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Analysis of transversely isotropic compressible and nearly-incompressible soft material structures by high order unified finite elements / Chiaia, Piero; Pagani, Alfonso; Cinefra, Maria; Carrera, Erasmo. - In: MECHANICS OF ADVANCED MATERIALS AND STRUCTURES. - ISSN 1537-6494. - (2023). [10.1080/15376494.2023.2273962]

Availability:

This version is available at: 11583/2983830 since: 2023-11-14T10:55:04Z

Publisher:

Taylor & Francis

Published

DOI:10.1080/15376494.2023.2273962

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1 Analysis of transversely isotropic compressible and
2 nearly-incompressible soft material structures
3 by high order unified finite elements

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5 **Abstract** *This work proposes high-order beam (1D) and plate (2D) finite element models for the large*
6 *strain analysis of compressible and incompressible transversely isotropic hyperelastic media, defined*
7 *within the Carrera Unified Formulation (CUF) framework. The strain energy density function adopted*
8 *in fiber-reinforced hyperelastic materials modeling is presented and expressed in terms of invariants*
9 *and pseudo-invariants of the right Cauchy-Green strain tensor. The explicit expression of the tangent*
10 *elasticity tensor is derived through the assumption of coupled formulation of strain energy functions.*
11 *Refined fully nonlinear beam and plate models are defined in a total Lagrangian formulation, deriving*
12 *the governing equations of the nonlinear static analysis through the Principle of Virtual Displacements*
13 *in terms of fundamental nuclei, in resulting expressions of internal and external forces vectors, and*
14 *tangent stiffness matrix independent of kinematic models and approximation theories adopted. The*
15 *iterative Newton-Raphson linearization scheme coupled with the arc-length constraint is adopted to*
16 *obtain actual numerical solutions. Different benchmark analyses in hyperelasticity are performed to*
17 *assess the capabilities of our proposed model, analyzing the three-dimensional stress field for moderate*
18 *to large strain states and comparing actual numerical results with exact closed-form solutions or results*
19 *available in the literature, demonstrating the capabilities and reliability of CUF models in the analysis*
20 *of fiber-reinforced soft materials and structures.*

21
22 **Keywords** Hyperelasticity; Fiber-reinforced hyperelastic materials; Soft structures; Unified Formula-
23 tion; Compressible hyperelastic models; Path-following methods.

24
25 **1 Introduction**

26 Bio-inspired material, soft rubber-like cross-ply, or multilayered biological tissues have been subjected
27 to intense studies in the last decades. Of particular relevance, the numerical simulation of mechanical
28 behavior of soft tissue is an actual challenging field in computational mechanics and fluid dynamics
29 since it allows a wide range of investigations to better understand the real nature of biological tis-
30 sues. In this framework, materials involved in muscular and cardiac tissue modeling deal with strong
31 anisotropy. Typically these are multilayered materials, and each sub-layer exhibits direction-dependent
32 mechanical properties and typical of fiber-reinforced materials, such as collagen fibers, muscular tissue,

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33 and blood vessels. For such materials, the mechanical behavior is described by transversely isotropic
34 hyperelastic constitutive law, for which direction-dependent mechanical properties are taken into ac-
35 count.

36 The enhanced elastic properties given by the hyperelastic behavior and the microstructural fiber-
37 reinforcement are the key properties of biological tissue. Fok *et al.* [1] analyzed multilayered arterial
38 cross-section by morpho-elasticity arguments. Holzapfel *et al.* [2] presented one of the most accurate
39 hyperelastic models in biological tissue modeling, the HGO (Holzapfel-Gasser-Ogden) model, consider-
40 ing transversely isotropic and orthotropic hyperelasticity, presenting a novel constitutive law for arterial
41 tissue modeling. A mathematical treatment of pseudoelastic stress-strain relations in hyperelasticity
42 has been presented by Fung *et al.* [3]. Some of the most remarkable hyperelastic models for rubber
43 and biological tissues have been validated by Ogden [4] and Gent [5], the last one later analyzed also
44 by Puglisi *et al.* [6].

45 In a general scenario, constitutive equations for anisotropic hyperelastic materials are well-established
46 transversely isotropic and orthotropic hyperelastic constitutive models are defined including large dis-
47 placements and strains formulation and a nonlinear stress-strain relation, both embedded in the classi-
48 cal strain energy function approach to hyperelastic material modeling. The availability of mathematical
49 models of hyperelastic materials allows the implementation of numerical procedures for simulations of
50 biological tissues. Due to the strongly nonlinear behavior of mathematical models, analytical solutions
51 are few and limited to very simple cases; for this reason, numerical procedures based on Finite Element
52 Method are one of the most common approaches. Arbind *et al.* [7] presented a general higher-order
53 shell theory for compressible hyperelastic materials. Amabili *et al.* [8] presented a finite element
54 model based on higher-order shell theories for the analysis of biological materials. Also Amabili *et al.*
55 [9] presented some experimental and numerical results on the characterization of human aortas.
56 Thin fiber-reinforced hyperelastic shells based on Reissner-Mindlin kinematics have been analyzed by
57 Balzani *et al.* [10]. Fiber-reinforced elastomers and their characterization by classical tension test have
58 been studied in a finite element scenario by Brown *et al.* [11]. Canales *et al.* [12] presented a nonlinear
59 optimization algorithm in the characterization of mechanical properties possessed by anisotropic
60 hyperelastic behavior modeled with the HGO model. Beheshti *et al.* [13] presented a general high-
61 order shell model for the analysis of compressible transversely isotropic materials. Zdunek *et al.* [14]
62 proposed a hybrid finite element formulation for transversely isotropic hyperelasticity.

63 In this work, we propose a new general finite element formulation for the analysis of transversely
64 isotropic (or continuous fiber-reinforced) hyperelastic materials based on Carrera Unified Formulation
65 (CUF). In this framework, the three-dimensional displacement field is expressed in terms of a recursive
66 index notation coupling kinematic models and approximation theories along the cross-section (1D
67 beam models) or thickness (2D plate/shell models), allowing the definition of matrix-form physical
68 quantities appearing in nonlinear governing equations in terms of fundamental nuclei independent
69 of the polynomial approximation employed in the definition of the finite element. The theoretical
70 framework of CUF and the definition of higher-order structural theories is presented in Carrera *et al.*
71 [15]. The accuracy of CUF models in the computation of accurate three-dimensional stress distributions
72 is established in many works as [16, 17, 18]. Higher-order CUF models have been extended more
73 recently to the geometrical nonlinear analysis of isotropic and composite structures [19, 20, 21] but
74 more recently the material nonlinearities of the hyperelastic constitutive law has been included in the
75 fully nonlinear beam, plate and hexahedral solid models as done in [22, 23, 24].

76 The present work is structured as follows (i) first, the mathematical formulations of kinematics,
77 hyperelastic constitutive law written under an invariant formulation and the tangent elasticity tensor
78 are presented in Section 2.1; (ii) second, unified CUF-based 1D and 2D models are discussed in Section
79 3; (iii) subsequently, we exploit the nonlinear governing equations by means of the Principle of Virtual
80 Displacements, defining the internal and external forces vector and tangent stiffness matrix fundamental
81 nuclei, presenting also the numerical iterative scheme employed in Section 4; (iv) numerical results
82 obtained by the present implementation of unified beam and plated models are presented in Section
83 5, establishing the capabilities of the present models in the case of compressible and incompressible
84 fiber-reinforced materials; (v) finally, we discuss about the main conclusions evinced in Section 6.

2 Constitutive law

2.1 Kinematics and strain measures

Figure 1 shows the undeformed Ω_0 and deformed configuration Ω of a continuum body in space, where $P_0 = (x^0, y^0, z^0)$ stands for a material point in the undeformed configuration and P the associated point in the actual deformed configuration. As the continuum body evolves during time, it occupies a continuous sequence of regions of the euclidean space. These regions occupied by the body at a certain time t are the *configurations* of the body. Starting from the reference configuration Ω_0 , the configuration of the body at the generic instant t is called the *current configuration*.

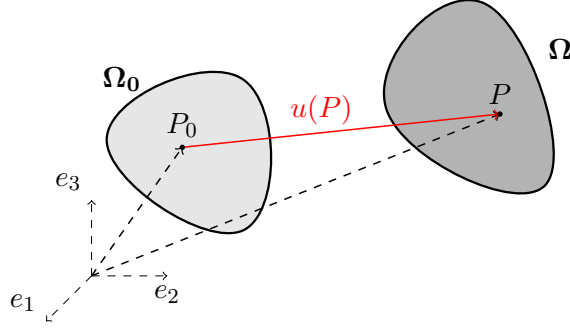


Figure 1: Reference and actual configuration of a deformable body.

The current position of the generic material point is described by the *deformation function*, that maps material point in the actual configuration to the related position in the reference configuration

$$\mathbf{P} = \mathbf{f}(\mathbf{P}_0) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 = f_1(x^0, y^0, z^0)\mathbf{e}_1 + f_2(x^0, y^0, z^0)\mathbf{e}_2 + f_3(x^0, y^0, z^0)\mathbf{e}_3 \quad (1)$$

thereafter, the classical strain measures commonly adopted in continuum mechanics, namely the deformation gradient $\mathbf{F} = \partial\mathbf{P}/\partial\mathbf{P}_0$ and the right Cauchy-Green strain tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, are automatically defined. Typically, isotropic hyperelastic mathematical models are defined starting from the invariants of the right Cauchy-Green strain tensor, to satisfy objectivity arguments and independence with respect to the reference frame, referring then to

$$I_1 = \text{tr}(\mathbf{C}) \quad (2)$$

$$I_2 = \frac{1}{2}((\text{tr}(\mathbf{C}))^2 - \text{tr}(\mathbf{C}^2)) = \text{tr}(\text{cof}(\mathbf{C})) \quad (3)$$

$$I_3 = \det(\mathbf{C}) = \det(\mathbf{F}^T\mathbf{F}) = J^2 \quad (4)$$

where $\text{tr}(\cdot)$ and $\det(\cdot)$ are the trace and the determinant operators, and $\text{cof}(\cdot)$ is the matrix of the cofactors. Hyperelastic anisotropy is studied by analyzing the mechanical behavior of the material when direction-dependent constitutive laws are provided. In this work, fiber-reinforced materials (transversely isotropic materials) will be considered. For such materials, supposing that $\mathbf{a}_0 = (a_x, a_y, a_z)^T$ is the single fiber-reinforcement direction, the dependence on the preferential direction of the mechanical response is embedded in the model considering two additional *pseudo-invariants* [25]

$$I_4 = \mathbf{a}_0 \cdot \mathbf{C}\mathbf{a}_0 \quad (5)$$

$$I_5 = \mathbf{a}_0 \cdot \mathbf{C}^2\mathbf{a}_0 \quad (6)$$

One can note that I_4 and I_5 are invariants only under a rotation with the respect of fiber axis.

110 2.2 Hyperelastic constitutive law in terms of invariants

111 The mechanical behavior of continuously fiber-reinforced soft materials is modeled in the classical
 112 hyperelastic framework based on strain energy functions. Considering a transversely isotropic hy-
 113 perelastic material, the classical strain energy function Ψ for isotropic materials (depending on the
 114 invariants (I_1, I_2, I_3) of right Cauchy-Green tensor \mathbf{C}) is extended embedding the dependence on the
 115 preferential direction is embedded in the definition. If, in a Cartesian reference frame, the vector of
 116 the fiber direction is $\mathbf{a}_0 = (a_x, a_y, a_z)^T$, the strain energy function is now expressed as a function of
 117 two tensorial quantities

$$\Psi = \Psi(\mathbf{C}, \mathbf{a}_0 \otimes \mathbf{a}_0) \quad (7)$$

118 where $(\cdot) \otimes (\cdot)$ stands for the dyadic product (tensor product) operator. The dependence on the special
 119 direction of the continuous fiber-reinforcement is expressed, in the three-dimensional space, by the
 120 structural tensor $\mathbf{a}_0 \otimes \mathbf{a}_0$ that modifies the constitutive equation. Due to the independence of the
 121 strain energy function with respect to the reference frame (objectivity argument), Ψ is expressed as a
 122 function of the three classical principal invariants of \mathbf{C} and two additional *pseudo-invariants*, in which
 123 the dependence of the fiber-reinforcement direction is incorporated and objectivity is still verified

$$\Psi = \Psi(I_1(\mathbf{C}), I_2(\mathbf{C}), I_3(\mathbf{C}), I_4(\mathbf{C}, \mathbf{a}_0), I_5(\mathbf{C}, \mathbf{a}_0)) \quad (8)$$

124 In nearly-incompressible hyperelastic materials, the volume ratio coefficient defined as $J = \det(\mathbf{F})$ is
 125 approaching the unity, thus also I_3 .

126 In literature, commonly adopted models refer to the *decoupled* formulation of strain energy functions,
 127 namely Ψ is written as the sum of a purely volumetric component, a purely isochoric component, and
 128 an additional anisotropic one

$$\Psi = \Psi_{vol}(J) + \bar{\Psi}_{iso}(\bar{I}_1, \bar{I}_2) + \bar{\Psi}_{aniso}(\bar{I}_1, \bar{I}_2, \bar{I}_4, \bar{I}_5) \quad (9)$$

129 where $\bar{\Psi}_{iso}$ and $\bar{\Psi}_{aniso}$ depend on the rescaled isotropic invariants, namely invariants $(\bar{I}_1, \bar{I}_2, \bar{I}_3)$ of
 130 $\bar{\mathbf{C}} = J^{-2/3}\mathbf{C}$ and anisotropic rescaled invariants $\bar{I}_4 = J^{-2/3}I_4$ and $\bar{I}_5 = J^{-4/3}I_5$. For a more detailed
 131 description of the model see Holzapfel [25].

132
 133 In our model, to simulate materials modeled with any possible strain energy function models, the
 134 expression of Ψ Eq.(8) is assumed. The constitutive law for transversely isotropic material is carried
 135 out by classical hyperelastic arguments. Once assigned the strain energy function, the stress-strain
 136 relation adopted is

$$\mathbf{S} = 2 \frac{\partial \Psi(\mathbf{C})}{\partial \mathbf{C}} \quad (10)$$

137 where \mathbf{S} is the PK2 (Piola-Kirchoff-2) stress tensor. Applying now the chain rule, the derivative of Ψ
 138 with respect to \mathbf{C} can be explicitly expressed as

$$\frac{\partial \Psi(\mathbf{C})}{\partial \mathbf{C}} = \frac{\partial \Psi(\mathbf{C})}{\partial I_1} \frac{\partial I_1}{\partial \mathbf{C}} + \frac{\partial \Psi(\mathbf{C})}{\partial I_2} \frac{\partial I_2}{\partial \mathbf{C}} + \frac{\partial \Psi(\mathbf{C})}{\partial I_3} \frac{\partial I_3}{\partial \mathbf{C}} + \frac{\partial \Psi(\mathbf{C})}{\partial I_4} \frac{\partial I_4}{\partial \mathbf{C}} + \frac{\partial \Psi(\mathbf{C})}{\partial I_5} \frac{\partial I_5}{\partial \mathbf{C}} \quad (11)$$

139 In this manner, the analytic expression of PK2 stress tensor is obtained by computing derivatives of Ψ
 140 with respect to invariants of the deformation process, since the derivatives of the invariant with respect
 141 to \mathbf{C} are easily computed starting from their definitions

$$\frac{\partial I_1}{\partial \mathbf{C}} = \frac{\partial \text{tr} \mathbf{C}}{\partial \mathbf{C}} = \frac{\partial (\mathbf{I} \mathbf{C})}{\partial \mathbf{C}} = \mathbf{I} \quad (12)$$

$$\frac{\partial I_2}{\partial \mathbf{C}} = \frac{1}{2} \left(\text{tr} \mathbf{C} \mathbf{I} - \frac{\partial \text{tr} \mathbf{C}^2}{\partial \mathbf{C}} \right) = I_1 \mathbf{I} - \mathbf{C} \quad (13)$$

$$\frac{\partial I_3}{\partial \mathbf{C}} = I_3 \mathbf{C}^{-1} \quad (14)$$

144

$$\frac{\partial I_4}{\partial \mathbf{C}} = \mathbf{a}_0 \otimes \mathbf{a}_0 \quad (15)$$

145

$$\frac{\partial I_5}{\partial \mathbf{C}} = \mathbf{a}_0 \otimes \mathbf{C}\mathbf{a}_0 + \mathbf{a}_0\mathbf{C} \otimes \mathbf{a}_0 \quad (16)$$

146 Substituting above expressions into Eq.(11), the most general expression of PK2 stress tensor depending
147 of the strain energy function is derived

$$\mathbf{S} = 2 \left[\left(\frac{\partial \Psi}{\partial I_1} + I_1 \frac{\partial \Psi}{\partial I_2} \right) \mathbf{I} - \frac{\partial \Psi}{\partial I_2} \mathbf{C} + I_3 \frac{\partial \Psi}{\partial I_3} \mathbf{C}^{-1} + \frac{\partial \Psi}{\partial I_4} \mathbf{a}_0 \otimes \mathbf{a}_0 + \frac{\partial \Psi}{\partial I_5} (\mathbf{a}_0 \otimes \mathbf{C}\mathbf{a}_0 + \mathbf{a}_0\mathbf{C} \otimes \mathbf{a}_0) \right] \quad (17)$$

148 In Eq.(17) the explicit dependence of the constitutive law with respect to the preferential direction of
149 the material is included by the additional terms referred to the derivatives with respect to I_4 and I_5 .

150 2.3 Incremental formulation and tangent elasticity tensor

151 In a total Lagrangian formulation of geometrical and material nonlinear problems, incremental formu-
152 lations are required. According to Holzapfel [25], the constitutive equation (17) can be rewritten in an
153 incremental form

$$\Delta \mathbf{S} = \mathbb{C} \frac{1}{2} \Delta \mathbf{C} \quad (18)$$

154 where \mathbb{C} is the well-known material Jacobian tensor or tangent elasticity tensor. In the linearized
155 version of governing equation

$$\mathbb{C} = 2 \frac{\partial \mathbf{S}(\mathbf{C})}{\partial \mathbf{C}} = \frac{\partial \mathbf{S}(\mathbf{E})}{\partial \mathbf{E}} = 4 \frac{\partial^2 \Psi}{\partial \mathbf{C} \partial \mathbf{C}} \quad (19)$$

156 Thereafter, by exploiting the derivatives of the strain energy function, one can obtain the analytic
157 closed-form expression of the tangent elasticity tensor here presented

$$\begin{aligned} \mathbb{C} = & 4 \left(\frac{\partial^2 \Psi}{\partial I_1^2} + 2I_1 \frac{\partial^2 \Psi}{\partial I_1 \partial I_2} + \frac{\partial \Psi}{\partial I_2} + I_1^2 \frac{\partial^2 \Psi}{\partial I_2^2} \right) \mathbf{I} \otimes \mathbf{I} + 4 \left(\frac{\partial^2 \Psi}{\partial I_1 \partial I_2} + I_1 \frac{\partial^2 \Psi}{\partial I_2^2} \right) (\mathbf{I} \otimes \mathbf{C} + \mathbf{C} \otimes \mathbf{I}) + \\ & + 4 \left(I_3 \frac{\partial^2 \Psi}{\partial I_1 \partial I_3} + I_1 I_3 \frac{\partial^2 \Psi}{\partial I_2 \partial I_3} \right) (\mathbf{I} \otimes \mathbf{C}^{-1} + \mathbf{C}^{-1} \otimes \mathbf{I}) + 4 \frac{\partial^2 \Psi}{\partial I_2^2} \mathbf{C} \otimes \mathbf{C} + \\ & - 4I_3 \frac{\partial^2 \Psi}{\partial I_2 \partial I_3} (\mathbf{C} \otimes \mathbf{C}^{-1} + \mathbf{C}^{-1} \otimes \mathbf{C}) + 4 \left(I_3 \frac{\partial \Psi}{\partial I_3} + I_3^2 \frac{\partial^2 \Psi}{\partial I_3^2} \right) \mathbf{C}^{-1} \otimes \mathbf{C}^{-1} + \\ & - 4I_3 \frac{\partial \Psi}{\partial I_3} \mathbf{C}^{-1} \odot \mathbf{C}^{-1} - 4 \frac{\partial \Psi}{\partial I_2} \mathcal{S} + 4 \left(\frac{\partial^2 \Psi}{\partial I_1 \partial I_4} + I_1 \frac{\partial^2 \Psi}{\partial I_2 \partial I_4} \right) (\mathbf{I} \otimes \mathbf{a}_0 \otimes \mathbf{a}_0 + \mathbf{a}_0 \otimes \mathbf{a}_0 \otimes \mathbf{I}) + \\ & - 4 \frac{\partial^2 \Psi}{\partial I_2 \partial I_4} (\mathbf{C} \otimes \mathbf{a}_0 + \mathbf{a}_0 \otimes \mathbf{a}_0 \otimes \mathbf{C}) + 4 \frac{\partial^2 \Psi}{\partial I_4^2} \mathbf{a}_0 \otimes \mathbf{a}_0 \otimes \mathbf{a}_0 \otimes \mathbf{a}_0 + \\ & + 4 \left(\frac{\partial^2 \Psi}{\partial I_1 \partial I_5} + I_1 \frac{\partial^2 \Psi}{\partial I_2 \partial I_5} \right) \left(\mathbf{I} \otimes \frac{\partial I_5}{\partial \mathbf{C}} + \frac{\partial I_5}{\partial \mathbf{C}} \otimes \mathbf{I} \right) - 4 \frac{\partial^2 \Psi}{\partial I_2 \partial I_5} \left(\mathbf{C} \otimes \frac{\partial I_5}{\partial \mathbf{C}} + \frac{\partial I_5}{\partial \mathbf{C}} \otimes \mathbf{C} \right) + \\ & + 4 \frac{\partial^2 \Psi}{\partial I_5^2} \left(\frac{\partial I_5}{\partial \mathbf{C}} \otimes \frac{\partial I_5}{\partial \mathbf{C}} \right) + 4 \frac{\partial^2 \Psi}{\partial I_4 \partial I_5} \left(\mathbf{a}_0 \otimes \mathbf{a}_0 \otimes \frac{\partial I_5}{\partial \mathbf{C}} + \frac{\partial I_5}{\partial \mathbf{C}} \otimes \mathbf{a}_0 \otimes \mathbf{a}_0 \right) + \\ & + 4 \frac{\partial \Psi}{\partial I_5} \frac{\partial^2 \Psi}{\partial \mathbf{C} \partial \mathbf{C}} + 4I_3 \frac{\partial^2 \Psi}{\partial I_3 \partial I_4} (\mathbf{a}_0 \otimes \mathbf{a}_0 \otimes \mathbf{C}^{-1} + \mathbf{C}^{-1} \otimes \mathbf{a}_0 \otimes \mathbf{a}_0) + \\ & + 4I_3 \frac{\partial^2 \Psi}{\partial I_3 \partial I_5} (\mathbf{a}_0 \otimes \mathbf{C}\mathbf{a}_0 \otimes \mathbf{C}^{-1} + \mathbf{a}_0\mathbf{C} \otimes \mathbf{a}_0 \otimes \mathbf{C}^{-1} + \mathbf{C}^{-1} \otimes \mathbf{a}_0 \otimes \mathbf{C}\mathbf{a}_0 + \mathbf{C}^{-1} \otimes \mathbf{a}_0\mathbf{C} \otimes \mathbf{a}_0) \end{aligned} \quad (20)$$

158 The full expressions of the terms $\mathbf{C}^{-1} \odot \mathbf{C}^{-1}$ and the fourth-order tensor $\mathcal{S} = (\mathbb{I} + \bar{\mathbb{I}})/2$ can be easily
159 found in reference textbooks.

3 Higher-order beam and plate CUF finite element

The Unified Formulation for isotropic hyperelastic materials has been already presented by Pagani et al. [22], in which the formulation of hyperelastic beam finite elements in CUF (Carrera Unified Formulation) framework is established. Lately, Augello *et al.* [23] presented the 2D-CUF plate models analyzing different benchmark problems in nearly-incompressible hyperelasticity. In this work, both refined beam (1D) models and plate (2D) models defined in the CUF framework are employed to study fiber-reinforced soft materials structures.

In general, the three-dimensional displacement field is expressed as a polynomial expansion of the *generalized* nodal displacements, coupling approximation expansion theories along the plate thickness or beam cross-section with kinematic models along the mid-surface or beam axis. This expansion technique allows the implementation of higher-order modes, exploited by means of a recursive index notation. In the classical orthonormal $\{x, y, z\}$ Cartesian reference frame, the three-dimensional displacement field for a beam and plate model is then expressed as

$$\text{Beam 1D models: } \mathbf{u}(x, y, z) = F_\tau(x, z)\mathbf{u}_\tau(y) \quad \tau = 1, \dots, M \quad (21)$$

$$\text{Plate 2D models: } \mathbf{u}(x, y, z) = F_\tau(z)\mathbf{u}_\tau(x, y) \quad \tau = 1, \dots, M \quad (22)$$

where M is the dimension of the polynomial expansion basis, related to the polynomial order expansion theory along the plate thickness or beam cross-section, and F_τ are the theory expansion functions that characterize the CUF model adopted, and finally \mathbf{u}_τ is the vector of generalized displacement components along the reference direction.

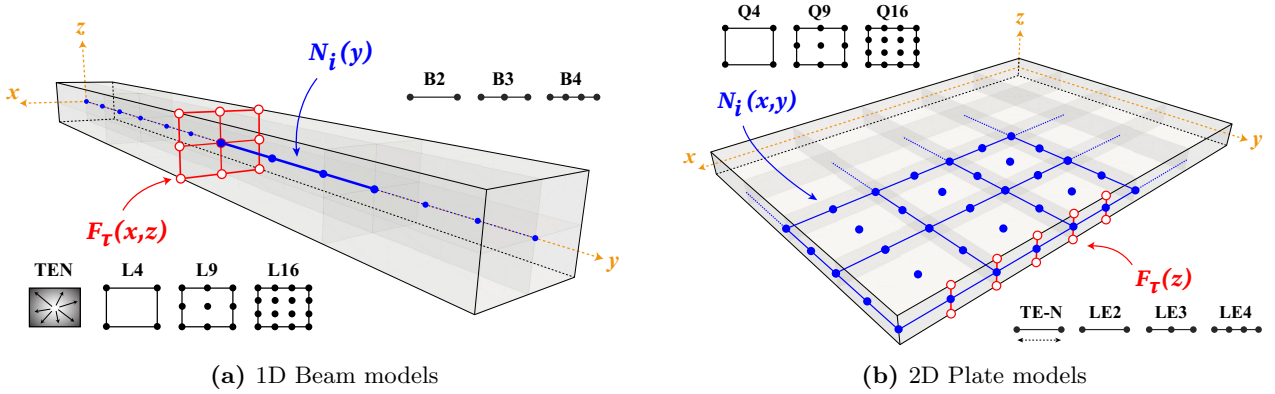


Figure 2: High order 1D and 2D CUF models.

Einstein's notation for repeated indices summation is considered in the definition of the displacement field in Eq. (22). This enables the implementation of higher-order structural theories by selecting the expansion basis function $F_\tau(x, z)$, which fully describes the model. In the present work, two distinct sets of expansion functions are considered for the cross-section expansion or thickness expansion the TE (Taylor Expansion) class and the LE (Lagrange Expansion) class. TE models utilize 1D or 2D MacLaurin polynomials, depending on the model, as basis functions for expanding the generalized reference displacement field (beam axis displacements or mid-surface plate displacements). Based on the chosen expansion order, higher-order theories are automatically defined in a hierarchical manner. As examples, we present the TE-1 linear expansion model, either for the 1D or 2D CUF models, which are

$$1\text{D: } \begin{cases} u_x(x, y, z) = u_{x_1}(y) + xu_{x_2}(y) + zu_{x_3}(y) \\ u_y(x, y, z) = u_{y_1}(y) + xu_{y_2}(y) + zu_{y_3}(y) \\ u_z(x, y, z) = u_{z_1}(y) + xu_{z_2}(y) + zu_{z_3}(y) \end{cases} \quad 2\text{D: } \begin{cases} u_x(x, y, z) = u_{x_0}(x, y) + u_{x_1}(x, y) \\ u_y(x, y, z) = u_{y_0}(x, y) + u_{y_1}(x, y) \\ u_z(x, y, z) = u_{z_0}(x, y) + u_{z_1}(x, y) \end{cases} \quad (23)$$

where u_{x_i} , u_{y_i} and u_{z_i} , in each definition, is the *generalized* displacement variable, unknown of the

188 problem, by which the whole three-dimensional displacement field is reconstructed by the expansion
 189 on F_τ expansion functions.

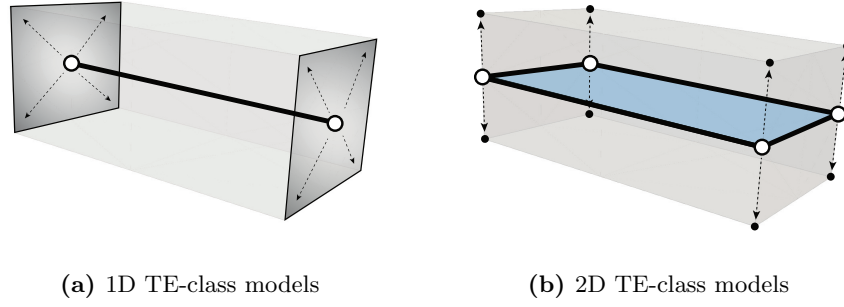


Figure 3: TE-class models, graphical representation of the mathematical model.

190 The TE-class models are generally referred to as ESLMs (Equivalent Single Layer Models), in
 191 which the displacement field of the structures is seen as an *homogenized* quantity combining each
 192 cross-section sub-components or a single layer of multilayered plates/shells in an equivalent but unique
 193 representative structure, taking into account the features of each sub-components.

194 In the case of LE-models instead, cross-section or thickness elements are defined starting from the
 195 set of Lagrange's polynomials adopted and the total number of finite nodes involved. The resulting
 196 model, in both cases, is a pure displacement-based model along the cross-section or thickness of the
 197 beam, by exploiting the isoparametric formulation. In the present work, linear, parabolic, and cubic
 198 expansion models will be adopted. From now on, 1D-CUF LE expansion models will be referred to as
 199 four-node linear L4, nine-node parabolic L9, and quadratic six-node cubic L16 cross-section expansion
 200 models, instead 2D-CUF expansion models will be addressed as linear LE2, parabolic LE3, and cubic
 201 LE4. As examples, we report the displacement field of a 1D-L9 and 2D-LE2 parabolic expansion model,
 202 expressed as

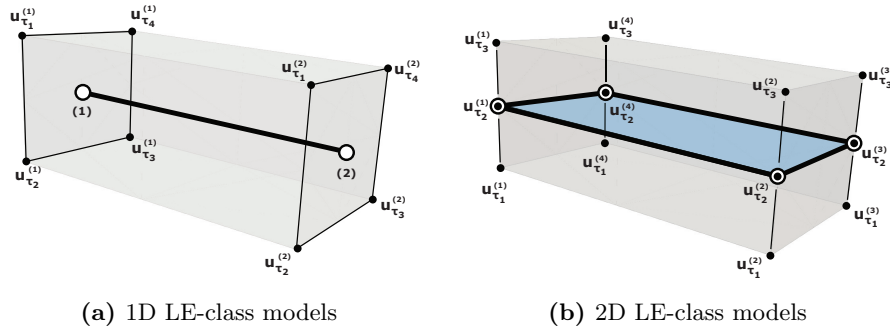


Figure 4: LE-class models, graphical representation of the mathematical model.

$$1D: \begin{cases} u_x(x, y, z) = F_1(x, z)u_{x_1}(y) + F_2(x, z)u_{x_2}(y) + F_3(x, z)u_{x_3} + \dots + F_9(x, z)u_{x_9}(y) \\ u_y(x, y, z) = F_1(x, z)u_{y_1}(y) + F_2(x, z)u_{y_2}(y) + F_3(x, z)u_{y_3} + \dots + F_9(x, z)u_{y_9}(y) \\ u_z(x, y, z) = F_1(x, z)u_{z_1}(y) + F_2(x, z)u_{z_2}(y) + F_3(x, z)u_{z_3} + \dots + F_9(x, z)u_{z_9}(y) \end{cases} \quad (24)$$

203

$$2D: \begin{cases} u_x(x, y, z) = F_1(z)u_{x_1}(x, y) + F_2(z)u_{x_2}(x, y) + F_3(z)u_{x_3}(x, y) \\ u_y(x, y, z) = F_1(z)u_{y_1}(x, y) + F_2(z)u_{y_2}(x, y) + F_3(z)u_{y_3}(x, y) \\ u_z(x, y, z) = F_1(z)u_{z_1}(x, y) + F_2(z)u_{z_2}(x, y) + F_3(z)u_{z_3}(x, y) \end{cases} \quad (25)$$

204 where the polynomial expansion basis considered is the set of 1D and 2D Lagrange parabolic poly-
 205 nomials. One key feature of LE expansion models is the independent local modeling of cross-section
 206 sub-components or multilayered plates/shells, by imposing displacements continuity of purely displac-
 207 e-ment components. A more detailed derivation of LE-class models and basis function adopted can be
 208 found in [26]. The capabilities of higher-order TE expansion models and LE models to deal with
 209 component-wise modeling of mechanical and aeronautical structures, the nonlinear static analysis, and
 210 pre-stressed vibration analysis are demonstrated in [27, 28, 29].

211 Independently of the expansion model adopted in the definition of a refined theory, the generalized
 212 displacement field components of the 1D CUF beam axis domain or 2D plate mid-surface domain are
 213 further discretized by adopting the classical FE approach

$$\text{Beam 1D models: } \mathbf{u}_\tau(y) = N_i(y)\mathbf{u}_{\tau i} \quad i = 1, \dots, N_n \quad (26)$$

$$\text{Plate 2D models: } \mathbf{u}_\tau(x, y) = N_i(x, y)\mathbf{u}_{\tau i} \quad i = 1, \dots, N_n \quad (27)$$

214 where the beam axis or plate mid-surface displacement components are formulated as a general linear
 215 combination adopting the N_i shape functions of the discrete nodal displacements $\mathbf{u}_{\tau i}$, unknowns of the
 216 model. In Eq. (27), the index i refers to the summation along the finite nodes per element adopted
 217 in the discretization of 1D beam axis or 2D plate mid-surface, and N_n refers to the expansion order
 218 governed by the total number of finite nodes involved. The final expression of the 3-D displacement
 219 field in the CUF domain is then a coupled expansion of structural theories, modeled with the expansion
 220 functions, and finite element approximation,

$$\text{Beam 1D models: } \mathbf{u}(x, y, z) = F_\tau(x, z)\mathbf{u}_\tau(y) = F_\tau(x, z)N_i(y)\mathbf{u}_{\tau i} \quad (28)$$

$$\text{Plate 2D models: } \mathbf{u}(x, y, z) = F_\tau(z)\mathbf{u}_\tau(x, y) = F_\tau(z)N_i(x, y)\mathbf{u}_{\tau i} \quad (29)$$

221 In our proposed model, the finite element approximation of the beam axis will be addressed as linear B2,
 222 parabolic B3, and cubic B4 finite elements, instead, the finite element approximation along the plate
 223 mid-surface will be referred to as linear Q4, parabolic Q9 and cubic Q16, indicating the total number
 224 of finite nodes adopted in the single element definition. Equation (29) is the most general expression
 225 of displacement field that immediately the definition of higher-order and refined finite element models
 226 for beam and plate structures in a hierarchical manner since it is independent of the polynomial basis
 227 adopted in the kinematics of the beam axis or plate mid surface and expansion theories considered.

228 The difference between these models from a finite element and assembling procedures point of view
 229 are shown in Fig. 5. The finite element matrices are combined differently depending on the expansion
 230 model chosen, by imposing the superposition of mechanical and stiffness properties in the TE-class
 231 case or imposing displacement continuity at interfaces in the LE-class case, in resulting finite element
 232 assembling procedures defined straightforwardly.

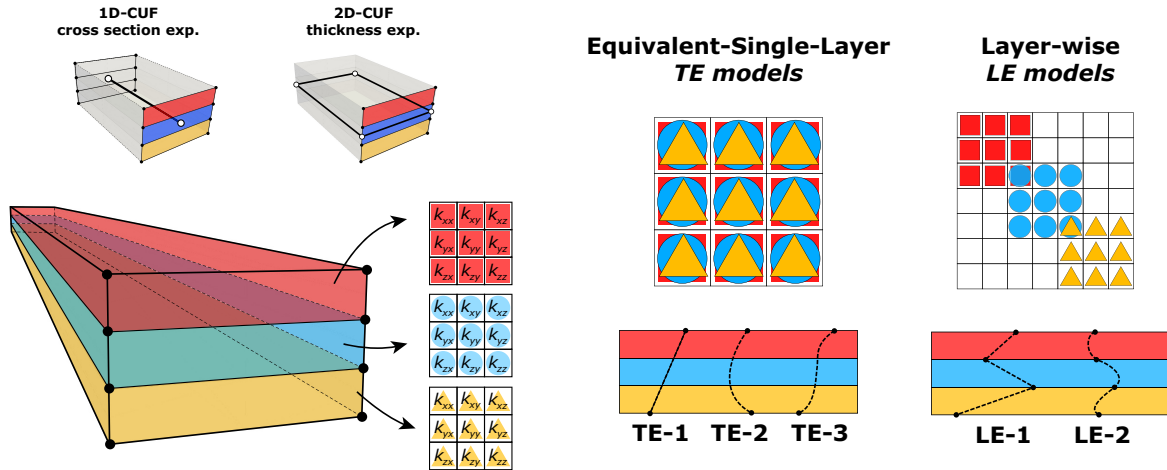


Figure 5: Unified models: Equivalent-Single-Layer and Layer-Wise models.

233 4 Nonlinear governing equations in matrix form

234 4.1 Internal and external force vector

235 In this work, governing equation in weak form are derived by means of the PVD (Principle of Virtual
236 Displacements). Considering a static nonlinear problem, supposing negligible body forces, PVD states

$$\delta\mathcal{L}_{int} = \delta\mathcal{L}_{ext} \quad (30)$$

237 where \mathcal{L}_{int} is the internal work, \mathcal{L}_{ext} is the external work and δ denotes the virtual variation. Analyt-
238 ically, these terms can be expressed as

$$(a) \delta\mathcal{L}_{int} = \int_{\Omega} \delta\mathbf{E}^T \mathbf{S} dV \quad (b) \delta\mathcal{L}_{ext} = \int_{\Omega} \delta\mathbf{u}^T \mathbf{f} dV \quad (31)$$

239 where \mathbf{S} is the PK2 stress tensor, \mathbf{E} is the Green-Lagrange strain tensor and \mathbf{f} is the vector of external
240 loads applied. The virtual quantities are introduced adopting the same index notation and polynomial
241 expansion but with independent indices, adopting the j index for virtual measures along the axis nodes
242 and the s index for the CUF expansion model, to obtain independent quantities with respect to real
243 ones. Since both 1D beam models and 2D plate models are considered, variable dependency will be
244 neglected from now on without loss of generality, since the derivation procedure is independent of the
245 model adopted. The generic virtual displacement is then defined as

$$\delta\mathbf{u}(x, y, z) = F_s \mathbf{u}_s = F_s N_j \delta\mathbf{u}_{sj} \quad j = 1, 2, \dots, N_n, \quad s = 1, \dots, M \quad (32)$$

246 Referring to the internal energy contribution, the full Green-Lagrange strain tensor can be rewritten
247 in terms of nodal displacement unknowns and expansion functions with the same index notation.
248 Introducing now *Voigt's notation* for the representation of physical symmetric quantities, stress, and
249 deformation tensors are rewritten in vector form as

$$\mathbf{S} = \{S_{xx}, S_{yy}, S_{zz}, S_{xz}, S_{yz}, S_{xy}\}^T \quad (33)$$

$$\mathbf{E} = \{E_{xx}, E_{yy}, E_{zz}, E_{xz}, E_{yz}, E_{xy}\}^T \quad (34)$$

251 Under the hypothesis of the fully nonlinear displacement-strain relation, the Green-Lagrange tensor
252 can be rewritten as done in Pagani *et al.* [19]

$$\mathbf{E} = (\mathbf{b}_1 + \mathbf{b}_{nl}) \mathbf{u} = (\mathbf{b}_1 + \mathbf{b}_{nl}) F_{\tau} N_i \mathbf{u}_{\tau i} = (\mathbf{B}_1^{\tau i} + \mathbf{B}_{nl}^{\tau i}) \mathbf{u}_{\tau i} \quad (35)$$

253 Applying the formal matrices of derivatives operator \mathbf{b}_1 and \mathbf{b}_{nl} to CUF expansion of the displacement
254 field, the algebraic matrices $\mathbf{B}_1^{\tau i}$ and $\mathbf{B}_{nl}^{\tau i}$ are obtained, and their explicit forms can be found in
255 [19, 20]. The virtual variation of the strain measure is written in compact form including the previously
256 introduced discretization of virtual displacement Eq.(32)

$$\delta\mathbf{E} = \delta((\mathbf{B}_1^{\tau i} + \mathbf{B}_{nl}^{\tau i}) \mathbf{u}_{\tau i}) = (\mathbf{B}_1^{sj} + 2\mathbf{B}_{nl}^{sj}) \delta\mathbf{u}_{sj} \quad (36)$$

257 Substituting now Eq.(36) into Eq.(31)(a)

$$\delta\mathcal{L}_{int} = \int_{\Omega} \delta\mathbf{u}_{sj}^T (\mathbf{B}_1^{sj} + 2\mathbf{B}_{nl}^{sj})^T \mathbf{S} dV = \delta\mathbf{u}_{sj}^T \mathbf{F}_{int}^{sj} \quad (37)$$

258 where \mathbf{F}_{int}^{sj} the 3x1 FN (Fundamental Nucleus) of the internal forces vector

$$\mathbf{F}_{int}^{sj} = \int_{\Omega} (\mathbf{B}_1^{sj} + 2\mathbf{B}_{nl}^{sj})^T \mathbf{S} dV \quad (38)$$

259 Referring to the external load contribution in the variational principle, the FN of the external load
260 vector is exploited by means of the same derivation procedure described for the internal energy con-
261 tribution. If \mathbf{f} is the vector of conservative loads applied to the structure, one has

$$\delta\mathcal{L}_{ext} = \int_{\Omega} \delta\mathbf{u}^T \mathbf{f} dV = \int_{\Omega} \delta\mathbf{u}_{sj}^T F_s N_j \mathbf{f} dV = \delta\mathbf{u}_{sj}^T \mathbf{F}_{ext}^{sj} \quad (39)$$

262 where \mathbf{F}_{ext}^{sj} the 3x1 FN of the external forces vector

$$\mathbf{F}_{ext}^{sj} = \int_{\Omega} F_s N_j \mathbf{f} dV \quad (40)$$

263 The main advantage in the adoption of CUF-based finite element models is herein noticeable by
 264 Eq.(38) and Eq.(39) the FNs of physical quantities are defined regardless of the specific kinematic
 265 model adopted in the discretization of finite nodes and approximation theory along the cross-section or
 266 thickness of the structure, therefore independent expressions of physical quantities for any arbitrarily
 267 polynomial expansion are found. The unique expressions of internal forces and external forces for
 268 specific combinations of models adopted are exploited by assigning the finite element shape functions
 269 N_i , N_j and theory of structure approximation F_τ and F_s of any order, and exploiting the summation
 270 over the recursive indices expansion (namely the summation over indices i and j , τ , and s). Finally,
 271 the PVD is written in compact notation as follow

$$\delta \mathbf{u}_{sj} : \mathbf{F}_{int}^{sj} - \mathbf{F}_{ext}^{sj} = 0 \quad (41)$$

272 Considering the summation over indices s and j , the global internal forces vector \mathbf{F}_{int} and external
 273 load vector \mathbf{F}_{ext} can be computed, following the CUF assembling procedure [15].

274
 275 In a finite element scenario for hyperelasticity, both large displacements and rotations (geometrical
 276 nonlinearities) and nonlinear constitutive law (material nonlinearities) must be taken into account for
 277 these reasons, equilibrium equations turn out to be strongly nonlinear, and common solution techniques
 278 are based on numerical iterative solvers.

279 4.2 Linearization of governing equations

280 The expanded nonlinear problem Eq.(41) is written as an equivalent optimization problem in the form
 281 of minimization of residual function [30]. Defining the *unbalanced nodal forces vector* as

$$\phi_{res} = \mathbf{F}_{int} - \mathbf{F}_{ext} \quad (42)$$

282 the solution of the nonlinear equilibrium problem is equivalent to finding the root of Eq.42 since,
 283 at equilibrium, due to balance the residual nodal forces vector is null ($\phi_{res} = 0$). In our formula-
 284 tion, a Newton-Raphson iterative scheme is employed, thus expanding Eq.(42) by considering Taylor's
 285 expansion around a known condition $(\mathbf{u}^0, \mathbf{F}_{ext}^0)$ truncated at the first order

$$\phi_{res}(\mathbf{u}^0 + \Delta \mathbf{u}, \mathbf{F}_{ext} + \Delta \mathbf{F}_{ext}) = \phi_{res}(\mathbf{u}^0, \mathbf{F}_{ext}^0) + \frac{\partial \phi_{res}}{\partial \mathbf{u}} \Delta \mathbf{u} + \frac{\partial \phi_{res}}{\partial \mathbf{F}_{ext}} \Delta \lambda \cdot \mathbf{F}_{ext}^{ref} = 0 \quad (43)$$

286 where the finite variation of the nodal load vector can be rearranged under the hypothesis of conserva-
 287 tive external load, in mathematical terms $\Delta \mathbf{F}_{ext} = \Delta(\lambda \mathbf{F}_{ext}^{ref}) = \Delta \lambda \mathbf{F}_{ext}^{ref}$. Defining the tangent stiffness
 288 matrix as $\frac{\partial \phi_{res}}{\partial \mathbf{u}} = \mathbf{K}_T$ and recognizing that $\frac{\partial \phi_{res}}{\partial \mathbf{F}_{ext}} = -\mathbf{I}$, Eq.(43) is written as

$$\mathbf{K}_T(\mathbf{u}^0) \Delta \mathbf{u} = \Delta \lambda \mathbf{F}_{ext}^{ref} - \phi_{res}(\mathbf{u}^0, \mathbf{F}_{ext}^0) \quad (44)$$

289 Considering the *loading scale parameter* λ as an additional variable of the problem, Eq.(44) must be
 290 coupled with a general constraint equation, to close algebraically the problem, thus

$$\begin{cases} \mathbf{K}_T(\mathbf{u}^0) \Delta \mathbf{u} = \Delta \lambda \mathbf{F}_{ext}^{ref} - \phi_{res}(\mathbf{u}^0, \mathbf{F}_{ext}^0) \\ c(\Delta \mathbf{u}, \Delta \lambda) = 0 \end{cases} \quad (45)$$

291 The constraint equation characterizes the numerical scheme adopted, one can implement displacement
 292 control, load control, and path-following methods by adopting a different constraint. In the present
 293 work, the path-following method proposed by Crisfield [31] is employed, and the implementation of
 294 such arc-length iterative solver in a CUF-based finite element scenario is described in detail in [19].

295 4.3 Tangent stiffness matrix

296 The analytic expression of the tangent stiffness matrix is here described in detail and exploited by
 297 linearization of equilibrium equation Eq.(42). Through the assumption of conservative loads only the
 298 virtual variation of the internal work has to be linearized, since the second variation of external loads
 299 will be identically null, therefore the finite variation of the internal strain energy contribution of the
 300 variational principle is

$$\Delta(\delta\mathcal{L}_{int}) = \int_{\Omega} \Delta(\delta\mathbf{E}^T \mathbf{S}) dV = \int_{\Omega} \delta\mathbf{E}^T \Delta\mathbf{S} dV + \int_{\Omega} \Delta(\delta\mathbf{E}^T) \mathbf{S} dV \quad (46)$$

301 The two integral terms appearing in Eq.(46) are now analyzed separately.

302 The first term represent the linearization of constitutive equation. Adopting Holzapfel formulation,
 303 Eq.(18) is rewritten in matrix form

$$\Delta\mathbf{S} = \mathbb{C} \frac{1}{2} \Delta\mathbf{C} = \mathbb{C} \Delta\mathbf{E} = \mathbb{C} (\mathbf{B}_1^{sj} + 2\mathbf{B}_{nl}^{sj}) \Delta\mathbf{u}_{sj} \quad (47)$$

304 Thus, the linearization of the constitutive equation can be written as

$$\begin{aligned} \int_{\Omega} \delta\mathbf{E}^T \Delta\mathbf{S} dV &= \int_{\Omega} \delta\mathbf{u}_{sj}^T (\mathbf{B}_1^{sj} + 2\mathbf{B}_{nl}^{sj})^T \mathbb{C} (\mathbf{B}_1^{\tau i} + \mathbf{B}_{nl}^{\tau i}) \Delta\mathbf{u}_{\tau i} dV = \\ &= \delta\mathbf{u}_{sj}^T \mathbf{K}_{ll}^{\tau sij} \Delta\mathbf{u}_{\tau i} + \delta\mathbf{u}_{sj}^T \mathbf{K}_{lnl}^{\tau sij} \Delta\mathbf{u}_{\tau i} + \delta\mathbf{u}_{sj}^T \mathbf{K}_{nll}^{\tau sij} \Delta\mathbf{u}_{\tau i} + \delta\mathbf{u}_{sj}^T \mathbf{K}_{nlnl}^{\tau sij} \Delta\mathbf{u}_{\tau i} = \\ &= \delta\mathbf{u}_{sj}^T \mathbf{K}_{ll}^{\tau sij} \Delta\mathbf{u}_{\tau i} + \delta\mathbf{u}_{sj}^T \mathbf{K}_{T_1}^{\tau sij} \Delta\mathbf{u}_{\tau i} \end{aligned} \quad (48)$$

305 where $\mathbf{K}_{T_1}^{\tau sij} = \mathbf{K}_{lnl}^{\tau sij} + \mathbf{K}_{nll}^{\tau sij} + \mathbf{K}_{nlnl}^{\tau sij}$ is the non linear contribution to the tangent stiffness matrix
 306 coming from the linearization of the constitutive equation and $\mathbf{K}_{ll}^{\tau sij}$ is the linear contribution.
 307 The second term of Eq.(46) is the linearization of geometrical relations. Exploiting the same derivation
 308 procedure described before, one can find the fundamental nucleus of the *geometrical* stiffness matrix
 309 $\mathbf{K}_{\sigma}^{\tau sij}$, the derivation will be not reported here but can be found explicitly in [19, 20].

$$\int_{\Omega} \Delta(\delta\mathbf{E})^T \mathbf{S} dV = \delta\mathbf{u}_{sj}^T \mathbf{K}_{\sigma}^{\tau sij} \Delta\mathbf{u}_{\tau i} \quad (49)$$

310 Substituting finally the expression of the contribution coming from the linearization of the constitutive
 311 equation Eq.(48) and the one coming from linearization of the geometrical relations Eq.(49), the
 312 fundamental nucleus of the tangent stiffness matrix is defined as

$$\begin{aligned} \Delta(\delta\mathcal{L}_{int}) &= \int_{\Omega} \delta\mathbf{E}^T \Delta\mathbf{S} dV + \int_{\Omega} \Delta(\delta\mathbf{E})^T \mathbf{S} dV = \\ &= \delta\mathbf{u}_{sj}^T \mathbf{K}_{ll}^{\tau sij} \Delta\mathbf{u}_{\tau i} + \delta\mathbf{u}_{sj}^T \mathbf{K}_{T_1}^{\tau sij} \Delta\mathbf{u}_{\tau i} + \delta\mathbf{u}_{sj}^T \mathbf{K}_{\sigma}^{\tau sij} \Delta\mathbf{u}_{\tau i} = \\ &= \delta\mathbf{u}_{sj}^T \mathbf{K}_T^{\tau sij} \Delta\mathbf{u}_{\tau i} \end{aligned} \quad (50)$$

313 As stressed in the previous theoretical derivation of the model, the FNs of sub-matrices of the tangent
 314 stiffness matrix are defined for any arbitrary polynomial expansion chosen. The unique expression of
 315 tangent stiffness matrix in the single CUF finite element is achieved by exploiting the summation over
 316 indices i and j, τ and s, thus on shape functions N_i , N_j and theory of structure approximation F_{τ} and
 317 F_s , but the resulting definition of FN is completely independent of the chosen kinematic models and
 318 approximation theory.

5 Numerical results

In this section, numerical results obtained via the present implementation of higher-order 1D beam and 2D plate CUF models for transversely isotropic hyperelastic materials are presented and validated adopting as reference results popular benchmark problems available in literature or simple cases in which closed-form solutions are known. We present different models and we investigate the capabilities of our models when different strain energy functions are considered, more complex geometries are involved, and in particular when different preferential directions of the fibers are chosen, analyzing standard problems involving Cartesian or curvilinear geometries.

5.1 Shear test of an incompressible block

The first problem is a simple shear test of a fiber-reinforced incompressible cubic block. This classical benchmark problem in hyperelasticity is a special case for which closed-form solutions are known. Geometry and boundary conditions are depicted in Fig. 6.

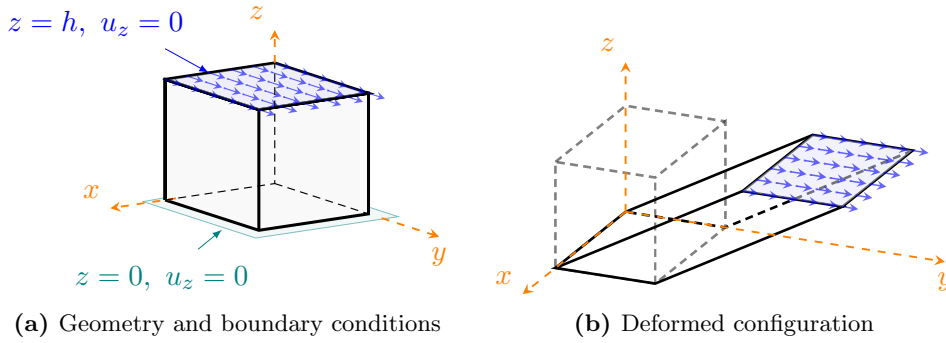


Figure 6: Simple shear tension test, description of the problem.

Material is modeled adopting a refined version of the classical HGO (Holzapfel-Gasser-Ogden) model for fiber-reinforced materials [2] and presented in Mendez et al. [32], in which the authors proposed this refined strain energy function to investigate the physical meaning of I_5 invariant

$$\Psi = \Psi_{vol}(I_3) + \Psi_{iso}(I_1, I_2, I_3) + \Psi_{ani}(I_3, I_4, I_5) \quad (51)$$

$$\Psi_{vol}(I_3) = \frac{k}{2}(J - 1)^2 = \frac{k}{2}(\sqrt{I_3} - 1)^2 \quad (52)$$

$$\Psi_{iso}(I_1, I_2, I_3) = \frac{c_1}{2}(\bar{I}_1 - 1) = \frac{c_1}{2}(I_1 I_3^{-1/3} - 1) \quad (53)$$

$$\begin{aligned} \Psi_{aniso}(I_3, I_4, I_5) &= \frac{c_2}{2c_3}(e^{c_3(\bar{I}_4 - 1)^2} - 1) + \frac{c_4}{2c_5}(e^{c_5(\bar{I}_5 - \bar{I}_4^2)^2} - 1) = \\ &= \frac{c_2}{2c_3}(e^{C_3(I_4 I_3^{-1/3} - 1)^2} - 1) + \frac{c_4}{2c_5}(e^{c_5(I_5 I_3^{-2/3} - I_4^2 I_3^{-2/3})^2} - 1) \end{aligned} \quad (54)$$

Material parameters adopted numerical constants considered are described in Table 1.

We now derive the explicit analytic expression of deformation gradient components, thus the analytic expression of the non-null terms of PK2 stress tensor. If $\{x^0, y^0, z^0\}$ are the coordinates of the generic material point of the cube in the reference configuration and $\{x, y, z\}$ are the coordinate in the deformed configuration, as previously shown in Fig. 1, the deformation field components for the shear problem in the $y - z$ plane shown in Fig. 6 are known, thus also the deformation gradient and the right Cauchy-Green strain tensor

$$(x, y, z) = f(x^0, y^0, z^0) \begin{cases} x = x^0 \\ y = y^0 + \gamma z^0 \\ z = z^0 \end{cases} \rightarrow \mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \mathbf{C} = \mathbf{F}^T \mathbf{F} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \gamma \\ 0 & \gamma & 1 + \gamma^2 \end{bmatrix}$$

345 where γ is a strictly positive real number. Supposing again that $\mathbf{a}_0 = (a_x, a_y, a_z)^T$, invariants and
 346 pseudo-invariants required for the computation of physical quantities are

$$\begin{cases} I_1 = 3 + \gamma^2 \\ I_2 = -3 - 2\gamma^2 \\ I_3 = 1 \end{cases} \quad \begin{cases} I_4 = a_x^2 + a_y(a_y + a_z\gamma) + a_z(a_y\gamma + a_z(1 + \gamma^2)) \\ I_5 = a_x^2 + a_y(a_y + a_z\gamma^2) + a_z(a_y\gamma^2 + a_z(1 + \gamma^2)^2) \end{cases}$$

347 One can note that $J = \det \mathbf{F} = 1$, so the mathematical enforcement of incompressibility is obtained
 348 by imposing the deformation gradient components. Computing then the derivatives of strain energy
 349 function Eq.(51), and computing the analytic expression of the stress tensor \mathbf{S} by Eq.(17), finally the
 350 analytic expression of Cauchy's "true" stress tensor function of the shear parameter γ is obtained

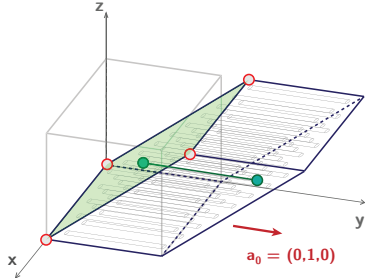
$$\boldsymbol{\sigma} = \frac{1}{J} \mathbf{F} \mathbf{S} \mathbf{F}^T \quad (55)$$

351 Due to the analytic complexity, here extended expression of PK2 stress components are not reported but
 352 they can be computed easily. In this study case, the capabilities of the present implementation of CUF-
 353 based models are investigated by analyzing the mechanical behavior when different fiber directions are
 354 considered. To validate the present implementation of hyperelastic 1D and 2D CUF models, the cubic
 355 specimen is analyzed by two independent models in the first case, the mathematical model adopted
 356 makes use of 1D beam elements, with one L4 linear element adopted for the cross-section discretization
 357 and one B2 linear element along the axis; in the case of 2D plate CUF models, only one Q4 linear
 358 element is adopted in the discretization of the mid-surface and one LE2 linear element is adopted for
 359 the thickness expansion theory. In particular, the mechanical response of the cube is analyzed in six
 360 different study cases, for each discretization adopted, in which different fiber vectors are considered.

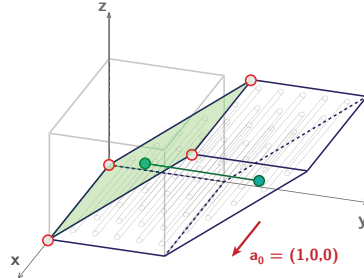
k [kPa]	c_1 [kPa]	c_2 [kPa]	c_3 [-]	c_4 [kPa]	c_5 [-]
$1 \cdot 10^8$	50	831.4	4.241	350.96	6.18

Table 1: Simple shear problem material properties

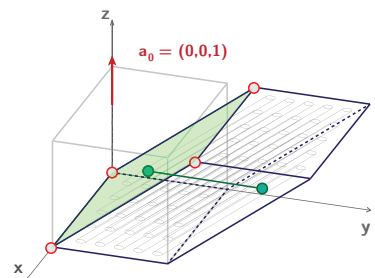
361 Figure 7(a), Fig. 7(b) and Fig. 7(c) depict, for each unitary versor \mathbf{a}_0 considered, the geometrical
 362 model of the fiber-reinforcement considered and the 1D CUF finite element adopted, instead Fig.
 363 7(d), Fig. 7(e) and Fig. 7(f) show the comparison between the analytic stress-stretch curve and the
 364 numerical results obtained by discretizing the specimen with 1LE2-1B2 beam CUF element, plotting
 365 the stress distribution versus the shear parameter γ in all cases, a perfect superposition of the numerical
 366 results is achieved. The same analysis is performed adopting the indicated 2D plate discretization of
 367 the cubic specimen, but for the sake of brevity, results are not reported here since actual numerical
 368 results perfectly match the one already presented. As another example, following the proposed study
 369 cases in Mendez *et al.* [32], the mechanical response of the cubic specimen is analyzed in the case of
 370 a preferential direction of the fibers laying in the $y - z$ plane, inclined of an angle θ with respect to
 371 the y-axis. Three different inclination conditions are proposed $\theta = 30^\circ$, $\theta = 45^\circ$ and $\theta = 60^\circ$. Figure
 372 7(g), Fig. 7(h) and 7(i) shows the comparison between analytic and numeric stress-stretch curves in
 373 the case of 1Q4-1B2 results are perfectly matching the analytic solution in all the cases.



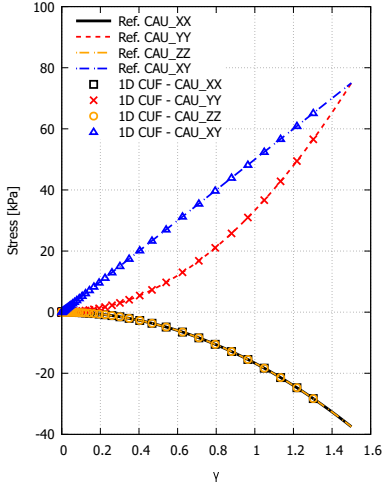
(a) Configuration $\mathbf{a}_0 = (0, 1, 0)$



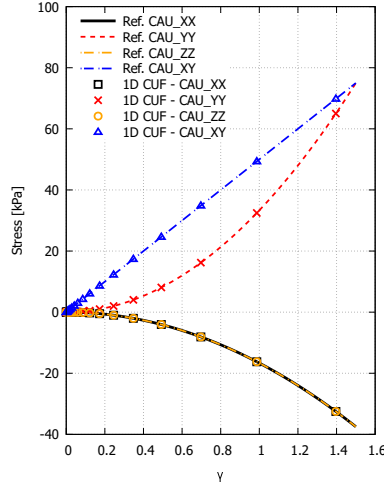
(b) Configuration $\mathbf{a}_0 = (1, 0, 0)$



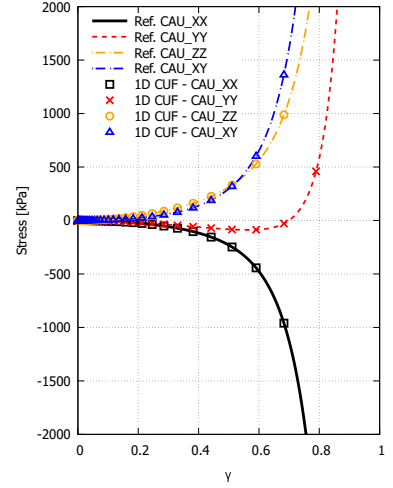
(c) Configuration $\mathbf{a}_0 = (0, 0, 1)$



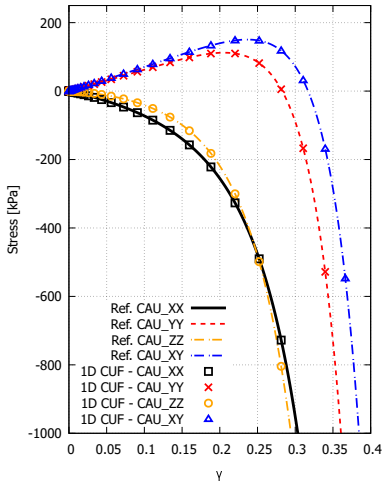
(d) $\mathbf{a}_0 = (0, 1, 0)$



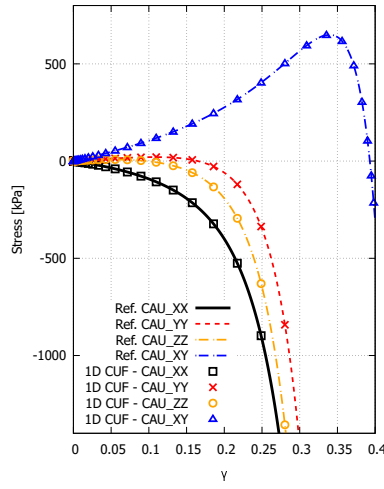
(e) $\mathbf{a}_0 = (1, 0, 0)$



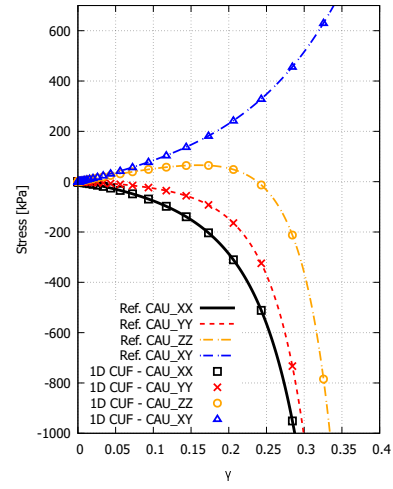
(f) $\mathbf{a}_0 = (0, 0, 1)$



(g) $\mathbf{a}_0 = (0, \cos 30^\circ, \sin 30^\circ)$



(h) $\mathbf{a}_0 = (0, \cos 45^\circ, \sin 45^\circ)$



(i) $\mathbf{a}_0 = (0, \cos 60^\circ, \sin 60^\circ)$

Figure 7: Shear tension test comparison between analytic and 1D beam CUF numerical solution

374 5.2 Circular plate under uniform transversal pressure

375 As a second numerical example, the bending of a circular plate presented by Beheshti *et al.* [13]
 376 is considered a benchmark case study. A circular plate of radius $R = 50$ mm and thickness $h = 5$
 377 mm clamped at the lateral surface is subjected to a vertical transversal pressure \mathbf{q}_z . The geometrical
 378 features and boundary conditions are depicted in Fig. 8(a).

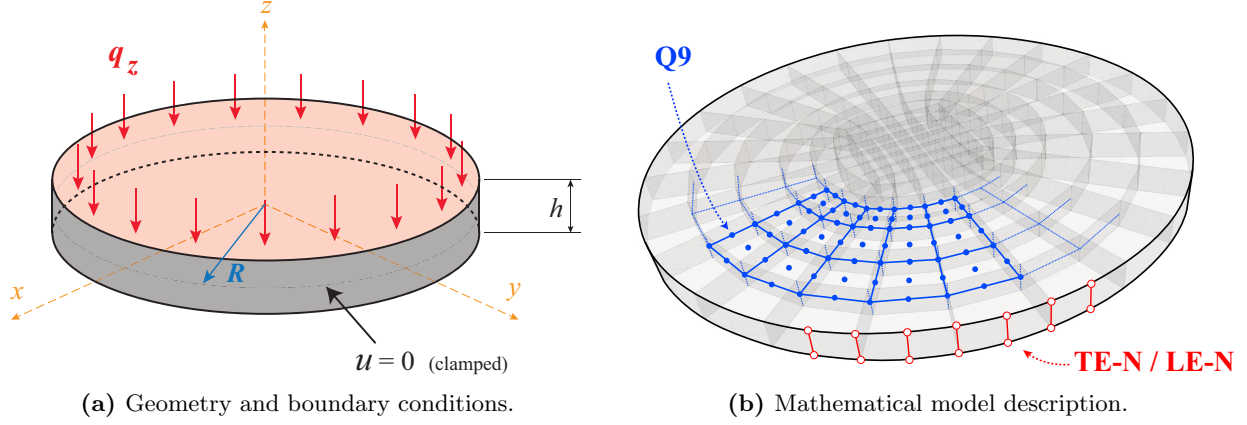


Figure 8: Circular plate configuration of the case study.

379 The mechanical response of the plate is investigated for different material conditions in the first
 380 case, an isotropic hyperelastic plate is analyzed, and thereafter a fiber-reinforced hyperelastic one in
 381 the same geometrical and load conditions. In the anisotropic case, two different fiber distributions are
 382 considered separately, a radial reinforcement and then a tangential reinforcement thanks to the numerical
 383 integration technique, the unitary vector \mathbf{a}_0 required for the computation of physical quantities is
 384 defined locally by the Gauss integration point in each element of the discretization.

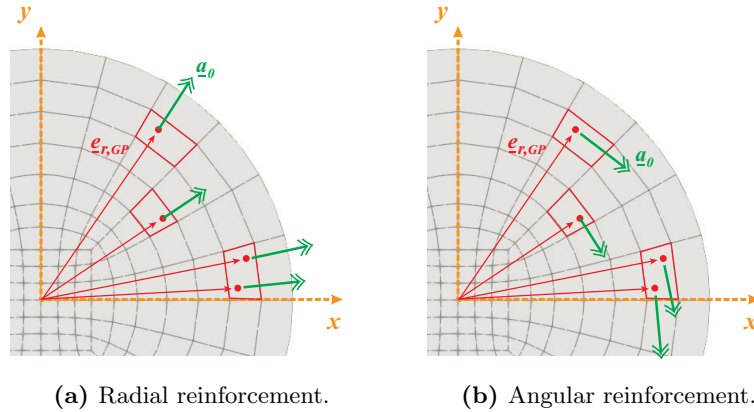


Figure 9: Circular plate, anisotropic case, fiber distribution in the definition of principal fiber direction, by taking advantage of the numerical integration scheme employed in stiffness matrices computation, the vector \mathbf{a}_0 is defined starting from the physical coordinates of the Gauss integration point, in this way a globally accurate distribution of fibers can be easily obtained.

385 Material is modeled with an isotropic Neo-Hookean model coupled with the standard reinforcement
 386 model for fiber-reinforced hyperelastic material and a stabilized volumetric logarithmic-power model

$$\Psi(\mathbf{C}) = \frac{\mu}{2}(I_1 - 3) + \frac{\lambda}{2}(J - 1)^2 - \mu \log J + \gamma(I_4 - 1)^2 \quad (56)$$

387 where the infinitesimal shear modulus is set to $\mu = 1$ MPa, the Lamé constant is set to $\lambda = 4$ MPa
 388 and the reinforcing model constant is $\gamma = 0.375$ MPa. In the case of an isotropic plate, the constant
 389 γ is set equal to zero.

390 As a preliminary investigation, a convergence analysis is carried out considering 2D plate CUF
 391 models, analyzing the influence of different kinematic models adopted in the discretization of plate
 392 mid-surface and the influence of the theory approximation along the thickness thus a convergence
 393 analysis for an increasing number of Q9 parabolic elements along the mid surface of the plate will
 394 be considered, instead the influence of the mathematical model in the expansion along the thickness
 395 will be investigated considering linear, quadratic, and cubic Taylor Expansion model and Lagrange
 396 Expansion model, that will be addressed as TE N (where N is the polynomial order), LE2 (Lagrange
 397 parabolic model) and LE3 (Lagrange cubic model). Actual numerical results obtained adopting 2D
 398 plate CUF elements are compared with the reference results, analyzing the structure in the specific
 399 configuration of transversely isotropic hyperelastic material with radial fiber distribution.

400 Figure 10(a) shows the equilibrium paths of the clamped plate in the radial fiber distribution
 401 condition analyzing the influence of the total number of finite elements adopted in the discretization on
 402 the numerical solution, instead of Fig. 10(b) shows the influence of the mathematical model adopted
 403 in the thickness expansion, plotting the vertical transversal displacement of the center of the plate
 404 (measured at the mid surface) versus the modulus of applied pressure. In all the cases, a perfect
 405 superposition of the numerical results is evidenced, and accurate predictions are obtained.

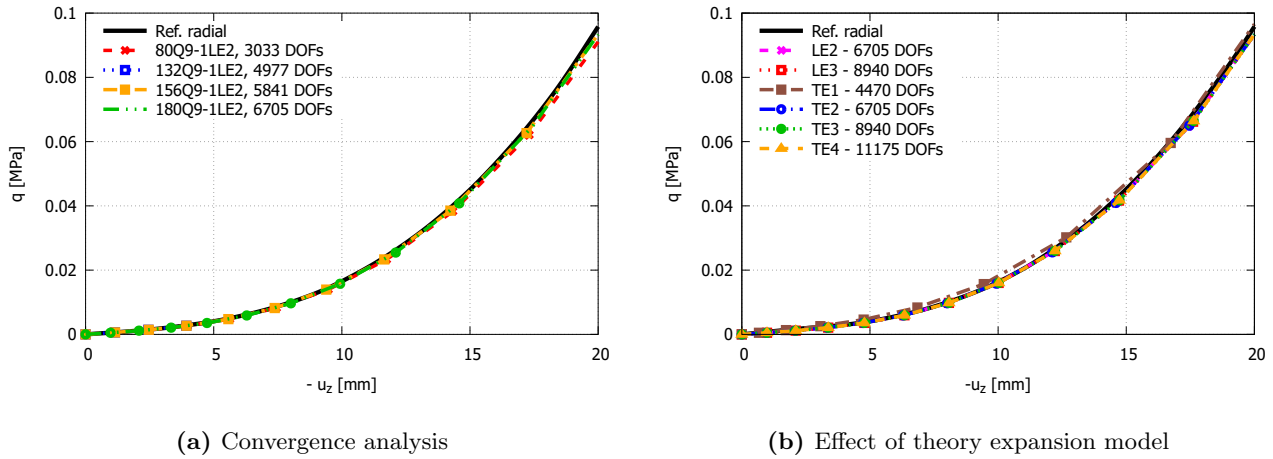


Figure 10: Circular plate equilibrium curve of the plate in the radial fiber distribution configuration.

406 In the following, the nonlinear static analysis of clamped plate in each described material con-
 407 figuration is carried out the isotropic and fiber-reinforced cases are studied adopting the convergent
 408 mathematical model previously described, employing 180 Q9 parabolic elements in the discretization
 409 of the plate mid-surface and a single LE3 element along the thickness. Figure 11 shows the equilibrium
 410 path for all material configurations, comparing the transversal displacement of the center of the plate
 411 (measured on the mid-surface) obtained by 2D CUF models with reference results. Actual numerical
 412 solutions are in perfect agreement with the reference solution. In particular, a stiffer behavior of the
 413 plate can be observed when a radial distribution of fibers is considered, instead in the case of tangen-
 414 tial/angular distribution the mechanical behavior is similar to the isotropic one. In the case of radial
 415 fiber distribution, Fig. 12(a) shows the transversal displacement distribution along the plate thickness
 416 (measured at the center), instead Fig. 12(b) shows the longitudinal displacement distribution along the
 417 diameter of the plate, measured at the mid surface. Again, the linear TE1 model, which corresponds
 418 to the first-order shear deformation theory, is not able to capture the correct transversal behavior due
 419 to the theoretical model assumptions. Figure 13 shows the deformed configurations of the anisotropic
 420 radial reinforced plate in different load conditions.

421

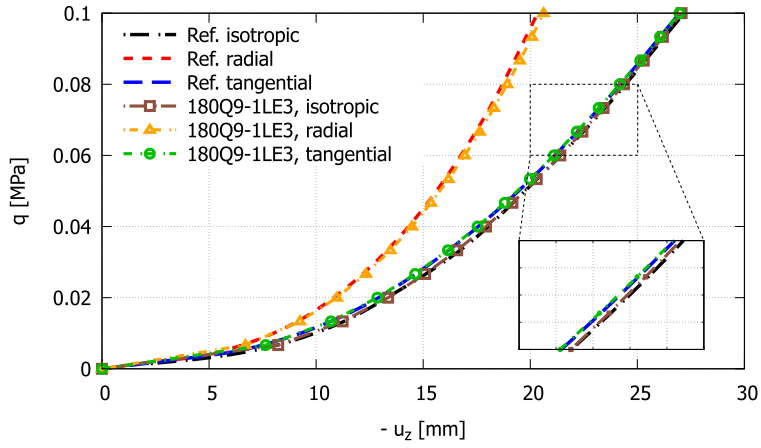


Figure 11: Circular plate equilibrium path for different fiber configurations.

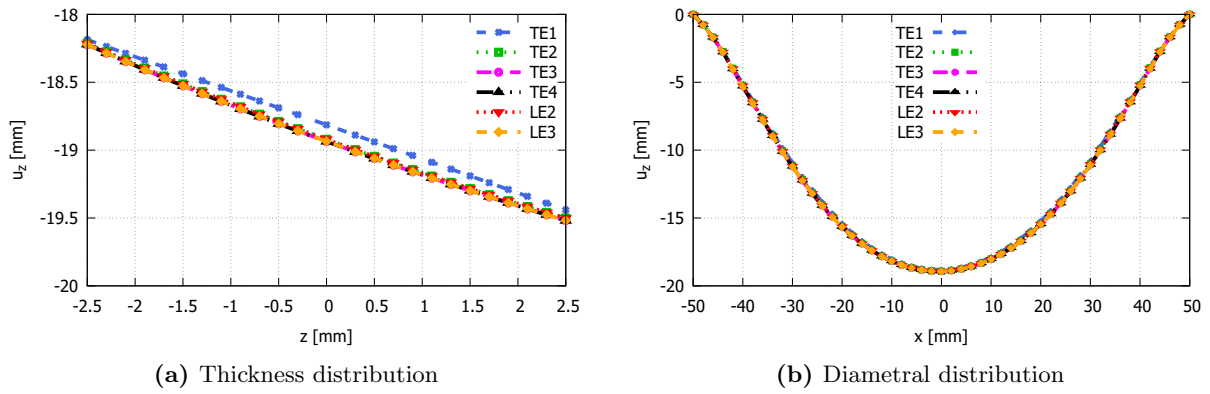


Figure 12: Circular plate displacement distribution along the thickness and the diameter of the plate, for various expansion theories, radial fiber distribution case.

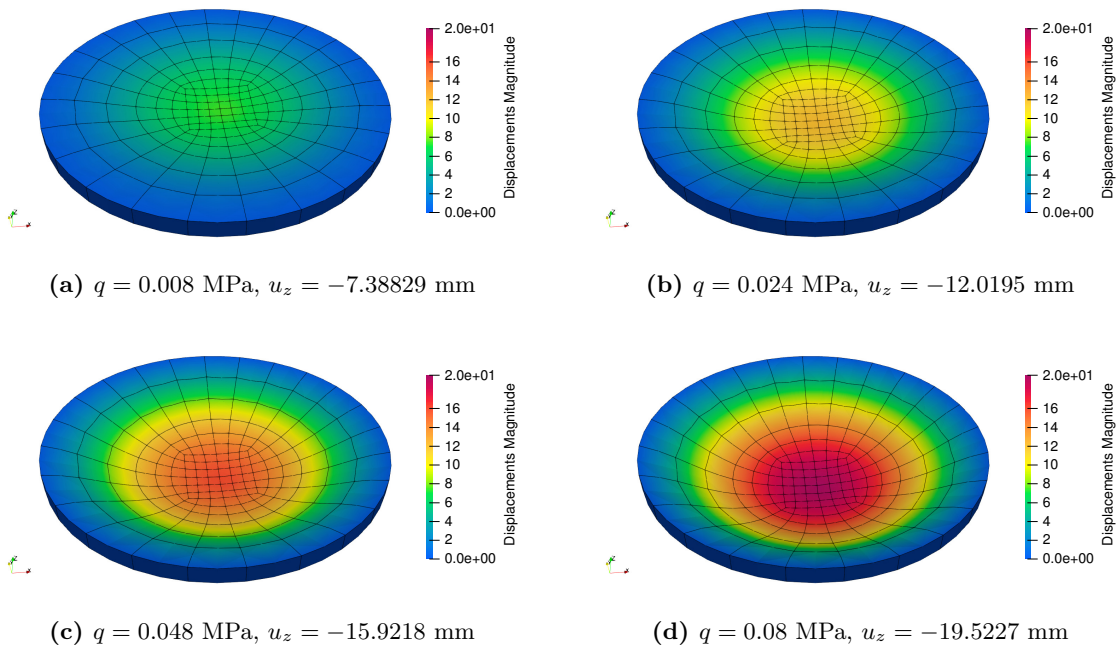


Figure 13: Circular plate deformed configurations for different value of applied pressure, 180Q9-1LE3

422 Furthermore, the influence of expansion approximation theory on the description of stresses is
 423 investigated. The clamped plate is considered now in the isotropic material condition, employing the
 424 previously convergent mesh made by 180 Q9 parabolic elements and different expansion models to
 425 analyze the through-the-thickness stress distributions. Figure 14 shows the distributions of PK2 stress
 426 components along the z direction, measured at the point A positioned at coordinates $(-3/4R, 0)$ mm on
 427 the plate mid-surface, when a transversal pressure $q_z = 0.1$ MPa is applied. In all the cases, differences
 428 are evidenced between models, which can be addressed to the higher accuracy of the deformation
 429 gradient and invariants computation thanks to higher-order expansions of the displacement field, which
 430 lead to more accurate predictions of the stress field components.

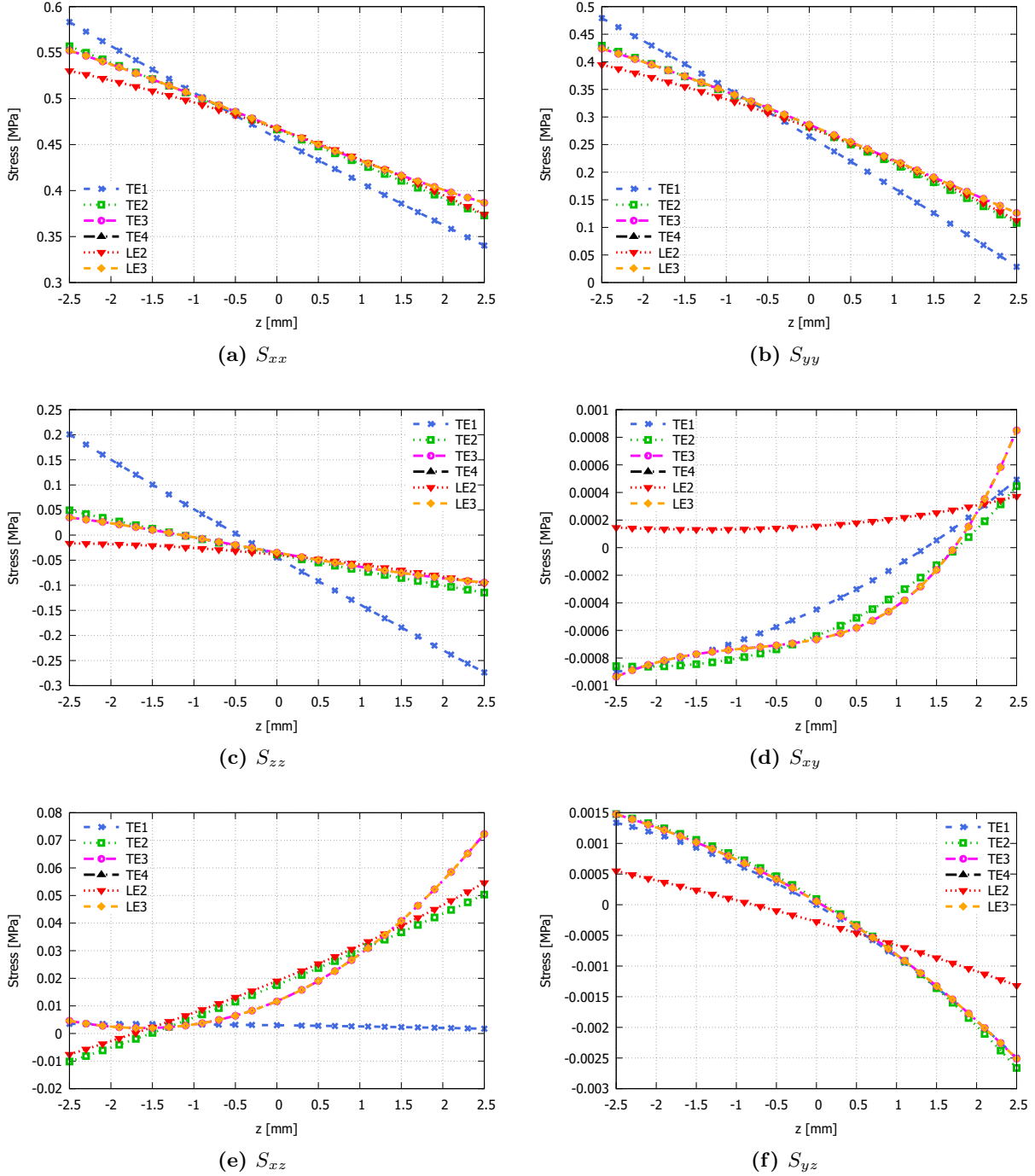


Figure 14: Circular plate effects of theory expansion on through-the-thickness stresses distribution, Piola-Kirchhoff 2 stresses for the isotropic case and load condition $\mathbf{q}_z = 0.1$ MPa.

431 The same comparison is considered to investigate Cauchy's stress, measured again at the same point
 432 and considering the same load configuration. Cauchy's stresses are computed thanks to the deformation
 433 gradient and the already available PK2 by Eq. (55). The through-the-thickness distribution of Cauchy's
 434 stresses obtained adopting the same expansion approximation theories is shown in Fig. 14. Again, the
 435 same considerations previously made are experienced in this last comparison.

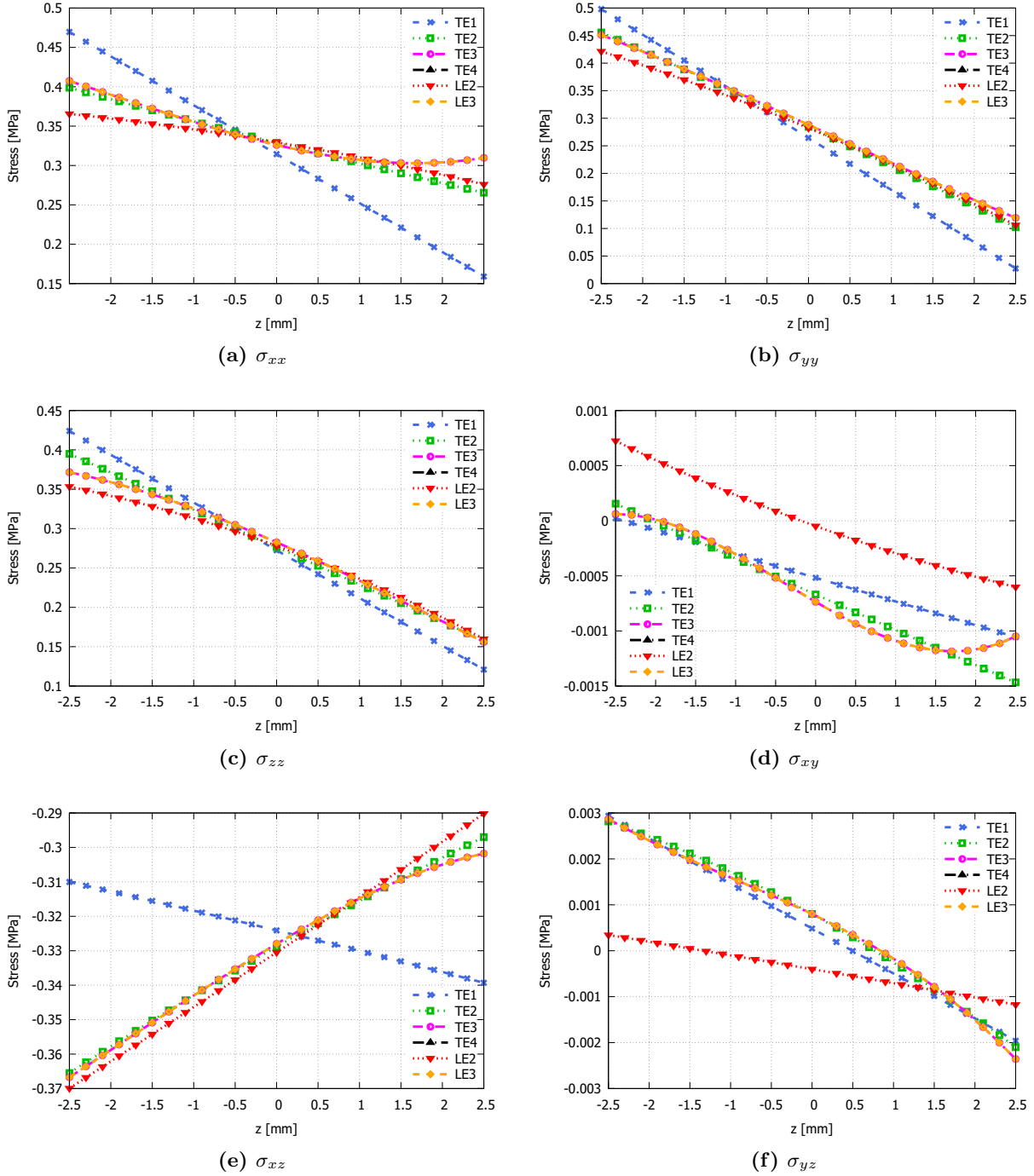


Figure 15: Circular plate effects of theory expansion on through-the-thickness stresses distribution, Cauchy's stresses for the isotropic case and load condition $q_z = 0.1$ MPa.

436 **5.3 Cantilever square plate under traction pressure**

437 In this third case study, the large strains analysis of a cantilever square plate, presented by Beheshti *et*
 438 *al.* [13], is carried out. A square plate of lateral length $a = 20$ mm and thickness $t = 1$ mm is subjected
 439 to a tensile pressure load at the free end of the plate. The geometrical features and boundary conditions
 440 are depicted in Fig. 16(a).

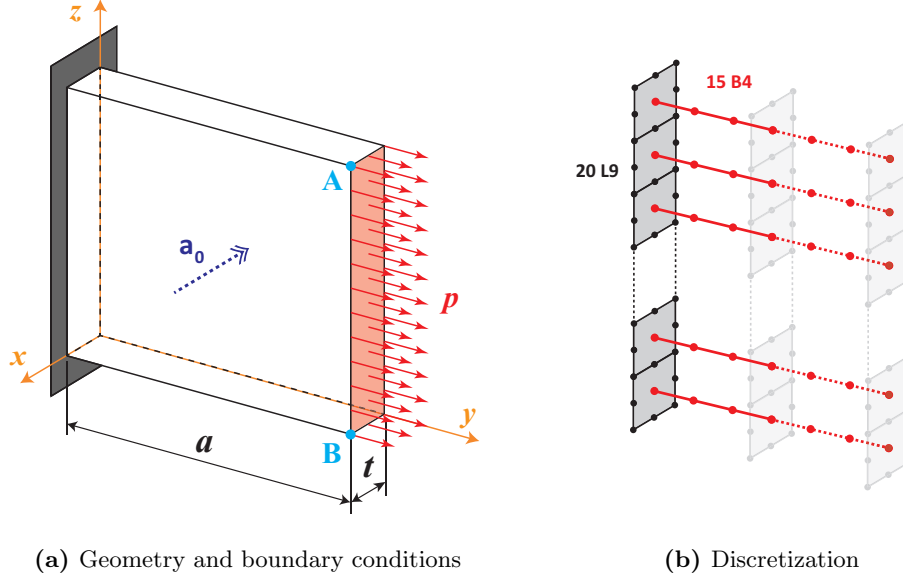
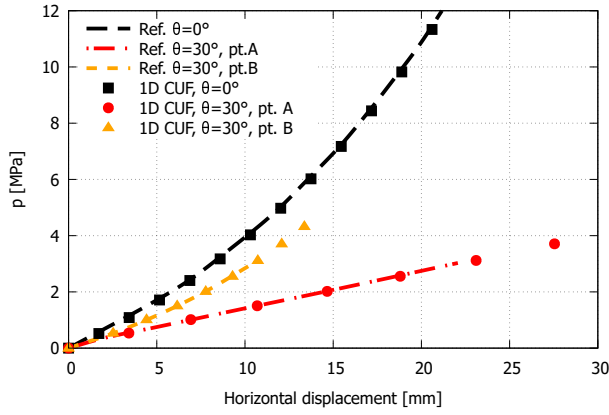


Figure 16: Clamped square plate configuration of the case study

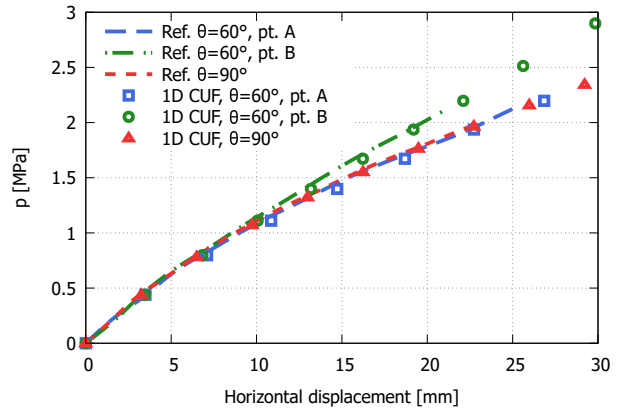
441 The square plate is made by a transversely isotropic hyperelastic material with a single-fiber preferred
 442 direction, defined as the unitary vector \mathbf{a}_0 in the plane $y - z$ inclined of an angle θ with respect
 443 to y -axis, thus $\mathbf{a}_0 = (0, \cos \theta, \sin \theta)$ the same plate is studied in different fiber configurations, with
 444 inclination angle varying from 0° to 90° . Material is modeled adopting the same strain energy function
 445 model Eq.(56) of the previous case study, with the same material constants as done in the reference
 446 case [13].

447
 448 The structure is discretized adopting 1D CUF models, employing 20 L9 parabolic elements along
 449 the clamped side of the plate and 15 B4 cubic elements along the longitudinal side of the beam in the
 450 y direction as shown in Fig. 16(b), for a total number of degrees of freedom equal to 16974.

451 The effects of anisotropy and the presence of a preferential direction are investigated by analyzing
 452 the horizontal displacement of points A and B, located at the tip free-end of the plate, for increasing
 453 the value of the tensile traction load applied. Figure 17(a) shows the load-displacement curve for fiber
 454 inclination angle $\theta = 0^\circ$ and $\theta = 30^\circ$ in this case, for $\theta = 0^\circ$ the horizontal displacement of the two-point
 455 are exactly coincident and, since the fiber direction is aligned with the load direction, the plate is much
 456 stiffer, instead in the case of $\theta = 30^\circ$ there is a transversal component of the preferential direction that
 457 affects the deformation process, and the final configuration results unsymmetric. Figure 17(b) shows
 458 instead the equilibrium path curve for fiber inclination angle $\theta = 60^\circ$ and $\theta = 90^\circ$. Differently with
 459 respect to the previously considered cases, since the preferential direction is more inclined, the plate
 460 is much less stiff; large displacement and strains of the plate are obtained with much lower values of
 461 applied pressure. Again, when $\theta = 90^\circ$ there is no component along the direction of the load thus the
 462 deformed structure is symmetric.



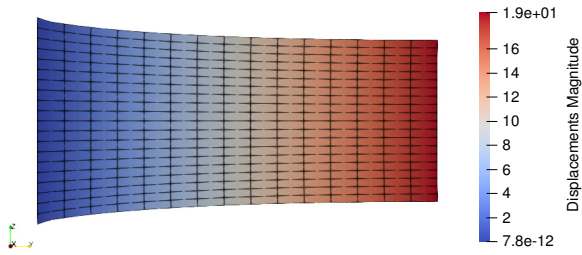
(a) Cases $\theta = 0^\circ$ and $\theta = 30^\circ$



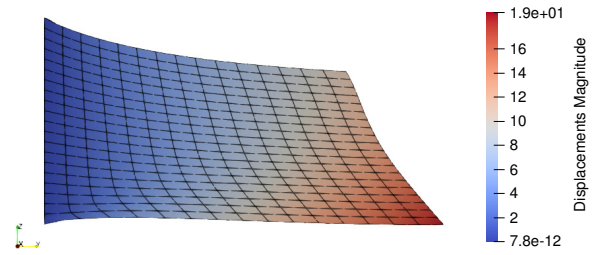
(b) Cases $\theta = 60^\circ$ and $\theta = 90^\circ$

Figure 17: Clamped square plate equilibrium curve for various fiber inclination

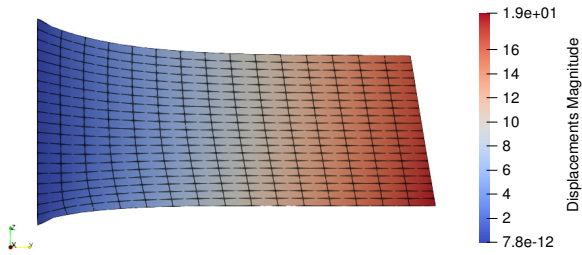
463 Figure 18 shows the deformed configuration for different fiber inclination configurations when the
 464 horizontal displacement of the point *B* is around 19 mm it can be clearly noted the strong influence
 465 of the mechanical behavior of the material with respect to anisotropy preferential direction.



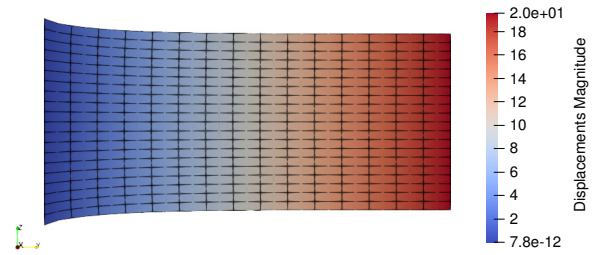
(a) Fiber angle $\theta = 0^\circ$



(b) Fiber angle $\theta = 30^\circ$



(c) Fiber angle $\theta = 60^\circ$



(d) Fiber angle $\theta = 90^\circ$

Figure 18: Clamped square plate deformed configuration representation

6 Conclusions

In this paper, we proposed the unified beam (1D) and plate (2D) CUF models for the large strain analysis of materials and structures in the hyperelastic compressible and incompressible regimes. In the domain of CUF, governing equations for the nonlinear static analysis of transversely isotropic hyperelastic materials are expressed in a compact notation starting from a generalized expansion of the 3D displacement unknowns coupling kinematic models and expansion theories, in a resulting expression of physical quantities (internal and external forces vector, tangent stiffness matrix) in matrix form, defining our fundamental nuclei independent of the polynomial expansion adopted. Numerical solutions are obtained by solving an algebraic system of equations with a Newton-Raphson linearized scheme. Our proposed results prove the capabilities of the present implementation of CUF 1D and 2D models to deal with large strains of fiber-reinforced hyperelastic structures, providing accurate results in terms of displacement and stress distributions, thanks to the higher-order three-dimensional description of the stress field guaranteed by the Unified formulation with adequate computational costs required for convergent solutions. The generalization of the constitutive law from isotropic hyperelastic to transversely isotropic hyperelastic materials is straightforward thanks to the Unified Formulation of tangent stiffness matrix in which the material Jacobian tensor is employed instead of the classical elasticity tensor, allowing us to rewrite the formulation of fully nonlinear finite element CUF models in the same framework without loss of generalities. Future works will deal with the extension of CUF hyperelastic models to shell structures, multilayered hyperelastic composites involved in biological tissue modeling (for which a suitable model for anisotropic behavior is required), the generalization of constitutive law adopted for orthotropic hyperelastic models in which the influence of two principal fiber directions are included, the stress analysis of multilayered composites made of linear elastic and hyperelastic layers, and finally the implementation of stabilization method for locking prevention, such as the *hybrid* formulation, in which the hydrostatic pressure (directly linked to the volumetric strains) is interpolated with an independent polynomial expansion.

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