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Design of the Impulsive Goodwin's Oscillator in 1-cycle ⋆

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Abstract: This paper presents a systematic approach to design a hybrid oscillator that admits an orbitally stable periodic solution of a certain type with pre-defined parameters. The parsimonious structure of the Impulsive Goodwin's oscillator (IGO) is selected for the implementation due to its well-researched rich nonlinear dynamics. The IGO is a feedback interconnection of a positive third-order continuous-time LTI system and a nonlinear frequency and amplitude impulsive modulator. A design algorithm based on solving a bilinear matrix inequality is proposed yielding the slope values of the modulation functions that guarantee stability of the fixed point defining the designed periodic solution. Further, assuming Hill function parameterizaton of the pulse-modulated feedback, the parameters of those rendering the desired stationary properties are calculated. The character of perturbed solutions in vicinity of the fixed point is controlled through localization of the multipliers. The proposed design approach is illustrated by a numerical example. Bifurcation analysis of the resulting oscillator is performed to explore the nonlinear phenomena in vicinity of the designed dynamics.

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Keywords: Hybrid and switched systems modeling, Event-based control, Control in systems biology, Controller constraints and structure, Application of nonlinear analysis and design.

1. INTRODUCTION

The impulsive Goodwin's oscillator (IGO) was introduced in Medvedev et al. (2006); Churilov et al. (2009) as a mathematical model of non-basal testosterone regulation in the male. The hybrid dynamics of the IGO portray the discrete control exercised by neurons of the brain on the hormone secretion of the endocrine glands of the organism by manipulating the amplitude and frequency of (release) hormone concentration pulses in blood.

In the IGO, the nonlinear static feedback of the original (continuous) Goodwin's oscillator, see Goodwin (1965), is substituted with a pulse modulation law. The impulsive feedback is constructed so that the IGO lacks equilibria and, therefore, overcomes the well-known limitations of its continuous predecessor in achieving sustained oscillation. It is also shown in Mattsson and Medvedev (2015) that solutions of the IGO can be fitted well to experimental data of measured testosterone and luteinizing hormone concentrations in healthy males. In contrast, the continuous Goodwin's oscillator remains purely a conceptional illustration of biological feedback regulation.

During the one and a half decennium since the inception of the IGO, numerous publications featuring generalizations of the concept to more dynamically complex continuous parts have appeared. For instance, the effects of time delay Churilov et al. (2014), exogenous driving signal Medvedev et al. (2018), and local feedback Taghvafard et al. (2019) have been studied. Yet, only analytical and numerical analysis of the IGO was addressed in these papers, focusing on periodical and non-periodical oscillating solutions.

The IGO in a periodic solution can also be perceived as a feedback structure to produce a repeating sequence of impulses exhibiting certain timing and weight. For an orbitally stable periodic solution, the attractivity of the orbit guarantees convergence to the nominal sequence after a deviation from the orbit has been introduced. The problem of selecting the parameters of the IGO to obtain a predefined solution is first considered in Medvedev et al. (2023). The introduced design approach employs necessary and sufficient stability conditions of a periodic solution to the IGO with a single firing of the pulse-modulated feedback in the least period, i.e. a 1-cycle. These stability conditions are difficult to generalize to solutions of higher periodicity and continuous dynamics of higher order.

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The present paper describes a solution to the IGO design problem based on a Bilinear Matrix Inequality (BMI), see e.g. VanAntwerp and Braatz (2000), for a 1-cycle with a pre-defined period and pulse weight. The proposed approach opens up for the use of established optimization tools for the IGO-design with respect to solutions of higher periodicity. Interestingly, the resulting computational problem is similar to what appears in output feedback stabilization of linear time-invariant systems, cf. Cao et al. (1998). Yet, in the case of IGO, the design is performed with respect to a certain periodic solution expressed as a fixed point and considered together with the continuous dynamics of the oscillator. Further, the matter of modulation functions parameterization is emphasized in the proposed design approach since the solution to the BMI and the desired solution parameters only provide interpolation points to the discrete part of the IGO.

The paper is organized as follows. First, the equations of the IGO are summarized. Necessary for the paper exposition facts about the dynamics of the IGO and, in particular, 1-cycle are then presented. Then, the proposed design procedure is introduced and illustrated by a numerical example. Further, to review the nonlinear dynamics of the designed IGO under parameter variation and in transients around the fixed point, bifurcation analysis is performed.

2. THE IMPULSIVE GOODWIN'S OSCILLATOR

The continuous part of the IGO, see Medvedev et al. (2006); Churilov et al. (2009), is given by

$$\dot{x}(t) = Ax(t), \quad z(t) = Cx(t), \tag{1}$$

where

$$A = \begin{bmatrix} -a_1 & 0 & 0 \\ g_1 & -a_2 & 0 \\ 0 & g_2 & -a_3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C^\intercal = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$a_1, a_2, a_3 > 0$ are distinct constants, $g_1, g_2 > 0$ are positive gains, z is the controlled output, and the state variables are $x = [x_1, x_2, x_3]^\top$. It is readily observed that the matrix A is Hurwitz stable and Metzler. Both properties agree well with the biological background of the model. Asymptotic stability of (1) corresponds then to the fact that biochemical substances decay with time and the positivity of x ensures an interpretation of the state variables in terms of concentrations. The chain structure of the continuous part portrays three substances represented by their concentrations where a preceding substance stimulates the production of the next one.

The impulsive feedback constitutes a difference equation

$$x(t_n^+) = x(t_n^-) + \lambda_n B, \quad t_{n+1} = t_n + T_n, \tag{2}$$
$$T_n = \Phi(z(t_n)), \quad \lambda_n = F(z(t_n)), \quad B^\intercal = [1\ 0\ 0],$$

where $n = 0, 1, \ldots$. The minus and plus in a superscript in (2) denote the left-sided and a right-sided limit, respectively. The instants t_n are called (impulse) firing times and λ_n represents the corresponding impulse weight. Notwithstanding the jumps in (2), $z(t)$ is a smooth function since

$$CB = 0, \, CAB = 0, \, CA^2B \neq 0. \tag{3}$$

Control law (2) constitutes a frequency and amplitude pulse modulation operator Gelig and Churilov (1998) implementing an output feedback over (1). The amplitude modulation function $F(\cdot)$ and frequency modulation function $\Phi(\cdot)$ are continuous and monotonic; $F(\cdot)$ is non-increasing and $\Phi(\cdot)$ is non-decreasing, also,

$$\Phi_1 \leq \Phi(\cdot) \leq \Phi_2, \quad 0 < F_1 \leq F(\cdot) \leq F_2, \tag{4}$$

where Φ_1, Φ_2, F_1, F_2 are positive constants. Despite of the impulsive nature of the feedback, its effect on continuous part (1) is, in a broad sense, similar to that of a conventional negative feedback. The firings of feedback law (2) are more sparse (larger T_n) and less prominent (lower weight λ_n) for higher values of the output $z(t)$.

3. STABLE 1-CYCLE IN THE IGO

Propagating from one firing time to the next one, the sequence of the state vector of the IGO obeys the impulse-to-impulse map Churilov et al. (2009)

$$X_{n+1} = Q(X_n), \tag{5}$$
$$Q(\xi) = \mathrm{e}^{A\Phi(C\xi)} \left(\xi + F(C\xi)B\right),$$

where $X_n = x(t_n^-)$. A periodic solution of the IGO with one firing of the feedback on the period is referred to as 1-cycle, see Zhusubaliyev and Mosekilde (2003), implying

$$X = Q(X). \tag{6}$$

As proved in Churilov et al. (2009), the IGO always has a unique 1-cycle, whose pulse weight λ and period T can be directly evaluated from the model parameters.

Introduce first and second divided difference de Boor (2005) of the exponential function

$$\mathrm{e}[a, b] = \frac{\mathrm{e}^a - \mathrm{e}^b}{a - b}, \tag{7}$$

$$\mathrm{e}[a, b, c] = \frac{\mathrm{e}[c, b] - \mathrm{e}[a, b]}{c - a}. \tag{8}$$

The transition matrix of (1) is then

$$\exp At = \left[\begin{array}{c:c:c} \mathrm{e}^{-a_1 t} & 0 & 0 \\ g_1 t \, \mathrm{e}[-a_1 t, -a_2 t] & \mathrm{e}^{-a_2 t} & 0 \\ g_1 g_2 t^2 \, \mathrm{e}[-a_1 t, -a_2 t, -a_3 t] & g_2 t \, \mathrm{e}[-a_2 t, -a_3 t] & \mathrm{e}^{-a_3 t} \end{array}\right].$$

The expression for the matrix exponential is essentially a consequence of Opitz's formula, see e.g. de Boor (2003). In virtue of the Mean Value Theorem for divided differences (de Boor, 2005, Corollary to Proposition 43), all divided differences of the exponential function are positive. This is well in line with the fact of A being Metzler.

For further use, introduce the partition

$$A = \begin{bmatrix} A_{11} & 0_{2\times 1} \\ A_{21} & -a_3 \end{bmatrix}. \tag{9}$$

Then

$$\exp(A_{11} t) = \begin{bmatrix} \mathrm{e}^{-a_1 t} & 0 \\ g_1 t \, \mathrm{e}[-a_1 t, -a_2 t] & \mathrm{e}^{-a_2 t} \end{bmatrix}.$$

In the proposition below and throughout the paper, vector inequalities are interpreted element-wise.

Proposition 1. (Medvedev et al., 2023) Given the parameters of 1-cycle $T > 0$, $\lambda > 0$, the fixed point $X^\intercal = [x_{01}, x_{02}, x_{03}] > 0$ of the map Q from (5) corresponding to the 1-cycle is calculated as

$$
\begin{aligned}
x_{01} &= \frac{\lambda \mathrm{e}^{-a_1 T}}{1-\mathrm{e}^{-a_1 T}},\\
x_{02} &= \frac{\lambda g_1 T\, \mathrm{e}[-a_1 T, -a_2 T]}{(1-\mathrm{e}^{-a_1 T})(1-\mathrm{e}^{-a_2 T})},\\
x_{03} &= \frac{\lambda g_1 g_2 T^2 \Big(\mathrm{e}[-a_1 T, -a_2 T, -a_3 T]}{(1-\mathrm{e}^{-a_1 T})(1-\mathrm{e}^{-a_2 T})(1-\mathrm{e}^{-a_3 T})}\\
&\quad + \mathrm{e}[-(a_1+a_2)T, -(a_1+a_3)T, -(a_2+a_3)T]\Big).
\end{aligned} \tag{10}
$$

The coordinates of X scale linearly with λ but depend on T in a highly nonlinear manner. Thus adjusting the cycle amplitude is simpler than changing its period.

From Churilov et al. (2009), a 1-cycle is known to be orbitally stable if only only if the Jacobian

$$
Q'(X) = \mathrm{e}^{A\Phi(z_0)} \left(I + F'(z_0)BC\right) + \Phi'(z_0)AXC, \quad z_0 = CX, \tag{11}
$$

is Schur stable. In order to reduce the problem of stabilizing the fixed point of (6) to a feedback control problem, employ the following form of the Jacobian above.

Proposition 2. (Medvedev et al. (2023)). Jacobian (11) admits the parameterization

$$
Q'(X) = \mathrm{e}^{A\Phi(z_0)} + \left(F'(z_0)J + \Phi'(z_0)D\right)E,
$$

where $E = [0\ 0\ 1]$, $J^\mathsf{T} = [j_1, j_2, j_3]$, $D^\mathsf{T} = [d_1, d_2, d_3]$ and $J > 0$, $D < 0$.

Exact expressions for J, D are provided in Medvedev et al. (2023). Here it suffices to mention that $J = \mathrm{e}^{A\Phi(z_0)} B > 0$ since A is Metzler and $B > 0$. Further, $D = AX < 0$ and the inequality is due to the structure of A and the expression for X in (10).

Using Proposition 2, the Jacobian can be written as

$$
Q'(X) = \mathrm{e}^{A\Phi(z_0)} + [J\ \ D] \begin{bmatrix} F'(z_0) \\ \Phi'(z_0) \end{bmatrix} E, \tag{12}
$$

where $JF'(z_0) + D\Phi'(z_0) < 0$, for all feasible values of $F'(z_0), \Phi'(z_0)$. The latter inequality highlights the role of pulse-modulated feedback (2) as negative feedback with respect to the IGO output $z(t)$.

From (12), it becomes apparent that the problem of obtaining a stable 1-cycle corresponding to the fixed point X in the IGO is similar to the classical problem of (static) output feedback stabilization of a discrete linear time-invariant (LTI) system, see e.g. Cao et al. (1998). More specifically, the latter solves the problem of finding a gain K_d to stabilize the closed-loop system

$$
\begin{aligned}
x_d(t+1) &= A_d x(t) + B_d u(t),\\
y(t) &= C_d x(t), \quad u(t) = K_d y(t).
\end{aligned}
$$

In the IGO, the slopes of the modulation functions play the role of feedback gains and control the eigenvalue spectrum of the Jacobian. Since the signs of $F'(z_0)$ and $\Phi'(z_0)$ are constrained by the structure of the IGO and the actual dynamics of the oscillator are highly nonlinear, the similarity is only partial and stabilization is considered with respect to a certain solution, i.e. a 1-cycle.

By analogy with the output feedback stabilization problem, the selection of $F'(z_0), \Phi'(z_0)$ to stabilize the fixed point X is equivalent to solving, with respect to K and P

$$
(A_\Phi + WKE)^\mathsf{T} P (A_\Phi + WKE) - P < 0, \tag{13}
$$

where

$$
A_\Phi = \mathrm{e}^{A\Phi(z_0)}, W = [J\ \ D], K^\mathsf{T} = [F'(z_0)\ \ \Phi'(z_0)], P > 0,
$$

and P, K are decision variables. The matrix inequality (13) is non-convex, yet software for solving BMIs through iterative procedures is available, e.g. Dinh et al. (2011), although there is no proof of convergence and the computational problem is in general known to be NP-complex.

The eigenvalues of the Jacobian $Q'(X)$, i.e. the multipliers of the fixed point X, define the character of the transient behavior in vicinity of it, cf. Section 5. Due to the structure of the Jacobian in (12), its eigenvalues cannot be freely assigned by choosing the parameters of pulse-modulated feedback (2). The analogy to output feedback stabilization of LTI systems does not provide much insight into the spectrum localization of the closed-loop system. The result below partially answers the question of how the eigenvalues of $Q'(X)$ can be placed by selecting the slopes of the modulation functions.

The principal minors of a real matrix A

$$
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},
$$

are $\det A$, the diagonal entries $a_{ii}, i = 1, 2, 3$, and the second-order complementary minors

$$
\begin{aligned}
m_{11}(A) &= a_{22}a_{33} - a_{23}a_{32},\\
m_{22}(A) &= a_{11}a_{33} - a_{31}a_{13},\\
m_{33}(A) &= a_{11}a_{22} - a_{21}a_{12}.
\end{aligned}
$$

A matrix is called P-matrix if all the principal minors of it are positive Fiedler and Pták (1962).

Proposition 3. Consider the IGO given by (1), (2) where $F(\cdot)$ is non-increasing and $\Phi(\cdot)$ is non-decreasing. Then the Jacobian $Q'(X)$ is a P-matrix if and only if

$$
0 < \mathrm{e}^{-a_3 T} + \gamma \Phi'(z_0), \tag{14}
$$

$$
0 < \mathrm{e}^{-a_3 T} + j_3 F'(z_0) + d_3 \Phi'(z_0), \tag{15}
$$

where

$$
\begin{aligned}
\gamma &= d_3 - L \exp\left(-A_{11}T\right) \begin{bmatrix} d_1 \\ d_2 \end{bmatrix},\\
L &= g_2 T \left[g_1 T\, \mathrm{e}[-a_1 T, -a_2 T, -a_3 T]\ \ \mathrm{e}[-a_2 T, -a_3 T] \right].
\end{aligned}
$$

Proof. Omitted.

The result below follows due to the theorem on the localization of the eigenvalues in Kellogg (1972).

Corollary 4. Let the Jacobian $Q'(X)$ be a P-matrix with the eigenvalues $\mu_i, i = 1, 2, 3$. Then it holds that

$$
|\arg \mu_i| < \frac{2\pi}{3}.
$$

Thus the eigenvalues of the Jacobian cannot, under the specified conditions, lie in a certain wedge around the negative axis in the complex plane.

4. DESIGN PROBLEM

Consider the problem of selecting the free parameters of the IGO defined by (1), (2) for the oscillator to exhibit a 1-cycle of the (least) period of T and with the weight λ.

Assume the parameter values $g_1 = 2.0$, $g_2 = 0.5$, $a_1 = 0.08$, $a_2 = 0.15$, and $a_3 = 0.2505$. Design now feedback

(2) ensuring that the IGO exhibits a 1-cycle with the parameters $\lambda = 5$, $T = 30$.

Making use of Proposition 1, the fixed point corresponding to the desired solution is

$$X^{\intercal} = [0.4988\ 12.6479\ 33.9716], \tag{16}$$

and thus $z_0 = 33.9716$. Design inequality (13) is satisfied for $F'(z_0) = -0.1$, $\Phi'(z_0) = 0.05$, and

$$P = \begin{bmatrix} 0.4055 & 0.1362 & 0.3581 \\ 0.1362 & 3.7646 & 8.8469 \\ 0.3581 & 8.8469 & 23.6682 \end{bmatrix}.$$

The spectral radius is then $\rho\,(Q'(X)) = 0.596181$.

Following Churilov et al. (2009), define the modulation functions in (2) as the Hill functions

$$\Phi(z) = k_1 + k_2\,\frac{(z/h_\Phi)^{p_\Phi}}{1+(z/h_\Phi)^{p_\Phi}}, \tag{17}$$

$$F(z) = k_3 + \frac{k_4}{1+(z/h_F)^{p_F}}.$$

Then the bounds in (4) are $F_1 = k_3, F_2 = k_3 + k_4, \Phi_1 = k_1, \Phi_2 = k_1 + k_2$. To obtain the desired 1-cycle parameters, the coefficients of the modulation functions have to satisfy

$$k_3 < \lambda < k_3 + k_4, \quad k_1 < T < k_1 + k_2. \tag{18}$$

The derivatives of modulation functions (17) are

$$\Phi'(z) = \frac{k_2 p_\Phi z^{p_\Phi - 1} h_\Phi^{-p_\Phi}}{\left(1+(z/h_\Phi)^{p_\Phi}\right)^2},\ F'(z) = -\frac{k_4 p_F z^{p_F-1} h_F^{-p_F}}{(1+(z/h_F)^{p_F})^2}.$$

Notice that, for $p_F, p_\Phi < 1$, both $\Phi'(z)$ and $F'(z)$ have a singularity in $z = 0$.

Frequency modulation: Introduce the auxiliary variables

$$\eta_\Phi = \left(\frac{z_0}{h_\Phi}\right)^{p_\Phi}, \quad \theta_\Phi = \frac{k_2 p_\Phi}{2 z_0 \Phi'(z_0)}.$$

Then, to render the desired value $\Phi'(z_0)$, θ_Φ has to satisfy the equation

$$\eta_\Phi^2 + 2(1-\theta_\Phi)\eta_\Phi + 1 = 0,$$

resulting in

$$\eta_{\Phi,1,2} = \theta_\Phi - 1 \pm \sqrt{\theta_\Phi(\theta_\Phi - 2)}.$$

Clearly, only positive real values of η_Φ are feasible, which fact limits the Hill function order of the frequency modulation function from below

$$p_\Phi > \frac{4 z_0 \Phi'(z_0)}{k_2}.$$

Keeping in mind (18), select $k_2 = 40$, yielding $p_\Phi > 0.1699$. When real, $\eta_{\Phi,1,2}$ are always positive. For $p_\Phi = 2$, one obtains $\eta_{\Phi,1} = 45.0760, \eta_{\Phi,2} = 0.0222$ and, from

$$h_{\Phi,1,2} = \frac{z_0}{\sqrt[p_\Phi]{\eta_{\Phi,1,2}}},$$

two feasible values of the scaling coefficient $h_{\Phi,1} = 5.0599$, $h_{\Phi,2} = 228.0806$ are calculated. The value of k_1 is obtained from the equation $\Phi(z_0) = T$. By substituting $h_{\Phi,1}$, $h_{\Phi,2}$ into the expression for $F(z)$, one has

$$k_{1,1} = -9.1319, \quad k_{1,2} = 29.1319.$$

Naturally, a negative time difference between impulses is not feasible and the positive value of k_1 has to be selected.

Amplitude modulation: Similarly, in terms of the auxiliary variables

$$\eta_F = \left(\frac{z_0}{h_F}\right)^{p_F}, \quad \theta_F = \frac{k_4 p_F}{2 z_0 F'(z_0)},$$

one has

$$\eta_F^2 + 2(1+\theta_F)\eta_F + 1 = 0,$$

and then

$$\eta_{F,1,2} = -(\theta_F + 1) \pm \sqrt{\theta_F(\theta_F + 2)}.$$

To obtain a real value of η_F, the order of the Hill function should be bounded from below

$$0 < -\frac{4 z_0 F'(z_0)}{k_4} < p_F. \tag{19}$$

To fulfill (18), select $k_4 = 0$, yielding $1.6986 < p_F$. Thus the choice $p_F = p_\Phi = 2$ is feasible and leads to $\eta_{F,1} = 2.2691, \eta_{F,2} = 0.4407$. Then the scaling factors of the amplitude modulation function are $h_{F,1} = 22.5521$, $h_{F,2} = 51.1734$. The value of k_3 is calculated from $F(z_0) = \lambda$, which gives for each $h_{F,1}, h_{F,12}$

$$k_{3,1} = 2.5529, \quad k_{3,2} = -0.5529.$$

Only $k_{3,1}$ is feasible since a negative k_3 allows (cf. (18)) for negative impulse weights in transients. To recapitulate, the following parameters of the modulation functions apply

Scenario 1
$\Phi(z):\ k_1 = 29.1319$, $k_2 = 40$, $p_\Phi = 2$, $h_\Phi = 228.0806$
$F(z):\ k_3 = 2.5529$, $k_4 = 8$, $p_F = 2$, $h_F = 22.5521$

The designed modulation functions $F(z)$, $\Phi(z)$, together with their derivatives are plotted in Fig. 1.

Obtaining a unique parameter set for the modulation functions given a continuous part of the IGO and a predefined 1-cycle is not always possible. To illustrate this, consider a case of $p_\Phi < 1$ resulting in the following parameter values for the same fixed point as before.

Scenario 2
$\Phi(z):\ k_1 = 27.7502$, $k_2 = 40$, $p_\Phi = 0.8$, $h_\Phi = 1.1537 \cdot 10^3$
$F(z):\ k_3 = \{1.1157, 0.8843\}$, $k_4 = 8$, $p_F = 1.7$, $h_F = \{32.8351, 35.1474\}$

This solution is characterized by a high value of h_Φ that makes the frequency modulation function less insensitive to changes in the output $z(t)$. Two similar feasible solutions for the amplitude modulation functions then arise.

5. BIFURCATION ANALYSIS

The dynamics of the IGO, as of any pulse-modulated system, are highly nonlinear and the design procedure outlined in the previous section is based on a desired stationary periodic solution. Therefore, numerical analysis of system behavior in deviation from the stationary solution and under parametric uncertainty is necessary.

In the analysis below, the parameters of Scenario 1 in Section 4 are assumed.

Fig. 2(a) shows the variation of the multipliers for fixed point (16) with respect to the parameter a_3 in (1). The latter defines the time constant of the last step of the first-order block chain and can be seen as a pre-filter introduced before feedback controller (2). Due to linearity of the continuous part of the IGO, similar bifurcation

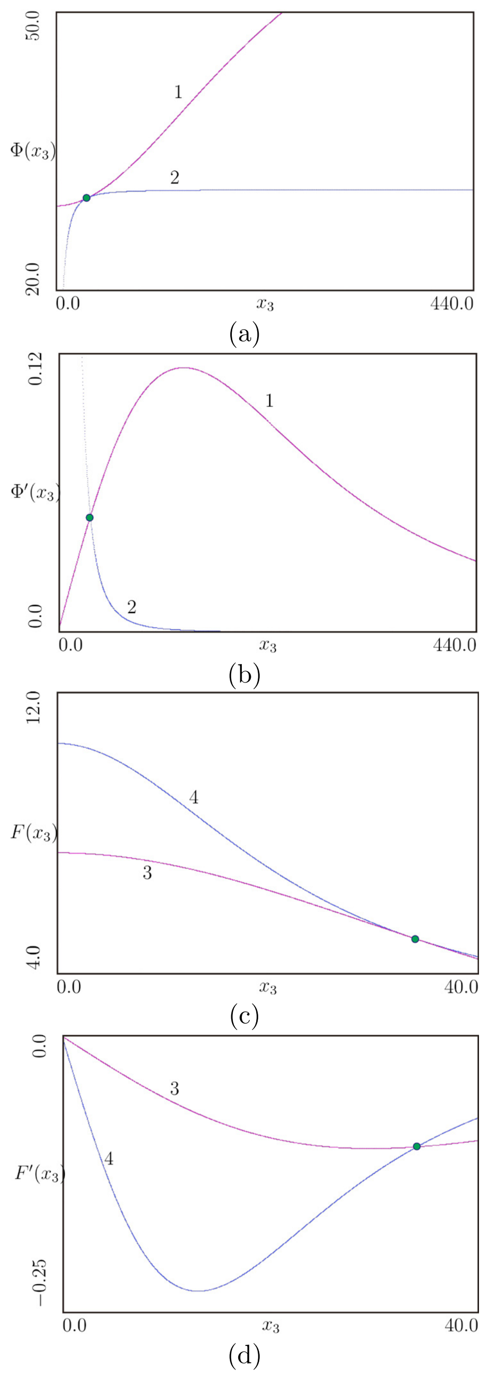

Fig. 1. The designed modulation functions and their derivatives. Function value in z_0 is marked by circle. The 1-cycle parameters $\lambda = 5$, $T = 30$. (a) $\Phi(x_3)$. (b) $\Phi'(x_3)$. Here $h_\Phi = 228.0806$, $k_1 = 29.1319$ for 1 and $h_\Phi = 5.0599$, $k_1 = -9.1319$ for 2. (c) $F(x_3)$. (d) $F'(x_3)$. Here $h_F = 22.5521$, $k_3 = 2.5529$ for 3 and $h_F = 51.1734$, $k_3 = -0.5525$ for 4.

phenomena arise under variation of a_1 and a_2. To better illustrate this diagram, a magnified part of the Fig. 2(a) that is outlined by the rectangle is presented in Fig. 2(b). As the parameter a_3 increases, two multipliers ρ_1 and ρ_2 of a stable fixed point may become complex: one of the real multipliers becomes equal to another one $\rho_1 = \rho_2 = \mu$ and results in the pair of complex-conjugated multipliers $\rho_{1,2} = \mu \pm i\,\omega$. Here μ and ω are the real and imaginary

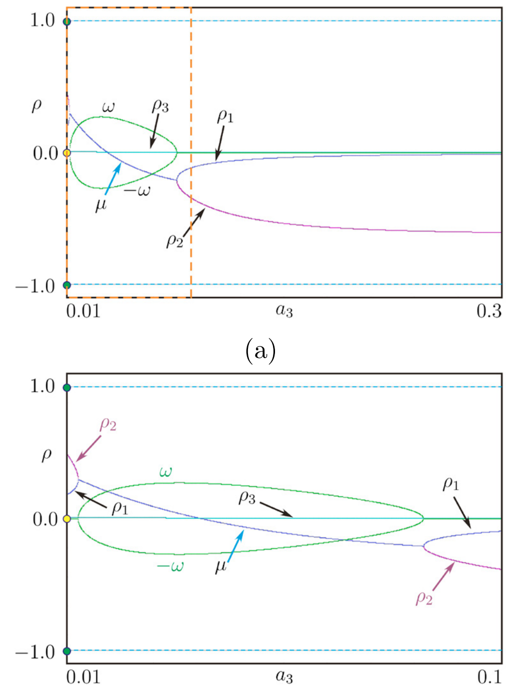

Fig. 2. (a) Variation of the multipliers $\rho_{1,2}$ for $0.01 < a_3 < 0.3$. $\lambda = 5.0$, $T_0 = 30.0$. (b) Magnified part of the multiplier diagram for the fixed point that is outlined by the rectangle in (a).

(green line) parts of the multipliers $\rho_{1,2} = \mu \pm i\,\omega$. As one can see from Fig. 2, the third multiplier ρ_3 is always real.

If one of the multipliers is real (in Fig. 2 ρ_3, $|\rho_3| < 1$) and the other two are complex-conjugated $\rho_{1,2}$ and $|\rho_{1,2}| = \sqrt{\mu^2 + \omega^2} < 1$, then a fixed point is called a stable focus. If all multipliers ρ_1, ρ_2, ρ_3 are real, and modulo less than one, then a fixed point $\mathcal{O}$ is called a stable node. With further increase in the value of a_3, a pair of complex-conjugated multipliers $\rho_{1,2} = \mu \pm i\omega$ become real again, and the stable focus fixed point transforms into a stable node one.

Figs. 3(a),(b) show the transient behavior when the two multipliers ρ_1, ρ_2 are real and negative, the third ρ_3 one is close to zero. In this case the convergence to the desired 1-cycle is non-monotonous and highly oscillating. This type of convergence can be beneficial in applications where mean values of the output are important as the transient behavior exhibits a train of interchanging over- and under-doses tending to the pre-defined λ.

Finally, Fig. 3(c),(d) covers the case when one of multipliers is real and the other two are complex-conjugated. Then the convergence is also non-monotonous but the transient process offers a compromise between the convergence rate and overshoot. This a type of transient that is often encountered in process control.

Notice that the multipliers cannot be placed arbitrarily due to the constrained structure of $Q'(X)$ given by (12).

6. CONCLUSION

The problem of designing a pulse-modulated feedback for a given continuous part of the IGO and pre-defined pa-

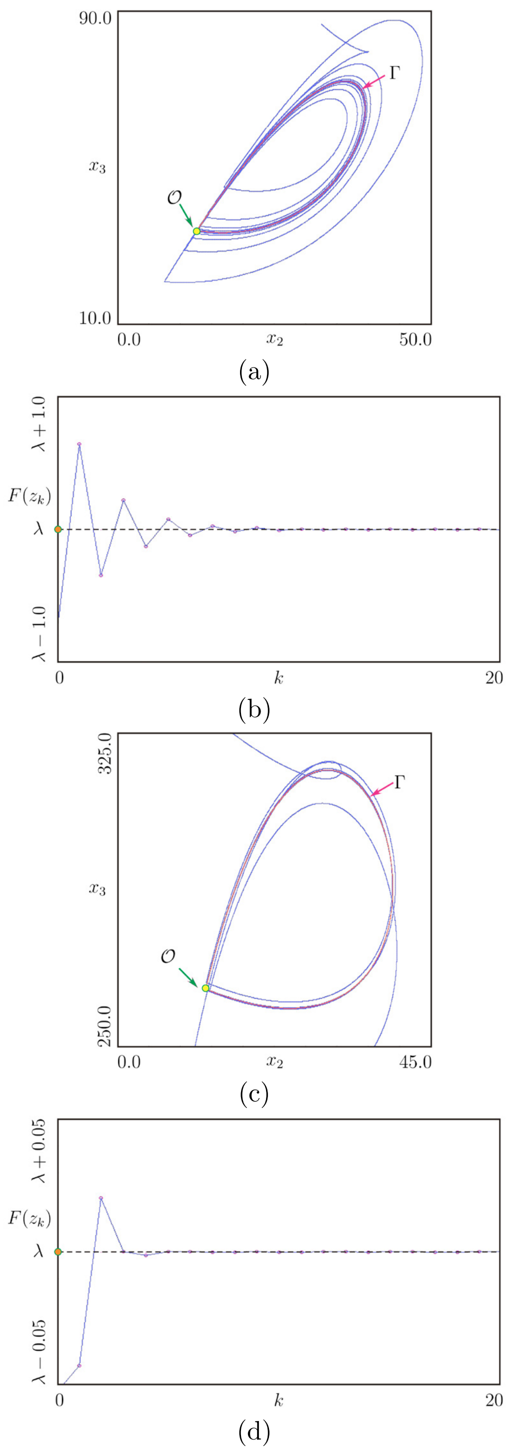

Fig. 3. (a) The parameter values $a_3 = 0.2505$ and multipliers: $\rho_1 = -0.0161654$, $\rho_2 = -0.596171$ and $\rho_3 = 5.07482 \cdot 10^{-5}$. (b) The convergence of the sequence $F(z_k)$ to the $\lambda = 5.0$. (c) The parameter values $a_3 = 0.0475$ and multipliers: $\rho_{1,2} = \mu \pm i\omega$, $\mu = -0.06864$ and $\omega = 0.245587$. (d) The convergence of the sequence $F(z_k)$ to $\lambda = 5.0$.

rameters of a 1-cycle is considered and solved for Hill-type modulation functions. The design problem involves calculating a fixed point of the discrete map that captures the evolution of the continuous state vector of the oscillator through the sequence of feedback firings. The fixed point and, therefore, the corresponding 1-cycle is stabilized by evaluating the slopes of the modulation functions at the fixed point as a solution of a BMI.

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