Note on Efron's monotonicity property under given copula structures.

Abstract . Given a multivariate random vector, Efron's marginal monotonicity (EMM) refers to the stochastic monotonicity of the variables given the value of their sum. Recently, based on the notion of total positivity of the joint density of the vector, Pellerey and Navarro (2021) obtained sufficient conditions for EMM when the monotonicity is in terms of the likelihood ratio order. We provide in this paper new sufficient conditions based on properties of the marginals and the copula. Moreover, parametric examples are provided for some of the results included in Pellerey and Navarro (2021) and in the present paper.

1 Introduction and background

Given a random vector of independent continuous marginals with logconcave densities, Efron (1965) studied the stochastic monotonicity of the marginals given the value of their sum, obtaining the following result.

Proposition 1 (Efron (1965))

Let $X_1, X_2, \cdots, X_n$ be n independent random variables with ILR densities, let $S = \sum_{i=1}^{n} X_i$ be their sum, and let $\Phi(x_1, x_2, \cdots, x_n)$ be a real measurable

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function, increasing in each of its arguments. Then, the function $s \mapsto E\left(\Phi\left(X_1, X_2, \cdots, X_n\right) \mid S = s\right)$ *is increasing.*

Throughout this paper, the term “increasing” is used for “non-decreasing” and “decreasing” is used for “non-increasing”. Recall that a continuous random variable X having density f is said to have the *increasing likelihood ratio* (ILR) property if $\frac{f(x+y)}{f(x)}$ decreases in x for all $y \geq 0$, i.e., if $\log f(x)$ is concave. X is said to have the *increasing proportional likelihood ratio* ($IPLR$) property if $f(\lambda x)/f(x)$ is increasing in x for any positive constant $\lambda < 1$. Equivalently, a random variable X with density f is $IPLR$ if and only if $x\ \eta(x)$ increases in x, where $\eta(x) = -f'(x)/f(x)$ (see Oliveira and Torrado (2015)). Note that ILR implies $IPLR$, but the reverse does not hold (Ramos and Sordo (2001)). Let us also recall that, given X and Y two continuous random variables with respective distribution functions F, G and densities f, g, respectively, X is said to be smaller than Y in the *usual stochastic order* ($X \leq_{st} Y$) if $F(x) \geq G(x)$ for all $x \in \mathbb{R}$. The order $X \leq_{st} Y$ holds if and only if, for all increasing functions $\phi : \mathbb{R} \to \mathbb{R}$, $\mathrm{E}[\phi(X)] \leq \mathrm{E}[\phi(Y)]$, provided that these expectations exist. Analogously, X is said to be smaller than Y in the *likelihood ratio order* ($X \leq_{lr} Y$) if $f(x)/g(x)$ is decreasing in the union of the supports (Shaked and Shanthikumar (2007)).

From now on, based in Proposition 1, we will refer as “Efron’s strong monotonicity” (ESM) to the monotonicity of $s \mapsto \{(X_1, X_2, \cdots, X_n) \mid S = s\}$ in terms of any stochastic order, and “Efron’s marginal monotonicity”(EMM) to the monotonicity of $s \mapsto \{X_i \mid S = s\}$. Efron’s monotonicity and its subsequent generalizations have been of great interest in different areas, as economics, combinatorial probability, dependence modeling and statistical theory. For a list of references on its applications, the interested reader may consult Saumard and Wellner (2018); Pellerey and Navarro (2021).

The results obtained in Efron (1965) have been extended in several ways. Lehmann (1966) showed that the conditions stated in Proposition 1 imply EMM in terms of the lr-order. More recently, Saumard and Wellner (2018) extended Efron’s results to bivariate vectors with non-independent variables, providing conditions in terms of the second derivatives of $-log f(x, y)$, which imply ESM and EMM in the usual stochastic order. In the same framework, Oudghiri (2021) provided sufficient conditions for an stronger ESM and EMM assumption, considering not only that $\phi(s) = E\left(\Phi\left(X_1, X_2, \cdots, X_n\right) \mid S = s\right)$ increases in s, but also that $\alpha(s)\phi(s)$ increases in s, for some functions α. Also in the bivariate setting, for non-independent vectors, Pellerey and Navarro (2021) give sufficient conditions which imply ESM in the usual stochastic order. They also generalize the result in Lehmann (1966), connecting the EMM monotonicity in the lr-order to the notion of total positivity.

Let us recall that, in the bivariate framework, a function $p : \mathbb{R}^2 \to \mathbb{R}^+$ is said to be *totally positive of order 2* (TP_2) if, for $x_1 < x_2$ and $y_1 < y_2$, it is verified that $p(x_2, y_2)p(x_1, y_1) \geq p(x_1, y_2)p(x_2, y_1)$. Let us note that, for such p, if at each point (x, y), the second order partial derivative $\frac{\partial^2}{\partial x \partial y} \log(p(x, y))$ exists, then p is TP_2 if and only if $\frac{\partial^2}{\partial x \partial y} \log(p(x, y)) \geq 0$ (Karlin (1967)).

Pellerey and Navarro (2021) provide the following results.

Proposition 2 (Pellerey and Navarro (2021))

Let the vector (X_1, X_2) have a joint density f. Then, the following conditions are equivalent:

1. *The function $f(x, s - x)$ is TP_2 in (x, s);*
2. *$[X_1 \mid S = s_1] \leq_{lr} [X_1 \mid S = s_2]$ whenever $s_1 \leq s_2$;*
3. *$[S \mid X_1 = x_1] \leq_{lr} [S \mid X_1 = x_2]$ whenever $x_1 \leq x_2$.*

Proposition 3 (Pellerey and Navarro (2021))

Let the vector (X_1, X_2) have a joint density f. If $f(x_1, x_2)$ is TP_2 in (x_1, x_2) and logconcave in x_2 (respectively, x_1) for every x_1 (respectively, x_2), then $f(x, s - x)$ ($f(s - x, x)$) is TP_2 in (x, s).

Let $\mathbf{X} = (X_1, X_2)$ be a random vector with joint density f and survival copula $\hat{C}$. Let $\hat{c}$ the second mixed partial derivative of $\hat{C}$ (here $\hat{c}$ is referred to as the density of the survival copula $\hat{C}$). As pointed out in Example 2.4 in Pellerey and Navarro (2021), $f(x_1, x_2)$ is TP_2 in (x_1, x_2) if and only if $\hat{c}(u_1, u_2)$ is TP_2. However, the condition $\hat{c}(u, z - u)$ is TP_2 in (u, z) does not imply that $f(x, s - x)$ is TP_2 in (x, s). This suggests to find conditions on $\hat{c}$ and the marginals implying that $f(x_1, x_2)$ is TP_2 in (x_1, x_2). This is what we do in Section 2 below. In Section 3, we provide a list of copulas that, when joined to exponential or uniform marginals, imply that $f(x, s - x)$ is TP_2 in (x, s). The final Section 4, instead, is devoted to an application of Proposition 3 in the context of generalized order statistics (GOSs).

2 Efron's marginal monotonocity in terms of the copula

Given a random vector (X_1, X_2) with exponential marginals and joint density f, our first result provides a conditions on the density of the corresponding survival copula $\hat{c}$ that ensures that $f(x, y - x)$ is TP_2 in (x, y). Note that $f(x_1, x_2) = \hat{c}(\bar{F}_1(x_1), \bar{F}_2(x_2))f_1(x_1)f_2(x_2)$.

Proposition 4 *Let the vector (X_1, X_2), with $X_1, X_2 \sim \exp(\lambda)$ have joint density f and density of the survival copula $\hat{c}$. If $\hat{c}(u, v/u)$ is TP_2 in (u, v), for all $0 < u < 1, 0 < v < u$, then $f(x, y - x)$ is TP_2 in (x, y).*

Proof The function $f(x, s-x)$ is TP_2 in (x, s) if

$$\frac{f(x, s_2 - x)}{f(x, s_1 - x)} = \frac{\hat{c}(\bar{F}_1(x), \bar{F}_2(s_2 - x))f_1(x)f_2(s_2 - x)}{\hat{c}(\bar{F}_1(x), \bar{F}_2(s_1 - x))f_1(x)f_2(s_1 - x)} = \frac{\hat{c}(\bar{F}_1(x), \bar{F}_2(s_2)/\bar{F}_2(x))}{\hat{c}(\bar{F}_1(x), \bar{F}_2(s_1)/\bar{F}_2(x))}$$

is increasing in x for all $s_1 < s_2$. The monotonicity follows from the fact that $\bar{F}_2(s-x) = \frac{\bar{F}_2(s)}{\bar{F}_2(x)}$, $\frac{f_2(s_2-x)}{f_2(s_1-x)}$ is constant in x and $\frac{\hat{c}(u,v_1/u)}{\hat{c}(u,v_2/u)}$ is decreasing in u for $v_1 < v_2$. This last condition is equivalent to say that $\hat{c}\,(u, v/u)$ is TP_2 in (u, v) , for all $0 < u < 1, 0 < v < u$. □

In Proposition 3, the joint density function $f(x_1, x_2)$ is required to be logconcave in x_2 for every x_1, which is equivalent to say that $\{X_2|X_1 = x_1\}$ is ILR for all x_1. This condition can be weakened when it is expressed in terms of the density of the survival copula whenever the marginal X_2 is exponential.

Proposition 5 *Let (X_1, X_2) be a random vector with joint density f and density of the survival copula $\hat{c}$. Let $X_2 \sim \exp(\lambda)$. If $\hat{c}\,(u, v)$ is TP_2 in (u, v) and $\{U_2|U_1 = u_1\}$ is $IPLR$, where $U_i \sim \bar{F}_i(X_i)$ for $i = 1, 2$ for all u_1, then $f(x, y-x)$ is TP_2 in (x, y).*

Proof Since $\hat{c}\,(u, v)$ is TP_2 in (x_1, x_2), then $f\,(x_1, x_2)$ is TP_2 in (x_1, x_2). By Proposition 3, it remains to see that $f(x_1, x_2)$ is logconcave in x_2 for all x_1. This is the same as proving that

$$\frac{f(x_1, x_2 + y)}{f(x_1, x_2)} = \frac{\hat{c}(\bar{F}_1(x_1), \bar{F}_2(x_2 + y))f_2(x_2 + y)}{\hat{c}(\bar{F}_1(x_1), \bar{F}_2(x_2))f_2(x_2)} = \frac{\hat{c}(\bar{F}_1(x_1), e^{-\lambda y}\bar{F}_2(x_2))}{\hat{c}(\bar{F}_1(x_1), \bar{F}_2(x_2))}$$

is decreasing in x_2 for all x_1 and $y \geq 0$, where we have used that $X_2 \sim \exp(\lambda)$. Taking into account that $f_{\{U_2|U_1=v\}}(u) = \hat{c}(v, u)$, this follows from the fact that $\{U_2|U_1 = u_1\}$ is $IPLR$ for all u_1, which is equivalent to say that $f_{\{U_2|U_1=v\}}(\alpha u)/f_{\{U_2|U_1=v\}}(u)$ increases in u for all v and any $0 < \alpha \leq 1$. □

It should be noted that the sufficient conditions in Proposition 4 neither imply, nor are implied, by the conditions in Proposition 5. For example, if $X_1, X_2 \sim \exp(\lambda)$ and $\hat{C}$ is the Ali-Mikhail-Haq copula with parameter $\theta \leq 1$ (see Subsection 3.3) then (X_1, X_2) satisfies the conditions on Proposition 4 but not those in Proposition 5. Similarly, if $X_2 \sim \exp(\lambda)$, $X_1 \nsim \exp(\lambda)$ and $\hat{C}$ is a Clayton copula (see Subsection 3.2) then (X_1, X_2) satisfies the conditions on Proposition 5 but not those in Proposition 4.

3 Examples of copulas

Given a copula C (joint distribution function of (U_1, U_2)) with density c, we consider the following properties:

- (P1) $c(u, s-u)$ is TP_2 in (u, s) for $0 < u < 1, u < s < 1 + u$.

- (P2) $c(u,v)$ is TP_2 in (u,v) for $0 < u < 1, 0 < v < 1$.
- (P3) $c(u,v)$ is logconcave in v for all u.
- (P4) $c(u,v/u)$ is TP_2 in (u,v) for $0 < u < 1, 0 < v < u$.
- (P5) $v\eta_{\{U_2|U_1=u\}}(v) = -(v\frac{\partial}{\partial v}c(u,v))/c(u,v)$ increases in v for all u.

Given $\mathbf{X} = (X_1, X_2)$ with copula C and survival copula $\hat{C}$, if any of these conditions hold:

- $X_1, X_2 \sim U(0,1)$ and C verifies (P1) (Proposition 2),
- $X_1, X_2 \sim U(0,1)$ and C verifies (P2) and (P3) (Proposition 3),
- $X_1, X_2 \sim \exp(\lambda)$ and $\hat{C}$ verifies (P4) (Proposition 4),
- $X_1 \sim \exp(\lambda)$ and $\hat{C}$ verifies (P2) and (P5) (Proposition 5),

then $f(x, y-x)$ is TP_2 in (x,y). Next, we provide examples of parametric families of copulas satisfying the property (P2) that also satisfy some of the other properties. Note that this is not the general case, for example, copulas (4.1.4) (Gumbel-Hougaard copula) and (4.1.2) in Nelsen (2007) satisfy (P2) but not (P1), (P3) (P4) or (P5).

3.1 Farlie-Gumbel-Morgenstern copula

Let consider the copula given by $C_\theta(u,v) = uv(1+\theta(1-u)(1-v))$, for $\theta \in [-1,1]$, $u, v \in [0,1]$. The copula density $c_\theta(u,v) = 1+\theta(1-2v)(1-2u)$ is TP_2 for $\theta \in [0,1]$, thus (P2) holds and, for such values, $\frac{d}{ds}\frac{d}{du}(\log(c_\theta(u,s-u)) \geq 0$ for all $0 < u < 1, u < s < u+1$, and (P1) holds. Moreover, $\frac{d^2}{dv^2}(\log(c_\theta(u,v)) \leq 0$, therefore c_θ verifies (P3). Analogously, as $\frac{d}{dv}\frac{d}{du}(\log(c_\theta(u,v/u))) \geq 0$ for all $\theta \in [0,1]$, (P4) is verified for such values. Finally, as $\frac{d}{dv}\left(\frac{-v\frac{\partial}{\partial v}c(u,v)}{c(u,v)}\right) \geq 0$ if and only if $(1-2u)\theta \geq 0$, (P5) does not hold for all u and $\theta \neq 0$.

3.2 Clayton copula

Let consider the copula 1 in Table 4.1 in Nelsen (2007), given by $C_\theta(u,v) = (u^{-\theta}+v^{-\theta}-1)^{-1/\theta}$ for $\theta > -1$ with $\theta \neq 0$ (note that, when θ tends to zero, this is the independence copula, which trivially verifies (P1)-(P5), as the density is 1). The density is given by $c_\theta(u,v) = u^{-1-\theta}v^{-1-\theta}\left(u^{-\theta}+v^{-\theta}-1\right)^{-2-\frac{1}{\theta}}(1+\theta)$. It is easy to see that, for $\theta > 0$, (P2) holds, but (P1) does not. Consequently, (P3) also fails to be satisfied (because (P2) and (P3) imply (P1)). It can be easily computed that both $\frac{d}{dv}\frac{d}{du}(\log(c_\theta(u,v/u)))$ and $\frac{d}{dv}\left(\frac{-v\frac{\partial}{\partial v}c(u,v)}{c(u,v)}\right)$ are positive for all $\theta > 0$, so, for such values, (P4) and (P5) holds. Here, we can see an example of the fact that $IPLR$ does not imply ILR. Considering a vector (U_1, U_2) with a Clayton copula, there are values of u_1 for which $\{U_2|U_1 = u_1\}$ is $IPLR$ but it is not ILR.

3.3 Ali-Mikhail-Haq copula

Let us now consider the copula 3 in Table 4.1 in Nelsen (2007), given by $C_\theta(u,v) = \frac{uv}{(1-\theta(1-u)(1-v))}$ for $\theta \in [-1,1)$. The density of the copula, given by

$$c_\theta(u,v) = \frac{1+\theta(u+v+uv-2+(1-u)(1-v)\theta)}{(1-(1-u)(1-v)\theta)^3}$$

is TP_2 for $\theta \in [0,1)$. Computing $\frac{d}{ds}\frac{d}{du}(\log(c_\theta(u,s-u)))$ and $\frac{d^2}{dv^2}(\log(c_\theta(u,v)))$, we see that, for all $\theta \in [0,1)$, $c(u,s-u)$ is TP_2 in (u,s) but $c(u,v)$ is not logconcave in v for all u. It can be also shown that $\frac{d}{dv}\frac{d}{du}(\log(c_\theta(u,v/u)))$ is positive for all $0 < u \leq 1$ and $0 \leq v \leq u$ if and only if $\theta \in [0,1/2]$ (which means that (P4) holds whenever $\theta \in [0,1/2]$). Finally, (P5) does not hold.

3.4 Frank copula

Given the copula $C_\theta(u,v) = -\frac{1}{\theta}\log\left(1+\frac{(e^{-u\theta}-1)(e^{-v\theta}-1)}{e^{-\theta}-1}\right)$, for $\theta \in \mathbb{R}\setminus\{0\}$ (copula 5 in Table 4.1 in Nelsen (2007)), the density is given by

$$c_\theta(u,v) = \frac{e^{(1+u+v)\theta}\left(e^\theta-1\right)\theta}{\left(e^{(u+v)\theta}-e^\theta\left(e^{u\theta}+e^{v\theta}-1\right)\right)^2}.$$

This function is TP_2 in (u,v) if $\theta \geq 0$ and, since $\frac{d^2}{dv^2}(\log(c_\theta(u,v))) < 0$ for all $\theta \neq 0$, (P3) and (P1) hold for $\theta \geq 0$. It can be verified that $\frac{d}{dv}\frac{d}{du}(\log(c_\theta(u,v/u)))$ is positive (and therefore (P4) holds) for all $0 < u \leq 1$ and $0 \leq v \leq u$ if and only if $\theta \in [0,1]$. Property (P5) does not hold whatever θ.

The results of this section can be summarized in Table 1:

	P1	P2	P3	P4	P5
FGM copula	$\theta \in [0,1]$	$\theta \in [0,1]$	$\theta \in [-1,1]$	$\theta \in [0,1]$	
Clayton copula		$\theta > 0$		$\theta > 0$	$\theta > 0$
AMH copula	$\theta \in [0,1)$	$\theta \in [0,1)$		$\theta \in [0,\frac{1}{2}]$	
Frank copula	$\theta > 0$	$\theta > 0$	$\theta \in \mathbb{R}\setminus\{0\}$	$\theta \in (0,1]$	

Table 1 Values of the parameters under which properties P1 to P5 are satisfied.

4 Other examples: Generalized Order Statistics (GOSs)

In this Section, we aim to provide further examples of parametric families where Proposition 3 can be applied. With this purpose, we introduce the notion of generalized order statistics (GOSs) (see Kamps (1995)).

Definition 1 Let $n \in \mathbb{N}, k \geq 1, m_1, \ldots, m_{n-1} \in \mathbb{R}, M_r = \sum_{j=r}^{n-1} m_j, 1 \leq r \leq n-1$, be parameters such that $\gamma_r = k + n - r + M_r \geq 1$ for all $r \in \{1, \ldots, n-1\}$, and let $\tilde{m} = (m_1, \ldots, m_{n-1})$ if $n \geq 2$ ($\tilde{m} \in \mathbb{R}$ arbitrary, if $n = 1$). If the random variables $U_{(r,n,\tilde{m},k)}$, $r = 1, \ldots, n$, possess a joint density of the form

$$h(u_1, \ldots, u_n) = k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{j=1}^{n-1} \left(1 - u_j\right)^{m_j} \right) (1 - u_n)^{k-1}$$

defined on $0 \leq u_1 \leq \ldots \leq u_n \leq 1$, then they are called GOSs. For a given distribution function F, the random variables $X_{(r,n,\tilde{m},k)} = F^{-1}\left(U_{(r,n,\tilde{m},k)}\right)$, for $r = 1, ..n$, are called the GOSs based on F.

Several models of ordered random variables are included in this model:

- Considering $m_i = 0$ for all $i = 1, \ldots, n-1$ and $k = 1$, we get the order statistics from a distribution F.
- Taking $m_i = -1$ for all $i = 1, \ldots, n-1$ and $k = 1$, we get the first n record values from a sequence of random variables with distribution F (or the first n epoch times of a nonhomogeneous Poisson process).
- A generalization of the previous model is the case in which $k \in N$, resulting in the so-called k-records.
- GOS also includes some other models of interest such as sequential order statistics and progressively type-II censored order statistics.

Proposition 6 *Let F be an absolutely continuous distribution function with logconcave density f. Let $(X_{(i,n,\tilde{m},k)}, X_{(j,n,\tilde{m},k)})$, $1 \leq i < j \leq n$, a bivariate random vector of GOSs from F. Assume that any of the following conditions is satisfied:*

(a) $k \geq 1, m_i \geq 0$.
(b) $k > 0, m_i \geq -1$ *and the failure rate function* $\lambda(\cdot)$ *of* F *is log-concave.*

Then, $\{X_{(i,n,\tilde{m},k)} | X_{(i,n,\tilde{m},k)} + X_{(j,n,\tilde{m},k)} = t\}$ *increases in* t *in the likelihood ratio order.*

Proof It is well-known that any random vector of GOS is MTP_2 (Belzunce et al. (2005)). Then, the bivariate vector $(X_{(i,n,\tilde{m},k)}, X_{(j,n,\tilde{m},k)}), 1 \leq i \leq j \leq n$, is TP_2. In order to obtain the result, by Proposition 2.4 in Pellerey and Navarro (2021), it is sufficient to prove that $\{X_{(j,n,\tilde{m},k)} | X_{(i,n,\tilde{m},k)} = t\}$ has a logconcave density function (or it is ILR). It is known that

$$\{X_{(j,n,\tilde{m},k)} \mid X_{(i,n,\tilde{m},k)} = t\} \simeq_{\text{st}} \{X_{(j-i,n-i,\tilde{m}',k)} \mid X_{(1,n-r+1,\tilde{m}',k)} > t\},$$

where $\tilde{m}' = (m'_1, \ldots, m'_{n-r})$ is such that $m'_j = m_{n-j}$ for $j = 1, \ldots, n-r$. If condition (a) or (b) holds, then $X_{(j-i,n-i,\tilde{m}',k)}$ has a logconcave density (Chen et al. (2009)) and it is easy to see that logconcavity is preserved by right truncations, that is, $\{X_{(j-i,n-i,\tilde{m}',k)} \mid X_{(1,n-r+1,\tilde{m}',k)} > t\}$ has also a logconcave density and the assertion follows. $\square$

In particular:

- If f is logconcave and $X_{i:n}, X_{j:n}$ are two order statistics with $1 \leq i < j \leq n$ then, $\{X_{i:n} | X_{i:n} + X_{j:n} = s\}$ increases in s in the lr-ratio order. In particular, it holds for $\{\min(X_1, X_2) | X_1 + X_2 = s\}$. It can be also checked that, when $n = 2$, $\{\max(X_1, X_2) | X_1 + X_2 = s\}$ increases in s in the lr-ratio order.
- If X_{L_n}, X_{L_m}, $n < m$ are two record values of F and f and the failure rate function $\lambda(t)$ are both logconcave, then $\{X_{L_n} | X_{L_n} + X_{L_m} = s\}$ increases in s in the likelihood ratio order.
- If $X_{L_n^k}, X_{L_m^k}$, $n < m$ are two k-record values of F, and $f(.)$ and $\lambda(.)$ are both logconcave, then $\{X_{L_n^k} | X_{L_n^k} + X_{L_m^k} = s\}$ increases in s in the lr-ratio order.

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