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Delay Robustness of Consensus Algorithms: Continuous-Time Theory

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***Abstract*—Consensus among autonomous agents is a key problem in multi-agent control. In this paper, we consider averaging consensus policies over time-varying graphs in presence of unknown but bounded communication delays. It is known that consensus is established (no matter how large the delays are) if the graph is periodically, or *uniformly* quasi-strongly connected (UQSC). The UQSC condition is often believed to be the weakest sufficient condition under which consensus can be proved. We show that the UQSC condition can actually be substantially relaxed and replaced by a condition that we call *aperiodic* quasi-strong connectivity (AQSC), which, in some sense, proves to be very close to the necessary condition (the so-called integral connectivity). Under the assumption of reciprocity of interactions (e.g., for undirected or type-symmetric graphs), a necessary and sufficient condition for consensus in presence of bounded communication delays is established; the relevant results have been previously proved only in the undelayed case.**

I. Introduction

Consensus policies are prototypic distributed algorithms for multi-agent coordination [1], [2] inspired by regular "intelligent" behaviors of biological and physical systems [3]–[5]. The most studied first-order consensus algorithms are based on the principle of *iterative averaging*. Averaging algorithms were first proposed in sociological literature [6], [7] and have found numerous applications in distributed computing [8]–[11]. Some consensus algorithms for general agents are squarely based on first-order averaging protocols [12], [13].

Consider a finite team of agents $\mathcal{V}$, each of which is associated with some value of interest $x_i \in \mathbb{R}$, $i \in \mathcal{V}$. In the discrete-time case, the agents simultaneously update their values to the average of their own value and the others' values:

$$x_i(t+1) = \sum\nolimits_{j\in\mathcal{V}} a_{ij}(t)x_j(t) \quad \forall i \in \mathcal{V}, t = 0, 1, \dots \qquad (1)$$

where $(a_{ij}(t)) \in \mathbb{R}^{\mathcal{V}\times\mathcal{V}}$ is a stochastic matrix. The continuous-time counterpart of the algorithm (1) is

$$\dot{x}_i(t) = \sum\nolimits_{j\in\mathcal{V}} a_{ij}(t)(x_j(t) - x_i(t)) \;\; \forall i \in \mathcal{V},\, t \geq 0 \qquad (2)$$

where the matrix $(a_{ij}(t)) \in \mathbb{R}^{\mathcal{V}\times\mathcal{V}}$ is nonnegative. In both cases, the entry $a_{ij}(t)$ is interpreted as the weight of influence of agent j on agent i at time t: the larger weight is, the stronger is attraction of agent i's value to agent j's value.

The central question regarding dynamics (1) and (2) is establishing eventual (global) *consensus*, that is, convergence of all values $x_i(t)$ to the same value $\bar{x} = \lim_{t\to\infty} x_i(t)\,\forall i$ (which depends on the initial condition). In the case of constant weights $a_{ij}(t) \equiv a_{ij}$, the consensus criterion is well-known [1], [14], [15]. Consensus in the continuous-time case (2) established if and only if the graph associated to matrix $A \triangleq (a_{ij})$ is quasi-strongly connected, that is, some node of the graph is connected (directly or indirectly) to all other nodes. In the discrete-time case (1), some aperiodicity conditions also arise [15], [16]. A more general behavior is a "partial" (group, cluster) consensus [17]–[19], i.e., splitting of the agents into several groups converging to different values.

Finding criteria ensuring consensus in the case of a general time-varying matrix $A(t) \triangleq (a_{ij}(t))$ is a difficult problem whose complete solution is still elusive. A well-known *necessary* condition for consensus is the so-called *integral* (essential, persistent) connectivity [20]–[22]: the arcs (j, i) such that

$$\sum\nolimits_{t=0}^{\infty} a_{ij}(t) = \infty \quad \text{or} \quad \int_0^{\infty} a_{ij}(t)dt = \infty \qquad (3)$$

should constitute a quasi-strongly connected graph. This condition, however, is far from being sufficient. Simple counter-examples [23], [24] show that the agents may fail to reach consensus (the solution may even fail to converge) even if the persistent graph is complete; necessary and sufficient consensus conditions are still elusive. In the discrete-time case, consensus is equivalent to ergodicity of the backward infinite products $A(t)\dots A(0)$ [25]–[27]. Necessary consensus conditions inspired by theory of inhomogeneous Markov chains were proposed in [28], [29] (the infinite flow, absolute infinite flow and jet infinite flow conditions); the verification of these properties is, however, a self-standing non-trivial problem.

Sufficient criteria for consensus can be divided into several groups. Conditions of the first type require the periodic, or *uniform* quasi-strong connectivity (UQSC) [1], [30], [31]: two numbers $T, \varepsilon > 0$ should exist such that the unions of the interacting graphs over each interval $[t, t+T]$, $t \geq 0$ are quasi-strongly connected, and this connectivity property persists if one removes "light" arcs whose weights are less than ε. The uniform connectivity is however only sufficient yet not necessary for consensus and implies, in fact, much stronger properties of consensus with uniform convergence [31], [32].

Consensus criteria of the second kind ensure consensus in presence of the integral connectivity and some conditions ensuring some balance of couplings. The simplest condition of this type is the coupling symmetry $a_{ij}(t) = a_{ji}(t)$ [33], which condition can be in fact relaxed to weight-balance, type-symmetry or cut-balance conditions [20]. All of these conditions guarantee *reciprocity* of interactions: if some group of agents $S \subset \{1, \dots, n\}$ influences the remaining agents from $S^c = \{1, \dots, n\}\setminus S$, then agents from S^c also influence agents

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from S, moreover, the mutual influences of groups S and S^c are commensurate. The most general criteria for consensus over reciprocal graphs were obtained in [24], [34]. Criteria of the third type impose the condition of *arc-balance* [22], [35], requiring that the weights of persistent arcs are commensurate.

An important question regarding consensus algorithms is robustness against *communication delays*. Such delays naturally arise in the situation where the agents have direct access to their own values, whereas the neighbors' values are subject to non-negligible time lags. Delays of this type are inevitable in networks spread over large distances (e.g., where the agents communicate via Internet), but also arise in many physical models [36], [37]. The UQSC property in fact ensures consensus robustness against arbitrary bounded delays [30], [36], [38], [39]. However, delay robustness *without* uniform connectivity of the graph has remained an open problem; the existing results are mostly limited to the discrete-time case and impose restrictive conditions on the matrix $A(t)$, e.g., the uniform positivity of its non-zero entries [40], [41]. There is a judgement that, dealing with delayed consensus algorithms, the "uniform quasi-strong connectivity is in fact the weakest assumption on the graph connectivity such that consensus is guaranteed for arbitrary initial conditions" [39]. As will be shown, this judgement is not actually correct, and the UQSC can be reduced to a much weaker condition, which we call *aperiodic* quasi-strong connectivity (AQSC); in the case of reciprocal interactions, the AQSC can be further relaxed.

We focus on delay robustness of linear continuous-time consensus algorithms (2) As shown in the extended version of this paper [42], the theory for discrete-time algorithms is similar. The main results of this paper are as follows.

First, we obtain a novel consensus criterion in the case of a general time-varying directed graph, extending the UQSC condition to a much weaker condition termed AQSC (Theorem 1). This result, in fact, is of interest even for the undelayed case and, as it will be shown, generalizes many consensus criteria available in the literature [22], [35], [39]. **Second,** we extend the reciprocity-based consensus criteria established in [20], [21], [24] to the case of communication delays. **Third**, we prove consensus robustness against some classes of unknown disturbances. Along with global consensus, we consider criteria for partial consensus between some agents.

The paper is organized as follows. Section II introduces preliminary concepts and notation. Section III provides the problem setup (delay-robust consensus in averaging algorithms). In Section IV, we discuss known necessary condition of consensus (persistent connectivity). Section V offers the first sufficient condition for consensus, applicable to a general directed graph and generalizing the commonly used UQSC condition. Another sufficient condition of consensus, applicable to non-instantaneously type-symmetric graphs, is introduced in Section VI. Section VII collects the technical proofs of the main results. Section VIII gives a numerical example, illustrating the main results.

II. Preliminaries

Throughout the text, symbol $\triangleq$ should be read as "defined as". For integers $m \leq n$, let $[m:n] \triangleq \{m, m+1, \dots, n\}$.

Given a finite set of indices $\mathcal{V}$, we use $\mathbb{R}^{\mathcal{V}}$ to denote the set of vectors $x = (x_i)_{i\in\mathcal{V}}$, where $x_i \in \mathbb{R}$. For such a vector, $\min x \triangleq \min_{i\in\mathcal{V}} x_i$ and $\max x \triangleq \max_{i\in\mathcal{V}} x_i$. As usual, $\|x\|_\infty \triangleq \max_{i\in\mathcal{V}} |x_i|$. For two vectors $x, y \in \mathbb{R}^{\mathcal{V}}$, we write $x \leq y$ if $x_i \leq y_i \,\forall i$. Similarly, we use $\mathbb{R}^{\mathcal{V}\times\mathcal{V}}$ to denote the set of matrices $A = (a_{ij})_{i,j\in\mathcal{V}}$, where $a_{ij} \in \mathbb{R}\,\forall i,j \in \mathcal{V}$.

The vectors of standard coordinate basis in $\mathbb{R}^{\mathcal{V}}$ are denoted by $\mathbf{e}^i \triangleq (\delta^i_j)_{j\in\mathcal{V}}$, where $\delta^i_i \triangleq 1\,\forall i$ and $\delta^i_j \triangleq 0\,\forall j \neq i$. Let $\mathbf{1}_{\mathcal{V}} \triangleq \sum_{i\in\mathcal{V}} \mathbf{e}^i$ denote the vector of ones and $I_{\mathcal{V}} \triangleq (\delta^i_j)_{i,j\in\mathcal{V}}$ be the identity matrix; the subscript $\mathcal{V}$ will be omitted when this does not lead to confusion.

A (directed) graph is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a finite set of *nodes* and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is the set of *arcs*. A *walk* from node $i \in V$ to node $j \in V$ is a sequence of arcs (v_0, v_1), (v_1, v_2),…,(v_{n-1}, v_n) starting at $i_0 = i$ and ending at $i_n = j$. A graph is *strongly connected* if every two nodes are connected by a walk and *quasi-strongly connected*[1] if some node (a *root*) is connected to all other nodes by walks.

A *weighted* graph, determined by a weight matrix $A \in \mathbb{R}^{\mathcal{V}\times\mathcal{V}}$, is a triple $\mathcal{G}[A] \triangleq (\mathcal{V}, \mathcal{E}, A)$, where $A \in \mathbb{R}^{\mathcal{V}\times\mathcal{V}}$ is a *nonnegative* matrix that is compatible with graph $(\mathcal{V}, \mathcal{E})$, that is[2], $\mathcal{E} \triangleq \{(j,i) \in \mathcal{V}\times\mathcal{V} : a_{ij} > 0\}$. Given $\varepsilon > 0$, denote

$$A^{[\varepsilon]} \triangleq (a^{[\varepsilon]}_{ij}), \quad a^{[\varepsilon]}_{ij} = \begin{cases} a_{ij}, & a_{ij} \geq \varepsilon, \\ 0, & a_{ij} < \varepsilon. \end{cases}$$

Graph $\mathcal{G}[A^{[\varepsilon]}]$ is thus obtained from $\mathcal{G}[A]$ by removing "lightweight" arcs of weight $< \varepsilon$; we call graph $\mathcal{G}[A]$ *(quasi)-strongly ε-connected* if $\mathcal{G}[A^{[\varepsilon]}]$ is (quasi-)strongly connected.

III. Problem Setup

Consider the *delayed* counterpart of system (2) as follows

$$\dot{x}_i(t) = \sum\nolimits_{j\neq i} a_{ij}(t)(\hat{x}^i_j(t) - x_i(t)),\ i \in \mathcal{V}, \tag{4}$$

Unlike the algorithm (4), at time agent i receives a *retarded* value of agent $j \neq i$ denoted by $\hat{x}^i_j(t) \triangleq x_j(t - h_{ij}(t))$. Here $a_{ij}(t) \geq 0$ and locally L_1-summable and $h_{ij}(t) \geq 0$ are measurable on $[0, \infty)$, satisfying the following assumption.

Assumption 1: The delays are bounded: a constant $\bar{h} \geq 0$ exists such that $h_{ij}(t) \in [0, \bar{h}]\,\forall t \geq 0$. Furthermore,

$$\mu \triangleq \sup_{\substack{t\geq 0 \\ i\neq j}} \int_t^{t+\bar{h}} a_{ij}(s)\,ds < \infty. \tag{5}$$

Obviously (5) holds for undelayed systems (4), where $\bar{h} = 0$. Generally, the knowledge of $\bar{h} > 0$ is not needed to verify (5): if the supremum in (5) is finite for *some* value $\bar{h} > 0$, then it is automatically finite *for all* $\bar{h} > 0$.

As will be shown in Appendix, none of the two conditions from Assumption 1 can be completely discarded. The first condition in Assumption 1 (delay boundedness) is standard for consensus literature; without this assumption, consensus can be proved only for *special* types of delays and special

[1] A quasi-strongly connected graph is also called a graph with a *directed spanning tree* [1] and a *rooted* graph [43].

[2] Note that in multi-agent control [1] the influence of agent j on agent i corresponds to arc (j,i), not (i,j).

graphs [44], [45]. The condition (5) holds, in particular, when the coefficients $a_{ij}(t)$ are uniformly bounded (such assumptions are typical in the literature [22], [46]).

Under the condition (5), the solution of (4) is uniquely determined [47] by the initial condition

$$x(t_*) = x_* \in \mathbb{R}^n, \quad x(t_* + s) = \varphi(s) \ \ \forall s \in [-\bar{h}, 0), \qquad (6)$$

with $t_* \geq 0$, $\varphi \in L_\infty([-\bar{h}, 0] \to \mathbb{R}^{\mathcal{V}})$. Without loss of generality, we assume that $x(\cdot)$ is right-continuous at $t = t_*$.

Problem setup: global and partial consensus

We start with the definition of consensus.

Definition 1: Algorithm (4) establishes consensus among agents i and j if for any initial condition (t_*, x_*, φ) in (6)

$$\bar{x}_i \triangleq \lim_{t\to\infty} x_i(t) = \bar{x}_j \triangleq \lim_{t\to\infty} x_j(t) \qquad (7)$$

(in particular, both limits exist[3]). The algorithm establishes consensus in the group of agents $\mathcal{V}' \subseteq \mathcal{V}$ if it establishes consensus between each two agents from $\mathcal{V}'$. If (7) holds for all $i, j \in \mathcal{V}$, we say that the *global* consensus is established.

The global consensus can also be defined as the relation

$$\Lambda(t) - \lambda(t) \xrightarrow[t\to\infty]{} 0, \qquad (8)$$

where $\lambda(t), \Lambda(t)$ are the maximal and minimal values of the agents over the time window $[t - \bar{h}, t]$:

$$\lambda(t) \triangleq \inf_{t-\bar{h}\leq s\leq t} \min x(s), \quad \Lambda(t) \triangleq \sup_{t-\bar{h}\leq s\leq t} \max x(s). \quad (9)$$

It can easily shown (see e.g. [36], [39] and Lemma 8 below) that λ and Λ are, respectively, non-decreasing and non-increasing. Hence, all solutions of (4) are bounded.

Problem 1: Find the conditions on $A(\cdot)$ ensuring that the algorithm (4) establishes global consensus (8) or, more generally, consensus between some pairs of agents i and j.

We conclude this section with two important remarks.

Remark 1: Notice that we formally confine ourselves to linear consensus algorithms with scalar values $x_i(t) \in \mathbb{R}$; the extension to vectors $x_i(t) \in \mathbb{R}^d$ is straightforward.

Remark 2: The sufficient consensus conditions derived below are also applicable to *state-dependent* coefficients a_{ij}, provided that the solution of such a nonlinear system is well-defined. The criteria from Theorems 1 and 2 presented below can be generalized, e.g., to algorithms examined in [39]

$$\dot{x}_i(t) = \sum\nolimits_{j\neq i} a_{ij}(t)\varphi_{ij}(\hat{x}^i_j(t), x_i(t)) \in \mathbb{R}^d, \qquad (10)$$

where continuous nonlinear couplings φ_{ij} are such that $\varphi_{ij}(y,z) = \psi_{ij}(y,z)(y-z)$, where $\psi_{ij}(y,z) \in \mathbb{R}\, \forall y, z \in \mathbb{R}^d$ is uniformly positive and bounded on each compact in $\mathbb{R}^{2m}$. Indeed, a solution of (10) also satisfies (4) with new weights $\hat{a}_{ij}(t) \triangleq a_{ij}(t)\psi_{ij}(\hat{x}^i_j(t), x_i(t))$. The matrix function $\hat{A}(\cdot)$, as can be shown [42], inherits all properties of the original matrix $A(\cdot)$ that ensure consensus. Theorems 1 and 2 allow to extend the result of [39], relaxing the UQSC assumption on the graph and discarding the uniform positivity of nonlinearities ψ_{ij}.

[3] Observe that due to time-varying weights and delays, the existence of limits (7) in the time-varying system is a non-trivial self-standing problem.

IV. NECESSARY CONSENSUS CONDITIONS

Necessary conditions for consensus (global or between some pairs of agents) are closely related to the following criterion of *robustness* against L_1-summable disturbances, which has been obtained for undelayed algorithms in [46] . Along with system (4), consider the "disturbed" dynamics

$$\dot{x}_i(t) = \sum\nolimits_{j\in\mathcal{V}} a_{ij}(t)(\hat{x}^i_j(t) - x_i(t)) + f_i(t),\ i \in \mathcal{V}. \quad (11)$$

If the functions f_i are locally L_1-summable, then the solution is determined by specifying initial condition (6).

Lemma 1: If the undisturbed algorithm (4) establishes consensus between two agents i and j, the same holds for (11) with an L_1-summable disturbance: $\int_0^\infty |f_m(t)|\, dt < \infty\, \forall m$.

In view of Lemma 1, one can expect that the "non-essential" interactions between the agents, corresponding to L_1-summable functions $a_{km}(\hat{x}_m - x_k)$, should not have any effect on consensus, that is, consensus (global or partial) depends only on *persistently interacting* pairs of agents.

Definition 2: Agent j *persistently interacts* with agent i if the second relations in (3) holds. Denoting the set of such pairs (j, i) by $\mathcal{E}_\infty$, graph $\mathcal{G}_\infty = (\mathcal{V}, \mathcal{E}_\infty)$ is said to be the graph of persistent interactions, or the *persistent graph* of algorithm (4).

Consider now a counter-part of algorithm (4) that is "cleaned" from non-persistent interactions

$$\begin{gathered}\dot{x}_i(t) = \sum\nolimits_{j\neq i} \tilde{a}_{ij}(t)(\hat{x}^i_j(t) - x_i(t)),\ i \in \mathcal{V},\\ \tilde{a}_{ij}(t) = \begin{cases} a_{ij}(t), & (j,i) \in \mathcal{E}_\infty \\ 0, & (j,i) \notin \mathcal{E}_\infty. \end{cases}\end{gathered} \qquad (12)$$

Corollary 1: Consensus between two agents (7) is established by (4) if and only if this consensus is ensured by (12).

Corollary 1 often simplifies analysis of consensus algorithms, because, in many situations, system (12) decomposes into several independent subsystems. This corollary implies, in particular, the following necessary consensus condition.[4]

Lemma 2: If algorithm (4) establishes consensus between two agents k and m, then at least one of the following statements is valid: a) $(k, m) \in \mathcal{E}_\infty$; b) $(m, k) \in \mathcal{E}_\infty$; c) there exists some agent $r \neq k, m$ such that k, m can be reached from r in the graph $\mathcal{G}_\infty$. If the *global* consensus is established, then $\mathcal{G}_\infty$ is quasi-strongly connected.

A gap between necessary and sufficient conditions

It is well-known that the necessary condition from Lemma 5 is *not* sufficient for consensus: a simple counterexample with $n = 3$ agents in [23, Section IV-C] demonstrates[5] that even the *complete* persistent graph $\mathcal{G}_\infty$ does not guarantee global or partial consensus (furthermore, limits in (7) may fail to exist). This simple yet instructive example, however, may be called "pathological" for several reasons. **First,** there are arbitrarily long periods of time when only one arc (either $(1, 2)$ or

[4] In the undelayed case, Lemma 2 was proved in [22].

[5] The example from [23] is for the discrete-time system (1), which, as shown in [42], may be considered as a special case of (4) with "sawtooth" delays. Also, one can construct a continuous-time system (2) with the same properties.

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$(3,2)$) is active. **Second,** due to the first effect, some persistent couplings are much stronger than others, e.g.,

$$\limsup_{T\to\infty}\frac{\int_0^T a_{12}(t)\,dt}{\int_0^T a_{31}(t)\,dt}=\infty. \tag{13}$$

Third, the interactions are *non-reciprocal*: agent 2 is constantly influenced by one of agents 1 and 3 yet does not influence them, except for very rare time instants. All consensus criteria, existing in the literature, somehow remove one of these "anomalies" and can thus be divided into three groups.

The **first** "pathology" is excluded by the classical UQSC assumption (see Definition 5 below): the union of graphs over each interval $[t,t+T]$, where $t\geq 0$, has to be quasi-strongly ε-connected (here $\varepsilon,T>0$ are some constants). This assumption prohibits, e.g., too long periods of time when only two agents interact. The **second** effect (13) is excluded by the arc-balance condition [35], [46]. These two groups of criteria are generalized by Theorem 1 as reported in Table I.

Finally, there are consensus conditions that forbid the **third** pathological effect and assume the *reciprocity* of interactions: if some group of agents $\mathcal{V}_0\subseteq\mathcal{V}$ influences the remaining agents from $\mathcal{V}_0^c\triangleq\mathcal{V}\setminus\mathcal{V}_0$, then group $\mathcal{V}_0^c$ influences on $\mathcal{V}_0$. Examples of such conditions are type-symmetry and cut-balance [24], [35]. For reciprocal graphs, the necessary consensus conditions from Lemma 2 become also *sufficient*. A novel consensus criterion of this type is offered by Theorem 2.

V. Consensus over general directed graphs: the aperiodic quasi-strong connectivity

In this section, we establish a sufficient condition for delay-robust consensus, based on the *aperiodic* quasi-strong connectivity (AQSC). We start with definitions and notation.

Definition 3: The union of graphs $\mathcal{G}[A(t)]$ over interval $[t_1,t_2]\subset[0,\infty)$ is the graph $\mathcal{G}[A_{t_1}^{t_2}]$ associated with the matrix

$$A_{t_1}^{t_2}\triangleq\int_{t_1}^{t_2}A(s)\,ds. \tag{14}$$

For an increasing sequence $\mathfrak{t}=(t_p)_{p=0}^{\infty}\subseteq[0,\infty)$, denote

$$\ell=\ell(A(\cdot),\mathfrak{t})\triangleq\sup_{\substack{p=0,1,\dots\\ i,j}}\int_{t_p}^{t_{p+1}}a_{ij}(s)\,ds. \tag{15}$$

Definition 4: The matrix function $A(\cdot)$ is *aperiodically* quasi-strongly connected (AQSC) if there exist $\varepsilon>0$ and an increasing sequence $t_p\to\infty$ satisfying the two conditions:

(i) the unions of graphs $\mathcal{G}[A_{t_p}^{t_{p+1}}]$ are quasi-strongly ε-connected for all $p=0,1,\dots$

(ii) the supremum in (15) is finite: $\ell(A(\cdot),\mathfrak{t})<\infty$.

The term "aperiodically" in Definition 4 emphasizes the difference with classical *uniform* (periodic) quasi-strong connectivity (see Definition 5 below).

Furthermore, it will be convenient to suppose that

$$t_{p+1}-t_p\geq\bar{h}, \tag{16}$$

which condition, as shown by the next lemma, can always be provided by passing to a subsequence (t_{pk}) with some $k\geq 1$.

Lemma 3: Assume that the matrix function $A(\cdot)$ is AQSC with some sequence (t_p) and obeys Assumption 1. Then, there exists an integer $k\geq 1$ such that $t_{p+k}-t_p\geq\bar{h}\,\forall p$.

A. Robust consensus under the AQSC condition

We are now ready to formulate our first consensus criterion.

Theorem 1: Suppose that algorithm (4) satisfies Assumption 1, the matrix-valued function $A(\cdot)$ is AQSC. Then, the algorithm establishes global consensus. Furthermore, if the corresponding sequence (t_p) satisfies (16) (which does not reduce generality), then a number $\theta\in(0,1)$ exists such that

$$\begin{gathered}\|x(t)-\bar{x}\|_\infty\leq\max x(t)-\min x(t)\leq\theta^k(\Lambda(t_*)-\lambda(t_*)),\\ \forall k\geq 0\quad\forall t:\,t_{r+2k(n-1)}\leq t\leq t_{r+2(k+1)(n-1)}.\end{gathered} \tag{17}$$

Here r is an index such that $t_r\geq t_*$ and $n=|\mathcal{V}|$.

Remark 3: Analysis of the proof shows that θ depends, in fact, on parameters ε (Definition 4), ℓ from (15), and $n=|\mathcal{V}|$, being independent of $A(\cdot)$, $h_{ij}(\cdot)$ and (t_p) (provided that the assumptions of Theorem 1 hold). Its explicit computation[6] (i.e., the estimation of the convergence speed of a consensus algorithm) is a very hard problem, which has been solved only in the undelayed case [22], [32], [34], [46], [48]. In some situations [49], the convergence rate of the delayed algorithm (4) can be estimated via the convergence rate of its undelayed counterpart (2). The result of [49], however, imposes many restrictive assumptions on the weights $a_{ij}(t)$.

Under the assumptions of Theorem 1, the robustness property from Lemma 1 admits the following generalization.

Lemma 4: Suppose that the assumptions of Theorem 1 hold, including (16). Then, for any solution to (11), one has

$$\lim_{t\to\infty}(\Lambda(t)-\lambda(t))\leq C\limsup_{p\to\infty}\int_{t_p}^{t_{p+1}}\|f(t)\|_\infty dt, \tag{18}$$

where $C=C(\theta)$ is a constant determined by θ from (17). In particular, algorithm (11) establishes global consensus when

$$\int_{t_p}^{t_{p+1}}|f_i(t)|\,dt\xrightarrow[p\to\infty]{}0\quad\forall i\in\mathcal{V}. \tag{19}$$

Although Theorem 1 primarily deals with *global* consensus, it can also be used to prove partial consensus in some situations, as illustrated by the following corollary.

Corollary 2: Assume that the set of agents $\mathcal{V}_0\subseteq\mathcal{V}$ is "closed" in the persistent graph, that is, *no arc* $(j,i)\in\mathcal{E}_\infty$ exists connecting agent $j\in\mathcal{V}\setminus\mathcal{V}_0$ to agent $i\in\mathcal{V}_0$. If the submatrix $A_0=(a_{ij})_{i,j\in\mathcal{V}_0}$ satisfies the AQSC condition, then algorithm (4) establishes consensus in the group $\mathcal{V}_0$.

B. AQSC vs. the necessary consensus condition

A natural question arises on how close are the sufficient condition from Theorem 1 and the necessary condition of global consensus from Lemma 2. The following lemma shows that the condition (i) in Definition 4 is in fact *equivalent* to the necessary condition. Hence, the gap between necessity and sufficiency is caused by the additional condition $\ell<\infty$.

[6]Note that θ could be found explicitly if one had a closed-form representation of function ρ, which is currently elusive.

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Lemma 5: The following statements are equivalent:

a) the persistent graph $\mathcal{G}_\infty$ is quasi-strongly connected;
b) there exist a sequence $t_p \to \infty$ and a constant $\varepsilon > 0$ satisfying the condition (i) from Definition 4;
c) for *every* $\varepsilon > 0$ there exists a sequence $t_p \to \infty$ that satisfies the condition (i) from Definition 4.

C. Discussion: alternative consensus conditions

The relation between our results and the previously published criteria for continuous-time consensus algorithms over time-varying directed graphs are summarized in Table I. Theorem 1 generalizes two groups of criteria that are based on the UQSC condition and on the arc-balance condition. We give the formal definitions for the reader's convenience.

In this subsection, we discuss the relations between Theorem 1 and some previously published consensus criteria.

1) AQSC vs. UQSC: As has been already mentioned, delay consensus algorithms have been examined only under the property of *uniform* quasi-strong connectivity (UQSC).

Definition 5: We call the graph UQSC if the conditions (i), (ii) from Definition 4 hold[7] for the *periodic* sequence $t_p = pT$.

Remark 4: The condition (17) ensures *exponentially fast* convergence to the ultimate vector $\bar{x}$; as shown in [46], in fact, the UQSC condition is *necessary* for this type of consensus.

Remark 5: In the special case where the stronger *uniform* quasi-strong connectivity holds ($t_p = pT$), condition (19) holds e.g. when the disturbance is *vanishing* $f(t) \xrightarrow[t\to\infty]{} 0$. The latter result in the undelayed case ($\bar{h} = 0$) was first established in [46]; it was also shown that the UQSC condition is *necessary* for this type of robustness.

2) Intermittent communication: The difference between AQSC and UQSC properties is prominently illustrated by networks with *intermittent* communication [50]:

$$a_{ij}(t) = \alpha(t)\bar{a}_{ij}, \quad \forall i \neq j \quad \forall t \geq 0. \tag{20}$$

where $\bar{a}_{ij} \geq 0$ are *constant* and $\alpha(t) \geq 0$.

Lemma 6: For the matrix $A(\cdot)$ from (20), the statements are equivalent:

(a) the AQSC condition holds;
(b) the persistent graph $\mathcal{G}_\infty$ is quasi-strongly connected;
(c) $\int_0^\infty \alpha(t)dt = \infty$ and $\mathcal{G}[\bar{A}]$ is quasi-strongly connected.

Notice that the condition from Lemma 6 allows arbitrarily long periods of "silence" when the network is unavailable ($\alpha(t) = 0$), which is incompatible with the UQSC.

3) The arc-balance condition: The network from Lemma 6 is a particular case of an *arc-balanced*[8] network [22].

Definition 6: The weighted graph $\mathcal{G}[A(\cdot)]$ is said to be *arc-balanced* if such a constant $K \geq 1$ exists that for all $t \geq 0$

$$K^{-1}a_{km}(t) \leq a_{ij}(t) \leq K a_{km}(t) \; \forall (m,k),(j,i) \in \mathcal{E}_\infty. \tag{21}$$

Lemma 7: Under the arc-balance condition (21), the AQSC holds if and only if $\mathcal{G}_\infty$ is quasi-strongly connected.

[7] Note that condition (ii) in this situation follows from Assumption 1 except for the undelayed case ($\bar{h} = 0$). Also, it holds when $a_{ij}(t)$ are bounded, which condition is typical in most works on consensus [30], [39], [46].

[8] For simplicity, we consider only the "anytime" arc-balance from [22]; Lemma 7 below remains valid for the "non-instantaneous" arc-balance [35].

VI. Consensus over type-symmetric graphs

Lemma 7 implies that Theorem 1 gives a *necessary and sufficient* consensus condition for arc-balanced graphs. Another situation where the gap between necessary and sufficient conditions is absence is the situation where interactions between the agents are *reciprocal*; the most general condition of reciprocity available in the literature is the non-instantaneous cut-balance [24], [34], [35], however, the relevant conditions cannot be easily verified. We confine ourselves to a stronger requirement of the *non-instantaneously type-symmetry (NITS)*, which, unlike the cut-balance, can be efficiently tested [24].

Definition 7: Matrix function $A(\cdot) : [0,\infty) \to \mathbb{R}^{\mathcal{V}\times\mathcal{V}}$ with entries $a_{ij} \geq 0$ is *type-symmetric* if $K \geq 1$ exists such that

$$K^{-1}a_{ji}(t) \leq a_{ij}(t) \leq K a_{ji}(t) \quad \forall i,j \in \mathcal{V} \, \forall t \geq 0. \tag{22}$$

Obviously, symmetric matrix ($A(t) = A(t)^\top \, \forall t \geq 0$) is type-symmetric (with $K = 1$). A generalization of (22) is the *non-instantaneous* type-symmetry introduced in [24].

Definition 8: The matrix function $A(\cdot) : [0,\infty) \to \mathbb{R}^{\mathcal{V}\times\mathcal{V}}$ with nonnegative entries $a_{ij} \geq 0$ possess the *non-instantaneous type-symmetry* (NITS) property if there exist an *increasing* sequence $t_p \xrightarrow[p\to\infty]{} \infty$ (where $p = 0,1,\ldots$) and a constant $K \geq 1$ such that the following two conditions hold:

1) for any $i,j = 1,\ldots,n$ one has
$$\int_{t_p}^{t_{p+1}} a_{ij}(t)\,dt \leq K \int_{t_p}^{t_{p+1}} a_{ji}(t)\,dt; \tag{23}$$
2) the supremum in (15) is finite: $\ell < \infty$.

Verification of the NITS condition may seem a non-trivial problem, however this condition can be efficiently tested if the weights a_{ij} are uniformly bounded (which also entails Assumption 1), see the proof of [24, Theorem 2] and the example in [24]. The NITS condition also holds in the case of type-symmetric graph, as shown by the following remark.

Remark 6: The NITS condition follows from (22) and Assumption 1 (it suffices to choose $t_p = p$).

Remark 7: The condition (23) implies, obviously, that the persistent graph $\mathcal{G}_\infty$ is undirected, that is, $(i,j) \in \mathcal{E}_\infty$ if and only if $(j,i) \in \mathcal{E}_\infty$. In particular, $\mathcal{G}_\infty$ is quasi-strongly connected if and only if it is strongly connected; otherwise, the graph $\mathcal{G}_\infty$ consists of several connected components.

We now formulate the main result of this section ensuring the delay robustness for NITS networks.

Theorem 2: Assume that the matrix-valued function $A(\cdot)$ with nonnegative entries $a_{ij}(t) \geq 0$ possess the NITS property and satisfies Assumption 1. Then, every solution to (4) converges: the limits (7) exist for all $i \in \mathcal{V}$. The algorithm establishes consensus between agents i and j if and only if i and j are connected by a walk in $\mathcal{G}_\infty$. The algorithm global consensus is established *if and only if* $\mathcal{G}_\infty$ is connected.

Remark 8: As illustrated by [24, Proposition 3], the condition $\ell < \infty$ cannot be discarded even in the undelayed case.

The result of Theorem 2 extends the results of [20], [21], [24], [33] on consensus over undirected and type-symmetric graphs to the delayed case. Notice however that, unlike those results, it does not guarantee that $\dot{x}_i \in L_1[0,\infty) \, \forall i \in \mathcal{V}$; the validity of the latter statement remains in fact an open problem.

Ref.	Graph properties	Extra assumptions	Remarks
[30]	A stronger form of the UQSC condition: graphs $\mathcal{G}[A_t^{t+T}]$ are quasi-strongly ε-connected and share a common root node;	$A(\cdot)$ is bounded and piecewise-continuous, equal and constant h_{ij}	Exponential convergence is claimed (but formally, the proof is available only for the undelayed case).
[36]	The UQSC condition	$A(t) = A_{\sigma(t)}$, where the switching signal $\sigma(t)$ is piecewise-constant and attains values in a finite set	Formally, the algorithms are nonlinear, but satisfy the conditions of our Remark 2. Asymptotic convergence is guaranteed without convergence rate estimates.
[39]	The UQSC condition	Same as [36], but $\sigma(t)$ also enjoys the positive dwell-time property	Same remarks as for [36]
[46]	The UQSC condition	a_{ij} continuous almost everywhere and uniformly bounded, no delay $\bar{h} = 0$	Exponential convergence was proved; the main results of [46] are *robustness* criteria similar to Lemma 4.
[50]	The intermittent communication (20), graph $\mathcal{G}[\bar{A}]$ is strongly connected	$\alpha(t)$ switches between 0 and a constant $\bar{\alpha} > 0$ with some restrictions on the switching policy, no delay ($\bar{h} = 0$)	In some situations, convergence rate can be estimated. The results of [50] are applicable to agents with high-order nonlinear dynamics.
[22]	The arc-balance condition (21)	$A(\cdot)$ is continuous almost everywhere, no delay ($\bar{h} = 0$)	Convergence rate is estimated explicitly.

TABLE I: Special cases of Theorem 1, available in the literature.

Reciprocity vs. repeated connectivity

Theorem 2 substantially differs from Theorem 1 and other results on delay-robust consensus over repeatedly quasi-strongly connected graphs. As can be seen from its proof (Section VII, see also [51]), it retains its validity for an arbitrary set of bounded functions $x_1(t), \dots, x_n(t)$ that obey the following system of *differential averaging inequalities*

$$\dot{x}_i(t) \leq \sum\nolimits_{j\in\mathcal{V}} a_{ij}(t)(\hat{x}_j^i(t) - x_i(t)),\ \forall i \in \mathcal{V},\ t \geq 0. \quad (24)$$

As shown in our previous works [41], [52], the averaging inequalities, have numerous applications in multi-agent control and social dynamics modeling. Whereas the consensus dynamics has some well-known contraction properties (in particular, the function $\Lambda(t) - \lambda(t)$ is non-increasing and can serve as a Lyapunov function), inequality (24) does not possess such properties: whereas $\Lambda(t)$ is non-increasing [51], $\lambda(t)$ need not be monotone. Theorem 1 is not valid for the inequalities, and estimates like (17) cannot be established for them. Theorem 2 thus requires some tools that are principally different from usual contraction analysis [22], [39], [46]; its proof is actually based on the seminal idea of a sorting permutation [20], [21].

VII. Technical Proofs

We start with several technical lemmas that will be used in the proof of main results and are concerned with the *evolutionary matrices* [47], [53] of system (4).

A. Evolutionary matrices and their properties

The *evolutionary matrix* $U(t, t_*) \in \mathbb{R}^{\mathcal{V}\times\mathcal{V}}$, where $t \geq t_* \geq 0$, is the matrix whose j-th column (for all $j \in \mathcal{V}$) is the solution of (4) determined by the initial condition $x(t_*) = \mathbf{e}_j$ and $x(t) \equiv 0\,\forall t \in [t_* - \bar{h}, t_*)$. Obviously, the solution with $x(t_*) = x_*$ and $x(t) \equiv 0\,\forall t < t_*$ is then given by $U(t, t_*)x_*$. The "variation of constants" formula retains its validity, which allows to compute the solution of the general system (11), (6):

$$\begin{aligned} x(t) &= U(t, t_*)x_* + \int_{t_*}^{t} U(t,\xi)[f(\xi) + g(\xi)]\,d\xi, \\ g_i(t) &\triangleq \sum\nolimits_{j\neq i} a_{ij}(t)\varphi_j(t - h_{ij}(t) - t_*) \quad \forall i \in \mathcal{V}. \end{aligned} \quad (25)$$

Here, to simplify notation, the function φ from (6) is extended from $[-\bar{h}, 0)$ to $[-\bar{h}, \infty)$, denoting $\varphi(\tau) \triangleq 0\,\forall\tau \geq 0$.

Lemma 8: For any solution of (4), (6), the inequalities hold

$$\begin{aligned} x_i(t) &\leq x_i(s)e^{-\int_s^t \alpha_i(\xi)d\xi} + \Lambda(s)\left[1 - e^{-\int_s^t \alpha_i(\xi)d\xi}\right], \\ x_i(t) &\geq x_i(s)e^{-\int_s^t \alpha_i(\xi)d\xi} + \lambda(s)\left[1 - e^{-\int_s^t \alpha_i(\xi)d\xi}\right], \quad (26) \\ \alpha_i(t) &\triangleq \sum\nolimits_{j\neq i} a_{ij}(t) \quad \forall i \in \mathcal{V}\,\forall t \geq s \geq t_*. \end{aligned}$$

and λ, Λ are, respectively, *non-decreasing* and *non-increasing*.

Proof: We prove only the statements involving $\Lambda(t)$ (as can be noticed, the proof works also for the inequalities (24)). The statement related to $\lambda(t)$ are trivially derived from them by considering the solution $(-x(t))$, which also obeys (4).

To prove that $\Lambda(t)$ is non-increasing, choose an instant $s \geq t_*$ and a constant $\Lambda' > \Lambda(t_*)$. We are going to show that $\max x(t) < \Lambda'$ for any $t \geq s$. Obviously, the latter inequality holds when t is close to s; let t' be the *first* instant $t > s$ when the inequality is violated, that is,

$$\max x(t) < \Lambda' \ \ \forall t \in [s - \bar{h}, t'), \quad x_i(t') = \Lambda' \quad \text{for some } i.$$

One arrives at a contradiction with (4), because

$$\begin{gathered} \dot{x}_i(t) \leq \alpha_i(t)[\Lambda' - x_i(t)] \ \ \forall t \in [s, t') \Longrightarrow \\ x_i(t') \leq e^{-\int_s^{t'} \alpha_i(\xi)d\xi}x_i(s) + \Lambda'\left(1 - e^{-\int_s^{t'} \alpha_i(\xi)d\xi}\right) < \Lambda' \end{gathered}$$

(α_i is defined in (26)). The contradiction proves that $\Lambda(t) < \Lambda'\ \ \forall t \geq s$. Since $\Lambda' > \Lambda(s)$ can be arbitrary, $\Lambda(t) \leq \Lambda(s)$ whenever $t \geq s$, which proves the monotonicity.

The first inequality in (26) is proved similarly. Indeed, when $t \geq s$, we have $\hat{x}_j^i(t) \leq \Lambda(t) \leq \Lambda(s)\,\forall j \neq i$. and hence

$$\dot{x}_i(t) \leq \alpha_i(t)[\Lambda(s) - x_i(t)] \quad \forall t \geq s \geq t_*,$$

which finishes the proof. ■

In the subsequent, the following corollary will be used.

Corollary 3: Let Assumption 1 hold and $t_p \to \infty$ be a sequence such that the supremum ℓ in (15) is finite. Let $s \in (t_{q-1} - \bar{h}, t_q - \bar{h}]$, where $q \geq 1$ is some index. Then

$$\begin{aligned} x_i(t) &\geq \theta_0 x_i(s) + (1 - \theta_0)\lambda(t)\,\forall t \in [s, t_q]\,\forall i \in \mathcal{V}, \\ x_i(t) &\leq \theta_0 x_i(s) + (1 - \theta_0)\Lambda(t)\,\forall t \in [s, t_q]\,\forall i \in \mathcal{V}, \end{aligned} \quad (27)$$

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where $\theta_0 \overset{\Delta}{=} e^{-(\mu+\ell)(n-1)}$. Similarly, if $s \in (t_{q-1}, t_q]$, then

$$\begin{aligned} x_i(t) \geq \theta_1 x_i(s) + (1-\theta_1)\lambda(t)\,\forall t \in [s,t_q]\,\forall i,\\ x_i(t) \leq \theta_1 x_i(s) + (1-\theta_1)\Lambda(t)\,\forall t \in [s,t_q]\,\forall i. \end{aligned} \tag{28}$$

where $\theta_1 \overset{\Delta}{=} e^{-\ell(n-1)}$.

Proof: To prove the first inequality (28), one may easily notice that $\int_s^{t_q} a_{ij}(\xi)d\xi \leq \ell$ for each index $j \neq i$, and thus $\int_s^t \alpha_i(\xi)d\xi \leq (n-1)(\mu+\ell)$ for all $t \in [s,t_q]$. The statement is now obvious from (26). The proof of the first inequality (27) is similar and uses the inequality

$$\int_s^{t_q} a_{ij}(\xi)d\xi = \int_s^{s+\bar{h}} a_{ij}(s)ds + \int_{s+\bar{h}}^{t_q} a_{ij}(s)ds \leq \mu+\ell,$$

which holds for all $j \neq i$ due to (5) and (15). The symmetric inequalities are proved by replacing $x(t)$ by $(-x(t))$. ∎

Lemma 8 implies the following important property of the evolutionary matrices.

Lemma 9: Matrix $U(t,s)$ is *substochastic*[9] for all $t \geq s \geq 0$. For any solution of (11) defined for $t \geq t_*$, one has

$$\begin{aligned} &\lambda(t_*)[\mathbf{1} - U(t,t_*)\mathbf{1}] \leq \\ &x(t) - U(t,t_*)x(t_*) - \int_{t_*}^t U(t,\xi)f(\xi)d\xi \leq \\ &\leq \Lambda(t_*)[\mathbf{1} - U(t,t_*)\mathbf{1}]. \end{aligned} \tag{29}$$

If Assumption 1 holds, then a constant $\psi > 0$ exists such that $U(t,s)\mathbf{1} \geq \psi\mathbf{1}$ whenever $t \geq s \geq 0$.

Proof: By construction of $U(t,t_*)$, the solution $x(t) = U(t,t_*)\mathbf{e}_i$ corresponds to $\lambda(t_*) = 0$ and hence $x_i(t) \geq 0\,\forall i \in \mathcal{V}\,\forall t \geq t_*$. Therefore, $U(t,t_*)$ is a nonnegative matrix. Similarly, the solution $x(t) = U(t,t_*)\mathbf{1}$ corresponds to $\Lambda(t_*) = 1$, and hence $U(t,t_*)\mathbf{1} \leq \mathbf{1}\,\forall t \geq t_*$, so $U(t,t_*)$ is substochastic whenever $t \geq t_* \geq 0$.

Consider now the solution $x(t) \equiv 1\,\forall t \geq t_*$, which corresponds to the initial condition (6) with $x_* = 1$ and

$$\varphi(s) = \mathbb{1}_{[-\infty,0)}(s) \overset{\Delta}{=} \begin{cases} 1, & s < 0 \\ 0, & s \geq 0. \end{cases}$$

$\varphi(s) = 1\,\forall s \in [-\bar{h},0)$. Using (25) with $f = 0$, one has

$$\mathbf{1} = U(t,t_*)\mathbf{1} + \int_{t_*}^t U(t,\xi)g^0(\xi)\,d\xi,$$

$$g_i^0(t) \overset{\Delta}{=} \sum\nolimits_{j\neq i} a_{ij}(t)\mathbb{1}_{[-\infty,0)}(t - h_{ij}(t) - t_*) \quad \forall i \in \mathcal{V}.$$

For a general solution to (11), we have $x_i(t) \leq \Lambda(t_*)$ for all $t \in [t_* - \bar{h}, t_*]$, and thus the functions $g_i(t)$ introduced in (25) obey the inequalities $g_i(t) \leq \Lambda(t_*)g_i^0(t)$. For this reason,

$$\begin{aligned} &x(t) - U(t,t_*)x(t_*) - \int_{t_*}^t U(t,\xi)f(\xi)d\xi \leq \\ &\leq \Lambda(t_*)\int_{t_*}^t U(t,\xi)g^0(\xi)\,d\xi = \Lambda(t_*)\left(\mathbf{1} - U(t,t_*)\mathbf{1}\right), \end{aligned}$$

which proves the second inequality in (29). The first one is proved similarly, replacing $\leq$ by $\geq$ and $\Lambda(t_*)$ by $\lambda(t_*)$.

To prove the final statement, the second inequality from (26) can be applied to the solution $x(t) = U(t,s)\mathbf{1}$. By construction, for this solution one has $\lambda(s) = 0$ and $x(s) = \mathbf{1}$. Hence for any $t \in [s, s+\bar{h}]$ one has $x_i(t) \overset{(5)}{\geq} \psi \overset{\Delta}{=} e^{-(n-1)\mu}\,\forall i \in \mathcal{V}$. In particular, $\lambda(s+\bar{h}) \geq \psi$, and thus (Lemma 8) $x(t) = U(t,s)\mathbf{1} \geq \psi\mathbf{1}$ also for $t \geq s+\bar{h}$. This finishes the proof. ∎

Remark 9: Non-negativity of matrices $U(t,t_*)$ implies, in view of (25), in particular, the monotonicity of dynamics (4). If $x, \tilde{x}$ are two solutions such that $x(t) \leq \tilde{x}(t)$ for $t \in [t_* - \bar{h}, t_*]$, then $x(t) \leq \tilde{x}(t)$ for $t \geq t_*$.

Consensus, similar to the undelayed case [1], admits a simple interpretation in terms of the evolutionary matrix.

Lemma 10: The algorithm (4) establishes consensus among agents i and j if and only if the limits exist and coincide:

$$\lim_{t\to\infty} \mathbf{e}_i^\top U(t,t_*) = \lim_{t\to\infty} \mathbf{e}_j^\top U(t,t_*) \quad \forall t_* \geq 0. \tag{30}$$

Global consensus is established if and only if the limit exists

$$\bar{U}_{t_*} \overset{\Delta}{=} \lim_{t\to\infty} U(t,t_*) = \mathbf{1}p_{t_*}^\top, \quad p_{t_*} \in \mathbb{R}^{\mathcal{V}} \tag{31}$$

(that is, $\bar{U}_{t_*}$ has equal rows) for any $t_* \geq 0$.

Proof: It suffices to prove the first statement. The "only if" part in the first statement is straightforward, because $U(t,t_*)x_*$ is a solution to (4) for each vector $x_* \in \mathbb{R}^{\mathcal{V}}$. Consensus between agents i,j implies that the limits exist $\lim_{t\to\infty} \mathbf{e}_i^\top U(t,t_*)x_* = \lim_{t\to\infty} \mathbf{e}_j^\top U(t,t_*)x_*$ for all x_*, which is equivalent to (30). To prove the "if" part, recall that an arbitrary solution to (4) can be found from (25), where $f \equiv 0$ and $g(t) = 0$ for $t > t_* + \bar{h}$. Hence, we have

$$x(t) = U(t,t_*)x(t_*) + \int_{t_*}^{\bar{h}} U(t,\xi)g(\xi)dt \;\; \forall t > t_* + \bar{h}.$$

Using the Lebesgue dominated convergence theorem, it can be easily shown that (30) entails

$$\begin{aligned} \bar{x}_i &= \lim_{t\to\infty} \mathbf{e}_i^\top x(t) = \lim_{t\to\infty} \mathbf{e}_i^\top U(t,t_*)x(t_*) + \\ &+ \int_{t_*}^{\bar{h}} \lim_{\xi\to\infty} \mathbf{e}_i^\top U(t,\xi)g(\xi)\,d\xi = \\ &= \lim_{t\to\infty} \mathbf{e}_j^\top U(t,t_*)x(t_*) + \int_{t_*}^{\bar{h}} \lim_{\xi\to\infty} \mathbf{e}_j^\top U(t,\xi)g(\xi)\,d\xi \\ &= \lim_{t\to\infty} \mathbf{e}_j^\top x(t) = \bar{x}_j, \end{aligned}$$

(where all limits exist), proving consensus among i and j. ∎

1) **Evolutionary matrices under the AQSC condition:** In the proof of Theorem 1, we will use an additional statement.

Lemma 11: Let the AQSC condition hold with $\varepsilon > 0$ and a sequence (t_p), which obeys (16). Then, matrices $U(t_{p+2},t_p)$ have uniformly positive diagonal entries $U(t_{p+2},t_p)_{ii} \geq \tilde{\eta} \overset{\Delta}{=} e^{-2(n-1)\ell}$ and their graphs are quasi-strongly $\tilde{\varepsilon}$-connected with $\tilde{\varepsilon} \overset{\Delta}{=} \varepsilon e^{-3(n-1)\ell}$. Here $n = |\mathcal{V}|$ and ℓ is from (15).

Proof: For a fixed $j \in \mathcal{V}$, consider the solution $x(t) = U(t_{p+2},t_p)\mathbf{e}_j$ of (4), which corresponds to initial conditions (6) with $t_* = t_p$, $x_* = \mathbf{e}_j$, $\varphi(s) \equiv 0\,\forall s < 0$. Using

[9]In the case undelayed case ($\bar{h} = 0$), the evolutionary matrix is known to be stochastic [1], [29]. This is the principal difference between the delayed and undelayed consensus dynamics, which makes it impossible to reduce delayed equations (4) to ergodicity of Markov chains and stochastic matrix products.

the second inequality (26) (where $s = t_p$, $\lambda(s) = 0$) and (15), one proves that for each $t \in [t_p, t_{p+2}]$

$$x_j(t) = U(t, t_p)_{jj} \geq e^{-\int_{t_p}^t \alpha_i(\xi)d\xi} \overset{(15)}{\geq} \tilde{\eta}. \tag{32}$$

Applying this for $t = t_{p+2}$, one proves the first statement.

For $t \in [t_{p+1}, t_{p+2}]$, one has $t - h_{ij}(t) \geq t_{p+1} - \bar{h} \geq t_p$. Recalling that $x(t) \geq 0$, for each $j \neq i$ we thus have

$$\dot{x}_i(t) \geq -\alpha_i(t)x_i(t) + a_{ij}(t)\hat{x}_j^i(t) \overset{(32)}{\geq} -\alpha_i(t)x_i(t) + a_{ij}(t)\tilde{\eta},$$

$$x_i(t_{p+2}) \geq \int_{t_{p+1}}^{t_{p+2}} e^{-\int_{t_{p+1}}^s \alpha_i(\xi)d\xi} a_{ij}(s)\hat{x}_j^i(s)ds \geq$$

$$\geq e^{-(n-1)\ell}\tilde{\eta} \int_{t_{p+1}}^{t_{p+2}} a_{ij}(s)ds.$$

In particular, if $(A_{t_{p+1}}^{t_{p+2}})_{ij} \geq \varepsilon$, then one has

$$x_i(t_{p+2}) = U(t_{p+2}, t_p)_{ij} \geq e^{-(n-1)\ell}\tilde{\eta}\varepsilon = e^{-3(n-1)\ell}\varepsilon = \tilde{\varepsilon}.$$

Thanks to the AQSC condition, there exists a spanning tree in the graph $\mathcal{G}[U(t_{p+2}, t_p)]$ whose arcs have weights $\geq \tilde{\varepsilon}$. ■

2) **Evolutionary matrices under the NITS condition:** The proof of Theorem 2 is based on another property of the evolutionary matrices, specific for type-symmetric graphs.

Lemma 12: If the continuous-time matrix $A(\cdot)$ obeys the NITS property and Assumption 1, then matrices $U(t, t_*)$ have uniformly positive diagonal elements, that is,

$$U(t, t_*)_{ii} \geq \varrho > 0 \quad \forall t \geq t_* \geq 0\, \forall i \in \mathcal{V}. \tag{33}$$

The constant $\varrho = \varrho(n, \ell, \mu, K)$ depends on $n = |\mathcal{V}|$ and the constants K from (23), ℓ from (15) and μ from (5).

Proof: Throughout the proof, the sequence (t_p) is same as in Definition 8. Without loss of generality, assume that $t_0 = 0$.

Step 1. Notice that (33) is entailed by the following:

(A) A constant $\tilde{\varrho} = \tilde{\varrho}(n, \ell, \mu, K) \in (0, 1)$ exists featured by the following property. If $x(t)$ is a solution of (4) such that $\lambda(t_q) \geq 0$ and $x_i(t) \geq 1$ for $t \in [t_q - \bar{h}, t_q]$ for some agent $i \in \mathcal{V}$ and some index q, then one has the inequality holds

$$x_i(t) \geq \tilde{\varrho} \quad \forall t \geq t_q. \tag{34}$$

Indeed, let (A) be valid. Consider solution $x(t) = U(t, t_*)\mathbf{e}_i$, where $i \in \mathcal{V}$ and $t_* \geq 0$. Thanks to Corollary 3 (applied to $s = t_*$, $x_i(s) = 1$ and $\lambda(s) = 0$), an index q exists such that $x_i(t) \geq \theta_0$ for all $t \in [t_*, t_q]$; also, $\lambda(t_q) \geq 0$. Applying (A) to the solution $\tilde{x}(t) = \theta_0^{-1}x(t)$, one shows that

$$x_i(t) = U(t, t_*)_{ii} \geq \varrho \overset{\Delta}{=} \theta_0\tilde{\varrho} = e^{-(\mu+\ell)(n-1)}\tilde{\varrho} \quad \forall t \geq t_*.$$

Step 2. We will show that statement (A), in turn, is implied by another condition (B) presented below. Given a solution to system (4), $\mathcal{V}_0 \subseteq \mathcal{V}$ and $t \geq 0$, denote

$$\lambda_{\mathcal{V}_0}(t) \overset{\Delta}{=} \inf_{s \in [t-\bar{h}, t]} \min_{i \in \mathcal{V}_0} x_i(s).$$

(B) A constant $\gamma = \gamma(n, \ell, \mu, K) \in (0, 1/2)$ exists featured by the following property. If $x(t)$ is a solution of (4) such that $\lambda(t_q) \geq 0$ and $\lambda_{\mathcal{V}_0}(t_q) \geq 1\, \forall i \in \mathcal{V}_0 \subseteq \mathcal{V}$ for some $q \geq 0$, then one of the following statements (i) and (ii) hold

(i) $\lambda_{\mathcal{V}_0}(t) > 1/2\, \forall t \geq t_q$;

(ii) a set $\mathcal{V}_1 \supsetneq \mathcal{V}_0$ and index $r = r(q) > q$ exist such that $\lambda_{\mathcal{V}_0}(t) \geq \gamma\, \forall j \in \mathcal{V}_0$ when $t \in [t_q, t_r]$ and $\lambda_{\mathcal{V}_1}(t_r) \geq \gamma$.

Indeed, assume that (B) holds and consider a solution satisfying the assumptions of statement (A). Applying (B) to this solution and set $\mathcal{V}_0 = \{j : x_j(t_q) \geq 1\} \ni \{i\}$, one of statements *(i)* and *(ii)* should hold.

If *(i)* holds, then $x_i(t) \geq \beta_0 \overset{\Delta}{=} 1/2$ for all $t \geq t_q$.

Assume that *(ii)* holds and let $q_1 = r(q_1) > q$, $\mathcal{V}_1 \supsetneq \mathcal{V}_0$ be the corresponding integer index and set of agents. Applying (B) to the solution $\tilde{x}(t) = \gamma^{-1}x(t)$, $\tilde{q} = q_1$ and $\tilde{\mathcal{V}}_0 = \mathcal{V}_1$, one shows that either condition *(i)* holds, and then $\lambda_{\mathcal{V}_1}(t) \geq \beta_1 \overset{\Delta}{=} \beta_0\gamma$ for $t \geq t_{q_1}$ or the scenario from *(ii)* is realized, and there exist such a set $\mathcal{V}_2 \supsetneq \mathcal{V}_1$ and an index $q_2 = r(q_1) > q_1$ that $\lambda_{\mathcal{V}_1}(t) \geq \gamma^2$ when $t \in [t_{q_1}, t_{q_2}]$ and $\lambda_{\mathcal{V}_2}(t_{q_2}) \geq \gamma^2$.

In the latter situation, we repeat the procedure and apply (B) to the solution $\hat{x}(t) = \gamma^{-2}x(t)$, $\hat{q} = q_2$ and $\hat{V}_0 = V_2$, showing that either $\lambda_{\mathcal{V}_2}(t) \geq \beta_2 \overset{\Delta}{=} \beta_0\gamma^2$ for $t \geq t_{q_2}$ or there exist a set $\mathcal{V}_3 \supsetneq \mathcal{V}_2$ and index $q_3 > q_2$ such that $\lambda_{\mathcal{V}_2}(t) \geq \gamma^3$ when $t \in [t_{q_2}, t_{q_3}]$ and $\lambda_{\mathcal{V}_3}(t_{q_3}) \geq \gamma^3$, in which situation we can again apply statement (B), and so on.

Since $n = |\mathcal{V}|$ is finite, this procedure terminates after $m \leq n-1$ steps, after which scenario *(i)* is realized and the set $\mathcal{V}_m$ cannot be constructed. By construction, $i \in \mathcal{V}_0 \subset \ldots \subset \mathcal{V}_{m-1}$, and hence $x_i(t) \geq \beta_{m-1} = \beta_0(\gamma)^{m-1} \geq \beta_0(\gamma)^{n-2} = \gamma^{n-2}/2$ for all $t \geq t_q$. Hence, (B) implies (A) with $\tilde{\varrho} = \gamma^{n-2}/2$.

Step 3. We are now going to prove statement (B) via induction on $n = |\mathcal{V}|$. The induction base $n = 1$ is trivial, in this situation the only agent obeys the equation $\dot{x} = 0$, so if $x(t_p) \geq 1$, then condition *(i)* holds automatically.

Suppose that (B) (and thus also statement of Lemma 12) has been proved for groups of $\leq n-1$ agents and $|\mathcal{V}| = n$. Notice that for any subgroup $\tilde{\mathcal{V}} \subsetneq \mathcal{V}$, the corresponding matrix $\tilde{A} = (a_{ij})_{i,j \in \tilde{\mathcal{V}}}$ obeys the NITS conditions and (5) with the same constant K, sequence (t_p) and constants ℓ, μ as A. We thus know that Lemma 12 is valid for each reduced system

$$\dot{x}_i = \sum\nolimits_{j \in \tilde{\mathcal{V}} \setminus \{i\}} a_{ij}(t)(\hat{x}_j^i(t) - x_i(t)),\ i \in \tilde{\mathcal{V}}. \tag{35}$$

Introducing the corresponding evolutionary matrix $\tilde{U}(t, t_*)$, we have $\tilde{U}(t, t_*)_{ii} \geq \varrho(|\mathcal{V}'|, \ell, \mu, K)$. We define

$$\varrho' = \rho'(\ell, \mu, K) \overset{\Delta}{=} \min_{k \leq n-1} \varrho(k, \ell, \mu, K) > 0.$$

Step 3a. Notice that the validity of one of the statements *(i), (ii)* needs to be proved only for the *special* solution $x(t), t \geq t_q$ that is determined by the initial condition

$$\begin{aligned} x_i(t) &\equiv 1\, \forall t \in [t_q - \bar{h}, t_q]\, \forall i \in \mathcal{V}_0 \\ x_i(t) &\equiv 0\, \forall t \in [t_q - \bar{h}, t_q]\, \forall i \notin \mathcal{V}_0. \end{aligned} \tag{36}$$

Indeed, if $\tilde{x}(t), t \geq t_q$ is some other solution with $\tilde{\lambda}_{\mathcal{V}_0}(t_q) \geq 1$ and $\tilde{\lambda}(t_q) \geq 0$, then one has $\tilde{x}(t) \geq x(t)\, \forall t \geq t_q$ in accordance with Remark 9. Obviously, if *(i)* holds for $x(t)$, then it also holds for $\tilde{x}(t)$, and the validity of *(ii)* for $x(t)$ entails the validity of *(ii)* for $\tilde{x}(t)$ (with same $\mathcal{V}_1, r, \gamma$).

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Step 3b. Consider now the solution that satisfies (36) and assume that *(i)* does not hold. We are going to show that *(ii)* holds with $\mathcal{V}_1 \triangleq \mathcal{V}_0 \cup \{j\}$, where $j \notin \mathcal{V}_0$ and

$$\gamma \triangleq c_2 e^{-(n-1)(\ell+\mu)}, \quad c_2 \triangleq \frac{e^{-(n-1)\ell}\varrho' C}{2(\varrho' C+1)}, \quad C \triangleq \frac{2}{Kn^2}. \quad (37)$$

Note that the constant $\gamma > 0$ is determined by n, ℓ, μ, K and does not depend on $\mathcal{V}_0$. Since *(i)* is not valid, the minimal $t' > t_q$ exists such that $\min_{i\in\mathcal{V}_0} x_i(t') = 1/2$. Let $s \geq q+1$ be such an index that $t_{s-1} < t' \leq t_s$. For each $i \in \mathcal{V}_0$ one has $x_i(t) \geq \frac{1}{2}$ when $t \in [t_q, t']$. Now (28) entails the inequality

$$x_i(t) \geq c_1 \triangleq \tfrac{1}{2} e^{-(n-1)\ell} > c_2 \quad \forall t \in [t_q, t_s]\, \forall i \in \mathcal{V}_0, \quad (38)$$

(which, e.g., is valid at $t = t''$). To prove *(ii)*, it suffices to find an index $j \notin \mathcal{V}_0$ such that $x_j(t'') \geq c_2$ a some instant $t'' \in [t_q, t_s]$: choosing index r in such a way that $t_{r-1} < t'' + \bar{h} \leq t_r$, (27) entails the inequality $\lambda_{\mathcal{V}_0\cup\{j\}}(t_q) \geq c_2\theta_0 = \gamma$.

The existence of the desired index j and the instant t'' will be proved by contradiction. Assume, on the contrary, that $x_j(t) < c_2$ on $[t_q, t_s]$ for all $j \notin \mathcal{V}_0$.

Denote $\mathcal{V}_0^c \triangleq \mathcal{V} \setminus \mathcal{V}_0$ and consider the functions

$$f_i(t) \triangleq \begin{cases} \sum_{j\in\mathcal{V}_0^c} a_{ij}(t)[\hat{x}_j^i(t) - x_i(t)], \; i \in \mathcal{V}_0 \\ \sum_{j\in\mathcal{V}_0} a_{ij}(t)[\hat{x}_j^i(t) - x_i(t)], \; i \in \mathcal{V}_0^c. \end{cases} \quad (39)$$

Due to (4), the function $x(t)$ obeys the system of equations

$$\dot{x}_i = \sum\nolimits_{j\in\mathcal{V}_0} a_{ij}(t)(\hat{x}_j^i(t) - x_i(t)) + f_i(t), \; i \in \mathcal{V}_0, \quad (40)$$

$$\dot{x}_i = \sum\nolimits_{j\in\mathcal{V}_0^c} a_{ij}(t)(\hat{x}_j^i(t) - x_i(t)) + f_i(t), \; i \in \mathcal{V}_0^c. \quad (41)$$

In view of (38) and our assumption, the inequalities hold

$$0 \leq x_j(t) < c_2 < c_1 \leq x_i(t) \leq \Lambda(t_q) = 1.$$

for any $j \in \mathcal{V}_0^c$, $m \in \mathcal{V}_0$ and $t \in [t_q, t_s]$. Hence,

$$\begin{aligned} 0 \geq f_i(t) &\geq -\sum\nolimits_{j\in\mathcal{V}_0^c} a_{ij}(t) \; \forall i \in \mathcal{V}_0 \, \forall t \in [t_q, t_s] \\ f_j(t) &\geq (c_1 - c_2) \sum\nolimits_{j\in\mathcal{V}_0} a_{ji}(t) \; \forall j \in \mathcal{V}_0^c \, \forall t \in [t_q, t_s]. \end{aligned} \quad (42)$$

We now introduce the subvectors $x^+ = (x_i)_{i\in\mathcal{V}_0}$ and $x^\dagger = (x_i)_{i\in\mathcal{V}_0^c}$ corresponding to the subvectors $f^+, f^\dagger$ of vector f. Denoting the evolutionary matrix of (40) by U^+ and applying the inequality in (29) to $t_* = t_q$, $x = x^+$, $f = f^+$, one has

$$x^+(t') \geq \lambda_{\mathcal{V}_0}(t_q)\mathbf{1}_{\mathcal{V}_0} + \int_{t_p}^{t'} U^+(t', t) f^+(t) dt \Longrightarrow$$

$$x_i(t') \geq 1 - \sum\nolimits_{k\in\mathcal{V}_0, j\in\mathcal{V}_0^c} \int_{t_p}^{t_q} a_{kj}(t) dt \quad \forall i \in \mathcal{V}_0.$$

By assumption, $x_i(t') = 1/2$ for some i. Taking into account that $|\mathcal{V}_0|\,|\mathcal{V}_0^c| \leq n^2/4$, there exist $k \in \mathcal{V}_0$ and $j \in \mathcal{V}_0^c$ such that

$$\int_{t_p}^{t_q} a_{kj}(t) dt \geq \frac{1/2}{n^2/4} = \frac{2}{n^2} \overset{(23)}{\Longrightarrow} \int_{t_p}^{t_q} a_{jk}(t) dt \geq \frac{2}{Kn^2} = C.$$

Denoting the evolutionary matrix of (41) by $U^\dagger$ and applying (29) to $t_* = t_q$, $x = x^+$, $f = f^+$, one has

$$x^\dagger(t_q) \geq \underbrace{\lambda_{\mathcal{V}_0^c}(t_q)}_{=0} \mathbf{1}_{\mathcal{V}_0^c} + \int_{t_p}^{t_q} U^\dagger(t_q, t) f^\dagger(t) dt.$$

Recalling that $U^\dagger(t,s)_{jj} \geq \varrho'$ because $|\mathcal{V}_0^c| < n$, one has

$$x_j(t_q) \geq \varrho' \int_{t_p}^{t_q} f_j(t) dt \overset{(42)}{\geq} \varrho'(c_1 - c_2) \int_{t_p}^{t_q} a_{jk}(t) dt \geq$$
$$\geq \varrho'(c_1 - c_2) C \overset{(37),(38)}{=} c_2,$$

leading thus to a contradiction with the assumption that $x_j(t) < c_2$ for all $t \in [t_q, t_s]$. The induction step is proved. ■

B. Proofs of Lemma 1, Corollary 1 and Lemma 2

1) Proof of Lemma 1: Assume that the algorithm (4) establishes consensus (7) between two agents i and j. In view of (25), every solution of (11) (starting at $t_* \geq 0$) is the sum of a solution of (4) and the "forced" solution

$$x^f(t) = \int_{t_*}^{t} U(t,\xi) f(\xi)\, d\xi = \int_{t_*}^{\infty} \hat{U}(t,\xi) f(\xi)\, d\xi, \quad (43)$$

where $\hat{U}(t,\xi) = 0$ if $\xi > t$ and $\hat{U}(t,\xi) = U(t,\xi)$ for $\xi \leq t$.

To prove that consensus between i and j is preserved when $f \in L_1([0,\infty) \to \mathbb{R}^{\mathcal{V}})$, it thus suffices to show that

$$\lim_{t\to\infty} x_i^f(t) = \lim_{t\to\infty} x_j^f(t)$$

(and both limits exist). To prove this, notice that (30) can be rewritten as follows: for each instant $t_* \geq 0$, the vector $u_{t_*}^\top$ exists such that $\lim_{t\to\infty} \mathbf{e}_i^\top \hat{U}(t,t_*) = \lim_{t\to\infty} \mathbf{e}_j^\top \hat{U}(t,t_*) = u_{t_*}^\top$. Recalling that matrices $\hat{U}(t,\xi)$ are uniformly bounded (Lemma 9) and f is L_1-summable, the Lebesgue dominated convergence theorem entails the existence of coincident limits

$$\lim_{t\to\infty} x_i^f(t) = \lim_{t\to\infty} \int_{t_*}^{\infty} \mathbf{e}_i^\top \hat{U}(t,t_*) f(\xi)\, d\xi = \int_{t_*}^{\infty} u_{t_*}^\top f(\xi) d\xi,$$
$$\lim_{t\to\infty} x_j^f(t) = \lim_{t\to\infty} \int_{t_*}^{\infty} \mathbf{e}_j^\top \hat{U}(t,t_*) f(\xi)\, d\xi = \int_{t_*}^{\infty} u_{t_*}^\top f(\xi) d\xi,$$

which finishes the proof of Lemma 1. ■

2) Proof of Corollary 1: If $x(t)$ is a solution of (12), then $x(t)$ is bounded (Lemma 8) and obeys (11), where $f_i(t) \triangleq -\sum_{j:(j,i)\notin\mathcal{E}_\infty} a_{ij}(t)(\hat{x}_j^i(t) - x_i(t))$ is L_1-summable. Similarly, every solution of (4) is bounded and obeys the equation

$$\dot{x}_i(t) = \sum\nolimits_{j\in\mathcal{V}} \tilde{a}_{ij}(t)(\hat{x}_j^i(t) - x_i(t)) + \tilde{f}_i(t), \; i \in \mathcal{V}, \quad (44)$$

where $\tilde{f}_i = \sum_{j:(j,i)\notin\mathcal{E}_\infty} a_{ij}(t)(\hat{x}_j^i(t) - x_i(t))$ is L_1-summable. The statement is now obvious from Lemma 1. ■

3) Proof of Lemma 2: In view of Corollary 1, it suffices to prove the statement for the algorithm (12). For $i \in \mathcal{V}$, let set $\mathcal{V}_i$ consist of i and all nodes $r \in \mathcal{V}$ that are connected to i in the graph $\mathcal{G}_\infty$ by walks. Lemma 2 can be restated as follows: consensus between k and m implies that $\mathcal{V}_k \cap \mathcal{V}_m \neq \emptyset$.

Each set $\mathcal{V}_i$ is "closed" in the graph $\mathcal{G}_\infty$: if $r \notin \mathcal{V}_i$, then $(r,j) \notin \mathcal{E}_\infty \, \forall j \in \mathcal{V}_i$; equivalently, $\tilde{a}_{jr}(t) \equiv 0 \, \forall j \in \mathcal{V}_i, r \notin \mathcal{V}_i$. Thus the algorithm (12) cannot establish consensus between k and m such that $\mathcal{V}_k \cap \mathcal{V}_m = \emptyset$, because the dynamics of subvectors $(x_i)_{i\in\mathcal{V}_k}$ and $(x_i)_{i\in\mathcal{V}_m}$ are fully decoupled. ■

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C. Proof of Lemma 3

Using notation introduced in (14) and (5), all entries of the matrices $A_t^{t+\bar{h}}$ do not exceed μ, and their sum does not exceed $n(n-1)\mu$. On the other hand, each graph $\mathcal{G}[A_{t_p}^{t_{p+1}}]$ contains a spanning tree that has $n-1$ arcs ($n=|\mathcal{V}|$) of weight $\geq \varepsilon$, and hence the total weight of its arcs is not less than $\varepsilon(n-1)$. The union of graphs over $[t_p, t_{p+k}]$, where $k \geq 1$, has the total weight of arcs $\geq k\varepsilon(n-1)$. Choosing k so large that $k\varepsilon > \mu n$, the interval $[t_p, t_{p+k}]$ should thus have length $t_{p+k} - t_p \geq \bar{h}$.

D. Proof of Lemma 5

We are going to prove the implications $c) \implies b) \implies a) \implies c)$. The first implication $c) \implies b)$ is straightforward.

Suppose that $b)$ holds and let $A_p \triangleq A_{t_p}^{t_{p+1}}$. By assumption, graphs $\mathcal{G}[A_p^\varepsilon]$ are quasi-strongly connected for all p, that is, each of those graphs contains a directed spanning tree [1]. Since the number of possible trees with n nodes is finite, at least one of these spanning trees belongs to an infinite sequence of graphs $\mathcal{G}^k \triangleq \mathcal{G}[A_{p_k}^\varepsilon]$, where $p_k \to \infty$. For each arc (j,i) of this tree, one thus has

$$\int_{t_{p_k}}^{t_{1+p_k}} a_{ij}(s)ds \geq \varepsilon,$$

which means that $\int_0^\infty a_{ij}(s)\,ds = \infty$. The graph $\mathcal{G}_\infty$ contains the common spanning tree of graphs $\mathcal{G}^k$, and *a)* is true.

Finally, assume that $a)$ holds. For a given $\varepsilon > 0$, consider the following sequence (t_p): $t_0 = 0$, and for all $p \geq 0$

$$t_{p+1} = \inf\left\{ t > t_p : \int_{t_p}^{t} a_{ij}(s)ds \geq \varepsilon \quad \forall (j,i) \in \mathcal{G}_\infty \right\}.$$

By construction, graphs $\mathcal{G}[A_p]$ are quasi-strongly ε-connected for all p. It remains to prove that $t_p \to \infty$. Indeed, $t_{p+1} > t_p$ by construction, and thus the limit $\bar{t} \triangleq \lim_{p\to\infty} t_p$ exists. If we had $\bar{t} < \infty$, then some functions $a_{ij}(\cdot)$ could not be locally L_1-summable, because for $(j,i) \in \mathcal{E}_\infty$ one would have

$$\int_{t_0}^{\bar{t}} a_{ij}(t)dt \geq \sum_{p=0}^{\infty} \int_{t_p}^{t_{p+1}} a_{ij}(t)dt = \infty.$$

This proves the remaining implication $a) \implies c)$. ∎

E. Proofs of Lemmas 6 and 7

1) Proof of Lemma 7: The "only if" part follows from Lemma 5. To prove the "if" part, choose $(j,i) \in \mathcal{G}_\infty$ and $\varepsilon > 0$. Since $a_{ij} \notin L_1$, for some sequence $t_p \to \infty$

$$\int_{t_p}^{t_{p+1}} a_{ij}(t)dt = K\varepsilon.$$

Then, using (21) we have

$$\varepsilon \leq \int_{t_p}^{t_{p+1}} a_{km}(t)dt \leq K^2\varepsilon \quad \forall (m,k) \in \mathcal{E}_\infty.$$

Recalling that $a_{kl} \in L_1$ for $(l,k) \notin \mathcal{E}_\infty$, one proves that matrices $A_{t_p}^{t_{p+1}}$ are bounded (the supremum in (15) is finite) and their graphs are quasi-strongly ε-connected (containing graph $\mathcal{G}_\infty$), that is, the AQSC condition holds. ∎

2) Proof of Lemma 6: The proof of implication $(b) \iff (c)$ is straightforward from the definition of $\mathcal{G}_\infty$. If *(b)* and *(c)* are both valid, then, obviously, the matrix (20) obeys the arc-balance condition (21), where K is found as

$$K = \max\left\{ \frac{\bar{a}_{ij}}{\bar{a}_{km}} : \bar{a}_{ij}, \bar{a}_{km} > 0 \right\}.$$

The implication $(b) \iff (a)$ now follows from Lemma 7. ∎

F. Proofs of Theorem 1, Lemma 4 and Corollary 2

Theorem 1 is based on Lemma 9 and the following lemma.

Lemma 13: Let $B^1, \ldots, B^{n-1} \in \mathbb{R}^{\mathcal{V}\times\mathcal{V}}$, where $n = |\mathcal{V}|$, be *substochastic* matrices with positive diagonal entries and quasi-strongly connected graphs $\mathcal{G}[B^i]$. Consider a sequence $z^1, \ldots, z^n \in \mathbb{R}^{\mathcal{V}}$, where $0 \leq z^1 \leq \mathbf{1}$ and

$$0 \leq z^k \leq B^{k-1}z^{k-1} + (\mathbf{1} - B^{k-1}\mathbf{1}), \quad k = 2, \ldots, n. \tag{45}$$

Then, $0 \leq \max z^n - \min z^n < 1$. Moreover, for each $\varepsilon, \eta > 0$ there exists $\rho = \rho(\varepsilon, \eta) \in (0,1)$ such that if $\mathcal{G}[B^i]$ are quasi-strongly ε-connected and $b_{ii}^1, \ldots, b_{ii}^{n-1} \geq \eta\, \forall i \in \mathcal{V}$, then $\max z^n - \min z^n \leq \rho(\varepsilon, \eta)$.

Proof: For a vector $x \in [0,1]^{\mathcal{V}} \triangleq \{(x_i)_{i\in\mathcal{V}} : x_i \in [0,1]\,\forall i\}$, denote $\mathbf{ZER}(x) \triangleq \{i : x_i = 0\} \subseteq \mathcal{V}$ and $\mathbf{ONE}(x) \triangleq \{i : x_i = 1\} \subseteq \mathcal{V}$. In view of (45) and substochasticity of B^k, we have $z^k \in [0,1]^{\mathcal{V}}$ for all $k = 1, \ldots, n$.

For any substochastic matrix B with positive diagonal entries and any $x \in [0,1]^{\mathcal{V}}$, the inequality $x_i > 0$ (respectively, $x_i < 1$) entails that $(Bx)_i > 0$ (respectively, $(Bx)_i < 1$). Hence, the sets $\mathbf{ZER}(z^k)$ and $\mathbf{ONE}(z^k)$ are nested: $\mathbf{ZER}(z^k) \subseteq \mathbf{ZER}(z^{k-1})$ and $\mathbf{ONE}(z^k) \subseteq \mathbf{ONE}(z^{k-1})$.

Furthermore, as can be easily seen, if $j \notin \mathbf{ONE}(z^k)$ and $b_{ij}^k > 0$, then $z_i^{k+1} < 1$. Similarly, if $j \notin \mathbf{ZER}(z^k)$ and $b_{ij}^k > 0$, then $z_i^{k+1} > 0$. If $\mathcal{G}[B^k]$ is quasi-strongly connected, with some node r, then either $r \notin \mathbf{ONE}(z^k)$ or $r \notin \mathbf{ZER}(z^k)$. In the first situation, we either have $\mathbf{ONE}(z^k) = \emptyset$ or a path connecting r to $\mathbf{ONE}(z^k)$ in $\mathcal{G}[B^k]$ should exist, that is, at least one arc comes to $\mathbf{ONE}(z^k)$ from outside. In this situation, $\mathbf{ONE}(z^{k+1}) \subsetneq \mathbf{ONE}(z^k)$. Similarly, if $r \notin \mathbf{ZER}(z^k)$, then either $\mathbf{ZER}(z^k) = \emptyset$ or $\mathbf{ZER}(z^{k+1}) \subsetneq \mathbf{ZER}(z^k)$. Since the cardinality of $\mathbf{ONE}(z^1) \cup \mathbf{ZER}(z^1)$ is not greater than $n = |\mathcal{V}|$, at least one of the sets $\mathbf{ONE}(z^n)$ or $\mathbf{ZER}(z^n)$ is empty, and hence $\max z^n - \min z^n > 0$.

The set $\mathfrak{B}(\eta, \varepsilon)$ of substochastic matrices B such that $a_{ii} \geq \eta$ and $\mathcal{G}[B]$ is quasi-strongly ε-connected is *compact*. Hence, the set of all sequences $(z^1, \ldots, z^n)$ obeying (45) is also compact. The continuous function $\max z^n - \min z^n$ thus reaches a maximum $\rho = \rho(\varepsilon, \eta) < 1$ on $\mathfrak{B}(\eta, \varepsilon)$. ∎

1) Proof of Theorem 1: Without loss of generality (Lemma 3), we assume that (16) holds and thus Lemma 11 is applicable. Also, one may suppose that $t_0 \geq t_*$ and $r = 0$.

For brevity, we denote $M(t) \triangleq \max x(t) \leq \Lambda(t)$, $m(t) \triangleq \min x(t) \geq \lambda(t)$ and $D(t) \triangleq \Lambda(t) - \lambda(t)$.

We are going to show that

$$D(t_{2n-1}) \leq \theta D(t_0), \quad \theta \triangleq 1 - \tilde{\eta} + \tilde{\eta}\rho(\tilde{\eta}, \tilde{\varepsilon}), \tag{46}$$

where $\tilde{\eta}, \tilde{\varepsilon}$ are from Lemma 11 and ρ is the function from Lemma 13. Note that θ depends on $n = |\mathcal{V}|$, ℓ and ε.

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In view of the system (4) linearity, it suffices to prove (46) for the solution such that $\lambda(t_0)=0$ and $\Lambda(t_0)=1$, so that $D(t_0)=1$. For this solution, one has $0\leq\lambda(t)\leq\bar{\Lambda}(t)\leq 1\,\forall t\geq t_0$ thanks to Lemma 8. Using (29) (where $t_*=t_p$ and $t=t_{p+2}$), one arrives at the inequality

$$0\leq x(t_{p+2})-U(t_{p+2},t_p)x(t_p)\leq\mathbf{1}-U(t_{p+2},t_p)\mathbf{1}.$$

In view of Lemma 11, matrices $B^j=U(t_{2j},t_{2j-2})$ have strongly positive diagonal entries and quasi-strongly $\tilde{\varepsilon}$-connected graphs. Applying Lemma 13 to matrices $B^j=U(t_{2j},t_{2j-2})$ and vectors $z^j=x(t_{2j-2})$, one arrives at

$$M(t_{2(n-1)})-m(t_{2(n-1)})\leq\rho=\rho(\tilde{\eta},\tilde{\varepsilon})<1, \tag{47}$$

where ρ is the function from Lemma 13. Substituting $t_*=t_{2n-2}$ into (29), we also have

$$0\leq x(t)-U(t,t_{2n-2})x(t_{2n-2})\leq\mathbf{1}-U(t,t_{2n-2})\mathbf{1},$$

and thus for $t\in[t_{2n-2},t_{2n-1}]$ inequalities hold

$$\begin{aligned}U(t,t_{2n-2})_{ii}m(t_{2n-2})&\leq U(t,t_{2n-2})_{ii}x_i(t_{2n-2})\leq x_i(t)\leq\\ &\leq U(t,t_{2n-2})_{ii}\,x_i(t_{2n-2})+1-U(t,t_{2n-2})_{ii}\leq\\ &\leq U(t,t_{2n-2})_{ii}M(t_{2n-2})+1-U(t,t_{2n-2})_{ii}.\end{aligned}$$

Using the inequality (32) (where $p=2n-2$) and recalling that $m(t_{2n-2})\geq 0, M(t_{2n-2})\leq 1$, one shows that

$$\tilde{\eta}m(t_{2n-2})\leq x_i(t)\leq\tilde{\eta}M(t_{2n-2})+1-\tilde{\eta}\ \ \forall i\in\mathcal{V}$$

for all $t\in[t_{2n-2},t_{2n}]$. Since $t_{2n-1}\geq t_{2n-2}+\bar{h}$, we have

$$\tilde{\eta}m(t_{2n-2})\leq\lambda(t_{2n-1})\leq\Lambda(t_{2n-1})\leq\tilde{\eta}M(t_{2n-2})+1-\tilde{\eta},$$

whence $D(t_{2n-1})\leq\tilde{\eta}(M(t_{2(n-1)})-m(t_{2(n-1)}))+1-\tilde{\eta}\overset{(47)}{\leq}\theta=\theta D(t_0)$, which proves (46).

Replacing t_0 by t_{2n-1}, the same argument shows that

$$D(t_{(2n-1)k})\leq\theta D(t_{(2n-1)(k-1)})\leq\ldots\leq\theta^k D(t_0), \tag{48}$$

which finishes the proof of the second inequality in (17). The first inequality is from Lemma 8: since $\Lambda(t)$ is non-increasing, one has $\bar{x}_i=\lim_{s\to\infty}x_i(s)\leq\Lambda(t)\,\forall t\geq 0$. Similarly, $\bar{x}_i=\lim_{s\to\infty}x_i(s)\geq\lambda(t)\,\forall t\geq 0$. Hence, for all $i\in\mathcal{V}$ we have $x_i(t),\bar{x}_i\in[\lambda(t),\Lambda(t)]$, and $\|x(t)-\bar{x}\|_\infty\leq\Lambda(t)-\lambda(t)$. ■

2) Proof of Lemma 4: Applying (17) to $t_*=s$ and solution $x(t)=U(t,s)v$, where $v\in\mathbb{R}^{\mathcal{V}}$ is an arbitrary vector with $\|v\|_\infty=1$, one proves that

$$\begin{gathered}\|U(t,s)-\bar{U}_s\|_\infty\leq(\Lambda(s)-\lambda(s))\theta^k=2\theta^k,\\ t\in[t_{r+2k(n-1)},t_{r+2(k+1)(n-1)}],\,t_r\geq s.\end{gathered} \tag{49}$$

For an arbitrary vector $x\in\mathbb{R}^{\mathcal{V}}$, denote $\delta(x)\overset{\Delta}{=}\max x-\min x$; obviously, $0\leq\delta(x)\leq 2\|x\|_\infty$ and $\delta(x+y)\leq\delta(x)+\delta(y)$. Similar to Lemma 1, it suffices to prove (18) for the "forced" solution (43). To this end, we introduce the function

$$x_*^f(t)\overset{\Delta}{=}\int_{t_*}^t\bar{U}_sf(s)\,ds.$$

It seems natural that, in view of (31), x^f and x_*^f are sufficiently close when t becomes large. To make the latter statement formal, choose an index $q\geq r$ (so that $t_q\geq t_*$). Obviously,

$$\begin{aligned}\left\|x^f(t)-x_*^f(t)\right\|_\infty\leq\int_{t_*}^{t_q}\|U(t,s)-\bar{U}_s\|_\infty\|f(s)\|_\infty ds+\\ +\int_{t_q}^t\|U(t,s)-\bar{U}_s\|_\infty\|f(s)\|_\infty ds\end{aligned}$$

The Lebesgue dominated convergence theorem ensures that the first integral vanishes as $t\to\infty$. In view of (49), the second integral can be estimated as

$$\sum_{k=0}^{\infty}\underbrace{2(n-1)\,2(1+\theta+\theta^2+\ldots)}_{=C_1}\sup_{p\geq q}\int_{t_p}^{t_{p+1}}\|f(t)\|_\infty dt.$$

One thus concludes that for every $q\geq r$ we have

$$\limsup_{t\to\infty}\left\|x^f(t)-x_*^f(t)\right\|_\infty\leq C_1\sup_{p\geq q}\int_{t_p}^{t_{p+1}}\|f(t)\|_\infty dt.$$

Since q can be arbitrary, one can now pass to the limit $q\to\infty$, replacing sup in the latter inequality by $\limsup_{p\to\infty}$. Hence,

$$\limsup_{t\to\infty}\delta(x^f(t))\leq\underbrace{\limsup_{t\to\infty}\delta(x_*^f(t))}_{=0}+\limsup_{t\to\infty}\delta(x^f(t)-x_*^f(t))$$

$$\leq 2\limsup_{t\to\infty}\left\|x^f(t)-x_*^f(t)\right\|_\infty=C\limsup_{p\to\infty}\int_{t_p}^{t_{p+1}}\|f(t)\|_\infty dt,$$

where $C\overset{\Delta}{=}2C_1$. This proves (18) for the "forced" solution $x^f(t)$, which implies (18) for an arbitrary solution of (11). ■

G. Proof of Theorem 2

In this subsection, we assume that the conditions of Theorem 2 hold. We will prove, in fact, that consensus is guaranteed for every bounded solution to the inequality (24) (in particular, for all solutions to (4)). For every such a solution, the function $\Lambda(t)$ is non increasing (Lemma 8), and thus the limit $\bar{\Lambda}\overset{\Delta}{=}\lim_{t\to\infty}\Lambda(t)>-\infty$ exists. Assume that $\Lambda(t_*)>\lambda(t_*)$ (otherwise, consensus is obvious due to Lemma 8).

The proof of Theorem 2 is based on the fruitful idea of a one-to-one *ordering mapping* $\sigma_t:[1:n]\to\mathcal{V}$ ($n=|\mathcal{V}|$), which sorts the vector is the ascending order

$$z_1(t)=x_{\sigma_t(1)}(t)\leq\ldots\leq z_n(t)=x_{\sigma_t(n)}(t). \tag{50}$$

Notice that, in general, $\sigma_t(i)$ is defined non-uniquely (if $x(t)$ has two or more equal components), however, the mapping σ_t can always be chosen *measurable* in t and the functions $z_i(t)$ are absolutely continuous on $[0,\infty)$ [20], [54].

1) The reduction to the global consensus case: As we have noticed in Section VI, the persistent graph $\mathcal{G}_\infty$ is undirected, consisting thus of several disjoint connected components. It suffices to prove Theorem 2 for the case where $\mathcal{G}_\infty$ is connected. Indeed, if $A(\cdot)$ satisfies the NITS condition, this condition also holds for the matrix $\tilde{A}(\cdot)$ from (12). The system (12) consists of fully decoupled subsystems, corresponding to connected components of $\mathcal{G}_\infty$. If consensus in each of these components is proved, Corollary 1 guarantees that this consensus is also guaranteed by protocol (4).

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2) Global consensus in terms of the sorted vector: Introducing the sorted state vector $z(t)$ (50), the global consensus is established if and only if

$$z_k(t_p) \xrightarrow[p\to\infty]{} \bar\Lambda = \lim_{t\to\infty} \Lambda(t) > -\infty \quad \forall k = 1,\dots,n. \quad (51)$$

Indeed, (51) is equivalent to

$$x(t_p) \xrightarrow[p\to\infty]{} \bar\Lambda\mathbf{1}. \quad (52)$$

If (52) is valid, then, choosing $t_{p-1} < t' < t_p$ and using Lemma 9 (where t_* in (29) is replaced by t'), one has $x(t_p) \le \psi x(t') + (1-\psi)\Lambda(t')\mathbf{1}$, and hence $[x(t_p) - (1-\psi)\Lambda(t')\mathbf{1}]\psi^{-1} \le x(t') \le \Lambda(t')\mathbf{1}$, where the leftmost and rightmost expressions both go to $\bar\Lambda\mathbf{1}$ as $p\to\infty$.

3) The proof of (51)*:* Assuming that the persistent graph $\mathcal{G}_\infty$ is connected, we are going to prove (51) using the backward induction on $k = n,\dots,1$.

Induction base. We will prove a stronger statement $z_n(t) = \max x(t) \xrightarrow[t\to\infty]{} \bar\Lambda$, using the inequality (26) and Assumption 1. Due to Assumption 1, one has $\exp(-\int_s^t \alpha_i(\xi)d\xi) \ge \delta \triangleq e^{-(n-1)\mu}$ for all $s\ge 0$ and $t\in[s, s+\bar h]$. Hence

$$\Lambda(s+\bar h) = \sup\nolimits_{t\in[s,s+\bar h]} \max x(t) \le \delta \max x(s) + (1-\delta)\Lambda(s)$$
$$\implies \delta^{-1}[\Lambda(s+\bar h) - (1-\delta)\Lambda(s)] \le \max x(s) \le \Lambda(s),$$

Therefore, $z_n(s) = \max x(s) \xrightarrow[s\to\infty]{} \bar\Lambda$. This also implies that

$$\limsup_{t\to\infty} z_r(t) \le \limsup_{t\to\infty} z_n(t) \le \bar\Lambda. \quad (53)$$

Induction base. Suppose that (51) has been proved for $k = r+1,\dots,n$; our goal is to prove (51) it for $k = r$. In view of (53), it suffices to show that

$$\liminf_{p\to\infty} z_r(t_p) \ge \bar\Lambda. \quad (54)$$

Assume, on the contrary, that a subsequence $\tau_m \triangleq t_{p(m)}$ exists, $m\to\infty$ such that $z_r(\tau_m) \le \Lambda(\tau_m) - \delta\,\forall m$. Passing to a subsequence, one can assume, without loss of generality, that $\mathcal{V}_0 \triangleq \sigma_{\tau_m}([1:r]) \subset \mathcal{V}$ does not depend on m (this set contains indices of the r smallest elements at time τ_m). By definition, $x_i(\tau_m) \le z_r(\tau_m) \le \Lambda(\tau_m) - \delta \quad \forall i\in\mathcal{V}_0\,\forall m$. Without loss of generality, we choose m so large that

$$z_{r+1}(t) > \Lambda(\tau_m) - \delta/3 \quad \forall t \ge \tau_m - \bar h. \quad (55)$$

We will show that the aforementioned assumptions contradict to the induction hypothesis, proving the existence of such $j\notin\mathcal{V}_0$, a sequence $\tilde\tau_m = t_{\tilde p(m)} \to\infty$ and $\tilde\delta < \delta$ that

$$\max_{i\in\mathcal{V}_0\cup\{j\}} x_i(\tilde\tau_m) \le \Lambda(\tau_m) - \tilde\delta. \quad (56)$$

The cardinality of $\mathcal{V}_0\cup\{j\}$ is $r+1$, so (56) entails that $z_{r+1}(t_{\tilde p(m)}) \le \Lambda(\tau_m) - \tilde\delta$ for each m, which is incompatible with the induction hypothesis. Informally speaking, some agent from $\mathcal{V}_0^c \triangleq \mathcal{V}\setminus\mathcal{V}_0$ comes "sufficiently close" to agents from $\mathcal{V}_0$ due to persistent interactions between $\mathcal{V}_0$ and $\mathcal{V}_0^c$.

The proof of (56) is based on Lemma 12, and, in fact, similar to Step 3 of its proof. We divide it into several steps.

Step 1. As we have seen in the proof of Lemma 12, introducing the functions f_i as in (39), the functions $x_i(t)$ obey the system (40),(41). One may easily notice that the submatrices $A^+ = (a_{ij})_{i,j\in\mathcal{V}^+}$ and $A^\dagger = (a_{ij})_{i,j\in\mathcal{V}^\dagger}$ obey the NITS condition (with the same sequence (t_p) and constant K), and hence Lemma 12 is applicable to the corresponding evolutionary matrices $U^+, U^\dagger$ of linear systems (40),(41).

Introducing the subvectors $x^+ = (x_i)_{i\in\mathcal{V}_0}$ and $x^\dagger = (x_i)_{i\in\mathcal{V}_0^c}$ and corresponding vectors $f^+, f^\dagger$ and using Lemma 9 (where $t_* = \tau_m, U = U^+$), we have

$$x^+(t) \le U^+(t,\tau_m)x^+(\tau_m) + \Lambda(\tau_m)(\mathbf{1}_{\mathcal{V}^+} - U^+(t,\tau_m)\mathbf{1}_{\mathcal{V}^+})$$
$$+ \int_{\tau_m}^t U^+(t,\xi)f^+(\xi)d\xi \quad \forall t\ge\tau_m.$$

Recalling that $x^+(\tau_m) \le (\Lambda(\tau_m) - \delta)\mathbf{1}_{\mathcal{V}^+}$, one arrives at

$$x^+(t) \le (\Lambda(\tau_m) - \delta)\mathbf{1}_{\mathcal{V}^+} + \int_{\tau_m}^t U^+(t,\xi)f^+(\xi)d\xi. \quad (57)$$

Recalling that $x^\dagger(\tau_m) \le \Lambda(\tau_m)\mathbf{1}_{\mathcal{V}^\dagger}$ and using Lemma 9 (with $\tau_m, U^\dagger$ instead of t_*, U), one also proves that

$$x^\dagger(t) \le \Lambda(\tau_m)\mathbf{1}_{\mathcal{V}^\dagger} + \int_{\tau_m}^t U^\dagger(t,\xi)f^\dagger(\xi)d\xi. \quad (58)$$

Step 2. Using the connectivity of $\mathcal{G}_\infty$, we will show that $\max x^+(t) > [\Lambda(\tau_m) - 2\delta/3]$ at some instant $t\ge\tau_m$. Indeed, assume that this statement is incorrect, and hence $x^+(t) \le [\Lambda(\tau_m) - 2\delta/3]\mathbf{1}_{\mathcal{V}_0}\,\forall t\ge\tau_m$. In view of (55), $\max x^+(t) < z_{r+1}(t)$ for $t\ge\tau_m$, that is, the set $\mathcal{V}_0$ contains the indices of r smallest elements of $x(t)$, whereas the indices of $n-r$ largest elements belong to $\mathcal{V}_0^c$ at *all* instants $t\ge\tau_m$ (and not only at τ_m). Therefore, $x_j(t) > \Lambda(\tau_m) - \delta/3\,\forall j\in\mathcal{V}_0^c$ and

$$f_i(t) \overset{(39)}{\le} -\frac{\delta}{3}\sum_{j\in\mathcal{V}_0} a_{ij}(t) \le 0 \quad \forall i\in\mathcal{V}_0^c \;\; \forall t\ge\tau_m - \bar h.$$

Recalling that the persistent graph is connected and $a_{ij}\notin L_1[0,\infty)$ for some $j\in\mathcal{V}_0$, we have $\int_{\tau_m}^\infty f_i(t)dt = -\infty\,\forall i\in\mathcal{V}_0^c$. Lemma 12 and (58) entail that $x_i(t) \xrightarrow[t\to\infty]{} -\infty \quad \forall i\in\mathcal{V}_0^c$, which leads us to a contradiction.

Step 3. Let $t'\ge\tau_m$ be the first instant such that $x_i(t') = \Lambda(\tau_m) - 2\delta/3$ for some $i\in\mathcal{V}_0$; find q such that $t_{q-1} < t' \le t_q$. Corollary 3 entails that

$$x_i(t) \le (1-\theta_1)\Lambda(t') + \theta_1 x(t'), \quad \forall t\in[t', t_q]\,\forall i\in\mathcal{V}_0,$$

Therefore, for all $i\in\mathcal{V}_0$ and $t\in[\tau_m, t_q]$ one has

$$x_i(t) \le \Lambda(\tau_m) - \delta_1, \quad \delta_1 \triangleq \frac{2\delta}{3}\theta_1. \quad (59)$$

On the other hand, by construction $x_i(t') - x_i(\tau_m) \ge \delta/3$. Using (57) and recalling that $U(t,\xi)$ is substochastic, one has

$$\sum_{l\in\mathcal{V}_0}\int_{\tau_m}^{t'} f_l(t)dt = \sum_{\substack{l\in\mathcal{V}_0,\\ j\in\mathcal{V}_0^c}}\int_{\tau_m}^{t'} a_{lj}(t)[\hat x_j^l(t) - x_l(t)]dt \ge \frac{\delta}{3}.$$

Since $\hat x_j^l(t) - x_l(t) \le D_0 \triangleq \Lambda(t_*) - \lambda(t_*)$ due to Lemma 8,

$$\sum_{\substack{l\in\mathcal{V}_0,\\ j\in\mathcal{V}_0^c}}\int_{\tau_m}^{t'} a_{lj}(t)dt \ge c_1 \triangleq \frac{\delta}{3D_0}.$$

Therefore, in view of the NITS condition and $t_q \geq t'$,

$$\sum_{\substack{l\in\mathcal{V}_0,\\ j\in\mathcal{V}_0^c}} \int_{\tau_m}^{t_q} a_{jl}(t)dt \geq c_2 \triangleq K^{-1}c_1. \tag{60}$$

Informally speaking, we have proved that the influence between $\mathcal{V}_0$ and $\mathcal{V}_0^c$ during $[\tau_m, t_q]$ is strong enough.

Step 4. Using (60), Lemma 12 and inequality (58), one can show that for some $j \in \mathcal{V}^\dagger$ and $t'' \in [\tau_m, t_q]$ we have $x_j(t'') \leq \Lambda(\tau_m) - \delta_2$, where, by definition

$$\delta_2 \triangleq \min\left(\frac{\varrho^\dagger c_2\delta_1}{\varrho^\dagger c_2 + n - r}, \frac{\delta}{3}\right), \tag{61}$$

c_2 is defined in (60) and $\rho^\dagger$ is the constant from (33), corresponding to $U^\dagger$.

Indeed, assume that $x_j(t) > \Lambda(\tau_m) - \delta_2$ for all $j \in \mathcal{V}^\dagger$ and $t \in [\tau_m, t_q]$. In view of (55), the latter inequality holds also for $t \in [\tau_m - \bar{h}, \tau_m]$ (recall that $\delta_2 < \delta/3$). Therefore, we have $\hat{x}_i^j(t) - x_j(t) > \delta_1 - \delta_2$ for $i \in \mathcal{V}^+, j \in \mathcal{V}^\dagger, t \in [\tau_m, t_q]$. Thus

$$\sum_{j\in\mathcal{V}^\dagger} \int_{t_p}^{t_q} f_j(t)dt \overset{(60)}{<} -c_2(\delta_1 - \delta_2) < 0.$$

Applying Lemma 12 to matrix $U^\dagger$, one has

$$\begin{aligned}(n-r)(\Lambda(\tau_m) - \delta_2) \leq \mathbf{1}_{\mathcal{V}^\dagger}^\top x^\dagger(t_q) \overset{(58)}{\leq} (n-r)\Lambda(\tau_m) + \\ + \int_{\tau_m}^{t_q} \mathbf{1}_{\mathcal{V}^\dagger}^\top U^\dagger(t_q, t) f^\dagger(t)dt < \\ < (n-r)\Lambda(\tau_m) - \rho^\dagger c_2(\delta_1 - \delta_2).\end{aligned}$$

The latter inequality, obviously, contradicts (61).

Step 5. Finally, one can find such an index $\tilde{p}(m) \leq q$ that $t'' \in (t_{\tilde{p}(m)-1}, t_{\tilde{p}(m)}]$. Corollary 3 entails that

$$x_j(t_{\tilde{p}(m)}) \leq \Lambda(\tau_m) - \tilde{\delta}, \quad \tilde{\delta} \triangleq \theta_1\delta_2.$$

The latter inequality, in combination with (59), obviously entails (56), where $\tilde{\tau}_m = t_{\tilde{p}(m)}$, which contradicts to the induction hypothesis. The induction step is proved, which finishes the proof of Theorem 2. ∎

VIII. Numerical simulation

In this section, we consider several numerical examples illustrating the difference between Theorem 1 and 2. In all experiments, $n = 4$ agents interact over a simple graph (Fig. 1), where all connections are bidirectional yet unequally weighted: $a_{ij}(t) \neq a_{ji}(t)$. In fact, we will have $\mathcal{G}[A(t)] = \mathcal{G}_\infty$. Three types of delays are used: (a) zero delays $h_{ij} \equiv 0\,\forall i, j$; (b) asymmetric constant delays $h_{ij}(t) = \tau_j^0\bar{h}$; (c) asymmetric time-varying delays $h_{ij}(t) = \bar{h}(\sin(t + \theta_j^0))^2$. The initial conditions and parameters of the delays in all tests are

$$x(t) \equiv x(0) = [-1, 0.8, -0.5, 1]^\top \quad \forall t \in [-\bar{h}, 0],$$
$$\tau^0 = [0.1, 0.9, 0.4, 0.2],\ \theta^0 = [\pi/4, \pi/3, 0, \pi],\ \bar{h} = 50s.$$

We also introduce the functions

$$a(t) = \frac{1}{0.01 + t},\ b(t) = \frac{\log(1.01 + t)}{0.01 + t}.$$

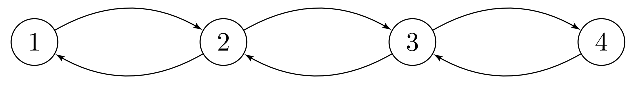

Fig. 1: The graph $\mathcal{G}[A(t)] = \mathcal{G}_\infty$ in experiments 1-3.

Experiment 1: Type-symmetric graph. Consider first the situation where $a_{12}(t) = b(t)$, $a_{21}(t) = 2b(t)$, $a_{34}(t) = 0.9b(t)$, $a_{43}(t) = b(t)$, whereas $a_{23}(t) = a(t), a_{32}(t) = 0.8a(t)$. All functions are bounded, and thus (5) holds.

Note that the links between agents 2 and 3 are much weaker than the other links, because $a(t)/b(t) \xrightarrow[t\to\infty]{} 0$. One can also note that Theorem 1 cannot be applied, because the AQSC condition is violated. Indeed, if the condition (i) in Definition 4 holds, then the graph $\mathcal{G}[A_{t_p}^{t_{p+1}}]$ contains at least one of arcs $2 \to 3$ or $3 \to 2$, and, by noticing that $b(t) > a(t)\log t$,

$$\int_{t_p}^{t_{p+1}} a(t)dt \geq \varepsilon \Longrightarrow \int_{t_p}^{t_{p+1}} b(t)dt \geq \varepsilon \log t_p \xrightarrow[p\to\infty]{} \infty,$$

so the supremum in (15) is infinite. However, the type-symmetry condition (22) holds, so consensus is ensured by Theorem 2 and Remark 6. The behaviors of the solutions in the three situations (no delay, static delays, time-varying delays) are shown in Fig. 2. Neither static nor time-varying delays destroy consensus, but they substantially reduce the convergence speed and alter the consensus value $\bar{x}_i$ from (7).

Experiment 2: None of Theorems works. Consider now the weights $a_{12}(t) = a(t)$, $a_{21}(t) = 2b(t)$, $a_{34}(t) = 0.9b(t)$, $a_{43}(t) = a(t)$, $a_{23}(t) = a(t), a_{32}(t) = 0.8a(t)$. None of Theorems 1 and 2 can be applied, because the AQSC condition remains violated, and type-symmetry is also destroyed. Although the absence of consensus has not been proved analytically, the simulation over a very long time interval (Fig. 3) reveals that the system fails to reach global consensus under constant delays (case (b)). This can be expected, because attraction between two "central" nodes of the graph 2 and 3 is much weaker than ties connecting them to the "peripheral" nodes 1 and 4. This example shows the importance of condition (ii) in Definition 4 and the insufficiency of the persistent graph's connectivity for consensus.

Experiment 3: Consensus guaranteed by Theorem 1. We now show that consensus is regained if asymmetric weights obey the AQSC condition. Consider now the weights $a_{12}(t) = a(t)$, $a_{21}(t) = 2b(t)$, $a_{34}(t) = 0.9b(t)$, $a_{43}(t) = b(t)$, $a_{23}(t) = a(t), a_{32}(t) = b(t)$. Unlike experiments 1 and 2, the conditions of Definition 4 are satisfied by the sequence (t_p), where $t_0 = 0$ and $t_{p+1} > t_p$ is the instant when

$$\int_{t_p}^{t_{p+1}} b(t)\,dt = 1 \quad \forall p = 0, 1 \ldots$$

Then, obviously, the graphs $\mathcal{G}[A_{t_p}^{t_{p+1}}]$ contain a chain $1 \mapsto 2 \mapsto 3 \leftrightarrow 4$ of arcs whose weight is ≥ 1, whereas the weights of other arcs in these graphs do not exceed $\leq (\log 1.01)^{-1}$, because $b(t) \geq (\log 1.01)a(t)\,\forall t \geq 0$. The behaviors of the solutions in all cases (no delay, static delays, time-varying delays) are shown in Fig. 4. Neither static nor time-varying delays destroy consensus, but they substantially reduce the convergence speed and alter the consensus value $\bar{x}_i$ from (7).

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Notice that consensus in this experiment does not follow from any of the results available in the literature, even in the undelayed case. The UQSC condition fails to hold, because the weights $a_{ij}(t)$ vanish as $t \to \infty$. The arc-balance condition (21) also fails to hold, because $a(t)/b(t) \to 0$ as $t \to \infty$.

CONCLUSION

First-order consensus protocols, based on the idea of iterative averaging (discrete time) or the Laplacian flow (continuous-time), are prototypic distributed algorithms that arise in many problems of multi-agent coordination. In spite of substantial progress in their analysis, some fundamental problems still remain open, in particular, there are no necessary and sufficient criteria of the algorithm's convergence in the situation where the interaction graph is time-varying. The latter problem becomes especially complicated in the case where communication among the agents is delayed. The most typical assumption under which the delay robustness of consensus is proved the uniform quasi-strong connectivity (UQSC).

In this paper, we demonstrate that the UQSC assumption is not necessary and can be substantially relaxed. Our first main result shows that, in the case of a general directed graph, it can be relaxed to the condition we refer to as the aperiodic quasi-strong connectivity (AQSC), which, as we show, is closely related to the well-known necessary consensus condition (integral connectivity). A special case of the AQSC graph is the arc-balanced graph, for such graphs there is no gap between necessary and sufficient conditions. The same holds for networks with *reciprocal* (e.g. undirected or type-symmetric) interactions; this constitutes the second main result of our work. As has been mentioned in Remark 3, the explicit estimation of convergence speed of consensus algorithms (even in the UQSC case) remains a non-trivial problem.

APPENDIX A
DISCUSSION ON ASSUMPTION 1

In this appendix, we demonstrate that Assumption 1 in fact cannot be omitted even for the case of two agents. Consider two agents whose values evolve in accordance with

$$\begin{aligned} \dot{x}_0(t) &= a(t)(x_1(t-\tau(t)) - x_0(t)), \\ \dot{x}_1(t) &= a(t)(x_0(t-\tau(t)) - x_1(t)). \end{aligned} \qquad (62)$$

Here $a(t) \geq 0$ is a locally L_1-summable function, however $a \notin L_1[0,\infty)$. As shown in Lemma 6, the AQSC condition is satisfied for some sequence (t_p); the NITS condition is also fulfilled with the same sequence. All conditions of Theorems 1 and 2 thus holds except for, possibly, Assumption 1.

Unbounded delay can destroy consensus

A trivial counterexample showing that the delay boundedness condition cannot be discarded is the case $\tau(t) = t$, $a(t) = 1$, in which situation (62) turns into

$$\dot{x}_0(t) = x_1(0) - x_0(t), \ \ \dot{x}_1(t) = x_0(0) - x_1(t).$$

Obviously, one has $x_i(t) \xrightarrow[t\to\infty]{} x_{1-i}(0)$ $(i = 0, 1)$, so consensus is impossible unless $x_0(0) = x_1(0)$. All existing results on consensus with unbounded delays [44], [45] thus assume that $t - \tau(t)$ is separated from 0 as $t \to \infty$. Even in this situation, quite restrictive assumptions on the graph $G[A(\cdot)]$ have to be imposed to prove consensus robustness.

Notice that a similar effect can be found by considering the case of very large constant delay $\tau(t) \equiv \tau$. The deviation between the agents' states $v(t) = x_0(t) - x_1(t)$ obeys the equation $\dot{v}(t) = -v(t) - v(t-\tau)$. Characteristic roots of the latter system, i.e., the solutions of the transcendental equation

$$s + 1 + e^{-s\tau} = 0, \quad s \in \mathbb{C} \qquad (63)$$

converge[10] [55] to the imaginary axis as $\tau \to \infty$. In this sense, the equation $\dot{v}(t) = -v(t) - v(t-\tau)$ loses asymptotic stability as the delay becomes infinite $\tau \to \infty$.

Unbounded integral in (5) can destroy consensus

We now show that the function $a(\cdot)$ can be chosen in such a way that the system (62) fails to establish consensus in spite of the delay boundedness. On each interval $[k, k+1)$ (where $k = 0, 1, \ldots$), we define the "saw-tooth" [53] delay $\tau(t) = t - k \, \forall t \in [k, k+1)$, so that $\hat{x}_j(t) = x_j(t - \tau(t)) = x_j(k)$. Therefore, the agents obey the following equations

$$\dot{x}_i(t) = a(t)(x_{1-i}(k) - x_i(t)), \ i = 0, 1, \ \forall t \in [k, k+1). \quad (64)$$

Denoting $a_k \triangleq \exp(-\int_k^{k+1} a(t)\, dt)$, one has

$$x_{1-i}(k) - x_i(k+1) = a_k(x_{1-i}(k) - x_i(k)),$$

which leads to a discrete-time consensus algorithm as follows

$$x_i(k+1) = (1 - a_k)x_{1-i}(k) + a_k x_i(k), \quad i = 0, 1, \quad (65)$$

which, as easily shown, ensures that $|x_1(k+1) - x_0(k+1)| = |1 - 2a_k|\,|x_1(k) - x_0(k)|$. If $a_k < 1/2$ and $\sum_k a_k < \infty$, then $\pi_* \triangleq \prod_{k=0}^{\infty}(1 - 2a_k) > 0$, and hence $|x_1(k) - x_0(k)| \xrightarrow[k\to\infty]{} \pi_*|x_1(0) - x_0(0)|$, so consensus is not established.

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[10]Using the terminology from [55], the equation (65) has only continuous pseudo-spectrum, which converges to the imaginary axis as $\tau \to \infty$.

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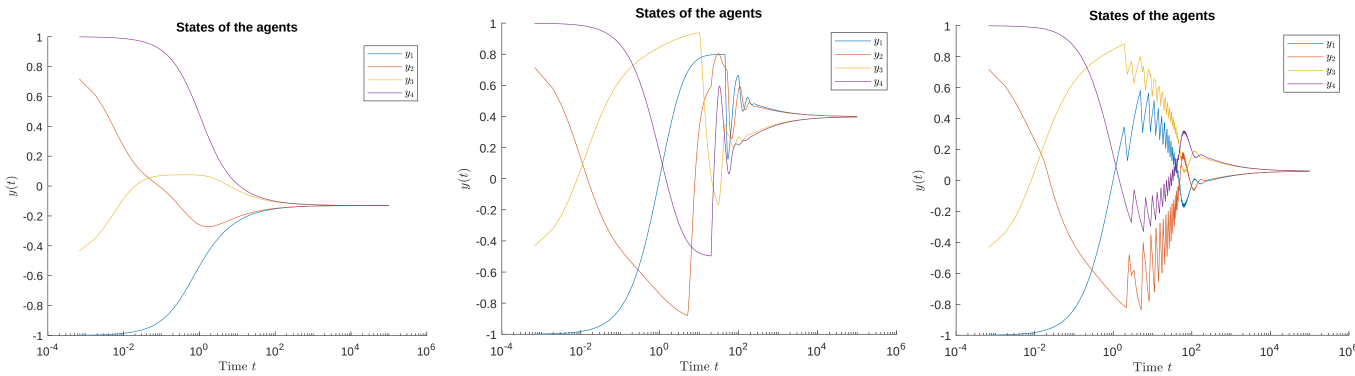

(a) Undelayed communication: $\bar{x}_i \approx -0.13$ (b) Static delays: $\bar{x}_i \approx 0.4$ (c) Time-varying delays: $\bar{x}_i \approx 0.06$

Fig. 2: Experiment 1: consensus over a type-symmetric graph

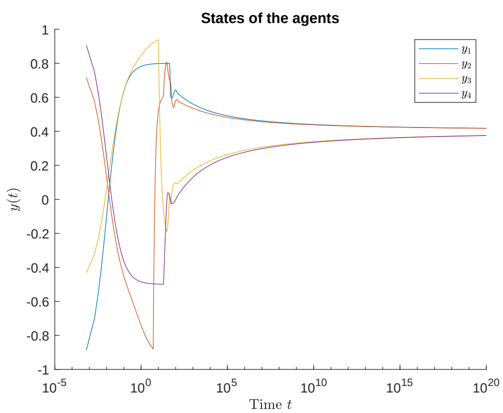

Fig. 3: Experiment 2: simulation shows absence of consensus in situation (b) (static delays) due to violation of the AQSC and NITS conditions.

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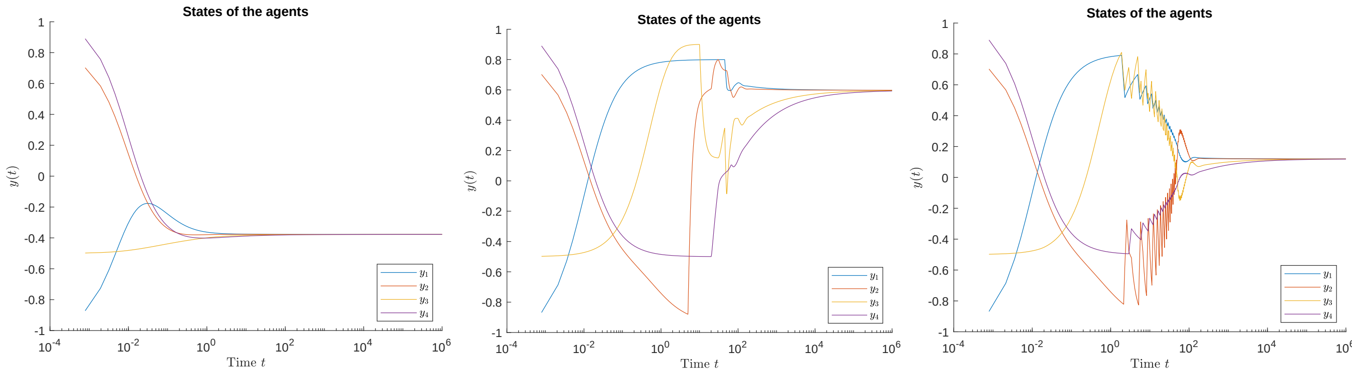

(a) Undelayed communication: $\bar{x}_i \approx -0.37$ (b) Static delays: $\bar{x}_i \approx 0.6$ (c) Time-varying delays: $\bar{x}_i \approx 0.12$

Fig. 4: Experiment 3: consensus over a AQSC graph

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