Optimal Clearing Payments in a Financial Contagion Model

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Abstract. Financial networks are characterized by complex structures of mutual obligations. These obligations are fulfilled entirely or in part (when defaults occur) via a mechanism called *clearing*, which determines a set of payments that settle the claims by respecting rules such as limited liability, absolute priority, and proportionality (pro-rated payments). In the presence of shocks on the financial system, however, the clearing mechanism may lead to cascaded defaults and eventually to financial disaster. In this paper, we first study the clearing model under pro-rated payments of Eisenberg and Noe, and we derive novel necessary and sufficient conditions for the uniqueness of the clearing payments, valid for an arbitrary topology of the financial network. Next, we observe that the proportionality rule is a factor that potentially concurs to the cascaded defaults effect, and that the aggregated systemic loss can be reduced if this rule is lifted. We thus shift the focus from the individual interest to the overall systemic interest to contain the adverse effects of cascaded failures, and we show that pro-rate-free clearing payments can be computed uniquely by solving suitable convex optimization problems.

Key words. Financial networks, systemic risk, clearing payments, linear programming, graph theory.

MSC codes. 91G45, 90C35, 90B10, 05C90

1. Introduction.

The financial industry is participated by organizations that are linked to each other by means of an intricate structure of mutual obligations. The behavior of such financial interconnected system has been extensively studied over the past years, see for instance [6, 19]. The interconnection among financial institutions creates potential channels of contagion, whereby a failure (financial default) of a single entity in the system can result in a threat to the stability of the entire global financial system. Recent examples of such a behavior include the collapse of Lehman Brothers, recognized as one of the reasons of the global financial crisis in 2008, and the government bailout of the giant insurance company AIG, [7]. Much effort has been invested in understanding the effects of *systemic risk*, that is, how stresses, such as bankrupts and failures, to one part of the system can spread to others, and eventually lead to avalanche breakdowns, see, e.g., [13, 15, 21, 22].

An important line of research pursued in systemic risk theory focuses on the development of realistic models of *clearing* procedures between financial institutions. Clearing is essentially a set of rules under which the participants to a financial network agree to settle payments, when these payments cannot meet the original liabilities due to defaults, [27]. The seminal

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work in [13] introduced a simple mathematical model of clearing in a financial network, in which financial institutions have two types of assets: the external assets (e.g., incoming cash flows) and the internal assets (e.g., funds that banks lend to one another). The model in [13] assumes that the obligations of all entities within the financial system are paid simultaneously and are determined by three fundamental rules: 1) limited liability, that is, the total payment of each node can not exceed its available cash flow; 2) the priority of the debt claims, that is, stockholders receive no value until the node is able to completely pay off all of its outstanding liabilities; 3) the proportionality, or pro-rata rule, that is, all debts have equal priority, so that all claimant institutions are paid proportionally to their nominal claims. Under these assumptions, the matrix of mutual interbank payments is uniquely determined by the so-called *clearing vector*, which is found as the fixed point of a nonlinear equation. This vector always exists [13], whereas its uniqueness has been proved under certain regularity assumptions, see [13, 21, 27, 34]; these uniqueness conditions are however only *sufficient*, but not necessary.

The basic model offered in [13] has been later extended in various directions, incorporating non-trivial features of real-world financial networks, see, e.g., the recent survey in [26]. The models presented in [16, 36], for instance, take into account cross-holdings and seniorities of liabilities, and the works of [11, 35] introduce the concept of liquidity risk. Other works considered also illiquid assets [4], cross-ownership of equities and liabilities [18], decentralized clearing processes [12], and multiple maturity dates [29].

The contribution of the present work is twofold. First, we give a full solution to the problem of the uniqueness of the clearing vector in the proportional payments case. From the financial perspective, uniqueness is important since it guarantees that no ambiguity exists in the payments, so that each entity must abide to one and only one clearing payment, with no possible controversy. From the computational viewpoint, uniqueness ensures that different methods for finding a clearing vector, such, e.g., the fictitious default algorithm [21] and more advanced methods from [29], return the same answer. The first sufficient graph-theoretical condition for the uniqueness of the clearing payments was obtained in [13] (see also the works [14, 27], giving a simpler and more elegant proof): the clearing vector is unique if the financial network is *regular*, which means that every bank either has an outside asset, or has a (direct or indirect) creditor with outside assets. Another sufficient condition for uniqueness is formulated in [21]: the clearing vector is unique if each node of the network has a chain of liabilities to the external sector. Both conditions are only sufficient yet not necessary. To the best of our knowledge, the only necessary and sufficient condition for the clearing vector's uniqueness applicable to an arbitrary financial network available in the literature is the very general result in [30], which examines the uniqueness of equilibria in a dynamical flow network with saturations. This criterion, primarily motivated by more general models of systemic risk proposed in [17], however, appears to be of limited practical use in the classical Eisenberg-Noe model, since it requires computation of some parameters that depend on the payment matrix (left and right Perron eigenvectors for its irreducible blocks). Unlike the procedure in [30], our proposed method allows to test the uniqueness of the clearing vector without knowledge of the payment matrix; only the graph of liability relations matters. Similar to [30], we are also able to find the whole set of clearing vectors. As a byproduct of the developed theory, we also derive some new properties of the maximal (or dominant) clearing vector that are of independent interest.

The second key aspect addressed in this paper is the lifting of the proportionality (pro-rata) rule. Such division rule is the most common in the literature on systemic risk and it was introduced since the seminal work of [13]. The proportional rule, however, may not be realistic in some situations, for instance when liabilities have different seniorities, see, e.g., [28]. Other division rules have thus been also considered in the literature; examples include constrained equal awards or constrained equal losses rules [37], welfare-maximizing rules [20], and general division rules [5,12]. In [12], the authors propose a decentralized clearing process in a discrete setup which assumes integer payments. Instead, [5] considers a multi-period liability clearing problem. Differently from the model considered in this work, [5] also makes the assumption that entities cannot pay other entities more than the cash they have on hand. Motivated by these works, in Section 5 a clearing scheme under unconstrained (i.e., non necessarily proportional) payments is analyzed. We show that an optimal clearing matrix can be computed by solving a linear program. Plain relaxation of the proportionality constraint leads as a downside to the loss of the clearing matrix uniqueness. Also, the feasible payment matrices do not constitute a complete lattice, in particular, even for very simple financial networks with $n = 3$ nodes the minimal and maximal payment matrix may fail to exist. However, we recover uniqueness by means of a simple two-stage optimization scheme whose result is the unique clearing payments matrix that *(a)* achieves the best possible deviation loss from the nominal liabilities, and *(b)* has the minimum "size" among all matrices achieving the optimal loss. The positive systemic effects obtained by releasing the pro-rata rule in favor of an aggregated performance approach are illustrated by means of a schematic example as well as via extensive numerical tests using a synthetic random network, which is similar to a "testbench" model from [31].

The practical implementation of an optimal unconstrained clearing payments scheme, however, currently faces several obstacles. Some of these obstacles are actually shared also by the pro-rated scheme and by other mathematical default schemes, and are due to unmodelled non-idealities, contract renegotiations, credit freeze, local laws and delays in their application. Another issue is that any clearing scheme should be contractualized ex-ante and all players should adhere to it. More relevantly, an optimal aggregated unconstrained clearing approach presupposes the existence of a central authority that has full knowledge of the inter-bank liability matrix. While this assumption may not be valid in the current situation, we believe that the technology exists for making the financial network more transparent, and that evidence of the systemic and social advantages of an aggregated approach to clearing may foster the progress in this area.

The remainder of this paper is organized as follows. Section 2 defines the notation. Section 3 introduces the Eisenberg-Noe model and related concepts. Section 4 presents a novel necessary and sufficient condition for the clearing vector's uniqueness, in the situation where the pro-rata constraint is adopted. In Section 5, we study the set of clearing matrices who do not necessarily satisfy the pro-rata rule. We demonstrate that one such matrix can always be found by solving a convex optimization problem aimed at minimizing a system-level loss. In general, however, the optimal clearing matrix is non-unique. We next consider specifically a system-level (i.e., aggregated) loss given by the sum of all deviations of the actual payments from the nominal liabilities. In this setting, we provide a two-stage procedure based on convex programming for finding the unique clearing payments matrix which achieves the best possible

system loss and which has at the same time the smallest possible Euclidean size. Section 6 presents a schematic example that shows how the proposed approach may lead to effective isolation of shocks and containment of the default contagion. This section also contains extensive simulation tests on random networks, with a comparison of the system losses and the number of default nodes in the cases where the proportionality rule is adopted and where it is discarded. Section 7 concludes the paper.

2. Preliminaries and notation. Given a finite set $\mathcal{V}$, the symbol $|\mathcal{V}|$ stands for its cardinality. For two families of real numbers $(a_\xi)_{\xi\in\Xi}, (b_\xi)_{\xi\in\Xi}$, the symbol $a \leq b$ (b dominates a, or a is dominated by b) denotes the element-wise relation $a_\xi \leq b_\xi$, $\forall \xi \in \Xi$. We write $a \lneq b$ if $a \leq b$ and $a \neq b$. The operations min and max are defined element-wise, e.g., $\min(a,b) \doteq (\min(a_\xi, b_\xi))_{\xi\in\Xi}$. These notation symbols apply to vectors (usually, $\Xi = \{1,\dots,n\}$) and matrices (usually, $\Xi = \{1,\dots,n\}\times\{1,\dots,n\}$).

Every nonnegative square matrix $A = (a_{ij})_{i,j\in\mathcal{V}}$ corresponds to a (weighted directed) graph $\mathcal{G}[A] = (\mathcal{V}, \mathcal{E}[A], A)$ whose nodes are indexed by $\mathcal{V}$ and whose set of arcs is defined as $\mathcal{E}[A] = \{(i,j) : a_{ij} > 0\}$. The value a_{ij} can be interpreted as a weight of arc (i,j), which is also denoted as $i \to j$. A sequence of arcs $i_0 \to i_1 \to \dots \to i_{s-1} \to i_s$ constitutes a *walk* between nodes i_0 and i_s in graph $\mathcal{G}[A]$. The set of nodes $J \subseteq \mathcal{V}$ is *reachable* from node i if $i \in J$ or a walk from i to some element $j \in J$ exists; J is called *globally reachable* in the graph if it is reachable from every node $i \notin J$.

A graph is strongly connected (strong) if every two nodes i, j are mutually reachable. A graph that is not strong has several strongly connected components (for brevity, we call them simply *components*). A component is said to be non-trivial if it contains more than one node. A component is said to be a *sink* component if no arc leaves it.

A nonnegative square matrix $A \in \mathbb{R}^{\mathcal{I}\times\mathcal{I}}$ is said to be *stochastic* if all its rows sum to 1: $\sum_{j\in\mathcal{I}} a_{ij} = 1$, $\forall i \in \mathcal{I}$ and *substochastic* if $\sum_{j\in\mathcal{I}} a_{ij} \leq 1$, $\forall i \in \mathcal{I}$. Introducing the vector of ones $\mathbf{1} \in \mathbb{R}^{\mathcal{I}}$, matrix $A \geq 0$ is stochastic if $A\mathbf{1} = \mathbf{1}$ and substochastic if $A\mathbf{1} \leq \mathbf{1}$.

3. Financial Networks. We henceforth use the notation introduced in [21], except for a few minor changes. A *financial network* is represented as a weighted digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \bar{\mathcal{P}})$ whose nodes stand for financial institutions (banks, funds, insurance companies, etc.) and whose weighted adjacency matrix $\bar{P} = (\bar{p}_{ij})$ represents the mutual liabilities among the institutions. Namely, entry $\bar{p}_{ij} \geq 0$ means that node i has an obligation to pay $\bar{p}_{ij}$ currency units to node j at the end of the current time period, and an arc $(i,j) \in \mathcal{E}$ from node i to node j exists if and only if $\bar{p}_{ij} > 0$. By definition, $\bar{p}_{ii} = 0\,\forall i$, so the graph contains no self-arcs.

Along with mutual liabilities, the banks have *outside assets*. The outside *asset* $\bar{c}_i \geq 0$ is the total payment due from non-financial entities (the external sector) to node i; these numbers constitute vector $\bar{c} = (\bar{c}_i)_{i\in\mathcal{V}}$.

For each node i, we define the nominal cash in-flow and out-flow (standing for the *asset* and *liability* sides of the balance sheet):

$$\bar{\phi}_i^{\text{in}} \doteq \bar{c}_i + \sum_{k\neq i} \bar{p}_{ki}, \quad \bar{p}_i \doteq \bar{\phi}_i^{\text{out}} \doteq \sum_{k\neq i} \bar{p}_{ik}. \tag{3.1}$$

Remark 1. Notice that the network may contain one or several *sink nodes*, that is nodes without outgoing arcs ($\bar{p}_i = 0$). These nodes represent banks with assets but without liabilities. One sink node is often introduced (see, e.g., [13, 21]) for accommodating liabilities to non-financial institutions: it represents a fictitious financial entity with no liabilities and whose assets are the liabilities of the remaining banks to the external sector.

In regular operations it will hold that $\bar{\phi}_i^{\text{in}} \geq \bar{\phi}_i^{\text{out}}$, meaning that each bank is able to pay its debts at the end of the period. The risk of financial contagion arises in the situation when a financial shock hits some nodes, meaning that the outside assets drop to smaller-than-expected values $c_i \in [0, \bar{c}_i)$. In this situation, it may happen that

$$c_i + \sum_{k \neq i} \bar{p}_{ki} < \bar{p}_i.$$

In this case, node i becomes unable to fully meet its payment obligations, and then *defaults*. When in default, a node pays out according to its capacity, thus reducing the amounts paid to the adjacent nodes, which in turn, for this reason, may also default and reduce their payments to other nodes, and so on in a cascaded fashion. As a result of default, the *actual* payment $p_{ij} \in [0, \bar{p}_{ij}]$ from node i to node j, in general, may be less than the nominal due payment $\bar{p}_{ij}$. A natural question arises: which matrices of actual payments $P = (p_{ij}) \leq \bar{P}$ may be considered as "fair" in the case of default? We shall see that the pro-rata rule is a commonly accepted rule for allocating payments in the case of default, but we shall also explore an alternative approach that aims at minimizing the aggregated loss over the financial system in Section 5. Denote the vectors of actual in-flows and out-flows by

$$\phi^{\text{in}} \doteq c + P^\top \mathbf{1}, \quad p \doteq \phi^{\text{out}} \doteq P\mathbf{1}. \tag{3.2}$$

The conditions to which the payments matrix $P = P(c, \bar{P})$ is subject to are as follows, [13]:

(i) **(limited liability)** The total payment of each node does not exceed its in-flow, that is, $\phi^{\text{in}} \geq \phi^{\text{out}}$;

(ii) **(absolute priority of debt claims)** Either node i pays its obligations in full ($p_i = \bar{p}_i$), or it pays all its value to the creditors ($p_i = \phi_i^{\text{in}}$).

Recalling that $P \leq \bar{P}$ and $p = P\mathbf{1} \leq \bar{p}$, conditions (i) and (ii) are reformulated compactly as

$$P\mathbf{1} = \min(\bar{p}, c + P^\top \mathbf{1}). \tag{3.3}$$

Definition 1. *A matrix P is called a* clearing matrix *(or matrix of clearing payments) corresponding to the vector of outside assets c, if $0 \leq P \leq \bar{P}$, and (3.3) holds.*

Notice that (3.3) is a system of $n \doteq |\mathcal{V}|$ nonlinear equations in n^2 variables p_{ij}. Hence, one cannot expect to find a unique solution, in general. To obtain uniqueness of the solution (in the generic situation), a third requirement is typically introduced, see, e.g., [13], known as the *proportionality* or *pro-rata* rule, which expresses the requirement that all debts have equal priority and must be paid in proportion to the initial claims. The imposition of this rule reduces the number of variables to $n = |\mathcal{V}|$. It is known that under the pro-rata rule a

clearing vector always exists; also, one such vector can be found by solving a convex optimization problem with n variables, applying a standard fixed-point iteration or a more advanced "fictitious default algorithm" [13, 21, 29]. In Section 4 we provide a necessary and sufficient criterion for the uniqueness of the clearing vector in the pro-rata case, and we also offer an algorithm for finding the set of all clearing vectors. The pro-rata rule reflects an underlying criterion of local fairness among neighboring nodes, and it is a convention enforced in many contracts. In Section 5 we discuss the case where the pro-rata rule is lifted and substituted by a system-level aggregated loss minimization criterion.

4. Pro-rata rule and clearing vectors. One standard approach to determine the clearing payments is based on imposing an additional restriction on the payments p_{ij}, stating that the payments of node i to the claimants should be proportional to the nominal liabilities $\bar{p}_{ij}$. It is convenient to introduce the matrix of normalized, or *relative* liabilities

$$A = (a_{ij}), \quad a_{ij} = \begin{cases} \frac{\bar{p}_{ij}}{\bar{p}_i}, & \text{if } \bar{p}_i > 0, \\ 1, & \text{if } \bar{p}_i = 0 \text{ and } i = j, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

By definition, matrix A is *stochastic*, that is, $a_{ij} \geq 0$ and $\sum_j a_{ij} = 1$ for all i or, equivalently, $A\mathbf{1} = \mathbf{1}$. The pro-rata rule can then be formulated as $P = \operatorname{diag}(P\mathbf{1})A$ or, equivalently,

$$p_{ij} = p_i a_{ij}, \quad \forall i, j \in \mathcal{V}. \tag{4.2}$$

Condition (4.2) is known as equal priority [13], *pro-rata* [21] or proportionality [34] rule. Under (4.2), it can be proved that

$$(P^\top \mathbf{1})_i = \sum_{j \in \mathcal{V}} (P^\top)_{ij} = \sum_{j \in \mathcal{V}} p_{ji} = \sum_{j \in \mathcal{V}} p_j a_{ji} = (A^\top p)_i \, \forall i,$$

which allows us to rewrite (3.3) in the equivalent vector form

$$p = \min(\bar{p}, c + A^\top p). \tag{4.3}$$

Definition 2. *A vector $p \geq 0$ is said to be a* clearing *vector if it satisfies* (4.3).

The existence of a clearing vector is usually proved by appealing to the general Knaster-Tarski fixed-point theorem [13, 27], applied to the non-decreasing mapping

$$p \mapsto \min(\bar{p}, c + A^\top p).$$

This theorem implies that the set of clearing vectors is non-empty and, furthermore, this set constitutes a complete lattice (with respect to the relation $\leq$), therefore, the *minimal* and *maximal* clearing vectors do exist. This monotonicity-based approach is convenient, because it allows to prove the existence of clearing vectors in more complicated models [4, 27].

At the same time, the Knaster-Tarski theorem does not give a full description of the set of all clearing vectors. One of the important questions that dates back to the original model

from [13] is whether this set is a singleton, that is, whether the three simple rules (absolute priority, limited liability and pro-rata payments) uniquely determine a clearing vector. Such a uniqueness guarantees that no ambiguity exists in the payments, so that each entity must abide to one and only one clearing payment, which is important from both the economical and the computational viewpoints. As it will be shown, in many aspects the maximal (or dominant) clearing vector is the most natural, because it minimizes the overall system loss. The computation of this maximal clearing vector is non-trivial, as usually one has to solve an LP with n variables and $2n$ nonlinear constraints; an alternative method from [34] finds the maximal clearing vector in no more than n steps, where at each step one has to solve a non-degenerate system of linear equations of the dimension $O(n)$. If the clearing vector is unique, then it can be computed by a more efficient "fictitious default algorithm" [13, 21].

Note that in degenerate situations the clearing vector may be non-unique. For instance, if $A = I_n$, $\bar{p} \neq 0$, and $c = 0$, every vector p such that $0 \leq p \leq \bar{p}$, obviously, satisfies (4.3). As it will be shown (Theorem 6), the existence of such a "closed" subgroup of banks independent of the remaining network and external sector is in fact the only reason for non-uniqueness of the clearing vector. Subsection 4.3 offers necessary and sufficient conditions for the clearing vector's uniqueness. We also show that, even when these conditions do not hold, still some of the clearing vector's elements are uniquely determined by A and c. In such situations, our Theorem 4 allows to describe the whole polytope of clearing vectors.

4.1. The dominant clearing vector – Extremal properties. Whereas the existence of a maximal clearing vector is usually proved via the Knaster-Tarski fixed-point theorem [13], we consider an alternative construction, which also clarifies the geometrical meaning of this vector. Consider the convex polyhedron

$$\mathcal{D} = [0, \bar{p}] \cap \{p : c + A^\top p \geq p\}. \tag{4.4}$$

The set $\mathcal{D}$ is non-empty (it contains, e.g., the null vector), and it can thus be represented as the convex hull of its extreme points (or vertices). The following lemma (see the appendix Section 8 for its proof) shows that one of these extreme points is the maximal (with respect to $\leq$ relation) element of $\mathcal{D}$, being also a clearing vector.

Definition 3. *$F : [0, \bar{p}] \to \mathbb{R}$ is non-increasing (respectively, decreasing) if $F(p^1) \geq F(p^2)$ (respectively, $F(p^1) > F(p^2)$) whenever $p^1, p^2 \in [0, \bar{p}]$ and $p^1 \leq p^2$ (respectively, $p^1 \lneq p^2$).*

Lemma 1. *The polyhedron (4.4) has the following properties:*

1. *a* maximal *point $p^* \in \mathcal{D}$ exists that dominates all other points $p^* \geq p$, $\forall p \in \mathcal{D}$;*
2. *p^* is a global minimizer in the optimization problem*

$$\min_x F(x) \quad \textit{subject to} \quad x \in \mathcal{D} \tag{4.5}$$

   whenever function $F : [0, \bar{p}] \to \mathbb{R}$ is non-increasing. If F is decreasing, then p^* is a unique *minimizer in (4.5).*
3. *p^* is a clearing vector for the financial network;*
4. *each strongly connected component of graph $\mathcal{G}[A]$, which is a sink (i.e., no arcs leave it), contains at least one node i such that $p_i^* = \bar{p}_i$;*

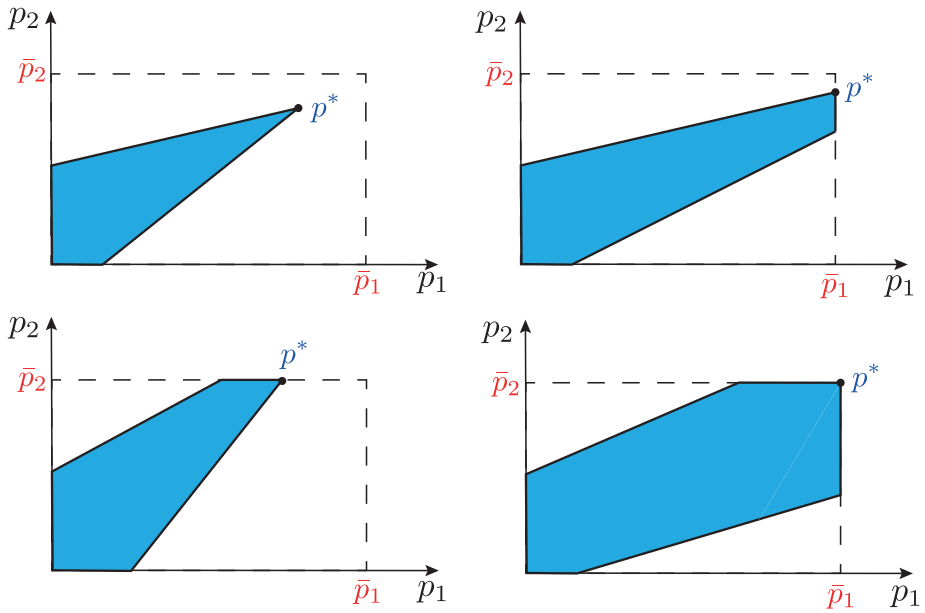

Figure 1. *Examples of possible shapes of $\mathcal{D}$ and the dominant clearing vector in the case of two nodes.*

5. *p^* is the only clearing vector that enjoys property 4.*

Remark 2. Statement (2) is a refinement of [13, Lemma 4] stating that if F is strictly decreasing, then every solution to (4.5) is a clearing vector. Notice that formally the result in [13] does not imply the uniqueness of a minimizer.

The clearing vector p^* from Lemma 1 is henceforth referred to as the *dominant* clearing vector, because it dominates all elements of $\mathcal{D}$ (e.g., all possible clearing vectors). Figure 1 illustrates possible structures of polyhedron $\mathcal{D}$ and the location of p^* in the case of $n = 2$.

Introducing the vector of losses (the discrepancies between nominal and actual assets)

$$\ell(p) = \bar{\phi}^{\text{in}} - \phi^{\text{in}} = (\bar{c} - c) + A^\top(\bar{p} - p) \geq 0,$$

statement (2) of Lemma 1 implies that a clearing vector can be found by solving a convex optimization problem such as, e.g., the convex QP

$$\begin{aligned} \min_{p \in \mathbb{R}^n} \quad & \|\ell(p)\|_2^2 \\ \text{s.t.:} \quad & \bar{p} \geq p \geq 0 \\ & c + A^\top p \geq p, \end{aligned} \tag{4.6}$$

where $\phi^{\text{in}} = c + A^\top p$, $\bar{\phi}^{\text{in}} = \bar{c} + A^\top \bar{p}$, or the conventional LP as follows

$$\begin{aligned} \min_{p \in \mathbb{R}^n} \quad & \mathbf{1}^\top \ell(p) \\ \text{s.t.:} \quad & \bar{p} \geq p \geq 0 \\ & c + A^\top p \geq p. \end{aligned} \tag{4.7}$$

Indeed, cost functions (4.6) and (4.7) are decreasing on $[0, \bar{p}]$. To prove this, consider $p^1, p^2 \in [0, \bar{p}]$ such that $p^1 \lneq p^2$. Then, $\mathbf{1}^\top(p^2 - p^1) = \sum_{i\in\mathcal{V}}(p_i^2 - p_i^1) > 0$. Recalling that A is a stochastic matrix, it is now obvious that (4.7) is strict decreasing:

$$\mathbf{1}^\top(\ell(p^1) - \ell(p^2)) = \mathbf{1}^\top A^\top(p^2 - p^1) = \mathbf{1}^\top(p^2 - p^1) > 0. \tag{4.8}$$

To prove that (4.6) is strict decreasing, notice that the vector $\ell(p^1) - \ell(p^2) = A^\top(p^2 - p^1)$ is nonnegative, furthermore, at least one its component is positive due to (4.8). On the other hand, $\ell(p^1), \ell(p^2)$ are also nonnegative, because $p^1, p^2 \leq \bar{p}$. This implies that

$$\ell(p^1)_i \geq \ell(p^2)_i \geq 0 \quad \forall i \in \mathcal{V},$$

and at least one of these inequalities is strict. Hence, $\|\ell(p^1)\|_2^2 > \|\ell(p^2)\|_2^2$.

According to Lemma 1, therefore, a unique minimizer $p = p^*$ exists in the optimization problems (4.6) and (4.7). Notice that usually uniqueness of a solution is guaranteed only for *strictly* convex function, whereas the linear function (4.7) is not strictly convex, and the function (4.6) fails to be strictly convex if $\det A = 0$.

Although the dominant clearing vector p^* depends on matrix A and vectors $c, \bar{p}$, and the closed-form analytic expressions for its elements are not available, some properties of this clearing vector in fact do not depend on the liability matrix and are determined only by the topology of graph $\mathcal{G}$. In particular, the following lemma can be proved (see the appendix Section 8 for its proof), which establishes the criterion for positivity of the dominant clearing vector's elements.

Lemma 2. *The element p_i^* of the dominant clearing vector is positive if and only if i is not a sink node ($\bar{p}_i > 0$) and at least one of the following conditions holds:*

1. *i has outside assets, that is, $c_i > 0$;*
2. *i is reachable from some node $j \neq i$ with $c_j > 0$;*
3. *the strongly connected component of graph $\mathcal{G}$ to which i belongs is a sink component.*

The result of Lemma 2 proves to be useful in *dynamic* (multi-step) models of interbank clearing that have been recently examined in [9, 10].

4.2. Uniqueness of the clearing vector: a sufficient condition. In this subsection, we offer a *sufficient* condition ensuring that the dominant clearing vector p^* is the unique clearing vector. In fact, we show that some elements of the clearing vector are always determined uniquely. We start by introducing some auxiliary notation. Let $C^+ \doteq \{i : c_i > 0\}$ stand for the set of nodes that receive nonzero outside assets, and let $S \doteq \{i : \bar{p}_i = 0\}$ stand for the set of *sink* nodes, who owe no liability payments. We introduce the set

$$I_0 \doteq C^+ \cup S = \{i : c_i > 0 \vee \bar{p}_i = 0\}. \tag{4.9}$$

The following lemma, whose proof is given in the appendix Section 8, establishes a sufficient condition for uniqueness of the clearing vector and generalizes results from [13, 14, 21].

Lemma 3. *Let $I_0' \supseteq I_0$ stand for the set of all nodes in the graph $\mathcal{G}$, from where I_0 can be reached. Then, for* every *clearing vector p we have $p_i = p_i^* \quad \forall i \in I_0'$. In particular, if set I_0 is* globally *reachable in the graph of a financial network $\mathcal{G}$, then the dominant clearing vector p^* is the only clearing vector corresponding to the vector of outside assets c.*

Remark 3. Since each path in the graph ends in one of the *sink* components, it can be easily proved that the uniqueness condition from the second part of Lemma 3 admits the following equivalent reformulation: each strongly connected component of graph $\mathcal{G}$ being a sink is either trivial (contains only one node) or contains node i such that $c_i > 0$. In this form, the uniqueness criterion from Lemma 3 becomes a special case of [24, Theorem 4.5]. The latter theorem deals with more general clearing systems, where the pro-rata rule is replaced by a more general division rule.

4.3. Uniqueness of the clearing vector: the general case. In this subsection we derive two criteria of the clearing vector's uniqueness. The first of them (Theorem 4) assumes that the dominant clearing vector p^* has been found. In this situation, we are able not only to check the uniqueness of the clearing vector, but also to describe the whole polytope of the clearing vectors without preprocessing the graph, e.g., computing the structure of its strongly connected components which is prerequisite for algorithms from [25, 30]. On the other hand, if one is interested only in the uniqueness of a clearing vector, and the structure of the graph is known, a simpler graph-theoretic criterion can be used (Theorem 6) that does not require knowledge of p^*. It should also be noticed that, unlike the initial results from [13, 21], the uniqueness criteria from Theorem 4 and Theorem 6 are not only sufficient, but also necessary.

Assume that the condition in Lemma 3 does not hold, that is, $I_0' \neq \mathcal{V}$. The banks corresponding to nodes from $\mathcal{V}_1 \doteq \mathcal{V} \setminus I_0'$ do not have outside assets ($\mathcal{V}_1 \cap C^+ = \emptyset$) and do not pay to nodes from I_0' (otherwise, a chain of liability from them to I_0 would exist). Hence, matrix $A^1 \doteq (a_{ij})_{i,j\in\mathcal{V}_1}$ is stochastic. At the same time, nodes from I_0' *can* have liability payments to nodes from $\mathcal{V}_1$, which depend only on the dominant vector p^* and constitute the vector

$$c^{(1)} \doteq (c_i^{(1)})_{i\in\mathcal{V}_1}, \quad c_i^{(1)} = c_i^{(1)}(p^*) \doteq \sum_{k\in I_0'} a_{ki}p_k^*, \quad i \in \mathcal{V}_1.$$

We can now apply Lemma 3 to a reduced financial network $\mathcal{G}_1$ with node set $\mathcal{V}_1$, normalized payment matrix A_1 and vector of external assets $c^{(1)}$. Introducing the set[1]

$$I_1 = \{i \in \mathcal{V}_1 : c_i^{(1)} > 0\}$$

and denoting $I_1' \supseteq I_1$ all nodes from which set I_1 is reachable (banks that are connected by chains of liability to nodes from I_1), Lemma 3 ensures that the elements of the reduced network's clearing vector $p_i, i \in I_1'$, are determined *uniquely*. The definition (4.3) entails that if p is a clearing vector for the original network, then its subvector $p^1 = (p_i)_{i\in\mathcal{V}_1}$ is a clearing vector for the reduced network $\mathcal{G}_1$. This also applies to p^*. Lemma 3 entails now that for each clearing vector p (in the original network) one has $p_i = p_i^* \, \forall i \in I_1'$.

If $I_0' \cup I_1' = \mathcal{V}$, we have uniqueness of the clearing vector. Otherwise, we have a group of banks $\mathcal{V}_2 = \mathcal{V} \setminus (I_0' \cup I_1')$ that are not in debt to the nodes from I_1' and I_0', however, they can receive liability payments from the group I_1'. For group $\mathcal{V}_2$, these payments may be treated as outside assets. Let

$$c^{(2)} \doteq (c_i^{(2)})_{i\in\mathcal{V}_2}, \quad c_i^{(2)} \doteq \sum_{k\in I_1} a_{ki}p_i^*.$$

[1]Notice that unlike I_0, set I_1 contains no sink nodes, because all sink nodes of the graph $\mathcal{G}$ belong to I_0.

If the set $I_2 = \{i \in \mathcal{V}_2 : c_i^{(2)} > 0\}$ is non-empty, one can consider the set $I_2' \supseteq I_2$ of all nodes from where I_2 can be reached. Lemma 1 implies that the elements p_i, , $i \in I_2'$ of the clearing vector are uniquely determined: $p_i = p_i^* \, \forall i \in I_2'$.

We arrive at the following iterative procedure, which allows to test the clearing vector's uniqueness (and, in fact, even to find the whole set of clearing vectors).

Algorithm 4.1 Clearing vector's uniqueness test.

Initialization. Compute the dominant clearing vector p^* (e.g., by solving the LP (4.6)). Set $q \leftarrow 0$, $I_0 \leftarrow C^+ \cup S = \{i : c_i > 0 \vee \bar{p}_i = 0\}$. Find the set $I_0' \supseteq I_0$ of all nodes, from which I_0 is reachable.

repeat

1) $q \leftarrow q + 1$;
2) $\mathcal{V}_q \leftarrow \mathcal{V} \setminus (I_0' \cup I_1 \ldots \cup I_{q-1}')$;
3) compute the vector of payments from I_{q-1}' to $\mathcal{V}_q$

$$c^{(q)} = (c_i^{(q)})_{i \in \mathcal{V}_q}, \quad c_i^{(q)} \doteq \sum_{k \in I_{q-1}'} a_{ki} p_k^* \quad \forall i \in \mathcal{V}_q;$$

4) find the set $I_q = \{i \in \mathcal{V}_q : c_i^{(q)} > 0\}$;
5) find the set $I_q' \supseteq I_q$ of nodes from $\mathcal{V}_q$, from where I_q can be reached in $\mathcal{G}$.

until $\mathcal{V}_q = \emptyset$ or $c^{(q)} = 0$.

Theorem 4. *Algorithm 4.1 stops after a finite number of steps $s \geq 0$. The elements of a clearing vector, corresponding to indices $i \in I_0' \cup I_1' \cup \ldots \cup I_s'$, are uniquely determined: $p_i = p_i^*$. The clearing vector is unique if and only if $\mathcal{V}_s = \emptyset$, otherwise, there are infinitely many clearing vectors. Precisely, p is a clearing vector if and only if*

$$p_i = \begin{cases} p_i^*, & i \in I_0' \cup I_1 \ldots \cup I_s', \\ \xi_i, & i \in \mathcal{V}_s, \end{cases} \tag{4.10}$$

where $\xi \in \mathbb{R}^{\mathcal{V}_s}$ is an arbitrary vector satisfying the constraints

$$B^\top \xi = \xi, \quad 0 \leq \xi_i \leq \bar{p}_i, \; \forall i \in \mathcal{V}_s, \quad B \doteq (a_{ij})_{i,j \in \mathcal{V}_s}. \tag{4.11}$$

A proof of Theorem 4 is offered in the appendix Section 8. Theorem 4 entails the following technical proposition, which is of independent interest. The *net* vector (or the vector of *equities*) corresponding to the clearing vector p is

$$\zeta \doteq \phi^{\text{in}} - \phi^{\text{out}} = c + A^\top p - p = \max(c + A^\top p - \bar{p}, 0) \geq 0. \tag{4.12}$$

The component ζ_i represents the *net worth* of node i, that is, the difference between the in- and the out-flows at that node.

Corollary 5. *[13, 27] The vector of equities ζ is the same for all possible clearing vectors p.*

The proof of Corollary 5 is straightforward from the representation (4.10). Indeed, the equities of banks from $\mathcal{V}_s$ are all zeros, whereas the remaining equities depend only on the subvector $(p_i)_{i\in I_0'\cup\ldots\cup I_s'}$, which is uniquely determined by $p^* = p^*(c, A, \bar{p})$.

Notice that although the subvector ξ in (4.10) is defined non-uniquely, some of its elements are in fact uniquely determined due to Lemma 2. As we know, $\xi_i = p_i^* = 0$ whenever i does not belong to a sink component and is not reachable from C^+. Combining Theorem 4 with Lemma 2, we can establish an alternative uniqueness criterion, which *does not* require knowledge of the vector p^*.

Theorem 6. *The following two conditions are equivalent:*

i) the clearing vector is unique;

ii) each non-trivial sink component of $\mathcal{G}$ either contains a node from C^+ or is reachable from C^+.

A proof of Theorem 6 is given in the appendix Section 8.

4.4. Discussion: Lemma 3 and Theorems 4 and 6 vs. previously known results. Comparing Theorem 4 to previously known results, several relevant features can be noticed. To find the dominant clearing vector p^*, one needs to know the matrix $\bar{P}$ (or, equivalently, $\bar{p}$ and A) and the vector of external assets $c \geq 0$. As we have already discussed, efficient algorithms do exist for computing this vector (e.g., by solving an LP or a convex QP). The computation of the sets $I_s, I_s', \mathcal{V}_s$ requires in fact very limited information. We do not need to compute the vectors $c^{(q)}$, it suffices to know which of their components are positive. A closer look at our algorithm shows that this depends on the set C^+ (nodes who receive external assets) and the topology of graph $\mathcal{G}$ (which determines, in view of Lemma 2, which elements of p^* are positive). Also, we give an explicit and simple description of *all* clearing vectors.

4.4.1. Criteria from [13] and [21]. The second part of Lemma 3 (the uniqueness of a clearing vector if I_0 is globally reachable) covers two well-known criteria of the clearing vector's uniqueness.

The uniqueness criterion from [13, Theorem 2] states that the clearing vector is unique if the set C^+ is globally reachable. In [13], this "regularity" property is formulated as follows: the "risk orbit" of each node i (that is, the set of nodes reachable from i) is a surplus set, that is, the orbit contains a node j with $c_j > 0$. Notice that formally this definition of regularity does not apply to a network with a sink node, whose "risk orbit" is empty. It can be proved that Theorem 2 in [13] retains its validity if one formally considers the degenerate orbit of a sink node as a surplus set. An advantage of the approach developed in [13] is the possibility to generalize the proof (with some variations) to some advanced models of systemic risk [17, 27].

The criterion from [21] guarantees that the clearing vector is unique if all nodes are connected to the external sector by chains of liabilities. Introducing the fictitious sink node, this can be formulated as follows: the set S of sink nodes contains only one node, which is globally reachable.

Note that the first part of Lemma 3, although it follows from the second part, is not easily available in the literature. This statement plays the central role in our uniqueness criteria (Theorems 4 and 6). Unlike our Theorem 4, the aforementioned results from [13, 21, 27] do not allow to parameterize all clearing vectors when the uniqueness fails to hold.

4.4.2. An extension of the Eisenberg-Noe model: the negative inflow vector. One of the limitations of the Eisenberg-Noe model is the assumption that banks cannot become *insolvent*, that is, the equity vector (3.2) is always nonnegative. As discussed in [17, 25] the latter assumption can be violated in practice, because some payments to the external sector (e.g., the bank's operation costs) are more senior than debts to other banks and cannot be reduced, even if the external assets drop. Hence, a generalization of the Eisenberg-Noe model has been introduced in [17] where the vector c can have both positive and negative components $c_i \in \mathbb{R}$, which stand for the difference between outside asset of bank i and its external liability. In this situation, the definition of the clearing vector has to be modified to

$$p = \min\left(\bar{p}, \max(A^\top p + c, 0)\right), \tag{4.13}$$

which is equivalent to (4.3) if $c \geq 0$.

The existence of clearing vectors in the case of sign-indefinite c can be derived from the Knaster-Tarski theorem, and the standard fixed-point iteration delivers one of the clearing vectors [17]. However, the uniqueness problem has been addressed quite recently. For strongly connected networks, the uniqueness was established in [1] and [33]. For the general situation, two different uniqueness criteria have been proposed in [25, Theorem 2.2] and [30, Theorem 2], which also provide a description of all possible clearing vectors. In the situation when $c \geq 0$, however, the cited criteria appear to be superfluous and more computationally demanding than our results. Notice that our Theorem 4 requires finding the maximal clearing vector p^* (which, in the case $c \geq 0$, is obtained by solving a convex QP or an LP), however, this theorem does not require knowledge of the structure of the strongly connected components of the graph (or, equivalently, of an irreducible decomposition of matrix A). On the other hand, if the structure of strongly connected sink components is known, Theorem 6 allows to test the clearing vector's uniqueness without finding the clearing vector itself. The general results from [25, 30] exploit the irreducible decomposition of matrix A (whose computation is a self-standing non-trivial procedure for a large-scale network), and do not allow to test the uniqueness without computing the complete clearing vector. Notice that the procedure of computing the clearing vector (even in the case of its uniqueness) is not made explicit in [25] (one of the option is the modified fictitious default algorithm [17]). The procedure of clearing vector computation in [30] is explicit, however, it requires finding the left and right Perron-Frobenius eigenvectors for each irreducible block, which is not needed in our results.

4.5. Examples.

Example 1. We first illustrate our uniqueness criterion on the synthetic financial network shown in Figure 2. This network includes one sink node, representing the external sector, and three non-trivial strongly connected components (i.e., cliques).

The strongly connected component 1 stands for a group of banks that works directly with the external sector, using also some financial services of banks from group 3 and thus owing some liabilities to them. The banks from group 2, constituting another strongly connected component, owe debts to banks from groups 1 and 3; these banks thus also have indirect liability to the external sector (through banks from group 1). The group of banks 3 constitutes a non-trivial *sink* component, owing neither direct nor even indirect liability to the financial

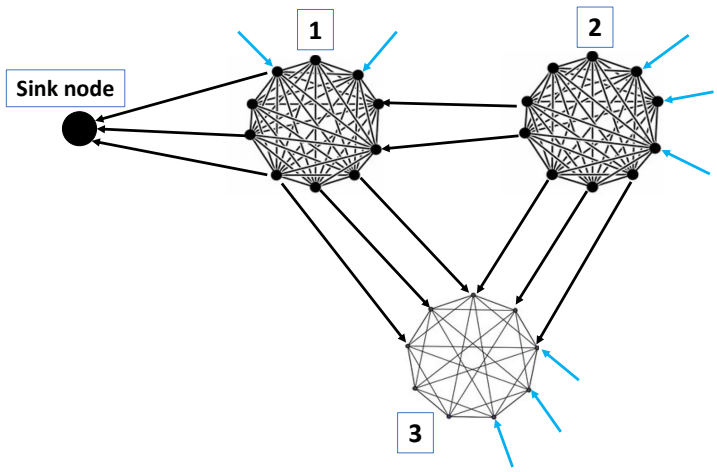

Figure 2. *Network with a sink node and three strongly connected components.*

sector. Notice that the presence of this component makes the criterion from [21] inapplicable: the set S of sink nodes is not globally reachable.

Denoting the vector of outside assets corresponding to group i by $c_{[i]} \geq 0$, the following situation are possible:

- If $c_{[3]} \neq 0$, we are in the situation of [13, Theorem 2]: the set C^+ is reachable from any node but for the sink node (as has been remarked, the result from [13] retains its validity in presence of sinks).
- If $c_{[3]} = 0$, then the result from [13] becomes inapplicable (group 3 is not connected to C^+). Nevertheless, Theorem 6 ensures the uniqueness of the clearing vector if $c_{[1]} \neq 0$ or $c_{[2]} \neq 0$, because component 3 is reachable from set C^+.
- Finally, if $c = 0$, the clearing vector is not unique. However, the subvector corresponding to the union of groups 1 and 2 is unique: $p_{[1]} = p^*_{[1]}$ and $p_{[2]} = p^*_{[2]}$ for each clearing vector p (here the subscript $[i]$ has the same meaning as for subvectors $c_{[i]}$). This is entailed by the first part of Lemma 3, because groups 1 and 2 are connected to the unique sink node (set S).

Example 2. Next, we illustrate Algorithm 4.1 by considering the network in Figure 3, which displays a network with $n = 15$ nodes that contains only one sink node ($S = \{0\}$) and three nodes with outside assets ($C^+ = \{1, 2, 3\}$), which together form the set I_0. Lemma 2 entails that in this situation $p^*_i > 0$, $\forall i \neq 0, 9$ (notice that nodes 13-15 constitute a sink component, satisfying thus condition 3) from Lemma 2). The set I'_0 contains I_0 and two nodes $4, 5$ that have liabilities towards nodes 0 and 2. The set I_1 contains nodes that have no liability to I'_0, however, should receive payments from 4 and 5. Hence, $c^{(1)}_6, c^{(1)}_7, c^{(1)}_8 > 0$. The nodes $6, 7, 8$ constitute the set I_1; the set I'_1 is obtained by adding node 9 who has liability to one of them. On the next iteration of the algorithm, one computes the sets $I_2 = \{10, 11\}$ and $I'_2 = \{12\} \cup I_2$. The next vector $c^{(3)}$ will be zero, because the remaining nodes of the graph constitute an isolated group. Hence, the clearing vector is not unique, but for each clearing

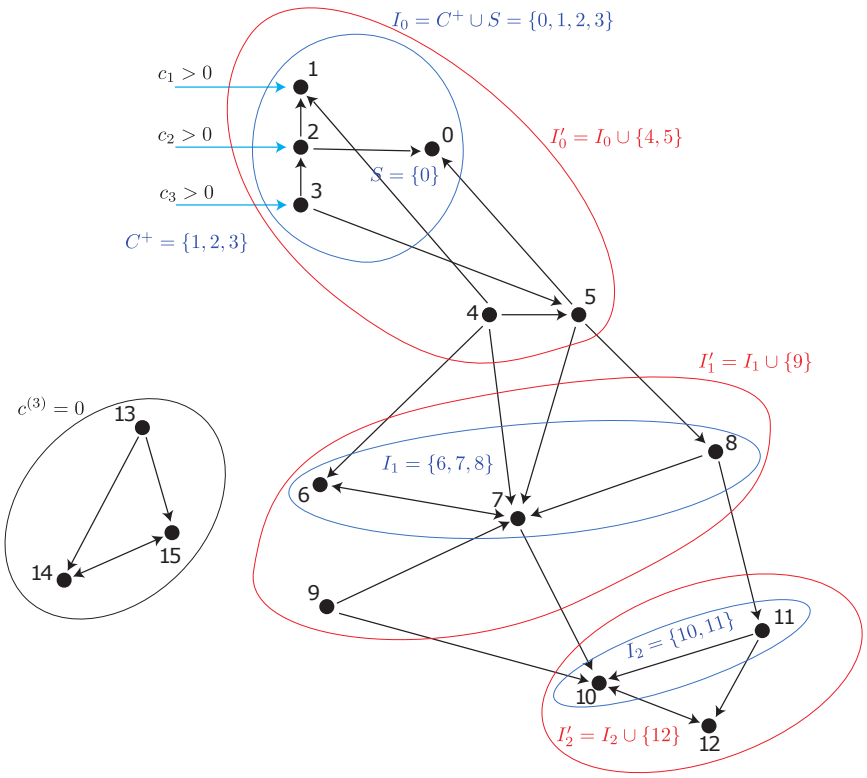

Figure 3. *The sets I_s (encircled by blue lines) and I_s' (encircled by red lines) for a special financial network with $n = 15$ nodes with $\bar{p}_i > 0$, $\forall i \neq 0$.*

vector p one has $p_i = p_i^*$, $\forall i = 0, \ldots, 12$.

Removing the isolated strongly connected component $\{13, 14, 15\}$ from the graph in Figure 3, one obtains a financial network that has only one clearing vector $(p_i^*)_{i=1}^{12}$ yet does not satisfy the sufficient condition from Lemma 3: set $C^+ \cup S$ cannot be reached from any of the nodes $6, \ldots, 12$. None of the criteria from [13] and [21] guarantees uniqueness in this situation.

Remark 4. In the situations considered in Examples 1 and 2 the outside assets of several banks drop simultaneously ($c_i = 0$ for several nodes i). This situation is common in actual financial networks where the banks are connected not only by mutual liabilities, but also by mutual exposures to external risky assets [2, 15]. A bankruptcy of some business or a forced sale of some asset at a low price in the case of the market's turbulence [11] has a negative effect on all shareholders, as illustrated by the housing market crash in 2008. Such "systematic" shocks are introduced in problems of portfolio compression [3] and have to be considered in any realistic stress-test simulation.

5. Lifting the pro-rata condition. In this section we propose a new clearing mechanism that does not hinge upon the proportional payments rule. While such new clearing mechanism is not yet in operation, we here advocate the idea that a more transparent knowledge of the inter-bank liability matrix, and a centralized authority in charge of defining the clearing

payments in case of default, may lead to important systemic and social benefits due to the avoidance or reduction of catastrophic default events.

Without the pro-rata constraint in place, we have potentially more freedom to appropriately select a payment matrix P that satisfies the conditions of Definition 1. These conditions do not identify the payment matrix univocally. Indeed, as we discussed extensively in the previous sections, one matrix that satisfies the definition is the one resulting from the pro-rata rule, but this may not be the only one, in general. This fact gives us some degrees of freedom that shall be used towards seeking clearing matrices who promote a virtuous behavior of the whole network. As it will be shown in Section 6, the proposed non-proportional clearing mechanism not only visibly reduces the overall total imbalance between the nominal and actual payments, but it also helps isolating defaults and preventing their cascaded spread over the network.

5.1. Optimal clearings. We next present an optimization-based approach for determining a clearing matrix that minimizes an appropriate system-level loss function. Recalling that $[0, \bar{P}] = \{P \in \mathbb{R}^{n\times n} : 0 \leq P \leq \bar{P}\}$, we consider the convex polyhedron in the space of matrices

$$\mathcal{D}_{n\times n} = [0, \bar{P}] \cap \{P \in \mathbb{R}^{n\times n} : c + P^\top \mathbf{1} \geq P\mathbf{1}\}.$$

We call a function $F : [0, \bar{P}] \to \mathbb{R}$ *decreasing* if $F(P_1) > F(P_2)$ whenever $P_1 \lneq P_2$. For any such function, consider the optimization problem

$$\min F(P) \quad \text{subject to} \quad P \in \mathcal{D}_{n\times n}. \tag{5.1}$$

The following result holds; see Section 8 in the Appendix for a proof.

Lemma 4. *For any decreasing function $F : [0, \bar{P}] \to \mathbb{R}$, every local minimizer in problem (5.1) is a clearing matrix (as defined in Definition 1).*

Notice that Lemma 4 does not require the function to be continuous. For a continuous function, the global minimum always exists due to compactness of $\mathcal{D}_{n\times n}$. Two examples of functions that are continuous and decreasing on $[0, \bar{P}]$ are

$$F_1(P) \doteq \|\bar{P} - P\|_1 = \sum_{i,j=1}^{n} |\bar{P}_{ij} - P_{ij}|, \tag{5.2}$$

$$F_2(P) \doteq \|\bar{P} - P\|_2^2 = \sum_{i,j=1}^{n} (\bar{P}_{ij} - P_{ij})^2. \tag{5.3}$$

Both these functions provide an aggregated, system-level, measure of the impact of defaults on the deviation of payments from their nominal liabilities. In particular, minimization of the performance index (5.2) is equivalent to the problem

$$\begin{aligned} F_1^* = \min_{P\in\mathbb{R}^{n,n}} \quad & \|\bar{P} - P\|_1 \\ \text{s.t.:} \quad & \bar{P} \geq P \geq 0 \\ & c + P^\top \mathbf{1} - P\mathbf{1} \geq 0, \end{aligned} \tag{5.4}$$

which can be readily recast in the form of an LP, whereas minimization of (5.3) leads to the convex QP problem

$$F_2^* = \min_{P \in \mathbb{R}^{n,n}} \quad \|\bar{P} - P\|_2^2 \quad \text{s.t.:} \quad \bar{P} \geq P \geq 0, \quad c + P^\top \mathbf{1} - P\mathbf{1} \geq 0. \tag{5.5}$$

While both (5.4) and (5.5) are valid ways for obtaining a clearing matrix P, from the financial point of view they have different characteristics. Notice in fact that the objective in (5.5) is strongly convex, so the optimal clearing matrix resulting from (5.5) is unique. However, the F_2 criterion is expressed in "squared currency," which may not have an immediate financial meaning. Further, this sum-of-squared losses function tends to avoid large individual losses, at the expense of possibly having many small losses. This is a well-known behavior of squared losses, which may be critical in the present context since default is an on/off process: a node goes into default as soon as its balance goes negative, irrespective of how large is the negative balance. Especially for this reason our focus in this work is on the F_1 criterion in (5.4). As it is widely known, this ℓ_1-norm criterion tends to be less sensitive to large residuals and promotes solutions that are *sparse*. Sparsity, in our context, is a very desirable property, since a sparse residual matrix $\bar{P} - P$ means that many of its entries are zero, which in turn means that many nodes meet their liabilities and thus do not default. We next focus on the F_1 criterion, which represents the total sum that is lost due to defaults (notice that in the case of no defaults we have $F_1^* = 0$).

5.2. Non-uniqueness of clearing matrices. In general, the set of clearing matrices defined via Definition 1 has more than one element. Further, this set has a non-trivial structure and fails to be a complete lattice[2]. Lemma 1 does not retain its validity in the class of all admissible clearing matrices, in particular, there exists no maximal clearing matrix, as exemplified by the following example.

Example 3. Consider a degenerate 3-node network whose nodes 1 and 2 are sinks ($\bar{p}_{1i} = 0, \bar{p}_{2i} = 0$, $\forall i$) and receive no outside assets ($c_1 = c_2 = 0$), whereas node 3 receives outside asset $c_3 > 0$ and owes to both 1 and 2 ($\bar{p}_{31}, \bar{p}_{32} > 0$). A clearing matrix P must then have the following structure

$$P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p_{31} & p_{32} & 0 \end{bmatrix}, \quad p_{31} \in [0, \bar{p}_{31}],\ p_{32} \in [0, \bar{p}_{32}]. \tag{5.6}$$

and equation (3.3) is equivalent to

$$p_{31} + p_{32} = \theta \doteq \min(\bar{p}_{31} + \bar{p}_{32}, c_3). \tag{5.7}$$

[2]It should be noticed that some alternative definitions of clearing matrices are possible, under which the set of clearing matrices becomes a complete lattice, see, e.g., a very general construction from [12] that deals with integer payments and allowing the banks to apply different bankruptcy policies.

Considering two clearing matrices

$$P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \theta & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \theta & 0 \end{bmatrix},$$

one may easily notice that no matrix $P \geq \max(P_1, P_2)$ can satisfy (5.7). In particular, the set of clearing matrices understood in the sense of Definition 1 has *no* maximal element. It can be easily verified that every matrix (5.6) that satisfies (5.7) is a *global* minimizer of problem (5.4). Hence, in spite of the strict monotonicity of the cost function in (5.4), the optimal solution to (5.4) is not unique. Contrary, if we introduce the proportionality rule, the unique (thanks to Lemma 3) clearing vector and the corresponding clearing matrix, result to be, respectively

$$p^* = (0, 0, \theta)^\top, \quad P^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\bar{p}_{31}\theta}{\bar{p}_{31}+\bar{p}_{32}} & \frac{\bar{p}_{32}\theta}{\bar{p}_{31}+\bar{p}_{32}} & 0 \end{bmatrix}.$$

5.3. A two-stage approach for uniqueness. Given a liability matrix $\bar{P}$ and an in-flow vector $c \geq 0$ our purpose is to establish a policy that permits determining uniquely a clearing matrix that globally minimizes the system-level loss F_1. Uniqueness is important since the clearing payments must be non controversial, and each player must be in condition to compute and agree univocally on the same payment values. As we discussed in the previous section, however, the LP problem in (5.4) may have multiple global optimal solutions, in general. The following lemma characterizes *all* optimal solutions to (5.4).

Lemma 5. *Let P^* be an optimal solution of problem (5.4). Then, the set of all optimal solutions to (5.4) is characterized as the polytope*

$$\mathcal{S}^* \doteq \{\tilde{P} = P^* + \Delta : \ 0 \leq \tilde{P} \leq \bar{P}, \, c + \tilde{P}^\top \mathbf{1} - \tilde{P}\mathbf{1} \geq 0, \, \mathbf{1}^\top \Delta \mathbf{1} = 0\}.$$

A proof of Lemma 5 is given in the appendix Section 8.

The next proposition illustrates how we can break the tie and specify a two-stage rule that results in a unique optimal clearing policy.

Proposition 1. *Let P^* be an optimal solution of problem (5.4), and let F_1^* be the corresponding globally optimal loss level. Let further Δ^* be such that*

$$\begin{aligned} \Delta^* = \arg \min_{\Delta \in \mathbb{R}^{n,n}} \quad & \|P^* + \Delta\|_2^2 \\ s.t.: \quad & \bar{P} \geq P^* + \Delta \geq 0 \\ & c + (P^* + \Delta)^\top \mathbf{1} - (P^* + \Delta)\mathbf{1} \geq 0 \\ & \mathbf{1}^\top \Delta \mathbf{1} = 0. \end{aligned} \tag{5.8}$$

Then:

(a) $\tilde{P}^* \doteq P^* + \Delta^*$ is a clearing matrix;
(b) $\tilde{P}^*$ achieves the globally optimal loss level F_1^*, that is $F_1(\tilde{P}^*) = F_1^*$;
(c) $\tilde{P}^*$ is the unique smallest-Euclidean-norm solution among all optimal solutions to (5.4).

A proof of Proposition 1 is given in the appendix Section 8.

We call this a two-stage approach for determining the unique optimal clearing matrix, since the method consists in solving two convex problems in sequence: first we solve the LP (5.4) and find any optimal solution P^*. Next, we solve the convex QP (5.8) and find its unique optimal Δ^*. Finally, our optimal clearing matrix of interest is $\tilde{P}^* = P^* + \Delta^*$, which is the unique minimum norm solution among all possible optimal solutions to problem (5.4).

6. Numerical Experiments. We next evaluate numerically the systemic improvement of the optimal matrix clearings (i.e., those computed according to Proposition 1) with respect to the standard pro-rated clearings (computed according to (4.7)). The improvement is evaluated both in terms of reduction of the systemic loss F_1 and in terms of containment of default contagion, as expressed by the percentage of defaulted nodes in the network. First, we propose in Section 6.1 a simple schematic example, and then we perform extensive randomized tests in Section 6.2 using synthetic random networks similar to ones proposed in [31].

6.1. An illustrative schematic example. We consider a variation on the simplified network discussed in [21]. This network is composed of $n = 5$ nodes (including one sink node), with liability matrix

$$\bar{P} = \begin{bmatrix} 0 & 180 & 0 & 0 & 180 \\ 0 & 0 & 100 & 0 & 100 \\ 90 & 0 & 0 & 100 & 50 \\ 150 & 0 & 0 & 0 & 150 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Suppose there is a nominal scenario where external cash flows are given as

$$c = c_{\text{nom}} \doteq [121,\ 21,\ 150,\ 204,\ 0]^\top.$$

It can be readily verified that in the nominal scenario all the nodes in the network remain solvent (no defaults), and the clearing payments coincide with the nominal liabilities. Consider next a situation in which a "shock" happens on the inflow at node 3, so that this inflow reduces from 150 to 130, that is

$$c = c_{\text{shock}} \doteq [121,\ 21,\ 130,\ 204,\ 0]^\top.$$

Under the pro-rata rule, the (unique) clearing payments, resulting from the solution of (4.7), are shown in smaller font below the nominal liabilities in the left panel of Figure 4: all nodes in the network default in a cascade fashion due to initial default of node 3. The total defaulted amount, defined as the sum of all the unpaid liabilities is in this case $F_1 = 13.98$.

Then, we lifted the pro-rata rule, and we computed the unique clearing payments according to Proposition 1. The results in this case are shown in the right panel of Figure 4: only node 3 defaults, while all other nodes manage to pay their liabilities in full. Not only we reduced the sum of all unpaid liabilities to $F_1 = 10$, but we also obtained *isolation* of the contagion, since the default is now limited to node 3 and did not spread to other parts of the network.

6.2. Random networks test. The random graphs used for simulations are constructed using a technique inspired by [31]. The topology of the graph is given by the standard Erdös-Renyi $\mathcal{G}(n, p)$ graph. The interbank liabilities $\bar{P}_{ij}$ for every edge (i, j) of the random graph

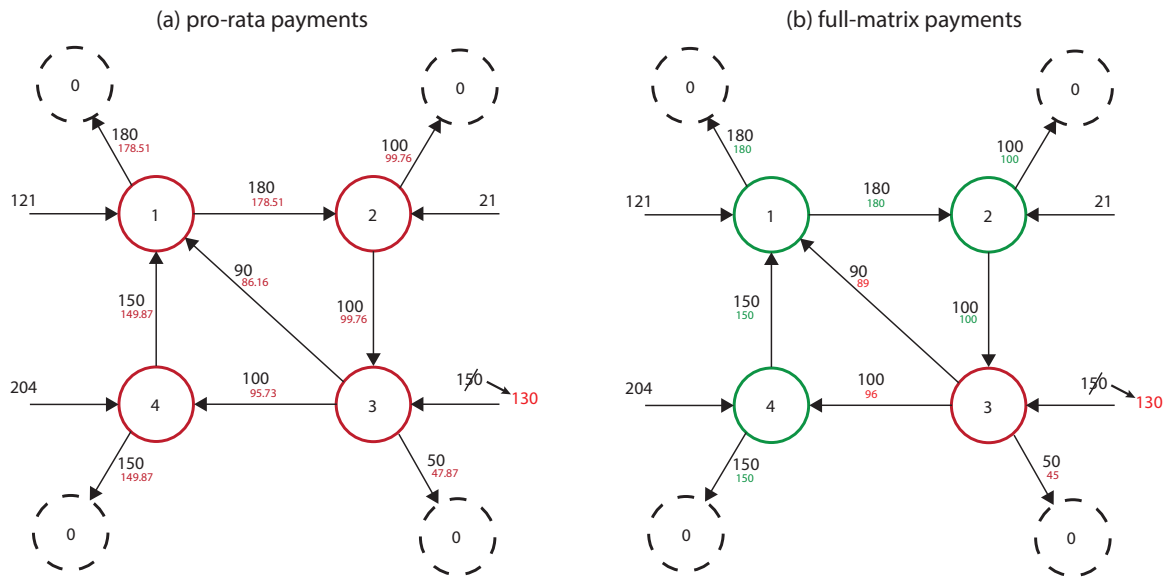

Figure 4. *A four-nodes network with a shock on the cash inflow at node 3. Panel (a) shows the pro-rated clearing payments, panel (b) shows the optimal matrix clearings.*

are then found by sampling from a uniform distribution $\bar{P}_{ij} \sim \mathcal{U}(0, P_{max})$, where P_{max} is the maximum possible value of a single interbank payment. In the experiments we set $P_{max} = 100$. Unlike [31], the values $\bar{P}_{ij}$ can thus be heterogeneous.

Following [31], we define the total amount of the external assets $E = \frac{\beta}{1-\beta} I$, where $I = \sum_{i,j=1}^{n} \bar{P}_{ij}$ is the total amount of the interbank liabilities and $\beta = E/(E + I)$ is a parameter representing the percentage of external assets in total assets at the system level; in our experiments $\beta = 0.05$. The nominal asset vector $\bar{c}$ is then computed in two steps. First, each bank is given the minimal value of external assets under which its balance sheet is equal to zero. At the second step, the remainder of the aggregated external assets is evenly distributed among all banks.

The financial shock is modeled by randomly choosing a subset of n_s banks of the system and nullifying their external financial assets.

6.2.1. A numerical study of the "price" of proportionality. To evaluate the "price" of imposing the pro-rata rule, we consider the F_1 performance index presented in (5.2). This is used as *system loss* also in [21]. Its minimal value over all matrices obeying the pro-rata constraint (4.2) is $F_1^{(\mathrm{pr})} = \sum_i (\bar{p}_i - p_i^*)$, where p^* is the dominating clearing vector from Lemma 1, which is found by solving problem (4.6). Relaxing the pro-rata constraint, we use the two-stage approach presented in Section 5.3 to find the globally optimal clearing matrix P^*, resulting in the system loss $F_1^{(\mathrm{nopr})} = \|\bar{P} - P^*\|_1$. The price, or global effect, of the

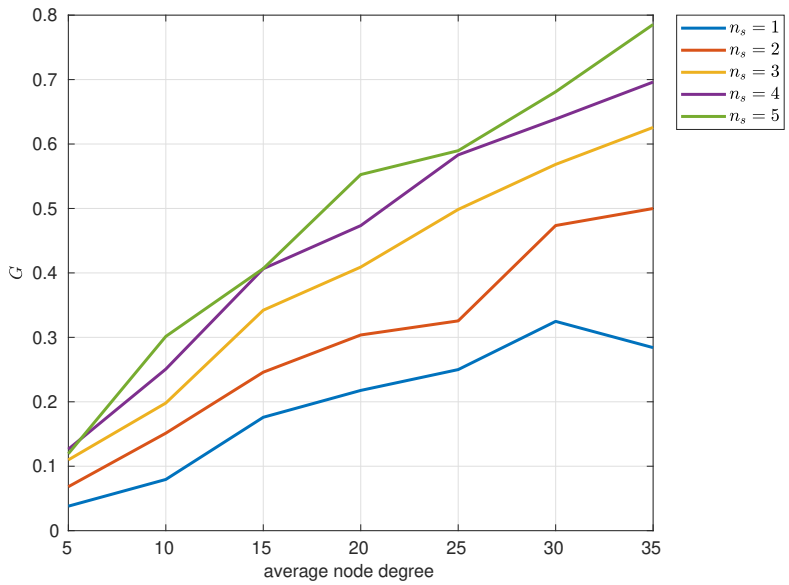

Figure 5. *The cost of introducing the pro-rata rule.*

pro-rata rule can thus be estimated by the following ratio

$$G = \frac{F_1^{(\mathrm{pr})} - F_1^{(\mathrm{nopr})}}{F_1^{(\mathrm{nopr})}} \in [0, 1].$$

If $G = 0$ (as, e.g., in Example 2, where all clearing matrices are optimal), the imposition of pro-rata constraint is "gratuitous" in the sense that it does not increase the aggregate system loss: $F_1^{(\mathrm{pr})} = F_1^{(\mathrm{nopr})}$. The larger value G we obtain, the more "costful" is the pro-rata restriction.

It seems natural that G is growing as the graph is becoming more dense, since in this situation the pro-rata rule visibly reduces the number of free variables in the optimization problem. We have tested this conjecture using the random model described above. The random graph contained $n = 50$ nodes, whereas the average node degree $d = np$ varied from 0 to 35. The number n_s of nodes that receive the shock varies from 1 to 5. The results were averaged over 50 runs. The resulting dependence between G and d is illustrated in Figure 5. We can see that the cost G can be up to 79%. We can observe that, when there is a stronger shock, the relaxation of the pro-rata rule has a larger impact.

To evaluate the price of the pro-rata rule, we also introduce another metric that measures the number of defaulted nodes. This metric evaluates the dimension of the failure cascade caused by the initial shock. Figure 6 compares the number of defaulted nodes ($\phi_i^{\mathrm{out}} < \bar{\phi}_i^{\mathrm{out}}$) with or without the pro-rata rule. We observe that relaxing the pro-rata rule significantly reduces the cascade failures and promotes isolation of the default contagion.

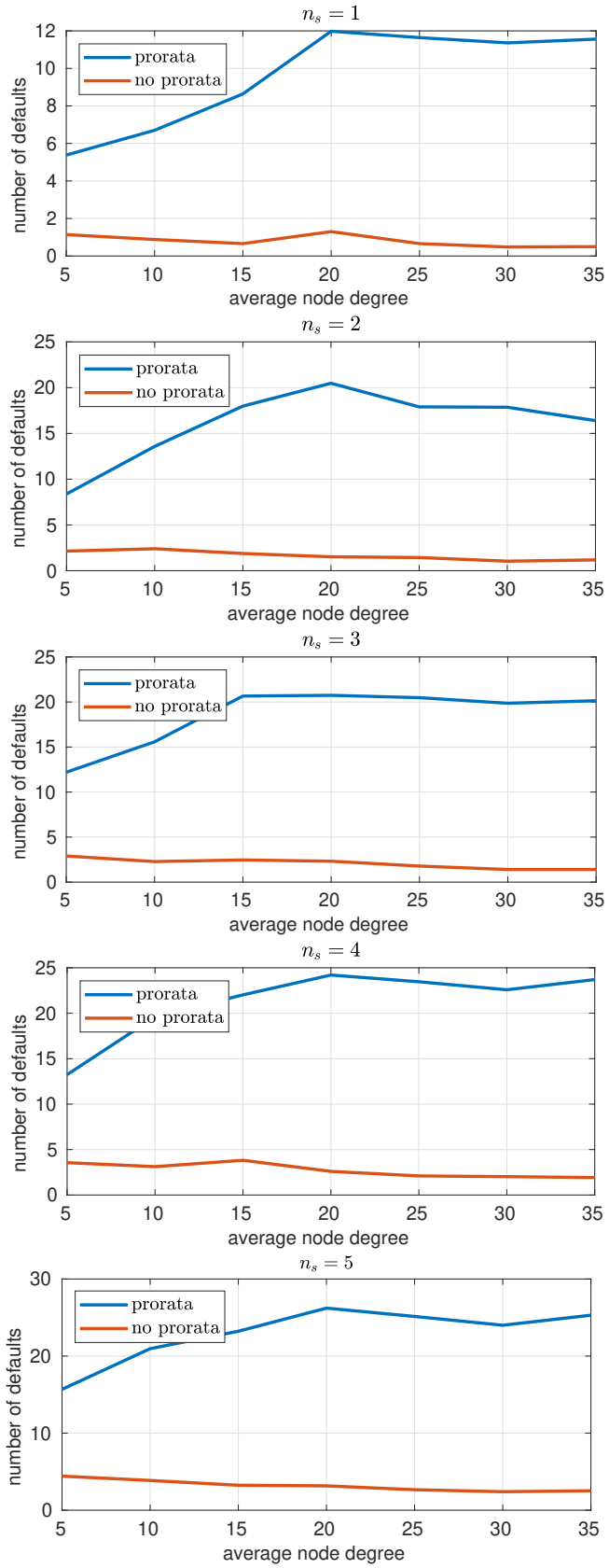

Figure 6. *Numbers of defaulting nodes with or without the pro-rata rule.*

7. Conclusions. Based on the financial networks model of [13], we explored in this paper the concept of a clearing vector of payments, and we developed new necessary and sufficient conditions for its uniqueness, together with a characterization of the set of all clearing vectors, see Theorem 4 and Theorem 6. Further, we examined matrices of clearing payments that naturally arise if one relaxes the pro-rata rule. Unique optimal clearing matrices can be computed efficiently via a two-stage convex optimization approach. Using numerical experiments with randomly generated synthetic networks, we showed that relaxation of the pro-rata rule allows to reduce significantly the aggregated systemic loss and the total number of defaulted nodes, thus providing effective default isolation.

A few aspects remain critical for the proposed approach. First and foremost, an existing gap from theory to practice is due to the fact that an aggregated unconstrained approach to the management of systemic risk should be accepted and contractualized ex-ante by the participant players. Further, the inter-bank liability structure should be made transparent and available to a central regulatory authority who decides the clearing payments in the case of defaults. Alternatively, further research could be devoted to exploring the feasibility of dispensing from centralized knowledge of the whole liability matrix and of developing a system-level but distributed and decentralized optimal clearing payment algorithm whose iterations are based only on *local* exchange of information among neighboring nodes.

REFERENCES

[1] D. Acemoglu, A. Ozdaglar, and A. Tahbaz-Salehi, *Systemic risk and stability in financial networks*, American Economic Review, 105 (2015), pp. 564–608.

[2] F. Allen, A. Babus, and E. Carletti, *Asset commonality, debt maturity and systemic risk*, Journal of Financial Economics, 104 (2012), pp. 519–534.

[3] H. Amini and Z. Feinstein, *Optimal network compression*, European Journal of Operational Research, 306 (2023), pp. 1439–1455.

[4] H. Amini, D. Filipović, and A. Minca, *To fully net or not to net: Adverse effects of partial multilateral netting*, Operations Research, 64 (2016), pp. 1135–1142.

[5] S. Barratt and S. Boyd, *Multi-period liability clearing via convex optimal control*, Available at SSRN 3604618, (2020).

[6] S. Battiston, J. B. Glattfelder, D. Garlaschelli, F. Lillo, and G. Caldarelli, *The structure of financial networks*, in Network Science, Springer, 2010, pp. 131–163.

[7] A. Blundell-Wignall and P. Slovik, *The EU stress test and sovereign debt exposures*, in OECD Working Papers on Finance, Insurance and Private Pensions, No. 4, OECD Financial Affairs Division, 2010, https://www.oecd.org/finance/financial-markets/45820698.pdf.

[8] G. Calafiore, G. Fracastoro, and A. Proskurnikov, *On optimal clearing payments in financial networks*, in IEEE Conf. Decision and Control, 2021, pp. 4804–4810.

[9] G. Calafiore, G. Fracastoro, and A. Proskurnikov, *Control of dynamic financial networks*, IEEE Control Systems Letters, 6 (2022), pp. 3206–3211.

[10] G. C. Calafiore, G. Fracastoro, and A. Proskurnikov, *Clearing payments in dynamic financial networks*, Automatica, 158 (2023), p. 111299.

[11] R. Cifuentes, G. Ferrucci, and H. S. Shin, *Liquidity risk and contagion*, Journal of the European Economic Association, 3 (2005), pp. 556–566.

[12] P. Csóka and P. Jean-Jacques Herings, *Decentralized clearing in financial networks*, Management Science, 64 (2018), pp. 4681–4699.

[13] L. Eisenberg and T. H. Noe, *Systemic risk in financial systems*, Management Science, 47 (2001), pp. 236–249.

[14] K. El Bitar, Y. M. Kabanov, and R. Mokbel, *On uniqueness of clearing vectors reducing the systemic*

risk, Inform. Primen., 11 (2017), pp. 109–118.
[15] M. Elliott, B. Golub, and M. O. Jackson, *Financial networks and contagion*, American Economic Review, 104 (2014), pp. 3115–53.
[16] H. Elsinger et al., *Financial networks, cross holdings, and limited liability*, Oesterreichische Nationalbank Austria, 2009.
[17] H. Elsinger, A. Lehar, and M. Summer, *Risk assessment for banking systems*, Management Science, 52 (2006), pp. 1301–1314.
[18] T. Fischer, *No-arbitrage pricing under systemic risk: Accounting for cross-ownership*, Mathematical Finance: An International Journal of Mathematics, Statistics and Financial Economics, 24 (2014), pp. 97–124.
[19] D. M. Gale and S. Kariv, *Financial networks*, American Economic Review, 97 (2007), pp. 99–103.
[20] A. Gallice, *Bankruptcy problems with reference-dependent preferences*, International Journal of Game Theory, 48 (2019), pp. 311–336.
[21] P. Glasserman and H. P. Young, *Contagion in financial networks*, Journal of Economic Literature, 54 (2016), pp. 779–831.
[22] A. Haldane and R. May, *Systemic risk in banking ecosystems*, Nature, 469 (2011), pp. 351–355.
[23] F. Harary, R. Norman, and D. Cartwright, *Structural models. An introduction to the theory of directed Graphs*, Wiley & Sons, New York, London, Sydney, 1965.
[24] P. J.-J. Herings and P. Csóka, *Uniqueness of clearing payment matrices in financial networks*, GSBE Research Memoranda, 014 (2021), pp. 1–29, https://doi.org/10.26481/umagsb.2021014.
[25] T. Hurd, *Contagion! Systemic Risk in Financial Networks*, Springer Nature, 2016.
[26] M. O. Jackson and A. Pernoud, *Systemic risk in financial networks: a survey*, Annual Review of Economics, 13 (2021), pp. 171–202.
[27] Y. M. Kabanov, R. Mokbel, and K. El Bitar, *Clearing in financial networks*, Theory of Probability & Its Applications, 62 (2018), pp. 252–277.
[28] M. M. Kaminski, *'hydraulic' rationing*, Mathematical Social Sciences, 40 (2000), pp. 131–155.
[29] M. Kusnetsov and L. A. M. Veraart, *Interbank clearing in financial networks with multiple maturities*, SIAM Journal on Financial Mathematics, 10 (2019), pp. 37–67.
[30] L. Massai, G. Como, and F. Fagnani, *Equilibria and systemic risk in saturated networks*, Mathematics of Operation Research, 47 (2022), pp. 1707–2545.
[31] E. Nier, J. Yang, T. Yorulmazer, and A. Alentorn, *Network models and financial stability*, Journal of Economic Dynamics and Control, 31 (2007), pp. 2033–2060.
[32] A. Proskurnikov, G. Calafiore, and M. Cao, *Recurrent averaging inequalities in multi-agent control and social dynamics modeling*, Annual Reviews in Control, 49 (2020), pp. 95–112.
[33] X. Ren and L. Jiang, *Mathematical modeling and analysis of insolvency contagion in an interbank network*, Operations Research Letters, 44 (2016), pp. 779–783.
[34] L. C. Rogers and L. A. Veraart, *Failure and rescue in an interbank network*, Management Science, 59 (2013), pp. 882–898.
[35] H. S. Shin, *Risk and liquidity in a system context*, Journal of Financial Intermediation, 17 (2008), pp. 315–329.
[36] T. Suzuki, *Valuing corporate debt: the effect of cross-holdings of stock and debt*, Journal of the Operations Research Society of Japan, 45 (2002), pp. 123–144.
[37] W. Thomson, *Game-theoretic analysis of bankruptcy and taxation problems: Recent advances*, International Game Theory Review, 15 (2013), p. 1340018.

8. Appendix.

8.1. Technical preliminaries. The proof of the main results of this paper are based on few technical propositions. The next proposition follows, e.g., from [23, Corollary 4.3a']

Proposition 2. *Each graph contains at least one sink strong component. Any node whose component is not a sink is a connected to one of the sink components by a path.*

We will also use a technical lemma establishing necessary and sufficient conditions for the

Schur stability of substochastic matrices. The spectral radius of a square matrix A is denoted by $\rho(A)$. We call a matrix A *Schur* stable if $\rho(A) < 1$. The Gershgorin disk theorem implies that $\rho(A) \leq 1$ for any substochastic matrix A. If A is stochastic, then $\rho(A) = 1$ and A cannot be Schur stable.

Lemma 6. *Let $A \in \mathbb{R}^{\mathcal{V}\times\mathcal{V}}$ be a substochastic matrix. Then three statements are equivalent:*

1. *Matrix A is Schur stable: $\rho(A) < 1$;*
2. *Submatrix $A^0 = (a_{i,j})_{i,j\in\mathcal{V}_0}$ is* not *stochastic for every subset of indices $\mathcal{V}_0 \subseteq \mathcal{V}$;*
3. *The set of nodes $\mathcal{V}_d = \{i : \sum_j a_{ij} < 1\}$ is non-empty and globally reachable in graph $\mathcal{G}[A]$;*
4. *Each strong component of $\mathcal{G}[A]$ being a sink contains at least one node from $\mathcal{V}_d$.*

Proof of Lemma 6. Implication 3 $\Longrightarrow$ 1 is a well-known fact, see e.g. [32, Lemma 7]. To prove 1$\Longrightarrow$2 notice that if submatrix $A^0 = (a_{i,j})_{i,j\in\mathcal{V}_0}$ is stochastic, then $0 \leq \sum_{k\notin\mathcal{V}_0} a_{ik} \leq 1 - \sum_{j\in\mathcal{V}_0} a_{jk} = 0\,\forall i \in \mathcal{V}_0$, that is, $a_{ik} = 0\,\forall i \in \mathcal{V}_0, k \notin \mathcal{V}_0$. Hence, A is decomposable as follows

$$A = \begin{pmatrix} A^0 & \mathbf{O} \\ * & * \end{pmatrix}, \tag{8.1}$$

where $\mathbf{O}$ is a block of zeros and $*$ stand for some other submatrices. Hence, 1 is an eigenvalue of A, in particular, $\rho(A) = 1$ and A is not Schur stable.

Implication 2$\Longrightarrow$4 is straightforward from the definition of a sink component. If $\mathcal{V}_0$ is the set of nodes belonging to such a strong component, then matrix A is decomposed as in (8.1), where $A^0 = (a_{i,j})_{i,j\in\mathcal{V}_0}$. In view of statement 2, A^0 is not a stochastic matrix, so the sum in at least one of its rows is less than 1, that is, $\mathcal{V}_0 \cap \mathcal{V}_d \neq \emptyset$.

Implication 3$\Longrightarrow$4 is straightforward from Proposition 2 and the definition of a strongly connected component (whose every two nodes are mutually reachable).

8.2. Proofs of the main results. Proof of Lemma 1. Since $\mathcal{D}$ is compact (closed and bounded), the projection map $p \mapsto p_i$ reaches a maximal value $p_i^* = \max_{p\in\mathcal{D}} p_i$. The vector $p^* \doteq (p_i^*)_{i\in\mathcal{V}}$, by construction, dominates every vector from $\mathcal{D}$. It remains to show that this vector belongs to $\mathcal{D}$. By definition, $p_i^* \in [0, \bar{p}_i]$, $\forall i$. Also, $c + A^\top p^* \geq c + A^\top p \geq p$, $\forall p \in \mathcal{D}$. Hence, for each index $i \in \mathcal{V}$ one has

$$(c + A^\top p^*)_i \geq p_i \quad \forall p \in \mathcal{D}.$$

Taking the maximum over all p, one shows that

$$(c + A^\top p^*)_i \geq p_i^* = \max_{p\in\mathcal{D}} p_i,\ \forall i \in \mathcal{V},$$

i.e., $c + A^\top p^* \geq p^*$, finishing the proof of statement 1.

Statement 2 is now straightforward from Definition 3.

Statement 3 now follows from [13, Lemma 4], stating that every minimizer in problem (4.5) with a decreasing function F is a clearing vector. We give here the proof for the reader's convenience. Assume, by the purpose of contradiction, that (4.3) fails to hold, that is, $p_i^* < \min\left(\bar{p}_i, c_i + \sum_{k\neq i} a_{ki} p_k^*\right)$ for some i. Consider the vector $\hat{p} \doteq p^* + \delta \mathbf{e}_i$, where $\delta > 0$ is

sufficiently small and $\mathbf{e}_i$ is the coordinate vector whose ith coordinate is 1 and others are null. By noticing that $\left(c + A^\top \hat{p}\right)_i > \hat{p}_i$ and

$$\left(c + A^\top \hat{p}\right)_j \geq \left(c + A^\top p^*\right)_j \geq p^*_j = \hat{p}_j \quad \forall j \neq i,$$

one shows that $\hat{p} \in \mathcal{D}$, which contradicts Statement 1.

To prove statement 4, denote the set of nodes of a sink component by $\mathcal{V}_0$. Then, stochastic matrix A decomposes as in (8.1), and submatrix $A^0 = (a_{i,j})_{i,j\in\mathcal{V}_0}$ is also stochastic. Furthermore, this matrix is irreducible due to the definition of a strongly connected component. Thanks to the Perron-Frobenius theorem, $A^\top$ has a positive eigenvector π^0 such that $A^\top \pi^0 = 0$. Denoting

$$\pi_i = \pi^0_i \, \forall i \in \mathcal{V}_0, \quad \pi_i = 0 \, \forall i \notin \mathcal{V}_0,$$

one obviously has $A^\top \pi = \pi$. Notice that vector $p^\varepsilon = p^* + \varepsilon\pi$ obeys the inequality $p^\varepsilon \leq c + A^\top p^\varepsilon$ for each $\varepsilon > 0$ and, furthermore, $p^\varepsilon_i = p^*_i \leq \bar{p}_i \, \forall i \notin \mathcal{V}_0$. Since p^* is the maximal element in $\mathcal{D}$, $p^\varepsilon \notin \mathcal{D} \, \forall \varepsilon > 0$, which is possible only when $p^*_i = \bar{p}_i$ for some $i \in \mathcal{V}_0$.

Statement 5 follows from Lemma 6 and the maximality of p^*. Let p be a clearing vector enjoying the property from statement 4 and $I = \{i : p_i = \bar{p}_i\}$. Notice that, in view of Proposition 2, the set I is globally reachable in graph $\mathcal{G}[A]$ (each node is connected to a sink strong component, and all nodes within this component are connected to some node from I). Denoting $I^c = \mathcal{V} \setminus I$, submatrix $\hat{A} = (a_{ij})_{i,j\in I^c}$, obviously, cannot contain stochastic submatrices, being thus Schur stable (Lemma 6). The corresponding subvector $\hat{p} = (p_j)_{j\in I^c}$ obeys the equation

$$\hat{p} = \hat{A}^\top p + \hat{c}, \quad \hat{c}_j \doteq c_j + \sum_{i\in I} a_{ij}\bar{p}_i \, \forall j \in I^c. \tag{8.2}$$

Recalling that $p^* \geq p$, one has $p^*_i = \bar{p}_i \, \forall i \in I$; for this reason, subvector $\hat{p}^* = (p^*_j)_{j\in I^c}$ also obeys (8.2), which means that $\hat{p} = \hat{p}^*$ and, therefore, $p = p^*$.

Proof of Lemma 2. We start with "if" part. If $c_i > 0$ and $\bar{p}_i > 0$, then polytope $\mathcal{D}$, obviously, contains the vector $\varepsilon\mathbf{e}_i$, where $\varepsilon \in (0, \min(c_i, \bar{p}_i))$. Since $p^* = \max \mathcal{D}$, we have $p^*_i > 0$ whenever condition 1) holds. Notice also that if $p^*_j > 0, p^*_i > 0$ and $a_{ji} > 0$, then (4.3) implies that $p^*_i > 0$. Hence, condition 2) also entails that $p^*_i > 0$. Assume now that condition 3) holds and let $\mathcal{I}$ be the set of nodes of the strongly connected component containing i. Since i is not a sink, this component is non-trivial (contains two or more nodes), and for each $j \in \mathcal{I}$ we have $\bar{p}_j > 0$. Also, the submatrix $\tilde{A} = (a_{jk})_{j,k\in\mathcal{I}}$ is stochastic and irreducible, hence, in view of the Perron-Frobenius theorem, there exist a *strictly positive* eigenvector $v \in \mathbb{R}^{\mathcal{I}}$ such that $A^\top v = v$. Rescaling, one may assume that $v_i \leq \bar{p}_i \, \forall i \in \mathcal{I}$. Hence, $\mathcal{D}$ contains the vector p, where

$$p_j \doteq \begin{cases} v_j, j \in \mathcal{I} \\ 0, j \notin \mathcal{I} \end{cases} \qquad \forall j \in \mathcal{V},$$

in particular, $p^*_i \geq p_i > 0$.

To prove the "only if" part, notice first that $p^*_i > 0$ entails that $\bar{p}_i > 0$, so i is not a sink node. Assume, on the contrary, that none of 1)-3) holds. Consider again the strongly

connected component $\mathcal{I} \ni i$ and the corresponding irreducible submatrix $\tilde{A} = (a_{jk})_{j,k\in\mathcal{I}}$. Since condition 3) is violated, we have $\rho \doteq \rho(\tilde{A}) < 1$. Introducing the *positive* Perron-Frobenius eigenvector $v \in \mathcal{I}$ such that $\tilde{A}^\top v = v$ and (4.3) entails that

$$0 < \sum_{j\in\mathcal{I}} v_j p_j^* \le \sum_{j\in\mathcal{I}} v_j \sum_{k\in\mathcal{I}} a_{kj} p_k^* + \sum_{j\in\mathcal{I}} v_j \sum_{k\notin\mathcal{I}} a_{kj} p_k^* \le$$
$$\le \rho \sum_{k\in\mathcal{I}} v_k p_k^* + \sum_{j\in\mathcal{I}} v_j \sum_{k\notin\mathcal{I}} a_{kj} p_k^*.$$

The latter inequality may hold only if $k \notin \mathcal{I}$ exists such that $a_{kj} p_k^* > 0$. In other words, there exists another strongly connected component $\mathcal{I}_1 \neq \mathcal{I}$ such that $p_j^* > 0\, \forall j \in \mathcal{I}_1$ and $\mathcal{I}_1$ is connected to $\mathcal{I}$ by at least one arc (this implies that no arc can go from $\mathcal{I}$ to $\mathcal{I}_1$). Since condition 2) is violated for i, conditions 1 and 2 are also violated for every $i_1 \in \mathcal{I}_1$. Now we can repeat the argument, replacing $\mathcal{I}$ by $\mathcal{I}_1$ and find another strongly connected component $\mathcal{I}_2 \neq \mathcal{I}_1$, such that at least one arc comes from $\mathcal{I}_2$ to $\mathcal{I}_1$ (in view of this $\mathcal{I}_2 \neq \mathcal{I}$), $p_j^* > 0\, \forall j \in \mathcal{I}_2$ and none element of $\mathcal{I}_2$ satisfies condition 1 or 2. Repeating this process, one could construct the infinite set of disjoint strongly connected components $\mathcal{I}_0 \doteq \mathcal{I}, \mathcal{I}_1, \mathcal{I}_2, \ldots$ such that $\mathcal{I}_s$ is connected to $\mathcal{I}_{s-1}$, $p_j^* > 0\, \forall j \in \mathcal{I}_s$ yet conditions 1 and 2 fail to hold for $i \in \mathcal{I}_s$. This leads to the contradiction (the set of nodes $\mathcal{V}$ is finite).

Proof of Lemma 3. Consider an arbitrary clearing vector p^0 and let $J \doteq \{i : p_i^0 < \bar{p}_i\}$. Our goal is to show that $p_i^0 = p_i^*\, \forall i \in I_0'$.

Step 1. In view of (4.3), the vector $p = p^0$ obeys the equations

$$\begin{aligned} p_i &= \bar{p}_i \quad \forall i \notin J, \\ p_i &= (c + A^\top p)_i = c_i + \sum_{k\notin J} a_{ki}\bar{p}_k + \sum_{j\in J} a_{ji} p_j \;\forall i \in J. \end{aligned} \tag{8.3}$$

Step 2. We are going to show that if (8.3) has at least one solution, then $p = p^*$ should also be a solution to (8.3). Indeed, every solution to system (8.3) is a global minimizer in the optimization problem (4.5) with the objective function

$$F(p) \doteq \sum_{k\notin J} (\bar{p}_k - p_k) + \sum_{j\in J} \left(c_j + \sum_{i\in\mathcal{V}} a_{ij} p_i - p_j \right). \tag{8.4}$$

If p^0 satisfies (8.3), then $F(p^0) = 0 \le F(p)\, \forall p \in \mathcal{D}$. The function (8.4) is non increasing, because it can be written as

$$F(p) = \beta - b^\top p, \quad b_i \doteq 1 - \sum_{j\in I} a_{ij} \ge 0, \quad \beta = \text{const},$$

Lemma 1 guarantees that p^* is a global minimizer in (4.5), and hence $F(p^*) = 0$. Recalling that $p^* \in \mathcal{D}$, this is possible only if all addends in the right-hand side of (8.4) vanish as $p = p^*$.

Step 3. We now show that the equations (8.3) uniquely determine the subvector $(p_i)_{i\in I_0'}$. The first equation in (8.3) entails that $p_i = \bar{p}_i\, \forall i \in I_0' \setminus J$. By construction, the set I_0' cannot

be reached from any other node $j \notin I_0'$ (otherwise, I_0 could also be reached from j), that is, $a_{ji} = 0$ whenever $i \in I_0'$ and $j \notin I_0'$. Hence, for all solutions to (8.3) one has

$$p_i = c_i + \sum_{k \notin J} a_{ki}\bar{p}_k + \sum_{j \in I_0' \cap J} a_{ji} p_j \quad \forall i \in I_0' \cap J. \tag{8.5}$$

We are now going to show that the matrix $\tilde{A} = (a_{ij})_{i,j \in I_0' \cap J}$ is Schur stable. Assume, by purpose of contradiction , that $\tilde{A}$ is not Schur stable and thus (see Lemma 6) contains a *stochastic* submatrix $(a_{ij})_{i,j \in K}$, where $K \subseteq I_0' \cap J$. Since A is also stochastic, one has $a_{ij} = 0 \, \forall i \in K, j \in \mathcal{V} \setminus K$. Denoting

$$\tilde{z} \doteq (z_i)_{i \in \mathcal{V}}, \quad z_i \doteq \begin{cases} 1, \, i \in K, \\ 0, \, i \notin K, \end{cases}$$

one obviously has $Az \geq z$ or, equivalently, $z^\top A^\top \geq z^\top$. The second equation in (8.3) entails that

$$\begin{aligned} z^\top p = \sum_{i \in K} z_i p_i = \sum_{i \in K} z_i \left(c_i + \sum_{j \in \mathcal{V}} a_{ji} p_j \right) &\overset{z_i = 0 \, \forall i \notin K}{=\!=\!=} \\ = \sum_{i \in \mathcal{V}} z_i \left(c_i + \sum_{j \in \mathcal{V}} a_{ji} p_j \right) &\geq z^\top c + z^\top p, \end{aligned}$$

whence $z^\top c = 0$ and thus $K \cap C^+ = \emptyset$. On the other hand, $K \subseteq J \subseteq \mathcal{V} \setminus S$, therefore, K and I_0 are disjoint. By construction, graph $\mathcal{G}$ contains no arcs connecting K to nodes from $\mathcal{V} \setminus K$, in particular, I_0 is not reachable from K. This contradicts to the definition of $I_0' \supseteq K$. The contradiction shows that $\tilde{A}$ is Schur stable, in particular, (8.5) has a unique solution. Since both p^0 and p^* satisfy (8.3) and (8.5), $p_i^0 = p_i^*$ for all $i \in I_0'$.

Proof of Theorem 4. The first statement is obvious, since the sets $I_0', I_1', \ldots$ are disjoint and the set of all nodes $\mathcal{V}$ is finite.

To prove the second statement, we show it via induction on $k = 0, 1, \ldots$ that for each clearing vector p and each k we have $p_i = p_i^* \, \forall i \in I_k'$. The induction base ($k = 0$) is proved in Lemma 3. Assume that the statement has been proved for $k = 0, \ldots, q-1$ (where $q \geq 1$); we have to prove it for $k = q$. By construction, banks from $\mathcal{V}_q$ ($q \geq 1$) have no liability to banks from $\mathcal{V} \setminus \mathcal{V}_q = I_0' \cup \ldots \cup I_{q-1}'$. Also, they neither have outside assets ($c_i = 0 \, \forall i \notin I_0$) nor receive payments from banks from $I_0', \ldots, I_{q-2}'$. Only banks from I_{q-1}' can pay liability to them. For each clearing vector p, (4.3) entails that

$$p_i = \min \left(\bar{p}_i, \sum_{s \in I_{q-1}'} a_{si} p_s + \sum_{k \in \mathcal{V}_q} a_{ki} p_k \right) \quad \forall i \in \mathcal{V}_q. \tag{8.6}$$

In view of the induction hypothesis, the first sum is nothing else than $c_i^{(q)}$ for all clearing vectors p. Hence, for any clearing vector p (of the original financial network) the subvector

$(p_i)_{i\in\mathcal{V}_q}$ serves a clearing vector in the *reduced* financial network, determined by the set of nodes $\mathcal{V}_q \neq \emptyset$, the respective submatrix $A^q = (a_{ij})_{i,j\in\mathcal{V}_q}$ and the vector $c^{(q)}$, standing for the "external" assets. This holds, in particular, for the dominant clearing vector $p = p^*$. Lemma 3 guarantees that the elements $p_i, i \in I'_q$ are determined uniquely, that is, $p_i = p^*_i \,\forall i \in I'_q$, which proves the induction step $k = q$. This finishes the proof of the second statement of Theorem 4.

Furthermore, it is now obvious that each clearing vector (that is, a solution to (4.3)) has the structure (4.10), where $\xi = (p_i)_{i\in\mathcal{V}_s} \geq 0$ is some vector. If $\mathcal{V}_s = \emptyset$ (and the component ξ is empty), the only clearing vector is $p = p^*$. Otherwise, ξ obeys (4.11). Indeed, the matrix B is stochastic by construction, so that $\mathbf{1}^\top B = \mathbf{1}^\top$. In view of (4.3), one has

$$\xi \leq B^\top \xi.$$

Multiplying the latter inequality by $\mathbf{1}^\top$, one shows that it can only hold if $\xi = B^\top\xi$. On the other had, $\xi_i \leq \bar{p}_i \,\forall i \in \mathcal{V}_s$. The inverse statement is also obvious: each vector (4.10) satisfying (4.11) is a solution to (4.3). To finish the proof of the theorem, it remains to show that there are infinitely many subvectors ξ obeying (4.11). Notice that B is a stochastic matrix, and hence $\rho(B) = 1$. Also, $\bar{p}_i > 0 \,\forall i \in \mathcal{V}_s$ (the nodes cannot be sinks). The Perron-Frobenius theorem entails the existence of at least one non-negative eigenvector $\xi^0 \in \mathbb{R}^{\mathcal{V}_s} \setminus \{0\}$, which obeys (4.11). Obviously, the set of all vectors satisfying (4.11) is a convex polytope, which contains a trivial vector $\xi = 0$ and, thus, also the whole line segment $[0, \xi^0]$. We have proved that the set of clearing vectors in infinite.

Proof of Theorem 6. Suppose that (i) holds and consider some non-trivial sink component whose set of nodes is $\mathcal{I}$. Obviously, $\bar{p}_i > 0 \,\forall i \in \mathcal{I}$. The submatrix $\tilde{A} = (a_{ij})_{i,j\in\mathcal{I}}$ is stochastic, let $v \in \mathbb{R}^{\mathcal{I}}$ stand for the Perron-Frobenius eigenvector of $A^\top$ such that $A^\top v = v$. Assume that $\mathcal{I}$ neither intersects C^+ nor is reachable from C^+. If $j \notin \mathcal{I}$ is connected to some node $i \in \mathcal{I}$, then $c_j = 0$ and j cannot be reached from C^+. In view of Lemma 2, $p^*_j = 0$ and, therefore, for all vectors from $\mathcal{D}$ we have $p_j = 0$. Therefore, every vector p such that

$$p = \begin{cases} p^*_i, i \notin \mathcal{I}, \\ \varepsilon v_i, i \in \mathcal{I} \end{cases}$$

satisfies (4.3) when $\varepsilon > 0$ is small enough. We arrive at the contradiction with the assumption of the clearing vector's uniqueness. Implication (i)$\Longrightarrow$(ii) is proved.

Suppose now that (ii) holds yet the clearing vector is not unique. Consider the final set $\mathcal{V}_s \neq \emptyset$ found by Algorithm 4.1 ($s \geq 1$). By construction, no arc leads from $\mathcal{V}_s$ to $I'_0, \ldots, I'_{s-1}$, therefore, $\mathcal{V}_s$ contains at least one sink component. This sink component cannot be trivial, because all sink nodes belong to $I_0 \subseteq I'_0$. For the same reason, this sink component contains no nodes from C^+. In view of (ii), a path from C^+ to $\mathcal{V}_s$ exists. In view of Lemma 2, for each node j on this path we have $p^*_j > 0$. Hence, there should exist an arc $j \to i$ connecting some $j \notin \mathcal{V}_s$ (with $p^*_j > 0$) to some $i \in \mathcal{V}_s$. This contradicts to the assumption that $\mathcal{V}_s$ is a final set, because $c^{(s)} > 0$. Implication (ii)$\Longrightarrow$(i) is proved.

Proof of Lemma 4. Assume, by the purpose of contradiction, that a local minimizer P^*

fails to satisfy (3.3). Hence, an index i exists such that

$$p_i^* \doteq (P^*\mathbf{1})_i = \sum_j p_{ij}^* < \min\left(\bar{p}_i, c_i + \sum_k p_{ki}^*\right).$$

Since $p_i^* < \bar{p}_i$, there exists some j such that $p_{ij}^* < \bar{p}_{ij}$. Then for small $\delta > 0$ matrix $\hat{P} = P^* + \delta \mathbf{e}_i \mathbf{e}_j^\top$ (obtained from P^* by the replacement $p_{ij}^* \mapsto p_{ij}^* + \delta$, all other entries being invariant) belongs to $\mathcal{D}_{n\times n}$. Indeed, $\hat{P} \in [0, \bar{P}]$ and

$$\hat{p} \doteq \hat{P}\mathbf{1} = p^* + \delta \mathbf{e}_i \leq c + (P^*)^\top \mathbf{1} \leq c + (\hat{P})^\top \mathbf{1}.$$

At the same time, $F(\hat{P}) < F(P^*)$. This contradicts to the assumption of local optimality.

Proof of Lemma 5. The proof is immediate by observing that if $\tilde{P}$ is optimal for (5.4) then it must be that $F_1(\tilde{P}) = F_1(P^*)$, where

$$\begin{aligned} F_1(\tilde{P}) &= \|\bar{P} - \tilde{P}\|_1 = \mathbf{1}^\top(\bar{P} - \tilde{P})\mathbf{1} \quad [\text{since } \bar{P} \geq \tilde{P}] \\ F_1(P^*) &= \|\bar{P} - P^*\|_1 = \mathbf{1}^\top(\bar{P} - P^*)\mathbf{1} \quad [\text{since } \bar{P} \geq P^*]. \end{aligned}$$

Therefore, $F_1(\tilde{P}) = F_1(P^*)$ implies that $\mathbf{1}^\top(\tilde{P} - P^*)\mathbf{1} = 0$. Defining $\Delta \doteq \tilde{P} - P^*$, we have that $\tilde{P} = P^* + \Delta$, with Δ such that $\mathbf{1}^\top \Delta \mathbf{1} = 0$. Since $\tilde{P}$ must be feasible (i.e., it should satisfy $0 \leq \tilde{P} \leq \bar{P}$, $c + \tilde{P}^\top \mathbf{1} - \tilde{P}\mathbf{1} \geq 0$), we immediately obtain that $\tilde{P} \in \mathcal{S}^*$.

Conversely, take any $\tilde{P} \in \mathcal{S}^*$. Then, by construction, $\tilde{P}$ is feasible for (5.4) and its objective is

$$\begin{aligned} F_1(\tilde{P}) &= \|\bar{P} - \tilde{P}\|_1 = \mathbf{1}^\top(\bar{P} - \tilde{P})\mathbf{1} \quad [\text{since } \bar{P} \geq \tilde{P}] \\ &= \mathbf{1}^\top(\bar{P} - P^* - \Delta)\mathbf{1} = \mathbf{1}^\top(\bar{P} - P^*)\mathbf{1} - \mathbf{1}^\top \Delta \mathbf{1} \\ &= \mathbf{1}^\top(\bar{P} - P^*)\mathbf{1} \quad [\text{since } \mathbf{1}^\top \Delta \mathbf{1} = 0 \text{ in the definition of } \mathcal{S}^*] \\ &= \|\bar{P} - P^*\|_1 = F_1(P^*). \end{aligned}$$

Therefore, $\tilde{P}$ is optimal for (5.4), which concludes the proof.

Proof of Proposition 1. Problem (5.8) is quadratic and strongly convex in its matrix variable Δ. Further, this problem is feasible, since it admits at least the point $\Delta = 0$, due to the fact that P^* is optimal, hence feasible, for (5.4). Therefore, Δ^* exists and it is unique. Consider then $\tilde{P}^* \doteq P^* + \Delta^*$. Since by construction $\bar{P} \geq \tilde{P}^* \geq 0$ and $c + \tilde{P}^{*\top}\mathbf{1} - \tilde{P}^*\mathbf{1} \geq 0$, we have that $\tilde{P}^*$ is feasible for problem (5.4). Furthermore, its objective value is

$$\begin{aligned} F_1(\tilde{P}^*) &= \|\bar{P} - \tilde{P}^*\|_1 = \mathbf{1}^\top(\bar{P} - \tilde{P}^*)\mathbf{1} \quad [\text{since } \bar{P} \geq \tilde{P}^*] \\ &= \mathbf{1}^\top(\bar{P} - P^* - \Delta^*)\mathbf{1} = \mathbf{1}^\top(\bar{P} - P^*)\mathbf{1} - \mathbf{1}^\top \Delta^* \mathbf{1} \\ &= \mathbf{1}^\top(\bar{P} - P^*)\mathbf{1} \quad [\text{since } \mathbf{1}^\top \Delta^* \mathbf{1} = 0 \text{ due to the constraint in (5.8)}] \\ &= \|\bar{P} - P^*\|_1 = F_1^*. \end{aligned}$$

Therefore, $\tilde{P}^*$ is optimal for (5.4), which proves point (b) in Proposition 1. From Lemma 4 we also immediately conclude that $\tilde{P}^*$ is a clearing matrix, which proves point (a). For point (c), consider that from Lemma 5 it holds that $P^* + \Delta$ in problem (5.8) spans over all optimal solutions to problem (5.4). Therefore, problem (5.8) provides the unique minimum Euclidean norm solution among all the possible optimal solutions to (5.4).