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Novel Non-Iterative Numerical Quadratures for Singularities in Multiphysics Formulations and Basis Functions

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***Abstract*—This paper presents a novel non-iterative numerical quadrature method designed for integrands with singular/irregular behaviors well-approximated by generalized polynomials with real exponents potentially combined with logarithmic singularities. The procedure can be implemented easily at negligible computational cost and memory in finite methods for Multiphysics computing based on PDE and IE formulations.**

I. Introduction

Fast, accurate numerical integration is a continuous bottleneck in high-performance computational physics implementing finite methods, especially in large multiscale FEM/BEM simulations. The computational cost rises for irregular or singular integrands that can be present in integral kernels (in (1) weakly-singular potential integral for div-conforming surface elements), additive singular basis functions in PDE/IE formulations modelling abrupt discontinuities (in (2) wedge type EM behavior of physical quantities), testing integrals (in (3) double surface integrals with the outer being the testing one) or a combination of them [1-5].

$$\mathcal{I}_S = \int_S \mathbf{\Lambda}(\mathbf{r}') \frac{\exp(-jkR)}{4\pi R} \mathrm{d}S' \tag{1}$$

$$\begin{aligned} \mathbf{E} &= -\frac{\nu B}{\rho^{1-\nu}} (\cos\nu\phi\hat{\boldsymbol{\phi}} + \sin\nu\phi\hat{\boldsymbol{\rho}}) + j\omega\mu_0 A\rho^{\nu} \sin\nu\phi\hat{\mathbf{z}} \\ \mathbf{H} &= \frac{\nu A}{\rho^{1-\nu}} (\sin\nu\phi\hat{\boldsymbol{\phi}} - \cos\nu\phi\hat{\boldsymbol{\rho}}) + j\omega\varepsilon_0 B\rho^{\nu} \cos\nu\phi\hat{\mathbf{z}} \\ \mathbf{J}_s &= \frac{\nu A}{\rho^{1-\nu}} \hat{\mathbf{z}} + j\omega\varepsilon_0 B\rho^{\nu} \hat{\boldsymbol{\rho}} \end{aligned} \tag{2}$$

$$\mathcal{I} = \int_S \mathbf{\Lambda}(\mathbf{r}) \int_{S'} \mathbf{\Lambda}(\mathbf{r}') g(\mathbf{r}, \mathbf{r}') dS' dS \tag{3}$$

As an alternative to quadratures for specific multivariate integrals over standard regions (see for instance [6-8]) and to high precision computationally expensive adaptive/iterative/transformed/generalized-Gauss quadrature techniques (see for non-exhaustive description [9-14]), we propose a novel non-iterative quadrature scheme (thus more efficient) based on ad-hoc transformations of Gauss-Legendre (GL) quadrature capable of securing prefixed precision up to machine precision for integrands well-approximated by generalized polynomials with real exponents potentially combined with logarithmic singularities [15-16]. The considered types of integrands are encountered in finite methods exhibiting singularities due to the physical models (2) and/or integral formulations (1) and (3) [1-5].

Starting from irregular behaviors like the ones of generalized polynomials with real exponents, we enlarge the set of integrands to be treated to the ones containing logarithmic singularities. The methodology is effective for all functions well-approximated via these sets of functions often encountered in Computational Electromagnetics.

The implementation [16] of the proposed quadratures is with a-priori formulas built at run-time with negligible computational cost and memory use, in particular with respect to adaptive/iterative schemes and capable of exceeding flexibility of other generalized Gauss type rules. The total cost of the quadrature is just the one due to transformations applied to tabulated GL points and weights.

II. Scope and Methodology

The scope of the algorithm is to compute multivariate integrals with irregular or singular behavior up to an a-priori prescribed precision, considering one-dimensional nested integrals using slice-by-slice integration, as often implemented in numerical integration if an appropriate multivariate quadrature is not available on specific domains. For this reason, we focus our attention on 1D quadratures.

In the following, we consider integrands that can be approximated using generalized polynomials built on monomials with real exponents and potentially combined with logarithmic singularities

$$f(x) = \sum_{k=1}^{K} c_k x^{\lambda_k} + \log(x) \sum_{h=1}^{H} d_h x^{\lambda_h}, \quad \lambda_k, \lambda_h \in (-1, \infty] \tag{4}$$

The design of the quadrature relies on the two asymptotic error estimation of Gauss-Legendre quadrature rule performed respectively on x^{λ} and $x^{\lambda}\log(x)$ and obtained with the help of complex analysis. The closed form error estimates are parametrized by $\lambda \in (-1, \infty]$ and Fig. 1 reports the one for x^{λ} that overlaps the actual relative errors obtained considering quadruple precision in Gauss-Legendre implementation with *n* samples. A similar analysis can be repeated for $x^{\lambda}\log(x)$.

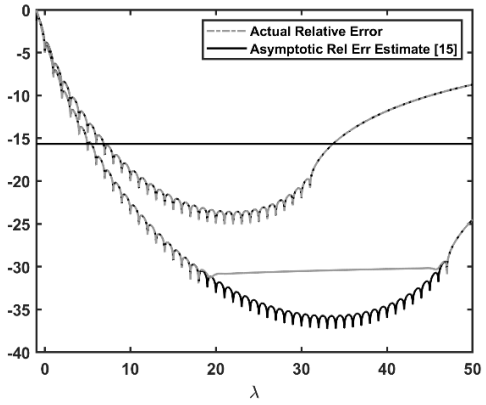


Figure 1. Plots of the actual relative error of GL quadrature integrating monomials x^{λ} with n=16, 24 samples in quadruple precision and its asymptotic estimates [15] reported in log10 scale. The curves coincidence except that the actual value is limited to 31-digits precision as expected. The constant black solid line is the double precision (IEEE format) relative accuracy level log10(eps)=−15.6536, eps=2E−52.

The design of the quadrature is based on

1) the definition of λ_{min} and λ_{max} in (4) for each kind of monomials

2) the analysis of performance of GL quadrature given a prescribed precision through asymptotic analysis of closed form error estimates

3) based on (1) and (2) the selection of the monomial transformation order r (5) such that the monomials with λ_{min} and λ_{max} fall in the region of the asymptotic estimates that is below the selected precision treeshold.

$$\gamma_r(x) = x^r \tag{5}$$

We obtain the quadrature $\{x_i,w_i\}$ (i=1,…,n) in the original domain x using

$$x_i = t_k^r, \quad w_i = r t_i^{r-1} w_i^{(t)} \tag{6}$$

where $\{t_i,w_i^{(t)}\}$ (i=1, . . . ,n) are points and weights of the classical GL quadrature in (0,1).

The algorithm for x^{λ} case is available in [16] and released as open source numerical library MTQR in C++ language for Windows and Linux platforms available at CPC Mendeley Library https://data.mendeley.com/datasets/276f78wzsw/1 and at the developer's repository https://github.com/MTQR/MTQR under GNU General Public License 3.

Examples of validation are available in [15-16] and will be presented at the conference. Implementations in FEM/MoM applications demonstrated efficacy in CEM as reported at [16],[4].

III Conclusion

The paper aims at presenting and distributing the concepts at the base of the algorithm and of the open source library MTQR for building fast and reliable quadrature rules at run-time capable of integrating singular and irregular behaviors with a prescribed precision in Multiphysics computational methods with negligible computational cost and memory use.

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References

[1] R. D. Graglia, G. Lombardi, D. R. Wilton and W. A. Johnson, "Modeling edge singularities in the method of moments," *2005 IEEE Antennas and Propagation Society International Symposium*, Washington, DC, USA, 2005, pp. 56-59 vol. 3A, doi: 10.1109/APS.2005.1552172.

[2] R. D. Graglia and G. Lombardi, "Singular higher order complete vector bases for finite methods," in *IEEE Transactions on Antennas and Propagation*, vol. 52, no. 7, pp. 1672-1685, July 2004, doi: 10.1109/TAP.2004.831292

[3] R. D. Graglia and G. Lombardi, "Machine precision evaluation of singular and nearly singular potential integrals by use of Gauss quadrature formulas for rational functions," IEEE Trans. Antennas Propag., vol. 56, pp. 981–998, 2008.

[4] R. D. Graglia and G. Lombardi, "Singular Higher Order Divergence-Conforming Bases of Additive Kind and Moments Method Applications to 3D Sharp-Wedge Structures," in *IEEE Transactions on Antennas and Propagation*, vol. 56, no. 12, pp. 3768-3788, Dec. 2008, doi: 10.1109/TAP.2008.2007390.

[5] G. Lombardi and R. D. Graglia, "Modeling Junctions in Sharp Edge Conducting Structures With Higher Order Method of Moments," in *IEEE Transactions on Antennas and Propagation*, vol. 62, no. 11, pp. 5723-5731, Nov. 2014, doi: 10.1109/TAP.2014.2355855.

[6] A. H. Stroud, Approximate Calculation of Multiple Integrals. Englewood Cliffs, NJ: Prentice Hall, 1971

[7] P. J. Davis and P. Rabinowitz, Methods of Numerical Integration. San Diego, CA: Academic, 1975

[8] R. Cools, "An encyclopedia of quadrature formulas," J. Complexity, vol. 19, pp. 445–453, 2003.

[9] W. Gautschi, "Numerical quadrature in the presence of a singularity," SIAM J. Numer. Anal., vol. 4, no. 3, pp. 357–362, 1967.

[10] G. Monegato, "Numerical evaluation of hypersingular integrals," J. Comput. Appl. Math., vol. 50, no. 1, pp. 9–31, 1994.

[11] J. Ma, V. Rokhlin, and S. Wandzura, "Generalized Gaussian quadrature rules for systems of arbitrary functions," SIAM J. Numer. Anal., vol. 33, no. 3, pp. 971–996, 1996, doi: 10.1137/0733048.

[12] P. Kolm and V. Rokhlin, "Numerical quadratures for singular and hypersingular integrals," Comput. Math. Appl., vol. 41, no. 3, pp. 327–352, 2001, doi: 10.1016/S0898-1221(00)00277-7.

[13] K. Pachucki, M. Puchalski, and V. Yerokhin, "Extended Gaussian quadratures for functions with an end-point singularity of logarithmic type," Comput. Phys. Commun., vol. 185, no. 11, pp. 2913–2919, 2014, doi: 10.1016/j.cpc.2014.06.018.

[14] G. V. Milovanović, T. Igić, and D. Turnić, "Generalized quadrature rules of Gaussian type for numerical evaluation of singular integrals," J. Comput. Appl. Math., vol. 278, pp. 306–325, 2015, doi: 10.1016/j.cam.2014.10.009.

[15] G. Lombardi, "Design of quadrature rules for Muntz and Muntz-logarithmic polynomials using monomial transformation," Int. J. for Numer. Methods Eng., vol. 80, no. 13, pp. 1687–1717, Dec. 2009, doi:10.1002/nme.2684

[16]G. Lombardi and D. Papapicco, "Quadrature of functions with endpoint singular and generalised polynomial behaviour in computational physics," Computer Physics Communications, vol. 299, p. 109124, Jun. 2024, doi: 10.1016/j.cpc.2024.109124.