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Research on Fast Computational Methods for Electromagnetic Scattering and Microwave Sensing Technologies

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Yuhao Zuo
Turin, August 8, 2026

Abstract

Numerical solution algorithms for electromagnetic scattering and propagation problems constitute an important research topic in computational electromagnetics. The Method of Moments (MoM) based on integral equations offers high computational accuracy; however, it generates dense system matrices, which typically require fast algorithms to improve computational efficiency. These fast algorithms compress the matrix blocks associated with far-field interactions, thereby accelerating the matrix-vector products involving the system matrix and the right-hand side vector, and have been widely employed in iterative solvers. Nevertheless, for electromagnetic problems involving complex objects, the resulting system matrices are often ill-conditioned, which may lead to convergence difficulties when iterative solvers are employed. Moreover, when solving Multiple Right-Hand Side (MRHS) problems, iterative methods generally require the iterative process to be restarted for different right-hand sides, resulting in relatively low computational efficiency. In contrast, direct solvers explicitly construct the inverse of the system matrix and therefore offer higher computational efficiency for MRHS problems, while being largely insensitive to convergence issues. Based on a kernel-independent fast algorithm, namely the Nested Equivalent Source Approximation (NESA), this dissertation conducts the following research:

1. A fast algorithm was developed for electromagnetic scattering problems involving complex objects composed of nonuniform dielectric and metallic materials. For Perfect Electric Conductors (PECs) and nonuniform dielectric media, Surface Integral Equations (SIEs) and Volume Integral Equations (VIEs) were respectively established, and a NESA-based Volume-Surface Integral Equation (VSIE) formulation was developed. The resulting VSIE system was efficiently solved using NESA. In the proposed method, equivalent surfaces and uniformly distributed equivalent sources are introduced. Based on the equivalence

principle, the surface and volume unknowns are mapped onto the equivalent sources, such that the coupling interactions between far-field groups are replaced by the interactions between equivalent sources. Compared with surface discretization, volume discretization generally introduces a substantially larger number of unknowns. In the NESAs-VSIE formulation, however, the volume unknowns are retained only at the finest level, while the coupling interactions involving volume unknowns at higher levels are replaced by the interactions among equivalent surface-source unknowns. Consequently, the dimension of the matrix system to be solved is significantly reduced, thereby improving the computational efficiency of solving VSIE problems.

2. A direct solver was developed for electromagnetic scattering problems in the mid- and low-frequency regimes. Based on the Low-Rank Approximation (LRA) form of NESAs, a system matrix structure similar to that of an H^2 -matrix was constructed, and elimination matrices were introduced. The entire system matrix was factorized using LU decomposition, thereby enabling the explicit computation of its inverse. During the factorization process, dense low-rank matrix blocks arise in the Schur complements. Singular Value Decomposition (SVD) was therefore employed to compress these matrix blocks, maintaining a sparse representation of the far-field matrix blocks. Compared with iterative solvers, the proposed direct solver achieves higher computational efficiency for problems with multiple excitation terms. Moreover, the introduction of equivalent sources further reduces the memory requirements compared with skeletonization-based algorithms. Theoretical analysis and numerical experiments demonstrate that both the time and memory complexities of the proposed algorithm are $O(N)$. Numerical simulations of the monostatic Radar Cross Section (RCS) of complex targets further demonstrate a significant improvement in computational efficiency over iterative solvers.

3. A direct solver was developed for high-frequency electromagnetic scattering problems. For high-frequency problems, the ranks of the matrix blocks increase with the number of unknowns, and uniformly distributed omnidirectional equivalent sources cannot effectively characterize the scattering fields of individual groups. Based on the matrix compression formulation of the Wideband Nested Equivalent Source Approximation (WNESA), the radiation matrices corresponding to different directions of each group are jointly considered during the system matrix factorization. Singular Value Decomposition (SVD) is then employed to extract a generalized radiation matrix, which is used to construct the elimination matrix for each group. The generalized radiation matrix collects the radiation information of a given group in all directions, thereby ensuring that the

resulting elimination matrix can effectively eliminate the far-field coupling blocks in all directions. Since WNESA imposes more stringent admissibility conditions, the indexing scheme for inadmissible blocks in the direct-solution procedure is also modified accordingly. Numerical examples are conducted at different frequencies to compare the computational results and verify the accuracy of the proposed algorithm in high-frequency scenarios. Theoretical analysis and numerical results demonstrate that the proposed direct solver for electrically large problems has a time complexity of $O(N^2)$ and a memory complexity of $O(N \log N)$.

4. An online microwave sensing system was designed for food contamination detection. The proposed direct solvers for electromagnetic scattering problems were applied to the development of a microwave sensing system for the detection of contaminants in food products. An online, real-time microwave sensing system was developed to extract variations in the electromagnetic characteristics inside food products from their scattered-field information, thereby enabling the detection and identification of potential contaminants. An antenna array was employed to focus and scan the electromagnetic beam, thereby improving the detection performance compared with conventional systems based on horn antennas. In addition, a machine learning model was developed to learn and classify the electromagnetic scattering characteristics of food products containing different types of contaminants.

In summary, this dissertation extends the application scope of the NESA-based low-rank approximation algorithm and develops fast algorithms for electromagnetic scattering problems involving PECs and complex nonuniform dielectric media. Direct solvers based on NESA are further developed for electromagnetic scattering problems in both low- and high-frequency regimes, enabling efficient solutions of Multiple Right-Hand Side (MRHS) problems. Finally, an online, real-time microwave sensing system for food contamination detection is developed based on electromagnetic scattering analysis.

Keywords: Volume-surface integral equation, Fast method, Direct solver, Nested equivalent source approximation, Microwave sensing.

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Time flies, and my doctoral journey is now coming to an end. Looking back on these years, I still remember the curiosity and anticipation I felt when I first joined the research group, knowing little about academic research yet eager to explore it. Today, as I complete this doctoral thesis, I realize how much this journey has involved not only exploration and achievement, but also difficulties, uncertainty, and moments of doubt. As I bring my doctoral studies to a close, I would like to express my sincerest gratitude to all those who have supported, helped, and accompanied me throughout this journey.

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*I would like to dedicate
this thesis to my loving
parents*

*I would like to dedicate this thesis to my
loving parents, whose unconditional love,
unwavering support, and constant
encouragement have always been my
greatest source of strength.*

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Chapter 1

Introduction

1.1 Research Background and Significance

In recent years, with the rapid development of the electronics and information technology industry, electromagnetic (EM) simulation has become an indispensable component of modern engineering design [1], [2]. Accurate analysis of the electromagnetic characteristics of target objects plays a vital role in a wide range of applications, including integrated circuit (IC) design, radar detection and aircraft stealth design, wireless communication systems, bioelectromagnetic, and electromagnetic compatibility (EMC). Compared with traditional analytical approaches, modern computational electromagnetic (CEM) methods overcome the limitations on target geometries and material configurations, enabling accurate modeling and simulation of increasingly complex electromagnetic structures. Consequently, many commercial EM simulation software packages, such as HFSS, FEKO, CST Studio Suite, and COMSOL Multiphysics, integrate multiple numerical algorithms to accommodate different electromagnetic scenarios, thereby significantly improving the efficiency of electromagnetic engineering design.

However, continuous advances in integrated circuit fabrication technologies and the expansion of operating frequency bands have led to increasingly complex electromagnetic structures characterized by intricate geometries, multiscale features, and heterogeneous material compositions. These developments impose more stringent requirements on the accuracy, computational efficiency, and applicability of electromagnetic algorithms [3]-[5]. For example, in the field of integrated circuit design [6]-[9], electromagnetic effects have become a dominant factor affecting circuit performance. At high operating frequencies, phenomena such as the skin effect, electromagnetic coupling, and crosstalk cannot be accurately characterized using conventional circuit models. Instead, rigorous solutions of Maxwell's equations based on computational electromagnetic algorithms are required to achieve the desired design accuracy. In addition,

parasitic effects and signal integrity issues have become increasingly critical in modern IC design. Serving as the essential link between Maxwell's equations and the physical implementation of integrated circuits, computational electromagnetics enables key tasks such as parasitic parameter extraction, power integrity analysis, signal integrity analysis, and advanced package modeling, making it one of the fundamental technologies underlying modern Electronic Design Automation (EDA) tools.

Another representative application is aircraft stealth design in the defence industry, which has become a critical aspect of modern electronic warfare. As the "eyes" of the battlefield, radar systems detect aircraft by receiving electromagnetic waves scattered from the target. Through optimized structural design and the application of radar-absorbing materials, aircraft can achieve an extremely low Radar Cross Section (RCS), thereby significantly reducing their detectability by radar systems. This, in turn, demands highly accurate electromagnetic modeling and simulation of complex platforms composed of metallic structures and inhomogeneous dielectric materials.

Computational electromagnetics has also become increasingly important in modern biomedical diagnostic systems [10], [11]. Conventional medical imaging techniques, such as X-ray imaging, computed tomography (CT), and magnetic resonance imaging (MRI), are associated with limitations including ionizing radiation, high equipment costs, and relatively long examination times. In contrast, microwave sensing systems based on computational electromagnetic algorithms detect and image pathological tissues by analyzing electromagnetic scattering fields generated by variations in the dielectric properties of biological tissues. Achieving reliable diagnostic performance requires electromagnetic algorithms capable of accurately modeling and simulating complex heterogeneous objects composed of multiple high-permittivity materials, thereby ensuring the accuracy and reliability of pathological tissue detection.

The above discussion demonstrates that the development of computational electromagnetics (CEM) has significantly enhanced the application value of electromagnetic engineering across a broad range of scientific and engineering disciplines. Meanwhile, increasingly sophisticated application scenarios and growing industrial demands have imposed more stringent requirements on the accuracy, efficiency, and robustness of electromagnetic algorithms, presenting substantial challenges to computational electromagnetics.

When the electromagnetic object under analysis consists of both metallic structures and inhomogeneous dielectric media, different discretization strategies must be adopted for the metallic surfaces and dielectric volumes in order to accurately characterize their electromagnetic behavior. Since surface discretization involves two-dimensional meshes, whereas volume discretization requires three-dimensional meshes, volume discretization generally produces a substantially larger number of unknowns than surface discretization for the same physical object. Consequently, the resulting system matrices are typically of very large scale, leading to excessive computational time and memory consumption as well as low computational efficiency. Furthermore, many practical

electromagnetic systems exhibit pronounced multiscale characteristics. Specifically, electrically large and geometrically smooth regions can be discretized using relatively coarse meshes, whereas electrically small or geometrically intricate features require much finer meshes to ensure modeling accuracy. Such multiscale discretization often results in severely ill-conditioned system matrices, causing iterative solvers to converge slowly or even fail to converge altogether, thereby compromising both computational efficiency and solution accuracy. Moreover, the computational complexity increases further in more challenging scenarios involving multiple excitation sources, making the efficient solution of electromagnetic problems considerably more difficult.

In view of these challenges, the development of accurate and efficient computational electromagnetic algorithms for complex structures and demanding application scenarios is of significant scientific and practical importance. The research presented in this dissertation is motivated by the need for accurate modeling and efficient numerical solutions of such complex electromagnetic problems.

This dissertation primarily focuses on the development of fast algorithms for electromagnetic scattering analysis. First, a fast solution technique for Volume-Surface Integral Equations (VSIEs) is developed based on the kernel-independent Nested Equivalent Source Approximation (NESA) algorithm [3], enabling accurate and efficient simulation of complex composite objects consisting of metallic conductors and inhomogeneous dielectric media. Subsequently, a direct solver based on the low-frequency NESA framework is proposed. By explicitly constructing the inverse of the system matrix, the proposed solver avoids the convergence issues commonly encountered by iterative methods while significantly improving the computational efficiency for Multiple Right-Hand Side (MRHS) problems. Furthermore, the proposed direct solver is extended to high-frequency electromagnetic scattering problems, resulting in a broadband direct solution framework that covers both low- and high-frequency regimes. Moreover, fast computational electromagnetics algorithms serve as an important bridge between electromagnetic theory and engineering applications. Their value lies not only in the development of theoretical models and the improvement of numerical solution efficiency, but also in providing reliable theoretical and computational support for the design, optimization, and performance analysis of complex electromagnetic systems. Finally, from the perspective of engineering applications, the fast electromagnetic scattering algorithms proposed in this dissertation are applied to guide the design of a microwave sensing system for food inspection. By exploiting the electromagnetic scattering information of food samples, potential internal contaminants can be accurately identified. Furthermore, through electromagnetic-wave focusing and scanning, the proposed system achieves improved detection performance compared with conventional horn-antenna-based systems.

1.2 Literature Review

1.2.1 Computational Electromagnetics Methods

Computational electromagnetics (CEM) is fundamentally based on Maxwell's equations, where the electromagnetic field distribution within a given domain is determined according to the governing boundary conditions and the equivalence principle [12]. Early research in computational electromagnetics primarily focused on objects with homogeneous material properties and simple geometries, such as spheres, cylinders, and infinitely large planar structures, all of which possess a high degree of symmetry. These problems could generally be solved using rigorous analytical methods but were inadequate for analyzing more complex electromagnetic structures. The rapid development of digital computers has greatly advanced numerical techniques for computational electromagnetics, enabling the simulation of increasingly sophisticated structures, including antennas, integrated circuits, radar targets, and other multiscale heterogeneous systems.

According to the treatment of Maxwell's equations, numerical electromagnetic methods can generally be classified into two categories: high-frequency asymptotic methods and full-wave methods. High-frequency asymptotic methods are primarily intended for electrically large objects. Their fundamental idea is to simplify Maxwell's equations by exploiting asymptotic approximations when the characteristic dimensions of the target are much larger than the operating wavelength. Representative high-frequency techniques include Geometrical Optics (GO) [13], Geometrical Theory of Diffraction (GTD) [14], Uniform Theory of Diffraction (UTD)[15], [16], Physical Optics (PO) [17], Physical Theory of Diffraction (PTD) [18], [19], and the Shooting and Bouncing Rays (SBR) method [20]. However, for electrically small objects or multiscale structures with complex geometrical features, these asymptotic techniques may suffer from significant accuracy degradation, making full-wave methods necessary.

Unlike high-frequency asymptotic methods, full-wave methods preserve both near-field and far-field coupling effects and therefore provide solutions that rigorously satisfy Maxwell's equations. Representative full-wave methods include the Finite Element Method (FEM)[21], the Finite-Difference Time-Domain (FDTD) method [22], [23], and the Method of Moments (MoM) [24]. Among these, FEM and FDTD are differential equation-based methods, whereas MoM is an integral equation-based method. Differential equation methods provide a local description of the electromagnetic field by modeling the field variation at every point within the computational domain. Through volumetric discretization, they are well suited for analyzing electromagnetic problems involving inhomogeneous dielectric materials and complex structures. However, these methods require artificial truncation of the computational domain through appropriate boundary conditions [25], [26], which inevitably introduces additional approximation errors and may reduce the overall solution accuracy.

In contrast, integral equation methods employ Green's functions (GFs) to express the electromagnetic field at any observation point as the superposition of contributions from all equivalent sources distributed throughout the domain. Consequently, the radiation condition is satisfied inherently, eliminating the need for artificial boundary truncation and generally resulting in higher numerical accuracy. According to the geometric support of the unknown equivalent sources, integral equations can be classified into Volume Integral Equations (VIEs), Surface Integral Equations (SIEs), and their hybrid formulation, the Volume-Surface Integral Equation (VSIE) [27]-[29]. Surface integral equations are primarily applicable to Perfect Electric Conductors (PECs) and piecewise homogeneous dielectric media, where equivalent sources are distributed on conductor surfaces or dielectric interfaces to model the electromagnetic behaviours of the target structures. Representative surface integral equation formulations for dielectric media include the Poggio–Miller–Chang–Harrington–Wu–Tsai (PMCHWT) equation [30], [31], the Müller integral equation [32], [33], and the Electric and Magnetic Current Combined-Field Integral Equation (JMCFIE) [34], [35]. Compared with volume integral equation formulations, surface integral equations generally involve significantly fewer unknowns. However, they are not applicable to arbitrarily inhomogeneous dielectric media. Consequently, Volume Integral Equations (VIEs) are commonly employed for modelling inhomogeneous dielectrics. For composite objects consisting of both PECs and inhomogeneous dielectric materials, Volume-Surface Integral Equations (VSIEs) are typically constructed by formulating surface integral equations on the PEC surfaces and volume integral equations within the dielectric regions. Unlike differential equation-based methods such as the Finite Element Method (FEM) and the Finite-Difference Time-Domain (FDTD) method, the Method of Moments (MoM) based on integral equations does not require discretization of the entire computational domain. Instead, according to the equivalence principle, only equivalent surface currents or volume currents distributed on the object need to be solved. Nevertheless, the system matrix generated by MoM is dense. For direct solvers, the memory and computational complexities are $O(N^2)$ and $O(N^3)$, respectively, whereas for iterative solvers they are $O(N^2)$ and $O(N_{iter}N^2)$, respectively, where N_{iter} denotes the number of iterations required for convergence. As a result, conventional MoM is generally limited to problems with relatively small numbers of unknowns and becomes computationally prohibitive for large-scale electromagnetic simulations. To further improve the computational capability of MoM and extend its applicability, extensive research has been devoted to the development of fast algorithms over the past several decades. These methods have significantly reduced both computational time and memory requirements while also enabling the development of a variety of efficient direct solvers.

Fast algorithms for the Method of Moments primarily exploit the physical and mathematical properties of Green's functions. They can generally be classified into three categories: Fast Fourier Transform (FFT)-based methods, Fast

Multipole Method (FMM)-based methods, and kernel-independent low-rank matrix compression methods. FFT-based approaches take advantage of the convolution structure of Green's functions in homogeneous media. By introducing auxiliary regular grids, far-field interactions are transformed into convolution operations, which can then be efficiently evaluated using the FFT. Representative methods include the Precorrected Fast Fourier Transform (P-FFT) [36], [37], the Adaptive Integral Method (AIM) [38], [39], and the Integral Equation Fast Fourier Transform (IE-FFT) method [40], [41]. Owing to their relatively simple implementation, excellent parallel scalability, and suitability for GPU acceleration, these methods have been widely applied to printed circuit board (PCB) analysis, electromagnetic compatibility (EMC), integrated circuit packaging, and high-speed interconnect analysis.

The Fast Multipole Method (FMM) is one of the most representative analytical acceleration techniques for the Method of Moments. Originally proposed by Rokhlin [42], [43], FMM introduces the concept of multipole expansion to transform pairwise interactions between distant particles into interactions between groups of particles. Owing to its outstanding computational performance, FMM was soon extended to the solution of the Helmholtz and Maxwell equations and has since become one of the most influential fast algorithms in computational electromagnetics. By exploiting the analytical properties of the free-space Green's function, FMM represents far-field interactions using truncated multipole expansions, thereby avoiding direct evaluation of interactions between all pairs of unknowns and enabling efficient propagation of far-field information.

To further improve computational efficiency for large-scale problems, W. C. Chew and his collaborators at the University of Illinois Urbana–Champaign introduced an octree-based hierarchical data structure and developed the Multilevel Fast Multipole Algorithm (MLFMA) [44]–[47]. By constructing a multilevel tree structure to perform hierarchical clustering and recursive translation operations, MLFMA reduces both the computational and memory complexities to $O(N \log N)$. In recent years, research on MLFMA has primarily focused on wideband MLFMA [55]–[58], parallel MLFMA [49]–[54], and hybrid fast algorithms that combine MLFMA with hierarchical matrix techniques [48].

Unlike the Fast Multipole Method (FMM), which relies on the analytical expansion of Green's functions to achieve matrix compression, low-rank matrix compression methods exploit the inherent low-rank property of far-field interaction matrix blocks arising from electromagnetic integral operators. Matrix compression and accelerated computation are achieved by constructing low-dimensional approximations of these matrix blocks. Since these methods primarily depend on the algebraic structure of the system matrix rather than the analytical form of the kernel function, they generally exhibit greater generality and applicability. However, their computational efficiency is typically lower than that of kernel-dependent methods. Representative low-rank compression techniques include the Adaptive Cross Approximation (ACA) [59], [60], the

Multilevel Matrix Decomposition Algorithm (MLMDA) [61], [62], the Multilevel UV (MLUV) method [63], the Interpolative Decomposition (ID) method [64], [65], and QR decomposition-based methods [66], [67]. In recent years, M. M. Li and co-workers at Nanjing University of Science and Technology proposed the Nested Equivalent Source Approximation (NESA) [68]-[70], a kernel-independent low-rank compression technique. In NESA, uniformly distributed equivalent sources are constructed on equivalent spheres for each group within an octree hierarchy. Far-field interactions are then transformed into interactions between the corresponding equivalent sources. Furthermore, because coupling matrices associated with equivalent-source groups having identical relative spatial configurations are identical, they can be reused throughout the hierarchical structure, thereby significantly reducing memory consumption. Each of the aforementioned methods possesses its own advantages and is suitable for different classes of electromagnetic simulation problems. Moreover, these approaches can be combined to complement one another, leading to further improvements in computational efficiency. These fast algorithms substantially accelerate the Matrix-Vector Product (MVP) and are therefore widely employed in conjunction with iterative methods for solving large-scale electromagnetic problems. Nevertheless, a major limitation of iterative solvers is their dependence on convergence behavior. For complex electromagnetic problems, the iterative process may converge very slowly or even fail to converge, resulting in significant degradation of both computational efficiency and solution accuracy. To alleviate these convergence issues, various acceleration techniques, such as preconditioning and the Domain Decomposition Method (DDM) [71], [72], have been extensively investigated.

Early studies on preconditioning mainly focused on general-purpose algebraic preconditioners. Representative approaches include the Block Diagonal preconditioner [45], the Symmetric Successive Over-Relaxation (SSOR) preconditioner [73], the Sparse Approximate Inverse (SAI) preconditioner [74]-[76], and the Incomplete LU (ILU) preconditioner [77]-[79]. Since the 1990s, physics-based preconditioning techniques have attracted increasing attention in computational electromagnetics. The Loop-Tree preconditioner [80], [81] constructs new basis functions through linear combinations of conventional basis functions. Specifically, divergence-free Loop basis functions are generated based on mesh vertices, while the remaining basis functions constitute an irrotational Tree structure. This decomposition effectively separates the magnetic vector potential from the electric scalar potential at low frequencies, thereby significantly alleviating the low-frequency ill-conditioning of integral equations. Subsequently, G. Vecchi proposed the Multiresolution (MR) preconditioner [82], [83], which is constructed using multiresolution basis functions. To address multiscale electromagnetic problems, F. P. Andriulli and G. Vecchi further developed a hierarchical multiresolution preconditioning technique [84]. Building upon this work, F. Vipiana and G. Vecchi proposed the hybrid Multiresolution-Incomplete LU (MR-ILU) preconditioner [85], which significantly improves the conditioning of system matrices arising from high-resolution multiscale problems.

To further overcome the dense-mesh breakdown encountered in integral equation formulations, the Calderón preconditioning technique proposed by F. P. Andriulli et al. [86] has become a milestone in the development of integral equation preconditioners owing to its rigorous theoretical foundation and excellent frequency stability. This technique was subsequently extended to the Combined Field Integral Equation (CFIE) [87] and Volume-Surface Integral Equation (VSIE) formulations [88], further broadening its applicability. Unlike the aforementioned preconditioning techniques, the Domain Decomposition Method (DDM) [89], [90] partitions the computational domain into multiple subdomains, each of which is solved independently. The subdomain solutions are coupled through interface boundary conditions to obtain the global solution of the entire system. This strategy provides considerable flexibility for handling multiscale problems with heterogeneous structural characteristics. For example, electrically large smooth regions and electrically small fine structures can be assigned to different subdomains, thereby improving the ill-conditioning of the system matrix caused by multiscale discretization. In recent years, DDM has become an active research topic for solving large-scale electromagnetic problems. Representative contributions have been made by the research groups led by Z. P. Nie at the University of Electronic Science and Technology of China (UESTC) [91], X. Q. Sheng at the Beijing Institute of Technology (BIT) [92], [93], and X. W. Zhao at Xidian University [94], [95]. In addition to iterative methods, another important class of solution techniques for electromagnetic system matrices is direct methods.

Unlike iterative solvers, direct methods explicitly construct an approximate inverse or factorization of the system matrix, allowing the current solution to be obtained directly through multiplication with the excitation vector. Consequently, direct methods are free from the convergence issues that often limit iterative solvers and have therefore become one of the major research directions in computational electromagnetics. Another significant advantage of direct methods is their superior efficiency in solving Multiple Right-Hand Side (MRHS) problems. Since the matrix factorization is performed only once, the solutions corresponding to multiple excitation vectors can be obtained directly without restarting an iterative procedure. As a result, direct methods are considerably more efficient than iterative methods for applications such as monostatic Radar Cross Section (RCS) simulations involving multiple incident directions. The primary limitation of direct methods is that they require an additional matrix factorization stage before the solution process. Consequently, for problems involving only a small number of right-hand-side vectors, direct methods may not offer a computational advantage over iterative approaches. Classical direct solvers, including LU factorization, Gaussian elimination, and QR factorization, have traditionally been restricted to relatively small-scale problems because of their prohibitively high computational complexity. In recent years, however, the integration of direct solvers with various matrix compression techniques has significantly reduced their computational cost. Depending on the adopted admissibility condition, the compressed system matrix exhibits different

hierarchical structures and matrix representations, leading to a variety of efficient direct solution frameworks. The admissibility condition determines whether the coupling matrix between two groups should be approximated. When all off-diagonal coupling matrices are approximated, that is, all non-self-interaction matrix blocks are compressed, the resulting criterion is referred to as the Weak Admissibility Condition. Under this condition, the hierarchical matrix structure is relatively simple because inadmissible blocks appear only along the diagonal of the system matrix. Representative direct solvers based on this matrix structure include the Hierarchically Semi-Separable (HSS) method [96]-[100] and the Hierarchically Off-Diagonal Low-Rank (HODLR) method [101]-[104]. Although these approaches accelerate the direct solution process to a certain extent, their computational efficiency is generally inferior to that of direct solvers based on the Strong Admissibility Condition. Since near-field interaction matrices generally possess higher numerical ranks than far-field interaction matrices, the Strong Admissibility Condition introduces a distance criterion between groups to determine whether a coupling matrix should be approximated. Consequently, only sufficiently well-separated interaction blocks are compressed, whereas nearby interaction blocks are computed directly. This strategy significantly improves the compression efficiency of low-rank matrix approximations. Compared with weak admissibility, the resulting hierarchical matrix structure is more complicated because inadmissible blocks are no longer confined to the diagonal of the system matrix and may appear at arbitrary locations.

Representative direct solvers based on the strong admissibility condition include butterfly-based direct solvers [105]-[107], the Hierarchical Matrix (H-matrix) method [59], [108]-[110], and its enhanced counterpart, the H^2 -matrix method [111], [122]. The H-matrix was first proposed by W. Hackbusch in 1999 as a hierarchical sparse matrix representation [108] and was subsequently applied to multiscale problems in 2000 [109]. In 2008, the research group led by D. Jiao at Purdue University pioneered the application of the H-matrix technique to electromagnetic integral equation problems [112]. Building upon the Matrix Decomposition Algorithm (MDA) developed for the improved Method of Moments, T. Wan proposed a kernel-independent LU factorization algorithm based on the H-matrix framework in 2011 [113], [128]. Compared with the H-matrix, the H^2 -matrix further reduces computational and memory complexities by introducing transfer matrices between adjacent hierarchical levels. These transfer matrices enable higher-level coupling matrices to be represented using lower-level coupling information, thereby achieving a nested representation of the entire system matrix through the reuse of cluster bases constructed at the leaf level. The H^2 -matrix framework was first introduced by S. Börm, W. Hackbusch, and their collaborators in 2002 [111]. Subsequently, D. Jiao and co-workers carried out extensive research on the application of H^2 -matrices to computational electromagnetics [123]-[126]. In 2009, they developed an H^2 -matrix-based integral equation solver for computational electrodynamics, further reducing the

computational complexity of direct electromagnetic analysis [114]. In 2018, M. M. Ma et al. proposed an error-controllable direct solver for volume integral equation formulations based on the H^2 -matrix representation [115]. This approach was further improved in 2019 by introducing a refined factorization procedure in which the cluster bases are progressively and completely updated throughout the factorization process, resulting in enhanced computational accuracy and efficiency [127].

The H-matrix-based direct solvers for electromagnetic problems discussed above generally achieve the compression of far-field matrix blocks through analytical expansions of Green's functions. Alternatively, kernel-independent purely algebraic techniques can also be employed, among which skeletonization based on Interpolative Decomposition (ID) [116], [117] is one of the most representative approaches. In 2012, the research group led by J. F. Lee introduced the skeletonization technique and developed a direct solver for integral equation formulations of electromagnetic scattering from Perfect Electric Conductors (PECs) [118]. In 2020, Z. Rong et al. further improved the efficiency of skeletonization by accelerating the skeleton selection process [119]. Subsequently, in 2021, the research group led by J. Hu proposed the Strong Admissibility Skeletonization Factorization (SASF) method for electromagnetic scattering from PEC objects [120], which further reduced both computational and memory complexities. In 2018, M. M. Li et al. combined skeletonization with the Nested Equivalent Source Approximation (NESA) to develop a hybrid matrix compression scheme [70], in which skeletonization is employed at the leaf level while equivalent-source approximation is adopted at higher levels, thereby improving the overall computational efficiency. Building upon this hybrid compression framework, Y. H. Zuo et al. incorporated the H^2 -matrix representation and proposed a direct solver for low- and intermediate-frequency electromagnetic scattering from PEC objects [121].

1.2.2 Microwave sensing technology

In engineering applications such as microwave detection, radar imaging, biomedical diagnosis, and industrial nondestructive testing, electromagnetic scattering analysis provides an important foundation for system design and performance evaluation. Microwave Sensing (MWS) exploits variations in the reflection, transmission, and scattering characteristics arising from the interaction between electromagnetic waves and media to detect and characterize the dielectric properties of target objects.

In recent years, microwave sensing technology has attracted considerable research interest in biomedical applications. Since different human tissues and tissues under different pathological conditions exhibit distinct complex permittivity and electrical conductivity, microwave signals undergo different levels of reflection, transmission, scattering, and resonance responses during propagation. Therefore, microwave sensing enables noninvasive detection and

characterization of biological tissues. Compared with ionizing-radiation-based imaging techniques such as X-ray and computed tomography (CT), microwaves are non-ionizing radiation and therefore offer relatively good biological safety. Reference [10] developed a microwave sensing device for stroke monitoring. The system employs an antenna array consisting of 24 antennas. When a stroke occurs, the distribution of blood within the head changes. By analyzing variations in the scattering parameters measured at the antenna ports, the system can continuously monitor pathological changes in the internal brain tissues. For breast tissue detection, Ref. [152] investigated a 16×16 shielded slot-antenna array for microwave breast imaging, which enables rapid electronic scanning and electromagnetic focusing for breast cancer detection.

It can be seen that the application of microwave sensing in biomedical fields primarily exploits the interaction between electromagnetic waves and biological tissues, identifying targets and assessing their physiological or pathological states by detecting changes in their dielectric properties. However, a major technical challenge in biomedical microwave sensing arises from the variability of physiological samples. Since the biological tissue structures and dielectric properties differ from one individual to another, each sample generally needs to be modeled and analyzed separately.

In addition, microwave sensing systems for food contaminant detection represent another important application of electromagnetic scattering theory. Unlike the complex and highly individualized targets encountered in biomedical applications, similar products in the food industry are generally manufactured using standardized production processes, resulting in a high degree of consistency in their geometric shapes and dimensional parameters. Consequently, the electromagnetic properties of food products are more amenable to modeling and parametric characterization, making it possible to establish representative electromagnetic models for standardized food samples. This provides a reliable modeling foundation for electromagnetic simulation of online microwave sensing systems, antenna optimization and design, and the development of target-defect detection algorithms, while also contributing to improved stability of the detection results in practical systems.

The detection principle of microwave sensing for food products is similar to that used in biomedical applications. Since food materials and contaminants generally exhibit different complex permittivities, variations in the internal composition of food products can alter the propagation characteristics of electromagnetic waves, enabling microwave sensing to perform nondestructive detection of the internal state and contamination of food products. Compared with optical detection methods, microwaves offer strong penetration into dielectric media and are less susceptible to illumination conditions and surface contamination. They also provide advantages such as high detection speed, ease of online integration, and relatively low cost. Consequently, microwave sensing has been extensively investigated in recent years for applications including food quality inspection, biomedical detection, and industrial process monitoring.

For example, electromagnetic simulations of the internal electric field of a cocoa cream can and the scattering parameters at the ports of a microwave sensing system can be used to achieve imaging of contaminants inside the food product [153]. The detection coverage for the food sample can be further increased by employing multiple pairs of antennas [154]. Since different food materials exhibit different electromagnetic transmission characteristics, while the strength of the internal electromagnetic scattering directly affects the imaging quality and detection performance of the sensing system, enhancing the electromagnetic scattering inside the food sample has become an important research direction in microwave sensing for food inspection. Reference [155] proposed an antenna-array-based microwave sensing system capable of focusing electromagnetic waves over a relatively large spatial region, thereby improving the system's ability to detect contaminants inside oil cans.

1.3 Research Contents and Contributions of This Dissertation

The main focus of this dissertation is the development of fast algorithms for Volume-Surface Integral Equations (VSIEs) based on the Nested Equivalent Source Approximation (NESA) and the development of direct solvers for surface integral equations based on NESA. Specifically, the equivalent-source concept is extended to composite electromagnetic scattering problems involving metallic objects and inhomogeneous dielectric media. By using a relatively small number of equivalent-source coupling interactions to represent the coupling among a large number of volume unknowns generated by volumetric discretization, the computational efficiency of VSIE solutions is significantly improved. Furthermore, based on the NESA-based surface integral equation framework, a direct solver for broadband electromagnetic problems is developed by exploiting the nested property of equivalent sources and combining it with the H^2 -matrix structure. Finally, the proposed fast electromagnetic scattering algorithms are applied to the design of a microwave sensing system for food inspection, achieving improved system detection performance. The main contributions and innovations of this dissertation are summarized as follows:

(1) A NESA-based fast solver for VSIEs (NESA-VSIE) is proposed. This work has been published in *IEEE Antennas and Wireless Propagation Letters*.

A fast solution framework for VSIEs is developed based on the NESA algorithm. For electromagnetic scattering problems involving composite objects consisting of metallic structures and inhomogeneous dielectric media, the coupling matrices of far-field interaction groups are efficiently compressed through equivalent-source approximation. At the leaf level of the octree structure, the system matrix is categorized into four types of interactions according to the physical properties of the unknown sources, including metallic surface currents and dielectric volume currents. At higher levels of the octree, only interactions between NESA equivalent surface sources are retained. Consequently, the number

of unknowns introduced by volumetric discretization is significantly reduced, thereby improving the efficiency of solving volume integral equation problems.

(2) A NESA-based direct solver for low- and intermediate-frequency electromagnetic problems is developed. This work has been published in *IEEE Transactions on Antennas and Propagation*.

After compressing far-field coupling matrices using NESA, the approximate inverse of the system matrix is represented as a sequence of matrix multiplications through a series of error-controllable factorization and elimination procedures. The proposed method improves the efficiency of solving Multiple Right-Hand Side (MRHS) problems and avoids the convergence issues that may occur in iterative solvers. Furthermore, Singular Value Decomposition (SVD) is employed to compress the Schur complement matrices generated during the factorization process, thereby maintaining high computational efficiency. The computational and memory complexities of the proposed matrix factorization-based direct solver are both $O(N)$.

(3) A wideband direct solver based on WNESA is proposed for electrically large electromagnetic problems.

To address electrically large problems, a broadband direct solver is developed based on the Wideband Nested Equivalent Source Approximation (WNESA). Since the radiation and reception matrices of group interactions vary with different directions in WNESA, elimination matrices cannot be directly constructed using a unified cluster basis during matrix factorization. By incorporating radiation and reception matrices from all directions, SVD is employed to extract their common column space, from which a generalized cluster basis is constructed. The elimination matrices generated based on this generalized cluster basis are therefore effective for all admissible interaction blocks. In addition, the near-field interaction index is modified according to the directional admissibility condition of WNESA, resolving the incomplete searching problem of inadmissible blocks during the direct solution process. The computational and memory complexities of the proposed matrix factorization-based direct solver are $O(N^2)$ and $O(N \log N)$, respectively.

(4) A design of an online microwave sensing system for food inspection based on target scattering information is proposed. This work has been published in *IEEE Transactions on Antennas and Propagation*.

An online microwave sensing system for food inspection is developed based on the scattering characteristics of the target. An antenna array is introduced to focus and scan the electromagnetic beam, thereby expanding the detection range and increasing the electric-field intensity within the detection region. This improves the detection performance of conventional horn-antenna-based microwave sensing systems for food contaminants. Furthermore, the online sensing system is fabricated and experimentally implemented, enabling real-time dynamic detection of food products as they move along a conveyor track.

1.4 Organization of This Dissertation

This dissertation is organized into six chapters, which are arranged as follows.

Chapter 1 provides an introduction to the research. The background and significance of the fast algorithms for the Method of Moments (MoM) and related direct solvers in computational electromagnetics are presented. In addition, the historical development and current research status of these methods are reviewed.

Chapter 2 introduces the fundamental theories of electromagnetic integral equation methods, including the Volume Integral Equation (VIE) and Surface Integral Equation (SIE) formulations. The principles of the low-frequency and high-frequency Nested Equivalent Source Approximation (NESA) fast algorithms employed in this dissertation are presented, together with their low-rank compression representations. Furthermore, the Loop–Tree preconditioning technique is investigated for alleviating the low-frequency breakdown problem encountered in the Method of Moments.

Chapter 3 focuses on the development of a NESA-based fast solver for Volume-Surface Integral Equations (VSIEs), aiming to accelerate the solution of electromagnetic scattering problems involving composite objects consisting of metallic structures and dielectric media.

Chapter 4 investigates a NESA-based direct solver for low- and intermediate-frequency electromagnetic problems. The far-field and near-field interactions are factorized separately, enabling the explicit construction of an approximate inverse of the system matrix for efficiently solving electromagnetic scattering problems with Multiple Right-Hand Sides (MRHSs).

Chapter 5 extends the direct solver developed in Chapter 4 to electrically large targets by developing a high-frequency direct solver based on Wideband Nested Equivalent Source Approximation (WNESA). By combining the algorithms presented in Chapters 4 and 5, a broadband direct solution framework is established. This chapter introduces the construction method of the generalized cluster basis in the WNESA-based direct solver framework and addresses the incomplete indexing problem of inadmissible blocks caused by the directional admissibility condition of WNESA. Subsequently, electromagnetic scattering problems involving electrically large targets are analyzed.

Chapter 6 focuses on the design of a microwave sensing system for food inspection based on electromagnetic scattering. An antenna array is employed to achieve electromagnetic-wave focusing and scanning, while a machine learning model is developed to learn the scattering information of the food samples under test, thereby improving the overall detection performance of the sensing system.

Chapter 7 summarizes the research contents and achievements of this dissertation and discusses potential directions for future research.

Chapter 2

Introduction to Electromagnetic Integral Equation Methods and Fundamental Principles of Nested Equivalent Source Approximation

2.1 Introduction

Electromagnetic integral equation methods are among the most important numerical techniques for solving electromagnetic scattering and radiation problems in computational electromagnetics. Unlike differential equation-based methods such as the Finite Element Method (FEM) and the Finite-Difference Time-Domain (FDTD) method, integral equation methods are established based on Maxwell's equations and the corresponding Green's functions (GFs). By transforming electromagnetic field problems into integral equation formulations, these methods inherently satisfy the radiation condition at infinity and avoid the errors introduced by artificial truncation boundaries and absorbing boundary conditions. Therefore, integral equation methods possess unique advantages for analyzing electromagnetic problems in open regions. According to the definition domain of the unknown quantities, electromagnetic integral equation methods can be mainly classified into two categories: Surface Integral Equation (SIE) methods [27] and Volume Integral Equation (VIE) methods [129]. The SIE method is based on the equivalence principle, where the electromagnetic effects inside the scatterer are represented by equivalent currents distributed on the object surface. Since only the boundary of the object needs to be discretized, SIE methods generally require fewer unknowns and provide higher computational efficiency. For Perfect Electric Conductor (PEC) targets, representative SIE formulations include the Electric Field Integral Equation (EFIE) [24], the Magnetic Field Integral Equation (MFIE) [130], [131], and the Combined Field Integral Equation

(CFIE) [87], [132]. For dielectric objects, surface integral formulations such as the Poggio–Miller–Chang–Harrington–Wu–Tsai (PMCHWT) equation [30], [31] and the Müller integral equation [32], [33] are commonly employed. Different from SIE methods, VIE methods directly employ equivalent volume currents or polarization currents inside the target as unknowns. By establishing integral relationships within the volumetric region, VIE methods describe the interaction between electromagnetic fields and material parameters. VIE methods are capable of accurately modeling inhomogeneous dielectric media and are not restricted by continuously varying material properties inside the object. However, since the entire volume region must be discretized, VIE formulations generally introduce a much larger number of unknowns compared with SIE formulations. For complex composite structures consisting of both inhomogeneous dielectric materials and metallic components, the advantages of SIE and VIE methods can be combined by constructing a Volume-Surface Integral Equation (VSIE) formulation. In this approach, surface integral equations are employed for metallic regions, while volume integral equations are used for dielectric regions, enabling accurate modelling of complex electromagnetic scattering problems involving multiple materials. For integral equation formulations, the Method of Moments (MoM) is commonly adopted for discretization and numerical solution. However, MoM suffers from the low-frequency breakdown problem in low-frequency regimes [144], [145]. Under such conditions, the system matrix becomes severely ill-conditioned or even singular, and conventional solution techniques may fail to obtain accurate results. For example, when applying the Multilevel Fast Multipole Algorithm (MLFMA) to low-frequency problems, specialized kernel compression techniques are required to achieve stable far-field approximations [146], [147]. Preconditioning techniques can effectively alleviate the ill-conditioning of the system matrix and improve the accuracy and stability of MoM solutions. On the other hand, for large-scale electromagnetic problems involving a large number of unknowns, fast algorithms are required to improve computational efficiency. The Nested Equivalent Source Approximation (NESA) constructs uniformly distributed equivalent sources on equivalent spheres for each group within an octree structure. Through a pair of inverse source processes, the far-field interactions between groups are transformed into interactions between their corresponding equivalent sources [3]. Since NESA is a kernel-independent low-rank matrix compression method, it does not rely on the analytical form of Green's functions and therefore exhibits strong adaptability. By mapping the coupling between volume unknowns into the coupling between equivalent surface sources, NESA provides an efficient framework for accelerating the solution of VSIE problems.

This chapter first reviews the fundamental principles of electromagnetic integral equation methods and presents the corresponding mathematical formulations and derivations. Subsequently, the solution procedure of integral equations based on the Method of Moments (MoM) is introduced. In addition, the low-frequency breakdown problem encountered in MoM is investigated. This problem is mainly attributed to the significant imbalance between the

contributions of the magnetic vector potential and the electric scalar potential in the integral equation formulations, which leads to a severely ill-conditioned system matrix. To address this issue, the Loop–Tree preconditioning technique is studied to improve the conditioning of the system matrix and enhance the stability of MoM solutions. Finally, the fundamental concepts and implementation principles of the fast algorithm Nested Equivalent Source Approximation (NESA) are introduced. This provides the theoretical foundation for the subsequent development of the NESA-based fast solver for Volume-Surface Integral Equations (VSIEs) and the NESA-based direct solvers for Surface Integral Equations (SIEs) presented in the following chapters.

2.2 Formulation of Integral Equations

The first step in the integral equation method is to establish the corresponding integral equations based on the boundary conditions of Perfect Electric Conductors (PECs) and the relationships between electromagnetic field sources and fields inside dielectric media. For composite targets consisting of PEC regions and inhomogeneous dielectric materials, Surface Integral Equations (SIEs) are formulated for the PEC regions, while Volume Integral Equations (VIEs) are constructed for the inhomogeneous dielectric regions. These two formulations are then combined to form a Volume-Surface Integral Equation (VSIE) formulation. This subsection provides a detailed introduction to the derivation procedures of various electromagnetic integral equation formulations.

2.2.1 Surface Integral Equation

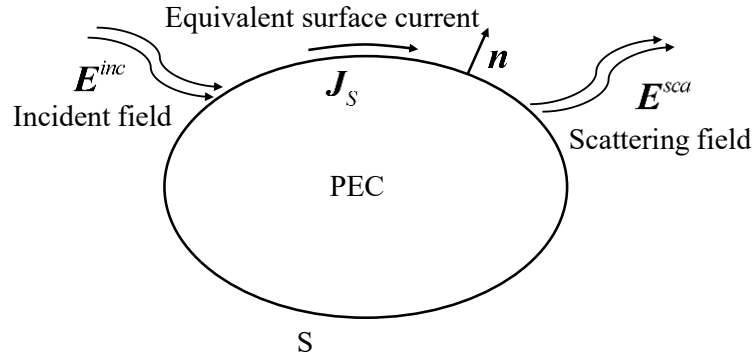


Fig. 2.1. Schematic illustration of electromagnetic scattering from a Perfect Electric Conductor (PEC).

As illustrated in Fig. 2.1, when a plane wave is incident on a Perfect Electric Conductor (PEC) object, the resulting scattered electromagnetic field can be equivalently represented as the radiation field generated by the induced surface currents on the object surface:

$$E^{sca}(\mathbf{r}) = -j\omega\mu_0 \int_S \mathbf{J}_s(\mathbf{r}') \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') dS \quad (2.1)$$

where $\mathbf{J}_s$ represents the equivalent surface current on the surface of the Perfect Electric Conductor (PEC) object, $\mathbf{r}'$ and $\mathbf{r}$ denote the source point and observation point in space, respectively, and $\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}')$ represents the dyadic Green's function in free space:

$$\bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') = \left(\bar{\mathbf{I}} + \frac{1}{k_0^2} \nabla \nabla \right) G(\mathbf{r}, \mathbf{r}') \quad (2.2)$$

In (2.2), $\bar{\mathbf{I}}$ denotes the unit dyadic, $k_0 = \omega \sqrt{\mu_0 \epsilon_0}$ represents the wavenumber in free space, ϵ_0 and μ_0 are the permittivity and permeability of free space, respectively, ω is the angular frequency, and $G(\mathbf{r}, \mathbf{r}')$ represents the scalar Green's function:

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk_0|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \quad (2.3)$$

The total field in space is equal to the sum of the incident field and the scattered field:

$$\mathbf{E}^{tot}(\mathbf{r}) = \mathbf{E}^{inc}(\mathbf{r}) + \mathbf{E}^{sca}(\mathbf{r}) \quad (2.4)$$

The boundary condition of the electric field on the surface of the Perfect Electric Conductor (PEC) gives:

$$\mathbf{n} \times \mathbf{E}^{tot}(\mathbf{r}) = 0, \quad \mathbf{t} \cdot \mathbf{E}^{tot}(\mathbf{r}) = 0 \quad (2.5)$$

where $\mathbf{t}$ denotes the tangential unit vector on the conductor surface, and $\mathbf{n}$ represents the outward normal unit vector. Therefore, by combining Eqs. (2.1), (2.4), and (2.5), the Electric Field Integral Equation (EFIE) for the surface of the Perfect Electric Conductor (PEC) can be derived as follows:

$$j\omega\mu_0 \int_S \mathbf{t} \cdot \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}_s(\mathbf{r}') dS = \mathbf{t} \cdot \mathbf{E}^{inc}(\mathbf{r}) \quad (2.6)$$

For Eq. (2.1), the scattered electric field can be expressed in terms of the magnetic vector potential and the electric scalar potential as follows:

$$\mathbf{E}^{sca}(\mathbf{r}) = -j\omega\mathbf{A}(\mathbf{r}) - \nabla\Phi(\mathbf{r}) \quad (2.7)$$

where:

$$\mathbf{A}(\mathbf{r}) = \mu_0 \int_S G(\mathbf{r}, \mathbf{r}') \mathbf{J}_s(\mathbf{r}') dS \quad (2.8)$$

$$\Phi(\mathbf{r}) = -\frac{1}{j\omega\epsilon_0} \int_S G(\mathbf{r}, \mathbf{r}') \nabla \cdot \mathbf{J}_s(\mathbf{r}') dS \quad (2.9)$$

Therefore, the electric field integral equation on the conductor surface can also be expressed as:

$$\mathbf{t} \cdot j\omega\mathbf{A}(\mathbf{r}) + \mathbf{t} \cdot \nabla\Phi(\mathbf{r}) = \mathbf{t} \cdot \mathbf{E}^{inc}(\mathbf{r}) \quad (2.10)$$

In addition, the corresponding Magnetic Field Integral Equation (MFIE) can be derived based on the magnetic field boundary condition on the surface of the conductor. Since the total magnetic field and the surface current satisfy the following relationship:

$$\mathbf{n} \times [\mathbf{H}^{sca}(\mathbf{r}) + \mathbf{H}^{inc}(\mathbf{r})] = \mathbf{J}_s(\mathbf{r}) \quad (2.11)$$

The scattered magnetic field $\mathbf{H}^{sca}(\mathbf{r})$ on the conductor surface can be expressed as:

$$\mathbf{H}^{sca}(\mathbf{r}) = \nabla \times \int_S G(\mathbf{r}, \mathbf{r}') \mathbf{J}_s(\mathbf{r}') dS \quad (2.12)$$

Substituting Eq. (2.12) into Eq. (2.11), the Magnetic Field Integral Equation (MFIE) for a closed Perfect Electric Conductor (PEC) object can be derived as follows:

$$\mathbf{J}_s(\mathbf{r}) - \mathbf{n} \times \nabla \times \int_S G(\mathbf{r}, \mathbf{r}') \mathbf{J}_s(\mathbf{r}') dS = \mathbf{n} \times \mathbf{H}^{inc}(\mathbf{r}) \quad (2.13)$$

After removing the singularity in Eq. (2.13), it can be expressed as:

$$\frac{1}{2} \mathbf{J}_s(\mathbf{r}) - \mathbf{n} \times \nabla \times P.V. \int_S G(\mathbf{r}, \mathbf{r}') \mathbf{J}_s(\mathbf{r}') dS = \mathbf{n} \times \mathbf{H}^{inc}(\mathbf{r}) \quad (2.14)$$

where $P.V.$ denotes the Cauchy principal value integral. Similarly, by representing Eq. (2.14) in terms of the magnetic vector potential, the following equation can be obtained:

$$\frac{1}{2} \mathbf{J}_s(\mathbf{r}) - \frac{1}{\mu_0} \mathbf{n} \times \nabla \times \mathbf{A}(\mathbf{r}) = \mathbf{n} \times \mathbf{H}^{inc}(\mathbf{r}) \quad (2.15)$$

For the electromagnetic scattering analysis of Perfect Electric Conductor (PEC) targets, both the Electric Field Integral Equation (EFIE) and the Magnetic Field Integral Equation (MFIE) are commonly employed. However, for closed conductor structures, inherent resonant modes exist inside the objects. When the operating frequency approaches the internal resonant frequencies, the numerical solution of the integral equation becomes ill-conditioned. Specifically, when the operating frequency is close to an electric resonant frequency, the matrix generated by the EFIE becomes nearly singular, resulting in degraded solution accuracy or even failure. Similarly, the MFIE suffers from the same problem when the operating frequency approaches a magnetic resonant frequency.

Since the electric and magnetic resonant frequencies of closed conducting structures generally do not coincide, the Combined Field Integral Equation (CFIE) can be constructed by linearly combining the EFIE and MFIE. The CFIE incorporates information from both the electric and magnetic field boundary conditions and can be expressed as follows:

$$CFIE = \alpha(EFIE) + (1 - \alpha)\eta_0(MFIE) \quad (2.16)$$

where $\alpha \in [0, 1]$ is the combination coefficient and $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ represents the intrinsic impedance of the medium. The CFIE effectively avoids the numerical instability of a single integral equation near internal resonant frequencies, thereby significantly improving the condition number of the matrix equation and enhancing the stability of the solution process. Therefore, CFIE has become one of the most important formulations for solving electromagnetic scattering problems involving closed conducting structures.

It should be noted that both CFIE and MFIE rely on the boundary conditions of closed surfaces, and their applications are mainly limited to closed conductor structures. In contrast, EFIE has a broader range of applications and can be used for the electromagnetic analysis of both closed and open conducting structures.

Considering the generality and applicability of the algorithm, the subsequent chapters of this dissertation mainly employ EFIE as the fundamental integral equation for analyzing electromagnetic characteristics of target objects.

2.2.2 Volume Surface Integral Equation

The previous subsection derived the formulations of surface integral equations for the electromagnetic scattering problem of Perfect Electric Conductors (PECs). In this subsection, the Volume Integral Equation (VIE) formulation for dielectric scattering problems is introduced, followed by the derivation of the Volume-Surface Integral Equation (VSIE) formulation by combining the surface integral equation with the volume integral equation.

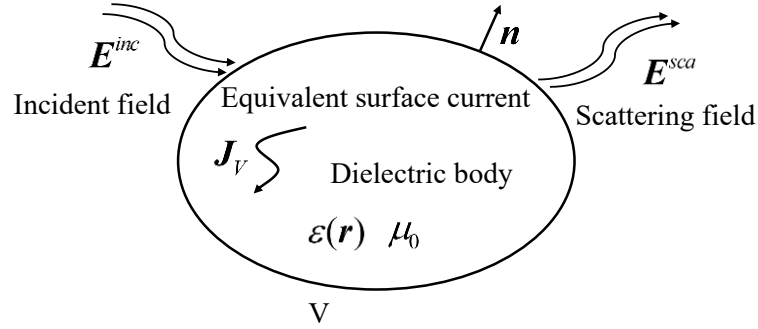


Fig. 2.2. Schematic illustration of electromagnetic scattering from a dielectric object.

As shown in Fig. 2.2, unlike the scattering problem involving PEC objects, when an electromagnetic wave is incident on a dielectric object, the resulting scattered field needs to be represented by the equivalent volume current $\mathbf{J}_V$ distributed inside the dielectric region. If the dielectric object has the electromagnetic parameters of $\epsilon(\mathbf{r}) = \epsilon_r(\mathbf{r})\epsilon_0(1 - j\sigma(\mathbf{r})/\omega)$, where ϵ_r denotes the relative permittivity, σ represents the conductivity, and μ_0 represents the permeability, the total electric field can similarly be expressed as the superposition of the incident field and the scattered field. Therefore, the volume integral equation inside the dielectric object can be obtained as follows:

$$\frac{\mathbf{J}_V(\mathbf{r})}{j\omega\epsilon_0(\epsilon_r - 1)} + j\omega\mu_0 \int_V \mathbf{J}_V(\mathbf{r}') \bar{\mathbf{G}}(\mathbf{r}, \mathbf{r}') dV = \mathbf{E}^{inc}(\mathbf{r}) \quad (2.17)$$

Equation (2.17) can also be expressed in terms of the magnetic vector potential and the electric scalar potential as follows:

$$\frac{\mathbf{J}_V(\mathbf{r})}{j\omega\epsilon_0(\epsilon_r - 1)} + j\omega\mathbf{A}(\mathbf{r}) + \nabla\Phi(\mathbf{r}) = \mathbf{E}^{inc}(\mathbf{r}) \quad (2.18)$$

where:

$$\mathbf{A}(\mathbf{r}) = \mu_0 \int_V \mathbf{G}(\mathbf{r}, \mathbf{r}') \mathbf{J}_V(\mathbf{r}') dV \quad (2.19)$$

$$\Phi(\mathbf{r}) = -\frac{1}{j\omega\epsilon_0} \int_V \mathbf{G}(\mathbf{r}, \mathbf{r}') \nabla \cdot \mathbf{J}_V(\mathbf{r}') dV \quad (2.20)$$

When dielectric materials and conducting objects coexist, they need to be treated separately. The equivalent volume currents are constructed inside the

dielectric region, while the equivalent surface currents are defined on the conductor surface, as illustrated in Fig. 2.3. Here, S denotes quantities associated with the conducting surface, and V represents quantities related to the dielectric volume.

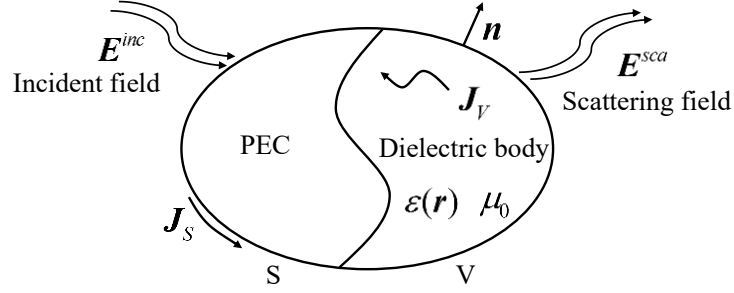


Fig. 2.3. Schematic illustration of electromagnetic scattering from a PEC-dielectric composite object.

At this point, the scattered field of the composite object is the sum of the scattered fields generated by the conducting and dielectric regions, which can be expressed as:

$$\mathbf{E}^{sca} = -[j\omega\mathbf{A}_V(\mathbf{r}) + \nabla\Phi_V(\mathbf{r}) + j\omega\mathbf{A}_S(\mathbf{r}) + \nabla\Phi_S(\mathbf{r})] \quad (2.21)$$

Since the total electric fields inside the dielectric region and on the conductor surface are respectively expressed as:

$$\mathbf{E}^{tot}(\mathbf{r}) = \frac{\mathbf{D}(\mathbf{r})}{\varepsilon(\mathbf{r})}, \mathbf{r} \in V \quad (2.22)$$

$$\mathbf{t} \cdot \mathbf{E}^{tot}(\mathbf{r}) = 0, \mathbf{r} \in S \quad (2.23)$$

By combining Eq. (2.21), the expressions of the incident electric field in the corresponding regions can be derived according to Eqs. (2.22) and (2.23), respectively. Thus, the Volume-Surface Integral Equation (VSIE) for the composite object can be obtained as follows:

$$\frac{\mathbf{D}(\mathbf{r})}{\varepsilon(\mathbf{r})} + j\omega\mathbf{A}_V(\mathbf{r}) + \nabla\Phi_V(\mathbf{r}) + j\omega\mathbf{A}_S(\mathbf{r}) + \nabla\Phi_S(\mathbf{r}) = \mathbf{E}^{inc}, \mathbf{r} \in V \quad (2.24)$$

$$[j\omega\mathbf{A}_V(\mathbf{r}) + \nabla\Phi_V(\mathbf{r}) + j\omega\mathbf{A}_S(\mathbf{r}) + \nabla\Phi_S(\mathbf{r})]_{\tan} = \mathbf{t} \cdot \mathbf{E}^{inc}, \mathbf{r} \in S \quad (2.25)$$

2.3 Solution of Integral Equations Using the Method of Moments

The electromagnetic integral equations are generally Fredholm integral equations, in which the unknown quantities appear in the integral operators in the form of continuous current or field distributions. For complex electromagnetic structures encountered in practical engineering applications, analytical solutions are usually unavailable due to the complexity of the target geometry, material properties, and boundary conditions. Therefore, numerical methods are required to discretize and solve these integral equations. The Method of Moments (MoM) is one of the most widely used numerical techniques for solving electromagnetic integral equations. Its fundamental idea is to transform continuous integral

equations into a finite-dimensional system of linear algebraic equations, which can then be solved using numerical linear algebra techniques to obtain the unknown current or field distributions.

The MoM was first introduced by K. K. Mei for solving electromagnetic integral equations [138], and was subsequently further developed and systematically established by R. F. Harrington [24]. The core idea of the MoM is to expand the unknown current using a finite set of known basis functions and to project the continuous integral equation onto a finite-dimensional function space through the Method of Weighted Residuals (MWR). By selecting appropriate testing functions and imposing constraints such as the Galerkin method, point matching, or least-squares matching, the integral equation can be discretized into a matrix equation. For Surface Integral Equations (SIEs), the unknowns generally correspond to the equivalent surface currents on the target surface, whereas for Volume Integral Equations (VIEs), they represent the polarization currents or equivalent volume currents within the dielectric region. In addition to naturally satisfying the radiation boundary condition inherent to integral equation formulations, the MoM offers high computational accuracy and has therefore become one of the most important numerical tools in computational electromagnetics for the analysis of electromagnetic scattering, radiation, and electromagnetic compatibility (EMC) problems. This section introduces the application of the MoM to solve the electromagnetic integral equations derived in Section 2.2.

2.3.1 Fundamental Principles of MoM

The integral equations introduced in Section 2.2 can be written in the following linear operator form: The The w:

$$L(f) = g \quad (2.26)$$

where L denotes the linear integral operator, f represents the unknowns to be solved (e.g., the equivalent current in the integral equation), and g is the known excitation function, such as the incident wave. Since it is impossible to obtain the exact solution of an infinite-dimensional integral equation directly, the target object must first be discretized, thereby transforming the original problem into a finite-dimensional one that can be solved numerically. Accordingly, the unknown function f is approximated by a linear combination of a finite set of basis functions (BFs) as follows:

$$f \approx \sum_{n=1}^N a_n f_n \quad (2.27)$$

where a_n denotes the unknown expansion coefficient associated with the n^{th} basis function, and N is the total number of basis functions. Substituting Eq. (2.27) into Eq. (2.26) and exploiting the linearity of the operator L yields:

$$\sum_{n=1}^N a_n L \cdot f_n \approx g \quad (2.28)$$

To determine the unknown expansion coefficients a_n in Eq. (2.28), a set of testing functions w_m is selected. Taking the inner product of each testing function with both sides of Eq. (2.28) yields:

$$\sum_{n=1}^N a_n \langle w_m, L \cdot f_n \rangle = \langle w_m, g \rangle, (m = 1, 2, \dots, N) \quad (2.29)$$

Equation (2.29) can be rewritten in the following matrix form:

$$[Z_{N \times N}] \{a_{N \times 1}\} = \{g_{N \times 1}\} \quad (2.30)$$

where:

$$\{g_m\} = \{\langle w_m, g \rangle\} \quad (2.31)$$

$$[Z_{m,n}] = [\langle w_m, L \cdot f_n \rangle] \quad (2.32)$$

In Eq. (2.30), Z is referred to as the impedance matrix or system matrix, g is known as the right-hand-side (RHS) vector or excitation vector, and a denotes the vector of unknown expansion coefficients. When the basis functions themselves are chosen as the testing functions, the resulting testing procedure is referred to as the Galerkin method.

Through the MoM, the continuous electromagnetic problem is ultimately transformed into a discrete matrix equation of the form given in Eq. (2.30), which can then be solved using either iterative methods or direct methods. It should be noted that the matrix generated by the MoM is generally dense, since nearly all of its entries are nonzero. Consequently, the conventional MoM requires a memory complexity of $O(N^2)$. Its computational complexity is $O(N^3)$ when solved using direct methods and $O(N^2)$ per iteration when solved using iterative methods. Therefore, fast algorithms are generally required to reduce both the computational and memory complexities, making the analysis of large-scale electromagnetic problems computationally feasible.

2.3.2 Selection of Basis Functions

As discussed in Section 2.3.1, the construction of the basis functions is a crucial step in the solution procedure. Various types of basis functions have been proposed for different problems and application scenarios [27],[129], [133]-[137]. For surface integral equations involving Perfect Electric Conductors (PECs), the Rao–Wilton–Glisson (RWG) basis function is the most widely used surface basis function because triangular elements can flexibly approximate objects with arbitrary and complex geometries [27]. As illustrated in Fig. 2.4, the RWG basis function is defined over a pair of adjacent triangular elements.

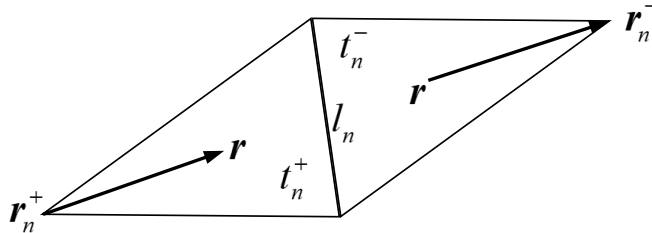


Fig. 2.4. Schematic diagram of an RWG basis function.

The RWG basis function is an edge-based basis function defined on a pair of adjacent triangular elements sharing a common edge. For the n^{th} RWG basis function, denoted by $\mathbf{f}_n^s$, let one triangle be the positive triangle t_n^+ and the other the negative triangle t_n^- . Their corresponding areas are A_n^+ and A_n^- , respectively, and the length of the common edge is l_n . The RWG basis function $\mathbf{f}_n^s$ is then defined as:

$$\mathbf{f}_n^s(\mathbf{r}) = \begin{cases} \frac{l_n}{2A_n^+}(\mathbf{r} - \mathbf{r}_n^+), \mathbf{r} \in t_n^+ \\ \frac{l_n}{2A_n^-}(\mathbf{r}_n^- - \mathbf{r}), \mathbf{r} \in t_n^- \end{cases} \quad (2.33)$$

where $\mathbf{r}_n^+$ and $\mathbf{r}_n^-$ denote the free vertices of the positive and negative triangles, respectively, i.e., the vertices opposite the common edge. As can be seen from the definition of the RWG basis function, the current represented by the basis function has no normal component on the non-shared edges, while the normal components on the shared edge are continuous across the two adjacent triangles. This property ensures the continuity of the surface current across the triangular pair.

Taking the divergence of the RWG basis function yields:

$$\nabla \cdot \mathbf{f}_n^s(\mathbf{r}) = \begin{cases} \frac{l_n}{A_n^+}, \mathbf{r} \in t_n^+ \\ -\frac{l_n}{A_n^-}, \mathbf{r} \in t_n^- \end{cases} \quad (2.34)$$

It can be observed that the divergence of the RWG basis function is constant within each of the two triangular elements. Furthermore, integrating the divergence over the areas of the positive and negative triangles yields a net charge of zero, indicating that no artificial accumulation or depletion of charge is introduced within the support of the basis function. This property ensures consistency with the charge continuity equation.

For dielectric objects, volume basis functions are required instead of surface basis functions. One of the most commonly used volume basis functions is the Schaubert–Wilton–Glisson (SWG) basis function [129], which is illustrated in Fig. 2.5.

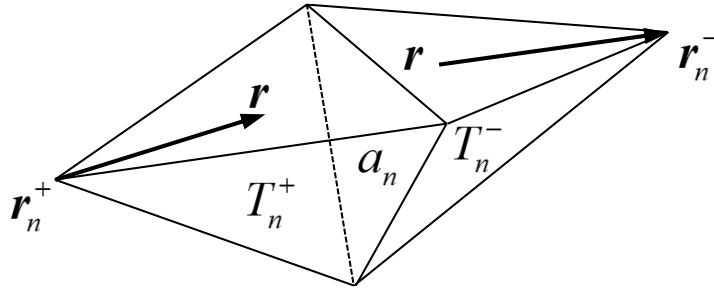


Fig. 2.5. Schematic diagram of an SWG basis function.

The SWG basis function is a face-based basis function. Similar to the RWG basis function, it is defined on a pair of adjacent tetrahedral elements sharing a common triangular face. For the n -th SWG basis function, denoted by $\mathbf{f}_n^v$, let one tetrahedron be the positive tetrahedron T_n^+ and the other be the negative tetrahedron T_n^- . Their corresponding volumes are V_n^+ and V_n^- , respectively, and the area of the common triangular face is a_n . The SWG basis function $\mathbf{f}_n^v$ is defined as:

$$\mathbf{f}_n^v(\mathbf{r}) = \begin{cases} \frac{a_n}{3V_n^+}(\mathbf{r} - \mathbf{r}_n^+), \mathbf{r} \in T_n^+ \\ \frac{a_n}{3V_n^-}(\mathbf{r}_n^- - \mathbf{r}), \mathbf{r} \in T_n^- \end{cases} \quad (2.35)$$

where $\mathbf{r}_n^+$ and $\mathbf{r}_n^-$ denote the vertices of the positive and negative tetrahedra, respectively, that are not located on the common face. Here, the SWG basis function is used to represent the electric flux density. Similarly, it can be observed that the SWG basis function has a normal component only on the common face, and this normal component is continuous across the adjacent tetrahedral elements. The divergence of the SWG basis function is expressed as:

$$\nabla \cdot \mathbf{f}_n^v(\mathbf{r}) = \begin{cases} \frac{a_n}{V_n^+}, \mathbf{r} \in T_n^+ \\ -\frac{a_n}{V_n^-}, \mathbf{r} \in T_n^- \end{cases} \quad (2.36)$$

It can be observed that the divergence of the SWG basis function is constant within each of the two tetrahedral elements. Moreover, the volume integration of the divergence over the positive and negative tetrahedra results in a net charge of zero. Therefore, the SWG basis function also satisfies charge conservation, preventing artificial charge accumulation or depletion within its support region.

2.4 Investigation of Loop-Tree Preconditioning for Low-Frequency Breakdown of the Method of Moments

In this section, the focus is placed on the ill-conditioned system matrix problem encountered by the Method of Moments (MoM) at extremely low frequencies, and the Loop-Tree preconditioning technique is investigated to alleviate the low-frequency breakdown of the MoM. It is well known that when the condition number of the system matrix becomes large, the number of iterations required by iterative solvers increases significantly. Therefore, for such ill-conditioned problems, direct methods generally outperform iterative methods in terms of both solution efficiency and accuracy. However, when the system matrix becomes nearly singular due to severe ill-conditioning, even direct solvers may produce solutions with considerable errors.

For example, at extremely low frequencies, electromagnetic problems become dominated by electrostatic effects, and the MoM suffers from the low-frequency breakdown problem. In 1983, D. R. Wilton first proposed a solution strategy for the low-frequency breakdown problem, which later became an important theoretical foundation for the development of the Loop-Tree preconditioning technique [148]. This section discusses the underlying mechanism of the MoM low-frequency breakdown and its corresponding mitigation approaches in detail.

2.4.1 Fundamental Principle of Loop-Tree Preconditioning

As shown in Eqs. (2.7)–(2.9) in Chapter 2, when the frequency ω becomes sufficiently low, the contribution of the magnetic vector potential term A in the integral equation is much smaller than that of the electric scalar potential term ϕ . In other words, at extremely low frequencies, the scattered field is mainly generated by the charge distribution, while the contribution from the current-induced field can be almost neglected. Due to the finite precision of computers, the numerical values associated with the A term may be dominated by numerical errors during the computation. Meanwhile, the system matrix becomes highly ill-conditioned, resulting in completely inaccurate solutions.

To overcome this problem, the Loop-Tree preconditioning method separates the vector potential term A and the divergence-related term into two independent components. It constructs two types of basis functions, namely solenoidal basis functions and non-solenoidal basis functions, corresponding to the current components with zero and nonzero divergence, respectively. Furthermore, the divergence term of the solenoidal basis functions is forced to be zero, thereby eliminating the numerical imbalance between the vector potential and scalar potential contributions. In the Loop-Tree preconditioning method, the two types of basis functions are illustrated in Fig. 2.6 [149].

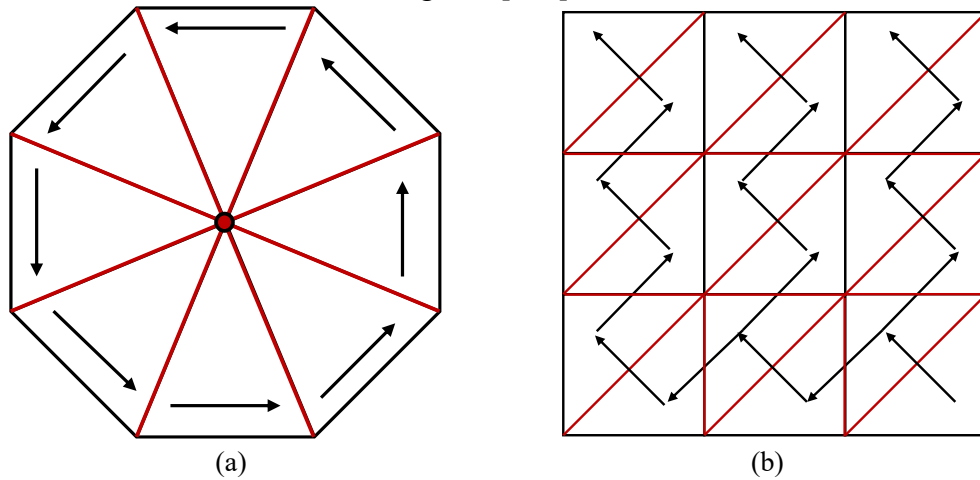


Fig. 2.6. (a) A Loop basis function composed of eight RWG basis functions; (b) A Tree basis function in a tree structure. The red lines represent the edges associated with the corresponding RWG basis functions, and the black arrows indicate the current flow directions.

The central point in Fig. 2.6(a) corresponds to a Loop basis function. It can be observed that the Loop basis function is a closed-loop basis function constructed

based on an internal vertex. Fig. 2.6(b) illustrates the selected Tree basis functions within a tree structure. The Tree basis functions are essentially still constructed from edge-based RWG basis functions. However, unlike Loop basis functions, all Tree basis functions do not form complete closed-loop structures, and each Loop basis function contains only a rotational component (curl component) without a divergence component. In terms of the system matrix, the transformation from RWG basis functions to Loop and Tree basis functions is achieved through multiplication by a transformation matrix. It should be noted that after the basis transformation, a diagonal preconditioning operation is still required for the transformed system matrix to effectively reduce its condition number. Therefore, the Loop-Tree preconditioning method transforms the original MoM system matrix equation into the following form:

$$DTZT^T DI = V \quad (2.37)$$

where T is the basis transformation matrix, which transforms the original RWG basis functions into the Loop and Tree basis functions. It can be expressed as:

$$T = \begin{bmatrix} T_{Loop} \\ T_{Tree} \end{bmatrix} \quad (2.38)$$

In Eq. (2.37), D is a diagonal matrix, whose diagonal elements are given by:

$$D_{ii} = \frac{1}{\sqrt{T_i Z T_i^T}} \quad (2.39)$$

Since each element of the system matrix can be decomposed into the contribution from the magnetic vector potential term A and the electric scalar potential term Φ , the entire system matrix can be expressed as:

$$Z = Z^A + Z^\Phi \quad (2.40)$$

Since the constructed Loop basis functions have zero divergence, all matrix elements associated with the electric scalar potential term Φ of the Loop basis functions should theoretically be zero. However, due to numerical errors in practical computations, these matrix elements may have nonzero values. Therefore, these elements need to be explicitly set to zero. Finally, after the basis transformation, the system matrix can be expressed as:

$$TZT^T = Z_{LoopTree} = \begin{bmatrix} Z_{Loop-Loop}^A & Z_{Loop-Tree}^A \\ Z_{Tree-Loop}^A & Z_{Tree-Tree}^A + Z_{Tree-Tree}^\Phi \end{bmatrix} \quad (2.41)$$

It can be observed from Eq. (2.41) that the application of Loop-Tree preconditioning decomposes the system matrix into two components corresponding to the magnetic vector potential term A and the electric scalar potential term Φ . The magnetic vector potential component is calculated normally, while the electric scalar potential component only needs to consider the contribution from the Tree basis functions. This operation can be achieved by directly setting the corresponding matrix elements T_{Loop} in the transformation matrix T to zero.

2.4.2 Numerical results of Loop-Tree Preconditioning

In this subsection, a metallic sphere model is used to demonstrate the improvement of the system matrix condition number achieved by the Loop-Tree preconditioning method, and the corresponding surface current results are compared. The radius of the metallic sphere is 1.8 m, and a fixed mesh size of approximately 0.15 m is adopted, resulting in 6534 unknowns. The variations of the condition numbers of four different system matrices are evaluated as the frequency decreases from 300 MHz to 30 Hz, including the original system matrix, the system matrix with only diagonal preconditioning, the system matrix after Loop-Tree preconditioning, and the system matrix after Loop-Tree preconditioning with the electric scalar potential term Φ associated with Loop basis functions removed, as shown in Fig. 2.7. Here, the condition number is obtained by performing singular value decomposition (SVD) on the entire system matrix and calculating the ratio between the maximum and minimum singular values.

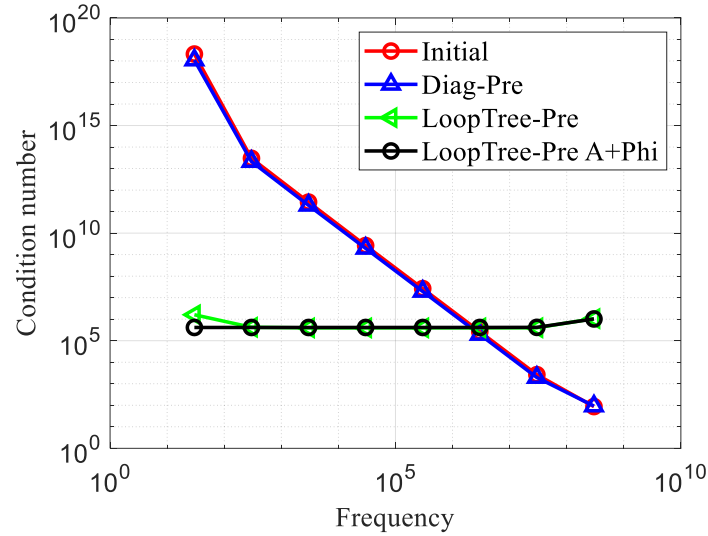


Fig. 2.7. Condition number variation of the metallic sphere system matrix with frequency under different preconditioning schemes.

As shown in Fig. 2.7, the condition number of the original system matrix gradually increases as the frequency decreases, eventually reaching the order of 10^{18} at 30 Hz. Meanwhile, diagonal preconditioning provides almost no improvement in reducing the condition number. After applying the Loop-Tree preconditioning, the condition number of the system matrix can be effectively maintained at a relatively low level of approximately 10^6 in the low-frequency range. However, the condition number still increases slightly at 30 Hz.

By further separating the magnetic vector potential term $\mathbf{A}$ and the electric scalar potential term Φ , and setting the matrix elements associated with the electric scalar potential term Φ of Loop basis functions to zero, the condition number can still be maintained below 10^6 even at the lowest frequency of 30 Hz.

Subsequently, the solution accuracy under different treatments is investigated. The system matrices are directly inverted through LU decomposition, and the obtained inverse matrices are multiplied by the incident electric field to obtain the surface current distributions on the metallic sphere, as shown in Fig. 2.8. The scales of the color bars in Fig. 2.8 are unified for a better comparison.

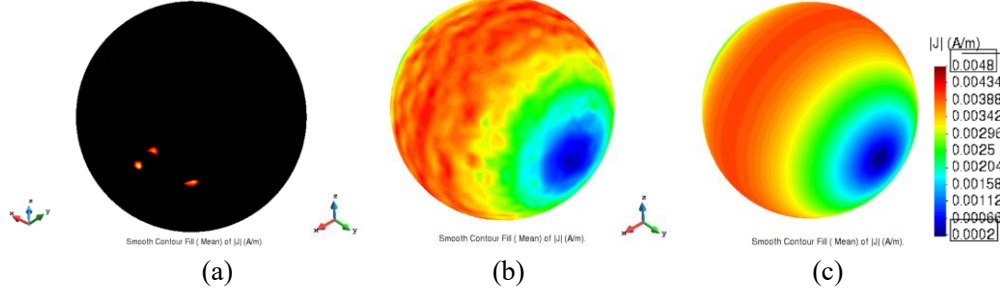


Fig. 2.8. Surface current distributions of the metallic sphere obtained by LU decomposition of the system matrix under different preconditioning schemes. (a) Without preconditioning; (b)

Loop-Tree preconditioning without separating the magnetic vector potential and electric scalar potential terms; (c) Loop-Tree preconditioning with the separation of the magnetic vector potential and electric scalar potential terms, where the electric scalar potential matrix components associated with Loop basis functions are set to zero.

The results shown in Fig. 2.8 indicate that, without Loop-Tree preconditioning, accurate current solutions cannot be obtained even when the system matrix is solved directly using LU decomposition, as illustrated in Fig. 2.8(a). When Loop-Tree preconditioning is applied without separating the magnetic vector potential term $\mathbf{A}$ and the electric scalar potential term Φ , the accuracy of the obtained current distribution is significantly improved; however, considerable numerical errors still remain, as shown in Fig. 2.8(b). By further separating the magnetic vector potential $\mathbf{A}$ and electric scalar potential term Φ based on the Loop-Tree decomposition, and setting the electric scalar potential matrix elements associated with Loop basis functions to zero, an accurate current distribution consistent with the expected result can finally be obtained, as presented in Fig. 2.8(c).

These results demonstrate that although direct solvers generally exhibit better performance than iterative solvers for moderately ill-conditioned problems, directly applying a direct solver may still lead to significant errors when the system matrix is nearly singular, such as in the low-frequency breakdown problem of the Method of Moments. Subsequently, the improvement of iterative solutions achieved by the Loop-Tree preconditioning method with the separation of the magnetic vector potential and electric scalar potential terms is further investigated, as shown in Fig. 2.9.

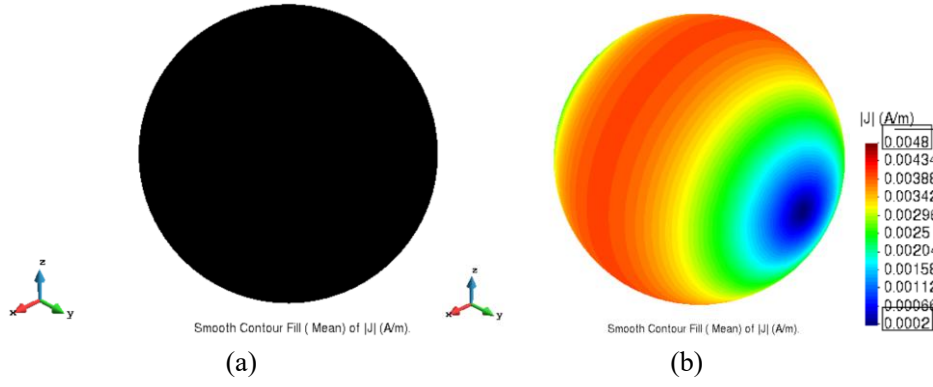


Fig. 2.9. Surface current distributions of the metallic sphere obtained by iterative solution of the system matrix under different preconditioning schemes. (a) Without preconditioning; (b) Loop-Tree preconditioning with the separation of the magnetic vector potential and electric scalar potential terms, where the electric scalar potential matrix elements associated with Loop basis functions are set to zero.

The iterative solver adopted in this example is the Generalized Minimal Residual (GMRES) method, with the convergence tolerance set to 10^{-5} . As shown in Fig. 2.9, for the iterative solution of the low-frequency breakdown problem, the result obtained without preconditioning is completely inaccurate. In contrast, the Loop-Tree preconditioning method with separation of the magnetic vector potential and electric scalar potential terms, together with setting the electric scalar potential matrix components associated with Loop basis functions to zero, successfully obtains the correct solution.

The variation of the iterative residuals with respect to the iteration number for the two approaches is presented in Fig. 2.10.

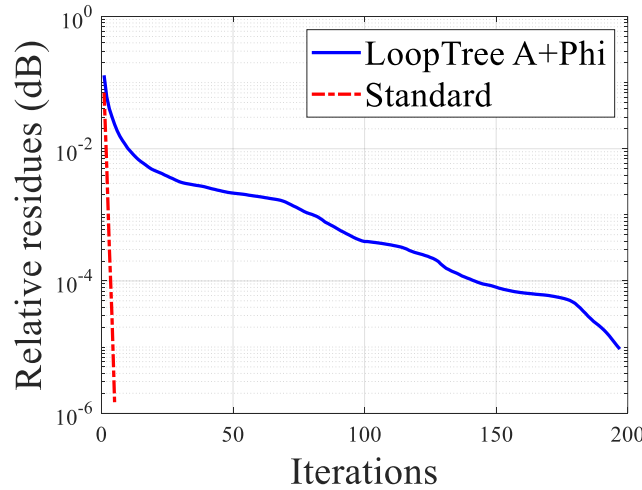


Fig. 2.10. The variation of the iterative residuals.

The results in Fig. 2.10 show that the iterative residual without preconditioning decreases rapidly, whereas the residual obtained by the Loop-Tree preconditioning method with separation of the magnetic vector potential term $\mathbf{A}$ and the electric scalar potential term Φ decreases relatively slowly. This indicates that, for the low-frequency breakdown problem, although the unpreconditioned iterative solver appears to converge rapidly, the extremely large

condition number of the system matrix causes numerical errors in the matrix elements and the right-hand side vector to dominate the solution process. As a result, the iterative solver converges to an inaccurate solution rather than the physically correct one.

In summary, applying preconditioning is essential for addressing the low-frequency breakdown problem in the Method of Moments. Furthermore, separating the magnetic vector potential term and the electric scalar potential term, and explicitly eliminating the electric scalar potential matrix components associated with Loop basis functions, can further improve the conditioning of the system matrix. This treatment is not only effective for iterative solvers but is also applicable to direct solvers when dealing with low-frequency breakdown problems.

2.5 NESAs for Low-Rank Compression

The matrices generated by the Method of Moments are dense, and fast algorithms are therefore required to improve computational efficiency. This subsection introduces the Nested Equivalent Source Approximation (NESA), a matrix low-rank compression method. NESA was proposed by the research group of M. M. Li [68]-[70]. Due to its kernel-independent property, NESA can be conveniently applied to various electromagnetic integral equation problems.

The basic idea of NESA is to construct equivalent sources for each group under an octree-based hierarchical partition. By ensuring that the equivalent sources and the actual sources within each group generate approximately identical radiation fields on the testing surface, the interactions between well-separated groups can be transformed into the interactions between their corresponding equivalent sources.

When the operating frequency increases or the electrical size of the target becomes larger, the radiation field generated by the sources varies more rapidly with spatial position, and the oscillatory behavior of the kernel function becomes more significant. Therefore, different equivalent source construction strategies are required for low-frequency and high-frequency problems. In this subsection, the fundamental principles of low-frequency NESA and high-frequency NESA are introduced separately, and the corresponding low-rank representations of far-field matrix blocks are derived.

2.5.1 NESAs for Low-Frequency Problems

Low-frequency NESA employs an octree-based hierarchical grouping scheme to partition the target into multiple levels of groups. It adopts the strong admissibility condition, under which only non-touching groups are regarded as far-field interaction groups, and their corresponding impedance matrix blocks can be approximated by low-rank representations. In contrast, the self-interactions within a group and the interactions between neighbouring groups are treated as near-field interactions, and their impedance matrix blocks are directly computed

using the Method of Moments. For electromagnetic problems at relatively low frequencies, the kernel function in the integral equation exhibits relatively mild oscillatory behaviour. Therefore, the testing surface can be constructed using an omnidirectional distribution strategy, as illustrated in Fig. 2.11.

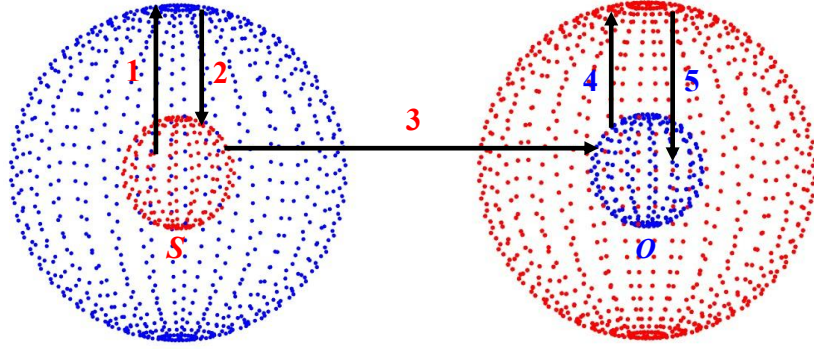


Fig. 2.11. Schematic diagram of the Low-frequency NESA for compressing far field couplings.

For the interaction between two far-field groups at the same hierarchical level, the source group is denoted by S , while the observation group (or field group) is denoted by O . The red dots represent the equivalent source points, and the blue dots represent the testing points. Both the source points and testing points are composed of three mutually orthogonal RWG basis functions. As shown in the figure, the external testing points are uniformly distributed over an omnidirectional spherical surface.

Processes 1 and 2 in Fig. 2.11 together constitute the radiation process. Specifically, the actual sources and the internal equivalent point sources are made to produce approximately identical field distributions at the external testing points. In this way, the actual sources within the source group are mapped onto the internal equivalent sources.

Processes 4 and 5 together constitute the receiving process. The fields generated by the external point sources at the internal equivalent testing points and the actual observation points are calculated separately, thereby mapping the fields at the actual observation points onto the equivalent testing points.

Process 3 directly calculates the coupling between the equivalent sources of the source group and the equivalent testing points of the observation group. Consequently, the impedance matrix describing the interaction between the source group and observation group can be expressed as [3]:

$$\begin{aligned} Z_{O,S} &= Z_{O,\sigma_O} Z_{\tau_O,\sigma_O}^\dagger Z_{\tau_O,\tau_S} Z_{\sigma_S,\tau_S}^\dagger Z_{\sigma_S,S} \\ &= U_O D_{O,S} V_S \end{aligned} \quad (2.42)$$

where σ and τ denote the testing sphere and equivalent sphere, respectively. The matrices Z_{O,σ_O} , $Z_{\tau_O,\sigma_O}^\dagger$, Z_{τ_O,τ_S} , $Z_{\sigma_S,\tau_S}^\dagger$ and $Z_{\sigma_S,S}$ correspond to Processes 1–5 in Fig. 2.11, respectively. Here, U_O denotes the receiving matrix generated by the receiving process, while V_S represents the radiation matrix generated by the radiation process. $D_{O,S}$ denote the translation matrices associated with the source

and observation groups. The $D_{O,S}$ matrix describing the coupling between the equivalent sources and testing points. For interactions between groups separated by larger distances, the coupling can be represented using equivalent sources at higher levels of the hierarchical tree. By introducing the transfer matrices B_{O,O^p} and $C_{S^p,S}$, the equivalent sources of child groups can be mapped onto the equivalent sources of their parent groups, thereby enabling the interaction information to be transferred upward and downward through the hierarchical tree:

$$Z_{O,S} = U_O B_{O,O^p} D_{O^p,S^p} C_{S^p,S} V_S \quad (2.43)$$

where p denotes the corresponding parent group.

2.5.2 NESAs for High-Frequency Problems

As the frequency increases, the kernel function associated with far-field interactions becomes increasingly oscillatory, and the rank of the corresponding impedance matrix is no longer approximately constant but instead increases with both the frequency and the number of unknowns. Such problems are generally referred to as electrodynamic problems, for which the objects under analysis are typically electrically large. Under these conditions, the testing-point distribution adopted in low-frequency NESAs cannot adequately sample the spatial field distribution generated by the source group. Although increasing the number of sampling points can improve the approximation accuracy, it also leads to a significant increase in the computational complexity of the algorithm.

To effectively address the increasingly oscillatory behaviour of the kernel function at high frequencies, the far-field interaction region can be further partitioned into subregions according to different spatial directions. Testing points are then constructed separately for each directional region to sample the characteristics of the field generated by the source group within the corresponding region. This approach is based on the fact that when the observation points are restricted to a certain angular range, the kernel function exhibits a relatively smooth and compressible behaviour. Consequently, within each directional region, the rank of the far-field interaction matrix block is approximately independent of the number of unknowns.

The high-frequency matrix-block compression algorithm developed for the upper levels of the octree, together with the low-frequency NESAs algorithm introduced in Section 2.5.1, constitutes a wideband NESAs (WNESAs) framework. For the high-frequency NESAs compression algorithm, a more stringent strong admissibility condition than that adopted in low-frequency NESAs is employed. Accordingly, far-field interactions are defined as follows:

$$\frac{R(s,t)}{\lambda} \geq \left(\frac{D_l}{\lambda} \right)^2 \quad (2.44)$$

where λ denotes the wavelength, $R(s,t)$ represents the distance between the interacting groups, and D_l denotes the group size at the l -th level. Furthermore, the far-field interaction groups are restricted to a cone with an angular width of

$O(\lambda/D_l)$. For high-frequency NESA, the interaction between two far-field groups within a specific direction is illustrated in Fig. 2.12.

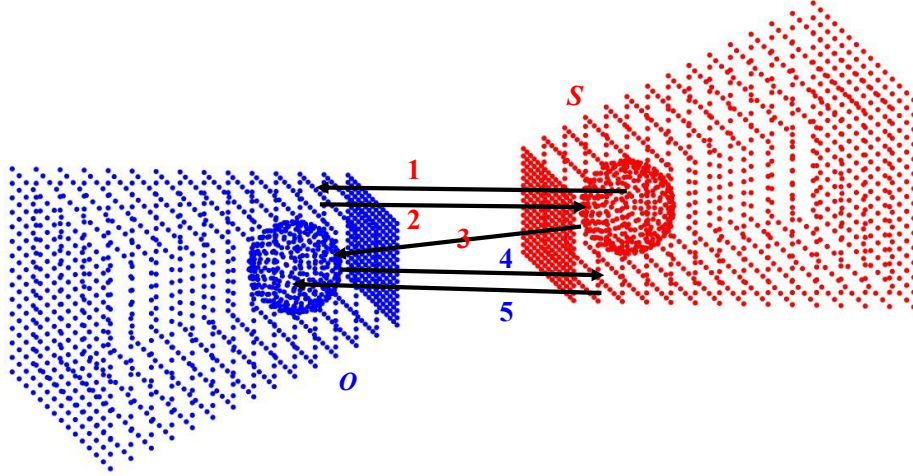


Fig. 2.12. Schematic diagram of the High-frequency NESA with directional equivalences for compressing far field couplings.

In the figure, the red dots represent the equivalent source points, while the blue dots represent the testing points. Processes 1–5 have physical meanings similar to those in the low-frequency NESA formulation. However, both the radiation and receiving matrices are constructed specifically for the corresponding direction. Therefore, the coupling matrix block between the far-field source group and observation group within direction Ω can be expressed as[3]:

$$\begin{aligned} Z_{O,S} &= Z_{O,\sigma_O^{-d}} Z_{\tau_O,\sigma_O^{-d}}^\dagger Z_{\tau_O,\tau_S} Z_{\sigma_S^d,\tau_S}^\dagger Z_{\sigma_S^d,S} \\ &= U_O^{-d} D_{O,S} V_S^d \end{aligned} \quad (2.45)$$

Here, $-d$ denotes the opposite direction of direction d . For interactions at greater distances, if the parent-level admissibility condition is satisfied, the corresponding far-field matrix blocks are compressed using the equivalent sources at the parent level. It can be observed that each small directional cone of the parent group containing the current group is fully enclosed within the corresponding larger directional cone of the parent group. Moreover, this directional admissibility condition naturally satisfies the nested relationship between adjacent hierarchical levels, which provides the fundamental basis for constructing the H^2 -matrix structure based on WNESA in subsequent sections.

Unless otherwise specified, this thesis defines objects with an electrical size smaller than six wavelengths as electrically small objects, which are solved using low-frequency NESA. Objects with an electrical size greater than or equal to six wavelengths are defined as electrically large objects, for which high-frequency NESA is introduced. This value can be determined by the error test introduced in Chapter 5, where it shows that when the electrical size is larger than six wavelengths, the RCS error will be significant. The resulting algorithm is referred to as WNESA (Wideband Nested Equivalent Source Approximation).

2.6 Summary

This chapter reviewed the fundamental principles of electromagnetic integral equation methods, including the formulation and derivation of surface integral equations and volume-surface integral equations. After establishing the integral equations, the Method of Moments (MoM) was introduced to discretize and solve these equations. Different basis functions were employed to discretize the integral equations for dielectric and conducting objects, respectively.

Subsequently, the low-frequency breakdown problem encountered by the MoM at extremely low frequencies was investigated, and the Loop-Tree preconditioning technique was introduced. Numerical results demonstrated that the Loop-Tree preconditioning method can effectively improve the accuracy of both direct and iterative solutions of the MoM system in low-frequency problems.

Finally, the Nested Equivalent Source Approximation (NESA), a low-rank compression method for electromagnetic problems at different frequency ranges, was introduced. The theoretical foundations and numerical techniques presented in this chapter provide the basis for the main research contributions of this thesis, including the NESA-based fast solution method for volume-surface integral equations and the NESA-based direct solution algorithms.

Chapter 3

Fast VSIE Solution Method for Complex Targets Based on Nested Equivalent Source Approximation

3.1 Introduction

Different types of integral equations can be formulated for different analysis objects by combining the corresponding electromagnetic boundary conditions with the equivalence principle. In general, surface integral equation (SIE) methods are mainly applicable to the electromagnetic scattering and radiation analysis of perfect electric conductor (PEC) objects and piecewise homogeneous dielectric targets. For PEC objects, commonly used formulations include the electric field integral equation (EFIE), magnetic field integral equation (MFIE), and combined field integral equation (CFIE). For dielectric objects, since both the electric and magnetic fields satisfy continuity conditions across material interfaces, equivalent electric and magnetic currents are generally introduced as the unknowns to formulate the corresponding integral equations. Among these formulations, the Poggio-Miller-Chang-Harrington-Wu-Tsai (PMCHWT) equation is one of the most widely used surface integral equations for dielectric objects. It combines the electric and magnetic field integral equations to construct a stable formulation and can accurately handle scattering from homogeneous and composite dielectric objects. The Müller integral equation employs a different combination of electromagnetic fields to effectively reduce the singularity of the kernel function and generally exhibits improved numerical stability for smooth dielectric objects and low-frequency problems. The electric and magnetic current combined field integral equation (JMCIE) is a combined-field formulation further developed from the PMCHWT equation. Similar to the CFIE formulation for PEC objects, JMCIE introduces a weighted combination of electric and magnetic field integral equations to improve iterative convergence and the condition number of the

system matrix, particularly near resonance frequencies. However, for more complex inhomogeneous dielectric media, surface integral equations cannot accurately characterize the electromagnetic properties within the entire object. In such cases, volume integral equations (VIEs) are required to model the electromagnetic interactions within the dielectric volume. Therefore, for complex configurations consisting of PEC and inhomogeneous dielectric materials, an SIE is generally formulated on the PEC surface, while a VIE is constructed within the inhomogeneous dielectric volume. The two formulations are then coupled to form a volume-surface integral equation (VSIE). As discussed in Section 2.3.2, SWG basis functions are required to discretize the VIE component of the VSIE. Since SWG basis functions are volumetric basis functions, volume discretization is required for their construction. Compared with surface discretization, volume discretization generally produces a substantially larger number of unknowns. Consequently, conventional MoM-based solutions of volume integral equations are strongly constrained by the electrical size and dielectric properties of the target, resulting in relatively low computational efficiency for large-scale volume discretization problems.

To address the aforementioned challenges, a fast solution method for VSIEs is proposed in this chapter. Following the concept of equivalent sources in NESAs, the volume sources within the dielectric region are mapped onto equivalent sources through external testing points. Since each equivalent source is composed of three pairs of mutually orthogonal RWG basis functions, the interactions between volume unknowns under the octree-based hierarchical structure are confined to the lowest level, whereas only interactions between surface equivalent sources remain at the higher levels. The compression of the volume interaction matrix into a surface interaction matrix is also mathematically justified. The rank of a matrix block generated by a volume integral equation increases approximately with the cube root of the number of unknowns, whereas the rank of a matrix block generated by a surface integral equation increases approximately with the square root of the number of unknowns[139]. Therefore, for targets with the same electrical size, although the matrix blocks generated by the volume integral equation have larger dimensions and consequently contain more unknowns, their ranks are comparable to those of the corresponding matrix blocks generated by the surface integral equation. During the radiation process, both volume and surface unknowns are involved. Therefore, separate radiation matrices are constructed to map the dielectric volume sources and PEC surface sources onto their corresponding equivalent sources. A similar treatment is adopted for the receiving process. Consequently, the overall system matrix can be decomposed into four components, which are independently compressed and stored.

The remainder of this chapter is organized as follows. First, the calculation of the VSIE matrix elements using the Method of Moments is introduced. Subsequently, the fundamental principles and compressed representation of dielectric objects based on NESAs are presented. The method is then extended to mixed PEC and inhomogeneous dielectric objects, and the corresponding

compressed VSIE formulation is established. Finally, numerical examples are provided to validate the accuracy and computational efficiency of the proposed method.

3.2 MoM Discretization of the VSIE for Mixed PEC and Dielectric Targets

For mixed PEC and inhomogeneous dielectric targets, the RWG basis functions are employed to represent the surface currents on the PEC:

$$\mathbf{J}_S = \sum_{n=1}^{N_M} I_n^M \mathbf{f}_n^S \quad (3.1)$$

while the SWG basis functions are used to represent the equivalent volume currents within the dielectric region:

$$\mathbf{D}_V = \sum_{n=1}^{N_D} I_n^D \mathbf{f}_n^V \quad (3.2)$$

In Eq. (3.1) and (3.2), $\mathbf{f}_n^S$ and $\mathbf{f}_n^V$ denote the RWG and SWG basis functions, respectively, while N_M and N_D represent the numbers of RWG and SWG basis functions, respectively. The corresponding expansion coefficients are denoted by I_n^M and I_n^D . By employing the Galerkin testing method and using the m -th SWG basis function $\mathbf{f}_m^V$ to test Eq. (2.24), the impedance matrix elements corresponding to the dielectric-to-dielectric interactions Z^{DD} and PEC-to-dielectric interactions Z^{DM} can be expressed as:w:

$$\begin{aligned} \int_V \mathbf{f}_m^V(\mathbf{r}) \cdot \mathbf{E}^{inc} dV = & \underbrace{\int_{V_m} \mathbf{f}_m^V(\mathbf{r}) \cdot \frac{\mathbf{D}(\mathbf{r})}{\boldsymbol{\varepsilon}(\mathbf{r})} dV + j\omega \int_{V_m} \mathbf{f}_m^V(\mathbf{r}) \cdot \mathbf{A}_V(\mathbf{r}') dV - \int_{V_m} \nabla \cdot \mathbf{f}_m^V(\mathbf{r}) \Phi_V(\mathbf{r}') dV + \int_{\Omega_m} \mathbf{n} \cdot \mathbf{f}_m^V(\mathbf{r}) \Phi_V(\mathbf{r}') dS}_{Z^{DD}} \\ & + \underbrace{j\omega \int_{V_m} \mathbf{f}_m^V(\mathbf{r}) \cdot \mathbf{A}_S(\mathbf{r}') dV - \int_{V_m} \nabla \cdot \mathbf{f}_m^V(\mathbf{r}) \Phi_S(\mathbf{r}') dV + \int_{\Omega_m} \mathbf{n} \cdot \mathbf{f}_m^V(\mathbf{r}) \Phi_S(\mathbf{r}') dS}_{Z^{DM}} \end{aligned} \quad (3.3)$$

where V_m denotes the volume occupied by the pair of tetrahedra associated with the m -th SWG basis function $\mathbf{f}_m^V$, Ω_m represents the outer boundary surface of the support region of $\mathbf{f}_m^V$, and $\mathbf{n}$ is the corresponding outward unit normal vector of Ω_m . Similarly, by employing the Galerkin testing method and using the m -th RWG basis function $\mathbf{f}_m^S$ to test (2.25), the impedance matrix elements corresponding to the dielectric-to-PEC interactions Z^{MD} and PEC-to-PEC interactions Z^{MM} can be expressed as:

$$\begin{aligned}
& \int_S \mathbf{f}_m^S(\mathbf{r}) \cdot \mathbf{E}_{\tan}^i dS = \\
& \underbrace{j\omega \int_{S_m} \mathbf{f}_m^S(\mathbf{r}) \cdot \mathbf{A}_V(\mathbf{r}') dS - \int_{S_m} \nabla \cdot \mathbf{f}_m^S(\mathbf{r}) \Phi_V(\mathbf{r}') dS}_{Z^{MD}} + \\
& \underbrace{j\omega \int_{S_m} \mathbf{f}_m^S(\mathbf{r}) \cdot \mathbf{A}_S(\mathbf{r}') dS - \int_{S_m} \nabla \cdot \mathbf{f}_m^S(\mathbf{r}) \Phi_S(\mathbf{r}') dS}_{Z^{MM}}
\end{aligned} \tag{3.4}$$

where S_m denotes the surface area occupied by the pair of triangles associated with the m -th RWG basis function $\mathbf{f}_m^S$. Next, the matrix-element expressions based on the interactions between the basis functions are derived. Within the dielectric volume, the relationship between the polarization current $\mathbf{J}_V(\mathbf{r})$ and the electric field $\mathbf{E}(\mathbf{r})$ is given by:

$$\mathbf{J}_V(\mathbf{r}) = j\omega(\varepsilon(\mathbf{r}) - \varepsilon_0)\mathbf{E}(\mathbf{r}) \tag{3.5}$$

Furthermore, $\mathbf{J}_V(\mathbf{r})$ can be expressed by electric flux $\mathbf{D}(\mathbf{r})$ as:

$$\mathbf{J}_V(\mathbf{r}) = j\omega K(\mathbf{r})\mathbf{D}(\mathbf{r}) \tag{3.6}$$

where:

$$\mathbf{D}(\mathbf{r}) = \varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r}), \tag{3.7}$$

$$K(\mathbf{r}) = \frac{\varepsilon(\mathbf{r}) - \varepsilon_0}{\varepsilon(\mathbf{r})} \tag{3.8}$$

Since the polarization charge density $\rho(\mathbf{r})$ is related to the divergence of the polarization current by:

$$j\omega\rho(\mathbf{r}) = -\nabla \cdot \mathbf{J}_V(\mathbf{r}) \tag{3.9}$$

With (3.2), Eq. (3.6), and Eq. (3.9) can be discretized using the SWG basis functions as:

$$\mathbf{J}_V(\mathbf{r}) = j\omega \sum_{n=1}^{N_D} K(\mathbf{r}) I_n^D \mathbf{f}_n^V(\mathbf{r}) \tag{3.10}$$

$$\rho(\mathbf{r}) = -I_n^D K(\mathbf{r}) \nabla \cdot \mathbf{f}_n^V(\mathbf{r}) - I_n^D \mathbf{f}_n^V(\mathbf{r}) \nabla \cdot K(\mathbf{r}) \tag{3.11}$$

Combining Eq. (3.10) and Eq. (3.11), the magnetic vector potential and electric scalar potential within the dielectric volume can be expressed in terms of the SWG basis functions. Similarly, for the PEC surface region, the corresponding expressions for the magnetic vector potential and electric scalar potential can be directly obtained in terms of the RWG basis functions by combining (3.2). Therefore, the specific matrix elements of the previously defined submatrices Z^{DD} , Z^{DM} , Z^{MD} , and Z^{MM} can be determined, and the resulting VSIE system can then be solved using the Method of Moments.

3.3 NESAs-Based Compressed Representation of the VIE for Dielectric Targets

First, the case of a target composed solely of dielectric materials without any metallic components is considered. In this case, NESAs is employed to compress the far-field interaction matrix blocks of the volume integral equation. As discussed in Section 2.4, NESAs is a purely algebraic compression method. Since it does not rely on an explicit expansion of the kernel function, it can be readily applied to the low-rank approximation of matrix blocks associated with different integral kernels. The conventional NESAs method was originally developed for SIEs and has been applied to metallic conductors and homogeneous dielectric objects. For the inhomogeneous dielectric objects considered in Section 3.2, interactions between the SWG basis functions and the RWG-based equivalent sources need to be constructed to achieve the low-rank approximation of the VIE matrix blocks. Fig. 3.1 illustrates the procedure for representing the far-field interactions between dielectric groups using equivalent sources.

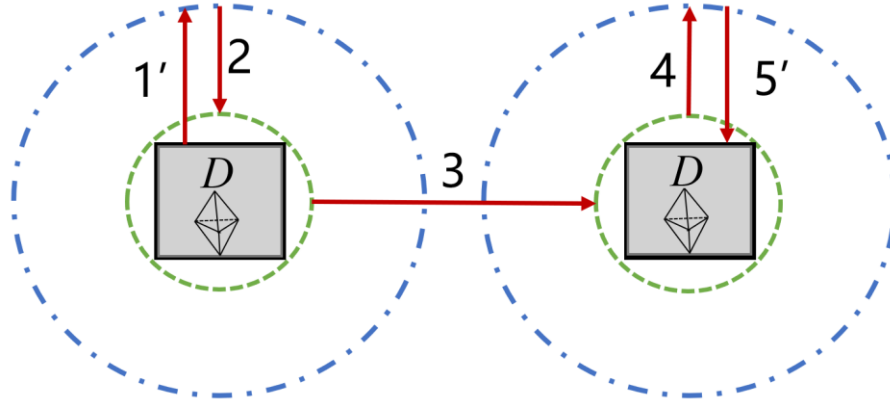


Fig. 3.1. Schematic illustration of the equivalent-source-based representation of far-field group interactions for dielectric targets. The region denoted by (D) represents the dielectric region. For the radiation process, the blue outer sphere represents the sphere on which the testing points are located, while the green inner sphere represents the sphere on which the equivalent sources are located.

In Fig. 3.1, Processes 1' and 2 constitute the radiation process for the dielectric region, in which the original SWG basis functions are replaced by equivalent sources to represent the radiation field generated by the dielectric region. Similarly, Processes 4 and 5' constitute the receiving process, in which the equivalent field on the equivalent surface is projected onto the original SWG basis functions. Consequently, the interactions between the SWG volumetric basis functions originally formulated by the volume integral equation are transformed into interactions between equivalent sources through the radiation and receiving processes. Since each equivalent source consists of three pairs of mutually orthogonal RWG basis functions, NESAs effectively transforms the volume unknowns into surface unknowns for the dielectric region.

Following the derivation presented in Section 2.4, the compressed representation of the far-field interaction matrix blocks for the dielectric region can be expressed as:

$$\begin{aligned} Z_{O,S}^{D,D} &= Z_{O,\sigma_O}^D Z_{\tau_O,\sigma_O}^\dagger Z_{\tau_O,\tau_S} Z_{\sigma_S,\tau_S}^\dagger Z_{\sigma_S,S}^D \\ &= U_O^D D_{O,S} V_S^D \end{aligned} \quad (3.12)$$

where the superscript D denotes the interactions associated with the dielectric region, σ and τ represent the testing sphere and equivalent sphere, respectively, and O and S denote the observation group and source group, respectively. Compared with Eq. (2.37), the main differences in Eq. (3.12) lie in U_O^D and V_S^D . Specifically, Z_{O,σ_O}^D and $Z_{\sigma_S,S}^D$ are calculated from the corresponding terms Z^{DM} and Z^{MD} in Eq. (3.3) and Eq. (3.4), respectively:

$$\begin{aligned} Z_{O,\sigma_O}^D &= j\omega \int_V \mathbf{f}^V(\mathbf{r}) \cdot \mathbf{A}_{S,eq}(\mathbf{r}') dV - \int_V \nabla \cdot \mathbf{f}^V(\mathbf{r}) \Phi_{S,eq}(\mathbf{r}') dV \\ &\quad + \int_\Omega \mathbf{n} \cdot \mathbf{f}^V(\mathbf{r}) \Phi_{S,eq}(\mathbf{r}') dS \end{aligned} \quad (3.13)$$

$$Z_{\sigma_S,S}^D = j\omega \int_S \mathbf{f}^{S,eq}(\mathbf{r}) \cdot \mathbf{A}_V(\mathbf{r}') dS - \int_S \nabla \cdot \mathbf{f}^{S,eq}(\mathbf{r}) \Phi_V(\mathbf{r}') dS \quad (3.14)$$

where the superscript “ eq ” denotes quantities associated with the equivalent surface sources, while $\mathbf{f}^S$ and $\mathbf{f}^V$ represent the RWG and SWG basis functions, respectively. Consistent with Eq. (2.37), $D_{O,S}$ represents the interactions between the equivalent sources.

3.4 NESAs-Based Compressed Representation of the VSIE for Mixed PEC and Dielectric Targets

Next, the case of mixed PEC and inhomogeneous dielectric objects is considered using NESAs. Since the target consists of both metallic and dielectric materials, the mappings from the RWG and SWG basis functions to the equivalent sources need to be treated separately. In this case, NESAs is employed simultaneously to approximate the low-rank representations of both the VIE and SIE matrix blocks. Fig. 3.2 illustrates the procedure for representing the far-field interactions between groups of mixed PEC and inhomogeneous dielectric objects using equivalent sources.

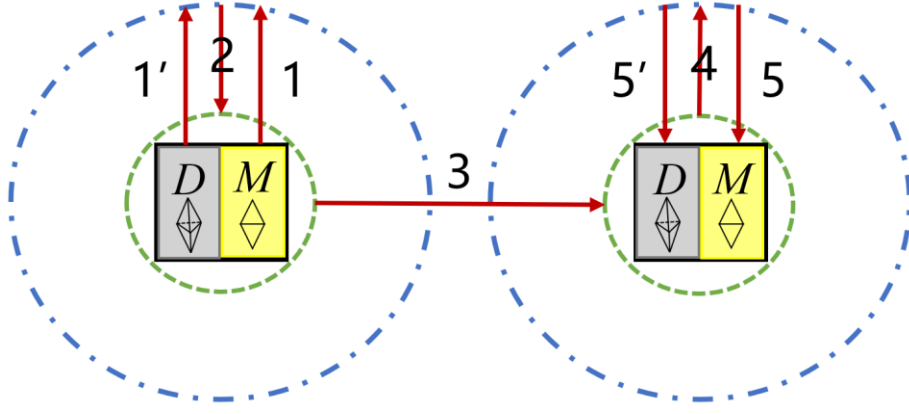


Fig. 3.2. Schematic illustration of the equivalent-source-based representation of far-field group interactions for mixed metallic and inhomogeneous dielectric targets. The region denoted by D represents the inhomogeneous dielectric region, while M represents the metallic conductor region. For the radiation process, the blue outer sphere represents the sphere on which the testing points are located, whereas the green inner sphere represents the sphere on which the equivalent sources are located. (© 2024 IEEE [151]).

In Fig. 3.2, the dielectric region is discretized into tetrahedral elements, on which SWG basis functions are constructed, while the metallic region is discretized into triangular elements, on which RWG basis functions are constructed. Taking the radiation process as an example, Processes 1 and 2 together constitute the radiation process for the metallic region, in which equivalent sources are used to represent the radiation field generated by the metallic region. Similarly, Processes 1' and 2 together constitute the radiation process for the dielectric region, in which equivalent sources are used to represent the radiation field generated by the dielectric region. Assuming that the actual radiation field generated jointly by the metallic and dielectric regions is equal to that generated by the corresponding equivalent sources, we have:

$$Z_{\sigma_S, \tau_S} I^{eq} = Z_{\sigma_S, M} I^M + Z_{\sigma_S, D} I^D \quad (3.15)$$

where τ and σ denote the equivalent sources and testing points, respectively, while I^M and I^D represent the metallic sources associated with the RWG basis functions and the dielectric sources associated with the SWG basis functions, respectively. Therefore, the left-hand side represents the field generated by the equivalent sources, whereas the right-hand side represents the field generated by the surface current sources on the metallic region and the volume current sources within the dielectric region. Accordingly, the equivalent sources can be expressed as:

$$I^{eq} = Z_{\sigma_S, \tau_S}^\dagger Z_{\sigma_S, M} I^M + Z_{\sigma_S, \tau_S}^\dagger Z_{\sigma_S, D} I^D \quad (3.16)$$

where $Z_{\sigma_S, \tau_S}^\dagger Z_{\sigma_S, M}$ and $Z_{\sigma_S, \tau_S}^\dagger Z_{\sigma_S, D}$ denote the radiation matrices associated with the metallic and dielectric regions, respectively. The receiving matrix is similarly constructed from the receiving processes of the metallic and dielectric regions. Physically, it maps the fields generated by the external equivalent sources at the internal testing points onto the metallic and dielectric regions, respectively,

and can therefore be regarded as the reverse process of radiation. Finally, the compressed representation of the VSIE impedance matrix can be expressed as:

$$\begin{aligned} Z_{O,S}^{x,y} &= Z_{O,\sigma_O}^x Z_{\tau_O,\sigma_O}^\dagger Z_{\tau_O,\tau_S} Z_{\sigma_S,\tau_S}^\dagger Z_{\sigma_S,S}^y \\ &= U_O^x D_{O,S} V_S^y \begin{cases} x = M \text{ or } D \\ y = M \text{ or } D \end{cases} \end{aligned} \quad (3.17)$$

where O and S denote the observation and source groups, respectively, while M and D represent the metallic and dielectric regions, respectively. When $x=M$, $y=M$, the target reduces to a purely metallic object, and (3.17) degenerates into the conventional NESAs for SIEs. Therefore, for the entire system matrix, the matrix blocks corresponding to far-field interactions can be divided into four parts according to the materials of the observation and source groups:

$$\begin{pmatrix} U_O^D D_{O,S} V_S^D & U_O^D D_{O,S} V_S^M \\ U_O^M D_{O,S} V_S^D & U_O^M D_{O,S} V_S^M \end{pmatrix} \cdot \begin{pmatrix} I^D \\ I^M \end{pmatrix} = \begin{pmatrix} b_D \\ b_M \end{pmatrix} \quad (3.18)$$

where b_D and b_M denote the excitation vectors associated with the dielectric and metallic regions, respectively. As shown in Fig. 3.2, at the lowest level of the octree, NESAs map both the metallic surface sources and dielectric volume sources onto the equivalent surface sources of each group. Therefore, at higher levels of the octree, only interactions between equivalent surface sources remain, which is consistent with the compression scheme of the conventional NESAs.

3.5 Numerical Results

A series of numerical examples are presented below to demonstrate the accuracy and computational complexity of the proposed NESAs-based VSIE fast algorithm. All numerical simulations were performed either on a server equipped with an Intel(R) Xeon(R) Silver 4114 CPU, 256 GB of RAM, and a clock frequency of 2.20 GHz, or on a personal computer equipped with an 11th-generation Intel(R) Core(TM) i7-11700F CPU with a clock frequency of 2.50 GHz.

3.5.1 Accuracy and Complexity Validation

First, the accuracy and computational complexity of the proposed fast algorithm are analyzed and verified. A three-layer dielectric-coated metallic sphere is constructed, where the radius of the inner metallic sphere is 0.5 m, the thickness of each dielectric layer is 0.05 m, and the relative permittivities of the three dielectric layers, from the outermost layer to the innermost layer, are 1.5-0.5j, 1.0-1.0j and 2.0 respectively. For the impedance matrix formed by two far-field interacting groups, the NESAs compression approximation is evaluated with different numbers of equivalent sources, and the results are compared with those obtained using the conventional Method of Moments as the reference solution. The matrix approximation error is quantified using the 2-norm error:

$$error_{i,j}^{NESA} = \frac{\|U_i^{NESA} \cdot D_{i,j}^{NESA} \cdot V_j^{NESA} - Z_{i,j}^{MoM}\|}{\|Z_{i,j}^{MoM}\|} \quad (3.19)$$

Where i and j are the group indexes of the far interactions. The approximation error curve is shown in Fig. 3.3.

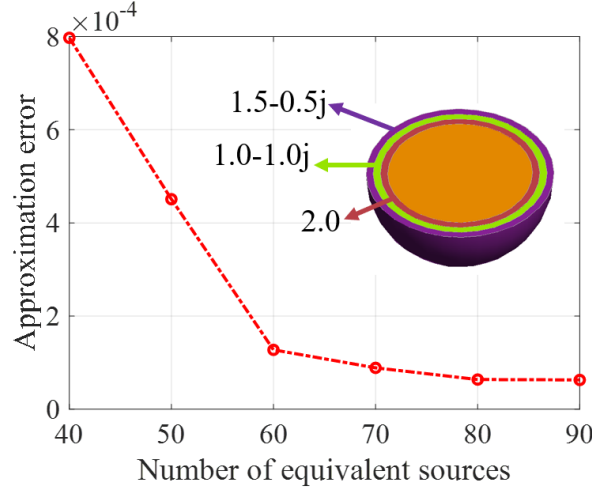


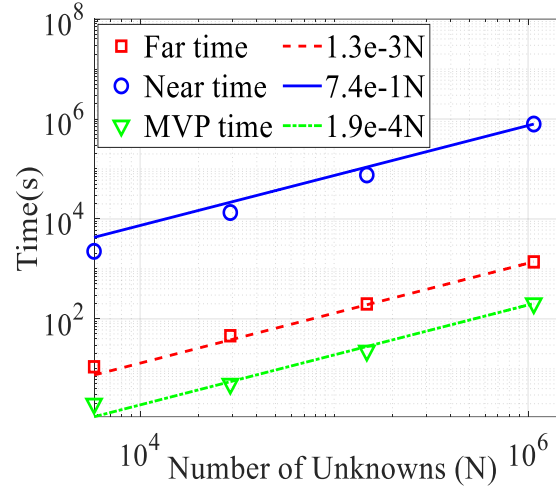
Fig. 3.3. NESAs compression error of the far-field impedance matrix block between two three-layer coated metallic sphere groups as a function of the number of equivalent sources. The inset illustrates the model geometry and the relative permittivity of each dielectric layer. (© 2024 IEEE [151]).

As shown in Fig. 3.3, the approximation error of NESAs for the VSIE matrix gradually decreases as the number of equivalent sources increases, demonstrating the accuracy of the proposed fast algorithm and confirming that its approximation error can be effectively controlled. The number of equivalent sources should be selected by balancing computational accuracy against resource consumption. A larger number of equivalent sources provides a more accurate representation of the spatial distribution of the radiation field generated by each group. However, it also increases the dimensions of the radiation, receiving, and translation matrices, thereby reducing computational efficiency and increasing memory consumption. Unless otherwise specified, 50 equivalent sources are used throughout this dissertation which has been discussed in [3].

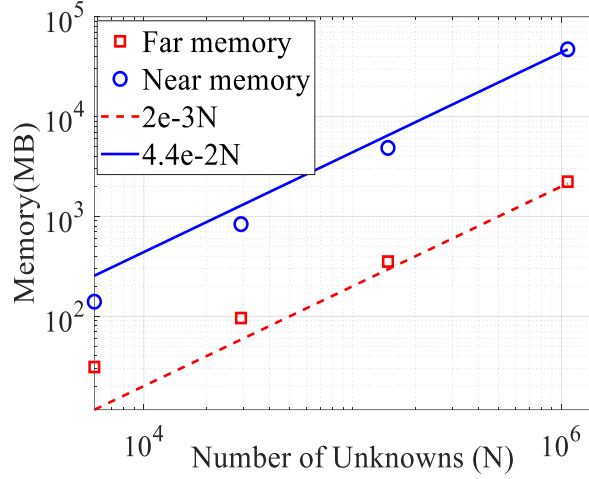
Next, the computational complexity of the proposed algorithm is investigated. As discussed in Section 3.3, the NESAs compression of VSIE matrix blocks at higher levels of the octree is consistent with the conventional compression scheme for SIE matrix blocks. The only difference is that the compression formulation at the lowest level must be modified according to whether the target region consists of metallic or dielectric materials. Therefore, the proposed VSIE algorithm has the same computational complexity as conventional NESAs. For low-frequency problems, its computational complexity is linear with respect to the number of unknowns, since modifying the type of basis functions at the lowest level does not alter the overall computational complexity of the algorithm.

To verify the linear computational complexity, the coated sphere model shown in Fig. 3.3 is considered. The numbers of unknowns are set to 5,790, 29,180, 147,513, and 1,070,289, respectively. The computational time and

memory consumption of the proposed algorithm are recorded for each problem size, as shown in Fig. 3.4.



(a)



(b)

Fig. 3.4. Computational and memory complexity of the NESAs-based VSIE fast algorithm. (a) Computational time; (b) Memory consumption. (© 2024 IEEE [151]).

It can be observed that both the computational time and memory complexity of the proposed algorithm are linear with respect to the number of unknowns. To further demonstrate the computational efficiency of the proposed method, the computational resource consumption for the coated-sphere simulations described above is recorded and compared with that of the multilevel fast multipole algorithm (MLFMA) for VSIE, as presented in Table 3.1.

TABLE 3.1. COMPARISON OF THE SIMULATION RESOURCE CONSUMPTION FOR THE COATED METALLIC SPHERE.

Method	Number of unknowns	Far/Near/Total memory (GB)	Computing time (s)	Group size at the leaf level
MLFMA	95738	14.72/0.55/15.27	698	0.23λ
NESA-VSIE	94669	2.16/0.58/2.74	448	0.08λ

As shown in the table, NESAs require substantially less memory than MLFMA for storing the near-field elements of the VSIE matrix. This is because MLFMA relies on the expansion of the integral kernel to compress far-field matrix blocks. Consequently, the group size at the lowest level is generally required to be larger than 0.25λ (λ is the wavelength), which limits the extent to which the octree can be further refined. For dielectric materials with different permittivities, the wavelength of the electromagnetic wave inside the material also varies. In particular, for high-permittivity dielectric materials, MLFMA must ensure that the wavelength within all groups satisfies the aforementioned group-size criterion. This can substantially increase the number of near-field elements in the system matrix.

In contrast, NESAs are kernel-independent matrix compression methods that do not rely on kernel-function expansions. Therefore, they are not subject to the 0.25λ group-size constraint. As a result, the octree can be further refined into more levels, allowing a larger number of matrix blocks to be compressed and thereby reducing the number of near-field elements. For example, in the present case, the finest-level group size in NESAs can be reduced to 0.08λ .

3.5.2 Electromagnetic Scattering Characteristics of Metallic and Inhomogeneous Dielectric Targets

In this subsection, the radar cross-section (RCS) of metallic and inhomogeneous dielectric targets is simulated to demonstrate the computational capability of the proposed algorithm in practical application scenarios. First, the scattering characteristics of a purely dielectric object are analyzed. A composite target consisting of a cone and a cylinder is constructed, as shown in Fig. 3.5. Both the cone and the cylinder have a base radius of 0.1 m. The cylinder has a height of 0.25 m, while the cone has a height of 0.22 m. The relative permittivities of the cone and cylinder are $1-2j$ and $2-4j$, respectively. The operating frequency is set to 1 GHz, resulting in 27,177 unknowns associated with the SWG basis functions. A plane wave is incident from the direction $\theta_i = 30^\circ, \varphi_i = 0^\circ$, and the bistatic RCS is calculated over the observation angular range of $\theta = 0^\circ \sim 180^\circ, \varphi = 0^\circ$. The results are compared with those obtained using the conventional Method of Moments (MoM), as shown in Fig. 3.5.

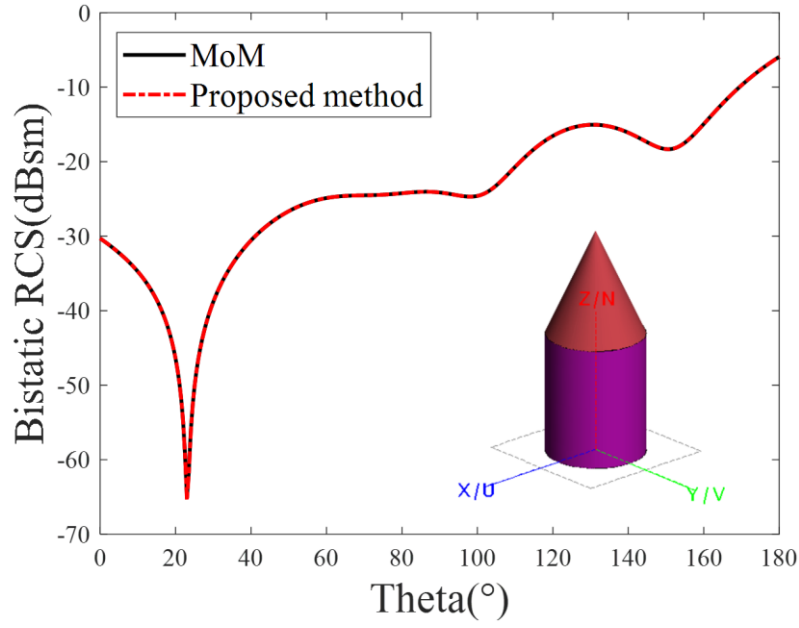


Fig. 3.5. Bistatic RCS simulation results for the composite dielectric target consisting of a cone and a cylinder..

As shown in Fig. 3.5, the results obtained using the proposed algorithm agree very well with those obtained using the conventional Method of Moments (MoM), demonstrating the accuracy of the proposed method.

Next, a metallic cone coated with two layers of inhomogeneous dielectric materials is considered. The cone has a height of 1.5 m and a base radius of 0.2 m. The relative permittivities ϵ_r of the inner and outer dielectric coatings are $4-0.04j$ and $1.5-0.03j$, respectively, and both coating layers have a thickness of 0.01 m. The operating frequency is set to 1 GHz. The incident-wave direction is $\theta_i = 90^\circ, \varphi_i = 0^\circ$, and the observation direction is $\theta = 0^\circ \sim 180^\circ, \varphi = 0^\circ$. The RCS results obtained using NESAs are compared with those calculated using the MLFMA, as shown in Fig. 3.6. It is worth noting that the WNESA is applied for the iterative solution [3] instead of the direct solution to be described in Chapter 6.

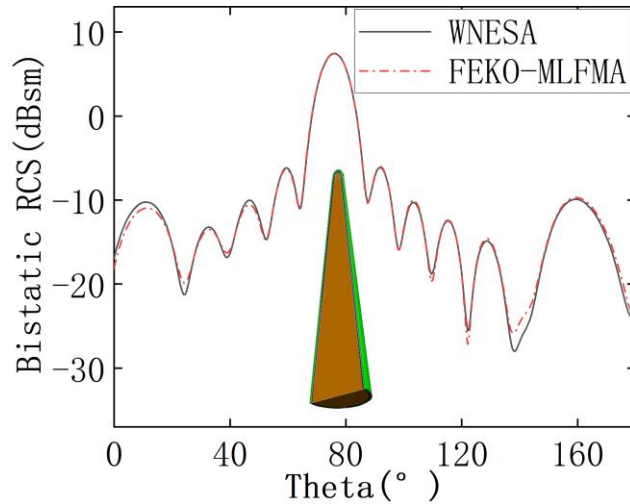


Fig. 3.6. Bistatic RCS simulation results for the metallic cone coated with two dielectric layers. (© 2024 IEEE [151]).

As shown in the figure, the bistatic RCS results obtained using the proposed algorithm agree very well with those calculated using MLFMA, further demonstrating the accuracy of the proposed method.

Finally, to investigate the electromagnetic scattering characteristics of a more complex metallic target surrounded by an inhomogeneous dielectric medium, a high-speed hypersonic vehicle is selected as the simulation target. When a metallic hypersonic vehicle travels at high speed through the atmosphere, a plasma sheath is generated around the vehicle [140], resulting in a spatially inhomogeneous distribution of relative permittivity. The metallic hypersonic vehicle considered here has a length of 0.2 m and a base radius of 0.1 m. When the vehicle travels at 10 Mach at an altitude of 30 km above the ground, the resulting plasma sheath has a length of 0.6 m and a base radius of 0.4 m. The corresponding distribution of relative permittivity is shown in Fig. 3.7.

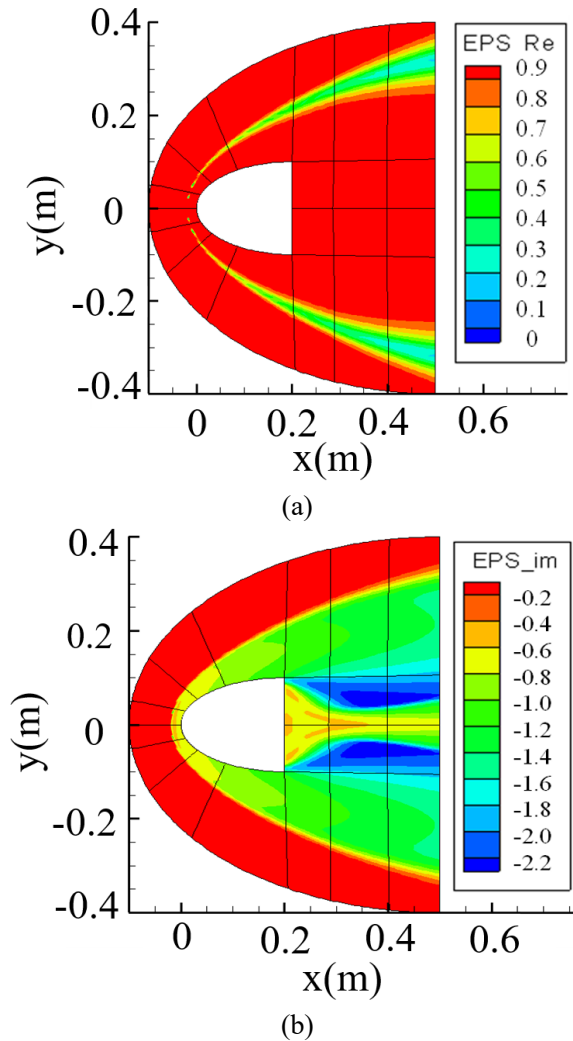


Fig. 3.7. Distribution of the dielectric permittivity of the plasma sheath surrounding the hypersonic vehicle. (a) Real part of the dielectric permittivity; (b) Imaginary part of the dielectric permittivity. (© 2024 IEEE [151]).

It can be observed that the dielectric permittivity distribution within the plasma sheath is highly inhomogeneous and cannot be accurately modeled using a simple piecewise-homogeneous dielectric model. Therefore, the SIE method

cannot be directly applied, and the VSIE formulation is employed to analyze its electromagnetic scattering characteristics. The operating frequency is set to 3 GHz, and the bistatic RCS is calculated. The incident-wave direction is $\theta_i = 90^\circ, \varphi_i = 180^\circ$, and the observation directions are $\theta = 0^\circ \sim 180^\circ, \varphi = 0^\circ$ and $\theta = 180^\circ \sim 0^\circ, \varphi = 180^\circ$. The surface discretization of the metallic vehicle produces 17,085 RWG basis functions, while the volume discretization of the plasma sheath produces 419,537 SWG basis functions. This result demonstrates that volume discretization indeed introduces substantially more unknowns than surface discretization. The calculated results are compared with those obtained using the conventional Method of Moments (MoM), as shown in Fig. 3.8. It can be seen that the WNESA calculation results are in good agreement with the MoM results. The WNESA results are obtained by the iterative solution [3] instead of the direct solution.

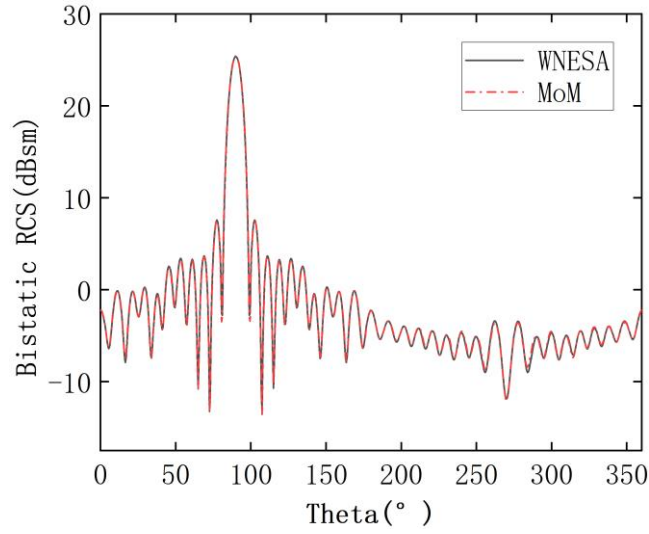


Fig. 3.8. Bistatic RCS of the metallic hypersonic vehicle and its plasma sheath. (© 2024 IEEE [151]).

3.6 Summary

This chapter investigates a NESA-based fast solver for the VSIE and extends the applicability of conventional NESA from metallic conductors or homogeneous dielectric media to mixed metallic and inhomogeneous dielectric targets. Based on the equivalence principle, the large number of unknowns generated by volume discretization is mapped onto equivalent sources, thereby significantly reducing the computational cost of VSIE analysis. Numerical examples demonstrate that, for electromagnetic scattering problems involving electrically small mixed metallic and inhomogeneous dielectric targets, the proposed algorithm achieves linear computational complexity, consistent with that of the conventional NESA for SIEs. Finally, the applicability of the proposed method to practical scenarios is further demonstrated through electromagnetic scattering analyses of mixed metallic and inhomogeneous dielectric targets, including a metallic projectile surrounded by a plasma sheath.

Chapter 4

NESA-Based Fast Direct Solution for EM Scattering at Low to Moderate Frequencies

4.1 Introduction

Integral equation problems can generally be solved using either iterative or direct methods. Iterative methods obtain the solution of a matrix equation by repeatedly evaluating the matrix–vector product between the system matrix and the unknown vector and comparing the resulting vector with the right-hand-side excitation vector to compute the iterative residual. When combined with fast algorithms for low-rank matrix blocks, iterative methods are often highly efficient for problems involving a small number of excitation terms. However, their efficiency strongly depends on the conditioning of the problem. When the system matrix has a large condition number, the iterative solution converges slowly. Moreover, for problems with multiple excitation terms, the iterative process must generally be restarted for each excitation term, which can substantially reduce the overall computational efficiency. In recent years, direct solvers have attracted considerable attention in the research community because they offer advantages in terms of convergence robustness and computational efficiency for multiple-right-hand-side (MRHS) problems.

Unlike iterative methods, direct solvers obtain an approximate inverse of the system matrix and then multiply it by the right-hand-side excitation vector to directly obtain the unknown vector. In general, direct solvers have a higher computational complexity than iterative methods. Their computational advantage mainly lies in the efficient multiplication between the computed inverse matrix and the right-hand-side vectors, whereas the preceding matrix inversion process is typically computationally expensive. To improve the efficiency of matrix inversion, direct solvers are often combined with low-rank matrix compression

techniques to accelerate the factorization and inversion process. Depending on the admissibility conditions adopted for matrix compression, these direct solvers can yield different hierarchical representations of the system matrix.

Direct solvers based on weak admissibility conditions include Hierarchically Semi-Separable (HSS) methods [96]-[100] and Hierarchically Off-Diagonal Low-Rank (HODLR) methods [101]-[104]. Their main advantage is a relatively simple matrix structure, since inadmissible blocks occur only along the diagonal. However, their computational complexity is generally higher than that of direct solvers based on strong admissibility conditions. Direct solvers based on strong admissibility conditions include butterfly-structured direct solvers [105]-[107], hierarchical matrix (H-matrix) methods [59], [108]-[110] and their improved variants [111], [112], as well as Strong Admissibility Skeletonization Factorization (SASF) methods [120]. In these approaches, inadmissible blocks may also occur off the diagonal, resulting in more complex hierarchical matrix structures.

Inspired by the SASF method and the H^2 -matrix method, this chapter proposes a direct solver for mid- and low-frequency problems based on the NESA matrix compression algorithm. The matrix structure generated by NESA is similar to that of the H^2 -matrix. By introducing transfer matrices between different levels, the radiation and receiving matrices at the lowest level can be reused, giving the system matrix a nested structure across different hierarchical levels. Since the NESA translation matrices are identical for child groups and their parent groups with the same relative spatial configuration, the proposed method requires less memory for matrix compression and approximation than SASF.

Furthermore, considering that the number of basis functions associated with groups at the lowest level of the octree is generally small, a hybrid compression strategy is introduced to further improve computational efficiency. The proposed direct solver combines skeleton compression with nested equivalent-source compression: skeleton decomposition is employed at the lowest level, while equivalent-source compression is adopted at higher levels. This results in a direct solver based on a Mixed-form NESA [70] representation that combines skeleton and equivalent-source compression. The direct solution procedure can essentially be regarded as a progressive LU factorization of the system matrix. However, the low-rank properties of far-field matrix blocks are exploited to approximately eliminate these blocks, thereby reducing the dimensions of the matrices involved in the factorization. Due to the presence of inadmissible off-diagonal blocks, dense Schur-complement blocks are generated during the LU factorization, which can significantly affect the sparse representation of the system matrix. To address this issue, the column spaces of these Schur-complement blocks are extracted using singular value decomposition (SVD), and the resulting subspaces are used to update the cluster bases during the direct solution process. The elimination matrices constructed from the updated cluster bases can simultaneously eliminate both the original admissible blocks and the Schur-complement blocks, allowing

the matrix factorization to proceed efficiently. Finally, the inverse of the system matrix can be approximately represented as a sequence of matrix products.

The remainder of this chapter is organized as follows. First, based on the low-rank compression representations of NESA matrix blocks at different hierarchical levels, the hierarchical structure of the NESA-based system matrix is derived, which provides the foundation for the level-by-level factorization of the system matrix in the proposed direct solver. The factorization procedure at the lowest level is then described in detail, followed by an introduction to the factorization of higher-level matrices using translation matrices. Subsequently, the time and memory complexities of the proposed mid- and low-frequency NESA-based direct solver are analyzed. Finally, numerical examples are presented to validate the accuracy and computational efficiency of the proposed method.

4.2 Analysis of the Hierarchical Matrix Structure Based on NESA

As discussed in Section 2.4.1, NESA primarily compresses matrix blocks by mapping the coupling between far-field groups onto the coupling between their equivalent sources. In the proposed direct solver, a skeleton decomposition method is employed at the lowest level of the octree. A set of basis-function skeletons is selected for each group, and the coupling between far-field groups is mapped onto the skeleton basis functions within the corresponding groups, as illustrated in Fig. 4.1(a).

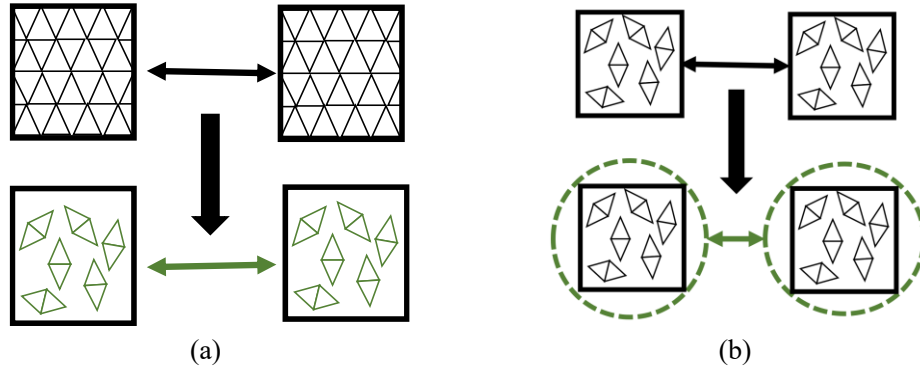


Fig. 4.1. (a) Schematic diagram of far-field interaction matrix compression at the lowest level; (b) Schematic diagram of far-field interaction matrix compression at higher levels.

The far field couplings can be compressed by:

$$\mathbf{Z}_{o,s} = \mathbf{U}_o^{skeleton} \mathbf{D}_{o,s}^{skeleton} \mathbf{V}_s^{skeleton} \quad (4.1a)$$

For more distant interactions, the coupling is represented using the equivalent sources of higher-level groups, thereby further mapping the coupling between the skeleton basis functions of child groups onto the equivalent sources of their parent groups, as illustrated in Fig. 4.1(b). The coupling matrix between more distant far-field groups can then be expressed as:

$$\mathbf{Z}_{o,s} = \mathbf{U}_o^{skeleton} \mathbf{B}_o^{equivalence} \mathbf{D}_{o,s}^{equivalence} \mathbf{C}_s^{equivalence} \mathbf{V}_s^{skeleton} \quad (4.1b)$$

In Eqs. (4.1a) and (4.1b), the matrices U and V denote the receiving and radiation matrices associated with the skeletons, respectively, while B and C represent the translation matrices associated with the equivalent sources. The advantages of the proposed hybrid low-rank compression strategy, which combines skeletons at the lowest level with equivalent sources at higher levels, can be illustrated through the following simple example.

A sphere model with 85,560 unknowns is considered and partitioned using a three-level octree. Different numbers of skeleton and equivalent-source levels are employed, and the memory consumption and computational time required for the cluster bases and coupling matrices associated with far-field interactions are recorded. The computational time for the system matrix–vector multiplication is also evaluated, as summarized in Table 4.1.

TABLE 4.1. MEMORY CONSUMPTION AND CONSTRUCTION TIME OF CLUSTER BASES AND COUPLING MATRICES, AND SYSTEM MATRIX–VECTOR MULTIPLICATION TIME FOR THE SPHERE MODEL WITH DIFFERENT SKELETON AND EQUIVALENT-SOURCE LEVELS.

Skeleton/Equivalence levels	Far field memory (MB)	Far field time (s)	MVM time (s)
0/3	560	56	0.92
1/2	1504	220	0.72
2/1	1781	251	0.68
3/0	1908	258	0.67

As shown in Table 4.1, when a pure equivalent-source strategy is employed, both the memory consumption and construction time required for building the cluster bases are substantially lower than those of the skeleton strategy. However, the corresponding system matrix–vector multiplication time is relatively high, and the memory consumption increases as the number of skeleton levels increases.

The savings in memory consumption and construction time achieved by the equivalent-source strategy can be explained from two aspects. First, for the cluster bases at a given hierarchical level, which correspond to the radiation matrices at the finest level or the translation matrices at higher levels in this work, the skeleton strategy requires separate cluster bases to be constructed for all groups at that level during the upward transfer to the next level. In contrast, for the equivalent-source strategy, there are only eight possible relative spatial configurations between a parent group and its child groups in an octree. Therefore, at most eight translation matrices need to be constructed to cover all possible configurations.

Second, for the coupling matrices, different interacting group pairs generally correspond to different coupling matrices when the skeleton strategy is employed, and these matrices therefore need to be constructed separately. In contrast, the equivalent sources have an identical spatial distribution for all groups. Consequently, interactions with the same relative spatial configuration can share

the same coupling matrix, enabling the reuse of coupling matrices and substantially reducing the associated memory and construction costs.

In summary, since the groups at the finest leaf level contain fewer basis functions than those at higher levels, skeleton decomposition is computationally more efficient than equivalent-source approximation at the lowest level. At higher levels, however, the equivalent-source translation matrices between parent and child groups with the same relative spatial configuration, as well as the coupling matrices corresponding to the same relative group configuration, can be reused. This allows the equivalent-source strategy to achieve lower memory consumption than skeleton decomposition at higher levels.

Therefore, the hybrid strategy combining equivalent sources and skeleton decomposition provides a practical compromise between the two approaches. It exploits the computational efficiency of skeleton decomposition at the finest level while taking advantage of the matrix reuse enabled by equivalent sources at higher levels, thereby maintaining relatively low memory consumption while achieving faster matrix factorization. Next, a two-level NESA matrix is considered as an example to illustrate its structural characteristics, as shown in Fig. 4.2.

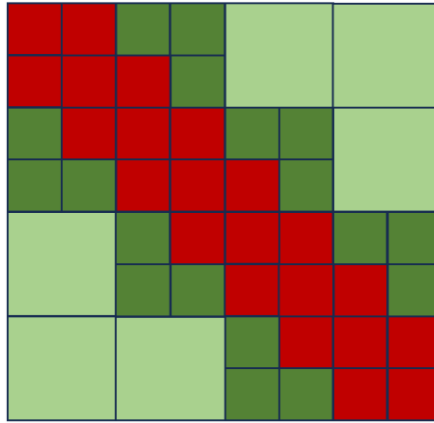


Fig. 4.2. Schematic diagram of the structure of a two-level NESA matrix.

In Fig. 4.2, the small blocks represent the matrix blocks associated with groups at the second level. The red blocks denote the inadmissible blocks at the second level, while the dark-green blocks denote the admissible blocks at the same level. The large light-green blocks represent the admissible blocks at the first level. The matrix elements of the red blocks at the second level are computed directly using the Method of Moments, whereas the dark-green blocks can be readily obtained from Eq. (4.1a). The expression of the light-green admissible blocks at the first level is more complicated. Assume that groups A and B at the first level are mutually far-field interacting groups, and that each group contains two child groups, with group A consisting of A1 and A2, and group B consisting of B1 and B2, as illustrated in Fig. 4.3.

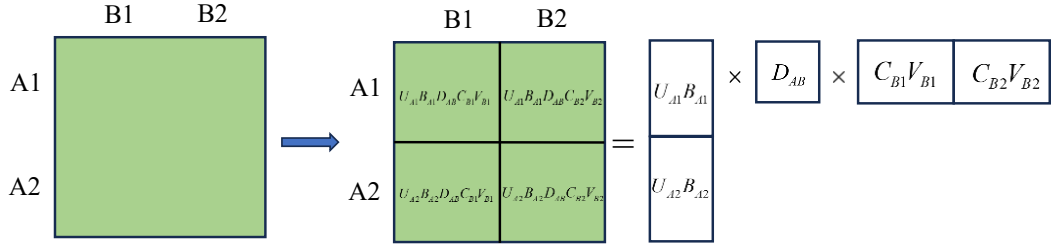


Fig. 4.3. Schematic diagram of the compression for admissible blocks at the first level.

Taking the receiving process as an example, U_{A1} and U_{A2} denote the processes of selecting skeletons from the original basis functions of the child groups A1 and A2, respectively, while B_{A1} and B_{A2} denote the processes of mapping the skeleton basis functions of A1 and A2 onto the equivalent sources of their parent group A, respectively. Therefore, the admissible block formed by groups A and B at the first level can also be decomposed into the product of three matrices. Unlike the lowest-level case, the radiation and receiving matrices at the first level additionally contain translation matrices that establish the nested relationships between different hierarchical levels.

As the hierarchical level increases by one, an additional translation matrix is introduced into the factorization of the far-field coupling matrix. Nevertheless, its compressed representation remains structurally similar to that of the admissible blocks at lower levels. This nested property provides the fundamental matrix structure for the direct solver introduced in the following sections. Hereafter, the lowest leaf level is denoted as level L , while the highest root level is denoted as level 1.

4.3 Implementation of the Direct Solution Based on NESAs

After analyzing the structural characteristics of the NESAs matrix, the system matrix can be factorized and inverted by exploiting its nested structure, following the framework of the H^2 -matrix method. The variables and parameters involved in the proposed direct solver are uniformly defined in Table 4.2.

TABLE 4.2. VARIABLES AND DEFINITIONS INVOLVED IN THE PROPOSED DIRECT SOLVER.

Variable	Definition
L	Total number of levels in the octree
m_i	Number of basis functions of group i
k_i	Number of skeletons of group i
U_i	Radiation matrix of group i
V_i	Receiving matrix of group i
R_i	Remaining part of U_i after SVD of group i
$D_{i,j}$	Initial coupling matrix between group i and j

Q_i	Elimination matrix for admissible block of group i
$F_{i,j}$	Fill-in blocks between group i and j
T_i^l	Transfer matrix of group i at l th level
P^l	Permuting matrix at l th level

Next, the direct solution procedure is described in detail using the system matrix corresponding to a practical rectangular metallic strip as an example. As shown in Fig. 4.4(a), the metallic strip is partitioned into two hierarchical levels. At level L , corresponding to the finest leaf level, the strip is divided into eight groups, while at the first level it is divided into four groups. The index of each group is indicated within the corresponding group.

The group indices are intentionally arranged in a non-sequential order to facilitate the subsequent explanation of the generation of Schur-complement blocks during the direct solution process and the treatment required when such blocks appear in admissible matrix blocks. Figure 4.4(b) illustrates the corresponding system matrix structure of the metallic strip.

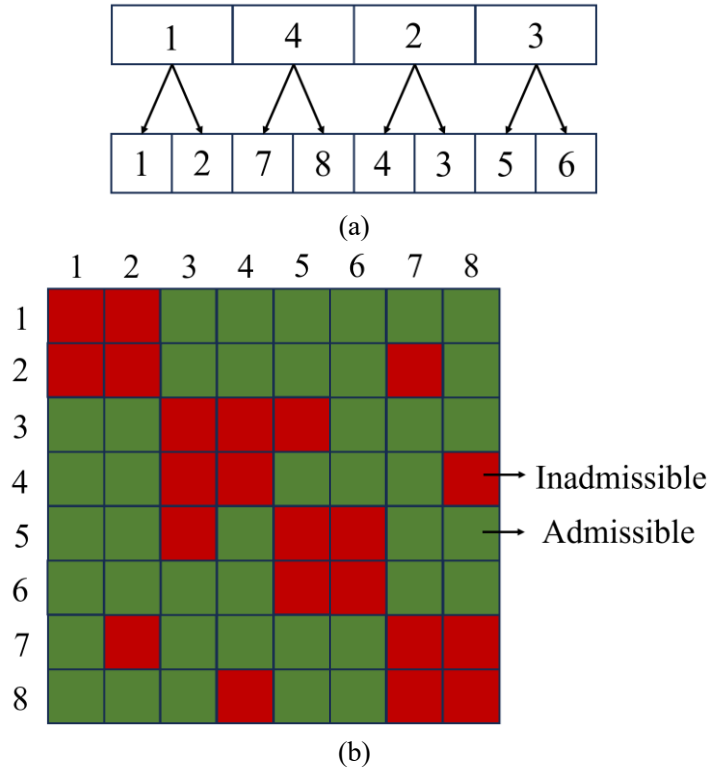


Fig. 4.4. (a) Grouping and numbering of the rectangular metallic strip; (b) Schematic diagram of the system matrix of the metallic strip at level (L) , where the red blocks denote inadmissible blocks corresponding to near-field interactions, and the green blocks denote admissible blocks corresponding to far-field interactions. (© 2026 IEEE [121]).

After obtaining the compressed representations of all admissible blocks at level L , the system matrix can be factorized and inverted level by level and group by group through a one-way bottom-up traversal. Since the receiving and radiation matrices constructed from the EFIE are transposes of each other, only

the receiving matrix (cluster basis) is considered here for illustration. Taking the system matrix shown in Fig. 4.4(a) as an example, the overall direct solution procedure is described in detail as follows.

4.3.1 Factorization of the System Matrix at the Leaf Level

The factorization process starts from level L . This subsection introduces the factorization procedure at level L , where admissible blocks are represented using skeleton decomposition. To further reduce the matrix dimensions and improve computational efficiency, the proposed algorithm first performs singular value decomposition (SVD) [141] on the original cluster bases and truncates the corresponding left and right singular vectors according to a prescribed threshold. Suppose that there are G nonempty groups at level L , with group indices $i(i=1,2,3\dots G)$. These groups are eliminated sequentially through the following two-step procedure, in which the corresponding admissible and inadmissible blocks are factorized in turn.

The first step is the factorization of admissible blocks. Taking the first group, $i=1$, as an example, suppose that it contains m_1 original basis functions and that k_1 skeleton basis functions are selected. The original cluster basis U_1 is replaced by its truncated representation $\tilde{U}_1$ obtained through SVD:

$$(U_1)_{m_1 \times k_1} \approx (\tilde{U}_1)_{m_1 \times \tilde{k}_1} (R_1)_{\tilde{k}_1 \times k_1} \quad (4.2)$$

where $\tilde{k}_1$ denotes the truncation rank of the SVD, which is smaller than k_1 . The matrix R_1 is obtained from the product of the corresponding singular values and right singular vectors after the SVD of the original cluster basis U_1 . Accordingly, the original coupling matrix $D_{1,j}$ associated with this group is updated to $\tilde{D}_{1,j}$ as follows:

$$(\tilde{D}_{1,j})_{\tilde{k}_1 \times \tilde{k}_j} = (R_1)_{\tilde{k}_1 \times k_1} (D_{1,j})_{k_1 \times k_j} (R_j^T)_{k_j \times \tilde{k}_j} \quad (4.3)$$

Here, $\tilde{k}_j$ denotes the truncated SVD rank of the cluster basis associated with the far-field group j of the first group. R_j^T is the transpose of R_j which arises from the transpose relationship between the receiving and radiation matrices. It can be observed that the dimension of the coupling matrix is also reduced through the SVD decomposition.

Subsequently, to eliminate the admissible blocks associated with the first group from the system matrix, a block-diagonal elimination matrix $Q_{1,\text{sys}}$, having the same dimension as the system matrix, is constructed. Its diagonal block corresponding to the first group is set to Q_1 , while the diagonal blocks corresponding to all other groups are identity matrices. The matrix Q_1 can be obtained directly by performing column transformations on the unitary matrix of left singular vectors resulting from the SVD decomposition:

$$\mathcal{Q}_1 = (\tilde{U}_1^\perp \quad \tilde{U}_1) \quad (4.4)$$

The column vectors of $\tilde{U}_1^\perp$ and $\tilde{U}_1$ are mutually orthogonal and complementary. Therefore, $\mathcal{Q}_1$ is also a unitary matrix. By multiplying the system matrix by the conjugate transpose of $\mathcal{Q}_{1,\text{sys}}$ from the left and multiplying it by $\mathcal{Q}_{1,\text{sys}}$ from the right, respectively, we obtain:

$$Z'_{\text{sys}} = \mathcal{Q}_{1,\text{sys}}^H Z_{\text{sys}} \bar{\mathcal{Q}}_{1,\text{sys}} \quad (4.5)$$

As a result, the first $m_1 - \tilde{k}_1$ rows and columns preceding the first group associated with the admissible blocks in the system matrix are approximately eliminated. This is because the corresponding part of the elimination matrix $\tilde{U}_1^\perp$ is orthogonal to the admissible blocks. The elimination of the admissible blocks associated with the remaining nonempty groups follows a similar procedure, except that the cluster bases of the respective groups need to be updated, which will be described in the following sections.

After the admissible blocks associated with the first group are eliminated, the system matrix in Fig. 4.4(b) is updated to the form shown in Fig. 4.5. Since the compressed representation of the admissible blocks involves approximation errors, and their elimination is achieved by constructing the corresponding complementary matrices through SVD, which also introduces truncation errors, the primary sources of error in the proposed algorithm originate from this first step.

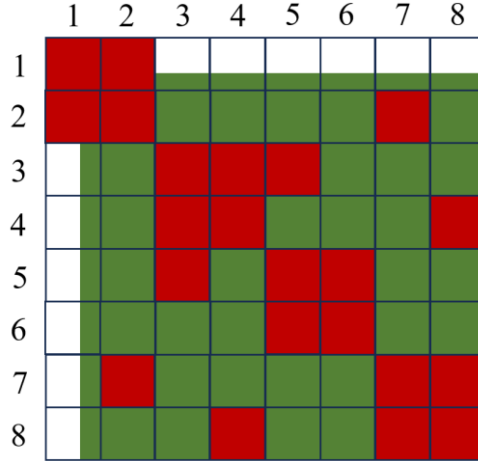


Fig. 4.5. Schematic diagram of the system matrix after eliminating the admissible blocks associated with the first group. All admissible blocks associated with the first group have been eliminated, leaving only the inadmissible block corresponding to its interaction with the second group. (© 2026 IEEE [121]).

The second step is the factorization of the inadmissible blocks. For the first group, LU decomposition is applied to the system matrix using the first $(m_1 - \tilde{k}_1) \times (m_1 - \tilde{k}_1)$ rows and columns of the first group's diagonal block as the pivot block. This process generates Schur-complement blocks in the system matrix, whose locations are determined by the positions of the inadmissible blocks associated with the corresponding group. A general representation is used here to

illustrate the generation of Schur-complement blocks. Consider a matrix composed of nine blocks (A,B,C,D,E,F,G,H,M). By performing LU decomposition on block A, the entire matrix can be factorized as follows:

$$\begin{pmatrix} E & F & G \\ H & A & B \\ M & C & D \end{pmatrix} = L \begin{pmatrix} E' & 0 & G' \\ 0 & I & 0 \\ M' & 0 & D' \end{pmatrix} U \quad (4.6)$$

where I is the identity matrix, L and U are the upper and lower triangle matrices:

$$L = \begin{pmatrix} I & FU_A^{-1} & 0 \\ 0 & L_A & 0 \\ 0 & CU_A^{-1} & I \end{pmatrix}, \quad U = \begin{pmatrix} I & 0 & 0 \\ L_A^{-1}H & U_A & L_A^{-1}B \\ 0 & 0 & I \end{pmatrix} \quad (4.7)$$

Matrix A is viewed as the pivot block for the LU factorization, e.g., $A = L_A U_A$, and:

$$E' = E - FU_A^{-1}L_A^{-1}H \quad (4.8)$$

$$G' = G - FU_A^{-1}L_A^{-1}B \quad (4.9)$$

$$M' = M - CU_A^{-1}L_A^{-1}H \quad (4.10)$$

$$D' = D - CU_A^{-1}L_A^{-1}B \quad (4.11)$$

Are the Schur complements. In addition, the second term on the right-hand side of the equation represents the dense fill-in matrix blocks generated during the LU factorization. Based on this observation, two elimination matrices are defined for the inadmissible blocks:

$$L^{-1} = \begin{pmatrix} I & -FU_A^{-1}L_A^{-1} & 0 \\ 0 & L_A^{-1} & 0 \\ 0 & -CU_A^{-1}L_A^{-1} & I \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} I & 0 & 0 \\ -U_A^{-1}L_A^{-1}H & U_A^{-1} & -U_A^{-1}L_A^{-1}B \\ 0 & 0 & I \end{pmatrix} \quad (4.12)$$

By left and right multiply the matrices from Eq. (4.12) for the initial matrix in Eq. (4.6), we can have:

$$L^{-1} \begin{pmatrix} E & F & G \\ H & A & B \\ M & C & D \end{pmatrix} U^{-1} = \begin{pmatrix} E' & 0 & G' \\ 0 & I & 0 \\ M' & 0 & D' \end{pmatrix} \quad (4.13)$$

It can be seen that both the row and column blocks associated with matrix (A) are eliminated, while the remaining blocks are updated by the corresponding fill-in matrix blocks. It should be noted that Eq. (4.6) provides a general representation of the second-step operation for the matrix. For the first group, since the pivot block for the LU decomposition is located at the upper-left corner of the system matrix, the blocks E, F, G, H, and M in Eq. (4.6) do not exist. Therefore, the matrix can be represented using a 2×2 block structure.

Based on the above derivation, the system matrix is factorized using the diagonal subblock corresponding to the first group as the pivot block, thereby eliminating all inadmissible blocks associated with the first group, as illustrated in Fig. 4.6.

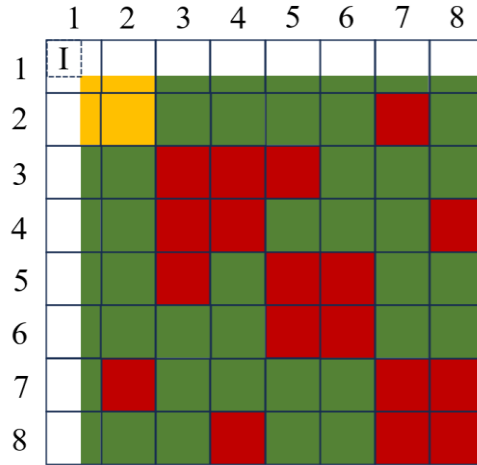


Fig. 4.6. Schematic diagram of the system matrix after eliminating the inadmissible blocks associated with the first group. All admissible and inadmissible blocks associated with the first group have been eliminated, and the diagonal pivot block corresponding to the first group is reduced to an identity block. (© 2026 IEEE [121]).

In Fig. 4.6, the yellow regions represent the portions updated by the fill-in matrix blocks generated from the corresponding Schur complements. Since all fill-in blocks generated during the elimination of the first group are added to inadmissible blocks, they can be directly accumulated into the corresponding matrix blocks.

However, during the subsequent elimination of the remaining groups, the generated fill-in blocks may also be added to admissible blocks. These admissible blocks are not explicitly stored as dense matrix blocks; instead, they are represented in compressed form through their cluster bases and coupling matrices. Therefore, directly adding the fill-in blocks and converting the resulting admissible blocks into inadmissible blocks would progressively increase the number of inadmissible blocks as the factorization proceeds, leading to a significant degradation in computational efficiency.

To preserve the compressed representation of the admissible blocks, the proposed algorithm temporarily stores the fill-in matrix blocks generated on admissible blocks until the factorization reaches the group corresponding to their positions. At that stage, all fill-in blocks associated with the group are compressed simultaneously, thereby maintaining the compressed representation of the system matrix throughout the factorization process.

It can be observed from the expressions of the fill-in matrix blocks in Eqs. (4.8)–(4.11) that these blocks are essentially low-rank matrices and can therefore be efficiently compressed. This procedure can be illustrated by the factorization of the third group. Assuming that the above two elimination steps are sequentially applied to each group, the system matrix is updated to the form shown in Fig. 4.7 after the factorization of the third group is completed.

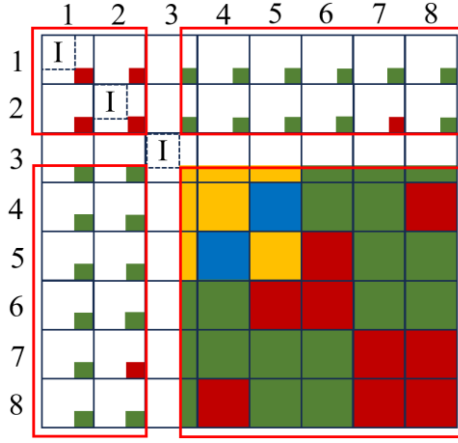


Fig. 4.7. Schematic diagram of the system matrix after completing the factorization of the third group. All matrix blocks associated with the first and second groups have been factorized, while the fill-in matrix blocks generated during the factorization of the third group are added to the admissible blocks in the blue regions. The four red boxes indicate the different locations of the Schur-complement blocks in the system matrix. The upper-left and upper-right boxes correspond to E and G in Eq. (4.6), respectively, while the lower-left and lower-right boxes correspond to M and D, respectively. (© 2026 IEEE [121]).

As shown in Fig. 4.7, the elimination of matrices $Z_{4,3}$, $Z_{5,3}$, $Z_{3,4}$, and $Z_{3,5}$ generates fill-in blocks $F_{4,5}$ and $F_{5,4}$, which are added to the admissible blocks. Taking the admissible block $Z_{4,5}$ as an example, the cluster bases of groups 4 and 5 need to be updated accordingly, so that the resulting $Z_{4,5}$ can still be represented in compressed form:

$$\hat{U}_4 \hat{D}_{4,5} \hat{V}_5 \approx \tilde{U}_4 \tilde{D}_{4,5} \tilde{V}_5 + F_{4,5} \quad (4.14)$$

Here, $\hat{U}_4$, $\hat{D}_{4,5}$, and $\hat{V}_5$ denote the updated representations of $\tilde{U}_4$, $\tilde{D}_{4,5}$, and $\tilde{V}_5$, respectively. Their specific expressions will be derived in detail in the following sections. Subsequently, during the factorization of the fourth group, the algorithm uses $\hat{U}_4$ rather than $\tilde{U}_4$ as the cluster basis and constructs the corresponding elimination matrix for the admissible blocks:

$$Q_4 = \begin{pmatrix} \hat{U}_4^\perp & \hat{U}_4 \end{pmatrix} \quad (4.15)$$

It can be shown that the elimination matrices Q_i ($i=1,2,3\dots G$) for the admissible blocks of all groups, are unitary matrices, and the accuracy of eliminating the admissible blocks in the first step is primarily determined by Eq. (4.14). From the above two steps, it can be observed that the elimination of admissible blocks in the first step involves truncation errors introduced by the SVD, and, more importantly, the approximation error associated with Eq. (4.14). In contrast, the elimination of inadmissible blocks in the second step is performed through an exact LU factorization and therefore introduces no approximation error. The approximation error arising in the first step constitutes the principal algorithmic error of the proposed direct solver. This error will be discussed in detail later, and it will be demonstrated that its impact on the final current solution remains controllable. The same two-step factorization procedure is then

sequentially applied to the remaining groups at the leaf level until all matrix blocks at this level have been factorized and eliminated.

Next, the treatment of the fill-in matrix blocks is introduced to obtain the representation in Eq. (4.14). The i -th group is taken as an example. To preserve the nested structure of the admissible blocks at different levels of NESA, the original cluster basis $\tilde{U}_i$ is retained as part of the new cluster basis $\hat{U}_i$ so that the original admissible blocks can still be represented in their original compressed form. However, the original cluster basis $\tilde{U}_i$ alone cannot accurately represent the newly generated fill-in matrix blocks, because its column space does not fully span the column space of the fill-in blocks, resulting in an unavoidable approximation error.

It should be noted that $\tilde{U}_i^\perp$ and $\tilde{U}_i$ generated by the SVD are strictly orthogonal and complementary. Consequently, the projection of the column space of the fill-in matrix blocks can be extracted solely from the $\tilde{U}_i^\perp$. Together with the column space of the original cluster basis $\tilde{U}_i$, these vectors can fully span the column space of all associated fill-in matrix blocks.

To achieve this, the Gram matrices of all fill-in matrix blocks associated with group i are first computed and accumulated. By multiplying the resulting matrix with the Gram matrix of $\tilde{U}_i^\perp$, the column spaces of the fill-in blocks are projected onto the complementary subspace of $\tilde{U}_i^\perp$. An SVD is then performed on the resulting matrix to extract the column space occupied by the fill-in matrix blocks, namely:

$$\tilde{U}_i^\perp (\tilde{U}_i^\perp)^H \left(\sum_{k=1}^{n_i} F_{i,j_k} F_{i,j_k}^H \right) \tilde{U}_i^\perp (\tilde{U}_i^\perp)^H \stackrel{SVD}{\approx} U_i^{aug} S (U_i^{aug})^H \quad (4.16)$$

where n_i denotes the number of fill-in matrix blocks associated with group i , and j denotes the corresponding column index. By setting the truncation threshold of the SVD to $\varepsilon_{fill-in}$, the extracted left singular vectors U_i^{aug} are selected as the augmented part of the updated cluster basis $\tilde{U}_i$.

It should be noted that, since the fill-in matrix blocks generated before the factorization of group i have already been partially factorized, the fill-in blocks remaining after the previous elimination steps, rather than the original fill-in blocks, should be used in Eq. (4.16).

In this work, the SVD truncation threshold $\varepsilon_{fill-in}$ in this step is adopted as the accuracy-control parameter of the proposed direct solver. A smaller $\varepsilon_{fill-in}$ results in a more accurate compressed representation of the fill-in matrix blocks, thereby making the matrix elements remaining after elimination of the admissible blocks closer to zero. Finally, the updated cluster basis of the i -th group can be expressed as:

$$\hat{U}_i = \begin{pmatrix} \tilde{U}_i & U_i^{aug} \end{pmatrix} \quad (4.17)$$

It can be inferred that $\tilde{U}_i$ and U_i^{aug} are also orthogonal, because U_i^{aug} is in the space of $\tilde{U}_i^\perp$.

4.3.2 Factorization of the System Matrix at Higher Levels

After completing the factorization at level L , the algorithm proceeds upward through the hierarchical levels and continues the factorization process level by level. This subsection introduces the factorization procedure at the higher levels. Since each group at a higher level generally contains more basis functions than those at the finest leaf level, the admissible blocks at higher levels are represented using equivalent sources. After completing the factorization at level (L) , the system matrix is updated to the form shown in Fig. 4.8.

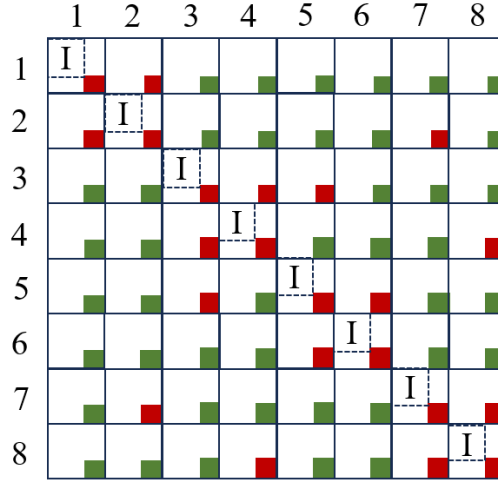
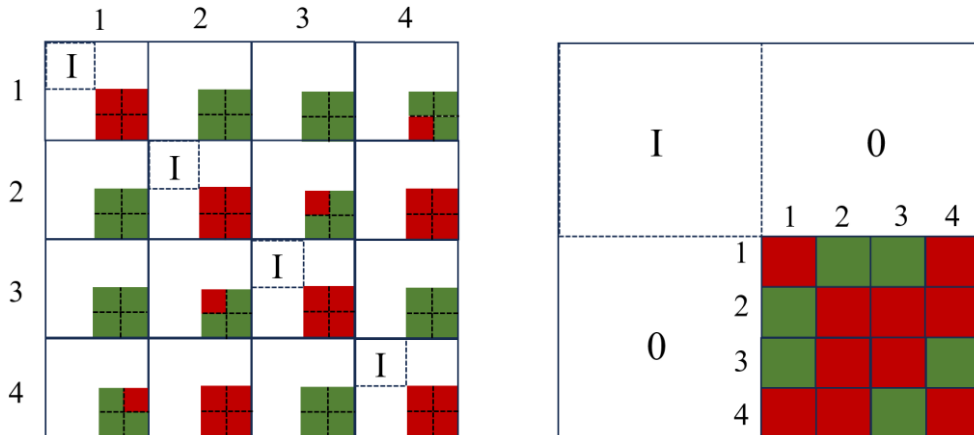


Fig. 4.8. Schematic diagram of the system matrix after completing the factorization at the finest leaf level. (© 2026 IEEE [121]).

As shown in Fig. 4.8, after the factorization at level L , most of the system matrix has been eliminated, leaving only a small matrix whose dimensions are consistent with those of the coupling matrices at level L . Before starting the factorization at level $L-1$, these remaining matrix blocks need to be aggregated and permuted to form a matrix structure similar to that at level L . The resulting system-matrix structures after the aggregation and permutation operations are shown in Fig. 4.9.



(a)

(b)

Fig. 4.9. Schematic diagram of the system matrix after completing the factorization at the finest leaf level: (a) system-matrix structure after aggregating the remaining matrix blocks associated with each group at the parent level; (b) system-matrix structure after rearranging the aggregated parent-level matrix blocks through permutation. (© 2026 IEEE [121]).

As shown in the figure, at the level $L-1$, if all interactions between the child groups of two groups are admissible, the corresponding matrix block at level $L-1$ is also admissible. Conversely, if any child-group interaction is inadmissible, the corresponding matrix block at the parent level is classified as inadmissible.

For the admissible blocks at level $L-1$, the cluster bases at level L have already been eliminated during the factorization at level L . Therefore, the admissible blocks at level $L-1$ are mainly represented in compressed form through the corresponding translation matrices. In other words, during the factorization at higher levels, the translation matrices at the current level play a role analogous to that of the cluster bases at level L .

More generally, considering the receiving process at level $l-1$, let the translation matrix associated with the receiving process of the i -th group at level l be denoted by T_i^l . Then, during the factorization of the system matrix, the cluster basis at level $l-1$ can be expressed as:

$$U_i^{l-1} = \begin{pmatrix} U_{i1}^l & & \\ & \ddots & \\ & & U_{i8}^l \end{pmatrix} \begin{pmatrix} T_{i1}^l \\ \vdots \\ T_{i8}^l \end{pmatrix} = \begin{pmatrix} \hat{U}_{i1}^l & & \\ & \ddots & \\ & & \hat{U}_{i8}^l \end{pmatrix} \begin{pmatrix} \begin{pmatrix} R_{i1}T_{i1}^l \\ 0 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} R_{i8}T_{i8}^l \\ 0 \end{pmatrix} \end{pmatrix} \quad (4.18)$$

In Eq. (4.18), the subscripts 1~8 indicate that in an octree, each nonempty parent group can contain up to eight nonempty child groups. R_i denotes the matrix obtained from the SVD decomposition in Eq. (4.2). It can also be observed that the original translation matrix is augmented by adding zero entries, thereby increasing its row dimension. This operation ensures that the dimension of the translation matrix is consistent with that of the augmented cluster basis at the child level. Therefore, the translation matrix at level $l-1$ can be expressed as:

$$\hat{T}_i^{l-1} = \begin{pmatrix} \begin{pmatrix} R_{i1}T_{i1}^l \\ 0 \end{pmatrix} \\ \vdots \\ \begin{pmatrix} R_{i8}T_{i8}^l \\ 0 \end{pmatrix} \end{pmatrix} \quad (4.19)$$

The remaining coupling matrices at level l must also be augmented accordingly to match the increased dimensions of the augmented cluster bases. Taking the coupling matrix associated with the (i, j) matrix block as an example, it can be expressed as:

$$\hat{D}_{i,j} = \begin{pmatrix} \tilde{D}_{i,j} & 0 \\ 0 & 0 \end{pmatrix} \quad (4.20)$$

If there are fill-in blocks added to the (i, j) matrix block, then the matrix can be expressed as:

$$\hat{U}_i \hat{D}_{i,j} \hat{V}_j + F_{i,j} = \hat{U}_i D'_{i,j} \hat{V}_j \quad (4.21)$$

Then the coupling matrix of the matrix block (i, j) is updated as:

$$D'_{i,j} = \hat{D}_{i,j} + (\hat{U}_i)^H F_{i,j} (\hat{V}_j)^H \quad (4.22)$$

The expression in Eq. (4.22) can also be illustrated by the schematic diagram in Fig. 4.10. Since the column and row spaces of $F_{i,j}$ are contained within $\hat{U}_i$ and $\hat{U}_j^T$, respectively, $F_{i,j}$ is orthogonal to the vectors in their complementary subspaces. Therefore, the corresponding matrix products are zero, as indicated by the blue regions in Fig. 4.10.

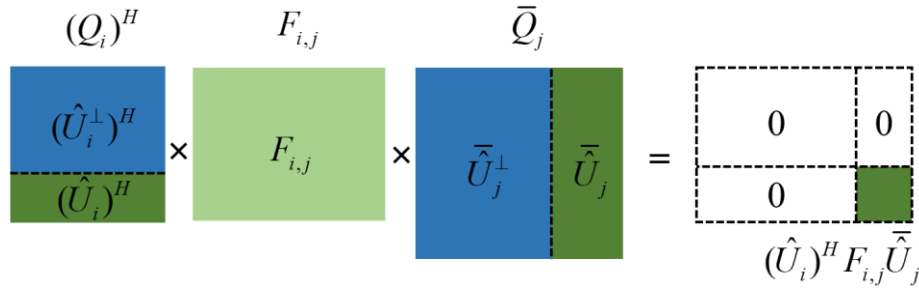


Fig. 4.10. Schematic diagram of the products between the fill-in matrix blocks and the elimination matrices. The products in the dark-green regions generate the updated coupling matrices, which correspond to the remaining admissible subblocks after the factorization and elimination at level (L) . The products in the blue regions vanish due to the orthogonality of the corresponding subspaces. (© 2026 IEEE [121]).

At this point, all the matrices required for the factorization at level $l-1$ have been obtained. The inadmissible blocks at level $l-1$ are formed by aggregating the remaining inadmissible subblocks from level l with the updated coupling matrices. The admissible blocks at level $l-1$ are represented in compressed form using the updated translation matrices at level $l-1$ and the corresponding coupling matrices. The factorization procedure at higher levels follows the same process as that at level $l-1$.

The above system-matrix factorization proceeds in a bottom-up traversal until the algorithm reaches the highest level, i.e., level 1. Finally, after LU factorization of the small matrix block remaining on the diagonal block at level 1, the entire system matrix Z_{sys} can be factorized hierarchically, group by group, as:

$$Z_{\text{sys}} = \left\{ \prod_{l=L}^1 \left[\left(\prod_{i=1}^{m^l} R_i^l \right) P_l^T \right] \right\} Z_{\text{diag}}^1 \left\{ \prod_{l=1}^L \left[P_l \left(\prod_{i=m^l}^1 S_i^l \right) \right] \right\} \quad (4.23)$$

where m^l denotes the number of nonempty groups at level l , $R_i^l = Q_{i,\text{sys}}^l L_i^l$ and $S_i^l = U_i^l (Q_{i,\text{sys}}^l)^T$ denote the products of the two types of elimination matrices for the

i -th group at level l . Here, L_i^l and U_i^l are the two elimination matrices generated by the LU factorization in Eq. (4.7) for the i -th group at level l .

P_l denotes the permutation matrix at the level l , as illustrated in Fig. 4.9, which aggregates and rearranges the remaining submatrix blocks at level l for processing at the next higher parent level. Z_{diag}^1 denotes the final diagonal block matrix remaining at level 1, e.g., the root level, which is directly factorized by LU decomposition as $Z_{diag}^1 = L^1 U^1$. According to Eq. (4.23), the inverse of the system matrix can then be derived as:

$$Z_{sys}^{-1} = \left\{ \prod_{l=L}^1 \left[\left(\prod_{i=1}^{m^l} (S_i^l)^{-1} \right) P_l^T \right] \right\} (Z_{diag}^1)^{-1} \left\{ \prod_{l=1}^L \left[P_l \left(\prod_{i=m^l}^1 (R_i^l)^{-1} \right) \right] \right\} \quad (4.24)$$

Where $(R_i^l)^{-1} = (L_i^l)^{-1} (Q_{i,sys}^l)^H$ and $(S_i^l)^{-1} = \bar{Q}_{i,sys}^l (U_i^l)^{-1}$, as $Q_{i,sys}^l$ is a unitary matrix. In summary, the proposed direct solver can be divided into three main steps.

The first step is to factorize and eliminate the admissible blocks. The basic idea is to determine the column space spanned by the admissible blocks and their corresponding fill-in matrix blocks, and then construct the corresponding complementary matrices as elimination matrices. Through matrix multiplication, the redundant components of the admissible blocks can be eliminated by exploiting the orthogonality and complementarity of the associated subspaces. This step involves updating the cluster basis matrices. An approximation error is introduced when the left singular vectors of the fill-in matrix blocks are extracted through SVD, and this approximation error can be controlled by the SVD truncation threshold $\mathcal{E}_{fill-in}$.

In the second step, the remaining inadmissible blocks after the first-step elimination are factorized and eliminated. Since this step involves LU factorization, the corresponding Schur complements are generated during the factorization of the system matrix. The resulting fill-in matrix blocks may be added to either the remaining admissible blocks or the inadmissible blocks. This step does not involve any approximation and therefore introduces no additional inversion error.

Finally, the system matrix is reduced to a simple matrix form containing only identity blocks and a small diagonal block matrix. After performing LU factorization and inversion of the remaining diagonal block, the inverse of the entire system matrix can be obtained through a sequence of matrix multiplications involving the elimination, permutation, and factorization matrices generated during the level-by-level and group-by-group factorization process.

It should be noted that, unlike the pure skeleton-based factorization algorithm, the mixed-form NESAs may encounter a matrix-dimension mismatch when transitioning from the finest skeleton level to the higher equivalent-source levels. Specifically, if the number of equivalent sources in a higher-level group exceeds the total number of skeleton basis functions of its child groups, the corresponding translation matrix has fewer rows than columns, and the matrix factorization

procedure cannot proceed. For this type of “overflow” in the aggregation of child-level skeleton bases, the proposed direct solver retains all matrix blocks associated with the corresponding group without further factorization and skips this group to proceed with the factorization of the next group. This treatment is reasonable because such a situation occurs only for groups containing a particularly small number of unknowns, for which the associated matrix blocks are of very small dimensions. In such cases, directly handling these small matrix blocks can be more computationally efficient than forcing further factorization.

This also constitutes an important distinction between the mixed-form NESA direct solver and the pure skeleton-based direct solver. In the pure skeleton-based approach, the parent-level source basis is constructed from the collection of skeleton bases selected from all child groups, and the parent-level skeleton basis is subsequently selected from this collection. Therefore, the number of parent-level skeleton basis functions is always less than or equal to the total number of skeleton basis functions of its child groups. A similar situation occurs when an augmented cluster basis of a group becomes a square matrix. In this case, there is no need to construct an additional elimination matrix for the factorization of the admissible blocks. This indicates that the column space associated with the admissible blocks, together with the column space occupied by the corresponding fill-in matrix blocks, already forms a complete space. Consequently, no complementary subspace can be constructed to eliminate the admissible blocks and fill-in blocks.

4.3.3 Complexity Analysis

This subsection provides a detailed analysis of the computational time and memory complexity of the proposed NESA-based direct solver. Assume that the target is discretized into N unknowns and hierarchically partitioned into L levels using an octree structure. Let m_l denote the number of nonempty groups at level l . For the i -th group, let n_i and f_i denote the numbers of near-field and far-field interaction groups, respectively. Let p_i denote the number of original basis functions associated with the i -th group, while k_l denotes the average number of selected skeleton basis functions at the finest leaf level or the number of equivalent sources at level l ($l \leq L-1$).

For the equivalent-source representation, k_l is a constant and is set to 50 by default, which is independent of the total number of unknowns N . This value has been discussed in [3]. Therefore, the computational time complexity at each level can be expressed as:

$$C_l = \sum_{i=1}^{m_l} n_i p_i^3 + (p_i - k_l)^3 + n_i^2 p_i^2 (p_i - k_l) + f_i k_l^3 \quad (4.25)$$

The first term in Eq. (4.25) represents the first step of the factorization process, namely, the elimination of the admissible blocks. In practice, this operation corresponds to updating the remaining inadmissible blocks and therefore involves only the near-field interaction groups. The second term represents the LU

factorization of the diagonal matrix blocks, which subsequently generates the corresponding Schur complements. The third term accounts for the computation of the dense fill-in matrix blocks, which are essentially the matrix blocks at the corresponding positions in the system matrix generated by the Schur-complement operation in the second step. Finally, the last term represents the update of the coupling matrices after the factorization at the current level is completed. Furthermore, Eq. (4.25) can be rewritten as:

$$\begin{aligned} C_l &= O(m_l n_l p_l^3 + m_l (p_l - k_l)^3 + m_l n_l^2 p_l^2 (p_l - k_l) + m_l f_l k_l^3) \\ &= O(m_l n_l p_l^3) \end{aligned} \quad (4.26)$$

Since p_l is always greater than or equal to k_l at each level, it can be inferred that the first term $m_l n_l p_l^3$ in Eq. (4.26) is the dominant term. Therefore, only the first term is retained for the subsequent time-complexity analysis. The complexity expressions at the finest leaf level and at the higher levels are analyzed separately. At the finest leaf level, the proposed algorithm imposes a prescribed constant to control the upper limit of the average number of basis functions in each group. Therefore, p_l is bounded by a constant. In addition, within the octree structure, n_l is also bounded by a fixed maximum value. Specifically, under the admissibility criterion adopted by NESAs, the maximum value of n_l is 26 for three-dimensional objects. Therefore, the computational complexity at the finest leaf level can be written as:

$$C_L = O(m_L n_L p_L^3) = O(m_L) O(n_L p_L^3) = C \cdot O(m_L) = O(N) \quad (4.27)$$

Since both p_l and n_l are bounded by fixed constants, the term $n_l p_l^3$ is also bounded by a constant. Therefore, the computational complexity at this level depends only on the number of groups m_L and scales linearly with the total number of unknowns N .

For the higher levels, excluding the highest level, the skeleton bases from the child groups are aggregated into their corresponding parent groups. The corresponding computational complexity can be expressed as:

$$C_{l=L-1-2} = \sum_{l=L-1}^2 [m_l n_l (8k_{l+1})^3] = O(N) \quad (4.28)$$

where $p_l = 8k_{l+1}$ denotes the number of skeleton basis functions aggregated from the child groups, and k_{l+1} is less than or equal to the number of equivalent sources at the corresponding parent level.

It should be noted that the number of levels L in the octree increases as $O(\log N)$ with N , while the number of non-empty groups m_l increases as $O(8^l)$ with l . Therefore, the complexity of the above term $C_{l=L-1-2}$ can be derived as $O(N)$. Finally, at the highest level, only one diagonal matrix block needs to be factorized using LU decomposition. Its dimension is $8k_2$, and the corresponding computational complexity is:

$$C_1 = O(p_1^3) = O(k_2^3) = O(C) \quad (4.29)$$

where C denotes a constant. By combining the computational complexities derived for all hierarchical levels above, the overall computational time complexity of the proposed direct solver can be obtained as:

$$C = O(N) + O(N) + O(C) = O(N) \quad (4.30)$$

Next, the memory complexity of the proposed algorithm is analyzed. It is first noted that the number of near-field elements at the finest leaf level scales linearly with the total number of unknowns. In addition, the cluster bases (receiving matrices) of all groups need to be stored at the finest leaf level, whereas at higher levels, only the translation matrices corresponding to the different relative positional configurations between parent and child groups need to be stored. Therefore, the storage complexity associated with the finest-level cluster bases and higher-level translation matrices $M_{cluster}$ can be expressed as:

$$\begin{aligned} M_{cluster} &= \sum_{i=1}^{m_L} p_L k_L + \sum_{l=L-1}^1 \sum_{i=1}^8 p_l k_l = O(m_L p_L k_L) + \sum_{l=L-1}^1 O(8k_l^2) \\ &= O(N) \end{aligned} \quad (4.31)$$

In addition, the coupling matrices need to be stored:

$$M_{coupling} = \sum_{l=L}^1 \sum_{i=1}^{m_l} k_l^2 = \sum_{l=L}^1 O(m_l k_l^2) = O(N) \quad (4.32)$$

Therefore, the overall memory complexity of the proposed algorithm can be expressed as $O(N)$. In summary, for problems with constant-rank far-field interactions, both the computational time and memory complexities of the proposed direct solver are $O(N)$.

Problems with approximately constant-rank interactions generally correspond to electrically small-scale electromagnetic problems at relatively low operating frequencies. As the frequency increases, the kernel function of the integral equation exhibits increasingly rapid spatial oscillations. If the testing points remain uniformly distributed over a spherical surface, the equivalent sources may no longer accurately reconstruct the electromagnetic field on the testing surface, resulting in a degradation in the accuracy of the NESA approximation.

To achieve accurate field reconstruction at higher frequencies, a directional testing-point distribution strategy will be introduced in the following subsection. Accordingly, the problems considered by the direct solver in this chapter are restricted to electrically small or moderate-sized targets with dimensions below 6λ . The time and memory complexity derived above will be further validated through numerical experiments in the subsequent sections.

4.4 Numerical Results

In this subsection, a series of numerical simulations are conducted to comprehensively evaluate the computational accuracy and efficiency of the proposed NESA-based direct solver. First, a metallic strip is considered, and the distribution of the system-matrix elements at different stages of the factorization process is explicitly presented, providing an intuitive illustration of the

progressive elimination procedure. The controllable accuracy of the direct solver is then investigated, demonstrating that more accurate current solutions can be obtained by adopting a smaller solver tolerance. Subsequently, the operating frequency is increased to increase the number of unknowns, thereby verifying the computational time and memory complexities of the proposed algorithm. Finally, the bistatic RCS results of several complex models are presented to further demonstrate the computational capability of the proposed method in practical electromagnetic scattering scenarios.

All numerical simulations are performed on either a server equipped with an Intel(R) Xeon(R) Silver 4114 CPU, 256 GB of memory, and a clock frequency of 2.20 GHz, or a personal computer equipped with an 11th-generation Intel(R) Core(TM) i7-11700F CPU operating at 2.50 GHz.

4.4.1 Demonstration of the System Matrix Factorization

First, a metallic strip is selected as the object of analysis and grouped according to the numbering scheme shown in Fig. 4.4(a). Only a single-level hierarchical structure is employed. The resulting system-matrix structure is similar to the diagonal-block structure in the lower-right corner of Fig. 4.9(b). Before the factorization, the distribution of the absolute values of the system-matrix elements is shown in Fig. 4.11.

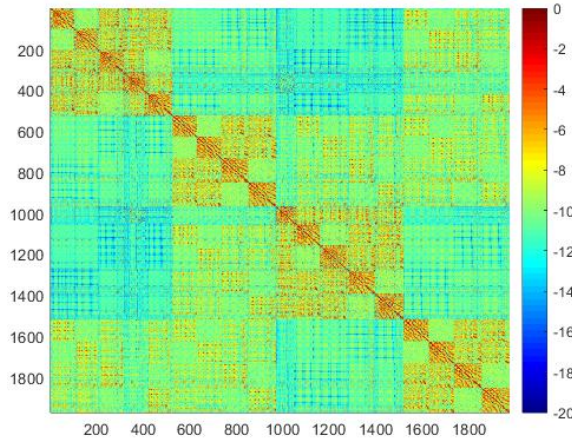


Fig. 4.11. Distribution of the system-matrix elements of the metallic strip before factorization. The absolute values of all matrix elements are shown in dB. (© 2026 IEEE [121]).

Subsequently, the system matrix is factorized and eliminated sequentially according to the proposed direct-solver procedure. For the four groups, the admissible blocks are first eliminated, followed by the elimination of the inadmissible blocks for each group. The corresponding system matrices after each elimination step are shown in Figs. 4.12(a)–4.12(h).

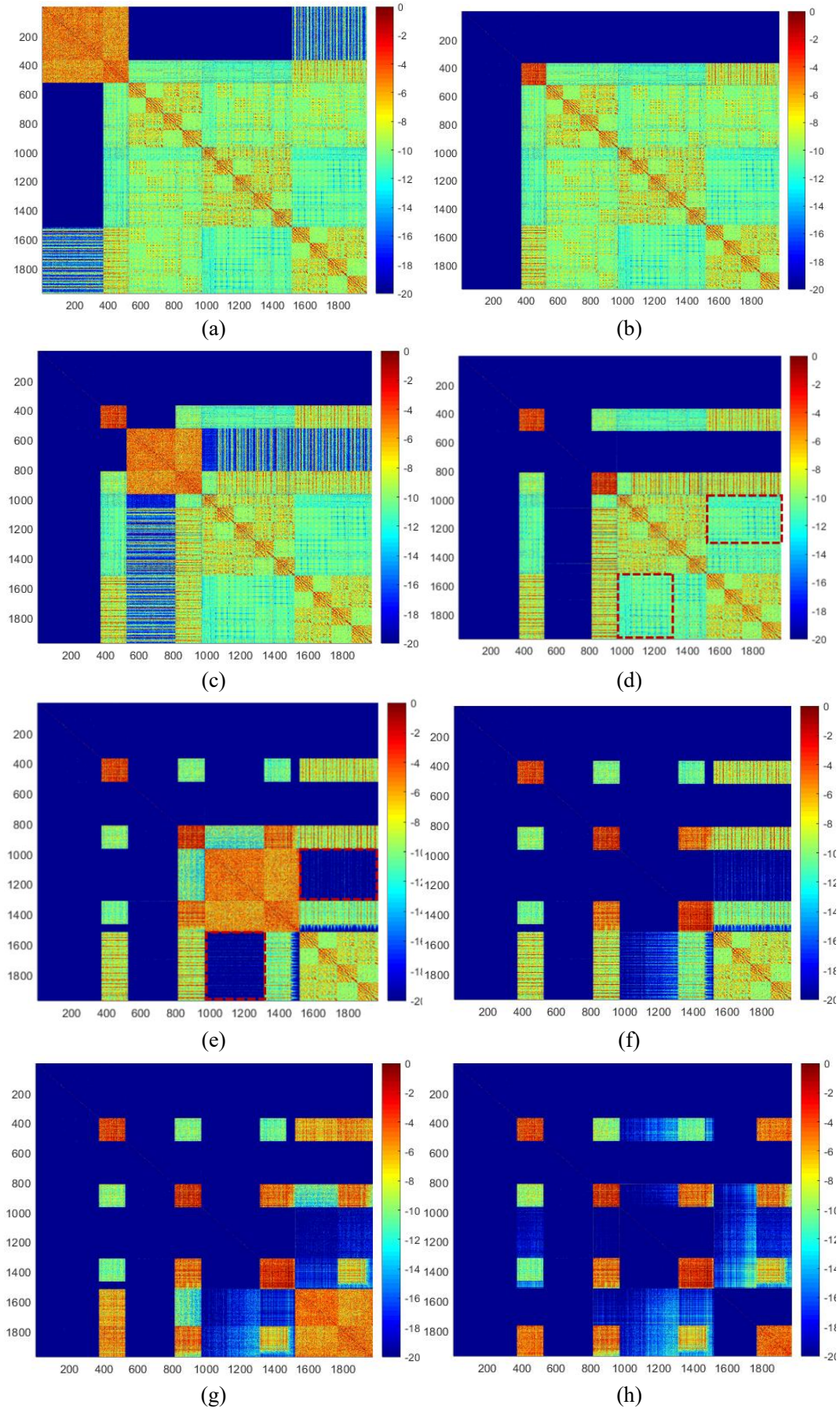


Fig. 4.12. Distribution of the system-matrix elements of the metallic strip after the factorization and elimination of each group. The elimination process proceeds sequentially from (a) to (h), where (a), (c), (e), and (g) correspond to the elimination of the admissible

blocks, while (b), (d), (f), and (h) correspond to the elimination of the inadmissible blocks. The elimination of the second group in (d) generates a fill-in matrix block that is added to the admissible block associated with the third group. Its location is marked by the red dashed box. As shown in (e), this fill-in matrix block is effectively eliminated. The absolute values of all matrix elements are shown in dB. (© 2026 IEEE [121]).

After the factorization is completed, the remaining small matrix blocks are aggregated and permuted so that they are collected into the diagonal block. The final form of the system matrix is shown in Fig. 4.13.

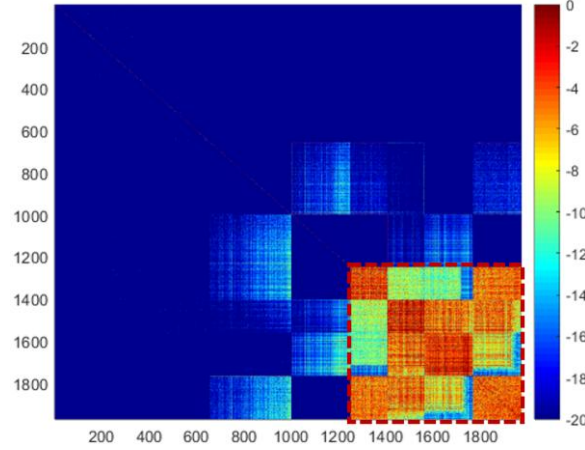


Fig. 4.13. Distribution of the system-matrix elements after the remaining small matrix blocks are aggregated and permuted following the completion of the factorization of all groups. The absolute values of all matrix elements are shown in dB. (© 2026 IEEE [121]).

Subsequently, the diagonal block matrix is factorized using LU decomposition to obtain the factorized form of the entire system matrix.

4.4.2 Accuracy Validation

First, a PEC sphere is considered to verify the correctness of the proposed direct solution. A metallic sphere with a radius of 1.8 m is illuminated by a plane wave at a frequency of 300 MHz. The bistatic RCS is calculated using the proposed direct solver and compared with the corresponding Mie series solution, as shown in Fig. 14.

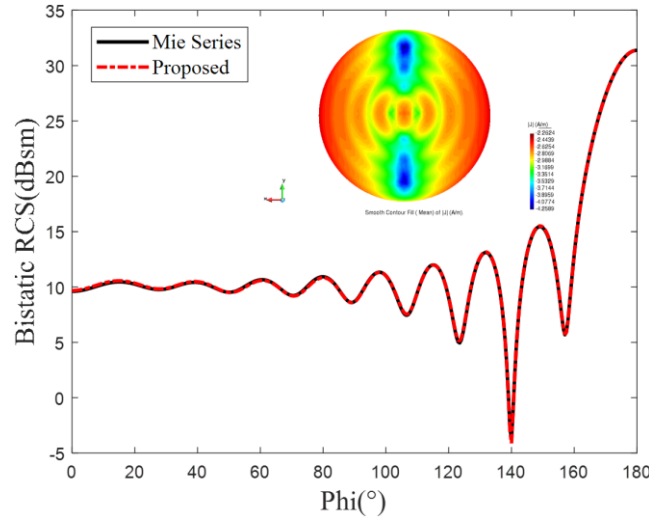


Fig. 4.14. Bistatic RCS of a metallic sphere with a radius of 1.8 m illuminated by a 300 MHz plane wave. The inset presents the induced surface-current distribution on the metallic sphere. (© 2026 IEEE [121]).

It can be observed that the bistatic RCS results obtained using the proposed NESAs-based direct solver agree very well with the analytical solution, thereby verifying the correctness of the proposed direct-solver algorithm.

The controllable accuracy of the direct solver is further investigated by evaluating the accuracy of the computed current distributions. Since the approximation accuracy of NESAs for low-rank matrix blocks has already been comprehensively investigated in [3], this work focuses primarily on the numerical accuracy introduced during the direct-solution process. As discussed in Section 4.3.1, the approximation error of the direct solver mainly originates from the compression of the fill-in matrix blocks in Eq. (4.16). By adopting a smaller truncation threshold for the SVD, the column space occupied by the fill-in matrix blocks can be captured more accurately. Consequently, the products between the corresponding orthogonal vectors in the elimination matrices and the fill-in matrix blocks become closer to zero, thereby reducing the approximation error introduced during the elimination process.

In addition, the skeleton bases are selected using the adaptive cross approximation (ACA) algorithm, whose truncation threshold also introduces an approximation error into the matrix-block representation. Therefore, the influence of both the skeleton-selection threshold and the SVD truncation threshold on the accuracy of the computed current is investigated.

First, the metallic sphere considered in Fig. 4.14 is used as the test case. For several different skeleton-selection thresholds, the variation of the current error with respect to the SVD truncation threshold $\epsilon_{fill-in}$ used for the compression of the fill-in matrix blocks in Eq. (4.16) is evaluated. The reference current solution is obtained by explicitly constructing the complete MoM system matrix and directly performing LU decomposition to solve the corresponding linear system. The current error is evaluated using the relative l_2 -norm, defined as:

$$error = \frac{\|x_{proposed} - x_{LU}\|_2}{\|x_{LU}\|_2} \quad (4.33)$$

The current computation error curve is given in Fig. 4.15.

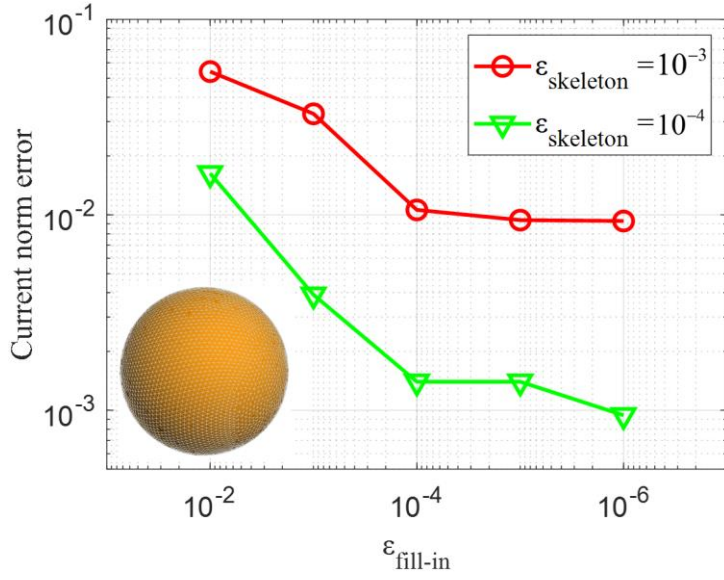


Fig. 4.15. Current errors of the metallic sphere under different skeleton-selection thresholds and direct-solver truncation thresholds. (© 2026 IEEE [121]).

As shown in Fig. 4.15, when the skeleton-selection thresholds are set to 10^{-3} and 10^{-4} , respectively, the current error gradually decreases as the SVD truncation threshold $\varepsilon_{\text{fill-in}}$ in Eq. (4.16) is reduced from 10^{-2} to 10^{-6} , and can ultimately be controlled below 10^{-3} . Moreover, a more accurate skeleton-selection procedure leads to a more accurate current solution. In other words, as the accuracy-control threshold of the proposed direct solver is progressively reduced, the relative error in the computed current decreases accordingly, demonstrating that the accuracy of the proposed direct solver can be effectively controlled.

The condition number of the system matrix in the above example is 135. To further investigate the controllable accuracy of the proposed algorithm for more ill-conditioned problems, a multiscale composite model consisting of a large frustum and a small cone is considered. The corresponding current-error curves are presented in Fig. 4.16.

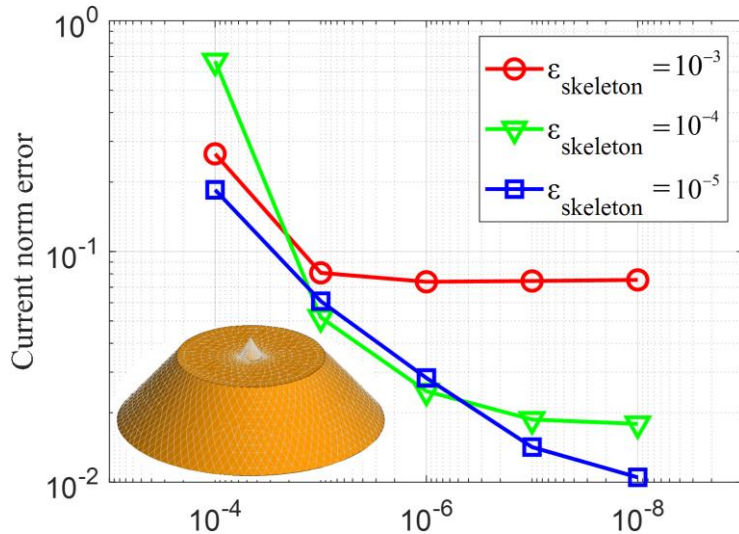


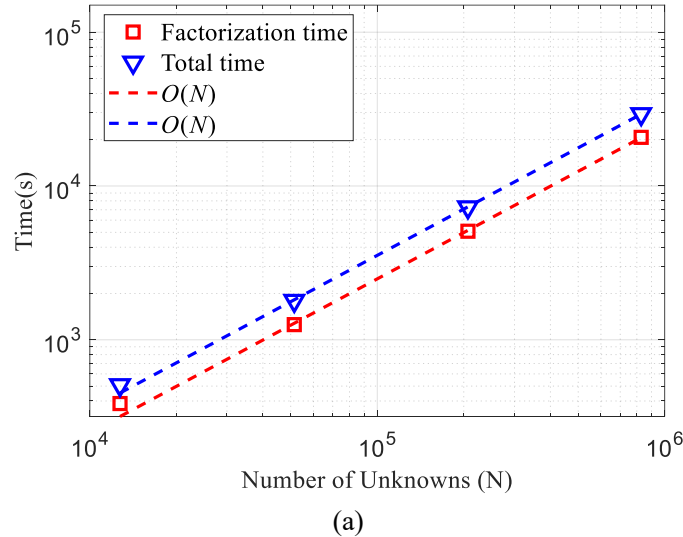
Fig. 4.16. Current errors of the multiscale composite cone structure under different skeleton-selection thresholds and direct-solver truncation thresholds. (© 2026 IEEE [121]).

As shown in Fig. 4.16, the mesh of the composite model is non-uniform, resulting in discretization elements with significantly different characteristic sizes. The corresponding system matrix has a condition number of 44,497, indicating that the problem is considerably more ill-conditioned than the metallic-sphere case. Similar to the metallic sphere, the current-solution accuracy gradually improves as both the skeleton-selection threshold and the accuracy-control threshold of the proposed direct solver are reduced.

However, unlike the metallic-sphere case, the overall accuracy obtained for the composite model is relatively lower. To achieve a comparable level of accuracy in the computed current, both the skeleton-selection threshold and the accuracy-control threshold need to be set to smaller values. This indicates that, although the proposed direct solver is relatively insensitive to the conditioning of the system matrix, more stringent accuracy-control thresholds are still required for highly ill-conditioned problems to obtain sufficiently accurate solutions.

4.4.3 Complexity Validation

To validate the linear computational time and memory complexities analyzed in Section 4.3.3, a series of numerical experiments are conducted in this subsection. A metallic sphere with a radius of 1.8 m is considered, and the mesh size is fixed at 0.1λ . The operating frequencies are set to 300 MHz, 600 MHz, 1.2 GHz, and 2.4 GHz, corresponding to 12,735, 51,432, 206,742, and 828,153 unknowns, respectively. The computational time and memory consumption of the proposed direct solver are recorded, as shown in Figs. 4.17(a) and 4.17(b), respectively. It can be seen that both the computational time and memory complexity of the proposed algorithm exhibit an $O(N)$ scaling behaviour, which is consistent with the theoretical complexity analysis presented above.



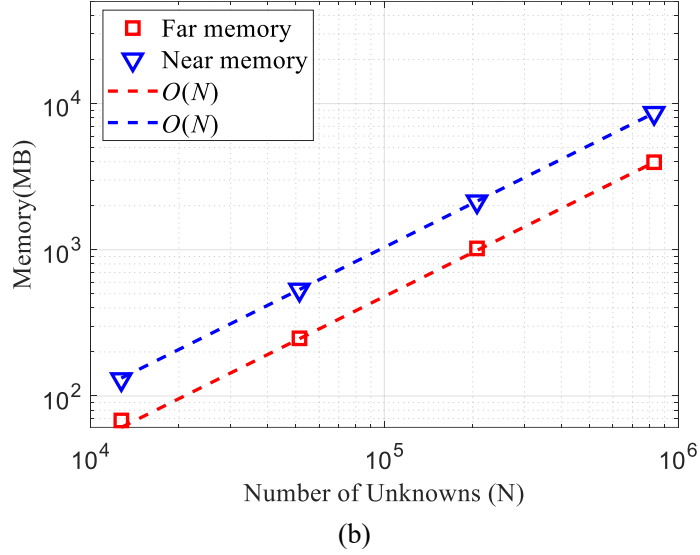


Fig. 4.17. Computational time and memory complexity of the NESAs-based direct solution. (a) System-matrix factorization time and total computational time; (b) memory consumption for near-field matrix elements and far-field matrix compression. (© 2026 IEEE [121]).

4.4.4 Analysis of Electromagnetic Scattering Characteristics of Complex Targets

Next, several complex target models are considered to evaluate the computational capability of the proposed algorithm in practical application scenarios. First, a metallic tank model is selected to calculate its surface-current distribution and bistatic RCS. The length, width, and height of the target are 4.00m × 1.94m × 1.12m, respectively. The operating frequency is set to 300 MHz, with the incident direction of the plane wave given by $\theta_i = 0^\circ, \varphi_i = 0^\circ$ and the observation direction given by $\theta = 0^\circ \sim 180^\circ, \varphi = 0^\circ$. The target is discretized with a mesh size of 0.1λ , resulting in a total of 14,604 unknowns. A three-level octree structure is employed to partition the target, with the group size at the finest leaf level set to 0.25λ . The results obtained using the proposed direct solver are compared with those obtained by directly factorizing the full MoM system matrix solved by LU decomposition, as shown in Fig. 4.18. The accuracy-control threshold of the proposed algorithm $\varepsilon_{fill-in}$ is set to 10^{-3} .

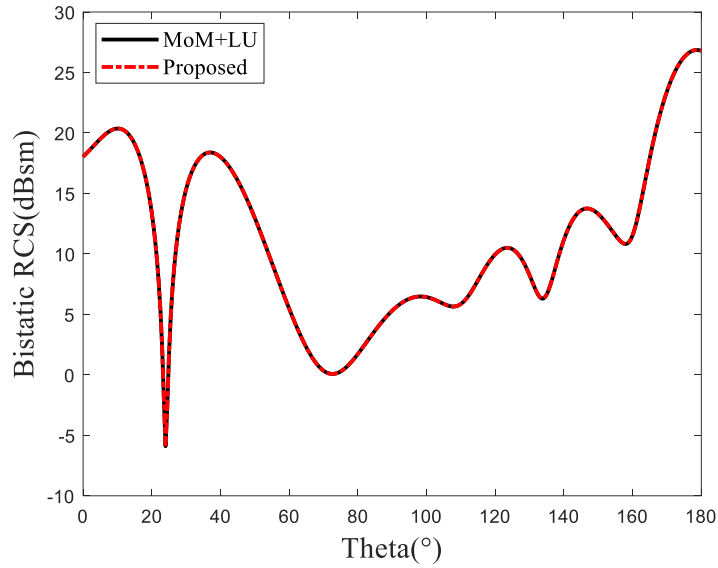


Fig. 4.18. Bistatic RCS of the tank model. (© 2026 IEEE [121]).

It can be observed that the results obtained using the proposed direct solver agree very well with the reference results. The surface-current distributions calculated by the two methods are compared in Fig. 4.19.

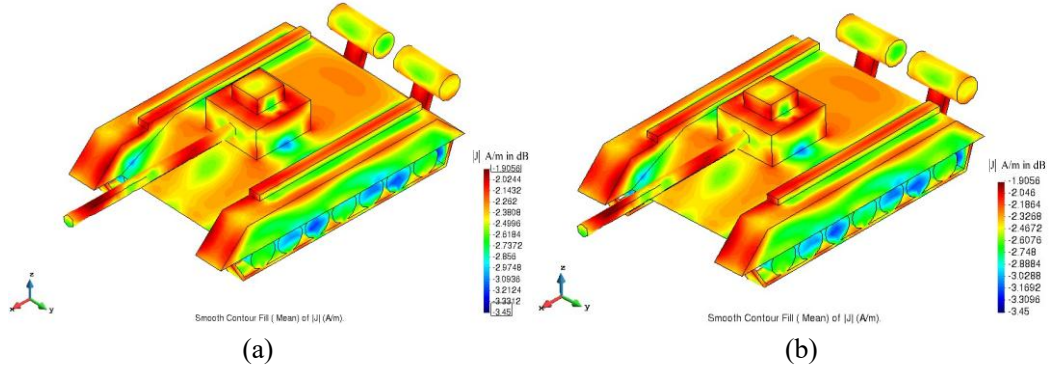


Fig. 4.19. Surface-current distributions of the tank model: (a) LU + MoM; (b) the proposed direct solver. The current magnitude is plotted in dB, and the same color scale is used for both subfigures. The norm error between (a) and (b) is 0.015. (© 2026 IEEE [121]).

Next, a fighter aircraft model is considered to further evaluate the proposed direct solver by calculating its surface-current distribution and bistatic RCS. The operating frequency is set to 800 MHz. After normalization by the wavelength, the length, width, and height of the fighter model are $5.6\lambda \times 1.5\lambda \times 4.0\lambda$, respectively. The incident direction of the electromagnetic wave is $\theta_i = 90^\circ, \varphi_i = 180^\circ$, and the observation direction is $\theta = 0^\circ \sim 180^\circ, \varphi = 180^\circ$. Several accuracy-control thresholds are considered, and the corresponding results are compared with those obtained using the conventional Method of Moments (MoM), as shown in Fig. 4.20.

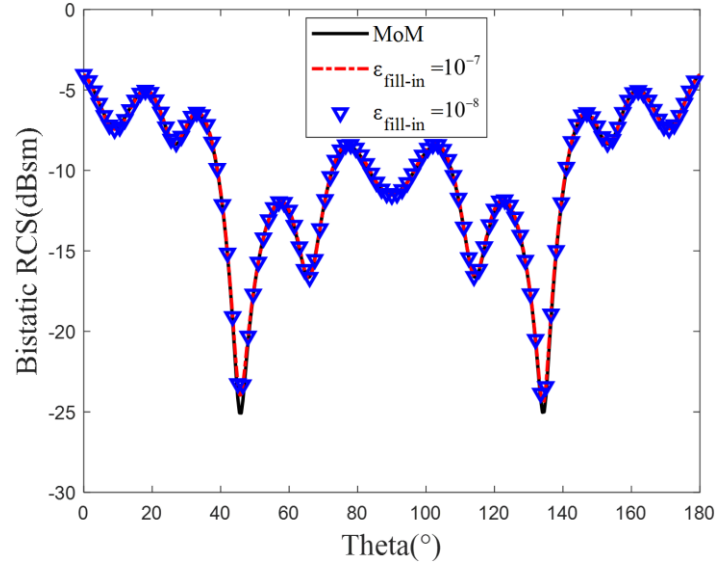


Fig. 4.20. Bistatic RCS of the fighter aircraft model. (© 2026 IEEE [121]).

The surface-current distributions obtained with different accuracy-control thresholds are presented in Fig. 4.21, demonstrating that the current-solution error of the proposed algorithm converges as the accuracy-control threshold is reduced. .

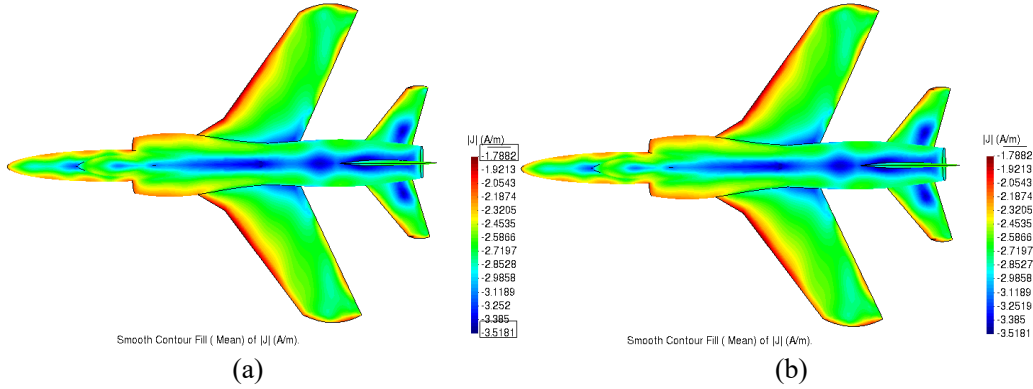


Fig. 4.21. Surface-current distributions of the fighter aircraft model under different accuracy-control thresholds: (a) 10^{-7} ; (b) 10^{-8} . The current magnitude is plotted in dB, and the same color scale is used for both subfigures. The norm error between (a) and (b) is 0.038. (© 2026 IEEE [121]).

For the same fighter aircraft model, the monostatic RCS is further calculated and compared with the results obtained using the conventional Method of Moments (MoM) and those reported in [70], as shown in Fig. 4.22. The incident and observation directions are varied over the range of $\theta_i = 90^\circ, \varphi_i = 0^\circ \sim 360^\circ$.

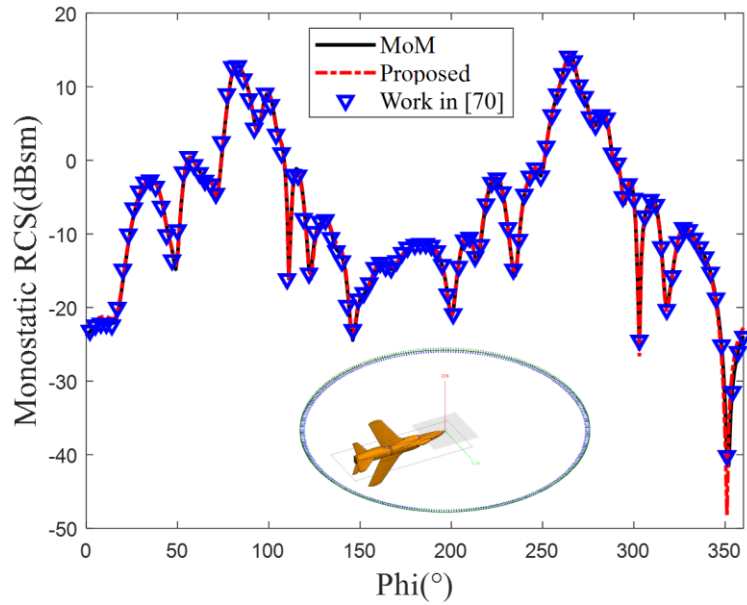


Fig. 4.22. Monostatic RCS of the fighter aircraft model. (© 2026 IEEE [121]).

The computational time and memory consumption of the proposed direct solver are further compared with those of the method presented in [70] for the fighter aircraft model, as summarized in Table 4.3.

TABLE 4.3. COMPUTATIONAL TIME AND MEMORY CONSUMPTION FOR MONOSTATIC RCS CALCULATION OF THE FIGHTER AIRCRAFT MODEL USING DIFFERENT METHODS.

Method	Factorization time (h)	Inversion time (h)	Total time (h)	Memory consumption (MB)
Method in [70]	0	0	96.9	1536.9
Proposed method	0.5	0.8	1.3	2046.6

The computational time and memory consumption listed in Table 4.3 show that, compared with the iterative solver presented in [70], the proposed direct solver requires additional time for system-matrix factorization and inversion. However, once the inverse of the system matrix is obtained, the computational cost associated with additional right-hand-side vectors becomes negligible, and therefore the proposed direct solver achieves higher efficiency for problems involving multiple excitations.

In terms of memory consumption, the proposed direct solver requires additional storage space during the matrix factorization process, mainly due to the temporary storage of fill-in matrix blocks generated during the elimination of compressed admissible blocks. Nevertheless, these intermediate storage requirements can be released after the completion of the factorization at each hierarchical level. Therefore, the overall memory consumption does not increase significantly with the number of unknowns.

Subsequently, a more complex EV55 aircraft model is considered to further evaluate the capability of the proposed algorithm for highly ill-conditioned problems. The length and wingspan of the aircraft are 14.2 m and 16.1 m, respectively. At an operating frequency of 100 MHz, these dimensions correspond to 4.73λ and 5.37λ , respectively. A plane wave with an incident direction of $\theta_i = 90^\circ, \phi_i = 180^\circ$ propagates along the positive x -axis and illuminates the nose of the aircraft. The model contains 171,763 unknowns and is hierarchically partitioned using a five-level octree structure. The bistatic RCS and surface-current distributions are calculated using both the method in [70] and the proposed direct solver. The obtained RCS results are shown in Fig. 4.23, and the corresponding surface-current distributions are presented in Fig. 4.24(a)-(c).

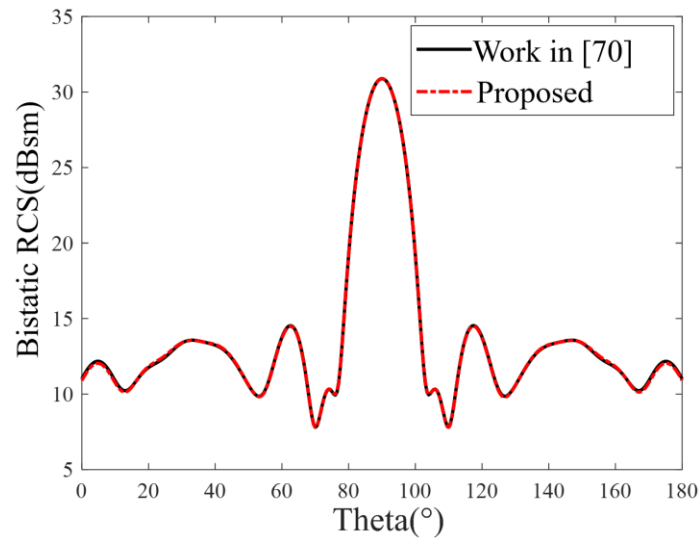
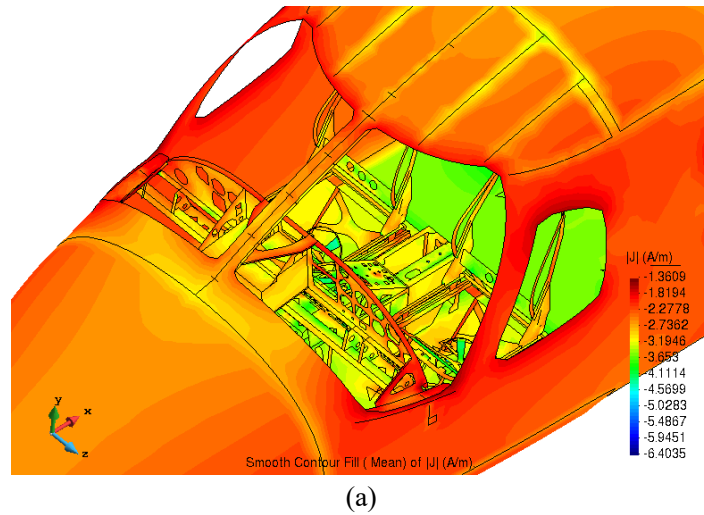


Fig. 4.23. Bistatic RCS of the EV55 aircraft model. (© 2026 IEEE [121]).



(a)

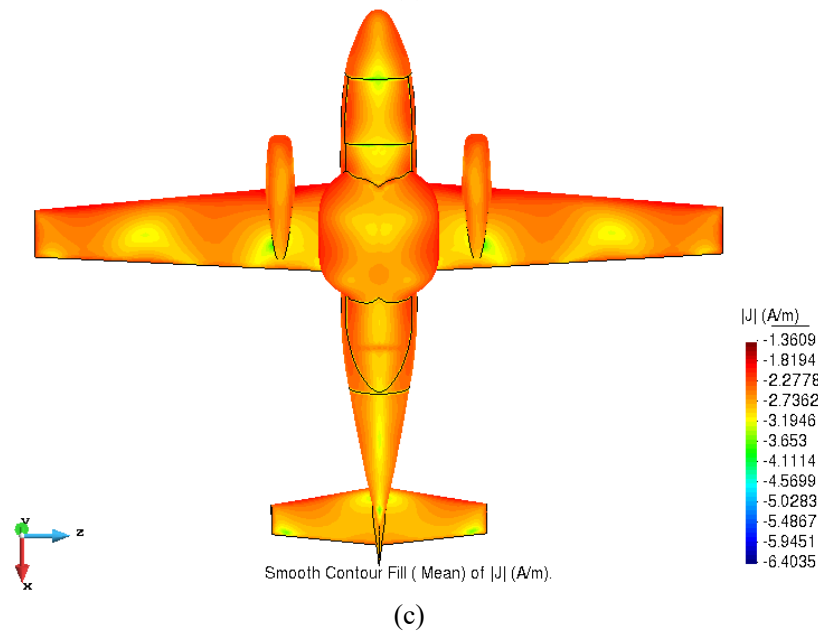
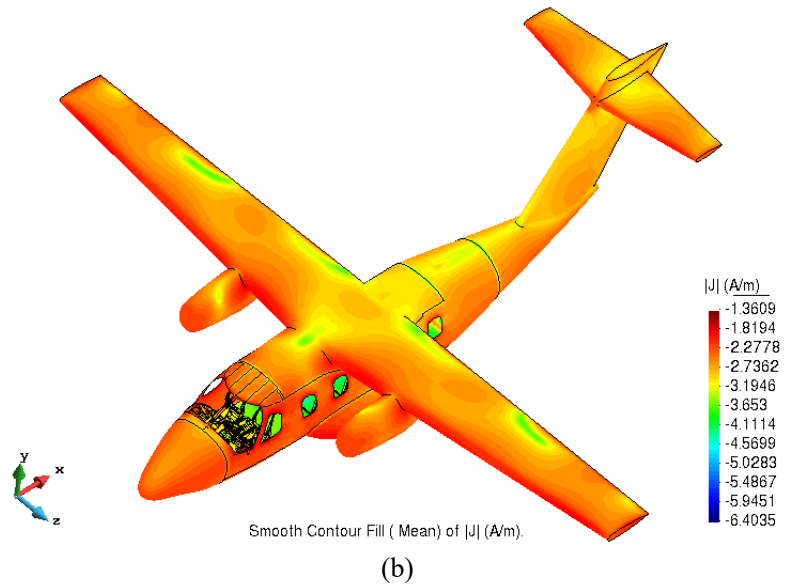


Fig. 4.24. Surface-current distributions of the EV55 aircraft model. (a) Detailed view of the cockpit area; (b) top view; (c) bottom view. (© 2026 IEEE [121]).

As shown in Fig. 4.24, the EV55 aircraft model has a considerably more complex geometry than the fighter aircraft model presented in Fig. 4.21. This increased structural complexity can be further illustrated by examining the iterative residual history of the corresponding iterative solution, as shown in Fig. 4.25.

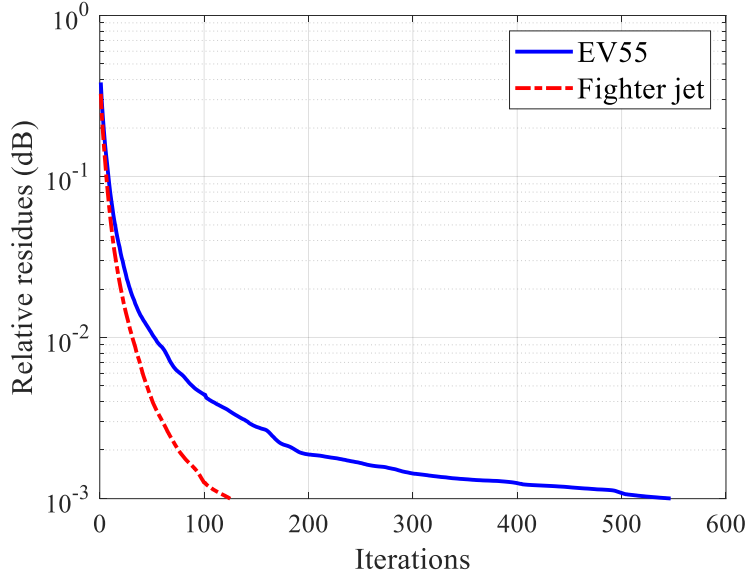


Fig. 4.25. Iterative residual curves for the fighter aircraft and EV55 aircraft models. (© 2026 IEEE [121]).

As shown in Fig. 4.25, when the iterative stopping residuals are set to 3×10^{-3} and 1×10^{-3} , the corresponding numbers of iterations for the EV55 aircraft model are 142 and 547, respectively, whereas those for the fighter aircraft model are 61 and 125, respectively. These results demonstrate that the proposed direct solver remains applicable to more ill-conditioned problems. For the EV55 aircraft model, the memory consumption of the method in [70] for storing the near-field matrix elements and compressed far-field matrices is 3604.6 MB and 85.3 MB, respectively. In addition, the proposed direct solver requires 117.6 MB of extra memory to temporarily store the fill-in matrix blocks generated at different hierarchical levels. In terms of computational time, for a single incident direction, the iterative solver in [70] requires 15.0 hours, whereas the proposed direct solver requires 20.4 hours for system-matrix factorization and only 0.1 hour for matrix inversion. It is worth noting that as the number of incident directions increases, the factorization time of the direct solver remains unchanged, while only the matrix-inversion time increases approximately linearly. Since the matrix-inversion time is substantially lower than the iterative solution time, the EV55 example further demonstrates that the proposed direct solver is more computationally efficient than the iterative solver in [70] for problems involving multiple right-hand sides. Moreover, the proposed direct solver can accurately solve even more ill-conditioned problems.

Finally, to further evaluate the capability of the proposed algorithm for large-scale problems, a $4 \times 4 \times 4$ spheres array, as shown in Fig. 4.26(a), is considered. The operating frequency is set to 300 MHz, and the electrical size of the spherical array is 4.3λ , resulting in 2,271,276 unknowns. The incident and observation directions are set to $\theta_i = 0^\circ, \varphi_i = 0^\circ$ and $\theta = 0^\circ \sim 360^\circ, \varphi = 0^\circ$, respectively. The bistatic RCS calculated using the proposed algorithm is compared with the result obtained using the MLFMA, as shown in Fig. 4.26(b).

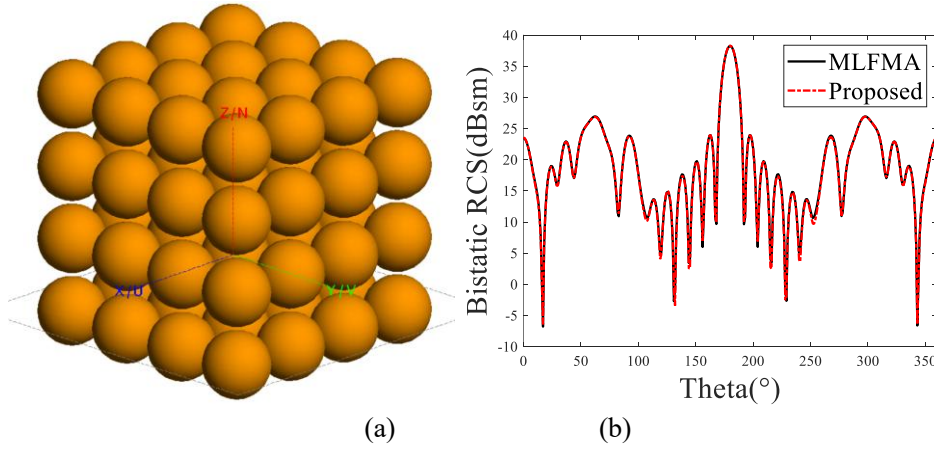


Fig. 4.26. (a) Schematic of the spherical array model; (b) bistatic RCS results. (© 2026 IEEE [121]).

It can be observed that the bistatic RCS results obtained using the proposed algorithm agree very well with those calculated by MLFMA, demonstrating that the proposed algorithm can still accurately solve problems with millions of unknowns and confirming its capability for large-scale electromagnetic problems. It should be noted that the algorithm presented in this chapter is primarily designed for mid- and low-frequency problems, for which the number of unknowns is generally below the million-unknown level. The numerical results presented above demonstrate that the proposed algorithm provides controllable computational accuracy, maintains high accuracy in practical electromagnetic scattering scenarios, and offers high computational efficiency for problems involving multiple right-hand-side vectors.

4.5 Summary

This chapter presents a direct solver based on the Nested Equivalent Sources Approximation (NESA) for solving low to moderate frequency electromagnetic scattering problems with constant rank interactions. The factorization and inversion procedures of the proposed direct solver are derived in detail. By exploiting the hierarchical nested representation of far-field matrix blocks in NESA, a system-matrix structure analogous to the H^2 -matrix is explicitly constructed and factorized into a sequence of matrix products, from which the inverse of the system matrix can be obtained. To efficiently handle the fill-in matrix blocks generated during the factorization process, the proposed algorithm compresses them using the complementary spaces of the cluster bases, thereby preserving the compressed representation of the far-field matrix blocks. The proposed algorithm achieves $O(N)$ for both computational time and memory complexities. A series of numerical examples validates its computational accuracy and efficiency.

Chapter 5

WNESA-Based Fast Direct Solution for EM Scattering at High Frequencies

5.1 Introduction

In recent years, the analysis of broadband electromagnetic problems has become one of the major research topics in computational electromagnetics. For electrically small and moderate-sized problems, the maximum rank of the inter-group coupling matrices in the system matrix remains approximately constant and does not increase with the number of unknowns. Such problems constitute the primary class investigated in Chapter 4. As the operating frequency increases, the number of unknowns grows with the electrical size of the target. Meanwhile, the increasingly oscillatory behavior of the kernel function in the integral equation causes the rank of the coupling matrices to increase accordingly. Consequently, the computational efficiency of low-rank compression methods gradually deteriorates when applied to electrically large problems [142].

To address the increasingly oscillatory behavior of the kernel function at higher frequencies, the WNESA method restricts the interaction between far-field groups to a prescribed angular range [3] and distributes the testing points over the surfaces of truncated pyramids with square bases oriented along different directions. The far-field coupling matrices are then compressed independently within each directional sector. This strategy is motivated by the fact that, when the interaction is restricted to a sufficiently narrow angular range, the kernel function becomes sufficiently smooth and amenable to low-rank compression [143].

Due to the highly oscillatory nature of the kernel function in high-frequency scenarios, the direct solver proposed in Chapter 4 has certain limitations. For conventional NESA, testing points distributed over a spherical surface cannot accurately sample the directional characteristics of the far-field electromagnetic

field. As a result, the equivalent-source coefficients cannot be accurately determined, leading to a significant degradation in the accuracy of the direct solver for electrically large targets. Furthermore, the direct solver constructs elimination matrices from the radiation and receiving matrices to perform system-matrix factorization. In WNESA, however, the radiation and receiving matrices associated with the same group differ across different directional sectors. Therefore, the construction of the elimination matrices must simultaneously account for the admissible blocks associated with all directional sectors of the group.

To address the above issues, this chapter proposes a direct solver based on the WNESA matrix-compression scheme. WNESA can accurately compress far-field group coupling matrices corresponding to different directions at higher levels of the octree structure. Building upon the direct-solver framework developed in Chapter 4, WNESA is introduced to construct cluster bases at higher levels. During the construction of the elimination matrices, the cluster bases associated with all directional sectors of the group and its parent groups are jointly considered, thereby constructing a generalized cluster basis capable of representing the cluster bases in all directions. In addition, WNESA employs a more stringent admissibility condition than NESA. Consequently, even spatially nonadjacent groups may form inadmissible coupling blocks. Therefore, the inadmissible blocks cannot be identified solely according to the spatial adjacency relationships of groups in the octree structure. To determine the indices of nonadjacent inadmissible blocks, this chapter adopts a traversal-based strategy that examines the interactions among groups at the child level. If all child-level group interactions are admissible at the current level, the corresponding coupling block between the parent groups is classified as inadmissible.

The remainder of this chapter is organized as follows. First, considering the more stringent admissibility condition required for high-frequency interactions, an indexing strategy for inadmissible groups under directional admissibility conditions is introduced for high-frequency electromagnetic scattering problems. Second, the construction of the elimination matrices for admissible blocks is improved. For cluster bases associated with different directions, a generalized cluster basis capable of representing all directional cluster bases is constructed. Subsequently, the system-matrix factorization procedure combining the high-frequency and low-frequency hierarchical decompositions is discussed, and the corresponding expression for the inverse of the system matrix based on the WNESA direct solver is derived. Finally, the time and memory complexities of the proposed algorithm are analyzed, and a series of numerical examples is presented to validate its computational accuracy and efficiency.

5.2 Implementation of the Direct Solution Based on WNESA

Since WNESA introduces a directional strategy at the higher levels of the octree structure to compress far-field coupling matrices based on NESA, its inter-level translation matrices retain the same nested property, while additionally incorporating directional information. Therefore, the overall strategy for system-matrix factorization and inversion remains similar to that described in Chapter 4: admissible blocks are first factorized and eliminated, followed by the elimination of inadmissible blocks. However, because WNESA employs a different compression scheme and a more stringent admissibility condition, the structure of the system matrix is modified, and the direct solver must therefore be appropriately revised.

First, due to the more stringent admissibility condition of WNESA, the inadmissible blocks at higher levels include not only the couplings between adjacent groups but also those between certain nonadjacent groups. These additional inadmissible couplings need to be identified according to the interaction characteristics of their child groups. Second, for the elimination of admissible blocks, the cluster bases associated with a given group at higher WNESA levels differ among different directional sectors. Since the elimination matrix for a group must eliminate all admissible blocks associated with that group in the system matrix, the cluster bases corresponding to all directional sectors of the group and its parent groups at the current and higher levels need to be collected to construct the elimination matrix. To this end, the receiving matrices corresponding to all directional sectors are accumulated, and the dominant column space is extracted through SVD to construct a generalized cluster basis for the group.

Mathematically, it can be inferred that any vector orthogonal to this generalized cluster basis is also orthogonal to the cluster bases corresponding to all directional sectors. Therefore, the elimination matrix constructed from the generalized cluster basis can eliminate all admissible blocks associated with the corresponding group. The following sections introduce the modifications required in the proposed direct solver to address these two issues, respectively.

5.2.1 Indexing of Inadmissible Groups Based on the Directional Admissibility Condition

In Section 2.4.2, the principle of far-field matrix-block compression based on high-frequency NESA and the corresponding admissibility condition have been briefly discussed. To illustrate this concept more clearly, Fig. 5.1 shows a schematic diagram of a child group at a higher level of the octree structure, together with its parent group and their respective admissible regions.

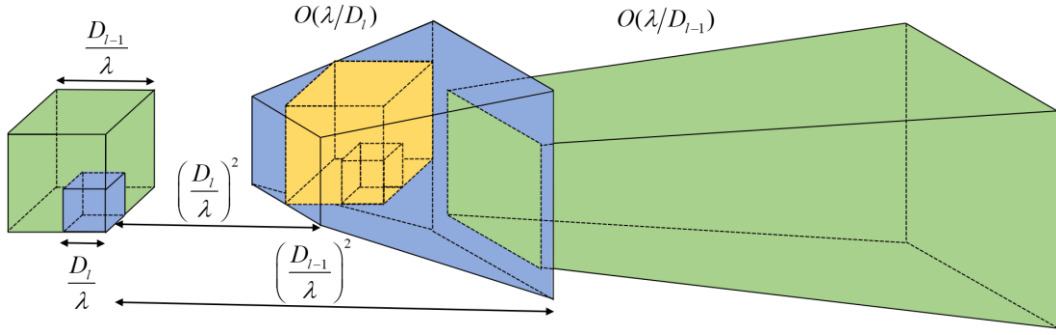


Fig. 5.1. Schematic of the directional admissible region in high-frequency NESAs. The green group is the parent group of the blue group, while the green and blue truncated pyramidal regions represent the admissible regions of the parent and child groups, respectively. The yellow parent group lies within the admissible region of the blue child group but outside the admissible region of the green parent group.

Each cube in the figure represents a group in the octree structure. For the green parent group and the blue child group, the pyramidal regions outlined in the corresponding colors represent their respective admissible regions in the considered direction. It can be observed that the directional admissible region of the parent group is nested within that of the corresponding child group.

For the yellow group shown in the figure, although it is not spatially adjacent to the green parent group, it is also outside the admissible region of the green parent group. Therefore, the strategy adopted in Chapter 4, which identifies inadmissible blocks by searching only adjacent groups, cannot identify the inadmissible block corresponding to the interaction between the yellow and green groups. This represents an important difference between the proposed direct solver and the iterative solver. For an iterative solver, it is sufficient to obtain the near-field matrix elements at the finest level and the compressed representations of far-field interactions at each hierarchical level to perform matrix-vector products involving the system matrix and the right-hand-side vectors. In contrast, the direct solver requires an explicit representation of the entire system matrix. Consequently, inadmissible blocks are not confined to the finest level but may also occur at every higher level. Therefore, at each hierarchical level, the corresponding inadmissible blocks must be identified and populated using the associated near-field interaction elements.

As shown in the figure, although the yellow group lies outside the admissible region of the green parent group, all child groups of the yellow group are located within the admissible region of the blue child group. This observation provides a criterion for identifying nonadjacent inadmissible blocks. Specifically, for each group that cannot be identified through the adjacency-based search, its child groups are traversed and their interactions with all child groups of the green group are examined. If the interactions between all child groups of the candidate group and all child groups of the green group are admissible at the child level, the corresponding interaction between the two parent groups is classified as an inadmissible block at the parent level. As illustrated in Fig. 5.2, when constructing the inadmissible blocks for high-frequency NESAs at higher levels, the proposed

algorithm therefore performs not only an adjacency-based search but also a traversal of child groups for the remaining candidate groups.

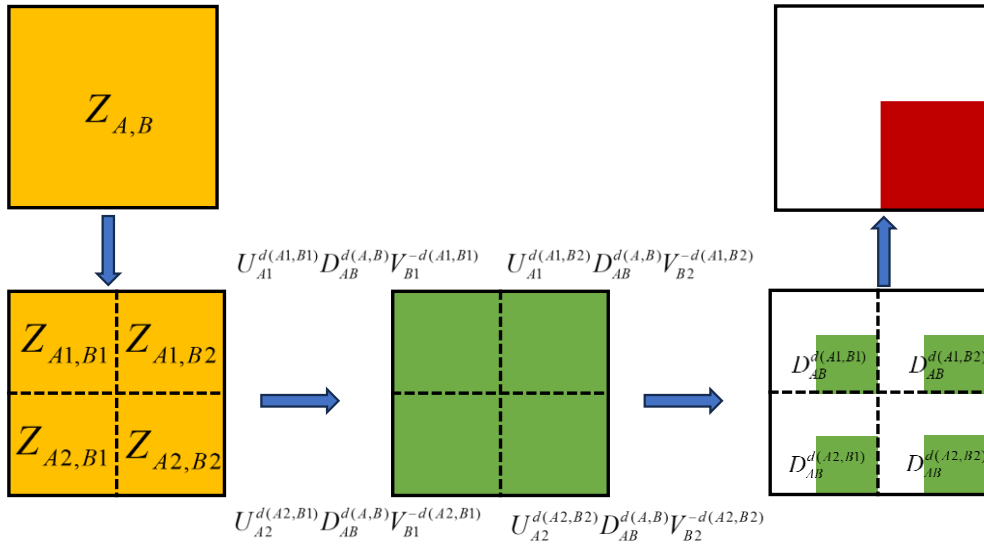


Fig. 5.2. Schematic of the identification and construction of nonadjacent inadmissible blocks in the WNESA direct solver.

Let A and B denote two parent-level groups, and let $A1$, $A2$, $B1$, and $B2$ denote their respective child groups. If the traversal of the interactions between the child groups shows that all child-level interactions are admissible, the corresponding parent-level coupling becomes an inadmissible block after the decomposition is completed, which is constructed by aggregating the directional translation matrices D associated with the interactions between the child groups.

Compared with the structure shown in Fig. 4.9, a key difference can be observed. For low-frequency NESAs, if all interactions between the child groups are admissible, the corresponding interaction between the parent groups also forms an admissible block at the parent level. Such an admissible block can be identified through the search for nonadjacent groups. In contrast, for high-frequency NESAs, the same child-level interaction pattern leads to an inadmissible block at the parent level. Therefore, the corresponding inadmissible block cannot be identified solely through the nonadjacent-group search. Instead, as illustrated in Fig. 5.2, it must be identified by traversing the interactions between the child groups.

5.2.2 Construction of the Generalized Cluster Basis

As discussed in Section 4.3, the elimination matrix for the admissible blocks associated with each group is constructed based on the cluster basis of that group. One important difference between high-frequency and low-frequency NESAs in the compression of admissible blocks is that, in high-frequency NESAs, the cluster bases associated with a given group are different for far-field groups located in different directions. Therefore, the elimination of all admissible blocks associated with a high-frequency NESAs group cannot be achieved using the cluster basis corresponding to only a single direction. In other words, in the high-frequency

regime, the rank of admissible blocks generally increases compared with that in the low-frequency regime, and the spatial information contained in the cluster basis of a single direction is insufficient to characterize the interactions in the remaining directions. Fundamentally, this is a consequence of the increasingly oscillatory behavior of the kernel function at higher frequencies.

We first derive the expression for the cluster basis in high-frequency NESAs. Let T denote the inter-level translation matrix, K denote the number of nonempty child groups of the i -th group at level l , and k denote the corresponding child-group index. Assume that the cluster basis associated with the k -th child group of group i is denoted by U_{ik}^{l+1} . For the transition level l_0 , which separates the high-frequency and low-frequency NESAs regions, the directional cluster basis of the i -th group can be expressed as:

$$U_{i,D}^{l_0} = \begin{pmatrix} ((U_{i1}^{l_0+1})^H U_{i1}^{l_0+1} T_{i1,D}^{l_0}) \\ \vdots \\ ((U_{iK}^{l_0+1})^H U_{iK}^{l_0+1} T_{iK,D}^{l_0}) \end{pmatrix} \quad (5.1)$$

Here, the superscript H denotes the conjugate transpose of a matrix, while the subscript D denotes a specific directional sector associated with group i . The matrix U_{ik}^{l+1} represents the direction-independent cluster basis of the child group ik , while $T_{ik,D}^{l+1}$ denotes the transfer matrix between the parent group i and its child group ik for direction D .

For a high-frequency level l above the transition level l_0 , assume that the generalized cluster basis of the child group ik is denoted by $\tilde{U}_{ik}^{l+1}$. Then, the directional cluster basis of group i corresponding to direction D can be expressed as:

$$U_{i,D}^l = \begin{pmatrix} ((\tilde{U}_{i1}^{l+1})^H U_{i1,d}^{l+1} T_{i1,D}^l) \\ \vdots \\ ((\tilde{U}_{iK}^{l+1})^H U_{iK,d}^{l+1} T_{iK,D}^l) \end{pmatrix} \quad (5.2)$$

Here, d denotes the set of directions associated with the interactions of the child groups of group i , while $U_{ik,d}^{l+1}$ represents the directional cluster basis of child group ik in direction d .

Since the elimination of admissible blocks is a global operation, a unified elimination matrix applicable to all admissible blocks in different directions must be constructed. For each group processed using high-frequency NESAs, the proposed algorithm collects all directions in which the group has interactions with other groups throughout the octree structure and combines the corresponding directional cluster bases, as illustrated in Fig. 5.3.

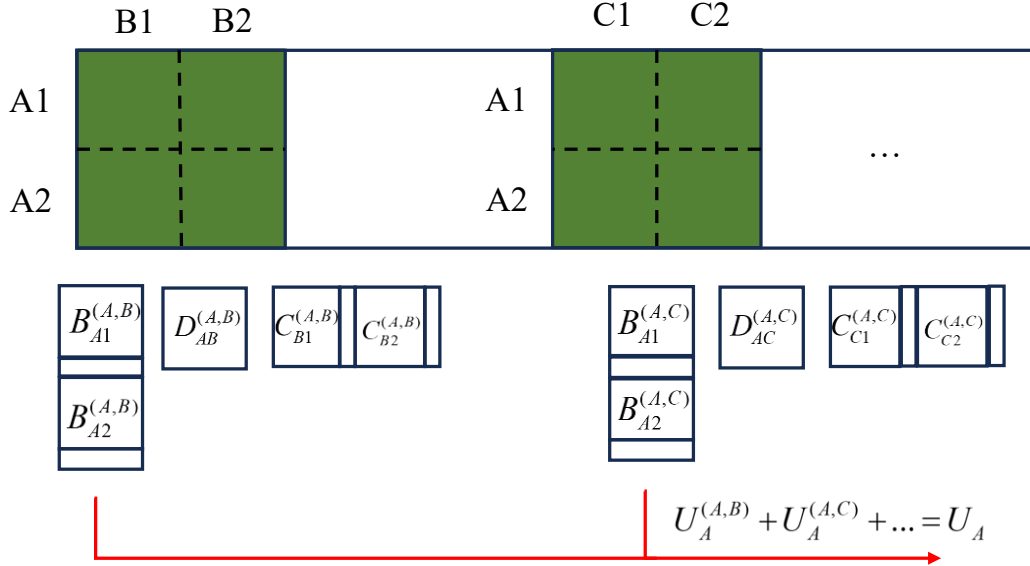


Fig. 5.3. Schematic of the accumulation of cluster bases associated with different directions in high-frequency NESA. The pairs (A,B) and (A,C) indicate that groups B and C are located in different directional sectors with respect to group A, respectively.

Then SVD is applied to extract the column space of these cluster bases, so for group i we have:

$$\sum_{m=1}^{M+M^p} U_{i,Dm}^l \stackrel{SVD}{\approx} \tilde{U}_i^l S_i (\tilde{V}_i^l)^T \quad (5.3)$$

Here, M and M^p denote the numbers of distinct directions associated with all interactions of group i and its parent groups, respectively. The matrices $\tilde{U}_i^l$ and $\tilde{V}_i^l$ are the left and right singular-vector matrices obtained after SVD truncation, while S_i denotes the diagonal matrix containing the corresponding singular values.

The unitary matrix composed of all retained left singular vectors can then be used to construct the elimination matrix for the admissible blocks associated with group (i), i.e., $Q_i^l = (\hat{U}_i^{l\perp} \quad \hat{U}_i^l)$ where $\hat{U}_i^l$ denotes the cluster basis $\tilde{U}_i^l$ augmented with the contributions from the fill-in matrix blocks.

Since the retained left singular vectors span the column space of the cluster bases associated with all relevant directions, the remaining left singular vectors after truncation are approximately orthogonal to all admissible blocks associated with group (i). Therefore, after the elimination operation performed through matrix multiplication, if fill-in matrix blocks are generated in the remaining admissible subblocks, these subblocks are updated as:

$$\tilde{Z}_{i,j}^{l,admissible} = (\tilde{U}_i^l)^H U_{i,d}^l D_{i,j}^l V_{j,-d}^l (\tilde{V}_j^l)^H + (\tilde{U}_i^l)^H F_{i,j} (\tilde{V}_j^l)^H \quad (5.4)$$

Here, $F_{i,j}$ denotes the fill-in matrix block added to the original $Z_{i,j}$ matrix block. These matrix subblocks are subsequently aggregated and transferred to the next hierarchical level following the procedure described in Chapter 4 for further factorization.

In Section 5.2.3, the system-matrix factorization formulation combining the high-frequency and low-frequency hierarchical decompositions will be derived. Subsequently, in Section 5.2.4, numerical examples will be presented to verify the correctness of the proposed algorithm and demonstrate its advantages over the algorithm developed in Chapter 4 for solving electromagnetic problems at higher frequencies.

5.2.3 Factorization of the System Matrix

The system-matrix factorization at the finest leaf level in the WNESA-based direct solver is consistent with that described in Chapter 4. Since the distances between groups at the finest level are relatively small, the corresponding far-field interactions can still be regarded as low-frequency interactions. For the factorization at higher hierarchical levels, however, the cluster bases are direction dependent, and the system-matrix factorization procedure therefore needs to be modified accordingly.

Specifically, for the low-frequency child levels below the transition level l_0 defined by NESA, the factorization follows the procedure described in Section 4.3. For the parent levels above l_0 , the system matrix is factorized and eliminated using the generalized cluster bases constructed in Section 5.2.2. Accordingly, the system matrix Z'_{sys} of an electrically large object can be factorized as:

$$Z'_{sys} = \left\{ \prod_{l=l_0-1}^1 \left[\left(\prod_{i=1}^{m^l} R_{i,d}^l \right) P_l^T \right] \prod_{l=L}^{l_0} \left[\left(\prod_{i=1}^{m^l} R_i^l \right) P_l^T \right] \right\} \times Z_{diag}^1 \\ \times \left\{ \prod_{l=l_0}^L \left[P_l \left(\prod_{i=m^l}^1 S_i^l \right) \right] \prod_{l=1}^{l_0-1} \left[P_l \left(\prod_{i=m^l}^1 S_{i,d}^l \right) \right] \right\} \quad (5.5)$$

Compared with Eq. (4.23), $R_{i,d}^l = Q_{i,sys,d}^l L_i^l$ and $S_{i,d}^l = U_i^l (Q_{i,sys,d}^l)^T$ denote the products of the two types of elimination matrices associated with group i at level l which is higher than l_0 . Here, $Q_{i,sys,d}^l$ is the elimination matrix for the admissible blocks constructed by considering all directions associated with group i , which is obtained from the generalized cluster basis.

Therefore, the inverse of the system matrix can therefore be derived as:

$$Z'^{-1}_{sys} = \left\{ \prod_{l=l_0-1}^1 \left[\left(\prod_{i=1}^{m^l} (S_{i,d}^l)^{-1} \right) P_l^T \right] \prod_{l=L}^{l_0} \left[\left(\prod_{i=1}^{m^l} (S_i^l)^{-1} \right) P_l^T \right] \right\} \times (Z_{diag}^1)^{-1} \\ \times \left\{ \prod_{l=l_0}^L \left[P_l \left(\prod_{i=m^l}^1 (R_i^l)^{-1} \right) \right] \prod_{l=1}^{l_0-1} \left[P_l \left(\prod_{i=m^l}^1 (R_{i,d}^l)^{-1} \right) \right] \right\} \quad (5.6)$$

A comparison of (5.5) and (5.6) with (4.23) and (4.24) shows that the system-matrix factorization for electrically large targets can be divided into two parts. The algorithm first performs the factorization from the finest level upward. Since the groups at the finest levels are relatively small, the far-field group interactions can still be regarded as low-frequency interactions and are therefore compressed

using the low-frequency NESAs scheme. As the factorization proceeds to higher levels, the far-field group interactions need to be compressed using equivalent sources within prescribed directional sectors. Accordingly, the elimination matrices involved in the factorization must also be reconstructed to account for the directional cluster bases.

In summary, the WNESA-based direct solver is designed for high-frequency electromagnetic problems involving electrically large targets. It modifies and improves the compression of far-field interactions at higher hierarchical levels and, by integrating these modifications with the NESAs-based direct-solver framework introduced in Chapter 4, forms a broadband direct solver capable of handling both low- and high-frequency electromagnetic problems.

5.2.4 Complexity Analysis

This subsection analyzes the computational complexity of the WNESA-based broadband direct solver. The complexity analysis follows a procedure similar to that presented in Chapter 4. Specifically, assume that the target is discretized into N unknowns and organized into L levels in an octree structure. Let m_l denote the number of nonempty groups at each level. For group i , let n_i and f_i denote the numbers of near-field and far-field groups, respectively. Unlike the case considered in Chapter 4, the number of near-field groups increases with the hierarchical level due to the directional admissibility condition adopted in WNESA.

Let p_i denote the number of original basis functions associated with group i , and let k_l denote the average number of selected skeletons at the finest leaf level or the number of equivalent sources at level l ($l \leq L-1$). Since the operations involving admissible matrix blocks are now directional, let ND_l denote the average number of directional sectors associated with each group at level l . The computational complexity at each level can then be expressed as:

$$C_l = \sum_{i=1}^{m_l} n_i p_i^3 + (p_i - k_l)^3 + n_i^2 p_i^2 (p_i - k_l) + ND_l f_i k_l^3 \quad (5.7)$$

The last term represents the update operation of the coupling matrices after the factorization at the corresponding level is completed. Since this term involves an additional directional factor and the third term n_i also increases accordingly, the overall computational complexity of the algorithm is expected to increase.

It can be observed that the variations of ND_l and n_i with respect to the levels scales as $O(4^{L-l})$. Therefore, the computational complexity at level l can be written as:

$$\begin{aligned} C_l &= \sum_{i=1}^{m_l} n_i p_i^3 + (p_i - k_l)^3 + n_i^2 p_i^2 (p_i - k_l) + ND_l f_i k_l^3 \\ &= \sum_{i=1}^{m_l} O(n_i^2 p_i^2 (p_i - k_l)) = O(m_l n_i^2 p_i^2 (p_i - k_l)) = O(m_l) O(4^{L-l})^2 \end{aligned} \quad (5.8)$$

By summing the computational costs over all hierarchical levels, the overall computational complexity of the proposed algorithm can be obtained as:

$$\sum_{l=L}^0 O(m_l)O(4^{L-l})^2 = \sum_{l=L}^0 O(4^{l+1}4^{L-l}4^{L-l}) = O(N^2) \quad (5.9)$$

Next, the memory complexity of the proposed algorithm is analyzed. First, at the finest level, the number of near-field matrix elements scales as $O(N)$ with the number of unknowns. For the far-field interactions, the cluster bases and coupling matrices associated with each group need to be stored separately. The memory complexity associated with these two types of matrices can therefore be expressed as:

$$M_{coupling} = \sum_{l=L}^1 \sum_{i=1}^{m_l} f_i k_i^2 = \sum_{l=L}^1 O(m_l f_i k_i^2) = O(N \log N) \quad (5.10)$$

$$\begin{aligned} M_{cluster} &= \sum_{l=L}^0 \sum_{i=1}^{m_l} N D_l p_l k_l = \sum_{l=L}^0 O(m_l)O(4^{L-l}) \\ &= O(N \log N) \end{aligned} \quad (5.11)$$

Therefore, by combining all the memory requirements discussed above, the overall computational time complexity of the proposed algorithm is $O(N^2)$, while its memory complexity is $O(N \log N)$. It can be observed that the complexity characteristics of the proposed algorithm differ from those of other direct solvers. In the proposed method, k_l is a constant and does not increase with the number of unknowns. However, as the number of unknowns increases, the depth of the hierarchical structure also increases, resulting in a larger number of directional sectors and consequently affecting the overall computational complexity.

5.3 Numerical Results

This subsection presents a series of numerical examples to validate the proposed algorithm. First, metallic sphere models at progressively increasing frequencies are simulated, and the results are compared with those obtained using the algorithm proposed in Chapter 4 to demonstrate the improved computational accuracy achieved by the proposed method. Subsequently, the computational time and memory complexities of the algorithm are evaluated, followed by simulations of several complex models to further demonstrate its computational performance. All numerical examples are performed under the same computational environment as that used in Chapter 4.

5.3.1 Accuracy Validation

First, to verify the correctness of the proposed algorithm, a metallic sphere with a radius of 1.8 m is considered as a benchmark example. Its bistatic RCS at a frequency of 600 MHz is calculated and compared with the corresponding Method of Moments (MoM) results. The surface current distribution on the metallic sphere is also presented, as shown in Fig. 5.4.

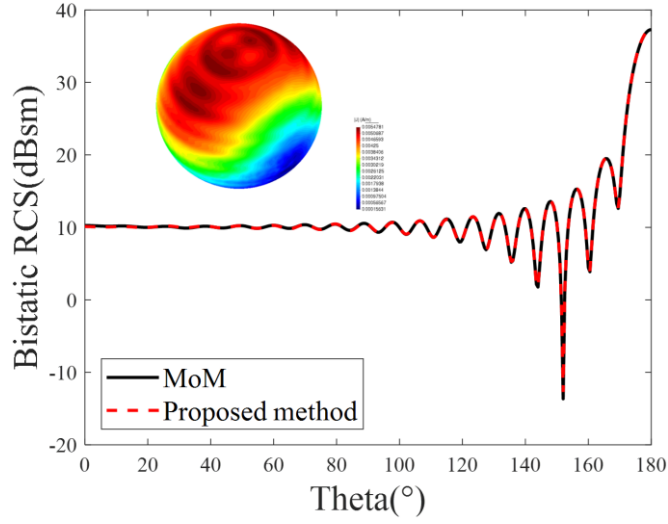
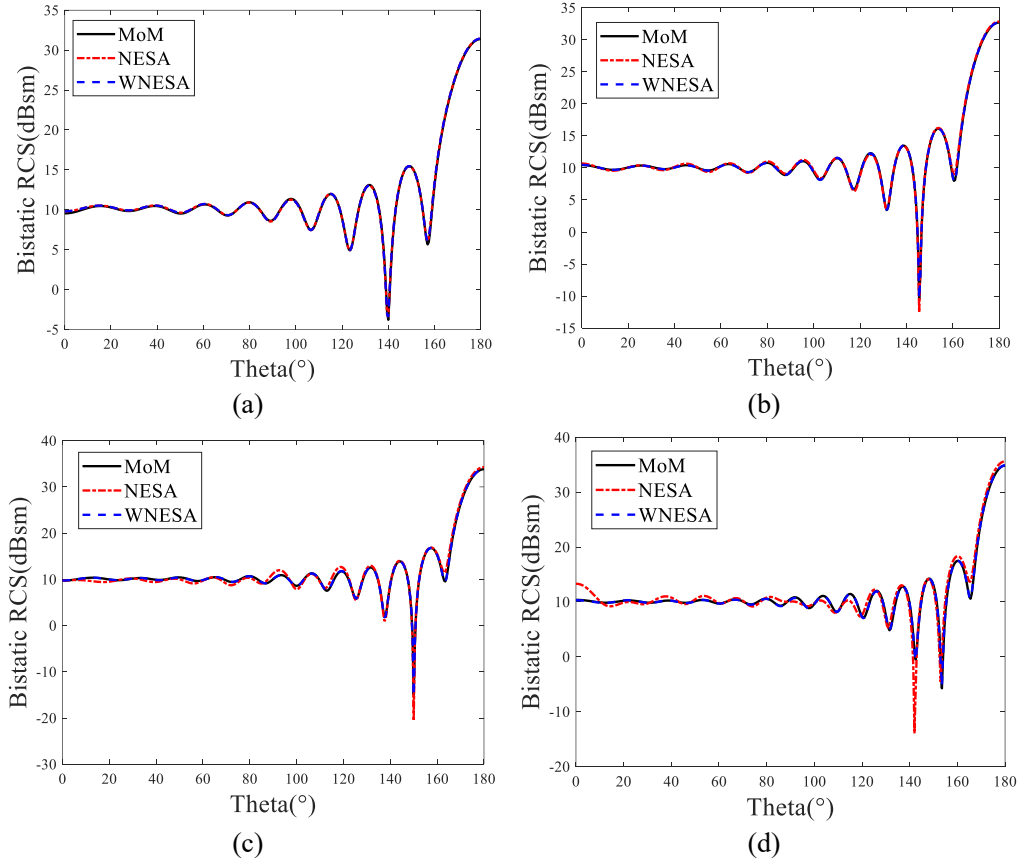


Fig. 5.4. Bistatic RCS and surface current distribution of the metallic sphere.

As shown in Fig. 5.4, the results obtained using the proposed algorithm agree very well with those obtained using the Method of Moments (MoM), demonstrating the correctness of the proposed method. To further investigate the improvement in computational accuracy achieved by the WNESA-based direct solver over the NESAs-based direct solver proposed in Chapter 4 for higher-frequency problems, the bistatic RCS of the metallic sphere is calculated at frequencies ranging from 300 MHz to 600 MHz. The results are obtained from direct solutions based on NESAs and WNESA and compared with the MoM solutions as the reference, as shown in Fig. 5.5.



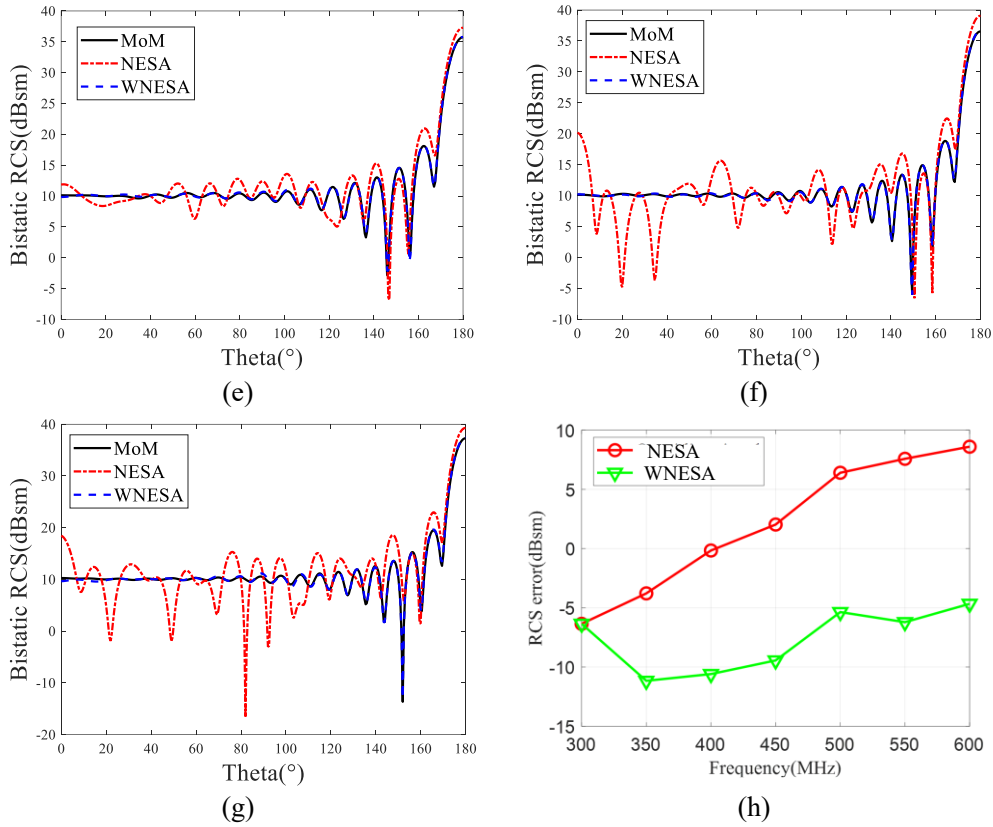


Fig. 5.5. Bistatic RCS of the metallic sphere at frequencies of (a) 300 MHz, (b) 350 MHz, (c) 400 MHz, (d) 450 MHz, (e) 500 MHz, (f) 550 MHz, and (g) 600 MHz. (h) RCS error of the proposed algorithm at different frequencies, with the MoM results used as the reference and the error expressed in dB.

As shown in Figs. 5.5(a)–(g), as the operating frequency increases, the computational error of the NESA-based direct solver gradually increases, whereas the error of the WNESA-based direct solver proposed in this chapter remains at a relatively low level, with its results remaining in close agreement with those obtained using the MoM. In Fig. 5.5(h), the RCS errors are further quantified by calculating the average RCS error at each frequency, providing a more intuitive demonstration of the accuracy advantage of the proposed method at higher frequencies.

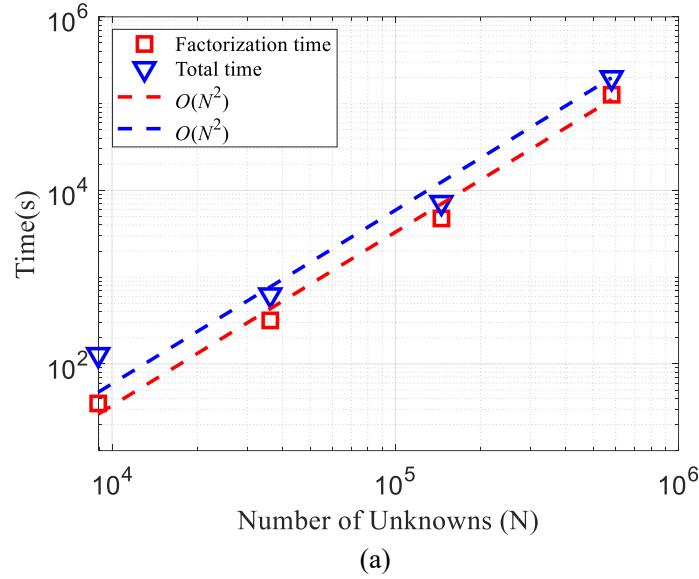
The results can be attributed to the increasingly oscillatory behavior of the integral-equation kernel as the frequency increases. Consequently, the electric field exhibits more rapid spatial variations over different locations on the testing surface. If the omnidirectional testing-point distribution strategy of NESA is still employed, the sampling of the electric field becomes insufficient to accurately characterize these variations. In other words, the equivalent-source coefficients obtained from the sampled electric field cannot accurately reconstruct an equivalent source whose scattered field is sufficiently close to that generated by the original source.

In contrast, when the testing surface is restricted to a specific directional sector, the oscillation of the integral-equation kernel within that sector is less pronounced, and the scattered field varies more smoothly. As a result, the electric field can be sampled and reconstructed more accurately, leading to more accurate

equivalent-source coefficients. This directional sampling strategy is therefore essential for maintaining the accuracy of NESAs-based compression at higher frequencies and constitutes the key advantage of WNESA over the conventional omnidirectional NESAs approach.

5.3.2 Complexity Validation

Next, the computational time and memory complexities analyzed in Section 5.2.4 are validated through numerical experiments. A metallic sphere with a diameter of 3.6 m is selected as the test model, and the operating frequency is increased successively from 300 MHz to 1.2 GHz, resulting in 8,934, 36,186, 145,500, and 580,470 unknowns, respectively. For the 300 MHz case, the electrical size of the model is relatively small, and therefore only NESAs is employed for matrix compression. For the higher-frequency cases, WNESA is introduced at the higher levels to account for the directional characteristics of the far-field interactions. The computational time and memory consumption of the proposed direct solver are recorded, as shown in Fig. 5.6. The results show that the observed computational time and memory consumption follow the same scaling trends as those predicted by the theoretical complexity analysis.



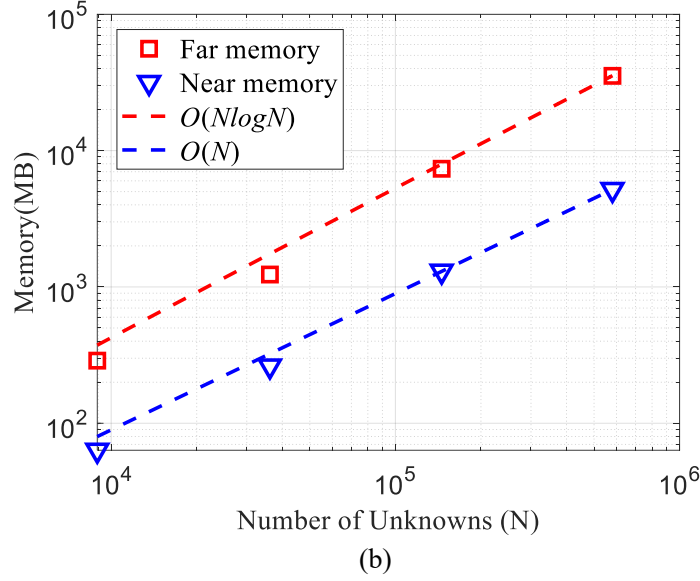


Fig. 5.6. Computational time and memory complexity of the WNEA-based direct solution. (a) System-matrix factorization time and total computational time; (b) memory consumption for near-field matrix elements and far-field matrix compression.

5.3.3 Analysis of Electromagnetic Scattering Characteristics of Complex Targets

Finally, the RCS of complex targets is simulated to further validate the capability of the proposed algorithm for solving electrically large problems in practical application scenarios. First, a missile model is considered. At an operating frequency of 5 GHz, the electrical dimensions of the missile are $26.7\lambda \times 3.9\lambda \times 3.9\lambda$, resulting in a total of 102,531 unknowns. The incident plane wave propagates along the direction opposite to the missile nose along the direction of $\theta_i = 90^\circ, \varphi_i = 0^\circ$. The receiving direction for the bistatic RCS is set to $\theta = 0^\circ \sim 180^\circ, \varphi = 0^\circ$. The calculated RCS results are compared with those obtained using the MLFMA, as shown in Fig. 5.7, while the corresponding surface current distribution of the missile is presented in Fig. 5.9(a).

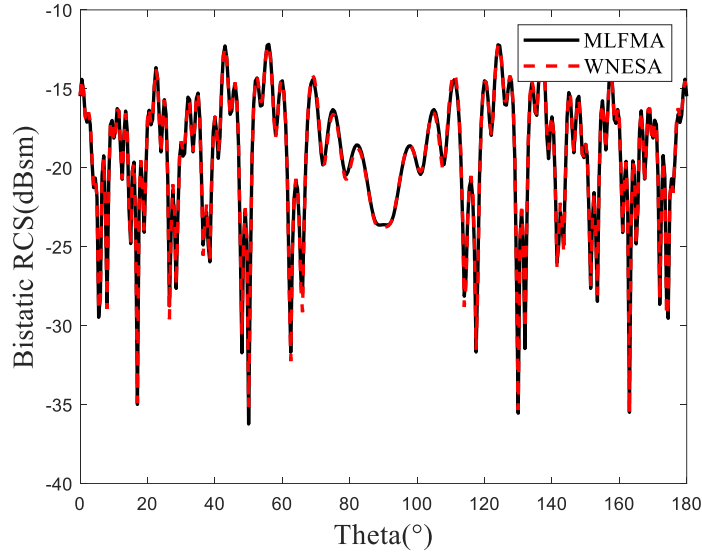


Fig. 5.7. Bistatic RCS of the missile at 5 GHz.

The monostatic RCS of the missile is further investigated for comparison. The operating frequency is set to 2.5 GHz, resulting in 67,521 unknowns. The monostatic RCS is calculated over an incident and receiving angular range of $\theta = 0^\circ \sim 360^\circ, \varphi = 0^\circ$, with an angular sampling interval of 0.5° , resulting in a total of 721 incident directions. The results obtained using the proposed algorithm are compared with those obtained using the MLFMA, as shown in Fig. 5.8.

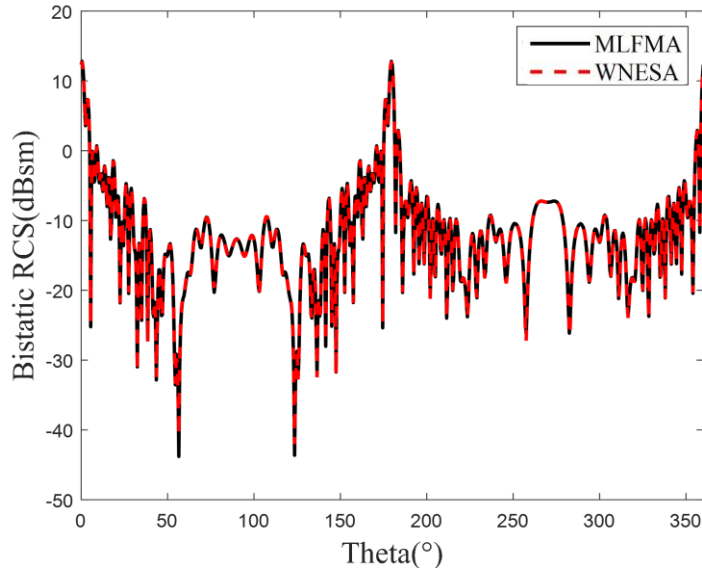


Fig. 5.8. Monostatic RCS of the missile at 2.5 GHz.

For this monostatic RCS example, the total computational time of the MLFMA is 1.0 h, with the majority of the time spent on the iterative solution process. In comparison, the proposed WNESA-based direct solver requires a total computational time of only 0.4 h, which is significantly faster than the iterative MLFMA solver. Furthermore, for the monostatic RCS calculation, the surface

current distribution corresponding to an incident direction of $\theta = 90^\circ, \varphi = 0^\circ$ is presented in Fig. 5.9(b).

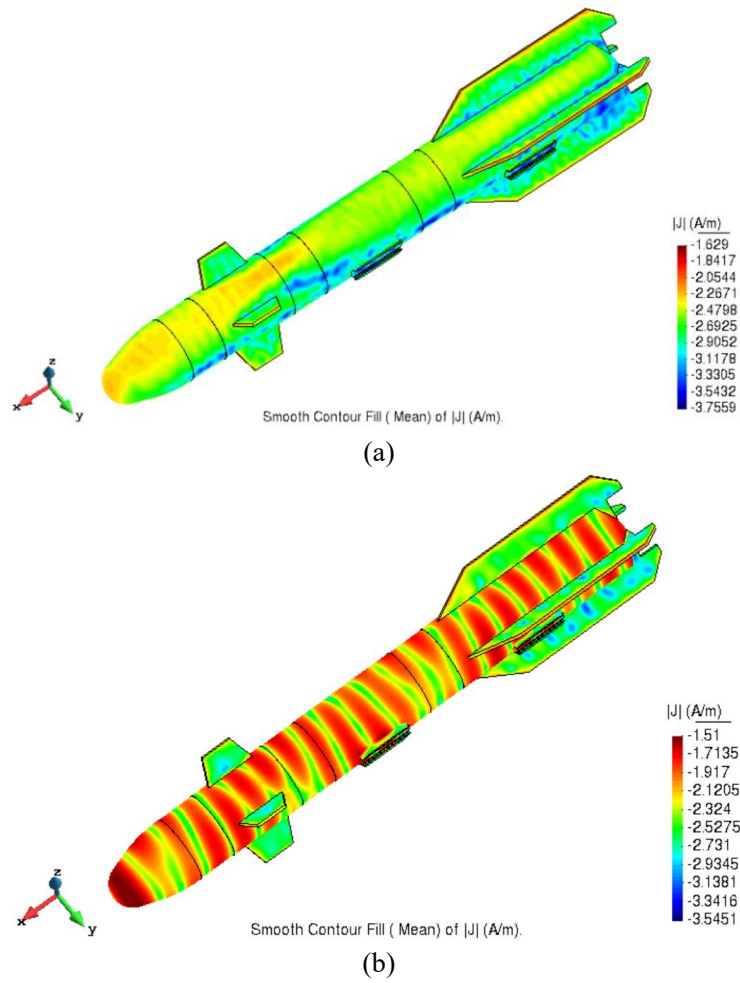


Fig. 5.9. Surface current distributions of the missile at (a) 5.0 GHz and (b) 2.5 GHz.

Next, an electromagnetic scattering example involving a fighter aircraft model is simulated to further validate the proposed algorithm. The operating frequency is set to 800 MHz, and the electrical dimensions of the fighter, normalized by the wavelength, are $15.1\lambda \times 10.9\lambda \times 3.2\lambda$. The monostatic RCS is calculated over the angular range of $\theta = 0^\circ \sim 360^\circ, \varphi = 0^\circ$ and compared with the results obtained using the MLFMA, as shown in Fig. 5.10.

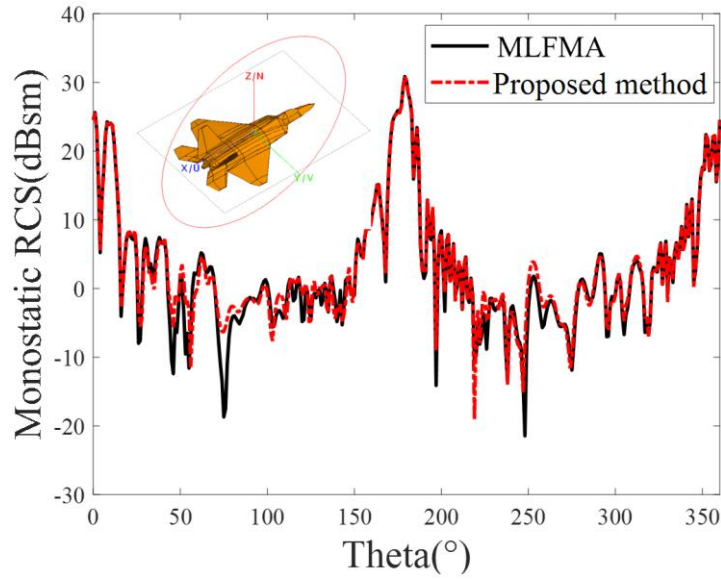


Fig. 5.10. Monostatic RCS of the fighter aircraft model.

The angular sampling interval for the monostatic RCS computation is 1° , generating 361 MRHs. Then the computational statistics of the simulation is compared below in Table 5.1.

TABLE 5.1. COMPUTATIONAL TIME AND MEMORY CONSUMPTION FOR THE MONOSTATIC RCS CALCULATION OF THE FIGHTER AIRCRAFT USING DIFFERENT METHODS.

Method	Factorization time (h)	Solution time (h)	Other time (h)	Total time (h)
MLFMA	0	1.07	0.01	1.08
WNESA direct solution	0.58	0.06	0.17	0.81

As shown in the table, compared with MLFMA, the proposed direct solver requires additional time for system matrix factorization. However, the time required for solving the induced currents is significantly lower than that of the iterative solver. Moreover, increasing the number of incident directions only leads to a proportional increase in the current-solution time, while the matrix factorization needs to be performed only once. Therefore, the proposed direct solver becomes increasingly advantageous in terms of computational efficiency as the number of incident directions increases, making it particularly suitable for problems involving multiple incident directions.

5.4 Summary

Building upon the direct solver presented in Chapter 4, this chapter introduces several improvements, including an indexing strategy for inadmissible blocks based on the more stringent directional admissibility condition, as well as a generalized cluster-basis construction that accounts for all relevant directions. These developments extend the NESA-based direct solver from the mid- and low-

frequency regime to high-frequency electrically large targets, enabling the efficient and accurate direct solution of electromagnetic scattering problems at higher frequencies. By combining the proposed high-frequency direct solver with the low-frequency solver developed in Chapter 4, a unified framework is established for the analysis and direct solution of broadband electromagnetic scattering problems. Finally, a series of numerical experiments validates the correctness, accuracy, and computational efficiency of the proposed algorithm.

Chapter 6

Design of a Microwave Sensing System for Food Inspection based on Electromagnetic Scattering

6.1 Introduction

In the preceding chapters, this dissertation has primarily focused on theoretical and computational methods for analyzing the electromagnetic scattering characteristics of objects. The research has centered on fast solution methods for electromagnetic integral equations, with particular emphasis on addressing the challenges associated with complex electromagnetic target modeling, the efficient solution of large-scale matrix systems, and the trade-off between computational efficiency and numerical stability. The ultimate goal is to improve the computational capability for the analysis of complex electromagnetic scattering problems. It should be noted that the development of efficient computational electromagnetics algorithms is not only of theoretical significance but also provides an important foundation for the design and practical implementation of complex microwave systems. Therefore, in this chapter, the electromagnetic computational algorithms developed in the preceding chapters are further applied to the analysis and design of a microwave sensing system, with the aim of exploring practical industrial applications based on the electromagnetic scattering characteristics of objects. This provides an engineering-oriented application validation of the proposed fast electromagnetic computational methods. Since the electromagnetic scattering characteristics of an object reflect information about its structural composition and material properties, they have significant application value in target detection, remote sensing imaging, biomedical applications, and industrial inspection. This chapter focuses on the application of electromagnetic scattering characteristics to food inspection in the food industry.

In recent years, food safety has received increasing attention from society. At present, food foreign-object detection mainly relies on optical inspection, X-ray inspection, metal detection, and chemical analysis. Although optical inspection offers advantages such as high detection speed and relatively low cost, it primarily relies on surface features and therefore has limited capability for detecting defects and foreign objects located inside packaging or within food products. X-ray inspection can penetrate food products and provides good detection performance for high-density foreign objects; however, its effectiveness is relatively limited for low-density contaminants such as plastics and wood fragments. In addition, X-ray systems are generally costly and require appropriate radiation safety management. Metal detectors, meanwhile, can only detect foreign objects made of metal, whereas food products may contain foreign objects composed of a wide variety of materials. Although chemical analysis methods generally provide high detection accuracy, they typically require sampling and complex sample-preparation procedures, making them difficult to implement for real-time, online inspection on production lines.

Therefore, developing a food quality inspection technology that combines nondestructive, efficient, real-time, and low-cost characteristics has become an important research direction in the field of food safety inspection. Microwave sensing based on electromagnetic scattering characteristics has gradually emerged as a research focus for food quality inspection. When illuminated by electromagnetic waves, food products produce specific scattering-field distributions. Their scattering characteristics depend not only on the external geometry of the food products, but also on their internal dielectric properties, structural composition, and the presence of foreign objects. When foreign objects such as metals, plastics, or glass are present inside a food product, or when quality defects such as spoilage, voids, or contamination occur, its scattering parameters (S-parameters) change accordingly. Therefore, analyzing variations in the scattering parameters of food products can provide effective information about their internal states, offering an important basis for food quality assessment and contaminant detection.

Existing microwave sensing systems for food contaminant detection generally employ multiple antennas to acquire the S-parameters of the food sample at different spatial positions, which are then used to image or identify the presence of contaminants. These microwave sensing systems typically use a single printed antenna or horn antenna as the transmitting and receiving element. Although a single antenna offers a simple structure and ease of implementation, horn antennas in particular provide relatively high directivity and gain, their radiation beams are primarily determined by the antenna geometry. As a result, the beam direction and energy distribution lack sufficient flexibility and cannot readily achieve dynamic focusing and beam scanning according to the position of the food sample under test. In contrast, an antenna array can achieve near-field focusing by adjusting the feeding phase of each array element, concentrating electromagnetic energy more effectively within the target region and thereby improving the detection sensitivity and efficiency of the microwave sensing system.

Based on these considerations, a microwave sensing (MWS) system based on an antenna array is developed for the detection of contaminants in canned edible oil, with the aim of further improving food contaminant detection capability. The proposed system enables electromagnetic-beam focusing, thereby enhancing the penetration of electromagnetic waves into the food sample under test. Moreover, the system can dynamically control the beam to focus on different positions within the food sample in real time, substantially expanding the detection coverage for food products on a production line. For the acquired S-parameters, a machine learning model is employed to classify the samples, enabling accurate identification of oil cans containing contaminants made of wood, glass, and stone.

The remainder of this chapter is organized as follows. First, the principle of food contaminant detection based on microwave sensing is investigated. Imaging simulations are conducted to verify the feasibility of detecting contaminants using S-parameters. In addition, the scattering characteristics of the food can are analyzed using the NESA-based direct solver, providing the motivation for designing a microwave sensing system that improves detection performance through electromagnetic-field focusing. Subsequently, the hardware design process of the microwave sensing system and the development and training of the machine learning model are presented, followed by experimental validation of the detection performance for different types of contaminants. Finally, the proposed system is deployed on a moving production line for real-time dynamic measurement experiments, demonstrating its potential for practical applications in food production lines.

6.2 Basic Principles of Microwave Detection of Food Contaminants

Assume that the detection system consists of a pair of transmitting and receiving antennas, which are used to measure the S-parameters of the edible-oil can at the same position under two conditions: without contaminants and with contaminants. The variation in the S-parameters can then be expressed as

$$\Delta S(\mathbf{r}_p, \mathbf{r}_q) \approx \frac{-j\omega\epsilon_b}{2a_p a_q} \int_R \mathbf{E}_b(\mathbf{r}_p, \mathbf{r}) \cdot \mathbf{E}_b(\mathbf{r}, \mathbf{r}_q) \Delta\chi(\mathbf{r}) d\mathbf{r} \quad (6.1)$$

where $\mathbf{r}_p$ and $\mathbf{r}_q$ denote the port positions of the receiving and transmitting antennas, respectively, a_p and a_q represent the known incident root-power waves at antenna ports p and q , respectively. ϵ_b denotes the complex relative permittivity of the edible oil inside the can, ω is the angular frequency, R represents the detection region, and $\Delta\chi = \Delta\epsilon / \epsilon_b$ denotes the dielectric contrast between the potential contaminant and the edible oil. $\mathbf{E}_b(\mathbf{r}_p, \mathbf{r})$ and $\mathbf{E}_b(\mathbf{r}, \mathbf{r}_q)$ denote the electric fields at position $\mathbf{r}$ within the detection region when the p - and q -antenna ports, respectively, act as radiation sources.

Since the size of potential contaminants is generally much smaller than that of the edible-oil can under test, the distorted Born approximation is introduced to

approximate the problem as the linear problem expressed in (6.1). It can be seen from (6.1) that, as long as there is a difference in the dielectric properties between the contaminant and the edible oil, the resulting dielectric contrast, together with the product of the internal electric fields, produces a variation in the S-parameters. Therefore, the presence of contaminants can be identified by measuring variations in the S-parameters. Moreover, a stronger internal electric field leads to a greater influence of the dielectric contrast on the S-parameter variation, thereby providing higher sensitivity to contaminants and improving the detection capability of the system.

The dissertation first investigates the imaging of contaminants. The problem described by (6.1) is essentially an inverse problem for a linear system, in which the unknown distribution of ΔS within the detection region can be reconstructed from the known $\Delta \chi$. This enables both the identification of contaminants inside the food product and the imaging of their locations. A pair of horn antennas is first employed as the transmitting and receiving antennas for the simulation study, as shown in Fig. 6.1. A cylindrical model with a bottom diameter of 8 cm and a height of 10 cm is constructed to represent the edible-oil can and is placed on the centerline between the two antennas. The can is sequentially moved along the z-axis from $z=6$ cm to $z=-6$ cm, with a spacing of 1 cm between adjacent positions, resulting in 13 simulated configurations.

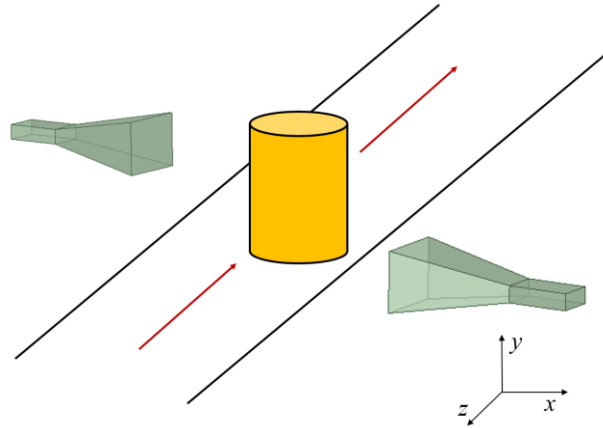


Fig. 6.1. Schematic diagram of the simulation setup for contaminant imaging.

A metallic sphere is placed at $(-2\text{cm}, 3\text{cm}, -1\text{cm})$. The operating frequencies are set to 9.5 GHz, 10 GHz, and 10.5 GHz, and S_{11} , S_{12} , and S_{22} are selected as the test objects. By discretizing (6.1), the following matrix equation can be obtained:

$$[S] = [Z][\chi] \quad (6.2)$$

Equation (6.2) is solved using the truncated singular value decomposition (TSVD) method. By selecting an appropriate number of retained singular values, the dielectric contrast distributions of the contaminant on the xoz , xoy and yoz planes are obtained, as shown in Fig. 6.2. .

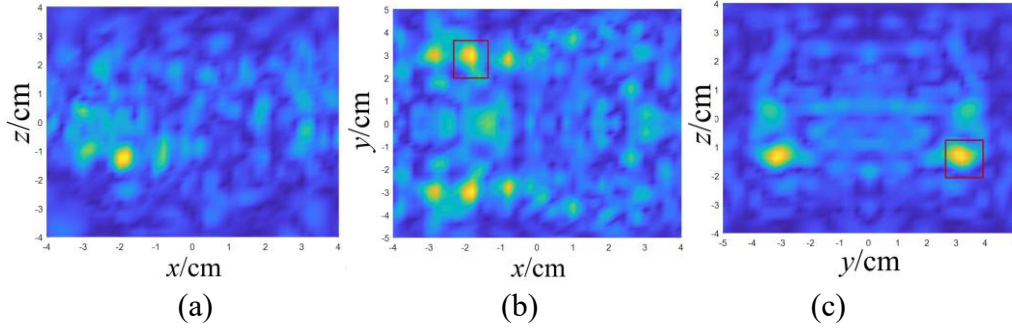


Fig. 6.2. Dielectric contrast distributions on the planes containing the contaminant with 50 retained singular values. The red boxes indicate the contaminant location: (a) xoz plane; (b) xoy plane; (c) yoz plane.

As shown in Fig. 6.2, a unique contaminant image is observed on the xoz plane, whereas multiple symmetric artifacts appear on the xoy and yoz planes. This is because the can is measured at different positions only along the z -axis, while its positions along the x - and y -axes remain unchanged. Therefore, the measurement scenarios lack sufficient spatial information about the can along the x and y directions, resulting in artifacts during the inverse imaging process. The above preliminary simulation results verify the effectiveness of microwave imaging and demonstrate the feasibility of detecting contaminants in edible oil using a microwave sensing system.

Furthermore, investigating the scattering characteristics of the can-shaped object under test can provide guidance for improving the sensing system and enhancing its contaminant detection performance. A representative edible-oil can model is constructed and discretized at 10 GHz, resulting in 64,254 unknowns. Since its electrical dimensions are $3.1\lambda \times 3.1\lambda \times 3.3\lambda$, the NESA-based direct solver proposed in Chapter 4 is employed to investigate its scattering characteristics for different receiving directions. A two-level octree structure is adopted for hierarchical grouping, with an average of 236 unknowns per group at the finest level. Since the conventional can model has a relatively simple structure without pronounced multiscale features, the accuracy threshold of the direct solver and the skeleton selection accuracy are set to $\dots$ and $\dots$, respectively. The can is placed vertically on the ground, and the incident electromagnetic wave impinges on the can along the direction normal to the ground plane. The RCS values for different horizontal observation directions are calculated, as shown in Fig. 6.3. The inset in Fig. 6.3 shows the corresponding surface current distribution on the can.

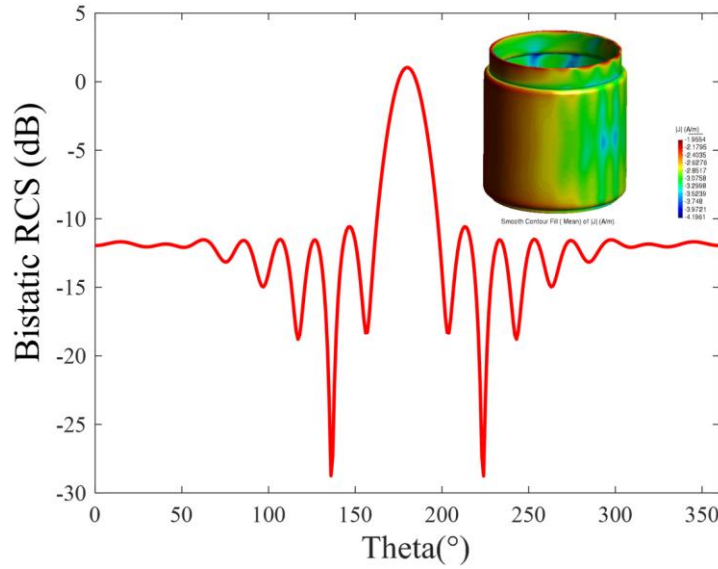


Fig. 6.3. Scattering characteristics and surface current distribution of the food can.

As observed in Fig. 6.3, the peak RCS occurs at 180° , corresponding to the direction directly opposite to the incident side of the can. This indicates that when the food can moves along the production line, the strongest scattering field can be detected when the can is located at the center of the transmitting and receiving antenna pair. When the can deviates from the center position within a certain range, the scattering field received by the receiving antenna gradually decreases as the observation angle increases. Therefore, for the transmitting and receiving ports of the microwave sensing system, adjusting the direction of the transmitted electromagnetic wave according to the position of the can is beneficial for generating a stronger scattering field. Meanwhile, the receiving direction should also be oriented toward the can to achieve more effective reception of the scattered electromagnetic waves.

To further improve the detection performance, the uniform electromagnetic field needs to be focused onto the target region, and the focal point should be dynamically controlled to track the can as it moves along the production line. Accordingly, edible oil cans are further considered as the target of interest. A microwave sensing system with real-time focusing and scanning capabilities is designed based on a metasurface antenna array, and multiple types of contaminants are subsequently identified using the proposed system.

6.3 Design and Validation of the Microwave Sensing System

Based on the analysis presented in Section 6.2, a schematic diagram of the microwave sensing system designed in this dissertation is shown in Fig. 6.4. The system employs a 10-GHz transmitarray metasurface antenna array as the transmitting and receiving ports, with a pair of horn antennas serving as the feed antennas. The horn antennas are connected to a two-port vector network analyzer (VNA) for measuring the S-parameters. The electromagnetic waves radiated by

the horn antennas can be focused at different spatial positions by means of the transmitarray metasurface antenna array.

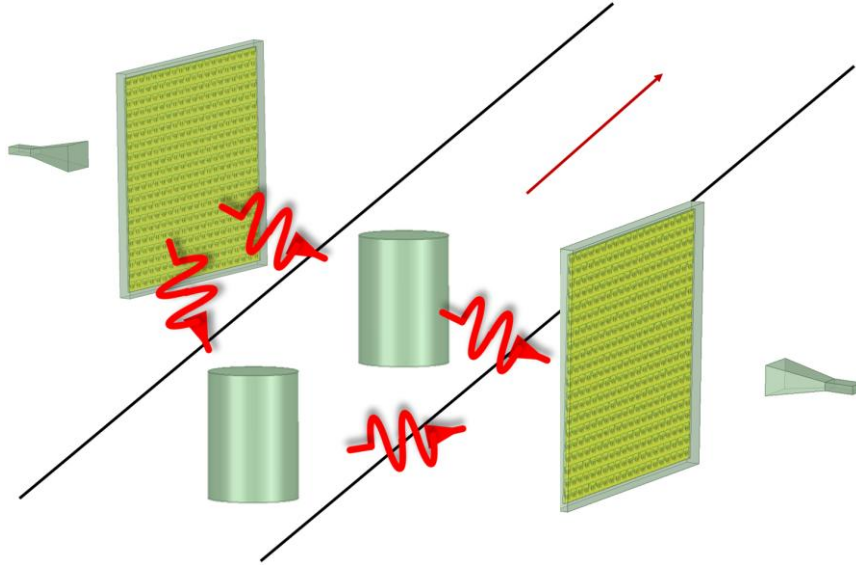


Fig. 6.4. Schematic diagram of the microwave sensing system for testing food cans on a moving production line. (© 2026 IEEE [155]).

As shown in the figure, the food products under test move along the production line, and different food products may require detection at different vertical heights due to variations in their shapes. Therefore, an antenna array can exploit its flexible control over electromagnetic waves to focus the electromagnetic beam at multiple spatial positions at different times.

First, it is necessary to determine whether the proposed system operates in the near-field focusing regime. In general, the boundary between the near-field and far-field regions of an antenna array can be determined by the distance $2D^2/\lambda$, where D is the aperture size of the antenna array and λ is the electromagnetic wavelength. For the system designed in this dissertation, the antenna-array aperture is 0.3m, while the wavelength is 0.03m, corresponding to a center operating frequency of 10 GHz. Therefore, the near-field/far-field boundary distance is calculated to be 6 m, which is greater than the separation between the metasurface antenna arrays and the target region in the designed experimental setup. It can thus be concluded that the detection region is located within the near-field region of the metasurface antenna array.

6.3.1 Design of the Microwave Sensing System

The designed transmitarray antenna consists of a 20×20 electrically reconfigurable array of unit cells loaded with PIN diodes. The design of the metasurface unit cell is first introduced. The 1-bit unit cell consists of two layers of F4BTMS350 dielectric substrates, two copper microstrip layers serving as the receiving and transmitting layers, respectively, two intermediate copper ground layers, and one layer of Rogers RO4450F bonding film. The receiving microstrip layer employs a side-fed ring-antenna structure, while the transmitting microstrip

layer consists of a ring antenna and two PIN diodes. The schematic configuration and dimensions of the unit cell are shown in Fig. 6.5.

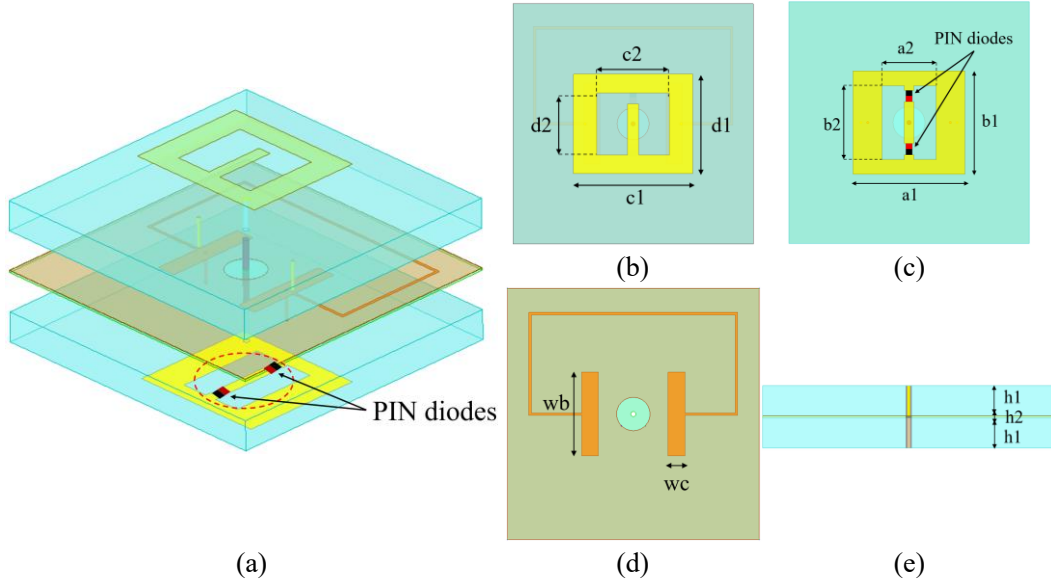


Fig. 6.5. Configuration of the transmitarray antenna unit cell: (a) overall structure; (b) top layer; (c) bottom layer; (d) intermediate layer; (e) front view. The dimensions are $a1=6.9$, $a2=3.4$, $b1=6.4$, $b2=4.6$, $c1=7.4$, $c2=4.5$, $d1=6.2$, $d2=3.9$, $wb=5$, $wc=1$, $h1=1.6$, $h2=0.2$, all in millimeters. (© 2026 IEEE [155]).

By controlling the ON and OFF states of all PIN diodes in the 400 unit cells, the transmitarray antenna can achieve electromagnetic-wave focusing at different positions in the near field. By tuning the transmission phase of each unit cell, the phase delay caused by the different spatial propagation distances between the feed antenna and the array elements can be compensated. For the array element located at the m -th row and n -th column, the required compensation phase $\phi_c(m, n)$ can be quantized as $\phi_c(m, n)$:

$$\phi_c(m, n) = \begin{cases} 0^\circ, -90^\circ + 2k\pi < \phi_c(m, n) \leq 90^\circ + 2k\pi \\ 180^\circ, 90^\circ + 2k\pi < \phi_c(m, n) \leq 270^\circ + 2k\pi \end{cases} \quad k = 0, \pm 1, \pm 2, \dots \quad (6.3)$$

Here, m and n denote the row and column indices of the array element, respectively. To illustrate the phase-control capability of the array elements, Fig. 6.6 presents the theoretically calculated compensation-phase distributions and the corresponding quantized compensation-phase distributions when the array is focused at different positions at 10 GHz.

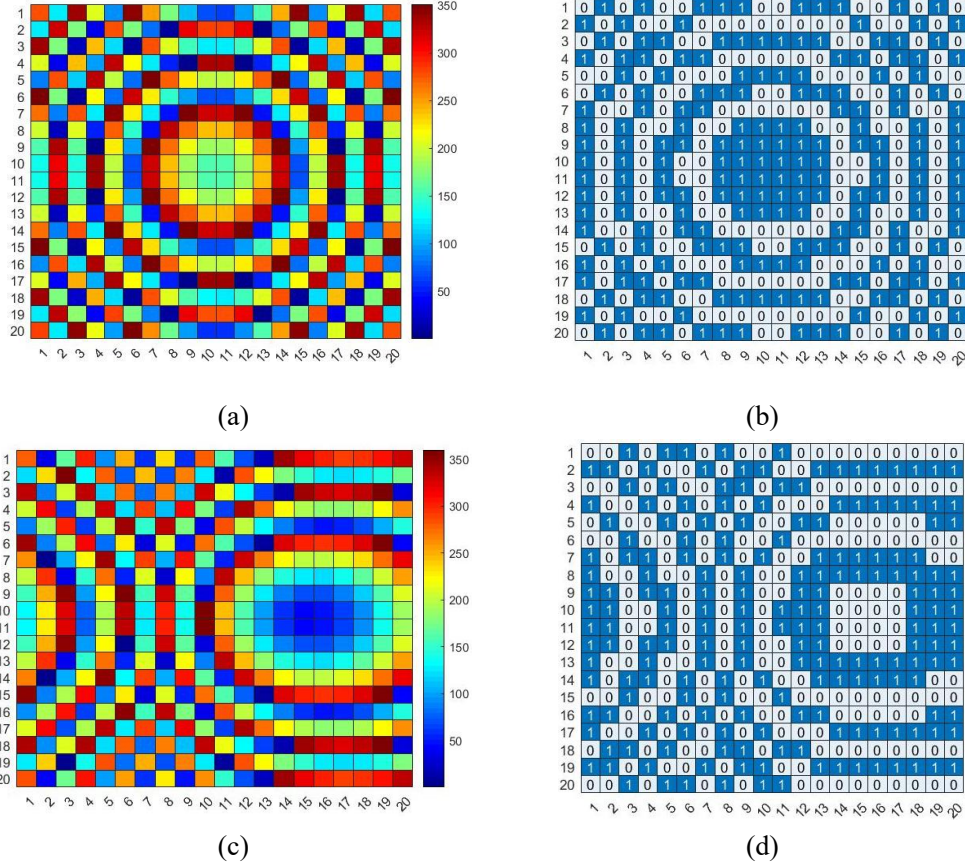


Fig. 6.6. (a) Theoretical phase distribution for focusing at the center position; (b) quantized phase distribution for focusing at the center position; (c) theoretical phase distribution for focusing at the right-side position; (d) quantized phase distribution for focusing at the right-side position. (© 2026 IEEE [155]).

In the figure, “0” denotes a quantized phase of 0° , while “1” denotes a quantized phase of 180° . The theoretical electric-field distributions corresponding to focusing at different positions under the quantized phase states are then evaluated by superposing the array factors ideally. The simulation plane is defined as a rectangular region with dimensions of $10\text{cm} \times 20\text{cm}$, which fully covers the cross section of the edible-oil can. The simulation plane is located 50 cm from the array, corresponding to the position of the cross section of the edible-oil can. As it is less than 6 m, the 50 cm is considered as the near field region of the array.

Taking the positive z -direction in Fig. 6.1 as the left side and the negative z -direction as the right side, the simulation plane is placed at positions 10 cm to the left, at the center, and 10 cm to the right, respectively. The focal points are set at 6 cm, 0 cm, and -6 cm, respectively, along the direction perpendicular to each corresponding simulation plane (i.e., the y -axis direction in Fig. 6.1). The electric-field distributions within the defined simulation planes are then calculated, as shown in Fig. 6.7.

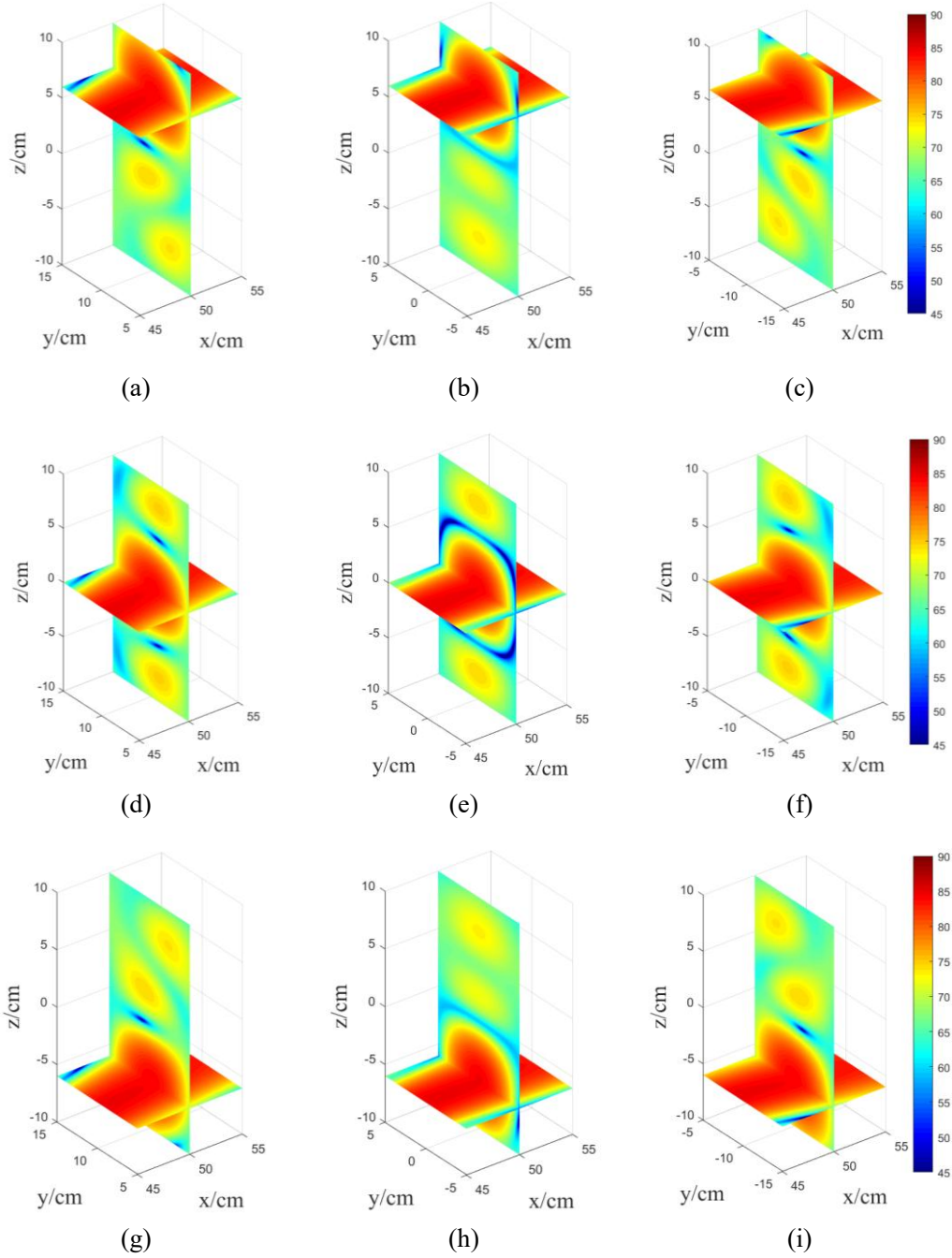


Fig. 6.7. Near-field electric-field distributions on the simulation planes obtained by array-factor superposition using equivalent ideal sources under quantized phase states: (a) upper-left position; (b) upper-center position; (c) upper-right position; (d) middle-left position; (e) center position; (f) middle-right position; (g) lower-left position; (h) lower-center position; (i) lower-right position. The electric-field magnitude is given in V/m, and a common color-bar range is used for all cases. (© 2026 IEEE [155]).

As shown in Fig. 6.7, the antenna array can effectively focus the electromagnetic field at different horizontal and vertical positions. Furthermore, a full-wave model of the actual array, including the associated feeding circuits, is constructed, with horn antennas used as the feed sources. The electric-field

distributions on the corresponding simulation planes for the different focal positions shown in Fig. 6.7 are then simulated, as presented in Fig. 6.8.

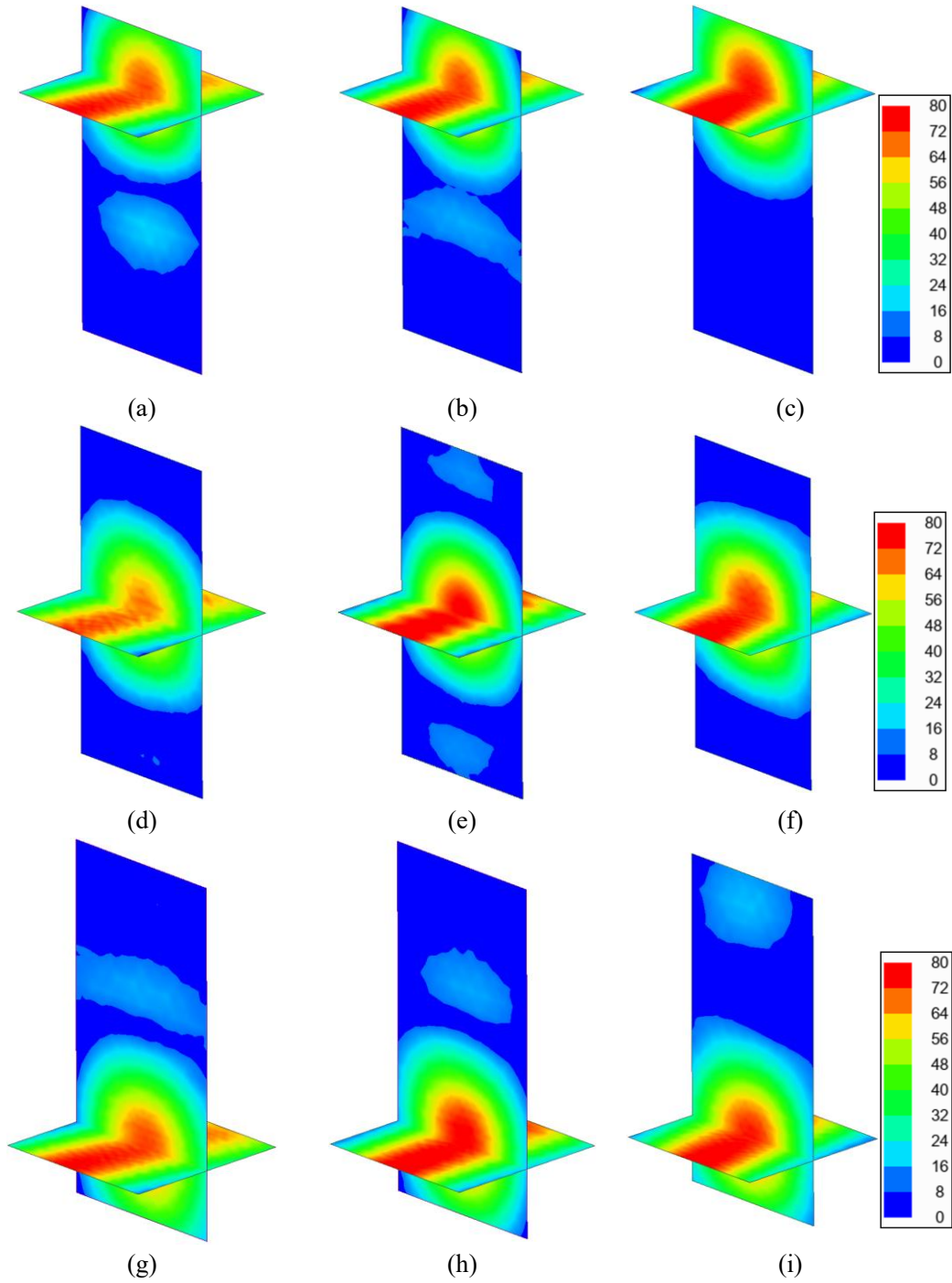


Fig. 6.8. Near-field electric-field distributions on the simulation planes obtained from the full-wave simulation of the actual array model using horn antennas as the feed sources: (a) upper-left position; (b) upper-center position; (c) upper-right position; (d) middle-left position; (e) center position; (f) middle-right position; (g) lower-left position; (h) lower-center position; (i) lower-right position. The electric-field magnitude is given in V/m, and a common color-bar range is used for all cases. (© 2026 IEEE [155]).

The simulation results in Fig. 6.8 are consistent with the theoretical results in Fig. 6.7, further validating the focusing capability of the proposed antenna array at

different positions. In practical production-line applications, the antenna array can dynamically scan and focus the electromagnetic beam on food products located at different positions at different times. Moreover, the focal distance in the vertical direction can be adjusted according to the height of the food product. This capability expands the detection region along the production line and increases the electric-field strength inside the food under test.

After completing the array simulations, the system was integrated and fabricated. Figure 6.9 presents the technical drawings and a photograph of the FPGA control board for the antenna array. The antenna array consists of 400 elements, each of which can be independently controlled, thereby enabling element-level beam control. The architecture of the proposed control system is shown in Fig. 6.9(a). The DC bias voltage of the system is supplied by a 12-V DC power source. According to the preset beam-control requirements, the host computer generates the corresponding beam-state encoding matrix and transmits it to the memory of the lower-level controller. The central processing unit then controls the FPGA, which regulates the operating states of the diodes in the antenna array, thereby dynamically controlling the electromagnetic response of each array element. The photograph of the fabricated system is shown in Fig. 6.9(b).

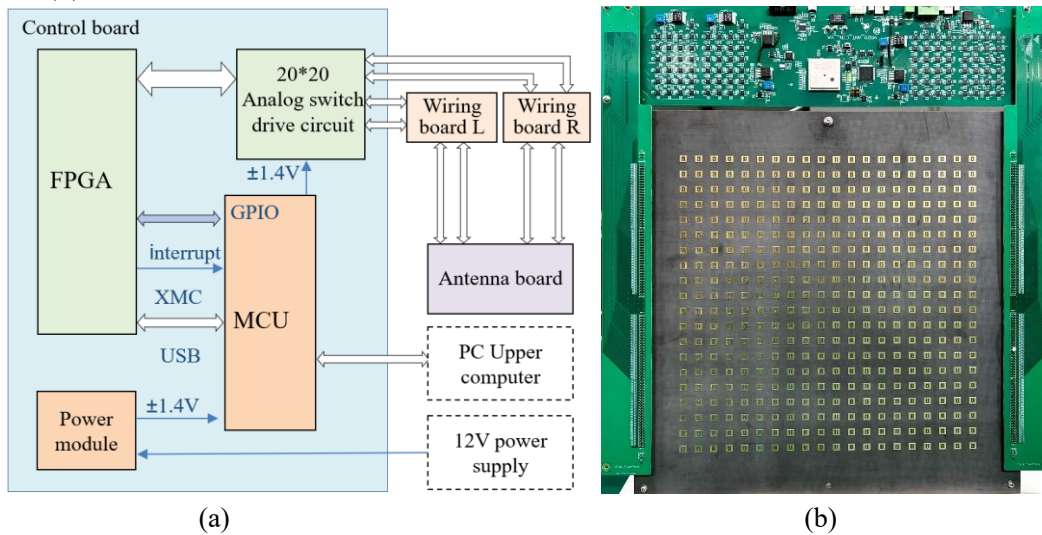


Fig. 6.9. (a) Technical drawing of the FPGA control board for the antenna array; (b) photograph of the fabricated system. (© 2026 IEEE [155]).

Next, the designed transmitarray antenna is experimentally characterized, and a microwave sensing system is assembled to experimentally identify edible-oil cans containing different types of contaminants, thereby validating the effectiveness of the proposed microwave sensing system.

6.3.2 Validation of the Microwave Sensing System

First, the focusing performance of the antenna array is experimentally evaluated at the different positions considered in Fig. 6.8. The experimental setup is shown in Fig. 6.10.

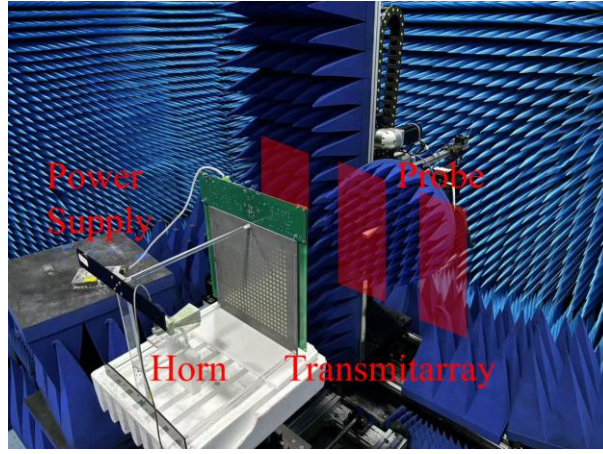
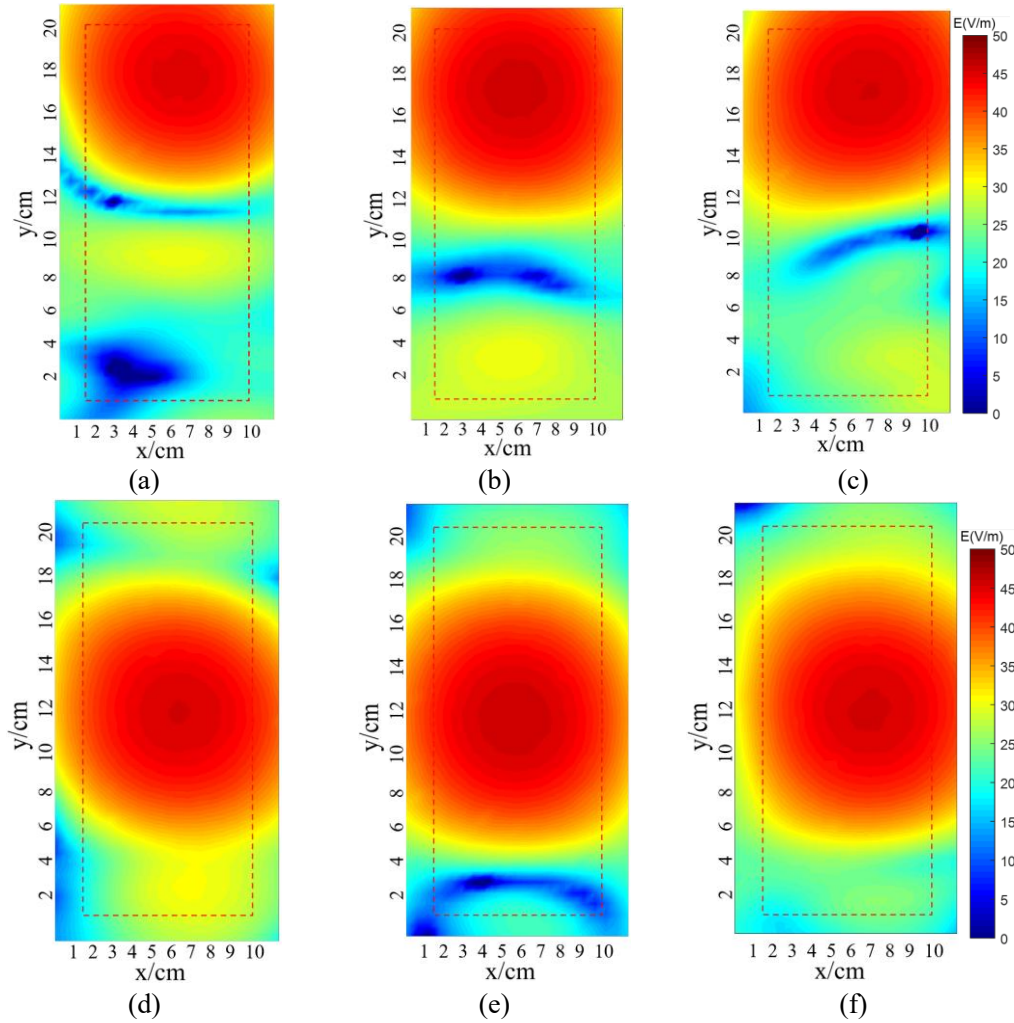


Fig. 6.10. Experimental setup for measuring the electric-field distribution at different near-field focusing positions of the antenna array in an anechoic chamber. (© 2026 IEEE [155]).

As shown in Fig. 6.10, the red rectangular region indicates the scanning area of the electric-field probe. The focusing positions of the antenna array are set to be consistent with those used in the simulation scenarios in Fig. 6.8. The experimentally measured electric-field distributions at the corresponding focal positions are presented in Fig. 6.11.



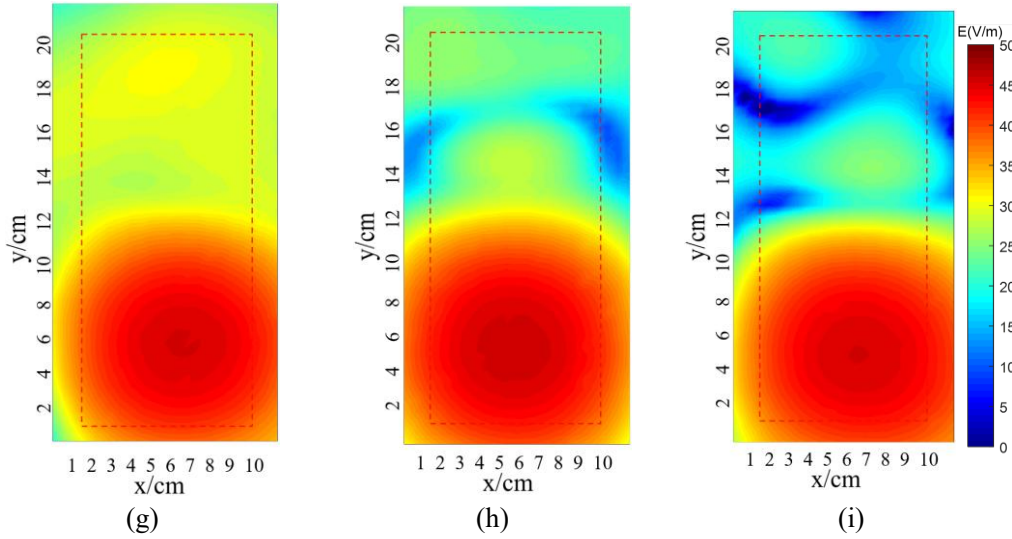


Fig. 6.11. Measured electric-field distributions at different near-field focusing positions of the antenna array in an anechoic chamber. The red dashed box has dimensions of 8cm×18cm, corresponding to the cross-sectional dimensions of the edible-oil can used in the subsequent experiments. (a) Upper-left position; (b) upper-center position; (c) upper-right position; (d) middle-left position; (e) center position; (f) middle-right position; (g) lower-left position; (h) lower-center position; (i) lower-right position. The electric-field magnitude is given in V/m, and a common color-bar range is used for all cases. (© 2026 IEEE [155]).

As shown in Fig. 6.11, the antenna array exhibits a pronounced focusing effect at the prescribed focal positions in the near-field experiment. A comparison with Fig. 6.8 shows that the experimentally measured electric-field distributions are consistent with the simulated results, thereby validating the near-field electric-field focusing capability of the proposed array. To further demonstrate the enhancement of the electric field achieved by the antenna array, the array was removed from the experimental setup, and the radiation electric field at the center position was measured using only the horn antenna, as shown in Fig. 6.12.

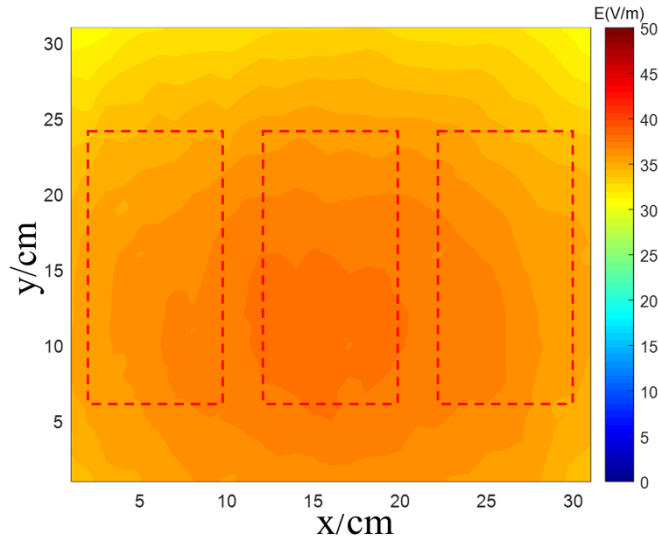


Fig. 6.12. Electric-field distribution radiated by the horn antenna at the center position. The measurement plane has dimensions of 30cm×30cm, and the red dashed box indicates the corresponding left, center, and right positions shown in Fig. 6.9. (© 2026 IEEE [155]).

Taking the center position as an example, a comparison of the electric-field intensity distributions in Fig. 6.12 and Figs. 6.11(d)–(f) shows that the maximum electric-field intensities in Figs. 6.11(d), (e), and (f) are increased by 20.7%, 21.1%, and 20.9%, respectively, compared with that in Fig. 6.12. This demonstrates that the electric field focused by the antenna array is significantly enhanced compared with the radiation field of the horn antenna. Moreover, the antenna array can achieve electric-field focusing over a wider spatial region.

6.4 Contaminant Experiment and Application of the Microwave Sensing System

In Section 6.3, the proposed microwave sensing system based on the transmitarray antenna was designed and experimentally validated in terms of its electric-field focusing and scanning capabilities. In the following section, the system is applied to the practical detection of contaminants in edible-oil cans, with the aim of identifying and classifying different contamination conditions. The experimental setup is shown in Fig. 6.13.

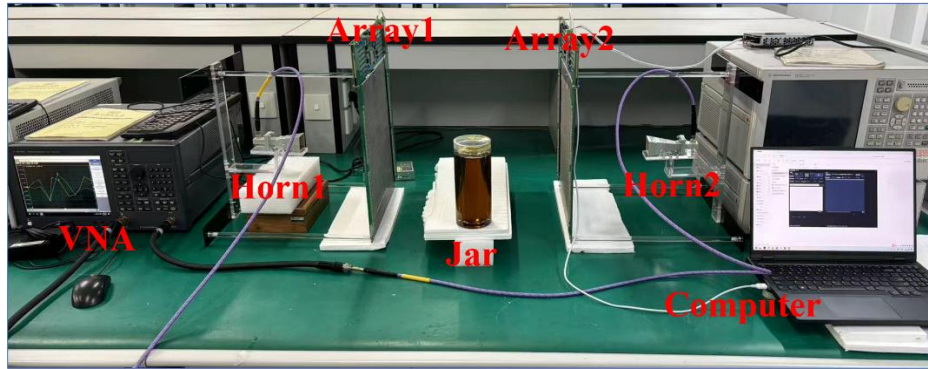


Fig. 6.13. Schematic diagram of the experimental setup for food contaminant detection using the microwave sensing system. The two antenna arrays are separated by 1 m. The edible-oil can under test is moved back and forth along the centerline between the two arrays through four successive positions to emulate its motion along a production line, with a displacement of 10 cm between adjacent positions. The host computer controls the antenna arrays to focus the electromagnetic beam at different positions, while the corresponding S-parameters are measured using the vector network analyzer (VNA). (© 2026 IEEE [155]).

In the experiment, a pair of transmitting and receiving antenna arrays are placed opposite each other, with the edible-oil can under test positioned along the centerline between the two arrays and sequentially moved along this line to emulate its motion on a production line. The focusing plane is set 50 cm away from each antenna array, resulting in an inter-array spacing of 1 m. The detection range is defined as cm from the center position between the antenna arrays. A beam-focusing operation and S-parameter measurement are performed each time the can moves by 10 cm, resulting in measurements at a total of nine positions.

Three types of contaminants, namely wood, glass, and stone, are used in the experiments. Each contaminant is placed at the upper, middle, and lower regions inside the edible-oil can. The dimensions and locations of the contaminants are shown in Fig. 6.14.

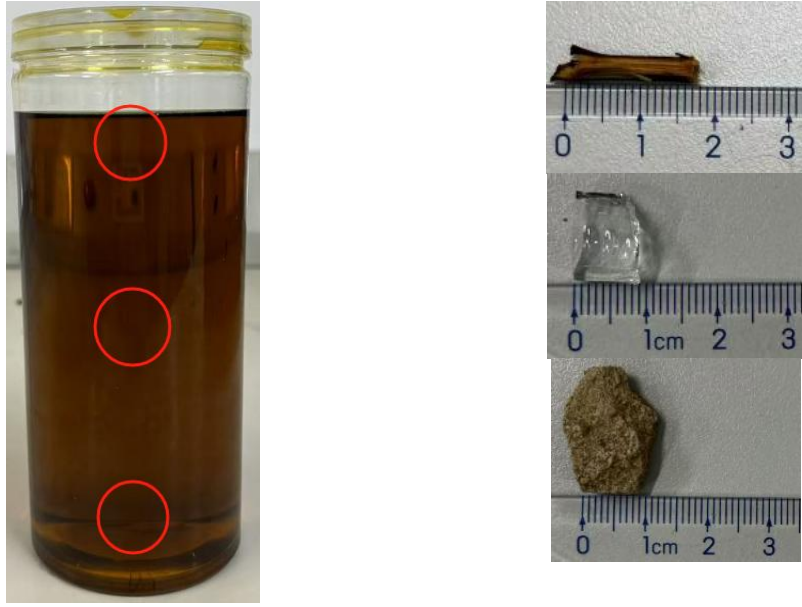


Fig. 6.14. Photographs of the edible-oil can and the contaminants used in the experiments. The red circles indicate the locations of the contaminants: wood placed in the upper region of the edible oil, glass placed in the middle region, and stone placed in the lower region of the can. (© 2026 IEEE [155]).

It should be noted that the dielectric constants of the contaminants considered in this study at 10 GHz must differ from that of the edible oil in order to produce measurable changes in the S-parameters, as investigated in [155]. In addition, the experiments in [153] measured the electromagnetic-wave penetration depth in edible oil over the frequency range from 9.5 to 10.5 GHz, which was approximately 3.2–4.5 cm. Since the radius of the edible-oil can used in the experiments is 4 cm, the electromagnetic-wave energy is significantly attenuated after penetrating through the can. The subsequent experimental results further demonstrate that the use of the antenna array can enhance the near-field electric-field intensity, thereby improving the contaminant detection performance of the proposed system.

6.4.1 Preliminary Experiments on Food Scattering Parameter Measurements

First, the microwave sensing system is used to measure and compare the S-parameters of the edible-oil can with and without contaminants. Taking the stone-contamination case shown in Fig. 6.14 as an example, the S₁₂ parameter of the edible-oil can is measured at nine positions over the frequency range from 9.5 GHz to 10.5 GHz, both with and without the stone contaminant. For each condition, 500 sets of sample data are collected, as shown in Fig. 6.15.

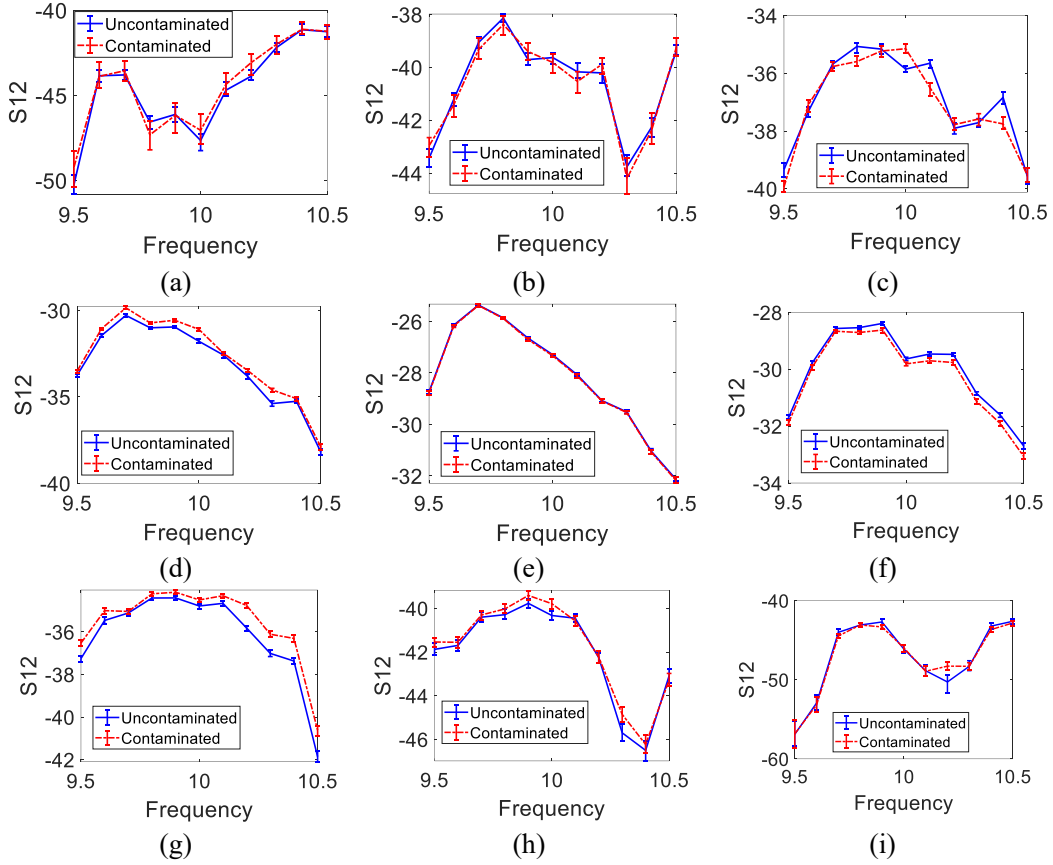


Fig. 6.15. Magnitude of the S12 parameter measured from 500 experimental samples. The red dashed line represents the mean value of all samples with stone contamination, while the blue solid line represents the mean value of all uncontaminated samples. Subplots (a)–(i) correspond to the measured S12-magnitude curves as a function of frequency at the nine test positions, respectively. The error bars at each data point indicate the maximum and minimum values measured across all samples at the corresponding position and frequency, reflecting the effects of experimental noise and operational errors. (© 2026 IEEE [155]).

As shown in Fig. 6.15, the S-parameter curves measured with and without stone contaminants differ at different positions. At some positions, such as that shown in Fig. 6.15(g), the difference between the S-parameters with and without contaminants is pronounced, indicating that the presence of the stone indeed produces a significant change in the S-parameters. In such cases, the presence of contaminants can be readily identified based on the distinct difference between the two measurement conditions. However, at other positions, such as that shown in Fig. 6.15(e), the two measured curves are very similar, making it difficult to distinguish the contaminated and uncontaminated cases directly. Therefore, based on the overall experimental results, it can be concluded that the S-parameters measured with and without contaminants exhibit differences, but these differences can be partially similar or difficult to distinguish at certain positions.

Consequently, machine-learning methods are employed to classify and identify the measurement data. Before performing the classification, the measurement error caused by noise in the experimental data and the S-parameter perturbation introduced by the stone contaminant are further quantified. Specifically, the S-parameter noise-extrema matrix for each group of

measurements and the S-parameter difference matrix obtained by subtracting the mean S-parameters of the uncontaminated and contaminated cases are analyzed, as shown in Fig. 6.16.

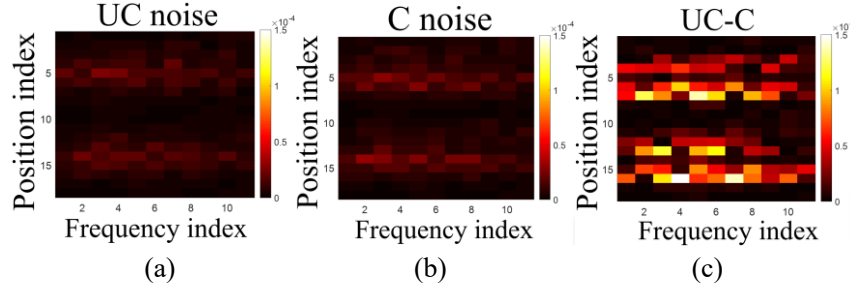


Fig. 6.16. Comparison of the maximum error matrix of (a) the uncontaminated case, (b) the contaminated case and (c) the difference between the uncontaminated mean value matrix and the contaminated mean value matrix. The abbreviation UC and C denote the uncontaminated and the contaminated respectively. (© 2026 IEEE [155]).

It can be observed that the values in Fig. 6.16(c) are generally larger than those in Figs. 6.16(a) and (b). Specifically, the 2-norms of the three matrices in Figs. 6.16(a), (b), and (c) are 1.37×10^{-4} , 1.18×10^{-4} , and 3.92×10^{-4} , respectively. This indicates that the variations in the S-parameters caused by the contaminant are greater than those caused by experimental measurement errors, thereby demonstrating the feasibility of identifying the presence of contaminants by measuring variations in the S-parameters.

6.4.2 Food Contaminant Classification Based on Machine Learning

Next, a machine-learning model is developed to classify the three types of contaminants shown in Fig. 6.14. Four cases are considered in the experiment: no contaminant, wood contamination, glass contamination, and stone contamination. Therefore, the classification task is formulated as a four-class problem. For each case, 500 sets of samples are repeatedly measured at each of the nine positions and then combined. Thus, a complete sample consists of the S-parameter data collected at all sampling frequencies across the nine measurement positions. The entire dataset is divided into training, validation, and test sets, accounting for 53%, 22%, and 25% of the dataset, respectively. The designed neural network is trained using the Adam optimizer with a batch size of 8. The initial learning rate is 0.0005.

To demonstrate the improvement in detection performance achieved by the microwave sensing system, the S-parameters of the cans under test are acquired using two different beam-focusing strategies based on the antenna array. These results are compared with those obtained using only the horn antennas after removing the antenna array. The model training accuracy curves and final classification results are shown in Fig. 6.17. Specifically, Figs. 6.17(a), (d), and (g) correspond to the horn-antenna control group; Figs. 6.17(b), (e), and (h) correspond to the microwave sensing system using only horizontal beam focusing; and Figs. 6.17(c), (f), and (i) correspond to the strategy employing simultaneous

horizontal and vertical beam focusing. In the confusion matrices and recall matrices, C0 denotes the case without contaminants, C1 denotes wood contamination, C2 denotes glass contamination, and C3 denotes stone contamination.

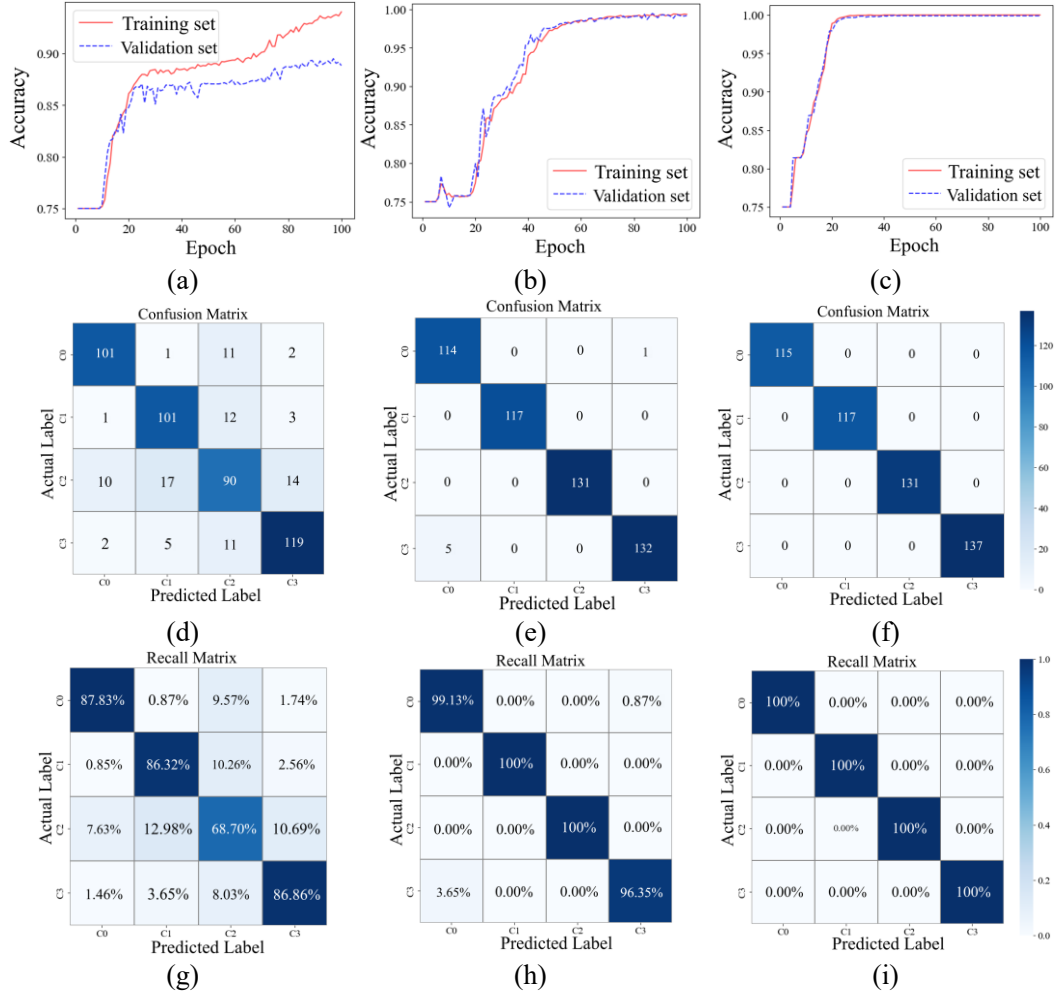


Fig. 6.17. Model training and classification results under different focusing strategies. (a)–(c) Accuracy curves during the training process of the machine-learning model; (d)–(f) confusion matrices; (g)–(i) recall matrices. The first column, (a), (d), and (g), corresponds to the horn-antenna-only strategy; the second column, (b), (e), and (h), corresponds to the microwave sensing system using horizontal beam focusing only; and the third column, (c), (f), and (i), corresponds to the microwave sensing system employing simultaneous horizontal and vertical beam focusing. (© 2026 IEEE [155]).

As shown in Fig. 6.17, the strategy employing simultaneous horizontal and vertical beam focusing achieves a final classification accuracy of 100%. In contrast, the strategy using horizontal focusing only produces a small number of misclassifications, while the overall accuracy remains close to 100%. As a comparison, when the microwave sensing system is not used and only the horn antennas are employed for detection, the classification error is considerably larger, with an accuracy below 90%.

These results demonstrate that the microwave sensing system enhances the electromagnetic energy inside the food by focusing the electromagnetic field, thereby producing larger variations in the S-parameters in the presence of

contaminants inside the food. As a result, the contaminants can be detected more readily. Furthermore, benefiting from the flexible beam-control capability of the antenna array, additional focusing in the vertical direction enables the system to probe a larger portion of the food volume, ultimately leading to improved detection performance.

The above experiments validate the effectiveness of the proposed microwave sensing system and demonstrate its improved contaminant-classification performance in practical detection scenarios compared with conventional horn-antenna-based sensing systems.

6.4.3 Application of the On-line Microwave Sensing System for Real-time Measurement

Finally, real-time dynamic measurements and detection are performed on food cans moving along a production line to evaluate the feasibility of applying the proposed system in a practical scenario. The experimental setup is shown in Fig. 6.18. Before the experiment, the detection range is first determined, and the can positions corresponding to the S-parameter acquisition points are preset within this range. A pair of photoelectric switches is placed on both sides of each measurement position to detect whether the can has moved to the corresponding position.

The can is placed on a fixed track, and the moving speed of the track is set to 15 cm/s. Once the experiment starts, when the can moves between the first pair of photoelectric switches, the host computer controls the two antenna arrays to focus the electromagnetic beam at the first preset measurement position. At the same time, the VNA is controlled to acquire the S-parameters at the corresponding position, completing the measurement at the first position. After the can leaves this measurement position, it continues to move toward the next pair of photoelectric switches, where the same measurement procedure is performed to obtain the S-parameters at the second position.

This process continues until the can leaves the entire detection region, completing the S-parameter measurements at all nine preset positions. The S-parameters obtained from the nine positions are then combined to form a complete data sample. To obtain a sufficient number of samples for constructing the machine-learning dataset, the above procedure is repeated 100 times for each contamination condition.

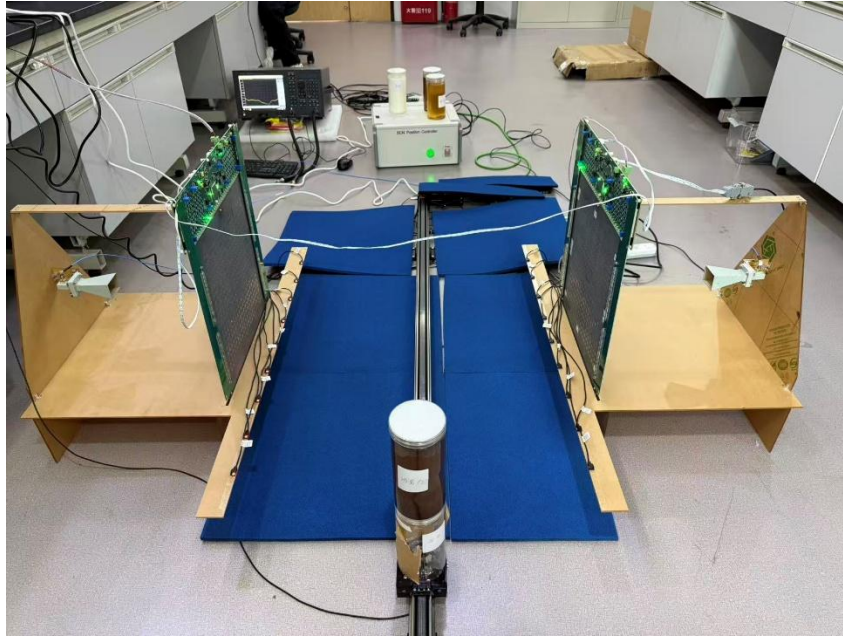


Fig. 6.18. Real-time microwave sensing system for food contaminant detection. Photoelectric switches are placed on both sides of the track to detect the position of the can at different times, while the food under test moves at a constant speed along the fixed track.

In the experiment, glass, metal, wood, and stone are used as contaminants for classification. Photographs of the contaminants used in the experiment are shown in Fig. 6.19.

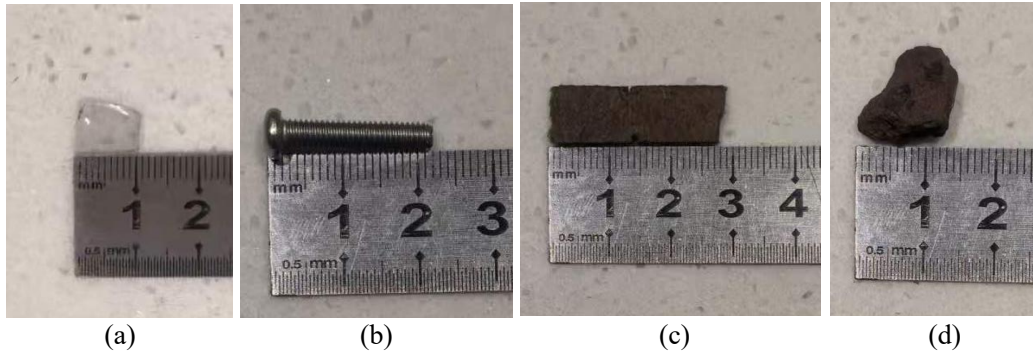


Fig. 6.19. Photographs of the contaminants used in the moving-production-line experiments with the microwave sensing system: (a) glass; (b) metal; (c) wood; (d) stone.

For comparison, the control group uses only the horn antennas as the transmitting and receiving ports to acquire the S-parameters. The classification results obtained by applying the machine-learning model to the two sets of experimental data are shown in Fig. 6.20. In the confusion and recall matrices, C0 denotes the glass-contamination case, C1 denotes the metal-contamination case, C2 denotes the wood-contamination case, and C3 denotes the stone-contamination case.

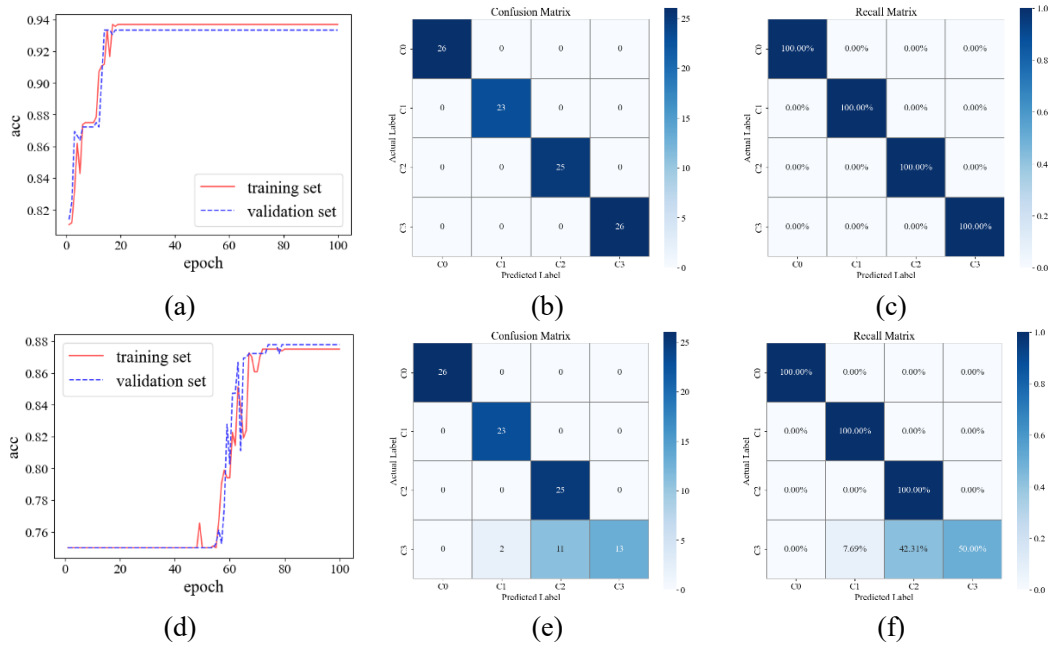


Fig. 6.20. Model training and classification results for the real-time dynamic experiments using the microwave sensing system and the horn-antenna control group. (a)–(c) Accuracy curves during the training process and classification results of the machine-learning model for the microwave sensing system; (d)–(f) Accuracy curves during the training process and classification results of the machine-learning model for the horn-antenna control group.

As shown in Fig. 6.20, the classification accuracy after training reaches 93.3% for the microwave sensing system, whereas the accuracy of the horn-antenna control group is 87.8%. This demonstrates that, under practical moving-production-line conditions, the microwave sensing system provides superior detection performance compared with the conventional horn-antenna configuration.

However, it can also be observed that the classification accuracy in the real-time detection scenario is lower than that obtained in the static experiments presented in Section 6.4.2. This reduction is mainly attributed to several sources of additional measurement error in the dynamic experiment. Specifically, the object may experience slight mechanical vibrations while moving along the track. In addition, a certain time delay exists in both the position detection by the photoelectric switches and the beam-focusing control of the antenna arrays. Consequently, the measurement error in the dynamic experiments is larger than that in the static experiments, resulting in a reduction in the final classification accuracy.

6.5 Summary

Building upon the theoretical foundations for calculating target electromagnetic scattering fields established in the previous five chapters, this chapter develops a microwave sensing system that utilizes electromagnetic scattering information for the detection and classification of food contaminants. Taking edible-oil cans as an example, the feasibility of detecting potential internal

contaminants based on the scattering field of the food is first validated through simulation-based imaging. An antenna array is then designed to focus the electromagnetic waves radiated by the horn antennas, thereby enhancing the detection performance of the system. Finally, experiments are conducted for edible-oil cans containing various types of contaminants, and a machine-learning model is developed based on the experimental data to identify and classify the contamination conditions inside the cans. The experimental results demonstrate that the proposed microwave sensing system achieves superior detection performance.

Furthermore, the proposed system is extended to a real-time detection system. Photoelectric switches are employed to detect the positions of the contaminants in real time, enabling automatic data acquisition on a moving production line. As introduced above, microwave sensing technology based on electromagnetic scattering information has become an active research topic in fields such as biomedical sensing and food inspection. Therefore, Table 6.1 compares the microwave sensing approaches related to the work presented in this thesis.

TABLE 6.1. COMPARISON OF THE PROPOSED MICROWAVE SENSING SYSTEM WITH SIMILAR SYSTEMS. (© 2026 IEEE [155]).

Application	Reference	Purpose	Antenna	Results
Biomedical	[152]	Breast imaging	Transmit-array	Array with a breast imager
Food Inspection	[153]	Contaminant imaging	Horn	The initial design, validation and teesting of a system for packaged food
Food Inspection	[154]	Contaminant imaging	Printed antenna	A new design of a multiantenna system with an illumination balancing algorithm
Food Inspection	[156]	Contaminant imaging	Vivaldi antenna	Imaging system for different food materials
Food Inspection	Proposed	Contaminant identification	Transmit-array	A new design of a transmit-array system with machine learning for identification

Chapter 7

Conclusions and Perspectives

7.1 Conclusions

With the rapid development of aerospace, radar detection, wireless communications, advanced electromagnetic materials, industrial inspection, and other related fields, the scale of numerical analysis for complex electromagnetic problems continues to increase, placing increasingly stringent demands on the computational efficiency, memory consumption, and numerical stability of computational electromagnetics methods. As an important numerical method for solving electromagnetic scattering and radiation problems in open regions, the Method of Moments (MoM) offers high computational accuracy and a rigorous theoretical foundation. However, conventional MoM suffers from high computational complexity and substantial memory requirements when applied to large-scale electromagnetic problems, which severely limits its application in practical engineering.

To address these challenges, this dissertation develops modeling and solution approaches for composite-media electromagnetic scattering problems involving both PEC and dielectric materials based on the Nested Equivalent Source Approximation (NESA) algorithm. Furthermore, to overcome the convergence difficulties of iterative solvers when the system matrix is ill-conditioned and to improve the computational efficiency for problems involving multiple right-hand sides, direct solvers are developed at low to moderate frequencies with NESA and at high frequencies with WNESA for electromagnetic scattering problems. A review of the entire dissertation reveals that the main research contents and contributions can be summarized as follows:

1. Electromagnetic Modeling and Fast Algorithms for Composite Objects Composed of Inhomogeneous dielectric bodies and PEC: This part corresponds to the research presented in Chapter 3. Conventional surface integral equation methods can generally characterize electromagnetic problems involving either perfect electric conductors or homogeneous media, but are not directly applicable

to composite objects containing both metallic and inhomogeneous dielectric regions. To address this issue, SWG and RWG basis functions are introduced to establish an electromagnetic scattering model for composite objects consisting of inhomogeneous media and metals. The compressed representation of the volume integral equation within the NESAs framework is derived, and an equivalent-source-based far-field matrix compression method is developed by combining the volume integral equation with the NESAs algorithm. The proposed method maps the coupling between volumetric unknowns onto the coupling between equivalent surface unknowns, thereby substantially reducing the dimensionality of the system to be solved and enabling the fast solution of electromagnetic scattering problems involving composite objects. Numerical results demonstrate that the computational accuracy can be controlled by adjusting the number of equivalent sources. Moreover, the proposed method effectively reduces the computational complexity of matrix-vector products and achieves time and memory complexities consistent with those reported in [3] for electromagnetic simulations of composite objects.

2. NESAs-Based Direct Solver: This part corresponds to the research presented in Chapter 4. Based on the far-field coupling compression and approximation of NESAs, a fast direct solver is developed for low-frequency electromagnetic problems. By exploiting the low-rank characteristics of far-field matrix blocks, elimination matrices are constructed to perform compressed factorization of admissible blocks, followed by the factorization of the remaining inadmissible blocks using LU decomposition. The system matrix is ultimately approximated as a product of a series of matrices, from which an approximate inverse of the system matrix can be obtained.

The computational error of the proposed direct solver can be controlled by adjusting both the compression accuracy of NESAs and the factorization accuracy of the fill-in matrix blocks generated during the elimination of inadmissible blocks. Therefore, the proposed direct solver provides controllable computational accuracy. Compared with direct solvers based on skeleton decomposition, the introduction of equivalent sources allows coupling matrices between groups with identical relative positions to be reused, thereby further reducing memory requirements. Numerical examples demonstrate the computational performance of the proposed algorithm for complex models, and the algorithm achieves a $O(N)$ complexity for both time and memory.

3. WNESAs-Based Broadband Direct Solver: This part corresponds to the research presented in Chapter 5. Compared with the work presented in Chapter 4, Chapter 5 further develops a fast direct solver for high-frequency electromagnetic problems based on the directional equivalent-source strategy. At the higher hierarchical levels, far-field coupling between groups is compressed within different directional sectors, resulting in direction-dependent cluster bases in the direct solver. This chapter considers the cluster bases associated with all relevant directions and constructs a generalized cluster basis for forming the elimination matrix of admissible blocks. In addition, the indexing strategy for inadmissible

blocks is improved to accommodate the more stringent directional admissibility condition.

The improved factorization algorithm for the high-frequency levels developed in Chapter 5, together with the low-frequency direct solver presented in Chapter 4, forms a unified broadband direct solver, namely, the WNESA-based direct solver. Numerical results demonstrate that the proposed algorithm achieves improved computational accuracy for high-frequency scenarios compared with the direct solver presented in Chapter 4. The proposed algorithm has a time complexity of $O(N^2)$ and a memory complexity of $O(N \log N)$.

4. Application of a Microwave Sensing System Based on Fast Electromagnetic Scattering Algorithms: This part corresponds to the research presented in Chapter 6. This chapter investigates a method for detecting food contaminants using electromagnetic scattering information and develops an antenna-array-based microwave sensing system to improve detection performance. Benefiting from the flexible control of electromagnetic waves provided by the antenna array, the microwave sensing system can focus the electromagnetic beam and scan the target as the food under test moves. Subsequently, a machine-learning model is developed to learn and classify the scattering parameters measured from food samples containing different types of contaminants. Since the microwave sensing system enhances the scattering field generated by the contaminants, it achieves higher classification accuracy than conventional horn-antenna-based systems. Finally, the system is applied to a real-time dynamic production line, where it also demonstrates good detection performance.

In summary, through a series of innovative studies on NESA-based computational electromagnetic algorithms, this thesis effectively improves the efficiency of electromagnetic scattering analysis and computation in complex scenarios. Furthermore, a microwave sensing system is designed based on the fast electromagnetic scattering algorithms, providing both theoretical support and a practical application scenario for the analysis of electromagnetic scattering characteristics in industrial inspection.

7.2 Perspectives

This dissertation has conducted an in-depth investigation into fast algorithms for electromagnetic scattering based on integral equations. Nevertheless, several aspects of the research remain to be further improved and complemented. Future research can be pursued in the following directions:

1. Improving the Computational Efficiency of the WNESA-Based Direct Solver: The WNESA-based direct solver presented in Chapter 5 still has a relatively higher computational and memory complexity than the MLFMA-based iterative solver. Consequently, its computational efficiency may deteriorate when solving problems with a larger number of unknowns. The Extended Admissibility Condition (EAC) [4], [150] can be introduced to reclassify some of the originally inadmissible blocks at higher hierarchical levels as extended admissible blocks

and represent them using compressed low-rank approximations. This strategy can reduce the costly computational overhead associated with the elimination of inadmissible blocks, thereby further improving the computational efficiency of the proposed direct solver.

2. Improving the Robustness of the Direct Solver for Ill-Conditioned Problems: Although the direct solver avoids the convergence difficulties encountered by iterative solvers and provides higher computational efficiency for ill-conditioned problems, its solution accuracy may still be affected when the system matrix is singular or nearly singular. Future research can investigate the incorporation of preconditioning techniques commonly used in iterative solvers into the direct-solver framework, with the aim of improving its robustness and solution accuracy for more severely ill-conditioned problems.

In addition, the direct and iterative approaches can be combined to exploit their respective advantages. For example, within the Domain Decomposition Method (DDM), the inner iterative solver can be replaced by the proposed direct solver, potentially improving the efficiency and robustness of the overall solution process.

3. Developing a Distributed Parallel Implementation for Large-Scale Problems: The direct solver proposed in this dissertation is currently implemented in a local serial manner. With the rapid development of high-performance computing, the parallel scalability of an algorithm has become as important as its computational complexity. Future research can therefore investigate distributed parallelization strategies for the proposed direct solver, with particular emphasis on the parallel processing of matrix factorization, fill-in matrix blocks, and multi-right-hand-side solutions. The resulting parallel implementation could be deployed on high-performance computing systems to further extend the proposed algorithm to larger-scale electromagnetic scattering problems.

4. For the microwave sensing system proposed in this thesis for food contaminant detection, the current machine-learning model requires offline training and classification. Developing a hardware system capable of real-time online classification and detection would facilitate the practical deployment of the microwave sensing system on food production lines.

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