^2)\,(1-\mathrm{j}^2)}}{n_-\,n_+\,\sqrt{1+2c_2-2\mathrm{j}^2}} - \frac{n_-+n_+}{n_-\,n_+}\right). \end{aligned} \tag{3.1.6}$$

In terms of the two pairs of dyonic charges of this solution, the entropy can be also expressed as

$$S_{BH} = \frac{\pi}{4}\left[-\chi(\Sigma) + \sqrt{\chi(\Sigma)^2 + 4(P_1P_2 + Q_1Q_2)}\right]. \tag{3.1.7}$$

Another physical quantity that can be computed for a rotating black hole is its angular momentum. Since the metric of the full black hole is not known, we shall adopt (a suitably modified version of) the prescription of [13], where the angular momentum is defined as a sort of Page charge. To define this, we first introduce an angle $\varphi = \frac{2\pi}{\Delta z}\,z$ and a Killing vector $k = \partial_\varphi$, in terms of which the angular momentum can then be expressed as

$$J(A_1, A_2) = \frac{1}{16\pi}\left[\int_{\Sigma} \star \mathrm{d}k + 2\,(k\cdot A_1)\,\mathscr{F}_1 + 2\,(k\cdot A_2)\,\mathscr{F}_2\right]. \tag{3.1.8}$$

Although the integrand here is not a closed form, so that this doesn't immediately lead to a conserved quantity, one can verify that $k \lrcorner \mathrm{d}$ applied to the integrand is zero. Assuming that the horizon Σ of the near horizon black hole solution and the corresponding copy of this surface Σ_∞ on the conformal boundary are the two boundary components of a k-invariant three-manifold, as one would expect for the black hole solution, it follows from Stokes' Theorem that (3.1.8) takes the same value integrated over either Σ or Σ_∞. However, being a type of Page charge, this angular momentum is not gauge invariant. We will evaluate it in the gauge given in (2.2.1), which is natural from the point of view of a near horizon solution as it is invariant under the isometries of AdS$_2$ [13]. We refer to the value of the angular momentum computed in this gauge as J_{AdS_2}, and we find

$$J_{AdS_2} = \frac{1-c_3^2}{4}\,\mathrm{j}\,\sqrt{1-\mathrm{j}^2}\left(\frac{\Delta z}{2\pi}\right)^2. \tag{3.1.9}$$

Note that we can also write

$$J_{AdS_2} = \frac{Q_1 Q_2}{4(Q_1+Q_2)}\sqrt{\chi(\Sigma)^2 + 4(P_1P_2 + Q_1Q_2)}\,, \tag{3.1.10}$$

and thus the entropy can be rewritten as

$$S_{BH} = \frac{\pi}{4}\left[\frac{4(Q_1+Q_2)}{Q_1Q_2}J_{AdS_2} - \chi(\Sigma)\right]. \tag{3.1.11}$$

3.2 First Law of Black Hole Thermodynamics

In this section, we will adopt a systematic approach to define and compute thermodynamic quantities. As anticipated, we can follow this procedure only in the presence of the full black hole solution. Therefore, we focus on the solution firstly appeared in [13].

It reads [1]

$$
\mathrm{d}s^2 = \frac{1}{H^2}\left\{-\frac{Q}{\Sigma}\left(\frac{1}{\kappa}\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi\right)^2 + \frac{\Sigma}{Q}\,\mathrm{d}r^2 \right.
$$
$$
\left. + \frac{\Sigma}{P}\,\mathrm{d}\theta^2 + \frac{P}{\Sigma}\sin^2\theta\left(\frac{a}{\kappa}\mathrm{d}t - \left(r^2+a^2\right)\mathrm{d}\phi\right)^2\right\} \tag{3.2.1}
$$

where

$$
P(\theta) = 1 - 2\alpha m\cos\theta + \left(\alpha^2\left(a^2+e^2+g^2\right) - a^2\right)\cos^2\theta, \tag{3.2.2}
$$
$$
Q(r) = \left(r^2 - 2mr + a^2 + e^2 + g^2\right)\left(1-\alpha^2 r^2\right) + r^2\left(a^2+r^2\right), \tag{3.2.3}
$$
$$
H(r,\theta) = 1 - \alpha r\cos\theta, \tag{3.2.4}
$$
$$
\Sigma(r,\theta) = r^2 + a^2\cos^2\theta, \tag{3.2.5}
$$

and the gauge field is given by

$$
A = -e\frac{r}{\Sigma}\left(\frac{1}{\kappa}\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi\right) + g\frac{\cos\theta}{\Sigma}\left(\frac{a}{\kappa}\mathrm{d}t - \left(r^2+a^2\right)\mathrm{d}\phi\right)
$$
$$
= A_t\,\mathrm{d}t + A_\phi\mathrm{d}\phi. \tag{3.2.6}
$$

Here, α, m, a, e, and g are the five fundamental parameters associated with acceleration, mass, rotation, electric and magnetic charge respectively. In the following, we will make use of $\Xi \equiv 1+\alpha^2(a^2+e^2+g^2)-a^2$ to shorten the expressions.

The standard holographic approach to defining conserved charges of black holes in AdS, involves first computing the boundary holographic energy-momentum tensor and conserved currents, the latter being associated with the global $U(1)$ symmetries dual to the gauge fields A_i in the bulk, $i=1,2$. Thus, the physical quantities will be defined as conserved charges associated with boundary conserved currents.

The first step of this analysis consists of writing the metric in a standard way, such that its behavior at the boundary becomes manifest. Thanks to Fefferman-Graham's theorem [58], we know that any asymptotically locally AdS solution admits this standard expansion. In the notation of [4], where $z = \frac{1}{r} - \alpha\cos\theta$ and $x^i = (t,\theta,\phi)$ are the coordinates, and the conformal boundary is located at $z=0$, the expansion of

[1] This solution was presented in [4]. It differs from the one in [13] for the normalization κ of the time coordinate. This normalization affects the definition of mass as discussed below.

(3.2.1) reads:

$$\mathrm{d}s^2 = N^2\mathrm{d}z^2 + h_{ij}\left(\mathrm{d}x^i + N^i\mathrm{d}z\right)\left(\mathrm{d}x^j + N^j\mathrm{d}z\right), \tag{3.2.7}$$

where N is the lapse function, N^i the shift vector and h_{ij} the induced metric on the conformal boundary. The outward-pointing unit vector n, normal to hypersurfaces of constant z, provides the splitting between the boundary coordinates and the coordinate z

$$n = \frac{1}{N}\left(N^i\partial_i + \partial_z\right). \tag{3.2.8}$$

The next essential component for the thermodynamic analysis is the total action. This includes the bulk action, the Gibbons-Hawking boundary term, and the holographic renormalization counter terms:

$$S_{\text{tot}} = \frac{1}{16\pi G_4}\left(\int \mathrm{d}^4x\sqrt{-g}\left(R + \frac{6}{l^2} - F^2\right) + \int \mathrm{d}^3x\sqrt{-h}(2K - 4 - R(h))\right), \tag{3.2.9}$$

where K is the trace, with respect the metric h, of the extrinsic curvature $K_{ij} = \frac{1}{2}\mathscr{L}_n g_{ij}$. Subsequently, the boundary stress tensor, the boundary gauge field and the boundary electric current can be computed. They are defined as

$$T_{ij} = \frac{1}{8\pi}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[-K_{ij} + h_{ij}K - 2h_{ij} + R_{ij}(h) - \frac{1}{2}h_{ij}R(h)\right]_{z=\varepsilon}, \tag{3.2.10}$$

$$A^{\text{bdy}} = \lim_{\varepsilon\to 0} A_i|_{z=\varepsilon}\mathrm{d}x^i, \tag{3.2.11}$$

$$j^i = -\frac{1}{4\pi}\lim_{\varepsilon\to 0}\left(\frac{1}{\varepsilon^3}n_\mu F^{\mu i}\right). \tag{3.2.12}$$

It is possible to show that for any given Killing vector k for the boundary geometry, there is an associated conserved current given by

$$\left(T^i_j + j^i A^{\text{bdy}}_j\right)k^j, \tag{3.2.13}$$

and hence a conserved charge. The equation (3.2.13) is the aim of the whole boundary construction above illustrated. With an appropriate choice of Killing vectors, mass, angular momentum, electric, and magnetic charge can be defined. The definition of mass deserves a brief comment. It is defined as the conserved charge associated with a time-like Killing vector k_t. However, the choice of time-like Killing vector is not unique. This ambiguity was resolved in [4] by requiring the first law of black hole thermodynamics to hold. Their analysis leads to the choice of $k_t = \partial_t + \Omega_\infty \frac{\Delta\phi}{2\pi}\partial_\phi$ as Killing vector, together with $\kappa = \frac{\sqrt{(\Xi+a^2)(1-\alpha^2\Xi)}}{1+a^2\alpha^2}$ as the normalization for the time coordinate. For brevity, we keep κ implicit in the following.

A resume of the computations of [4] for the black hole (3.2.1) is

$$\begin{aligned} M &= \frac{m\Delta\phi}{2\pi\kappa}\frac{\left(\Xi+a^2\right)\left(1-\alpha^2\Xi\right)}{\Xi\left(1+\alpha^2a^2\right)}, & J &= am\left(\frac{\Delta\phi}{2\pi}\right)^2, \\ Q_e &= \frac{e\Delta\phi}{2\pi}, & Q_m &= \frac{g\Delta\phi}{2\pi}, \\ S_{\mathrm{BH}} &= \frac{\Delta\phi}{2}\frac{r_+^2+a^2}{1-\alpha^2r_+^2}, & T &= \frac{Q'\left(r_+\right)}{4\pi\kappa\left(r_+^2+a^2\right)}, \end{aligned} \tag{3.2.14}$$

where r_+ is the radius of the black hole. Here the entropy is computed with the Bekenstein-Hawking formula while the temperature is defined as $T = \frac{\kappa_{\mathrm{sg}}}{2\pi}$, with κ_{sg} the surface gravity. This is the standard definition of temperature in the black hole contest. The surface gravity can be computed starting from the null-vector generating the black hole horizon which, in our case, is $V = \partial_t + \frac{1}{\kappa}\frac{a}{r_+^2+a^2}$. Thus we have $\kappa_{\mathrm{sg}} = \sqrt{-\frac{1}{2}\nabla_\mu V_\nu \nabla^\mu V^\nu}$.

Entropy, electric charge and magnetic charge computed in section 3.1 perfectly reduces to the ones given in (3.2.14). The angular momentum J is the conserved charge associated with the Killing vector $k_J = -\frac{\Delta_\phi}{2\pi}\partial_\phi$. However, as explained in 3.1, it is not a gauge-invariant quantity. Therefore, we can introduce another angular momentum, denoted as J_{AdS_2}, computed in a gauge that is natural from the perspective of the near horizon. The relation between the angular momenta in the two gauges is given by

$$J_{\mathrm{AdS}_2} = J + \frac{Q_e}{4}\chi. \tag{3.2.15}$$

The angular momentum computed in 3.1 reduces to J_{AdS_2} after taking the correct limit. Although there is no counterpart of J in the near horizon solution of Chapter 2, in Section 3.4, a conjecture for the renormalized on-shell action will lead to a generalization of (3.2.15).

Before checking the first law of black hole thermodynamics, the chemical potentials associated with the conserved charges need to be computed. The electrostatic potential Φ_e and the magnetic potential Φ_m are defined as

$$\Phi_e = -\lim_{r\to r_+} \iota_V A, \qquad \Phi_m = -\lim_{r\to r_+} \iota_V * A. \tag{3.2.16}$$

The angular velocity associated with the angular momentum J is denoted with Ω, while the angular velocity of the horizon is Ω_H. The computations of [4] lead to

$$\Phi_e = \frac{er_+}{\kappa\left(r_+^2+a^2\right)}, \qquad \Phi_m = \frac{gr_+}{\kappa\left(r_+^2+a^2\right)}, \tag{3.2.17}$$

$$\Omega_\infty = -\frac{2\pi}{\kappa\Delta\phi}\frac{a\left(1-\alpha^2\Xi\right)}{\Xi^2\left(1+a^2\alpha^2\right)}, \qquad \Omega_H = \frac{2\pi}{\kappa\Delta\phi}\frac{a}{r_+^2+a^2},$$

$$\Omega = \Omega_H - \Omega_\infty. \tag{3.2.18}$$

With these quantities, it is possible to check the first law of black hole thermodynamics:

$$\mathrm{d}M = T\mathrm{d}S_{\mathrm{BH}} + \Phi_e \mathrm{d}Q_e + \Omega \mathrm{d}J. \tag{3.2.19}$$

3.3 Supersymmetric and Extremal Limit

In this section, we briefly summarize a key result from [4] due to its significant impact on the setup chosen for the *SQFT* in Chapter 4. The solution (3.2.1) has five free parameters. An observation, initially made in [13], is that these five parameters are subject to three constraints: two arising from the requirement of supersymmetry and one from the requirement of extremality. Recall that a black hole is extremal if the function defining the black hole horizon, denoted as $Q(r)$ in (3.2.2), has a double root at the horizon solution r_+. The three equations are:

$$g = \alpha m \tag{3.3.1}$$

$$0 = \alpha^2(e^2+g^2)(\Xi+a^2)-(g-a\alpha e)^2 \tag{3.3.2}$$

$$0 = ag^2(a\alpha e-g)(e+a\alpha g)+\alpha^3 e^2(e^2+g^2)^2, \tag{3.3.3}$$

where the extremality condition is the last one. In this section, our focus is on studying supersymmetric yet non-extremal solutions. To elucidate this choice, we must revisit the connection between the gravitational theory and the field theory within the holographic framework. From the perspective of the field theory, we have a supersymmetric gauge theory with a gauge group of $U(N)$. This theory is dually related to the sum of gravitational (or string) objects in AdS that share the same boundary conditions. The supersymmetric and extremal AdS black hole represents the leading-order solution in the expansion as $N \to \infty$. This underscores why we attribute the extremal black hole to being dual to a field theory in the large N expansion. To gain insights beyond the large N regime, one can relax the extremality condition to encompass a more general boundary setup, making it relevant to the dual field theory. The outcomes of the supersymmetric but non-extremal analysis will, therefore, serve as the rationale for the chosen background in Chapter 4.

While implementing the first supersymmetry condition is straightforward, solving the second equation of (3.3.1) proves to be challenging. To tackle this, the analysis in [4] introduces a new set of parameters:

$$b = \frac{e}{g}, \qquad c = \frac{a}{g\alpha}, \qquad s = a\alpha, \tag{3.3.4}$$

$$\mu = \frac{n_+ + n_-}{n_+ - n-}, \qquad r_+ = \frac{s}{\alpha}\rho. \tag{3.3.5}$$

This parametrization is valid for $\alpha, a \neq 0$. The essence of the last equation is to treat the horizon radius as a new parameter ρ, which must satisfy $Q(\rho) = 0$ and also $Q'(\rho) = 0$ for the extremality condition to hold. The second supersymmetry condition can now be solved for c, yielding:

$$c = \frac{2(1+b^2)s}{1-2bs-s^2}\mu. \tag{3.3.6}$$

With this new parametrization, the physical charges in (3.2.14) become:

$$M = \frac{Q_m\sqrt{(2cQ_m - \chi s)(2csQ_m + \chi)}}{\chi\sqrt{s}} \tag{3.3.7}$$

$$J = cQ_m^2 \tag{3.3.8}$$

$$Q_e = bQ_m\,. \tag{3.3.9}$$

By combining the last identity with (3.3.6), we obtain a highly non-trivial relation between physical charges:

$$M = \frac{2}{\chi} J + Q_e. \tag{3.3.10}$$

This formula is known as the BPS relation as it directly descends from the structure of supersymmetry.

We can now focus on the equation for the horizon radius, $Q(\rho) = 0$. This equation can be solved for b instead of ρ, yielding the solution:

$$b = b_{\pm} \equiv \frac{2\mu\rho}{\rho^2 - 1} + \frac{\left(1 - s^2 \pm 2\mathrm{i}\mu s\right) B(\rho, s)}{2s(\rho^2 - 1)(\rho^2 s^2 - 1 \mp \mathrm{i}\mu s(\rho^2 + 1))} \tag{3.3.11}$$

where

$$B(\rho, s) \equiv \left(1 - \rho^2\right)\left(1 - \rho^2 s^2\right) + 2\mu\left(1 + \rho^2\right)\rho s. \tag{3.3.12}$$

This equation plays a crucial role, indicating that, after imposing supersymmetry, we cannot demand that ρ, b, s are all real parameters. Consequently, the boundary metric assumes complex values, a factor that must be carefully considered in the background analysis of the dual field theory made in Chapter 4.

Following the analysis of [4], we require $\rho \in \mathbb{R}$. The extremality condition $Q'(\rho) = 0$ will set $B(\rho, s)$ and hence

$$b_* = \frac{2\mu\rho}{\rho^2 - 1}, \tag{3.3.13}$$

$$s_* = \frac{\mu(1+\rho^2)\sqrt{\mu^2(1+\rho^2) + (\rho^2 - 1)^2}}{\rho(\rho^2 - 1)}, \tag{3.3.14}$$

where the subscript $*$ denotes the supersymmetric and extremal values of the parameters. Next, we construct two new complexified chemical potentials, defined as follows

$$\omega \equiv \frac{1}{T}(\Omega - \Omega_*), \qquad \varphi = \frac{1}{T}(\Phi_e - \Phi_{e,*}). \tag{3.3.15}$$

It's noteworthy that these potentials do not trivially approach zero in the extremal limit since the extremal temperature is also zero, i.e., $T_* = 0$. These two chemical potentials are interesting as they satisfy the following relation

$$\varphi - \frac{\chi}{4}\omega = \pm \mathrm{i}\pi. \tag{3.3.16}$$

The last constraint on chemical potentials is crucial as it will be obtained in Chapter 4 in the dual field theory setup. This serves as a highly non-trivial check of the holographic duality.

3.4 Entropy function

As mentioned in the introduction, the black hole entropy is the logarithm of a partition function in a microcanonical ensemble, while the renormalized on-shell action is dual to a field theory defined in a grand canonical ensemble. A direct computation, showing that the two quantities are related by a Legendre transform, was carried out in [4], where the authors could directly compute the renormalized Euclidean on-shell action. They obtained

$$I_E^{\min} \equiv -iS_{\text{tot}}|_{\text{on-shell}} = -S_{\text{BH}} - \omega J - \varphi Q_e = \pm \frac{1}{2\mathrm{i}}\left(\frac{\varphi^2}{\omega} + \omega Q_m^2\right), \tag{3.4.1}$$

which is valid for any of the complex supersymmetric solutions introduced at the end of the last section. Inspired by the results of [4], we conjecture in [1] a formula for the holographically renormalized on-shell action of the full supersymmetric, accelerating, rotating and multi dyonically charged black holes. It reads:

$$I = I(\omega, \varphi_1, \varphi_2) = \pm \frac{1}{2\mathrm{i}}\left(16\frac{\varphi_1\varphi_2}{\omega} + \frac{1}{4}P_1P_2\,\omega\right), \tag{3.4.2}$$

where we have introduced a rotational chemical potential ω, and the electric chemical potentials φ_i for the two gauge fields A_i, $i = 1,2.$, in complete anology with the last part of section (3.3). These chemical potentials are furthermore required to satisfy the constraint

$$2(\varphi_1 + \varphi_2) - \frac{\chi(\Sigma)}{4}\omega = \pm \mathrm{i}\pi\,. \tag{3.4.3}$$

More precisely, (3.4.2) should be the holographically renormalized on-shell action for a complex locus of supersymmetric solutions, that arise as an analytic continuation of the real black hole solutions we have conjecture.

Given the on-shell action (3.4.2), we may write down the following associated *entropy function*

$$\mathscr{S} \equiv -I(\omega,\varphi_1,\varphi_2) - (\omega J_{BH} + \varphi_1 Q_1 + \varphi_2 Q_2)\,. \tag{3.4.4}$$

Here the rotational chemical potential ω is conjugate to the black hole angular momentum J_{BH}. According to the AdS/CFT conjecture, the black hole entropy should then be obtained by extremizing (3.4.4) over the chemical potentials ω, φ_i, where the latter are subject to the constraint (3.4.3). This of course then implements the Legendre transform. We thus write

$$S(J_{BH},Q_1,Q_2) = \mathrm{ext}_{\{\omega,\varphi_1,\varphi_2,\Lambda\}}\left[\mathscr{S} - \Lambda\left(2(\varphi_1+\varphi_2) - \frac{\chi(\Sigma)}{4}\omega \mp \mathrm{i}\pi\right)\right]. \tag{3.4.5}$$

The extremization imposes

$$-\frac{\partial I}{\partial \omega} = J_{BH} - \frac{\chi(\Sigma)}{4}\Lambda\,, \qquad -\frac{\partial I}{\partial \varphi_i} = Q_i + 2\Lambda\,, \tag{3.4.6}$$

and we find the solution

$$\begin{aligned}
\Lambda &= \Big\{ - Q_1 - Q_2 \pm \mathrm{i}\chi(\Sigma) + \mathrm{i}\eta\Big[\chi(\Sigma)^2 - (Q_1-Q_2)^2 + 4P_1P_2 \\
&\qquad\qquad \pm 2\mathrm{i}(Q_1+Q_2)\chi(\Sigma) \pm 32\mathrm{i}J_{BH}\Big]^{1/2}\Big\}\,, \\
\omega &= \frac{4\pi\mathrm{i}\eta}{\sqrt{\chi(\Sigma)^2 - (Q_1-Q_2)^2 + 4P_1P_2 \pm 2\mathrm{i}(Q_1+Q_2)\chi(\Sigma) \pm 32\mathrm{i}J_{BH}}}\,, \\
\varphi_1 &= \pm\frac{\pi\eta[Q_2 - Q_1 \pm \mathrm{i}\chi(\Sigma)]}{4\sqrt{\chi(\Sigma)^2 - (Q_1-Q_2)^2 + 4P_1P_2 \pm 2\mathrm{i}(Q_1+Q_2)\chi(\Sigma) \pm 32\mathrm{i}J_{BH}}} \pm \frac{\mathrm{i}\pi}{4}\,.
\end{aligned} \tag{3.4.7}$$

Here φ_2 is determined by the constraint (3.4.3), and $\eta = \pm 1$ arises as a choice of sign in taking square roots when solving the equations. Imposing that the entropy is real, while assuming all conserved charges are real, we find

$$J_{BH} = \frac{Q_1+Q_2}{16}\left[-\chi(\Sigma)+\sqrt{\chi(\Sigma)^2+4(P_1P_2+Q_1Q_2)}\right], \tag{3.4.8}$$

where the sign of the square root in J_{BH} has been fixed by requiring $J_{BH} > 0$. Moreover, we then obtain the extremal value of (3.4.5) to be

$$S(J_{BH}, Q_1, Q_2) = \pm \mathrm{i}\pi\Lambda, \tag{3.4.9}$$

which gives

$$\begin{aligned} S(J_{BH}, Q_1, Q_2) &= \frac{4\pi}{(Q_1+Q_2)} J_{BH} \\ &= \frac{\pi}{4}\left[-\chi(\Sigma)+\sqrt{\chi(\Sigma)^2+4(P_1P_2+Q_1Q_2)}\right]. \end{aligned} \tag{3.4.10}$$

This precisely agrees with the black hole entropy S_{BH} in (3.1.7) computed from the near horizon solutions.

Notice that the two angular momenta (3.1.10), (3.4.8) are related via

$$J_{AdS_2} - \frac{4Q_1Q_2}{(Q_1+Q_2)^2} J_{BH} = \frac{Q_1Q_2}{4(Q_1+Q_2)}\chi(\Sigma). \tag{3.4.11}$$

Since we do not have the full black hole solutions it is not immediate to define and compute J_{BH} directly. However, we note that both J_{AdS_2} and J_{BH} were computed for the minimal gauged supergravity solutions with $Q_1 = Q_2$ (and $P_1 = P_2$) in [13], and the relation (3.4.11) reduces to the corresponding relation in this reference.

The entropy function can be expressed as the Legendre transform of

$$I(\omega, \varphi_1, \varphi_2) = \pm\frac{2}{\mathrm{i}\pi\,\omega}\left[F_{S^3}(\varphi_i - \tfrac{1}{8}\omega P_i) + F_{S^3}(\varphi_i + \tfrac{1}{8}\omega P_i)\right], \tag{3.4.12}$$

where $F_{S^3}(\Delta_1, \Delta_2) = 4\Delta_1\Delta_2 F_{S^3}$ is the large N S^3 free energy as a function of the trial R-symmetry, with Δ_i satisfying $\Delta_1 + \Delta_2 = 1$. Recall here that the free energy on the three-sphere is $F_{S^3} = \frac{\pi}{2G_4} = \frac{\sqrt{2}\pi N^{3/2}}{3}$. This is consistent with the general expectations for $d = 3$, $\mathscr{N} = 4$ SCFT with holographic duals, discussed in [59]. It is worth mentioning that, firstly in [60] and then in [22], a highly general gravitational block formula for the supersymmetric action has been developed. In that context, the authors were able to reproduce our (3.4.12) as a special case.

In the next chapter, we will compute the partition function for a supersymmetric quantum field theory for a generic N. This result, after performing the correct large N limit, should match the expression in (3.4.12). The partition function will be computed using a localization technique and can be organized as the sum of two terms arising from contributions at the two poles of the spindle. This mirrors the same behavior shown in (3.4.12), providing a strong indication that the duality between the two theories can be directly proven.

Chapter 4

Supersymmetric Field Theory

4.1 General Complex Backgrounds

As anticipated in the introduction 1.3, we consider a general class of rigid supersymmetric backgrounds of Euclidean new minimal supergravity where it is possible to accommodate the $\mathbb{\Sigma} \times S^1$ coming from the boundary orbifold geometry of the black hole. These backgrounds preserve two Killing spinors $(\zeta, \widetilde{\zeta})$ with R-charges ± 1, respectively.

The Killing spinor equations (KSEs), which the background fields must satisfy in order to preserve supersymmetry, are

$$\begin{aligned}(\nabla_\mu - \mathrm{i}A_\mu)\zeta &= -\frac{H}{2}\gamma_\mu \zeta - \mathrm{i}V_\mu \zeta \mp \varepsilon_{\mu\nu\rho}\frac{V^\nu}{2}\gamma^\rho \zeta\,, \\ (\nabla_\mu + \mathrm{i}A_\mu)\widetilde{\zeta} &= -\frac{H}{2}\gamma_\mu \widetilde{\zeta} + \mathrm{i}V_\mu \widetilde{\zeta} \mp \varepsilon_{\mu\nu\rho}\frac{V^\nu}{2}\gamma^\rho \widetilde{\zeta}\,, \end{aligned} \tag{4.1.1}$$

where A_μ is the R-symmetry background gauge field, V_μ is a globally defined co-closed one-form and H a scalar, a priori all complex-valued. We emphasise that $(\zeta, \widetilde{\zeta})$ are not related by charge conjugation and, consistently with the analysis of section 3.3, the metric $g_{\mu\nu}$ can be complex-valued.

Following the conventions of [45], we introduce

$$v = \zeta\, \widetilde{\zeta}\,, \quad K^\mu = \zeta \gamma^\mu\, \widetilde{\zeta}\,, \tag{4.1.2}$$

$$P^\mu = \zeta\gamma^\mu\zeta/v\,, \quad \widetilde{P}^\mu = \widetilde{\zeta}\gamma^\mu\widetilde{\zeta}/v \tag{4.1.3}$$

where $(P^\mu)^* \neq \widetilde{P}^\mu$. The KSEs imply that K^μ is a complex Killing vector. Thanks to Fierz identities $\left(K, P, \widetilde{P}\right)$ is a canonical complex frame generating the line element

$$\mathrm{ds}^2 = \frac{1}{v^2}K^2 - P\widetilde{P} \,, \tag{4.1.4}$$

with non-zero contractions being

$$\iota_K K = v^2 \,, \qquad \iota_{\widetilde{P}} P = \iota_P \widetilde{P} = -2 \,. \tag{4.1.5}$$

Using the KSEs, it is possible to determine the background fields A_μ, V_μ, H in terms of $\left(K, P, \widetilde{P}\right)$, as shown in [3]. In particular, we get

$$\nabla_{(\mu}K_{\nu)} = 0, \tag{4.1.6}$$

which ensures that K is a complex Killing vector. Thus, it can be parametrized as

$$K = \partial_\psi + \omega \partial_\varphi \,, \tag{4.1.7}$$

where ω is a complex constant and φ, ψ are 2π-periodic angular coordinates.

The most general metric on $\mathscr{M}_3$ invariant under two real Killing vectors $\partial_\psi, \partial_\varphi$ can be written as

$$\mathrm{ds}^2 = \mathrm{f}^2\mathrm{dx}^2 + \mathrm{h_{ij}}\mathrm{d}\psi_\mathrm{i}\mathrm{d}\psi_\mathrm{j} \quad \text{with} \quad \psi_1 = \psi \,, \psi_2 = \varphi \,, \tag{4.1.8}$$

where the complex-valued functions $f(x)$ and $h_{ij}(x)$, $i, j = 1, 2$ depend only on the coordinate x. In these coordinates we have

$$\begin{aligned} K &= (h_{11} + \omega h_{12})\mathrm{d}\psi + (\mathrm{h}_{12} + \omega \mathrm{h}_{22})\mathrm{d}\varphi \,, \\ P &= \mathrm{e}^{2\mathrm{i}\theta}\big(f\mathrm{dx} + \mathrm{i}(\sqrt{\mathrm{h}}/\mathrm{v})(-\omega \mathrm{d}\psi + \mathrm{d}\varphi)\big) \,, \\ \widetilde{P} &= \mathrm{e}^{-2\mathrm{i}\theta}\big(-f\mathrm{dx} + \mathrm{i}(\sqrt{\mathrm{h}}/\mathrm{v})(-\omega \mathrm{d}\psi + \mathrm{d}\varphi)\big) \,, \\ v^2 &= h_{11} + 2\omega h_{12} + \omega^2 h_{22} \,, \end{aligned} \tag{4.1.9}$$

where $h = \det(h_{ij})$ and $\theta \equiv \frac{\alpha_1\psi+\alpha_2\varphi}{2}$, with α_1, α_2 two real constants that we shall discuss momentarily. Defining $A^C \equiv A - \frac{3}{2}V$, the background fields read

$$\begin{aligned} V &= \frac{1}{v}\Big[\mathrm{i}HK - \star\mathrm{dK}\Big]\,, \\ A^C &= \frac{v^3}{4f\sqrt{h}}\Big[\frac{1}{\omega}\Big(\frac{h_{11}}{v^2}\Big)'\mathrm{d}\psi - \Big(\frac{\mathrm{h}_{22}}{\mathrm{v}^2}\Big)'\mathrm{d}\varphi\Big] + \mathrm{d}\theta\,, \end{aligned} \tag{4.1.10}$$

where a prime denotes derivative with respect to x. The function H satisfies $\mathscr{L}_K H = 0$ and is otherwise arbitrary; however, it will enter in the localization computation only though the following combinations:

$$\begin{aligned} h_R &\equiv \iota_K V - \mathrm{i}vH = -\frac{1}{2v}\star(K\wedge\mathrm{dK})\,, \\ \Phi_R &\equiv \iota_K\Big(A^C + V\Big) - \mathrm{i}vH = (\alpha_1 + \omega\alpha_2)/2\,. \end{aligned} \tag{4.1.11}$$

Taking $\gamma^1 = \sigma^2$, $\gamma^2 = \sigma^3$, $\gamma^3 = \sigma^1$, with σ^i being the Pauli matrices, in the frame where

$$e^1 = -f\,\mathrm{d}x\,, \quad \mathrm{e}^2 = \sqrt{\frac{\mathrm{h}}{\mathrm{h}_{11}}}\mathrm{d}\varphi\,, \quad \mathrm{e}^3 = \sqrt{\mathrm{h}_{11}}\big(\mathrm{d}\psi + \frac{\mathrm{h}_{12}}{\mathrm{h}_{11}}\mathrm{d}\varphi\big)\,, \tag{4.1.12}$$

the Killing spinors satisfying (4.1.1) take the form

$$\zeta_\alpha = e^{\mathrm{i}\theta}\begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}_\alpha\,, \qquad \widetilde{\zeta}_\alpha = -e^{-\mathrm{i}\theta}\begin{pmatrix} u_2 \\ u_1 \end{pmatrix}_\alpha\,, \tag{4.1.13}$$

where

$$u_{1,2} = 2^{-1/2}\sqrt{v \mp \omega\sqrt{h/h_{11}}}\,. \tag{4.1.14}$$

4.1.1 $\Sigma \times S^1$ with twist and anti-twist

On $\Sigma \times S^1$ we adopt the line element

$$\mathrm{d}s^2 = f^2\mathrm{d}x^2 + \big(1-x^2\big)(\mathrm{d}\varphi - \Omega\mathrm{d}\psi)^2 + \beta^2\mathrm{d}\psi^2\,, \tag{4.1.15}$$

where the function $f = f(x)$ satisfies

$$\lim_{x\to\pm 1} f \to \frac{n_\pm}{\sqrt{2(1\mp x)}} \ . \tag{4.1.16}$$

Wet take ψ to parameterize S^1 and $x \in [-1,1], \varphi$ as coordinates on $\mathbb{\Sigma}$. We require that f is regular on the complementary domain $(-1,+1)$, but we leave arbitrary the specific profile of f. The positive integers $n_\pm$ are coprime and encode the conical singularities $\mathbb{C}/\mathbb{Z}_{n_\pm}$ at the poles of the spindle. The dimensionless complex parameter Ω induces a refinement of the partition function by a fugacity for the angular momentum on the spindle, as it happens on the round sphere [28]. The parameter β is the ratio between the radius of S^1 and the radius of the equatorial circle of $\mathbb{\Sigma}$.

We use an orthonormal frame explicitly realizing $\mathbb{\Sigma}\times S^1$ as a topologically trivial $U(1)$ fibration over the spindle base:

$$\begin{aligned}
e^1 &= -f\mathrm{d}x \ , \\
e^2 &= \beta\sqrt{\frac{1-x^2}{\beta^2+\Omega^2(1-x^2)}}\mathrm{d}\varphi \ , \\
e^3 &= \sqrt{\beta^2+\Omega^2(1-x^2)}\Big(A^{(1)}+\mathrm{d}\psi\Big) \ , \qquad A^{(1)} = \frac{-\Omega\big(1-x^2\big)}{\beta^2+\Omega^2(1-x^2)}\mathrm{d}\varphi \ ,
\end{aligned} \tag{4.1.17}$$

where the flux of $A^{(1)}$ through $\mathbb{\Sigma}$ is vanishing, as it behooves a trivial circle fibration over a spindle. Near the north pole of $\mathbb{\Sigma}$ at $x=+1$ we have

$$e^1_{\mathscr{U}_+} = -\frac{n_+\mathrm{d}x}{\sqrt{2(1-x)}} = \mathrm{d}\rho_+ \ , \qquad e^2_{\mathscr{U}_+} = \sqrt{2(1-x)}\mathrm{d}\varphi = \frac{\rho_+}{n_+}\mathrm{d}\varphi \ , \quad \rho_+ = n_+\sqrt{2(1-x)} \ . \tag{4.1.18}$$

Hence, an open neighbourhood of $x=+1$ can be parametrized by the complex coordinate z_+ satisfying

$$z_+ = \rho_+ e^{\mathrm{i}\varphi/n_+} \ , \qquad z_+ \sim w_+ z_+ \ , \qquad w_+ = e^{2\pi\mathrm{i}/n_+} \ , \tag{4.1.19}$$

where the identification involving the root of unity w_+ makes manifest that $\mathscr{U}_+$ is isomorphic to $\mathbb{C}/\mathbb{Z}_{n_+}$. On the other hand, near the south pole of the spindle at $x=-1$

we have

$$e_S^1 = -\frac{n_- \mathrm{d}x}{\sqrt{2(1+x)}} = -\mathrm{d}\rho_+ \,, \qquad e_S^2 = \sqrt{2(1+x)}\mathrm{d}\varphi = \frac{\rho_-}{n_-}\mathrm{d}\varphi \,, \quad \rho_- = n_-\sqrt{2(1+x)} \,. \tag{4.1.20}$$

Thus, a proper coordinate in an open neighbourhood $\mathscr{U}_-$ of $x=-1$ is

$$z_- = \rho_- e^{-\mathrm{i}\varphi/n_-} \,, \qquad z_- \sim w_-^{-1} z_- \,, \qquad w_- = e^{2\pi\mathrm{i}/n_-} \,, \tag{4.1.21}$$

meaning that $\mathscr{U}_- \cong \mathbb{C}/mathbbZ_{n_-}$. The minus sign in the exponential of $z_- = \rho_- e^{-\mathrm{i}\varphi/n_-}$ takes into account the reversed orientation of $\mathscr{U}_-$ with respect to that of $\mathscr{U}_+$. The frame $(K, P_\pm)$, background fields A, V, H and Killing spinors $(\zeta, \widetilde{\zeta})$ are then completely determined by our general formulae (4.1.9), (4.1.10), (4.1.13) in terms of the metric functions $f(x), \Omega, \beta$ and the parameter ω.

By using (4.1.10) we find

$$\frac{1}{2\pi}\int_\Sigma \mathrm{dA} = -\frac{1}{2}\left(\frac{s_+}{n_+} + \frac{s_-}{n_-}\right) \,, \tag{4.1.22}$$

where $s_\pm$ denote the signs of the function $v/\sqrt{h_{11}}$ at the north and south poles, respectively.

We recall that supersymmetry on Σ is preserved by an R-symmetry background gauge field A satisfying only one [15] of the following conditions:

$$\int_\Sigma \frac{\mathrm{d}A}{2\pi} = \frac{1}{2}\left(\frac{1}{n_-} + \frac{\sigma}{n_+}\right) \equiv \frac{\chi_\sigma}{2} \,, \tag{4.1.23}$$

with $\sigma = \pm 1$ being *twist* and *anti-twist* configurations, respectively. In this notation,

$$\chi_+ = \frac{1}{n_-} + \frac{1}{n_+} = \frac{1}{4\pi}\int_\Sigma \sqrt{g}R \tag{4.1.24}$$

is the orbifold Euler characteristic of the spindle, previously defined in 2.4.8. In the case of $\sigma = +1$, the magnetic flux of the R-symmetry background field equals half of the orbifold Euler characteristic of the spindle χ_+. This configuration is the topological twist as it is the orbifold generalization of the analogous setup found in spheres [28, 61] and Riemann surfaces [62]. In the special case where $n_+ = n_- = 1$ the spindle become a smooth sphere, and the anti-twisted $\Sigma \times S^1$

configuration reduces to the generalized superconformal index background studied in [48, 49], characterized by the absence of magnetic flux for the R-symmetry connection. Eventually, after shrinking to zero the radius of S^1 in $\Sigma \times S^1$ and keeping $n_+ = n_- = 1$, the anti-twisted spindle becomes the supersymmetric sphere with no R-symmetry twist explored in [63].

By comparing 4.1.22 and 4.1.23, it is clear that the type of supersymmetry-preserving twist is completely encoded in the behaviour of the function v. Given a generic metric on $\Sigma \times S^1$ and a parameter ω, we regard the third equation in (4.1.9) as a definition of the function v. From this, it follows that generically the function $v/\sqrt{h_{11}}$ has the same sign at both poles, corresponding to the twist case. Instead, the anti-twist is realized if the function $v/\sqrt{h_{11}}$ has opposite sign at the poles. In this case the metric h_{ij} and the parameter ω need be fine-tuned. As $v^2 = (1-x^2)(\omega-\Omega)^2+\beta^2$, if no relation is imposed between ω, β and Ω, then the R-symmetry background field realizes the *twist*. As a special case, the standard topological twist corresponds to $\omega = \Omega$, yielding $v/\beta = -1$, so that in (4.1.23) we have $\sigma = +1$. The Killing spinors corresponding to the twisted $\Sigma \times S^1$ geometry, occurring if $\sigma = +1$, are given by 4.1.13, with

$$u_{1,2} = \sqrt{-\frac{\beta}{2}\left(1 \pm \Omega\sqrt{\frac{1-x^2}{\beta^2+\Omega^2(1-x^2)}}\right)} \,, \tag{4.1.25}$$

which satisfy the conformal Killing spinor equations provided that the background fields A^C, V, A take the form

$$\begin{aligned} A^C &= \mathrm{d}\theta - \frac{x}{2f\sqrt{1-x^2}}(\mathrm{d}\varphi - \Omega \mathrm{d}\psi) \,, \\ V &= -\mathrm{i}\beta H \mathrm{d}\psi \,, \\ A &= A^C + \frac{3}{2}V \,. \end{aligned} \tag{4.1.26}$$

Conversely, the *anti-twist* is realized by choosing $\Omega = \omega \pm \mathrm{i}\beta$. In this case we can take $v/\beta = x$, so that in (4.1.23) we have $\sigma = -1$.The Killing spinors associated with the anti-twisted $\Sigma \times S^1$ geometry exhibit a formal similarity to the Killing spinors corresponding to the twisted background. However, they differ in terms of their components $u_{1,2}$:

$$u_{1,2} = \sqrt{\frac{\beta}{2}\left[x \mp (\Omega - \mathrm{i}\beta)\sqrt{\frac{1-x^2}{\beta^2+\Omega^2(1-x^2)}}\right]}\ , \tag{4.1.27}$$

fulfilling the conformal Killing spinor equations if the background fields A^C, V, A are

$$\begin{aligned}
A^C &= \mathrm{d}\theta + \frac{1}{2f\sqrt{1-x^2}}[\mathrm{d}\varphi - (\Omega + \mathrm{i}\beta)\mathrm{d}\psi]\ ,\\
V &= \big(x^{-1}+x\big)H\mathrm{d}\varphi + \mathrm{i}\left\{\frac{\beta}{f\sqrt{1-x^2}} + x^{-1}\big[\beta + \mathrm{i}\Omega\big(1-x^2\big)\big]H\right\}\mathrm{d}\psi\ ,\\
A &= A^C + \frac{3}{2}V\ .
\end{aligned} \tag{4.1.28}$$

Requiring $(\zeta, \widetilde{\zeta})$ to be (anti-)periodic under $\psi \sim \psi + 2\pi$ implies that $\alpha_1 = n \in \mathbb{Z}$. Finally, one can see that at the north and south poles of Σ the spinors behave as

$$\zeta_\alpha^{\mathrm{N/S}} \sim \begin{pmatrix} 1 \\ -s_\pm \end{pmatrix}_\alpha\ , \qquad \widetilde{\zeta}^{\mathrm{N/S}} \sim \begin{pmatrix} s_\pm \\ 1 \end{pmatrix}_\alpha\ , \tag{4.1.29}$$

so that indeed they have the same 2d-chirality for the twist and the opposite 2d-chirality for the anti-twist [15].

4.2 Localization

Let G be a semi-simple Lie group with Lie algebra $\mathfrak{g}$, with $\mathfrak{R}_\mathrm{G}$ a generic representation of G and Ad_G its adjoint representation. For a three-dimensional $\mathscr{N} = 2$ gauge theory, the supersymmetry transformations of a vector multiplet $(\mathscr{A}, \sigma, \lambda, \widetilde{\lambda}, D) \in \mathrm{Ad}_\mathrm{G}$ and those of a chiral multiplet $(\phi, \psi, F) \in \mathfrak{R}_\mathrm{G}$ can be written as a cohomological complex [47]. Solving the BPS equations $\delta\psi = \delta\widetilde{\psi} = \delta\lambda = \delta\widetilde{\lambda} = 0$ yields the localizion locus of classical configurations contributing by Z_{class} to the partition function. Also, this formulation allows for recasting the computation of vector and chiral multiplets 1-loop determinants, $Z_{\text{1-L}}^{\mathrm{VM}}, Z_{\text{1-L}}^{\mathrm{CM}}$, as a cohomological problem. The explicit supersymmetry variations, as well as the cohomological fields, are provided in appendix C.

On general grounds, if $\mathrm{G} = \mathrm{G}_g \times \mathrm{G}_f$, the partition function of a theory on $S^1 \times \mathbb{\Sigma}$ with gauge group G_g and flavour group G_f reads

$$Z_{S^1\times\mathbb{\Sigma}}(\omega, u_f, \mathfrak{f}_f) = \sum_{\mathfrak{f}_g \in \Gamma_{\mathfrak{h}_g}} \oint_{\mathscr{C}} \frac{\mathrm{d}u_g}{|W_g|} \widehat{Z}(u_g, \mathfrak{f}_g | \omega, u_f, \mathfrak{f}_f) \,, \tag{4.2.1}$$

where $\mathfrak{h}_g$ is the maximal Cartan-subalgebra of $\mathfrak{g}_g$, $\Gamma_{\mathfrak{h}_g}$ the corresponding co-root lattice and W_g its Weyl group; while $u_{g,f} \in \mathfrak{h}_{g,f}$ and $\mathfrak{f}_{g,f} \in \Gamma_{\mathfrak{h}_{g,f}}$ denote gauge/flavour holonomies and fluxes, respectively; $\widehat{Z}$ is the product of Z_{class}, $Z^{\text{VM}}_{\text{1-L}}$ and $Z^{\text{CM}}_{\text{1-L}}$ and $\mathscr{C}$ is a suitable integration-contour for u_g. This expression also depends on the spindle data $(n_\pm, \sigma)$. $Z_{S^1\times\mathbb{\Sigma}}$ is also a G_f-flavoured Witten-index of the theory quantized on $\mathbb{\Sigma}$,

$$Z_{S^1\times\mathbb{\Sigma}} = \mathrm{Tr}_{\mathscr{H}[\mathbb{\Sigma}]}\left[\mathrm{e}^{-\omega J - \varphi R - \sum_i \varphi_i F_i}\right] \,, \tag{4.2.2}$$

where J, R, F_i generate angular momentum, R-symmetry and flavour symmetries, respectively; while $\mathscr{H}[\mathbb{\Sigma}]$ is the Hilbert space of states on the spindle, with either twist or anti-twist. The partition function (4.2.1) will be further elucidated in the following sections, providing explicit computations for each component. We anticipate that the fugacities ω and φ are not independent, but are related by

$$\varphi - \frac{\chi_{-\sigma}}{4}\omega = \mathrm{i}\pi n \,, \qquad n \in \mathbb{Z} \,. \tag{4.2.3}$$

For the anti-twist ($\sigma = -1$), taking $n = \pm 1$ so that the spinors are anti-periodic on S^1, this reproduces the relation (3.3.16) for the dual accelerating black holes.

4.2.1 BPS locus

We parametrize an Abelian gauge or flavour 1-form field on $\mathbb{\Sigma} \times S^1$ as

$$\mathscr{A} = \mathscr{A}_\varphi(x)\mathrm{d}\varphi + \mathscr{A}_\psi(x)\mathrm{d}\psi \,, \quad \mathfrak{f}_G = \frac{1}{2\pi}\int_{\mathbb{\Sigma}} \mathrm{d}\mathscr{A} = \frac{\mathtt{m}}{n_+ n_-} \,, \qquad \exp\left(\mathrm{i}\oint_{S^1}\mathscr{A}\right) = \mathfrak{h} \,, \tag{4.2.4}$$

with $\mathtt{m} \in \mathbb{Z}$ and $\mathfrak{h} \in U(1)$, where we removed the component $\mathscr{A}_x(x)$ along $\mathrm{d}x$ via a gauge transformation.

The 1-form in (4.2.4) is the representative of an $\mathscr{O}(-\mathtt{m})$ orbibundle on the spindle $\mathbb{\Sigma}$ and of the $U(1)$ holonomy group of S^1. In particular, the component $\mathscr{A}_\varphi(x)$ evaluated at the north pole at $x=1$ and at the south pole at $x=-1$ of the spindle reads

$$\mathscr{A}_\varphi(+1) = \frac{\mathfrak{m}_+}{n_+} \; , \qquad \mathscr{A}_\varphi(-1) = \frac{\mathfrak{m}_-}{n_-} \; , \qquad n_+\mathfrak{m}_- - n_-\mathfrak{m}_+ = \mathtt{m} \; , \tag{4.2.5}$$

where the integers $(\mathfrak{m}_+, \mathfrak{m}_-)$ can be expressed in terms of $\mathtt{m}$ and two other integers $(\mathfrak{a}_+, \mathfrak{a}_-)$ satisfying

$$\mathfrak{m}_+ = \mathtt{m}\,\mathfrak{a}_+ \; , \qquad \mathfrak{m}_- = \mathtt{m}\,\mathfrak{a}_- \; , \qquad n_+\mathfrak{a}_- - n_-\mathfrak{a}_+ = 1 \; . \tag{4.2.6}$$

Especially, given a pair $(\mathfrak{a}_+, \mathfrak{a}_-)$ satisfying the constraint above, any pair $(\mathfrak{a}_+ + n_+\delta\mathfrak{a}, \mathfrak{a}_- + n_-\delta\mathfrak{a})$ with $\delta\mathfrak{a} \in \mathbb{Z}$ fulfils the constraint too. Physical observables are supposed to be independent of $\delta\mathfrak{a}$.

The 1-form (4.2.4) is consistent with the gauge fields used in [15] provided that

$$\mathscr{A}^N = \mathscr{A}^N_{(0)} + \frac{\mathfrak{m}_N}{n_N}\mathrm{d}\varphi = \mathscr{A}^S_{(0)} + \left(\mathrm{p} + \frac{\mathfrak{m}_S}{n_S}\right)\mathrm{d}\varphi = \mathscr{A}^S + \mathrm{p}\,\mathrm{d}\varphi \; , \tag{4.2.7}$$

with $\mathrm{p} \in \mathbb{Z}$ modelling a gauge transformation in the overlap of the two patches (N, S), where the connections $\left(\mathscr{A}^N, \mathscr{A}^S\right)$ are well defined. The dictionary between our notation and that of [15] is

$$\begin{aligned}
\mathscr{A}^N_{(0)} &= \mathscr{A} - \frac{\mathfrak{m}_+}{n_+}\mathrm{d}\varphi \; , \qquad n_N = n_+ \; , \qquad \mathfrak{m}_N = \mathfrak{m}_+ \; , \\
\mathscr{A}^S_{(0)} &= \mathscr{A} - \frac{\mathfrak{m}_-}{n_-}\mathrm{d}\varphi \; , \qquad n_S = n_- \; , \qquad \mathfrak{m}_S = \mathfrak{m}_- - n_-\mathrm{p} \; .
\end{aligned} \tag{4.2.8}$$

In this language, the flux of $\mathscr{A}$ reads [15]

$$\frac{1}{2\pi}\int_{\mathbb{\Sigma}} \mathrm{d}\mathscr{A} = \mathrm{p} + \frac{\mathfrak{m}_S}{n_S} - \frac{\mathfrak{m}_N}{n_N} = \frac{\mathtt{m}}{n_+ n_-} \; , \tag{4.2.9}$$

in agreement with (4.2.4). If we add a flat connection $(\beta_2 \mathrm{d}\varphi)$ to $\mathscr{A}$ in (4.2.4) as

$$\mathscr{A} \to \mathscr{A}' = \left(\mathscr{A}_\varphi + \beta_2\right)\mathrm{d}\varphi + \mathscr{A}_\psi \mathrm{d}\psi \; , \tag{4.2.10}$$

we can make $\mathscr{A}$ vanish at the poles of Σ at $x = \pm 1$ by setting $\beta_2|_{\mathscr{U}_+} = -\mathfrak{m}_+/n_+$ in the northern patch $\mathscr{U}_+$ including $x = +1$ and $\beta_2|_{\mathscr{U}_-} = -\mathfrak{m}_-/n_-$ in the southern patch $\mathscr{U}_-$ covering $x = -1$.

The vector-multiplet BPS equations are obtained by setting to zero the gauginos $\left(\lambda, \widetilde{\lambda}\right)$ and their variations $\left(\delta\lambda, \delta\widetilde{\lambda}\right)$, giving

$$\begin{aligned}
&\Phi_G = \iota_K \mathscr{A} - \mathrm{i}\nu\sigma = \varphi_G = \text{constant} , \\
&D + \frac{\mathrm{i}}{2} P^\mu \widetilde{P}^\nu \mathscr{F}_{\mu\nu} + \frac{\mathrm{i}}{\nu}\sigma[\iota_K V - \mathrm{i}\nu H] = 0 .
\end{aligned} \tag{4.2.11}$$

We do not impose any reality conditions on fields, for the moment. Inserting the gauge field (4.2.4) into the BPS equations above yields the BPS values of the scalar σ and the auxiliary scalar D. Specifically, if $\sigma = +1$, we have topologically twisted $\Sigma \times S^1$ and the BPS locus

$$\sigma|_{\text{BPS}} = \mathrm{i}\beta^{-1}\left[-\varphi_G + \omega\mathscr{A}_\varphi(x) + \mathscr{A}_\psi(x)\right] , \quad D|_{\text{BPS}} = -\frac{\mathscr{A}'_\varphi(x)}{f\sqrt{1-x^2}} , \tag{4.2.12}$$

whereas, if $\sigma = -1$, we have anti-twisted $\Sigma \times S^1$ and

$$\begin{aligned}
&\sigma|_{\text{BPS}} = -\mathrm{i}\left[-\varphi_G + \omega\mathscr{A}_\varphi(x) + \mathscr{A}_\psi(x)\right]/(\beta x) , \\
&D|_{\text{BPS}} = \frac{\beta\,\sigma|_{\text{BPS}} + \left[\beta + \mathrm{i}\Omega\left(1-x^2\right)\right]\mathscr{A}'_\varphi(x) + \mathrm{i}\left(1-x^2\right)\mathscr{A}'_\psi(x)}{\beta x f\sqrt{1-x^2}} .
\end{aligned} \tag{4.2.13}$$

Such supersymmetric values of (σ, D) provide a non-trivial classical contribution to the partition function of the corresponding gauge theory thanks to the presence of (possibly mixed) Chern-Simons and Fayet–Iliopoulos terms, for example. Furthermore, the profiles $(\sigma|_{\text{BPS}}, D|_{\text{BPS}})$ implicitly affect the one-loop determinant of supersymmetric fluctuations via the gauge fugacity φ_G and the flux $\mathfrak{f}_G$.

Analogously, the BPS locus for the chiral multiplet is found by setting to zero the spinor fields $(\psi, \widetilde{\psi})$ together with their supersymmetric variations $(\delta\psi, \delta\widetilde{\psi})$. The outcome of this procedure is the following set of BPS equations:

$$\begin{aligned}
&(\mathscr{L}_K - \mathrm{i}r\Phi_R - \mathrm{i}q_G\Phi_G)\phi = 0 , \qquad (\mathscr{L}_K + \mathrm{i}r\Phi_R + \mathrm{i}q_G\Phi_G)\widetilde{\phi} = 0 , \\
&F + \mathrm{i}L_{\widetilde{P}}\phi = 0 , \qquad \widetilde{F} + \mathrm{i}L_P\widetilde{\phi} = 0 .
\end{aligned} \tag{4.2.14}$$

For arbitrary values of (Φ_R, Φ_G) the unique solution to the chiral-multiplet BPS equations is

$$\phi|_{\rm BPS} = \widetilde{\phi}|_{\rm BPS} = F|_{\rm BPS} = \widetilde{F}|_{\rm BPS} = 0\ . \tag{4.2.15}$$

As a consequence, we do not expect classical contributions to the partition function to come from F-terms or superpotentials, for instance. In this setup, matter fields affect the theory at the quantum level only, via one-loop determinants.

4.2.2 1-loop determinants

The chiral-multiplet one-loop determinant entering the partition function can be obtained by exploiting the cancellations between bosonic and fermionic degrees of freedom due to supersymmetry. This procedure is implemented by the formula

$$Z^{\rm CM}_{\text{1-L}} = \frac{\det_{{\rm Ker} L_P} \delta^2}{\det_{{\rm Ker} L_{\widetilde{P}}} \delta^2} = \frac{\det_{{\rm Ker} L_P} (L_K + \mathscr{G}_{\Phi_G})}{\det_{{\rm Ker} L_{\widetilde{P}}} (L_K + \mathscr{G}_{\Phi_G})}\ . \tag{4.2.16}$$

Indeed, the operators L_P and $L_{\widetilde{P}}$ pair bosonic and fermionic fields in the supersymmetry transformations written in cohomological form. The formula above follows from the fact that the squared supersymmetry variation $\delta^2 \propto (L_K + \mathscr{G}_{\Phi_G})$ commutes with such pairing operators. Thus, the unpaired eigenvalues surviving the cancellations and contributing to $Z^{\rm CM}_{\text{1-L}}$ are those corresponding to the eigenfunctions spanning the kernels of L_P and $L_{\widetilde{P}}$.

The 1-loop determinant of a vector multiplet, $Z^{\rm VM}_{\text{1-L}}$, includes the contribution of BRST-ghosts compatible with supersymmetry [47]. However, a standard argument implies that formally $Z^{\rm VM}_{\text{1-L}} = Z^{\rm CM}_{\text{1-L}}(r=2)|_{\mathfrak{R}_{\rm G} = {\rm Ad}_G}$ [28], so in the following we shall focus on chiral multiplets.

One-loop determinants of supersymmetric partition functions can be obtained from equivariant index theory [47]. For instance, in the case of a chiral multiplet on a manifold $\mathscr{M}$ and of charge r with respect to the $U(1)_R$ R-symmetry and charge q_G with respect to a gauge or flavour group $U(1)_G$, the eigenvalues contributing to $Z^{\rm CM}_{\text{1-L}}$

are encoded by the formula

$$\mathrm{ind}\big(L_{\widetilde{P}};\widehat{g}\big) = \sum_{p\in\mathscr{M}^{\widehat{g}}} \frac{\mathrm{tr}_{\Gamma_0}\widehat{g} - \mathrm{tr}_{\Gamma_1}\widehat{g}}{\det(1-J_p)} \ , \tag{4.2.17}$$

which is the index of the operator $L_{\widetilde{P}}$ with respect to the action of the group element $\widehat{g} = \exp\left(-\mathrm{i}\varepsilon\delta^2\right)$ induced by the square of the supersymmetry variation δ^2 and tuned by the equivariant parameter ε. The summation in (4.2.17) spans across the fixed-point set $\mathscr{M}^{\widehat{g}}$ of the transformation $\widehat{g}$ on the manifold $\mathscr{M}$. At each fixed point $p\in\mathscr{M}$, the group element $\widehat{g}$ induces an action on the manifold $\mathscr{M}$, thereby effecting a mapping from the coordinate z_p to z'_p. This transformation generates a non-trivial Jacobian denoted as $J_p = \partial z'_p/\partial z_p$, appearing in the denominator of (4.2.17). In addition, $\widehat{g}$ acts upon both Γ_0 and Γ_1, where Γ_0 represents the space of sections of the $U(1)_R\times U(1)_G$-valued line bundle L over $\mathscr{M}$, while Γ_1 is the image of Γ_0 under by $L_{\widetilde{P}}$. These combined contributions yield the numerator of (4.2.17). In the case of our interest $\mathscr{M} = \Sigma\times S^1$ and the neighborhoods $\mathscr{U}_p$ covering the fixed points $p\in\mathscr{M}^{\widehat{g}}$ are isomorphic to $\mathbb{C}/\mathbb{Z}_{n_p}$. In order for the computation to accomodate such new structures, the orbifold version of (4.2.17) is required. The fixed point formula in the framework of the equivariant orbifold index theorem is expressed as follows:

$$\mathrm{ind}_{\mathrm{orb}}\big(L_{\widetilde{P}};\widehat{g}\big) = \sum_{p\in\mathscr{M}^{\widehat{g}}} \frac{1}{n_p} \sum_{\widehat{w}\in\mathbb{Z}_{n_p}} \frac{\mathrm{tr}_{\Gamma_0}(\widehat{w}\,\widehat{g}) - \mathrm{tr}_{\Gamma_1}(\widehat{w}\,\widehat{g})}{\det(1-W_pJ_p)} \ , \tag{4.2.18}$$

where W_p is the Jacobian resulting from the action of $\widehat{w}$ on the coordinates z_p within $\mathscr{U}_p$. The elements $\widehat{w}\in\mathbb{Z}_p$ act on both sections and coordinates akin to the action of $\widehat{g}$, but their action is weighted by the roots of unity $w_p^\ell = e^{2\pi\mathrm{i}\,\ell/n_p}$, with $\ell = 0,\ldots,(n_p-1)$, rather than by the equivariant parameter ε. Specifically, on $\Sigma\times S^1$ we have

$$\begin{aligned} \delta^2 &= -2\mathrm{i}(L_K + \mathscr{G}_{\Phi_G}) = -2\mathrm{i}(\mathscr{L}_K - \mathrm{i}q_R\Phi_R - \mathrm{i}q_G\Phi_G) \ , \\ \mathscr{L}_K &= \left(\omega\partial_\varphi + \partial_\psi\right) \ , \end{aligned} \tag{4.2.19}$$

where (Φ_R, Φ_G) are to be evaluated on the BPS locus. The operator $\widehat{g}$ factorizes as $\widehat{g} = \widehat{g}_\Sigma \widehat{g}_{S^1}$, with

$$\begin{aligned} \widehat{g}_\Sigma &= \exp\left[-2t\left(\omega\partial_\varphi - \mathrm{i}q_R\Phi_R - \mathrm{i}q_G\Phi_G\right)\right] , \\ \widehat{g}_{S^1} &= \exp\left(-2t\partial_\psi\right) , \end{aligned} \tag{4.2.20}$$

being group actions corresponding to the spindle Σ and to the circle S^1, respectively. In particular, the fixed-point set $\mathscr{M}^{\widehat{g}_{S^1}}$ is empty as $\widehat{g}_{S^1}$ acts freely on S^1. Instead, $\mathscr{M}^{\widehat{g}_\Sigma}$ is non-trivial and it contains the poles of the spindle at $x = \pm 1$. The actions of $\widehat{g}_\Sigma$ and $\widehat{w}$ upon the complex coordinates defined in (4.1.19) and (4.1.21) is

$$\begin{aligned} \widehat{g}_\Sigma z_+ &= q_+ z_+ , & q_+ &= \exp\left(2\mathrm{i}\varepsilon\omega/n_+\right) , \\ \widehat{g}_\Sigma z_- &= q_-^{-1} z_- , & q_- &= \exp\left(2\mathrm{i}\varepsilon\omega/n_-\right) , \\ \widehat{w}_+ z_+ &= w_+ z_+ , & w_+ &= \exp\left(2\pi\mathrm{i}/n_+\right) , \\ \widehat{w}_- z_- &= w_-^{-1} z_- , & w_- &= \exp\left(2\pi\mathrm{i}/n_-\right) . \end{aligned} \tag{4.2.21}$$

The form of $L_{\widetilde{p}}$ as well as the action of $\widehat{g}_\Sigma$ and $\widehat{w}$ on sections in Γ_0 and Γ_1 exhibit slight differences for twisted and anti-twisted spindles. Consequently, we analyze these two cases separately. We keep the notation compact by defining the following quantities:

$$\begin{aligned} \mathfrak{p}_+ &= q_G p_+ - \sigma\frac{r}{2} , & \mathfrak{p}_- &= q_G p_- + \frac{r}{2} , \\ \mathfrak{b} &= 1 + \sigma\left\lfloor \sigma\frac{\mathfrak{p}_+}{n_+}\right\rfloor + \left\lfloor -\frac{\mathfrak{p}_-}{n_-}\right\rfloor , & \mathfrak{c} &= \frac{[\![-\mathfrak{p}_-]\!]_{n_-}}{n_-} - \sigma\frac{[\![\sigma\,\mathfrak{p}_+]\!]_{n_+}}{n_+} , \\ \gamma_R &= -\frac{\alpha_3}{2} + \frac{\omega}{4}\chi_{-\sigma} , & \gamma_G &= -\varphi_G + \frac{\omega}{2}\left(\frac{p_-}{n_-} + \frac{p_+}{n_+}\right) , \\ q &= e^{2\pi\mathrm{i}\omega} , & y &= q^{\mathfrak{c}/2} e^{2\pi\mathrm{i}(r\gamma_R + q_G\gamma_G)} , \end{aligned} \tag{4.2.22}$$

where by $[\![\bullet]\!]_\diamond$ we indicate the reminder of the integer division of $\bullet$ by $\diamond$. In particular, $(\mathfrak{b}-1)$ is the degree of the line orbibundle $\mathscr{O}(-\mathfrak{p}) = \mathscr{O}(-n_+\mathfrak{p}_- + n_-\mathfrak{p}_+) = \mathscr{O}(-(r/2)(n_+ + n_-) - q_G\mathfrak{m})$, with $\mathfrak{p} = (n_+\mathfrak{p}_- - n_-\mathfrak{p}_+)$. Also, ω, γ_G and γ_R are effective fugacities for angular momentum, gauge and R-symmetry, respectively. The objects written in (4.2.22) are valid for a theory invariant under a $U(1)$ gauge or flavour group. For a general, possibly non-Abelian, Lie group G, the substitutions

$$\mathfrak{m}_{\pm} \to \rho(\mathfrak{m}_{\pm}) , \qquad \gamma_G \to \rho(\gamma_G) , \tag{4.2.23}$$

are in order, with ρ being the weight of the representation $\mathfrak{R}_G$ that we defined previously.

Topological twist

In computing the contribution to the index formula (4.2.18) coming from the north and south pole of the spindle we work with 1-form fields whose components on Σ vanish at the origin of the patches $\mathscr{U}_+$ and $\mathscr{U}_-$. This makes the calculation simple as $L_{\widetilde{P}}$ becomes a pure differential operator, for instance. Especially, the component φ of the R-symmetry field vanishes at the origin of $\mathscr{U}_+$ and $\mathscr{U}_-$ if

$$\alpha_2|_{\mathscr{U}_+} = \frac{1}{n_+} , \qquad \alpha_2|_{\mathscr{U}_-} = -\frac{1}{n_-} . \tag{4.2.24}$$

Analogously, the component $\mathscr{A}_\varphi$ of the flavour field (4.2.10) vanishes at $x = \pm 1$ if

$$\beta_2|_{\mathscr{U}_+} = -\frac{p_+}{n_+} , \qquad \beta_2|_{\mathscr{U}_-} = -\frac{p_-}{n_-} , \tag{4.2.25}$$

in the patches $\mathscr{U}_+$ and $\mathscr{U}_-$, respectively. In this setting,

$$L_{\widetilde{P}}|_{\mathscr{U}_+} = -2\mathrm{i}\partial_+ , \qquad L_{\widetilde{P}}|_{\mathscr{U}_-} = 2\mathrm{i}\partial_- , \tag{4.2.26}$$

where $\overline{\partial}_\pm$ is the complex conjugate of $\partial_\pm = \partial_{z_\pm}$. In particular, the operators $L_{\widetilde{P}}|_{\mathscr{U}_+}$ and $L_{\widetilde{P}}|_{\mathscr{U}_-}$ are transversally elliptic with respect to the free action of $\widehat{g}_{S^1} \in U(1)$. Then, according to [64], the index of $L_{\widetilde{P}}|_{\mathscr{U}_+}$ on $\Sigma \times S^1$ is given by the weighted sum of the index on the quotient $\left(\Sigma \times S^1\right)/S^1 \cong \Sigma$ over all irreducible representations of $U(1)$, where the weights are the characters $\chi_{\mathfrak{R}_{U(1)}}(\widehat{g}_{S^1}) = \chi_n(\widehat{g}_{S^1}) = e^{-2\mathrm{i}\varepsilon n}$ with n being the label of the representative $\phi_n \in \Gamma_0$, where the latter is the space of sections of the line orbibundle

$$\mathrm{L} = \mathscr{O}(-\mathfrak{p}) = \mathscr{O}(-\mathfrak{p}) = \mathscr{O}(-n_+\mathfrak{p}_- + n_-\mathfrak{p}_+) = \mathscr{O}(-(r/2)(n_+ + n_-) - q_G\mathfrak{m}) . \tag{4.2.27}$$

The operator $\widehat{g}_\Sigma$ acts as

$$\widehat{g}_\Sigma|_{\mathscr{U}_+}\phi_n = \xi\, q_+^{-q_G p_+ + (r/2)}\phi_n = \xi\, q_+^{-\mathfrak{p}_+}\phi_n\ ,$$
$$\widehat{g}_\Sigma|_{\mathscr{U}_-}\phi_n = \xi\, q_-^{-q_G p_- - (r/2)}\phi_n = \xi\, q_-^{-\mathfrak{p}_-}\phi_n\ , \qquad \xi = e^{2\mathrm{i}\varepsilon[(r/2)\alpha_3 + q_G\varphi_G]}\ . \tag{4.2.28}$$

Moreover, the action of $\widehat{g}_\Sigma$ on sections in the image Γ_1 reads

$$\widehat{g}_\Sigma|_{\mathscr{U}_+}\partial_+\phi_n = \xi\, q_+^{-\mathfrak{p}_+ -1}\phi_n\ ,$$
$$\widehat{g}_\Sigma|_{\mathscr{U}_-}\partial_-\phi_n = \xi\, q_-^{-\mathfrak{p}_- +1}\phi_n\ . \tag{4.2.29}$$

Combining all these ingredients in the fixed point formula (4.2.18) yields

$$\mathrm{ind}_{\mathrm{orb}}\big(L_{\widetilde{P}};\widehat{g}\big) = \sum_{n\in\mathbb{Z}} \mathrm{ind}_{\mathrm{orb}}\big(L_{\widetilde{P}};\widehat{g}_\Sigma\big)\chi_n(\widehat{g}_{S^1})\xi\ ,$$
$$\mathrm{ind}_{\mathrm{orb}}\big(L_{\widetilde{P}};\widehat{g}_\Sigma\big) = \frac{1}{n_+}\sum_{j=0}^{n_+-1}\frac{w_+^{-j\mathfrak{p}_+}q_+^{-\mathfrak{p}_+}}{1-w_+^j q_+} + \frac{1}{n_-}\sum_{j=0}^{n_--1}\frac{w_-^{-j\mathfrak{p}_-}q_-^{-\mathfrak{p}_-}}{1-w_-^{-j}q_-^{-1}}\ , \tag{4.2.30}$$

where we factored out the flavour fugacity ξ. The identities

$$\frac{1}{n_p}\sum_{j=0}^{n_p-1}\frac{w_p^{j\nu_p}}{1-w_p^j q_p} = \frac{1}{n_p}\sum_{j=0}^{n_p-1}\frac{w_p^{-j\nu_p}}{1-w_p^{-j} q_p} = \frac{q_p^{[\![-\nu_p]\!]_{n_p}}}{1-q_p^{n_p}}\ , \qquad \frac{\bullet}{n_p} = \left\lfloor\frac{\bullet}{n_p}\right\rfloor + \frac{[\![\bullet]\!]_{n_p}}{n_p}\ , \tag{4.2.31}$$

imply

$$\mathrm{ind}_{\mathrm{orb}}\big(L_{\widetilde{P}};\widehat{g}\big) = \sum_{n\in\mathbb{Z}} \mathrm{ind}_{\mathrm{orb}}\big(L_{\widetilde{P}};\widehat{g}_\Sigma\big)\chi_n(\widehat{g}_{S^1})\xi\ ,$$
$$\mathrm{ind}_{\mathrm{orb}}\big(L_{\widetilde{P}};\widehat{g}_\Sigma\big) = \frac{\mathrm{q}^{-\lfloor\mathfrak{p}_+/n_+\rfloor}}{1-\mathrm{q}} + \frac{\mathrm{q}^{\lfloor-\mathfrak{p}_-/n_-\rfloor}}{1-\mathrm{q}^{-1}} = \frac{\mathrm{q}^{-\lfloor\mathfrak{p}_+/n_+\rfloor} - \mathrm{q}^{1+\lfloor-\mathfrak{p}_-/n_-\rfloor}}{1-\mathrm{q}}\ , \tag{4.2.32}$$

with $\mathrm{q} = q_+^{n_+} = q_-^{n_-}$. If

$$-\lfloor\mathfrak{p}_+/n_+\rfloor = 1 + \lfloor-\mathfrak{p}_-/n_-\rfloor \iff \mathfrak{b} = 0\ , \tag{4.2.33}$$

then $\operatorname{ind}_{\text{orb}}\left(L_{\widetilde{P}};\widehat{g}_{\mathbb{\Sigma}}\right)=0$, where $\mathfrak{b}=(1+\deg \mathrm{L})$ defined in (4.2.22) naturally appeared. Instead, if

$$-\lfloor \mathfrak{p}_+/n_+\rfloor < 1+\lfloor -\mathfrak{p}_-/n_-\rfloor \iff \mathfrak{b}\geq 1\ , \tag{4.2.34}$$

then

$$\operatorname{ind}_{\text{orb}}\left(L_{\widetilde{P}};\widehat{g}_{\mathbb{\Sigma}}\right)=\mathrm{q}^{-\lfloor \mathfrak{p}_+/n_+\rfloor}\frac{1-\mathrm{q}^{\mathfrak{b}}}{1-\mathrm{q}}=\mathrm{q}^{-\lfloor \mathfrak{p}_+/n_+\rfloor}\left(1+\mathrm{q}+\cdots+\mathrm{q}^{\mathfrak{b}-1}\right)\ , \tag{4.2.35}$$

where the overall factor $\mathrm{q}^{-\lfloor \mathfrak{p}_+/n_+\rfloor}$ is the *lift of the equivariant action* [65] at the north pole of the spindle. In fact, given that $L_{\widetilde{P}}$ equals the Dolbeault operator ∂ in the vicinity of any fixed point, $\operatorname{ind}_{\text{orb}}\left(L_{\widetilde{P}};\widehat{g}_{\mathbb{\Sigma}}\right)$ represents the equivariant orbifold index of ∂ on $\mathbb{\Sigma}$ in presence of the line bundle L. This index assigns a monomial q^{ℓ} to each element within the basis of anti-holomorphic sections of L in the kernel of ∂, where $\ell\in\mathbb{N}$, $\mathrm{q}^0=1$ corresponds to the constant section and the polynomial's degree is $(\mathfrak{b}-1)=\deg\mathrm{L}$. This generalizes what occurs in the case of the equivariant index of the Dolbeault operator on manifolds, where, modulo a sign, the degree of a line bundle equals the first Chern class of its representatives. Especially, in the non-equivariant limit $\mathrm{q}\to 1$ the index becomes

$$\operatorname{ind}_{\text{orb}}\left(L_{\widetilde{P}}\right)=\operatorname{ind}_{\text{orb}}(\partial)=\mathfrak{b}=1+\deg\mathrm{L}\ , \tag{4.2.36}$$

reproducing the Riemann-Roch-Kawasaki theorem for orbifolds of genus zero [66]. We can rewrite the index as

$$\begin{aligned}\operatorname{ind}_{\text{orb}}\left(L_{\widetilde{P}};\widehat{g}\right)|_{\mathfrak{b}\geq 1}&=\sum_{n\in\mathbb{Z}}e^{-2\mathrm{i}\varepsilon n}e^{2\mathrm{i}\varepsilon[(r/2)\alpha_3+q_G\varphi_G]}\sum_{\ell=-\lfloor \mathfrak{p}_+/n_+\rfloor}^{\lfloor -\mathfrak{p}_-/n_-\rfloor}\mathrm{q}^{\ell}\ ,\\ &=\sum_{n\in\mathbb{Z}}\sum_{\ell=-\lfloor \mathfrak{p}_+/n_+\rfloor}^{\lfloor -\mathfrak{p}_-/n_-\rfloor}e^{2\mathrm{i}\varepsilon[-n+(r/2)\alpha_3+q_G\varphi_G+\omega\ell]}\ ,\end{aligned} \tag{4.2.37}$$

and then exploiting the rule

$$\operatorname{ind}\left(\mathscr{D};e^{-\mathrm{i}\varepsilon\delta^2}\right)=\sum_j d_j e^{-\mathrm{i}\varepsilon\lambda_j}\qquad\to\qquad Z_{\text{ind}}=\prod_j\lambda_j^{-d_j}\ , \tag{4.2.38}$$

relating the index of a differential operator $\mathscr{D}$ to the determinant Z_{ind}, with j being a multi-index, λ_j the j-th eigenvalue of δ^2 and d_j the degeneracy of λ_j. Applying the prescription (4.2.38) to $\mathrm{ind}_{\mathrm{orb}}\big(L_{\widetilde{P}};\widehat{g}\big)|_{\mathfrak{b}\geq 1}$ provides the infinite product

$$\begin{aligned}
Z_{\mathrm{ind}}|_{\mathfrak{b}\geq 1} &= \prod_{n\in\mathbb{Z}} \prod_{\ell=-\lfloor \mathfrak{p}_+/n_+\rfloor}^{\lfloor -\mathfrak{p}_-/n_-\rfloor} \left[2(n-(r/2)\alpha_3 - q_G\varphi_G - \omega\ell)\right]^{-1} , \\
&= \prod_{n\in\mathbb{Z}} \prod_{j=-\lfloor -\mathfrak{p}_-/n_-\rfloor}^{\lfloor \mathfrak{p}_+/n_+\rfloor} \left[2\Big(n+\omega j - \frac{r}{2}\alpha_3 - q_G\varphi_G\Big)\right]^{-1} , \\
&= \prod_{n\in\mathbb{Z}} \prod_{j=0}^{\mathfrak{b}-1} \left[2\left(n+\omega\left(j+\frac{1-\mathfrak{b}+\mathfrak{c}}{2}\right) + r\gamma_R + q_G\gamma_G\right)\right]^{-1} , \\
&= \prod_{n\in\mathbb{Z}} \prod_{j\in\mathbb{N}} \frac{n+\omega\big(j+\frac{1+\mathfrak{b}+\mathfrak{c}}{2}\big) + r\gamma_R + q_G\gamma_G}{n+\omega\big(j+\frac{1-\mathfrak{b}+\mathfrak{c}}{2}\big) + r\gamma_R + q_G\gamma_G} = \frac{\big(-y^{-1}\big)^{\mathfrak{b}/2}}{\big(q^{(1-\mathfrak{b})/2}y^{-1};q\big)_{\mathfrak{b}}} .
\end{aligned} \tag{4.2.39}$$

Finally, if

$$-\lfloor \mathfrak{p}_+/n_+\rfloor > 1 + \lfloor -\mathfrak{p}_-/n_-\rfloor \iff \mathfrak{b}\leq -1 , \tag{4.2.40}$$

then

$$\begin{aligned}
\mathrm{ind}_{\mathrm{orb}}\big(L_{\widetilde{P}};\widehat{g}_{\Sigma}\big) &= \mathrm{q}^{\lfloor -\mathfrak{p}_-/n_-\rfloor}\frac{\mathrm{q}^{1-\mathfrak{b}}-\mathrm{q}}{1-\mathrm{q}} = -\mathrm{q}^{1+\lfloor -\mathfrak{p}_-/n_-\rfloor}\Big(1+\mathrm{q}^2+\cdots+\mathrm{q}^{-\mathfrak{b}-1}\Big) , \\
&= -\mathrm{q}^{-\lfloor \mathfrak{p}'_-/n_-\rfloor}\Big(1+\mathrm{q}^2+\cdots+\mathrm{q}^{\deg \mathrm{L}'}\Big) ,
\end{aligned} \tag{4.2.41}$$

with

$$\begin{aligned}
\mathfrak{p}'_+ &= \mathfrak{p}_+ + 1 = q_G p_+ - \frac{r-2}{2} , \\
\mathfrak{p}'_- &= \mathfrak{p}_- - 1 = q_G p_- + \frac{r-2}{2} , \\
\mathrm{L}' &= \mathscr{O}\big(-n_+\mathfrak{p}'_- + n_-\mathfrak{p}'_+\big) = \mathscr{O}\left[q_G\mathfrak{m} + \frac{r-2}{2}(n_+ + n_-)\right] , \\
-(\mathfrak{b}+1) &= \left\lfloor -\frac{\mathfrak{p}'_+}{n_+}\right\rfloor + \left\lfloor \frac{\mathfrak{p}'_-}{n_-}\right\rfloor = \deg \mathrm{L}' .
\end{aligned} \tag{4.2.42}$$

The overall factor $\mathrm{q}^{\lfloor \mathfrak{p}'_-/n_- \rfloor}$ is the lift of the equivariant action at the south pole of $\mathbb{\Sigma}$. Indeed, by writing

$$\mathrm{ind}_{\mathrm{orb}}\big(L_{\widehat{p}};\widehat{g}\big)|_{\mathfrak{b}\le -1} = -\sum_{n\in\mathbb{Z}} \sum_{\ell=1+\lfloor -\mathfrak{p}_-/n_- \rfloor}^{-1-\lfloor \mathfrak{p}_+/n_+ \rfloor} e^{2\mathrm{i}\varepsilon[-n+(r/2)\alpha_3+q_G\varphi_G+\omega\ell]} , \tag{4.2.43}$$

and applying (4.2.38), we obtain

$$\begin{aligned} Z_{\mathrm{ind}}|_{\mathfrak{b}\le -1} &= \prod_{n\in\mathbb{Z}} \prod_{\ell=1+\lfloor -\mathfrak{p}_-/n_- \rfloor}^{-1-\lfloor \mathfrak{p}_+/n_+ \rfloor} 2\Big(n - \frac{r}{2}\alpha_3 - q_G\varphi_G - \omega\ell\Big) , \\ &= \prod_{n\in\mathbb{Z}} \prod_{j=0}^{-1-\mathfrak{b}} \left[n + \omega\left(j + \frac{1+\mathfrak{b}+\mathfrak{c}}{2}\right) + r\gamma_R + q_G\gamma_G \right] , \\ &= \prod_{n\in\mathbb{Z}} \prod_{j\in\mathbb{N}} \frac{n+\omega\big(j+\frac{1+\mathfrak{b}+\mathfrak{c}}{2}\big)+r\gamma_R+q_G\gamma_G}{n+\omega\big(j+\frac{1-\mathfrak{b}+\mathfrak{c}}{2}\big)+r\gamma_R+q_G\gamma_G} = \frac{\big(-y^{-1}\big)^{\mathfrak{b}/2}}{\big(q^{(1-\mathfrak{b})/2}y^{-1};q\big)_{\mathfrak{b}}} . \end{aligned} \tag{4.2.44}$$

We then conclude that, on topologically twisted $\mathbb{\Sigma}\times S^1$, the one-loop determinant for a chiral multiplet with weight ρ in a representation $\mathfrak{R}_G$ of a gauge group G is

$$Z^{\mathrm{CM}}_{\text{1-L}} = \prod_{\rho\in\mathfrak{R}_G} \frac{\big(-y^{\rho}\big)^{\mathfrak{b}/2}}{\big(q^{(1-\mathfrak{b})/2}y^{\rho};q\big)_{\mathfrak{b}}} = \prod_{\rho\in\mathfrak{R}_G} \frac{\big(-y^{-\rho}\big)^{\mathfrak{b}/2}}{\big(q^{(1-\mathfrak{b})/2}y^{-\rho};q\big)_{\mathfrak{b}}} . \tag{4.2.45}$$

The property $(\bullet;q)_0 = 1$ enjoyed by the finite q-Pochhammer symbol ensures that $Z^{\mathrm{CM}}_{\text{1-L}}|_{\mathfrak{b}=0} = 1$, as anticipated. In the degenerate case $n_+ = n_- = 1$ the spindle becomes a smooth sphere and the simplifications

$$\begin{aligned} &\mathfrak{b} = 1 - q_G\mathfrak{m} - r = 1 - q_G\mathfrak{f}_G - r\mathfrak{f}_R , && \mathfrak{c} = 0 , \\ &2\gamma_R = -\alpha_3 \in \mathbb{Z} , && y = (-1)^{r\alpha_3} e^{2\pi\mathrm{i} q_G\gamma_G} , \end{aligned} \tag{4.2.46}$$

occurr. In turn, (4.2.45) becomes identical to the one-loop determinant for a chiral multiplet on topologically twisted $S^2\times S^1$ computed in [28], where $\alpha_3 = 0$ was assumed.

The one-loop determinant (4.2.45) can also be interpreted as the partition function of a weakly gauged chiral multiplet, namely a chiral multiplet coupled to a non-dynamical background vector multiplet. Such a partition function is not invariant under large gauge transformations shifting the effective fugacity γ_G by an integer n:

$$y \to e^{2\pi \mathrm{i} n} y \qquad \Longrightarrow \qquad Z^{\mathrm{CM}}_{\text{1-L}} \to (-1)^{n\mathfrak{b}} Z^{\mathrm{CM}}_{\text{1-L}} \ . \tag{4.2.47}$$

This multi-valuedness of $Z^{\mathrm{CM}}_{\text{1-L}}$ is usually associated to a parity anomaly affecting the matter sector of the theory. Such an anomaly can be cured by considering the classical contribution $Z^{\mathrm{CS}}_{\mathrm{eff}}$ of an effective Chern-Simons term with half-integer level, whose anomaly cancels that of $Z^{\mathrm{CM}}_{\text{1-L}}$. In the case of $S^2 \times S^1$ the counterterm needed is $Z^{\mathrm{CS}}_{\mathrm{eff}} = y^{\pm \mathfrak{f}_G/2} = y^{\pm \mathfrak{m}/2}$, while for $Z^{\mathrm{CM}}_{\text{1-L}}$ on $\mathbb{\Sigma} \times S^1$ it is

$$Z^{\mathrm{CS}}_{\mathrm{eff}} = (-y)^{\pm \mathfrak{b}/2} \ . \tag{4.2.48}$$

We discuss the differences between effective and canonical Chern-Simons terms in Section 4.2.3, where we compute the relevant classical contributions to the partition function of a gauge theory on $\mathbb{\Sigma} \times S^1$.

Anti-twist

In the case of anti-twisted $\mathbb{\Sigma} \times S^1$ the φ-component of the R-symmetry background field vanishes at the poles of the spindle if

$$\alpha_2|_{\mathscr{U}_+} = -\frac{1}{n_+} \ , \qquad\qquad \alpha_2|_{\mathscr{U}_-} = -\frac{1}{n_-} \ , \tag{4.2.49}$$

while flavour field (4.2.10) behaves as in the case of the topological twist. The pairing operator $L_{\widetilde{P}}$ is again transversally elliptic with respect to the free action of $\widehat{g}_{S^1} \in U(1)$, and near the poles of $\mathbb{\Sigma}$ assumes the value

$$L_{\widetilde{P}}|_{\mathscr{U}_+} = -2\mathrm{i}\overline{\partial}_+ \ , \qquad\qquad L_{\widetilde{P}}|_{\mathscr{U}_-} = 2\mathrm{i}\partial_- \ . \tag{4.2.50}$$

Repeating the methodology employed in the case of the topologically twisted $\mathbb{\Sigma} \times S^1$ yields

$$\begin{aligned} \mathrm{ind}_{\mathrm{orb}}\big(L_{\widetilde{P}};\widehat{g}\big) &= \sum_{n\in\mathbb{Z}} \mathrm{ind}_{\mathrm{orb}}\big(L_{\widetilde{P}};\widehat{g}_{\mathbb{\Sigma}}\big)\,\chi_n(\widehat{g}_{S^1})\,\xi \ , \\ \mathrm{ind}_{\mathrm{orb}}\big(L_{\widetilde{P}};\widehat{g}_{\mathbb{\Sigma}}\big) &= \frac{\mathrm{q}^{\lfloor -\mathfrak{p}_+/n_+ \rfloor}}{1-\mathrm{q}^{-1}} + \frac{\mathrm{q}^{\lfloor -\mathfrak{p}_-/n_- \rfloor}}{1-\mathrm{q}^{-1}} = \frac{\mathrm{q}^{\lfloor -\mathfrak{p}_+/n_+ \rfloor}}{1-\mathrm{q}^{-1}} - \frac{\mathrm{q}^{1+\lfloor -\mathfrak{p}_-/n_- \rfloor}}{1-\mathrm{q}} \ . \end{aligned} \tag{4.2.51}$$

Expanding $\mathrm{ind}_{\mathrm{orb}}\big(L_{\widetilde{P}};\widehat{g}\big)$ in powers of q gives

$$\mathrm{ind}_{\mathrm{orb}}\big(L_{\widetilde{P}};\widehat{g}\big)=\sum_{n\in\mathbb{Z}} e^{2\mathrm{i}\varepsilon[-n+(r/2)\alpha_3+q_G\varphi_G]}\sum_{\ell\in\mathbb{N}}\left(\mathrm{q}^{-\ell+\lfloor-\mathfrak{p}_+/n_+\rfloor}-\mathrm{q}^{\ell+1+\lfloor-\mathfrak{p}_-/n_-\rfloor}\right), \tag{4.2.52}$$

which, according to the rule (4.2.38), corresponds to the infinite product

$$\begin{aligned} Z_{\mathrm{ind}} &= \prod_{n\in\mathbb{Z}}\prod_{\ell\in\mathbb{N}}\frac{-n+(r-2)(\alpha_3/2)+q_G\varphi_G+\omega(\ell+1+\lfloor-\mathfrak{p}_-/n_-\rfloor)}{-n+(r/2)\alpha_3+q_G\varphi_G+\omega(-\ell+\lfloor-\mathfrak{p}_+/n_+\rfloor)}\,, \\ &= \prod_{n\in\mathbb{Z}}\prod_{\ell\in\mathbb{N}}\frac{-n+\omega\big(\ell+\frac{\mathfrak{b}+1-\mathfrak{c}}{2}\big)-r\gamma_R-q_G\gamma_G-\alpha_3}{-n+\omega\big(\ell+\frac{\mathfrak{b}-1+\mathfrak{c}}{2}\big)+r\gamma_R+q_G\gamma_G}\,, \\ &= \big(-y^{-1}\sqrt{q}\big)^{\frac{1}{2}(\mathfrak{b}-1)}\frac{\Big(q^{\frac{1}{2}(\mathfrak{b}+1)}y^{-1};q\Big)_\infty}{\Big(q^{\frac{1}{2}(\mathfrak{b}-1)}y;q\Big)_\infty}\,. \end{aligned} \tag{4.2.53}$$

Finally, the identities enjoyed by q-Pochhammer symbols allows for the derivation of a unique expression for $Z^{\mathrm{CM}}_{\text{1-L}}$ valid for both topologically twisted and anti-twisted $\Sigma\times S^1$:

$$Z^{\mathrm{CM}}_{\text{1-L}}=\prod_{\rho\in\mathfrak{R}_G}\big(-y^{-\rho}\big)^{\frac{1}{4}(2\mathfrak{b}-1+\sigma)}q^{\frac{1}{8}(1-\sigma)(\mathfrak{b}-1)}\frac{\Big(q^{\frac{1}{2}(\mathfrak{b}+1)}y^{-\rho};q\Big)_\infty}{\Big(q^{\frac{\sigma}{2}(\mathfrak{b}-1)}y^{-\sigma\rho};q\Big)_\infty}\,. \tag{4.2.54}$$

Once we have the 1-loop determinant for the chiral multiplet, we can exploit the relation $Z^{\mathrm{VM}}_{\text{1-L}}=Z^{\mathrm{CM}}_{\text{1-L}}(r=2)|_{\mathfrak{R}_\mathrm{G}=\mathrm{Ad}_G}$ to obtain the 1-loop determinant of the vector multiplet. The fields of the vector multiplet transform in the adjoint representation of the gauge group. After the root decomposition, the 1-loop determinant of an adjoint vector multiplet turns out to be manifestly independent of the R-symmetry twist and, up to a regularization-dependent sign, it reads:

$$\begin{aligned} Z^{\mathrm{VM}}_{\text{1-L}} &= \prod_{\alpha>0}\prod_{I=\pm}\left(\mathrm{z}^{-\alpha/2}-q^{\frac{\alpha_+}{2n_+}+\frac{\alpha_-}{2n_-}-\left\lfloor\frac{\alpha_I}{n_I}\right\rfloor}\mathrm{z}^{\alpha/2}\right)^{\mu_I} \\ &\quad\times q^{\frac{1}{8}(\mu_--\mu_+)\alpha(\mathfrak{f}_G)}\,, \end{aligned} \tag{4.2.55}$$

where $\alpha_\pm=\alpha(m_\pm)$, $\mathrm{z}^\alpha=\mathrm{e}^{2\pi\mathrm{i}\alpha(u)}$, α is the weight of the adjoint representation and $\mu_I\equiv 1+\lfloor\alpha_I/n_I\rfloor+\lfloor-\alpha_I/n_I\rfloor$.

4.2.3 Canonical and effective classical terms

General supersymmetric gauge theories include Chern-Simons and Fayet-Iliopoulos terms. The canonical Chern-Simons action at level k for a non-Abelian vector multiplet in the adjoint representation of a gauge group G is

$$S_{\text{CS}} = \frac{\mathrm{i}\,\mathrm{k}}{4\pi}\int \mathrm{d}^3x\,\mathrm{e}\,\mathrm{tr}\left[\varepsilon^{\mu\nu\rho}\left(\mathscr{A}_\mu\partial_\nu\mathscr{A}_\rho - \frac{2\mathrm{i}}{3}\mathscr{A}_\mu\mathscr{A}_\nu\mathscr{A}_\rho\right) + 2\mathrm{i}D\sigma + 2\widetilde{\lambda}\lambda\right]\ , \quad \mathrm{e} = \varepsilon_{x\varphi\psi}\ , \tag{4.2.56}$$

which, once evaluated on the BPS locus, contributes to the classical part of the partition function as follows:

$$Z^{\text{CS}}_{\text{class}}|_{\text{BPS}} = e^{-S_{\text{CS}}}|_{\text{BPS}} = e^{2\pi\mathrm{i}\mathrm{k}\gamma_G\mathfrak{f}_G} = e^{2\pi\mathrm{i}\,\mathrm{k}\,\gamma_G\,\mathfrak{m}/(n_+n_-)}\ . \tag{4.2.57}$$

The latter can be generalized to a mixed Chern-Simons term at level k_{ij} that combines two vector multiplets in the adjoint of different gauge groups $G_{(i)}$ and $G_{(j)}$ with $i \neq j$:

$$S_{\text{MCS}} = \frac{\mathrm{i}\mathrm{k}_{ij}}{2\pi}\int \mathrm{d}^3x\,e\left(\varepsilon^{\mu\nu\rho}\mathscr{A}^{(i)}_\mu\partial_\nu\mathscr{A}^{(j)}_\rho + \mathrm{i}D^{(i)}\sigma^{(j)} + \mathrm{i}D^{(j)}\sigma^{(i)} + \widetilde{\lambda}^{(i)}\lambda^{(j)} + \widetilde{\lambda}^{(j)}\lambda^{(i)}\right)\ , \tag{4.2.58}$$

yielding

$$Z^{\text{MCS}}_{\text{class}}|_{\text{BPS}} = e^{2\pi\mathrm{i}\mathrm{k}_{ij}\left(\gamma_{G_i}\mathfrak{f}_{G_j} + \gamma_{G_j}\mathfrak{f}_{G_i}\right)}\ . \tag{4.2.59}$$

The gauge symmetry can also be mixed with the R-symmetry via an R-symmetry-gauge Chern-Simons term, whose action reads [45]

$$S_{\text{RCS}} = \frac{\mathrm{i}\mathrm{k}_R}{2\pi}\int \mathrm{d}^3x\,e\left[\varepsilon^{\mu\nu\lambda}\mathscr{A}_\mu\partial_\nu\left(A_\lambda - \frac{1}{2}V_\lambda\right) + \mathrm{i}D\sigma + \frac{\mathrm{i}\,\sigma}{4}\left(R + 2V_\mu V^\mu + 2H^2\right)\right]\ , \tag{4.2.60}$$

giving

$$Z^{\text{RCS}}_{\text{class}}|_{\text{BPS}} = e^{2\pi\mathrm{i}\mathrm{k}_R(\gamma_R\mathfrak{f}_G + \gamma_G\mathfrak{f}_R)}\ , \tag{4.2.61}$$

which depends on the twist through the R-symmetry fugacity γ_R defined in (4.2.22) and the flux $\mathfrak{f}_R = \chi_\pm/2$. Finally, for any Abelian factor in a gauge group G, a classical contribution to the partition function can also descend from Fayet-Iliopoulos terms with parameter ζ_{FI} and action

$$S_{\text{FI}} = \frac{\zeta_{\text{FI}}}{2\pi} \int \mathrm{d}^3 x\, e \big(D - \mathscr{A}_\mu V^\mu - \sigma H \big) \ . \tag{4.2.62}$$

In fact, thanks to the relation [45]

$$V^\mu = -\mathrm{i} \varepsilon^{\mu\nu\lambda} \partial_\nu \mathscr{C}_\lambda \ , \tag{4.2.63}$$

the Fayet-Iliopoulos term can be interpreted as a mixed Chern-Simons term coupling fields in an Abelian vector multiplet to the background scalar H and the graviphoton field $\mathscr{C}_\mu$. Hence,

$$Z^{\text{FI}}_{\text{class}}|_{\text{BPS}} = e^{2\pi\mathrm{i}\, \zeta_{\text{FI}} (\gamma_{\mathscr{C}} \mathfrak{f}_G + \gamma_G \mathfrak{f}_{\mathscr{C}})} \ , \tag{4.2.64}$$

where $\gamma_{\mathscr{C}}$ and $\mathfrak{f}_{\mathscr{C}}$ are the fugacity and the flux associated to $\mathscr{C}_\mu$, respectively.

In general, the classical contributions obtained from evaluating on the BPS locus the canonical actions above are not invariant under large gauge transformations along S^1. For example, under a large gauge transformation acting as $\gamma_G \to \gamma_G + \ell$, with $\ell \in \mathbb{N}$, the classical part of the canonical Chern-Simons partition function term behaves as follows:

$$Z^{\text{CS}}_{\text{class}}|_{\text{BPS}} = e^{2\pi\mathrm{i}\,\mathrm{k}\,\gamma_G \mathfrak{m}/(n_+ n_-)} \qquad \to \qquad \widetilde{Z}^{\text{CS}}_{\text{class}}|_{\text{BPS}} = e^{2\pi\mathrm{i}\,\mathrm{k}\ell\mathfrak{m}/(n_+ n_-)} Z^{\text{CS}}_{\text{class}}|_{\text{BPS}} \ . \tag{4.2.65}$$

In the case in which the spindle is a sphere, then $n_+ = n_- = 1$ and the partition function exhibits invariance under large gauge transformations as $\widetilde{Z}^{\text{CS}}_{\text{class}}|_{\text{BPS}} = Z^{\text{CS}}_{\text{class}}|_{\text{BPS}}$. Conversely, for a general spindle $\mathbb{\Sigma}$, the parameters (n_+, n_-) are arbitrary coprime integers and $\widetilde{Z}^{\text{CS}}_{\text{class}}|_{\text{BPS}} \neq Z^{\text{CS}}_{\text{class}}|_{\text{BPS}}$ unless either the Chern-Simons level k or $\mathfrak{m}$ is a multiple of $(n_+ n_-)$.

A similar circumstance arises in supersymmetric theories on the manifold with boundary $D^2 \times S^1$, where D^2 represents a two-dimensional disk or hemisphere. The canonical Chern-Simons term lacks supersymmetry and gauge invariance on $D^2 \times S^1$. Restoring these properties requires taking into account degrees of freedom localized at the boundary $\partial\left(D^2 \times S^1\right) = T^2$, with $T^2 = S^1 \times S^1$ being a two-dimensional torus. A convenient method for promptly deriving the correct expression for the effective Chern-Simons term on $D^2 \times S^1$ was elucidated in [67], where it was demonstrated that the partition function for a Chern-Simons term at level $\mathrm{k} = \pm 1$ equals the collective partition function of two anomaly-free chiral multiplets $\left(\phi_{(1)}, \phi_{(2)}\right)$ coupled by a superpotential $W\left(\phi_{(1)}, \phi_{(2)}\right) = m_0 \phi_{(1)} \phi_{(2)}$, where m_0 encodes a mass. Such a procedure allowed to show that the Chern-Simons partition function on $\partial\left(D^2 \times S^1\right) = T^2$ is given by the theta function $\theta(\mathbf{x};\mathbf{q}) = \left(-\sqrt{\mathbf{q}}/\mathbf{x};\mathbf{q}\right)_\infty\left(-\sqrt{\mathbf{q}}\,\mathbf{x};\mathbf{q}\right)_\infty$ depending on gauge and flavour fugacities $\mathbf{x}$, as well as on an equivariant parameter ε appearing in the fugacity $\mathbf{q} = e^{\varepsilon}$ for the angular momentum on D^2. In particular, gauge invariance of such an effective Chern-Simons term is ensured by the analytic properties of the Theta function.

We can apply the same procedure to the case of $\mathbb{\Sigma} \times S^1$. Two chiral multiplets $\left(\phi_{(1)}, \phi_{(2)}\right)$ coupled by the aforementioned superpotential $W\left(\phi_{(1)}, \phi_{(2)}\right) = m_0 \phi_{(1)} \phi_{(2)}$ have charges satisfying

$$r^{(1)} + r^{(2)} = 2 \,, \qquad\qquad q_G^{(1)} + q_G^{(2)} = 0 \,, \tag{4.2.66}$$

which, in the case of the twisted spindle[1], imply

$$\mathfrak{b}^{(1)} + \mathfrak{b}^{(2)} = 0 \,, \qquad \mathfrak{c}^{(1)} + \mathfrak{c}^{(2)} = -\chi_- \,, \qquad y_{(1)} y_{(2)} = 1 \,, \tag{4.2.67}$$

where $\left(\mathfrak{b}^{(i)}, \mathfrak{c}^{(i)}, y_{(i)}\right)$ are $(\mathfrak{b}, \mathfrak{c}, y)$ evaluated at $r = r^{(i)}$ and $q_G = q_G^{(i)}$ with $i = 1, 2$. In particular, $y_{(1)} y_{(2)} = 1$ follows from the constraint $\gamma_R - (\omega \chi_-/4) \in \mathbb{Z}$. As in [67], we consider anomaly-free chiral multiplets, whose partition function is obtained from $Z^{\mathrm{CM}}_{\text{1-L}}$ by removing the phase factors and considering the q-Pochhammer part only. For instance, in the case of the topologically twisted $\mathbb{\Sigma} \times S^1$, we have

$$\begin{aligned} Z^{(2)} &= \frac{1}{\left(q^{(1-\mathfrak{b}_2)/2} y_{(2)}^{-1}; q\right)_{\mathfrak{b}_2}} = \frac{1}{\left(q^{(1+\mathfrak{b}_1)/2} y_{(1)}; q\right)_{-\mathfrak{b}_1}} \,, \\ &= \left(-y_{(1)}\right)^{\mathfrak{b}_1} \left(q^{(1-\mathfrak{b}_1)/2} y_{(1)}^{-1}; q\right)_{\mathfrak{b}_1} = \left(-y_{(1)}\right)^{\mathfrak{b}_1} / Z^{(1)} \,, \end{aligned} \tag{4.2.68}$$

[1] Similar relations hold in the case of the anti-twisted spindle $\mathbb{\Sigma}$.

yielding

$$Z^{(1)}Z^{(2)} = \left(-y_{(1)}\right)^{\mathfrak{b}_1} . \tag{4.2.69}$$

Then, according to [67] the partition function of the effective Chern-Simons term at level $\mathrm{k} = \pm 1$ has the form $Z^{\text{CS}}_{\text{eff}} = y^{\pm\mathfrak{b}}$. Since we have fixed only the sum of the R-charges $\left(q^{(i)}_G, r^{(i)}\right)$ but not their specific values, $Z^{\text{CS}}_{\text{eff}}$ contains information on all global and gauge symmetries affecting the chirals $\left(\phi_{(1)}, \phi_{(2)}\right)$. Indeed, if $n_+ = n_- = 1$, the spindle $\mathbb{\Sigma}$ boils down to a two-sphere and

$$Z^{\text{CS}}_{\text{eff}}|_{n_+=n_-=1} = e^{2\pi \mathrm{i} q_G \gamma_G (1-r-q_G \mathfrak{m})} . \tag{4.2.70}$$

At arbitrary level k we have

$$Z^{\text{CS}}_{\text{eff}} = y^{\mathrm{k}\,\mathfrak{b}} , \tag{4.2.71}$$

More generally, an effective Chern-Simons term mixing two gauge groups $\left(G_i, G_j\right)$ reads

$$Z^{\text{MCS}}_{\text{eff}} = y_{(i)}^{\mathrm{k}_{ij}\,\mathfrak{b}_j} y_{(j)}^{\mathrm{k}_{ij}\,\mathfrak{b}_i} . \tag{4.2.72}$$

The partition function (4.2.71) is manifestly invariant under large gauge transformations:

$$Z^{\text{CS}}_{\text{eff}}|_{\text{BPS}} \qquad \rightarrow \qquad \widetilde{Z}^{\text{CS}}_{\text{eff}}|_{\text{BPS}} = e^{2\pi \mathrm{i}\,\mathrm{k}\ell\mathfrak{b}} Z^{\text{CS}}_{\text{eff}}|_{\text{BPS}} = Z^{\text{CS}}_{\text{eff}}|_{\text{BPS}} . \tag{4.2.73}$$

In presence of matter, the Chern-Simons level k is affected by non-trivial corrections: for instance, a chiral multiplet causes an half-integer shift $\mathrm{k} \rightarrow \mathrm{k}' = \mathrm{k} + (1/2)$, as we see from 4.2.54.

In [3], drawing an analogy from the case of $D^2 \times S^1$, we have conjectured that the modifications to the canonical Chern-Simons theory's stem from degrees of freedom localized at the conical singularities of the spindle $\mathbb{\Sigma}$.

4.3 Examples and dualities

The results presented in the previous sections can find some direct applications. In [3], they were used to calculate the vacua of an effective Chern-Simons theory and also to verify the validity of a well-known duality between two different SQFTs.

4.3.1 Effective $U(1)$ Chern-Simons theory at level k

As a first example let us calculate the effective partition function of an Abelian Chern-Simons theory with gauge group $U(1)$ at level k. The corresponding contour integral reads

$$Z = -\sum_{\mathfrak{m}\in\mathbb{Z}} \int_{\mathscr{C}} \frac{\mathrm{d}y}{2\pi \mathrm{i} y} y^{\mathrm{k}\mathfrak{b}} y^{\mathfrak{n}} z^{\mathfrak{b}} \;, \tag{4.3.1}$$

where y is the gauge fugacity, $\mathfrak{m}$ the label of the gauge flux $\mathfrak{f}_G = \mathfrak{m}/(n_+ n_-)$, while z and $\mathfrak{n}$ are background flux and fugacity, respectively. The integer $\mathfrak{b}$ is given by

$$\mathfrak{b} = \mathfrak{b}(\mathfrak{m}) = 1 + \left\lfloor \frac{p_+}{n_+} \right\rfloor + \left\lfloor -\frac{p_-}{n_-} \right\rfloor = 1 + \left\lfloor \frac{\mathfrak{m}\,\mathfrak{a}_+}{n_+} \right\rfloor + \left\lfloor -\frac{\mathfrak{m}\,\mathfrak{a}_-}{n_-} \right\rfloor \;, \quad n_+\mathfrak{a}_- - n_-\mathfrak{a}_+ = 1 \;. \tag{4.3.2}$$

We write $\mathfrak{m} \in \mathbb{Z}$ as

$$\mathfrak{m} = n_+ n_- \mathfrak{m}' + \mathfrak{r} \;, \qquad \mathfrak{r} = 0, \dots, (n_+ n_- - 1) \;, \tag{4.3.3}$$

with $\mathfrak{m}' \in \mathbb{Z}$. The integral then becomes

$$\begin{aligned} Z &= -\sum_{\mathfrak{m}\in\mathbb{Z}} \int_{\mathscr{C}} \frac{\mathrm{d}y}{2\pi \mathrm{i} y} y^{\mathrm{k}\mathfrak{b}+\mathfrak{n}} z^{\mathfrak{b}} \;, \\ &= -\sum_{\mathfrak{r}=0}^{n_+n_- -1} \sum_{\mathfrak{m}'\in\mathbb{Z}} \int_{\mathscr{C}} \frac{\mathrm{d}y}{2\pi \mathrm{i} y} y^{-\mathrm{k}\mathfrak{m}' + \mathrm{k}\mathfrak{b}(\mathfrak{r}) + \mathfrak{n}} z^{-\mathfrak{m}' + \mathfrak{b}(\mathfrak{r})} \;, \\ &= -\sum_{\mathfrak{r}=0}^{n_+n_- -1} \sum_{\mathfrak{m}''\in\mathbb{Z}} \int_{\mathscr{C}} \frac{\mathrm{d}y}{2\pi \mathrm{i} y} y^{\mathrm{k}\mathfrak{m}'' + \mathfrak{n}} z^{\mathfrak{m}''} \;. \end{aligned} \tag{4.3.4}$$

The latter is non-vanishing only if the contour $\mathscr{C}$ surrounding $y = 0$ and if

$$\mathrm{k}\mathfrak{m}'' + \mathfrak{n} = 0\ , \tag{4.3.5}$$

which is a Diophantine equation that can be solved by conveniently splitting $\mathfrak{n}$ as

$$\mathfrak{n} = \mathrm{k}\mathfrak{n}' + [\![\mathfrak{n}]\!]_{\mathrm{k}}\ . \tag{4.3.6}$$

Indeed, plugging the expression above into the Diophantine equation gives

$$\mathfrak{m}'' = -\mathfrak{n}'\ , \qquad\qquad [\![\mathfrak{n}]\!]_{\mathrm{k}} = 0\ , \tag{4.3.7}$$

which yields the final result

$$Z = n_+ n_-\, z^{-\mathfrak{n}'}\, \delta_{[\![\mathfrak{n}]\!]_{\mathrm{k}},0} = n_+ n_-\, z^{-\lfloor \mathfrak{n}/\mathrm{k} \rfloor}\, \delta_{[\![\mathfrak{n}]\!]_{\mathrm{k}},0}\ , \tag{4.3.8}$$

meaning that $Z \neq 0$ only if $[\![\mathfrak{n}]\!]_{\mathrm{k}} = 0$. Especially, if $\mathfrak{n} = 0$, we find $Z = (n_+ n_-)$, suggesting that the effective Chern-Simons theory on $\mathbb{\Sigma}$ has $(n_+ n_-)$ vacua. Technically, this degeneracy stems from the fact that for any $\overline{\mathfrak{b}} \in \mathbb{Z}$ there are $(n_+ n_-)$ values of $\mathfrak{m} \in \mathbb{Z}$ such that $\mathfrak{b}(\mathfrak{m}) = \overline{\mathfrak{b}}$.

4.3.2 Effective $U(1)$ Chern-Simons theory coupled to a chiral multiplet

As another example we compute the partition function of a chiral multiplet coupled to an effective Chern-Simons term with gauge group $U(1)$ at level $\mathrm{k} = 1$. We also add a mixed Chern-Simons term with flavour fugacity $(-z)$ and effective flux $\widetilde{\mathfrak{b}}$. We again consider a countour $\mathscr{C}$ such that all contributions to the integral comes from the pole at $y = 0$. The collective partition function is

$$Z = \sum_{\mathfrak{m} \in \mathbb{Z}} \int_{\mathscr{C}} \frac{\mathrm{d}y}{2\pi\mathrm{i}\, y} \frac{(-z)^{\mathfrak{b}} y^{\widetilde{\mathfrak{b}}} y^{\mathfrak{b}-1}}{\left(y q^{(1-\mathfrak{b})/2}; q\right)_{\mathfrak{b}}}\ , \tag{4.3.9}$$

where the finite q-Pochhammer symbol corresponds to a chiral multiplet on topologically twisted $\mathbb{\Sigma} \times S^1$, while $y^{\mathfrak{b}-1}$ is the spindle counterpart of $e^{2\pi\mathrm{i}\gamma_G \mathfrak{m}}$, which is the

Chern-Simons term on $S^2 \times S^1$. It is convenient to expand the integrand as

$$Z = \sum_{\mathfrak{m}\in\mathbb{Z}} \int \frac{\mathrm{d}y}{2\pi \mathrm{i}y} \frac{(-z)^{\mathfrak{b}} y^{\widetilde{\mathfrak{b}}+\mathfrak{b}-1}}{\left(y q^{(1-\mathfrak{b})/2};q\right)_{\mathfrak{b}}} = \sum_{\mathfrak{m}\in\mathbb{Z}} \sum_{\ell\in\mathbb{N}} \int \frac{\mathrm{d}y}{2\pi \mathrm{i}y} (-z)^{\mathfrak{b}} y^{\widetilde{\mathfrak{b}}+\mathfrak{b}-1+\ell} \frac{\left(q^{\mathfrak{b}};q\right)_\ell}{(q;q)_\ell} q^{\ell(1-\mathfrak{b})/2} \,, \tag{4.3.10}$$

where we employed the q-binomial theorem

$$\frac{1}{(t;q)_{\mathfrak{b}}} = \frac{\left(t q^{\mathfrak{b}};q\right)_\infty}{(t;q)_\infty} = \sum_{\ell\in\mathbb{N}} \frac{\left(q^{\mathfrak{b}};q\right)_\ell}{(q;q)_\ell} t^\ell \,, \tag{4.3.11}$$

valid for both finite and infinite q-Pochhammer symbols. The contour integral providing Z is non-vanishing if

$$\widetilde{\mathfrak{b}} + \mathfrak{b} - 1 + \ell = 0 \,, \tag{4.3.12}$$

which is an integer-valued equation that can be solved by writing $\mathfrak{m}$ as

$$\mathfrak{m} = n_+ n_- \mathfrak{m}' + \mathfrak{r} \,, \qquad \mathfrak{m}' \in \mathbb{Z} \qquad \mathfrak{r} = 0, \dots, (n_+ n_- - 1) \,, \tag{4.3.13}$$

yielding

$$\widetilde{\mathfrak{b}} - \mathfrak{m}' + \mathfrak{b}(\mathfrak{r}) - 1 + \ell = 0 \,. \tag{4.3.14}$$

Inserting the latter in the expression for Z gives

$$\begin{aligned} Z &= -\sum_{\mathfrak{r}=0}^{n_+n_--1} \sum_{\ell\in\mathbb{N}} (-z)^{1-\widetilde{\mathfrak{b}}-\ell} \frac{\left(q^{1-\widetilde{\mathfrak{b}}-\ell};q\right)_\ell}{(q;q)_\ell} q^{\ell(\mathfrak{n}+\ell)/2} \,, \\ &= -n_+n_- \sum_{\ell\in\mathbb{N}} (-z)^{1-\widetilde{\mathfrak{b}}-\ell} \frac{\left(q^{\widetilde{\mathfrak{b}}};q\right)_\ell}{(q;q)_\ell} q^{\ell\left(\widetilde{\mathfrak{b}}+\ell\right)/2} \left(-q^{1-\widetilde{\mathfrak{b}}-\ell}\right)^\ell q^{\ell(\ell-1)/2} \,, \\ &= -n_+n_- \frac{(-z)^{1-\widetilde{\mathfrak{b}}}}{\left(q^{\left(1-\widetilde{\mathfrak{b}}\right)/2} z^{-1};q\right)_{\widetilde{\mathfrak{b}}}} \,, \\ &= n_+n_- \frac{z}{\left(z q^{\left(1-\widetilde{\mathfrak{b}}\right)/2};q\right)_{\widetilde{\mathfrak{b}}}} \,, \end{aligned} \tag{4.3.15}$$

where in the second, fourth and fifth equalities we exploited the identity

$$(t;q)_\ell = \left(t^{-1}\,q^{1-\ell};q\right)_\ell \left(-t\,q^{(\ell-1)/2}\right)^\ell . \tag{4.3.16}$$

Therefore, we find that on $\Sigma \times S^1$ the partition function of a chiral multiplet coupled to an effective Chern-Simons theory is dual to a free chiral multiplet, interpreted as describing a monopole operator. The overall factor (n_+n_-) originates from a sum over $\mathfrak{r}$ in the integer range going from 0 to $(n_+n_- - 1)$, spanning degenerate sectors of the original theory.

Chapter 5

Conclusions

In this thesis, I have synthesized the key findings from my research. The incorporation of orbifold geometry has yielded intriguing outcomes in both supergravity and supersymmetric field theory. While these results were achieved within a holographic research framework, their significance extends beyond holography and holds relevance in both of the aforementioned research scopes.

In Chapter 2, I presented the solution found in [1], which generalizes the analysis of [13] to non-minimal $D = 4$ gauged supergravity, particularly in theories with multiple gauge fields and, consequently, multiple conserved electric and magnetic charges. However, beyond these solutions, relatively little is known about supersymmetric accelerating black holes in other theories or supersymmetric accelerating black objects more generally. Nevertheless, it is clear that such solutions exhibit interesting properties and extended thermodynamics, as shown in [4] and explained in Chapter 3. For instance, the solution presented in Chapter 2, has also been used as a proving example for the validity of very general formulas in [60, 22].

In Chapter 4, I have summarized the results of [2, 3], where we demonstrated that three-dimensional $\mathscr{N} = 2$ SQFTs can be defined on $\mathbb{Σ} \times S^1$, endowed with both types of supersymmetry-preserving twists and that the corresponding partition functions give rise to the novel spindle index, generalizing and unifying the superconformal and topologically twisted indices. We have also shown that the generalization of the index theorem for orbifolds can find a direct application in physics, just as its more famous version for manifolds does. For instance, in [3], we have shown that the formalism of Chapter 4 can be apply to other orbifold as the branched lens space $L_{b_1,b_2,n_+,n_-}(n,1)$. This is a three-dimensional orbifold encompassing the squashed three-sphere, the branched sphere and the squashed lens space. The branched lens space is a $\mathcal{O}(-nn_+n_-)$ circle fibration over the spindle or, equivalently, can be viewed as the free quotient $S^3_{b_1,b_2,n_+,n_-}/\mathbb{Z}_n$ acting upon the S^1 fiber of the branched, squashed three-sphere. One more interesting point is the investigation of supersymmetric dualities, aiming to gain a deeper understanding of the quantum moduli space pertaining to supersymmetric quantum field theories on orbifolds. For instance, in the study of gauge theories on manifolds, the phenomenon of partition functions factorizing into holomorphic blocks is well-known [68, 67, 69, 70]

Regarding holography and the aim of reproducing 3.4.4, much work has been done, and much remains to be accomplished. In order to extract the microscopic entropy of the accelerating black holes in four-dimensional Anti-de Sitter spacetime, we need to compute the large-N limit of the spindle index of $\mathcal{N}=2$ theories with AdS$_4$ duals. More generally, the large-N limit of the spindle index should reproduce the entropy functions presented in [60], valid for an extensive class of three-dimensional $\mathcal{N}=2$ theories with gravity duals.

Appendix A

Spindle Geometry

In this appendix we will introduce the geometrical features of the so called "spindle". A mathematical introduction to orbifolds can be found in [71]. The main reference for orbi-bundle over spindle in the physical literature is [15]. Other useful material can be found, for example in [66, 72].

A.1 Weighted Projective Space

The term 'spindle' refers to the weighted projective space in complex dimension one, denoted as $\Sigma = \mathbb{WCP}^1_{[n_+,n_-]}$. Topologically, it is a 2-sphere S^2, while its orbifold structure can be obtained by taking a suitable quotient of S^3 by $U(1)$. Let z_1 and z_2 be the standard complex coordinates of the embedding of S^3 inside $\mathbb{C}^2$ as the unit sphere $S^3 = \left\{|z_1|^2 + |z_2|^2 = 1\right\} \subset \mathbb{C}^2$. The quotient is achieved by applying a weighted circle action given by

$$(z_1, z_2) \to (\lambda^{n_+} z_1, \lambda^{n_-} z_2) \tag{A.1.1}$$

where $\lambda = \mathrm{e}^{\mathrm{i}\theta} \in U(1), n_\pm \in \mathbb{N}$. This action is effective since $\mathrm{hcf}(n_+, n_-) = 1$. All powers of the primitive n_+-th root of unity $\lambda = \mathrm{e}^{2\pi\mathrm{i}/n_+}$ act trivially on $(z_1, 0)$, while the primitive n_--th root of unity $\lambda = \mathrm{e}^{2\pi\mathrm{i}/n_-}$ act trivially on $(0, z_2)$.

We recall the definition of orbifold charts [73] $(\tilde{\mathscr{U}}, \Gamma, \phi)$ for a 2-dimensional orbifold Σ. The underlying Hausdorff topological space is indicated by $|\Sigma|$. We require:

- $\tilde{\mathscr{U}}$ to form a basis of open sets in $\mathbb{R}^2$,
- Γ to be a finite group acting on $\tilde{\mathscr{U}}$ by linear transformations,
- $\phi : \tilde{\mathscr{U}} \longrightarrow |\Sigma|$ to be a continuous Γ-invariant map inducing a homeomorphism from $\tilde{\mathscr{U}}/\Gamma$ to $\mathscr{U} := \phi(\tilde{\mathscr{U}}) \subseteq |\Sigma|$ such that:
  - the $\mathscr{U}$ form a basis of open sets in $|\Sigma|$;
  - both charts satisfy the following compatibility condition: If $\mathscr{U}_1 \subseteq \mathscr{U}_2$, then there is an injection $\kappa : \left(\tilde{\mathscr{U}}_1, \Gamma_1, \phi_1\right) \longrightarrow \left(\tilde{\mathscr{U}}_2, \Gamma_2, \phi_2\right)$ which, by definition, amounts to:
    * a diffeomorphism $k : \tilde{\mathscr{U}}_1 \longrightarrow k\left(\tilde{\mathscr{U}}_1\right) \subseteq \tilde{\mathscr{U}}_2$
    * and a group isomorphism $K : \Gamma_1 \longrightarrow K(\Gamma_1) \subseteq \Gamma_2$ such that $\phi_1 = \phi_2 \circ k$ and k is K-equivariant $k \circ g = K(g) \circ k, \quad$ for all $g \in \Gamma_1$.

We cover the spindle with two charts in complete analogy with the sphere. The north and sud patch charts are given by $(\mathbb{C}, \mathbb{Z}_{n_+}, \phi_+)$ and $(\mathbb{C}, \mathbb{Z}_{n_-}, \phi_-)$ respectively, where $\mathbb{Z}_{n_\pm}$ are cyclic groups. The action of these group on coordinates $z_\pm \in \mathbb{C}$ is:

$$z_\pm \to \omega_\pm z_\pm \tag{A.1.2}$$

where $\omega_\pm \in \mathbb{Z}_{n_\pm}$. In future we will use the explicit notation $\omega_\pm^j = \mathrm{e}^{2\pi\mathrm{i}\frac{j}{n_\pm}}$ with $j = 0, 1, .., n_\pm - 1$. The $\mathbb{Z}_{n_+}$-invariant map ϕ_+ is given by

$$\phi_+ : z_+ \to z_+^{n_+} \tag{A.1.3}$$

while the $\mathbb{Z}_{n_-}$-invariant map ϕ_- is given by

$$\phi_- : z_- \to z_-^{-n_-} \tag{A.1.4}$$

On the intersection of the two charts, the transition function $\phi_\pm : \mathbb{C} \to \mathbb{C}$ is given by:

$$\phi_\pm : z_+ \to z_-^{n_-} = z_+^{-n_+} \tag{A.1.5}$$

Note that this presentation is a 'weighted' version of the well-known stereographic projection. Similar to manifolds, we can define orbifold coverings [71]. A covering of a smooth orbifold O is a pair $(\widehat{O}, \rho)$, where $\widehat{O}$ is another orbifold, and ρ is a surjective smooth map $\widehat{O} \to O$ satisfying the following conditions:

- For each $x \in |O|$, there is a chart $(\widetilde{U}, H, \varphi)$ over x such that $|\rho|^{-1}(U)$ is a disjoint union of open subsets V_i,
- Each V_i admits an orbifold chart of the type $\left(\widetilde{U}, H_i, \varphi_i\right)$, where $H_i < H$, such that ρ lifts to the identity $\widetilde{\rho}_i : \widetilde{U} \to \widetilde{U}$, with $\bar{\rho}_i : H_i \hookrightarrow H$.

Notice that, in general, $|\rho|$ is not a covering between the underlying topological spaces. When an orbifold can be covered by a manifold, it is referred to as a 'good-orbifold.' Conversely, if a manifold cover is not possible, it is termed a 'bad-orbifold'. Good-orbifold are global quotients of manifold by a finite group while bad-orbifold are generic quotient by Lie groups. The spindle is an example of a bad-orbifold, and it is one of the simplest instances to study in this category since it only has two orbifold points.

A.2 $U(1)$-Orbibundles

To specify a principal $U(1)$-orbibundle, we need to choose, in each chart, a homomorphism h from the local orbifold group into the fibre group $U(1)$. For the spindle, we will have two of these homomorphisms characterized by the choice of two integers m_+ and m_-.

$$h_+ : \mathbb{Z}_{n_+} \to U(1), \quad \text{where} \quad h_+(\omega) = \omega^{m_+} \tag{A.2.1}$$

$$h_- : \mathbb{Z}_{n_-} \to U(1), \quad \text{where} \quad h_-(\omega) = \omega^{m_-}. \tag{A.2.2}$$

Note that, since $\omega_\pm^{n_\pm} = 1$, we have $m_\pm \in \mathbb{Z}_{n_\pm}$. The homomorphisms h encode the action of the finite groups on fibers, completely determining the quotient on each local trivialization:

$$\left(\mathbb{C} \times S^1\right)/\mathbb{Z}_{n_\pm}, \quad \text{where } \omega_\pm \cdot (z_\pm, \xi_\pm) = \left(\omega z_\pm, \omega_\pm^{m_\pm} \xi_\pm\right). \tag{A.2.3}$$

Over the intersection of the charts, the two fiber coordinates are related by

$$t_\pm : \xi_+ \to \xi_- = \xi_+^{\frac{m_+ n_- - m_- n_+}{n_+ n_-}} \equiv \xi_+^{\frac{d}{n_+ n_-}}, \tag{A.2.4}$$

where we have defined $d = m_+n_- - m_-n_+$. It is customary to refer to the orbibundle as $\mathscr{O}(d)$, as the integer d specifies the orbibundle and determines the local modes near the orbifold points.

Appendix B

Uplift to $D = 11$

As already commented at the end of section 2.1, the solutions (2.2.1) can automatically be locally uplifted on S^7 to supersymmetric solutions of $D = 11$ supergravity. The relevant uplifting formulas are given in [53]. In this appendix we briefly comment on the conditions required for global regularity of these $D = 11$ solutions.

The uplifted $D = 11$ metric is given by

$$\begin{aligned} L^{-2}\mathrm{d}s_{11}^2 = &\ (UV)^{1/3}\mathrm{d}s_4^2 + 4(UV)^{1/3}\Big\{\mathrm{d}\eta^2 + \frac{\cos^2\eta}{4V}\big[\mathrm{d}\theta_1^2 + \sin^2\theta_1\mathrm{d}\phi_1^2 \\ & + (\mathrm{d}\psi_1 + \cos\theta_1\mathrm{d}\phi_1 - A_1)^2\big] + \frac{\sin^2\eta}{4U}\big[\mathrm{d}\theta_2^2 + \sin^2\theta_2\mathrm{d}\phi_2^2 \\ & + (\mathrm{d}\psi_2 + \cos\theta_2\mathrm{d}\phi_2 - A_2)^2\big]\Big\}\,. \end{aligned} \tag{B.0.1}$$

Here we have introduced an overall constant length scale $L > 0$, and have defined the functions

$$U \equiv (\mathrm{e}^{-\xi} + \chi^2\mathrm{e}^{\xi})\sin^2\eta + \cos^2\eta\,, \qquad V \equiv \mathrm{e}^{\xi}\cos^2\eta + \sin^2\eta\,. \tag{B.0.2}$$

Here the metric in curly brackets is a metric on S^7, where one views $S^7 \subset \mathbb{C}^2 \oplus \mathbb{C}^2$ as unit sphere, with the metrics in square brackets being metrics on the two copies of $S^3 \subset \mathbb{C}^2$. It follows that $\theta_i \in [0, \pi]$, $\eta \in [0, \frac{\pi}{2}]$, while ϕ_i have period 2π and ψ_i have period 4π, $i = 1, 2$. The gauge fields A_i then fibre the two three-spheres over the $D = 4$ spacetime, effectively gauging the Hopf $U(1)$ isometry of each S^3. The formula for the $D = 11$ four-form flux G is rather more involved, and can be found in [53].

For the spinning spindle solutions (2.2.1) the gauge fields are not in general globally defined one-forms on $\mathrm{AdS}_2 \times \Sigma$, as must be the case since the magnetic fluxes in (3.1.1) are generically non-zero. On the other hand, these gauge fields fibre the internal S^7 over this spacetime via (B.0.1), and this will lead to a globally well-defined $D = 11$ spacetime only if the P_i satisfy certain quantization conditions. Specifically, this requires

$$P_i = \frac{2\,\mathrm{p}_i}{n_- n_+}, \tag{B.0.3}$$

where $\mathrm{p}_i \in \mathbb{Z}$ are integers coprime to $n_\pm$. Imposing (B.0.3) the $D = 11$ spacetime is then the total space of an S^7 fibration over $\mathrm{AdS}_2 \times \Sigma$, with this total space being free from orbifold singularities.

The $D = 11$ solution can be understood as the near horizon limit of N M2-branes wrapped on the spindle Σ, where the flux number N is defined by

$$N = \frac{1}{(2\pi \ell_p)^6} \int_{S^7} \star_{11} G + \frac{1}{2} C \wedge G. \tag{B.0.4}$$

Here the S^7 is a copy of the fibre, at any point in the $D = 4$ spacetime. We find that in turn this fixes the constant L via

$$\frac{L^6}{(2\pi \ell_p)^6} = \frac{N}{128\pi^4}, \tag{B.0.5}$$

while the $D = 4$ Newton constant is

$$\frac{1}{G_{(4)}} = \frac{2\sqrt{2}}{3} N^{3/2}. \tag{B.0.6}$$

Appendix C

Supersymmetry and cohomology

C.1 Vector multiplet

C.1.1 Supersymmetry transformations.

Given a gauge group G, a non-Abelian $\mathscr{N}=2$ vector multiplet in three dimensions contains a gauge field $\mathscr{A}_\mu$, a scalar field σ, the two-component spinors λ_α, $\widetilde{\lambda}_\alpha$ and an auxiliary field D, all transforming in the adjoint representation of G. The fields $(\mathscr{A}_\mu, \sigma, \lambda, \widetilde{\lambda}, D)$ respectively have R-charges $(0,0,1,-1,0)$. The corresponding supersymmetry transformations are

$$
\begin{aligned}
\delta\mathscr{A}_\mu &= -\mathrm{i}\zeta\gamma_\mu\widetilde{\lambda} - \mathrm{i}\widetilde{\zeta}\gamma_\mu\lambda \ , \\
\delta\sigma &= -\zeta\widetilde{\lambda} + \widetilde{\zeta}\lambda \ , \\
\delta\lambda &= -\frac{\mathrm{i}}{2}\varepsilon^{\mu\nu\rho}\gamma_\rho\zeta\mathscr{F}_{\mu\nu} + \mathrm{i}\zeta(D+\sigma H) - \gamma^\mu\zeta(\mathrm{i}\mathrm{D}_\mu\sigma - V_\mu\sigma) \ , \\
\delta\widetilde{\lambda} &= -\frac{\mathrm{i}}{2}\varepsilon^{\mu\nu\rho}\gamma_\rho\widetilde{\zeta}\mathscr{F}_{\mu\nu} - \mathrm{i}\widetilde{\zeta}(D+\sigma H) + \gamma^\mu\widetilde{\zeta}(\mathrm{i}\mathrm{D}_\mu\sigma + V_\mu\sigma) \ , \\
\delta D &= \mathrm{D}_\mu\Big(\zeta\gamma^\mu\widetilde{\lambda} - \widetilde{\zeta}\gamma^\mu\lambda\Big) - \mathrm{i}V_\mu\Big(\zeta\gamma^\mu\widetilde{\lambda} + \widetilde{\zeta}\gamma^\mu\lambda\Big) - \Big[\sigma, \zeta\widetilde{\lambda} + \widetilde{\zeta}\lambda\Big] - H\Big(\zeta\widetilde{\lambda} - \widetilde{\zeta}\lambda\Big) \ ,
\end{aligned}
\tag{C.1.1}
$$

where $\mathscr{F}_{\mu\nu} = \partial_\mu\mathscr{A}_\nu - \partial_\nu\mathscr{A}_\mu - \mathrm{i}\left[\mathscr{A}_\mu, \mathscr{A}_\nu\right]$, while the gauge-covariant derivative D_μ acts on a field Ψ of R-charge q_R, central charge q_Z and in the representation $\mathscr{R}$ of the gauge group G as follows:

$$\mathrm{D}_\mu \Psi = \left[\nabla_\mu - \mathrm{i}q_R\left(A_\mu - \frac{1}{2}V_\mu\right) - \mathrm{i}q_Z\mathscr{C}_\mu - \mathrm{i}\mathscr{A}_\mu \circ_{\mathscr{R}}\right]\Psi \ , \tag{C.1.2}$$

with, for instance,

$$\mathscr{A} \circ_{\text{adjoint}} \Psi = [\mathscr{A},\Psi] \ , \qquad \mathscr{A} \circ_{\text{fundamental}} \Psi = \mathscr{A}\Psi \ , \qquad \mathscr{A} \circ_{\text{anti-fundamental}} \Psi = -\Psi\mathscr{A} \ . \tag{C.1.3}$$

The fields belonging to the vector multiplet have vanishing central charge, hence $\mathscr{C}_\mu$ appears in (C.1.1) only implicitly, via V_μ.

C.1.2 Cohomological complex.

The supersymmetry variation $\delta = \delta_\zeta + \delta_{\widetilde{\zeta}}$ is an equivariant differential. Indeed, the supersymmetry transformations for the vector multiplet can be rewritten as a cohomological complex containing the gauge field $\mathscr{A}$, the scalar σ, the Grassmann-odd 0-form $\chi = \mathrm{i}(\widetilde{\zeta}\lambda + \zeta\widetilde{\lambda})/(2v)$ and their δ-differentials:

$$\begin{aligned}
\delta\mathscr{A} &= \Lambda \ , & \delta\Lambda &= -2\mathrm{i}(L_K + \mathscr{G}_{\Phi_G})\mathscr{A} \ , \\
\delta\sigma &= -(\mathrm{i}/v)\iota_K\Lambda \ , & \delta\Phi_G &= 0 \ , \\
\delta\chi &= \Delta \ , & \delta\Delta &= -2\mathrm{i}(L_K + \mathscr{G}_{\Phi_G})\chi \ ,
\end{aligned} \tag{C.1.4}$$

with L_K being the covariant Lie derivative

$$\begin{aligned}
L_K &= \mathscr{L}_K - \mathrm{i}q_R\left(\iota_K A - \frac{1}{2}\iota_K V\right) - \mathrm{i}q_Z(\iota_K\mathscr{C}) + v(q_Z - q_R H) \ , \\
&= \mathscr{L}_K - \mathrm{i}q_R\Phi_R - \mathrm{i}q_Z(\iota_K\mathscr{C} + \mathrm{i}v) \ .
\end{aligned} \tag{C.1.5}$$

In fact, $L_K = \mathscr{L}_K$ in (C.1.4) as all fields in (C.1.4) have vanishing central charge q_Z and R-charge q_R. The cohomological complex of the vector multiplet is independent of the $U(1)_R$ bundle. In (C.1.4) also appears $\mathscr{G}_{\Phi_G}$, which is a gauge transformation with parameter Φ_G, respectively acting on $\mathscr{A}$ and on a field $X \neq \mathscr{A}$ in the representation $\mathscr{R}$ of the gauge group as

$$\mathscr{G}_{\Phi_G}\mathscr{A} = -\mathrm{d}_A\Phi_G = -\mathrm{d}\Phi_G + \mathrm{i}[\mathscr{A},\Phi_G] \ , \qquad \mathscr{G}_{\Phi_G}X = -\mathrm{i}\Phi_G \circ_{\mathscr{R}} X \ . \tag{C.1.6}$$

Explicitly, the Grassmann-odd 1-form Λ_μ, the Grassmann-odd 0-form χ and the Grassmann-even 0-forms Φ_G, Δ are defined by

$$
\begin{aligned}
\Lambda_\mu &= -\mathrm{i}\left(\zeta\gamma_\mu\widetilde{\lambda} + \widetilde{\zeta}\gamma_\mu\lambda\right) , \\
\chi &= \frac{\mathrm{i}}{2\nu}\left(\widetilde{\zeta}\lambda + \zeta\widetilde{\lambda}\right) , \\
\Phi_G &= \iota_K\mathscr{A} - \mathrm{i}\nu\sigma , \\
\Delta &= D + \frac{1}{\nu}\iota_K(\star\mathscr{F}) + \frac{\mathrm{i}}{\nu}\sigma[\iota_K V - \mathrm{i}\nu H] = D + \frac{\mathrm{i}}{2}P^\mu\widetilde{P}^\nu\mathscr{F}_{\mu\nu} + \frac{\mathrm{i}}{\nu}\sigma[\iota_K V - \mathrm{i}\nu H] ,
\end{aligned}
\tag{C.1.7}
$$

with the map from (Λ, χ) to $(\lambda, \widetilde{\lambda})$ being

$$
\lambda = \mathrm{i}\left(\frac{1}{2\nu^2}\iota_K\Lambda + \chi\right)\zeta - \frac{\mathrm{i}}{2\nu}(\iota_P\Lambda)\widetilde{\zeta} , \quad \widetilde{\lambda} = \mathrm{i}\left(\frac{1}{2\nu^2}\iota_K\Lambda - \chi\right)\widetilde{\zeta} - \frac{\mathrm{i}}{2\nu}\left(\iota_{\widetilde{P}}\Lambda\right)\zeta .
\tag{C.1.8}
$$

C.2 Chiral multiplet

C.2.1 Supersymmetry transformations.

A three-dimensional $\mathscr{N}=2$ chiral multiplet of R-charge r, central charge z and in a representation $\mathscr{R}$ of a gauge group G contains a complex scalar field ϕ, a two-component spinor ψ_α and an auxiliary field F. The fields (ϕ, ψ, F) respectively have R-charges $(r, r-1, r-2)$ and central charge z. The corresponding supersymmetry transformations are

$$
\begin{aligned}
\delta\phi &= \sqrt{2}\zeta\psi , \\
\delta\psi &= \sqrt{2}\zeta F + \mathrm{i}\sqrt{2}(\sigma + rH - z)\widetilde{\zeta}\phi - \mathrm{i}\sqrt{2}\gamma^\mu\widetilde{\zeta}\mathrm{D}_\mu\phi , \\
\delta F &= \mathrm{i}\sqrt{2}[z - \sigma - (r-2)H]\widetilde{\zeta}\psi - \mathrm{i}\sqrt{2}\mathrm{D}_\mu\left(\widetilde{\zeta}\gamma^\mu\psi\right) + 2\mathrm{i}(\widetilde{\zeta}\widetilde{\lambda})\phi .
\end{aligned}
\tag{C.2.1}
$$

Analogously, a three-dimensional $\mathscr{N}=2$ anti-chiral multiplet of R-charge $-r$, central charge $-z$ and in the conjugate representation $\overline{\mathscr{R}}$ of a gauge group G contains a complex scalar field $\widetilde{\phi}$, a two-component spinor $\widetilde{\psi}_\alpha$ and an auxiliary field $\widetilde{F}$. The fields $(\widetilde{\phi}, \widetilde{\psi}, \widetilde{F})$ respectively have R-charges $(-r, -r+1, -r+2)$ and central charge

$-z$. In this case, the supersymmetry transformations read

$$\begin{aligned}
&\delta\widetilde{\phi} = -\sqrt{2}\widetilde{\zeta}\widetilde{\psi}\ ,\\
&\delta\widetilde{\psi} = \sqrt{2}\widetilde{\zeta}\widetilde{F} - \mathrm{i}\sqrt{2}\widetilde{\phi}(\sigma + rH - z)\zeta + \mathrm{i}\sqrt{2}\gamma^{\mu}\zeta \mathrm{D}_{\mu}\widetilde{\phi}\ ,\\
&\delta\widetilde{F} = \mathrm{i}\sqrt{2}(\zeta\widetilde{\psi})[z - \sigma - (r-2)H] - \mathrm{i}\sqrt{2}\mathrm{D}_{\mu}(\zeta\gamma^{\mu}\widetilde{\psi}) + 2\mathrm{i}\widetilde{\phi}(\zeta\lambda)\ .
\end{aligned} \tag{C.2.2}$$

C.2.2 Cohomological complex.

Similarly to the case of the vector multiplet, the supersymmetry variation $\delta = \delta_{\zeta} + \delta_{\widetilde{\zeta}}$ acts on the fields of the chiral multiplet as an equivariant differential, which induces the following cohomological complex:

$$\begin{aligned}
&\delta\phi = C\ , && \delta C = -2\mathrm{i}(L_K + \mathscr{G}_{\Phi_G})\phi\ ,\\
&\delta B = \Theta\ , && \delta\Theta = -2\mathrm{i}(L_K + \mathscr{G}_{\Phi_G})B\ ,
\end{aligned} \tag{C.2.3}$$

where the Grassmann-odd 0-forms B, C and the Grassmann-even 0-form Θ are defined by

$$\begin{aligned}
&B = -\frac{\widetilde{\zeta}\psi}{\sqrt{2}v}\ ,\\
&C = \sqrt{2}(\zeta\psi)\ ,\\
&\Theta = F + \mathrm{i}\left[\mathscr{L}_{\widetilde{\mathrm{P}}} - \mathrm{i}r\left(\iota_{\widetilde{\mathrm{P}}}A - \frac{1}{2}\iota_{\widetilde{\mathrm{P}}}V\right) - \mathrm{i}z(\iota_{\widetilde{\mathrm{P}}}\mathscr{C}) - \mathrm{i}(\iota_{\widetilde{\mathrm{P}}}\mathscr{A})\right]\phi\ .
\end{aligned} \tag{C.2.4}$$

The inverse map from the Grassmann-odd scalars (B, C) to the spinor ψ is

$$\psi = \sqrt{2}B\zeta + \frac{C}{\sqrt{2}v}\widetilde{\zeta}\ . \tag{C.2.5}$$

For completeness, we report the intermediate form of the supersymmetry transformations, the one obtained before introducing Θ:

$$\begin{aligned}
\delta\phi =& C\ ,\\
\delta C =& -2\mathrm{i}\left[\mathscr{L}_K - \mathrm{i}r\left(\iota_K A - \frac{1}{2}\iota_K V\right) + v(z - rH) - \mathrm{i}z(\iota_K\mathscr{C}) - \mathrm{i}\Phi_G\right]\phi\ ,\\
\delta B =& F + \mathrm{i}\left[\mathscr{L}_{\widetilde{\mathrm{P}}} - \mathrm{i}r\left(\iota_{\widetilde{\mathrm{P}}} A - \frac{1}{2}\iota_{\widetilde{\mathrm{P}}} V\right) - \mathrm{i}z(\iota_{\widetilde{\mathrm{P}}}\mathscr{C}) - \mathrm{i}(\iota_{\widetilde{\mathrm{P}}}\mathscr{A})\right]\phi\ ,\\
\delta F =& -2\mathrm{i}\left[\mathscr{L}_K - \mathrm{i}(r-2)\left(\iota_K A - \frac{1}{2}\iota_K V\right) - \mathrm{i}z\iota_K\mathscr{C} + v[z - (r-2)H] - \mathrm{i}\Phi_G\right]B\\
&-\mathrm{i}\left[\mathscr{L}_{\widetilde{\mathrm{P}}} - \mathrm{i}r\left(\iota_{\widetilde{\mathrm{P}}} A - \frac{1}{2}\iota_{\widetilde{\mathrm{P}}} V\right) - \mathrm{i}z(\iota_{\widetilde{\mathrm{P}}}\mathscr{C}) - \mathrm{i}(\iota_{\widetilde{\mathrm{P}}}\mathscr{A})\right]C - (\iota_{\widetilde{\mathrm{P}}}\Lambda)\phi\ . \qquad \text{(C.2.6)}
\end{aligned}$$

Analogously, the cohomological complex of the anti-chiral multiplet reads

$$\begin{aligned}
\delta\widetilde{\phi} &= \widetilde{C}\ , & \delta\widetilde{C} &= -2\mathrm{i}(L_K + \mathscr{G}_{\Phi_G})\widetilde{\phi}\ ,\\
\delta\widetilde{B} &= \widetilde{\Theta}\ , & \delta\widetilde{\Theta} &= -2\mathrm{i}(L_K + \mathscr{G}_{\Phi_G})\widetilde{B}\ , \qquad \text{(C.2.7)}
\end{aligned}$$

where the Grassmann-odd 0-forms $\widetilde{B}, \widetilde{C}$ and the Grassmann-even 0-form $\widetilde{\Theta}$ are defined by

$$\begin{aligned}
\widetilde{B} &= \frac{\zeta\widetilde{\psi}}{\sqrt{2}v}\ ,\\
\widetilde{C} &= -\sqrt{2}(\widetilde{\zeta}\widetilde{\psi})\ ,\\
\widetilde{\Theta} &= \widetilde{F} + \mathrm{i}\left[\mathscr{L}_P + \mathrm{i}r\left(\iota_P A - \frac{1}{2}\iota_P V\right) + \mathrm{i}z(\iota_P\mathscr{C})\right]\widetilde{\phi} - \widetilde{\phi}(\iota_P\mathscr{A})\ , \qquad \text{(C.2.8)}
\end{aligned}$$

with the inverse map from the Grassmann-odd scalars $(\widetilde{B}, \widetilde{C})$ to the spinor $\widetilde{\psi}$ being

$$\widetilde{\psi} = \sqrt{2}\widetilde{B}\widetilde{\zeta} + \frac{\widetilde{C}}{\sqrt{2}v}\zeta\ . \qquad \text{(C.2.9)}$$

The intermediate step in the derivation of the cohomological complex of the anti-chiral multiplet is given by the transformations

$$
\begin{aligned}
\delta\widetilde{\phi} =& \widetilde{C}\ , \\
\delta\widetilde{C} =& -2\mathrm{i}\left[\mathscr{L}_K + \mathrm{i}r\left(\iota_K A - \frac{1}{2}\iota_K V\right) + v(-z + rH) + \mathrm{i}z(\iota_K\mathscr{C})\right]\widetilde{\phi} + 2\widetilde{\phi}\Phi_G\ , \\
\delta\widetilde{B} =& \widetilde{F} + \mathrm{i}\left[\mathscr{L}_P + \mathrm{i}r\left(\iota_P A - \frac{1}{2}\iota_P V\right) + \mathrm{i}z(\iota_P\mathscr{C})\right]\widetilde{\phi} - \widetilde{\phi}(\iota_P\mathscr{A})\ , \\
\delta\widetilde{F} =& -2\mathrm{i}\left[\mathscr{L}_K + \mathrm{i}(r-2)\left(\iota_K A - \frac{1}{2}\iota_K V\right) + \mathrm{i}z(\iota_K\mathscr{C}) + v[-z - (-r+2)H]\right]\widetilde{B} + 2\widetilde{B}\Phi_G \\
& - \mathrm{i}\left[\mathscr{L}_P + \mathrm{i}r\left(\iota_P A - \frac{1}{2}\iota_P V\right) + \mathrm{i}z(\iota_P\mathscr{C})\right]\widetilde{C} + \widetilde{C}(\iota_P\mathscr{A}) + \widetilde{\phi}(\iota_P\Lambda)\ . \qquad \text{(C.2.10)}
\end{aligned}
$$

List of Figures

1.1 Summary of AdS_4 black holes with either spherical or spindle horizons in $D = 4$, $\mathscr{N} = 4$ gauged supergravity. The solutions in the red frames admit a supersymmetric and extremal limit and their near horizon $AdS_2 \times \Sigma$ geometries are represented pictorially. From bottom-left to top-right: a spinning sphere, a spinning spindle, and a non-spinning spindle. In all cases the reference on the left refers to the non-extremal black holes, and that on the right refers to the near horizon solution in the supersymmetric limit. 4

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