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REGULAR ARTICLE

Generalized FGM dependence: geometrical representation and convex bounds on sums

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Abstract

Building on the one-to-one relationship between generalized FGM copulas and multivariate Bernoulli distributions, we prove that the class of multivariate distributions with generalized FGM copulas is a convex polytope. Therefore, we find sharp bounds in this class for many aggregate risk measures, such as value-at-risk, expected shortfall, and entropic risk measure, by enumerating their values on the extremal points of the convex polytope. This is infeasible in high dimensions. We overcome this limitation by considering the aggregation of identically distributed risks with generalized FGM copula specified by a common parameter p. In this case, the analogy with the geometrical structure of the class of Bernoulli distribution allows us to provide sharp analytical bounds for convex risk measures.

Keywords Multivariate Bernoulli distributions · GFGM copulas · Huang-Kotz FGM copulas · Risk measures · Convex order

1 Introduction

Finding bounds for aggregated risks with partial information on their joint distribution is a widely addressed problem in insurance and finance. The available information on the multivariate dependence is often modeled using a copula. Our main contribution is to solve the problem of finding analytical bounds for aggregated risks under the assumption that their dependence is modeled using a generalization of the Farlie-Gumbel-Morgenstern (FGM) copula. The aim of our work is beyond the appropriateness of adopting generalized FGM copulas in any specific application. We

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study their mathematical, and geometrical properties, that make them a powerful tool to deal with aggregated risks.

The FGM copulas have been first studied by Eyraud (1936), Farlie (1960), Gumbel (1960) and Morgenstern (1956). Due to its simple analytical formulation, FGM copulas are used to build multivariate models in many areas, including natural hazard - drought (Saghafian and Mehdikhani 2014), transport (Zou et al. 2014), marketing (Kim et al. 2022), and insurance. FGM copulas allow both positive and negative dependence but do not cover the complete range of dependence. This led to a wide variety of extensions of bivariate FGM copulas being proposed, such as the well-known Huang–Kotz FGM copulas (Huang and Kotz 1999), but much less in higher dimensions. Recently, researchers have integrated the Huang–Kotz FGM copulas and other extensions of that family as building blocks within models for directional dependence of genes in bioinformatics (Kim et al. 2008), bivariate failure time models with competing risks (Shih and Emura 2018, 2019; Shih and Shih 2019), copula-based stress-strength models in reliability (Domma and Giordano 2013; Hudaverdi and Susam 2023), and models that account for asymmetric dependence between variables in biomedicine (Li 2023).

This paper focuses on the generalization introduced by Blier-Wong et al. (2024a), that they name generalized FGM (GFGM). The family of GFGM copulas has been defined through a stochastic representation involving multivariate Bernoulli vectors. They extend a stochastic representation of FGM copulas introduced in Blier-Wong et al. (2022b), which is key to grasping the dependencies between the components of a random vector and to greatly facilitate the sampling procedure. The stochastic representations create a link between FGM copulas and multivariate symmetric Bernoulli vectors and between GFGM copulas and multivariate Bernoulli vectors with mean vector $\boldsymbol{p} = (p_1, \ldots, p_d)$ (Blier-Wong et al. 2024a). The importance of this link lies in the fact that the class of GFGM copulas inherits the geometrical properties of the class $\mathcal{B}_d(\boldsymbol{p})$ of probability mass functions (pmfs) of d-dimensional Bernoulli vectors with mean vector $\boldsymbol{p}$. Contrary to all the numerical studies and applications in references cited in this introduction that are restricted to two dimensions, this link provides tools for new applications in high dimensions and with a broader scope of dependence. Moreover, the vector $\boldsymbol{p}$ becomes a vector parameter for GFGM copulas. Although some results are presented for the general case, we focus on the case $p = p_1 = \ldots = p_d$, for two reasons: some results cannot be generalized in a straightforward manner to have different values within $\boldsymbol{p}$; and we aim at keeping the model parsimonious in terms of the number of parameters. This last motivation gives us a better insight into the role of the parameter p driving the dependence. Actually, we do more. We also prove that the class $\mathcal{G}_d^p(F)$ of joint distributions with a common univariate margin F and GFGM(p) copula shares the same geometrical structure as the class of multivariate Bernoulli distributions. This analogy allows us to efficiently work with sums $S = X_1 + \cdots + X_d$ of random variables with joint distribution in $\mathcal{G}_d^p(F)$, applying the results of Fontana et al. (2021). We also show that the convex order is preserved from the elements of the class of sums of components of random vectors following Bernoulli distributions to our class of interest $\mathcal{S}_d^p(F)$, that is the class of distributions of sums of random vectors with cdf in $\mathcal{G}_d^p(F)$. This contribution is our main result and allows us to find the vectors in $\mathcal{G}_d^p(F)$ whose sums are minimal in convex order, and this is important because they are the vectors where the lower bounds for convex risk measures are reached. We considered some convex risk measures, the expected shortfall and the entropic risk measure, and also the value-at-risk. We analytically find their bounds in the exponential and discrete margins cases and provide numerical illustrations in these two special cases in high dimensions. Lastly, building on the geometrical structure of the joint distributions behind the sums, we address another important open issue and exhibit some possible alternative dependence structures corresponding to minimal aggregated risks.

We also present some simple but new results for the GFGM dependence without assuming a scalar parameter p and identically distributed risks. We find a stochastic representation that generalizes the correspondence proved in Blier-Wong et al. (2024a) to any random vector $\boldsymbol{X} = (X_1, \ldots, X_d)$ with a dependence structure defined by a GFGM copula. Building on this representation, we also prove that the class $\mathcal{G}_d^{\boldsymbol{p}}(F_1, \ldots, F_d)$ of joint distributions of random vectors $\boldsymbol{X}$ with univariate marginals $F_1, \ldots, F_d$ and GFGM copula with parameters $\boldsymbol{p} = (p_1, \ldots, p_d)$ is a convex polytope, that is a convex hull of a finite set of points, called extremal points. However, the number of extremal points is huge and computational limitations exist in finding them in high dimensions. Overcoming these theoretical limitations is left for future work. Although this case's theoretical investigation is beyond this paper's scope, we also discuss some numerical examples of GFGM dependence without assuming identically distributed risks. In this general case, we can enumerate the extremal points up to dimension $d = 5$.

The paper is structured as follows. Section 2 introduces the preliminary notions about multivariate Bernoulli distributions and GFGM copulas with a common parameter and their link. A new stochastic representation for GFGM copulas is provided in Sect. 3. We study the geometrical structure and the convex order in the class of uniform vectors with GFGM copulas with a common parameter p in Sect. 4. In the subsequent section, we provide sharp bounds for the convex risk measures and the value-at-risk, together with numerical illustrations. Section 6 summarizes some existing applications of FGM copulas in the literature and outlines possible estimation procedures based on their geometrical representation. The last Sect. 7 presents an example of the generalization to different values of p for future research purposes and concludes.

2 Preliminaries and state of the art

In this section, we recall some notions on the set $\mathcal{B}_d$ of d-dimensional pmfs which have Bernoulli univariate marginal distributions, and on the class $\mathcal{C}_d$ of GFGM copulas.

2.1 Multivariate Bernoulli distributions and convex polytopes

Let us consider the Fréchet class $\mathcal{B}_d(\boldsymbol{p}) = \mathcal{B}_d(p_1, \ldots, p_d)$ of multivariate Bernoulli distributions with Bernoulli marginal distributions with means $p_j, j \in \{1, \ldots, d\}$. We assume throughout the paper that p_j are rational, that is $p_j \in \mathbb{Q}$, $j \in \{1, \ldots, d\}$.

Since $\mathbb{Q}$ is dense in $\mathbb{R}$, this is not a limitation in applications. We denote by $\mathcal{B}_d(p)$ the class of multivariate Bernoulli distributions with identical Bernoulli marginal distributions with mean p, meaning $p_1 = \cdots = p_d = p$. We assume that vectors are column vectors and we denote by $A^\top$ the transpose of a matrix A.

If $\boldsymbol{I} = (I_1, \ldots, I_d)$ is a random vector with joint pmf f in $\mathcal{B}_d$, we denote the column vector which contains the values of f over $\mathcal{X}_d = \{0, 1\}^d$ by $\boldsymbol{f} = (f_{\boldsymbol{x}} : \boldsymbol{x} \in \mathcal{X}_d) := (f(\boldsymbol{x}) : \boldsymbol{x} \in \mathcal{X}_d)$; we make the non-restrictive assumption that the set $\mathcal{X}_d$ of 2^d binary vectors is ordered according to the reverse-lexicographical criterion. As an example, we consider $d = 3$ and we have $\mathcal{X}_3 = \{000, 100, 010, 110, 001, 101, 011, 111\}$. The notations $\boldsymbol{I} \in \mathcal{B}_d(\boldsymbol{p})$ and $\boldsymbol{f} \in \mathcal{B}_d(\boldsymbol{p})$ indicate that $\boldsymbol{I}$ has joint pmf $f \in \mathcal{B}_d(\boldsymbol{p})$. In Fontana and Semeraro (2018), the authors prove that $\mathcal{B}_d(\boldsymbol{p})$ is a convex polytope (see as a standard reference De Berg et al. (1997)); it means that $\mathcal{B}_d(\boldsymbol{p})$ admits the following representation:

$$\mathcal{B}_d(\boldsymbol{p}) = \{\boldsymbol{f} \in \mathbb{R}_+^{2^d} : H\boldsymbol{f} = 0, \sum_{\boldsymbol{x} \in \mathcal{X}_d} f_{\boldsymbol{x}} = 1\},$$

where H is a $d \times 2^d$ matrix whose rows, up to a non-influential multiplicative constant, are $((\mathbf{1} - \boldsymbol{x}_j)^\top - \frac{(1-p_j)}{p_j}\boldsymbol{x}_j^\top)$, $j \in \{1, \ldots, d\}$, and where $\boldsymbol{x}_j$ is the vector which contains only the jth element of $\boldsymbol{x} \in \mathcal{X}_d$, $j \in \{1, \ldots, d\}$, e.g., for the bivariate case $\boldsymbol{x}_1^T = (0, 1, 0, 1)$ and $\boldsymbol{x}_2^T = (0, 0, 1, 1)$. Therefore, there are joint pmfs $\boldsymbol{r}_k \in \mathcal{B}_d(\boldsymbol{p})$, $k \in \{1, \ldots, n_{\boldsymbol{p}}^{\mathcal{B}}\}$, and for any $\boldsymbol{f} \in \mathcal{B}_d(\boldsymbol{p})$, there exist $\lambda_1, \ldots, \lambda_{n_{\boldsymbol{p}}^{\mathcal{B}}} \geq 0$ summing up to one such that

$$\boldsymbol{f} = \sum_{k=1}^{n_{\boldsymbol{p}}^{\mathcal{B}}} \lambda_k \boldsymbol{r}_k. \tag{2.1}$$

We call the vectors $\boldsymbol{r}_k$, $k \in \{1, \ldots, n_{\boldsymbol{p}}^{\mathcal{B}}\}$, the extremal points of $\mathcal{B}_d(\boldsymbol{p})$, and r_k the corresponding joint pmfs of the random vector $\boldsymbol{R}_k$. Here, $n_{\boldsymbol{p}}^{\mathcal{B}}$ is the number of extremal points of $\mathcal{B}_d(\boldsymbol{p})$ which depends on $\boldsymbol{p}$ and obviously on d. For low dimension d, the extremal points $\boldsymbol{r}_k$ can be found using for example the software `4ti2` (see 4ti2 team (2018)). However, their number $n_{\boldsymbol{p}}^{\mathcal{B}}$ increases rapidly with the dimension d, as discussed in Section 2 of Fontana and Semeraro (2024).

We need to introduce the following classes of distributions as they are building blocks of one of our main results. Let $\mathcal{E}_d(p) \subseteq \mathcal{B}_d(p)$ be the class of exchangeable d-dimensional Bernoulli distributions with mean $p \in [0, 1]$. Note that $f \in \mathcal{E}_d(p)$ if $f \in \mathcal{B}_d(p)$ and $f(\boldsymbol{x}) = f(\boldsymbol{x}_\sigma)$, where $\boldsymbol{x}_\sigma$ is any permutation of $\boldsymbol{x}$, for every $\boldsymbol{x} \in \mathcal{X}_d$. We also denote by $\mathcal{D}_d(dp)$ the class of univariate discrete distributions with support on $\{0, 1, \ldots, d\}$ and mean dp. If D is a discrete random variable with pmf f_D in $\mathcal{D}_d(dp)$, we denote the column vector containing the values of its pmf f_D over $\{0, \ldots, d\}$ by $\boldsymbol{f}^D = (f_1^D, \ldots, f_{d+1}^D)^\top$. In other words, $f_D(k) = f_{k+1}^D$, $k \in \{0, 1, \ldots, d\}$. The notations $D \in \mathcal{D}_d(dp)$ and $\boldsymbol{f}^D \in \mathcal{D}_d(dp)$ indicate that the discrete random variable D has pmf $f_D \in \mathcal{D}_d(dp)$. The authors of Fontana et al. (2021) prove that $\mathcal{D}_d(dp)$ is a convex polytope: it means that $\boldsymbol{f}^D \in \mathcal{D}_d(dp)$ if and only if there exist $\lambda_1, \ldots, \lambda_{n_p^{\mathcal{D}}} \geq 0$ summing up to 1 such that

$$\boldsymbol{f}^D = \sum_{k=1}^{n_p^{\mathcal{D}}} \lambda_k \boldsymbol{r}_k^D,$$

where $\boldsymbol{r}_k^D$, $k \in \{1, \ldots, n_p^{\mathcal{D}}\}$, are the extremal points of $\mathcal{D}_d(dp)$ and $n_p^{\mathcal{D}}$ is their number which depends on p and obviously on d (see Corollary 4.6 in Fontana et al. (2021) for the computation of $n_p^{\mathcal{D}}$). We denote by R_k^D a random variable with pmf $r_{D,k}$, $k \in \{1, \ldots, n_p^{\mathcal{D}}\}$. The extremal points have at most two non-zero components. Let k_1 and k_2 with $k_1 = 0, 1, \ldots, k_1^{\vee}$, $k_2 = k_2^{\wedge}, k_2^{\wedge} + 1, \ldots, d$, where $k_1^{\vee}$ is the largest integer lower than *dp* and $k_2^{\wedge}$ is the smallest integer greater than *dp*. The extremal pmfs $r_{D,k}$ have support on $\{k_1, k_2\}$ and have the following analytical expression:

$$r_{D,k}(y) = \begin{cases} \frac{k_2 - dp}{k_2 - k_1} & y = k_1, \\ \frac{dp - k_1}{k_2 - k_1} & y = k_2, \\ 0 & \text{otherwise.} \end{cases}$$

If *dp* is an integer, we also have an extremal point with support on the point *pd*, that is

$$r_{D,dp}(y) = \begin{cases} 1 & y = dp, \\ 0 & \text{otherwise.} \end{cases}$$

In Fontana et al. (2021), the authors show that the following relationship between classes of distributions holds:

$$\mathcal{E}_d(p) \leftrightarrow \mathcal{D}_d(dp), \tag{2.2}$$

that is, the class $\mathcal{D}_d(dp)$ has a one-to-one relationship with the class of exchangeable Bernoulli distributions $\mathcal{E}_d(p)$. In Fontana et al. (2021), the authors show that given $D \in \mathcal{D}_d(dp)$ there is one and only one exchangeable element $\boldsymbol{I}^e \in \mathcal{B}_d(p)$ such that $\sum_{j=1}^d I_j^e \stackrel{\mathcal{L}}{=} D$, where the notation $\stackrel{\mathcal{L}}{=}$ indicates equality in distribution. It follows that $\mathcal{E}_d(p)$ is a convex polytope and that the extremal points of $\mathcal{E}_d(p)$ are the exchangeable pmfs corresponding to the extremal points of $\mathcal{D}_d(dp)$. We denote by $\boldsymbol{e}_k$ extremal points or extremal pmfs of $\mathcal{E}_d(p)$, $k \in \{1, \ldots, n_p^{\mathcal{D}}\}$. We denote by $\boldsymbol{E}_k$ a random vector with pmf e_k. In Table 1, we provide each convex polytope with its generators.

2.2 Bernoulli distributions and GFGM copulas

The authors of Blier-Wong et al. (2024a) introduced the GFGM copulas through a stochastic representation that we provide below. This stochastic representation builds on a multivariate Bernoulli vector $\boldsymbol{I} \in \mathcal{B}_d(p)$ establishing a link with the class $\mathcal{B}_d(p)$ on which we establish our results.

Let $\boldsymbol{I} \in \mathcal{B}_d(\boldsymbol{p})$ be a d-variate Bernoulli random vector and let $\boldsymbol{U}_0 = (U_{0,1}, \ldots, U_{0,d})$ and $\boldsymbol{U}_1 = (U_{1,1}, \ldots, U_{1,d})$ be vectors of d independent uniform random variables.

Table 1 For each polytope of pmfs the second column provides the name of extremal points and their dimension, the third column provides the name of its corresponding random variable, and the last column the number of generators

Polytope	pmf-generator	rv-generator	Number of generators
$\mathcal{B}_d(p)$	$\boldsymbol{r}_k \in \mathbb{R}^{2^d}$	$\boldsymbol{R}_k$	$n_p^{\mathcal{B}}$
$\mathcal{D}_d(dp)$	$\boldsymbol{r}_{D,k} \in \mathbb{R}^{d+1}$	$R_{D,k}$	$n_p^{\mathcal{D}}$
$\mathcal{E}_d(p)$	$\boldsymbol{e}_k \in \mathbb{R}^{2^d}$	$\boldsymbol{E}_k$	$n_p^{\mathcal{D}}$

Assume the random vectors $\boldsymbol{I}$, $\boldsymbol{U}_0$, and $\boldsymbol{U}_1$ to be independent and define the random vector $\boldsymbol{U}$ with the following representation:

$$\boldsymbol{U} \overset{\mathcal{L}}{=} \boldsymbol{U}_0^{1-\boldsymbol{p}} \boldsymbol{U}_1^{\boldsymbol{I}} = (U_{0,1}^{1-p_1} U_{1,1}^{I_1}, \ldots, U_{0,d}^{1-p_d} U_{1,d}^{I_d}). \tag{2.3}$$

The joint cdf of the random vector $\boldsymbol{U}$ in (2.3) is the GFGM copula C with vector of parameters $\boldsymbol{p}$ as defined below.

Definition 2.1 A d-variate GFGM copula C with vector of parameters $\boldsymbol{p} = (p_1, \ldots, p_d) \in (0,1)^d$ has the following expression:

$$C(\boldsymbol{u}) = \prod_{m=1}^{d} u_m \left(1 + \sum_{k=2}^{d} \sum_{1 \le j_1 < \cdots < j_k \le d} \nu_{j_1 \ldots j_k} \left(1 - u_{j_1}^{\frac{p_{j_1}}{1-p_{j_1}}}\right) \cdots \left(1 - u_{j_k}^{\frac{p_{j_k}}{1-p_{j_k}}}\right)\right), \tag{2.4}$$

for $\boldsymbol{u} \in [0,1]^d$, where, for $1 \le j_1 < \cdots < j_k \le d$, $k \in \{2, \ldots, d\}$,

$$\nu_{j_1 \ldots j_k} = \mathrm{E}\left[\prod_{n=1}^{k} \frac{I_{j_n} - p_{j_n}}{p_{j_n}}\right],$$

where $\boldsymbol{I} = (I_1, \ldots, I_d) \in \mathcal{B}_d(\boldsymbol{p})$.

If one lets $b_j = \frac{p_j}{1-p_j}$, for $j \in \{1, \ldots, d\}$, the expression of the copula C in (2.4) becomes

$$C(\boldsymbol{u}) = \prod_{m=1}^{d} u_m \left(1 + \sum_{k=2}^{d} \sum_{1 \le j_1 < \cdots < j_k \le d} \nu_{j_1 \ldots j_k} \left(1 - u_{j_1}^{b_{j_1}}\right) \cdots \left(1 - u_{j_k}^{b_{j_k}}\right)\right), \tag{2.5}$$

for $\boldsymbol{u} \in [0,1]^d$. The shape parameters p_j (or b_j), $j \in \{1, 2, \ldots, m\}$, govern the dependence relation between the components of $\boldsymbol{U}$ in regard to the symmetry or asymmetry. When $p_j = p$, implying that $b_j = b = \frac{p}{1-p}$, $j \in \{1, 2, \ldots, m\}$, the copula C in (2.5) corresponds to the multivariate version of the symmetric bivariate Huang-Kotz FGM copula introduced and studied in Section 2 of Huang and Kotz (1999). In Bairamov et al. (2001), the authors propose an asymmetric bivariate Huang-Kotz FGM copula, while Bekrizadeh et al. (2012) suggests a multivariate version of the symmetric Huang-Kotz FGM copula; without however providing lower and upper bounds on the set of dependence parameters. The stochastic representation in (2.3) allows the authors of Blier-Wong et al. (2024a) to provide a multivariate extension of the Huang-Kotz FGM family of copulas with constraints on the dependence parameters; it is a key result to investigate dependence properties within this copula family and it easily provides a sampling algorithm. In the present paper, we build on the stochastic representation of the (2.3) to provide results on bounds of dependent aggregated risks. For a review on various extensions of the family of FGM copulas, including the Huang-Kotz FGM copulas, see Saminger-Platz et al. (2021) and Blier-Wong et al. (2024a).

We denote by $\mathcal{C}_d^{\boldsymbol{p}}$ and $\mathcal{C}_d^{p}$ the classes of d-variate GFGM copulas with parameters $\boldsymbol{p} = (p_1, \ldots, p_d)$ and with a common parameter p, respectively. The notation $\boldsymbol{U} \in \mathcal{C}_d^{\boldsymbol{p}}$ indicates that the joint cdf C of the random vector $\boldsymbol{U}$ with uniform margins is a copula $C \in \mathcal{C}_d^{\boldsymbol{p}}$. In the special case $p = \frac{1}{2}$, $\mathcal{C}_d^{1/2} \equiv \mathcal{C}_d^{\mathrm{FGM}}$, where $\mathcal{C}_d^{\mathrm{FGM}}$ is the class of d-variate Farlie-Gumbel-Morgenstern (FGM) copulas; see, e.g., Section 6.3 of Durante and Sempi (2015) and Blier-Wong et al. (2022b).

The authors of Blier-Wong et al. (2024a) provide a preliminary analysis of the possible dependence structures in each class $\mathcal{C}_d^{\boldsymbol{p}}$ in the bivariate case. In Figs. 3 and 4 of their paper, which we report in Appendix A, the authors show the density functions of the bivariate GFGM copulas that correspond to maximal and minimal dependence, for different values of $p_1, p_2 \in (0, 1)$. Their analysis concludes that for $p_1, p_2 \in (0.5, 1)$ the maximal dependence copulas have more mass in the upper tail, while for $p_1, p_2 \in (0, 0.5)$ in the lower tail. On the contrary, due to the moderate dependence of the FGM copula, the minimal dependence copulas have mass almost uniformly on the unit square, with less mass in the upper tail when $p_1, p_2 \in (0, 0.5)$, and less mass in the lower tail when $p_1, p_2 \in (0.5, 1)$. In Fig. 1 of Blier-Wong et al. (2024a), instead, they analyze the case of a common p and compare the maximal and minimal values of Spearman's rho in classes $\mathcal{C}_d^{p}$ for different values of the common parameter p, as the dimension d varies. They show that the range of correlation spanned by the copula increases as p increases. In Appendix A, we illustrate the effect of p on the obtainable dependence ranges discussing some examples.

In Blier-Wong et al. (2024a), the authors mention in Remark 1 that the class $\mathcal{C}_d^{\boldsymbol{p}}$ shares the geometrical structure of $\mathcal{B}_d(\boldsymbol{p})$. In particular, any $F_{\boldsymbol{U}} \in \mathcal{C}_d^{\boldsymbol{p}}$ is a convex combination of $F_{\boldsymbol{U}^{(\boldsymbol{R}_k)}}$, where $\boldsymbol{U}^{(\boldsymbol{R}_k)}$ is built from $\boldsymbol{R}_k$ according to (2.3), $k \in \{1, \ldots, n_{\boldsymbol{p}}^{\mathcal{B}}\}$. Indeed, by using the stochastic representation in (2.3), there is a multivariate Bernoulli vector $\boldsymbol{I} \in \mathcal{B}_d(\boldsymbol{p})$ such that $\boldsymbol{U} = \boldsymbol{U}^{(\boldsymbol{I})} = \boldsymbol{U}_0^{1-\boldsymbol{p}} \boldsymbol{U}_1^{\boldsymbol{I}}$, and we have

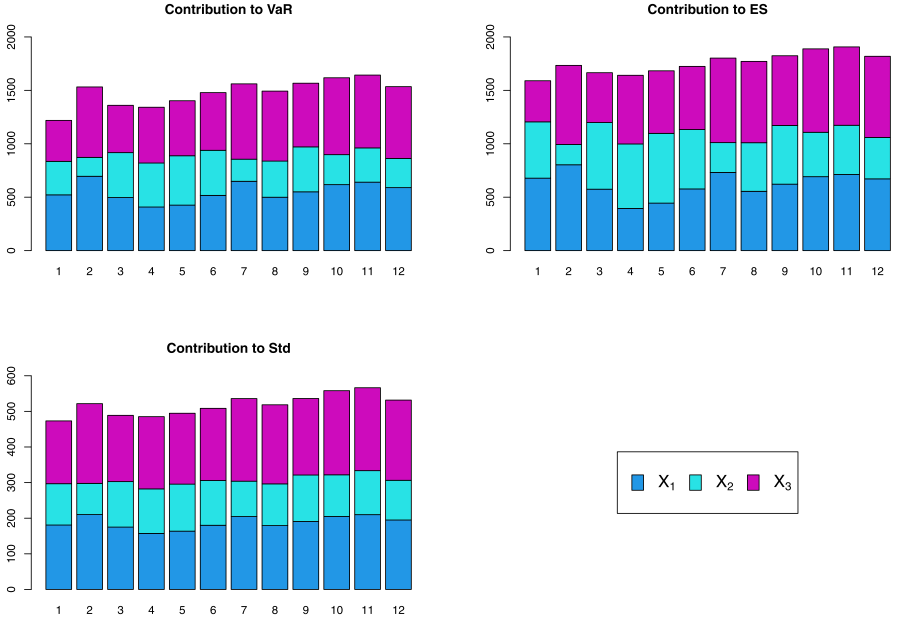


Fig. 1 Contributions of $X_j, j \in \{1, 2, 3\}$ to the $\mathrm{VaR}_{0.95}(S)$ (top-left), to the $\mathrm{ES}_{0.95}(S)$ (top-right), and to the standard deviation of S (bottom-left), based on Euler's rule

$$
\begin{aligned}
F_{\boldsymbol{U}^{(\boldsymbol{I})}}(\boldsymbol{x}) &= \Pr\left(U_1^{(\boldsymbol{I})} \leq x_1, \ldots, U_d^{(\boldsymbol{I})} \leq x_d\right) \\
&\stackrel{*}{=} \sum_{\boldsymbol{i} \in \{0,1\}^d} \Pr\left(U_{0,1}^{1-p_1} U_{1,1}^{i_1} \leq x_1, \ldots, U_{0,d}^{1-p_d} U_{1,d}^{i_d} \leq x_d\right) f_{\boldsymbol{I}}(\boldsymbol{i}) \\
&= \sum_{k=1}^{n_p^{\mathcal{B}}} \lambda_k \sum_{\boldsymbol{i} \in \{0,1\}^d} \Pr\left(U_{0,1}^{1-p_1} U_{1,1}^{i_1} \leq x_1, \ldots, U_{0,d}^{1-p_d} U_{1,d}^{i_d} \leq x_d\right) r_k(\boldsymbol{i}) \\
&\stackrel{**}{=} \sum_{k=1}^{n_p^{\mathcal{B}}} \lambda_k F_{\boldsymbol{U}^{(\boldsymbol{R}_k)}}(\boldsymbol{x}), \qquad \boldsymbol{x} \in [0,1]^d,
\end{aligned}
$$

where equalities $\stackrel{*}{=}$ and $\stackrel{**}{=}$ respectively follow from the independence of $\boldsymbol{U}_0$, $\boldsymbol{U}_1$ and $\boldsymbol{I}$ and of $\boldsymbol{U}_0, \boldsymbol{U}_1$ and $\boldsymbol{R}_k$, $k \in \{1, \ldots, n_{\boldsymbol{p}}^{\mathcal{B}}\}$. It follows that any GFGM copula C admits the representation

$$
C(\boldsymbol{u}) = \sum_{k=1}^{n_{\boldsymbol{p}}^{\mathcal{B}}} \lambda_k C_{\boldsymbol{R}_k}(\boldsymbol{u}), \quad \boldsymbol{u} \in [0,1]^d, \tag{2.6}
$$

where $C_{\boldsymbol{R}_k}$ is the copula associated to $\boldsymbol{U}^{(\boldsymbol{R}_k)}$, $k \in \{1, \ldots, n_{\boldsymbol{p}}^{\mathcal{B}}\}$. The class $\mathcal{C}_d^{\boldsymbol{p}}$ of copulas is a convex polytope.

We conclude this section with a simple but general new result for a class of multivariate distributions with dependence structure built with a family of copulas being a convex polytope. Our aim is to later investigate the specific class $\mathcal{G}_d^{\boldsymbol{p}}(F_1, \ldots, F_d)$ of distributions with marginal distributions $F_1, \ldots, F_d$ and with a copula in the class $\mathcal{C}_d^{\boldsymbol{p}}$.

Proposition 2.1 *Let $\mathcal{C}$ be a class of copulas. Let $\mathcal{F}_d(F_1, \ldots, F_d)$ be a class of multivariate distributions with marginal distributions $F_1, \ldots, F_d$ and with a copula in the class $\mathcal{C}$. If $\mathcal{C}$ is a convex polytope with extremal points $\widetilde{C}_1, \ldots, \widetilde{C}_n$, then $\mathcal{F}_d(F_1, \ldots, F_d)$ is a convex polytope with extremal points $\widetilde{F}_1, ..., \widetilde{F}_n$, where*

$$\widetilde{F}_k(\boldsymbol{x}) = \widetilde{C}_k(F_1(x_1), ..., F_d(x_d)),$$

for $\boldsymbol{x} \in \mathbb{R}^d$ and $k \in \{1, \ldots, n\}$.

Proof Consider $F_{\boldsymbol{X}} \in \mathcal{F}_d(F_1, \ldots, F_d)$. Then, for $\boldsymbol{x} \in \mathbb{R}^d$, we have

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = C(F_1(x_1), ..., F_d(x_d)) = \sum_{k=1}^{n} \lambda_k \widetilde{C}_k(F_1(x_1), ..., F_d(x_d)),$$

for some $\lambda_1 \geq 0, \ldots, \lambda_n \geq 0$ such that $\sum_{k=1}^{n} \lambda_k = 1$. Define $\widetilde{F}_k(\boldsymbol{x}) = \widetilde{C}_k(F_1(x_1), ..., F_d(x_d))$, for $k \in \{1, \ldots, n\}$, and the desired result directly follows. □

Corollary 2.1 *The class $\mathcal{G}_d^{\boldsymbol{p}}(F_1, \ldots, F_d)$ of distributions is a convex polytope with extremal points the distributions associated to the extremal points $\boldsymbol{r}_k$ of $\mathcal{B}_d(\boldsymbol{p})$, $k \in \{1, \ldots, n_{\boldsymbol{p}}^{\mathcal{B}}\}$.*

3 A new representation theorem and consequences

We generalize the results of Sect. 2.2 and introduce a stochastic representation for any random vector $\boldsymbol{X}$ with distribution in $\mathcal{G}_d^{\boldsymbol{p}}(F_1, \ldots, F_d)$. We then use this representation in two particular cases, more precisely with exponential and discrete marginals for which the expression of the distribution of the sum is analytical. The family of mixed Erlang distributions could also be considered. The notation $\boldsymbol{X} \in \mathcal{G}_d^{\boldsymbol{p}}(F_1, \ldots, F_d)$ indicates that the cdf of $\boldsymbol{X}$, $F_{\boldsymbol{X}}$, belongs to the class $\mathcal{G}_d^{\boldsymbol{p}}(F_1, \ldots, F_d)$.

Theorem 3.1 *Let $V_{0,1}, \ldots, V_{0,d}$, $V_{1,1}, \ldots, V_{1,d}$ be independent random variables, where $V_{0,j} \sim Beta\left(\frac{1}{1-p_j}, 1\right)$, $p_j \in (0, 1)$, and $V_{1,j} \sim U(0, 1)$, for $j \in \{1, \ldots, d\}$. Fix some margins $F_1, \ldots, F_d$ and, for $h \in \{0, 1\}$, let $\boldsymbol{Z}_h = (Z_{h,1}, \ldots, Z_{h,d})$ be vectors* *of independent random variables with* $Z_{0,j} \stackrel{\mathcal{L}}{=} F_j^{-1}(V_{0,j})$ *and* $Z_{1,j} \stackrel{\mathcal{L}}{=} F_j^{-1}(V_{0,j}V_{1,j})$, *for all* $j \in \{1, \ldots, d\}$. *Define the random vector*

$$\boldsymbol{X} = (\boldsymbol{1} - \boldsymbol{I})\boldsymbol{Z}_0 + \boldsymbol{I}\boldsymbol{Z}_1, \tag{3.1}$$

where $\boldsymbol{I} \in \mathcal{B}_d(\boldsymbol{p})$, $\boldsymbol{p} = (p_1, \ldots, p_d)$. *Then we have* $F_{\boldsymbol{X}} \in \mathcal{G}_d^{\boldsymbol{p}}(F_1, \ldots, F_d)$, *that is the distribution of the random vector* $\boldsymbol{X}$ *has margins* $F_1, \ldots, F_d$ *and copula* $C \in \mathcal{C}_d^{\boldsymbol{p}}$.

Proof Let $\boldsymbol{U} \in \mathcal{C}_d^{\boldsymbol{p}}$ and hence $\boldsymbol{U} = \boldsymbol{U}_0^{1-\boldsymbol{p}}\boldsymbol{U}^{\boldsymbol{I}}$, where $\boldsymbol{I} \in \mathcal{B}_d(\boldsymbol{p})$, with $\boldsymbol{p} = (p_1, \ldots, p_d)$. In the proof of Theorem 2 of Blier-Wong et al. (2024a), we have

$$F_{U_j}(u) = \Pr(U_{0,j}^{1-p_j}U_{1,j}^{I_j} \leq u) = \Pr(I_j = 0)F_{U_j|I_j=0}(u) + \Pr(I_j = 1)F_{U_j|I_j=1}(u),$$

where

$$F_{U_j|I_j=0}(u) = u^{\frac{1}{1-p_j}},$$

and

$$F_{U_j|I_j=1}(u) = \frac{u}{p_j} - \frac{1-p_j}{p_j}u^{\frac{1}{1-p_j}}, \quad j \in \{1, \ldots, d\}.$$

Thus, $(U_j|I_j = 0) \stackrel{\mathcal{L}}{=} V_{0,j}$ and $(U_j|I_j = 1) \stackrel{\mathcal{L}}{=} V_{0,j}V_{1,j}$, $j \in \{1, \ldots, d\}$, and the assert follows. □

Obviously, the special case of uniform margins leads to a stochastic representation of the GFGM family of copulas equivalent to (2.3). We notice that for $p = \frac{1}{2}$ we find the stochastic representation of FGM copulas in Blier-Wong et al. (2024b).

Theorem 3.1 provides a very useful representation of the random vector $\boldsymbol{X}$. This helps us to derive plenty of results such as examining the distribution of any integrable function of $\boldsymbol{X}$ and analyzing pairwise dependence properties of $\boldsymbol{X}$. The following corollary considers the expectation of functionals of $\boldsymbol{X}$ for which we derive bounds, in Sect. 5, building on the geometrical structure of Bernoulli vectors, the distribution of the sum $S^{(\boldsymbol{X})} = X_1 + \cdots + X_d$ which may represent aggregate risks in a portfolio and some results on correlation between components of $\boldsymbol{X}$ to analyze dependence corresponding to minimal aggregate risks.

Corollary 3.1 *Let* $\boldsymbol{X} \in \mathcal{G}_d^{\boldsymbol{p}}(F_1, \ldots, F_d)$, *and let* r_k, $k \in \{1, \ldots, n_{\boldsymbol{p}}^{\mathcal{B}}\}$, *be the extremal pmfs of* $\mathcal{B}_d(\boldsymbol{p})$. *The following holds.*

1. *Let* $\varphi : \mathbb{R}^d \to \mathbb{R}$ *be a real-valued function for which the expectation exists, we have*

$$E[\varphi(X_1,\ldots,X_d)] = \sum_{k=1}^{n_{\boldsymbol{p}}^{\mathcal{B}}} \lambda_k \sum_{\boldsymbol{i}\in\{0,1\}^d} r_k(\boldsymbol{i}) E[\varphi(Z_{i_1,1},\ldots,Z_{i_d,d})]. \tag{3.2}$$

2. *The distribution of the sum* $S^{(\boldsymbol{X})} = \sum_j^d X_j$ *is given by*

$$F_{S^{(\boldsymbol{X})}}(y) = \sum_{k=1}^{n_{\boldsymbol{p}}^{\mathcal{B}}} \lambda_k \sum_{\boldsymbol{i}\in\{0,1\}^d} r_k(\boldsymbol{i}) F_{\sum_{j=1}^d Z_{i_j,j}}(y), \quad y \in \mathbb{R}. \tag{3.3}$$

3. *The covariance between each pair of components* (X_{j_1}, X_{j_2}) *of* $\boldsymbol{X}$ *is*

$$Cov(X_{j_1}, X_{j_2}) = Cov(I_{j_1}, I_{j_2})\gamma_{j_1,j_2} = \nu_{j_1,j_2} p_{j_1} p_{j_2} \gamma_{j_1,j_2}, \tag{3.4}$$

where

$$\gamma_{j_1,j_2} = (E[Z_{1,j_1}] - E[Z_{0,j_1}])(E[Z_{1,j_2}] - E[Z_{0,j_2}]), \quad 1 \le j_1 < j_2 \le d.$$

The sharp bounds of the covariance are

$$\begin{aligned}&(\max(p_{j_1} + p_{j_2} - 1, 0) - p_{j_1} p_{j_2})\gamma_{j_1,j_2} \\ &\quad \le Cov(X_{j_1}, X_{j_2}) \le p_{j_1}(1 - p_{j_2})\gamma_{j_1,j_2}, \\ &\quad 1 \le j_1 < j_2 \le d.\end{aligned}$$

4. *If* $\boldsymbol{X}$ *has continuous marginal distributions, Spearman's rho between each pair of components* (X_{j_1}, X_{j_2}) *of* $\boldsymbol{X}$, $1 \le j_1 < j_2 \le d$, *is*

$$\rho_S(X_{j_1}, X_{j_2}) = \frac{3\,Cov(I_{j_1}, I_{j_2})}{(2 - p_{j_1})(2 - p_{j_2})} = \frac{3\,\nu_{j_1,j_2} p_{j_1} p_{j_2}}{(2 - p_{j_1})(2 - p_{j_2})},$$

where $\boldsymbol{I} = (I_1, \ldots, I_d)$ *is the Bernoulli random vector corresponding to* $\boldsymbol{X}$ *of Theorem 3.1. The sharp bounds of the Spearman's rho are given by*

$$\frac{3(\max(p_{j_1} + p_{j_2} - 1, 0) - p_{j_1} p_{j_2})}{(2 - p_{j_1})(2 - p_{j_2})} \le \rho_S(X_{j_1}, X_{j_2}) \le \frac{3 p_{j_1}(1 - p_{j_2})}{(2 - p_{j_1})(2 - p_{j_2})}.$$

Proof From the stochastic representation in (3.1) of Theorem 3.1, we have, for $j \in \{1, \ldots, d\}$,

$$X_j = \begin{cases} Z_{1,j} & \text{if } I_j = 1, \\ Z_{0,j} & \text{if } I_j = 0. \end{cases}$$

Thus, $X_j = Z_{I_j,j}$, $j \in \{1, \ldots, d\}$, and by conditioning on $\boldsymbol{I}$, the expectation of a function φ of $\boldsymbol{X}$ is given by

$$E[\varphi(X_1,\ldots,X_d)] = \sum_{\boldsymbol{i}\in\{0,1\}^d} f_{\boldsymbol{I}}(\boldsymbol{i})E[\varphi(Z_{i_1,1},\ldots,Z_{i_d,d})], \tag{3.5}$$

assuming that the expectations exist.

1. By replacing (2.1) in (3.5), (3.2) follows.
2. By choosing $\varphi(\boldsymbol{x}) = \mathbf{1}\{x_1+\cdots+x_d \le y\}$, where $\mathbf{1}\{A\} = 1$, if A is true, and $\mathbf{1}\{A\} = 0$, otherwise, (3.3) follows.
3. Covariance between X_{j_1} and X_{j_2} in (3.4), $1 \le j_1 < j_2 \le d$, follows by considering $\varphi(\boldsymbol{x}) = x_{j_1}x_{j_2}$ in (3.5).
4. Spearman's rho ρ_S for any pair of continuous rvs (X_{j_1}, X_{j_2}), $1 \le j_1 < j_2 \le d$, is given by

$$\rho_S(X_{j_1}, X_{j_2}) = \rho_P(U_{j_1}, U_{j_2}) = \frac{E[U_{j_1}U_{j_2}] - E[U_{j_1}]E[U_{j_2}]}{\sqrt{Var(U_{j_1})Var(U_{j_2})}}, \tag{3.6}$$

where ρ_P indicates the Pearson's correlation and $Var(U_{j_1}) = Var(U_{j_2}) = \frac{1}{12}$. Using the representation in (2.3), the expression for $E[U_{j_1}U_{j_2}]$, $1 \le j_1 < j_2 \le d$, is

$$E[U_{j_1}U_{j_2}] = E[U_{0,j_1}^{1-p_{j_1}}U_{0,j_2}^{1-p_{j_2}}U_{1,j_1}^{I_{j_1}}U_{1,j_2}^{I_{j_2}}] = E[U_{0,j_1}^{1-p_{j_1}}]E[U_{0,j_2}^{1-p_{j_2}}]E[U_{1,j_1}^{I_{j_1}}U_{1,j_2}^{I_{j_2}}],$$

where

$$E[U_{0,j_1}^{1-p}]E[U_{0,j_2}^{1-p}] = \frac{1}{2-p_{j_1}}\frac{1}{2-p_{j_2}},$$

and

$$E[U_{1,j_1}^{I_{j_1}}U_{1,j_2}^{I_{j_2}}] = \sum_{(i_{j_1},i_{j_2})\in\{0,1\}^2} f_{\boldsymbol{I}}^{(j_1,j_2)}(i_{j_1},i_{j_2})E[U_{1,j_1}^{i_{j_1}}]E[U_{1,j_2}^{i_{j_2}}], \tag{3.7}$$

where $f_{\boldsymbol{I}}^{(j_1,j_2)}$ denotes the (j_1, j_2)-marginal pmf of $\boldsymbol{I} = (I_1,\ldots,I_d)$. Finally, replacing (3.7) in (3.6), the expression of the Spearman's rho is given by

$$\begin{aligned}
&\rho_S(X_{j_1}, X_{j_2}) \\
&\quad = \frac{12f_{\boldsymbol{I}}^{(j_1,j_2)}(0,0) + 6(f_{\boldsymbol{I}}^{(j_1,j_2)}(0,1) + f_{\boldsymbol{I}}^{(j_1,j_2)}(1,0)) + 3f_{\boldsymbol{I}}^{(j_1,j_2)}(1,1)}{(2-p_{j_1})(2-p_{j_2})} - 3 \\
&\quad = \frac{3(f_{\boldsymbol{I}}^{(j_1,j_2)}(1,1) - p_1p_2)}{(2-p_{j_1})(2-p_{j_2})} \\
&\quad = \frac{3\,Cov(I_{j_1}, I_{j_2})}{(2-p_{j_1})(2-p_{j_2})},
\end{aligned}$$

for $1 \leq j_1 < j_2 \leq d$. Assume $p_{j_1} \leq p_{j_2}$. In Section 3.2 of Fontana and Semeraro (2018), the authors found the bounds for the covariance between (I_{j_1}, I_{j_2}) for any pair of Bernoulli variables. The maximum value is $Cov(I_{j_1}, I_{j_2}) = p_{j_1}(1 - p_{j_2})$, while the minimum value is $Cov(I_{j_1}, I_{j_2}) = \max(p_{j_1} + p_{j_2} - 1, 0) - p_{j_1}p_{j_2}$. In the first case, $E[I_{j_1}I_{j_2}] = \Pr(I_{j_1} = 1, I_{j_2} = 1) = p_{j_1}$ and $Cov(I_{j_1}, I_{j_2}) = p_{j_1}(1 - p_{j_2})$. In the second case, $E[I_{j_1}I_{j_2}] = \Pr(I_{j_1} = 1, I_{j_2} = 1) = \max(p_{j_1} + p_{j_2} - 1, 0)$ and $Cov(I_{j_1}, I_{j_2}) = \max(p_{j_1} + p_{j_2} - 1, 0) - p_{j_1}p_{j_2}$. Therefore, we have the following bounds for Spearman's rho of any pair of continuous rvs

$$\frac{3(\max(p_{j_1} + p_{j_2} - 1, 0) - p_{j_1}p_{j_2})}{(2 - p_{j_1})(2 - p_{j_2})} \leq \rho_S(X_{j_1}, X_{j_2})$$
$$= \rho_P(U_{j_1}, U_{j_2}) \leq \frac{3p_{j_1}(1 - p_{j_2})}{(2 - p_{j_1})(2 - p_{j_2})},$$

for $1 \leq j_1 < j_2 \leq d$. □

The relation (3.2) in particular holds for $\varphi(\boldsymbol{X}) = \phi(S^{(\boldsymbol{X})})$ and implies that the bounds of $E[\phi(S^{(\boldsymbol{X})})]$, for any ϕ for which the expectation exists, can be found by enumerating their values on the sums $S^{(\boldsymbol{R}_k)}$, $k \in \{1, \ldots, n_p^{\mathcal{B}}\}$, and this is computationally feasible in low dimension.

We end this section by considering two examples for $\mathcal{G}_d^{\boldsymbol{p}}(F_1, \ldots, F_d)$, where the margins are discrete in the first one and the margins are exponential in the second one.

Example 1 (Discrete margins) Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be defined as a d-dimensional random vector, where, for every $j \in \{1, \ldots, d\}$, X_j is a discrete random variable taking values on the set $A_j = \{0, 1, \ldots, n_j\}$, $n_j \in \mathbb{N}$, and with cdf F_j. When the joint distribution of $\boldsymbol{X}$ belongs to the class $\mathcal{G}_d^{\boldsymbol{p}}(F_1, \ldots, F_d)$, $\boldsymbol{X}$ admits the representation in (3.1). Moreover, for all $j \in \{1, \ldots, d\}$, to derive the values of the pmf of the discrete rvs $Z_{i,j}$, $i = 0, 1$, we observe that, for each $k \in A_j$, we have

$$f_{Z_{0,j}}(k) = \Pr(Z_{0,j} = k) = \Pr(F_j^{-1}(V_{0,j}) = k) = F_{V_{0,j}}(F_j(k)) - F_{V_{0,j}}(F_j(k-1)),$$

and

$$f_{Z_{1,j}}(k) = \Pr(Z_{1,j} = k) = \Pr(F_j^{-1}(V_{0,j}V_{1,j}) = k)$$
$$= F_{V_{0,j}V_{1,j}}(F_j(k)) - F_{V_{0,j}V_{1,j}}(F_j(k-1)),$$

where

$$F_{V_{0,j}}(u) = u^{\frac{1}{1-p_j}}, \quad u \in (0, 1),$$
$$F_{V_{0,j}V_{1,j}}(u) = \frac{u}{p_j} - \frac{1 - p_j}{p_j} u^{\frac{1}{1-p_j}}, \quad u \in (0, 1),$$

and $F_j(-1) := 0$. For all $j \in \{1, \ldots, d\}$, it follows that the expectations of the discrete rvs $Z_{i,j}$, $i = 0, 1$, are

$$E[Z_{0,j}] = n_j - \sum_{k=0}^{n_j-1} F_{V_0,j}(F_j(k)),$$

and

$$E[Z_{1,j}] = n_j - \sum_{k=0}^{n_j-1} F_{V_0V_1,j}(F_j(k)).$$

Finally, we consider the sum $S^{(\boldsymbol{X})} = X_1 + \cdots + X_d$, which can be rewritten using the representation in (3.1), as

$$S^{(\boldsymbol{X})} = \sum_{j=1}^{d} (1 - I_j) Z_{0,j} + I_j Z_{1,j}. \tag{3.8}$$

From (3.8), the expression of the probability generating function (pgf) of $S^{(\boldsymbol{X})}$ is given by

$$\mathcal{P}_{S^{(\boldsymbol{X})}}(s) = \sum_{\boldsymbol{i} \in \{0,1\}^d} f_{\boldsymbol{I}}(\boldsymbol{i}) \prod_{j=1}^{d} \mathcal{P}_{Z_{i_j,j}}(s), \quad s \in [-1, 1], \tag{3.9}$$

where the pgf of $Z_{i_j,j}$ is

$$\mathcal{P}_{Z_{i_j,j}}(s) = E[s^{Z_{i_j,j}}] = \sum_{k=0}^{n_j} f_{Z_{i_j,j}}(k) s^k, \quad s \in [-1, 1],$$

for $i_j \in \{0, 1\}$ and $j \in \{1, \ldots, d\}$. One uses the Fast Fourier Transform (`FFT`) algorithm of Cooley and Tukey (1965) to extract the values of the pmf of $S^{(\boldsymbol{X})}$ from its pgf in (3.9). Details about that efficient approach is explained in Chapter 30 of Cormen et al. (2009). See also Embrechts et al. (1993) for FFT applications in actuarial science and quantitative risk management. This procedure is illustrated in Example 6, within Sect. 7. □

Example 2 (Exponential margins) Assume that $\boldsymbol{X} \in \mathcal{G}_d^{\boldsymbol{p}}(F_1, \ldots, F_d)$, where F_j is the cdf of an exponential distribution with mean $\frac{1}{\lambda_j}$, $j \in \{1, \ldots, d\}$. In Blier-Wong et al. (2024a), they present an alternative stochastic representation of $\boldsymbol{X}$ equivalent to (3.1) that allows finding an analytical expression for the distribution of the sum of the components of the random vector $\boldsymbol{X}$. This other stochastic representation is

$$\boldsymbol{X} = \boldsymbol{W}_1 + \boldsymbol{I}\boldsymbol{W}_2, \tag{3.10}$$

where the components $W_{1,j}$ within $\boldsymbol{W}_1$ and $W_{2,j}$ within $\boldsymbol{W}_2$ are independent, with $W_{1,j}$ following an exponential distribution with mean $\frac{1-p_j}{\lambda_j}$ and $W_{2,j}$ also following an exponential distribution but with mean $\frac{1}{\lambda_j}$, for $j \in \{1, \dots, d\}$. The random vectors $\boldsymbol{I}$, $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ are independent.

From (3.10), Pearson's coefficient becomes

$$\rho_P(X_{j_1}, X_{j_2}) = \frac{Cov(I_{j_1}, I_{j_2})E[W_{2,j_1}]E[W_{2,j_2}]}{\sqrt{Var(X_{j_1})Var(X_{j_2})}} = Cov(I_{j_1}, I_{j_2}) \tag{3.11}$$

since $E[W_{2,j_1}] = \frac{1}{\lambda_{j_1}}$, $E[W_{2,j_2}] = \frac{1}{\lambda_{j_2}}$, $Var(X_{j_1}) = \frac{1}{\lambda_{j_1}^2}$, and $Var(X_{j_2}) = \frac{1}{\lambda_{j_2}^2}$, $1 \leq j_1 < j_2 \leq d$.

Considering now the sum $S^{(\boldsymbol{X})}$ of the components of $\boldsymbol{X}$, the representation in (3.10) allows us to express it as follows:

$$S^{(\boldsymbol{X})} \stackrel{\mathcal{L}}{=} W_{1,1} + I_1 W_{2,1} + \cdots + W_{1,d} + I_d W_{2,d} = \sum_{j=1}^{d} W_{1,j} + \sum_{j=1}^{d} I_j W_{2,j}. \tag{3.12}$$

To identify the distribution of $S^{(\boldsymbol{X})}$, we find from (3.12), the expression of the Laplace-Stieltjes transform (LST) of $S^{(\boldsymbol{X})}$, denoted by $\mathcal{L}_{S^{(\boldsymbol{X})}}$ and given by

$$\mathcal{L}_{S^{(\boldsymbol{X})}}(t) = \prod_{j=1}^{d} \mathcal{L}_{W_{1,j}}(t) \left(\sum_{\boldsymbol{i} \in \{0,1\}^d} f_{\boldsymbol{I}}(\boldsymbol{i}) \prod_{j=1}^{d} \left(\mathcal{L}_{W_{2,j}}(t)\right)^{i_j} \right), \quad t \geq 0. \tag{3.13}$$

Using techniques explained in Willmot and Woo (2007) and Cossette et al. (2013), the LST of S admits the representation given by

$$\mathcal{L}_{S^{(\boldsymbol{X})}}(t) = \sum_{k=0}^{\infty} \eta_k \left(\frac{\beta}{\beta + t} \right)^{d+k}, \quad t \geq 0,$$

where $\beta = \max\left(\lambda_1, \dots, \lambda_d, \frac{\lambda_1}{1-p_1}, \dots, \frac{\lambda_d}{1-p_d}\right)$, $\eta_k \geq 0$ for $k \geq 0$, and $\sum_{k=0}^{\infty} \eta_k = 1$. From the expression of its LST in (3.13), it follows that the rv S follows a mixed Erlang distribution. An application of (3.13) is provided in Example 5. Details about the computation of the sequence of probabilities $\{\eta_k, k \in \mathbb{N}_0\}$ are explained in Willmot and Woo (2007) and Cossette et al. (2013). □

In Example 2, we find the distribution of S, which comes with an analytical expression for F_S because we assume that the margins are exponential. In most cases, if the margins do not belong to a class of distributions closed under convolution, one must resort to numerical approximations. One of them is to use discretization techniques as explained in Section 4.3 of Blier-Wong et al. (2023) jointly with the method explained in Example 1.

4 Main result: minimal convex sums under GFGM dependence

Based on the results recalled and highlighted in Sect. 2, we provide in this section the geometrical structure embedded within the class of sums of components of random vectors whose joint distribution belongs to the class of d-variate GFGM copulas with a common parameter p, that is, the class $\mathcal{C}_d^p$. Then, we are able to present a new result about convex order, as stated in Theorem 4.2.

Let us begin with the following proposition, which is a direct consequence of the one-to-one relationship given in (2.2).

Proposition 4.1 *Let $\boldsymbol{I} \in \mathcal{B}_d(p)$ and let $S^{(\boldsymbol{I})} = \sum_{j=1}^{d} I_j$. Then, there exists an exchangeable Bernoulli random vector $\boldsymbol{I}^e \in \mathcal{E}_d(p)$ such that $S^{(\boldsymbol{I}^e)} = \sum_{j=1}^{d} I_j^e \in \mathcal{D}_d(dp)$ has the same distribution as $S^{(\boldsymbol{I})}$.*

Proof Given $\boldsymbol{I} \in \mathcal{B}_d(p)$, the sum of the components $S^{(\boldsymbol{I})} = \sum_{j=1}^{d} I_j$ is a random variable that takes values in the set $\{0, 1, \ldots, d\}$ and whose mean is $\mathrm{E}[S^{(\boldsymbol{I})}] = dp$, that is $S^{(\boldsymbol{I})} \in \mathcal{D}_d(dp)$. Therefore, from (2.2), it follows that there exists one exchangeable Bernoulli random vector $\boldsymbol{I}^e \in \mathcal{E}_d(p)$ such that $S^{(\boldsymbol{I}^e)} = \sum_{j=1}^{d} I_j^e \stackrel{\mathcal{L}}{=} S^{(\boldsymbol{I})}$. □

Let $\mathcal{S}_d^p$ denote the class of sums of components of random vectors with multivariate distributions in $\mathcal{C}_d^p$. Let us indicate with $S^{(\boldsymbol{U},\boldsymbol{I})} = \sum_{j=1}^{d} U_j^{(\boldsymbol{I})}$, where $\boldsymbol{U}^{(\boldsymbol{I})}$ is the random vector whose joint cdf is the copula C associated to $\boldsymbol{I} \in \mathcal{B}_d(p)$, as defined in (2.3). In order to study the geometrical structure of $\mathcal{S}_d^p$, we need to investigate the correspondence between distributions belonging to this class and multivariate Bernoulli distributions of the class $\mathcal{B}_d(p)$.

Lemma 4.1 *Let $\boldsymbol{I}, \boldsymbol{I}' \in \mathcal{B}_d(p)$ and let $S^{(\boldsymbol{I})} = \sum_{j=1}^{d} I_j$ and $S^{(\boldsymbol{I}')} = \sum_{j=1}^{d} I_j'$. Let C and C' be the GFGM(p) copulas corresponding to $\boldsymbol{I}$ and $\boldsymbol{I}'$, respectively. Finally, let $\boldsymbol{U}$ and $\boldsymbol{U}'$ be uniform random vectors with joint cdfs C and C', respectively. If $S^{(\boldsymbol{I})} \stackrel{\mathcal{L}}{=} S^{(\boldsymbol{I}')}$, then $\sum_{j=1}^{d} U_j \stackrel{\mathcal{L}}{=} \sum_{j=1}^{d} U_j'$.*

Proof Let F and F' be the cdfs of $\sum_{j=1}^{d} U_j$ and $\sum_{j=1}^{d} U_j'$, respectively. From (2.3), we have the following stochastic representation:

$$U_j = U_{0,j}^{1-p} U_{1,j}^{I_j}, \quad j \in \{1, \ldots, d\},$$

where $\boldsymbol{U}_0 = (U_{0,1}, \ldots, U_{0,d})$ and $\boldsymbol{U}_1 = (U_{1,1}, \ldots, U_{1,d})$ are vectors of independent standard uniform random variables and $\boldsymbol{U}_0$, $\boldsymbol{U}_1$, and $\boldsymbol{I} = (I_1, \ldots, I_d)$ are independent. Then, we have

$$\begin{aligned} F(x) &= \Pr\left(\sum_{j=1}^{d} U_j \leq x\right) = \Pr\left(\sum_{j=1}^{d} U_{0,j}^{1-p} U_{1,j}^{I_j} \leq x\right) \\ &= \sum_{\boldsymbol{i}\in\{0,1\}^d} \Pr\left(\sum_{j=1}^{d} U_{0,j}^{1-p} U_{1,j}^{i_j} \leq x\right) f_{\boldsymbol{I}}(\boldsymbol{i}) \\ &= \sum_{\boldsymbol{i}\in\{0,1\}^d} \Pr\left(\sum_{j:i_j=1} U_{0,j}^{1-p} U_{1,j} + \sum_{j:i_j=0} U_{0,j}^{1-p} \leq x\right) f_{\boldsymbol{I}}(\boldsymbol{i}), \quad x \in [0, d]. \end{aligned} \tag{4.1}$$

Since $\boldsymbol{U}_0$ and $\boldsymbol{U}_1$ are independent vectors of independent and identically distributied (iid) random variables, the distribution of the sum $\sum_{j:i_j=1} U_{0,j}^{1-p} U_{1,j} + \sum_{j:i_j=0} U_{0,j}^{1-p}$ does not depend on the position of the 1's in $\boldsymbol{i}$, but only on the number of 1's, that is, on the sum of the components $\sum_{j=1}^{d} i_j$. Hence, the cdf in (4.1) becomes

$$F(x) = \sum_{k=0}^{d} \sum_{\boldsymbol{i}\in\mathcal{X}_d^k} \Pr\left(\sum_{j=1}^{k} U_{0,j}^{1-p} U_{1,j} + \sum_{j=k+1}^{d} U_{0,j}^{1-p} \leq x\right) f_{\boldsymbol{I}}(\boldsymbol{i}), \tag{4.2}$$

where $\mathcal{X}_d^k = \{\boldsymbol{i} \in \{0,1\}^d : \sum_{j=1}^{d} i_j = k\}$ and we set $\sum_{j=1}^{0} U_{0,j}^{1-p} U_{1,j} := 0$ and $\sum_{j=d+1}^{d} U_{0,j}^{1-p} := 0$. It follows that

$$F(x) = \sum_{k=0}^{d} \Pr\left(\sum_{j=1}^{k} U_{0,j}^{1-p} U_{1,j} + \sum_{j=k+1}^{d} U_{0,j}^{1-p} \leq x\right) f_{S^{(\boldsymbol{I})}}(k), \quad x \in [0, d]. \tag{4.3}$$

Therefore, given that the first probability in the summation of (4.3) does not depend on $\boldsymbol{i}$ for $k \in \{0, \ldots, d\}$, we conclude that $S^{(\boldsymbol{I})} \stackrel{\mathcal{L}}{=} S^{(\boldsymbol{I}')}$ implies $F(x) = F'(x)$, for every $x \in [0, d]$. □

We can now prove the following theorem that characterizes $\mathcal{S}_d^p$.

Theorem 4.1 *The class $\mathcal{S}_d^p$ is a convex polytope and its extremal points are the cdfs $F_{S^{(\boldsymbol{U},\boldsymbol{E}_k)}}$ of the random variables $S^{(\boldsymbol{U},\boldsymbol{E}_k)} = \sum_{j=1}^{d} U_j^{(\boldsymbol{E}_k)}$, where $\boldsymbol{U}^{(\boldsymbol{E}_k)} \stackrel{\mathcal{L}}{=} \boldsymbol{U}_0^{1-p} \boldsymbol{U}_1^{(\boldsymbol{E_k})}$, and $\boldsymbol{E}_k$ is an extremal point of $\mathcal{E}_d(p)$, for $k \in \{1, \ldots, n_p^{\mathcal{D}}\}$.*

Proof Consider any $S \in \mathcal{S}_d^p$, then there exists $\boldsymbol{I} \in \mathcal{B}_d(p)$ such that $S = \sum_{i=1}^{d} U_i^{(\boldsymbol{I})}$. By Proposition 4.1, there exists a unique $\boldsymbol{I}^e \in \mathcal{E}_d(p)$ with $\sum_{j=1}^{d} I_j^e \stackrel{\mathcal{L}}{=} \sum_{j=1}^{d} I_j$. Thus by Lemma 4.1, $S \stackrel{\mathcal{L}}{=} \sum_{j=1}^{d} U_j^e$, where $\boldsymbol{U}^e \in \mathcal{C}_d^p$ is the uniform random vector with copula corresponding to $\boldsymbol{I}^e$. In other words, $\boldsymbol{U}^e$ admits the representation $\boldsymbol{U}^e \stackrel{\mathcal{L}}{=} \boldsymbol{U}_0^{1-p}\boldsymbol{U}_1^{\boldsymbol{I}^e}$, where $\boldsymbol{I}^e \in \mathcal{E}_d(p)$. Let F_S be the cdf of S. Then, we have

$$
\begin{aligned}
F_S(x) &= \Pr\left(\sum_{j=1}^{d} U_j^e \le x\right) = \Pr\left(\sum_{j=1}^{d} U_{0,j}^{1-p} U_{1,j}^{I_j^e} \le x\right) \\
&= \sum_{\boldsymbol{i}\in\{0,1\}^d} \Pr\left(\sum_{j=1}^{d} U_{0,j}^{1-p} U_{1,j}^{i_j} \le x\right) f_{\boldsymbol{I}^e}(\boldsymbol{i}) \\
&= \sum_{k=1}^{n_p^{\mathcal{D}}} \lambda_k \sum_{\boldsymbol{i}\in\{0,1\}^d} \Pr\left(\sum_{j=1}^{d} U_{0,j}^{1-p} U_{1,j}^{i_j} \le x\right) f_{\boldsymbol{E}_k}(\boldsymbol{i}) \\
&= \sum_{k=1}^{n_p^{\mathcal{D}}} \lambda_k \Pr\left(\sum_{j=1}^{d} U_j^{(\boldsymbol{E}_k)} \le x\right) \\
&= \sum_{k=1}^{n_p^{\mathcal{D}}} \lambda_k F_{S^{(\boldsymbol{U},\boldsymbol{E}_k)}}(x),
\end{aligned}
$$

where $F_{S^{(\boldsymbol{U},\boldsymbol{E}_k)}}$ is the cdf of $S^{(\boldsymbol{U},\boldsymbol{E}_k)} = \sum_{j=1}^{d} U_j^{(\boldsymbol{E}_k)}$. □

From Theorem 4.1, we can complete the relationship between classes in (2.2) as follows:

$$\mathcal{S}_d^p \leftrightarrow \mathcal{E}_d(p) \leftrightarrow \mathcal{D}_d(dp).$$

We now study the convex order of sums of the components of random vectors with joint distribution described by a GFGM(p) copula. We first recall the definition of the convex order.

Definition 4.1 Given two random variables X and Y with finite means, X is said to be smaller than Y in the convex order (denoted $X \preceq_{cx} Y$) if $E[\phi(X)] \le E[\phi(Y)]$, for all real-valued convex functions ϕ for which the expectations exist.

In the proof of the following Theorem 4.2, we need to recourse to the supermodular order that we recall below (see Definition 3.8.5 in Müller and Stoyan (2002)). A function $\varphi : \mathbb{R}^d \to \mathbb{R}$ is said to be supermodular if $\varphi(\boldsymbol{x}) + \varphi(\boldsymbol{y}) \le \varphi(\boldsymbol{x} \vee \boldsymbol{y}) + \varphi(\boldsymbol{x} \wedge \boldsymbol{y})$, where the operators $\wedge$ and $\vee$ denote coordinatewise minimum and maximum respectively.

Definition 4.2 We say that $\boldsymbol{U}$ is smaller than $\boldsymbol{U}'$ under the supermodular order, denoted $\boldsymbol{U} \preceq_{sm} \boldsymbol{U}'$, if $E[\varphi(\boldsymbol{U})] \le E[\varphi(\boldsymbol{U}')]$ for all supermodular functions φ, given that the expectations exist.

Theorem 4.2 *Let $\boldsymbol{I}, \boldsymbol{I}' \in \mathcal{B}_d(p)$ be such that $\sum_{j=1}^{d} I_j \preceq_{cx} \sum_{j=1}^{d} I'_j$. Let C and C' be the GFGM copulas associated to $\boldsymbol{I}$ and $\boldsymbol{I}'$ and let $\boldsymbol{U}$ and $\boldsymbol{U}'$ be uniform random vectors with joint cdf C and C', respectively. Then, $\sum_{j=1}^{d} U_j \preceq_{cx} \sum_{j=1}^{d} U'_j$.*

Proof By Proposition 4.1, there exist two exchangeable Bernoulli random vectors $\boldsymbol{I}^e$ and $\boldsymbol{I}^{e'}$ of the class $\mathcal{E}_d(p)$ such that $\sum_{j=1}^{d} I_j \stackrel{\mathcal{L}}{=} \sum_{j=1}^{d} I_j^e$ and $\sum_{j=1}^{d} I'_j \stackrel{\mathcal{L}}{=} \sum_{j=1}^{d} I_j^{e'}$. Therefore, by Lemma 4.1, we have

$$\sum_{j=1}^{d} U_j \stackrel{\mathcal{L}}{=} \sum_{j=1}^{d} U_j^e \quad \text{and} \quad \sum_{j=1}^{d} U'_j \stackrel{\mathcal{L}}{=} \sum_{j=1}^{d} U_j^{e'}, \tag{4.4}$$

where $\boldsymbol{U}^e$ and $\boldsymbol{U}^{e'}$ are uniform random vectors with joint distributions given by the GFGM copulas associated to $\boldsymbol{I}^e$ and $\boldsymbol{I}^{e'}$, respectively. Moreover, since $\sum_{j=1}^{d} I_j \preceq_{cx} \sum_{j=1}^{d} I'_j$ by hypothesis, then $\sum_{j=1}^{d} I_j^e \preceq_{cx} \sum_{j=1}^{d} I_j^{e'}$. However, as a consequence of results in Section 3 of Frostig (2001), the following double implication holds:

$$\sum_{j=1}^{d} I_j^e \preceq_{cx} \sum_{j=1}^{d} I_j^{e'} \iff \boldsymbol{I}^e \preceq_{sm} \boldsymbol{I}^{e'}.$$

Furthermore, by Theorem 4.2 of Blier-Wong et al. (2022b), $\boldsymbol{I}^e \preceq_{sm} \boldsymbol{I}^{e'}$ implies $\boldsymbol{U}^e \preceq_{sm} \boldsymbol{U}^{e'}$. Since, given a convex function $\phi : \mathbb{R} \to \mathbb{R}$, the function $\psi(\boldsymbol{x}) = \phi(x_1 + \ldots + x_d)$ is supermodular, $\boldsymbol{I}^e \preceq_{sm} \boldsymbol{I}^{e'}$ implies, in particular, $\sum_{j=1}^{d} U_j^e \preceq_{cx} \sum_{j=1}^{d} U_j^{e'}$. Finally, by the equality in distribution in (4.4), we have $\sum_{j=1}^{d} U_j \preceq_{cx} \sum_{j=1}^{d} U'_j$. □

Theorem 4.2 obviously holds for FGM copulas by setting $p = \frac{1}{2}$.

4.1 Class $\mathcal{G}_d^p(F)$ of distributions

Let us introduce the class $\mathcal{G}_d^p(F)$ of joint cdfs with a copula in the class $\mathcal{C}_d^p$ and with the same marginal cdfs F. Consider $\boldsymbol{X} \in \mathcal{G}_d^p(F)$. From Theorem 3.1, there exists $\boldsymbol{I} \in \mathcal{B}_d(p)$ such that $\boldsymbol{X}$ is built from the Bernoulli random vector $\boldsymbol{I}$, according to the stochastic representation (3.1). Using this representation, we can generalize Lemma 4.1 as follows.

Lemma 4.2 *Let $\boldsymbol{I}, \boldsymbol{I}' \in \mathcal{B}_d(p)$ be such that $S^{(\boldsymbol{I})} = \sum_{j=1}^{d} I_j$ and $S^{(\boldsymbol{I}')} = \sum_{j=1}^{d} I'_j$ are equal in distribution. Let $\boldsymbol{X}, \boldsymbol{X}' \in \mathcal{G}_d^p(F)$ be the random vectors corresponding to $\boldsymbol{I}$ and $\boldsymbol{I}'$, respectively. Then, $\sum_{j=1}^{d} X_j$ and $\sum_{j=1}^{d} X'_j$ have the same distribution.*

Proof Similarly to the proof of Lemma 4.1, Lemma 4.2 is proved observing that $\boldsymbol{Z}_0$ and $\boldsymbol{Z}_1$ have independent and identically distributed components. □

Again, the stochastic representation in (3.1) helps us to find the following generalization of Theorem 4.1.

Theorem 4.3 *The class $\mathcal{S}_d^p(F)$ of distributions of sums of components of vectors with distribution in $\mathcal{G}_d^p(F)$ is a convex polytope and its extremal points are the distributions of $S^{(\boldsymbol{X},\boldsymbol{E}_k)} = \sum_{j=1}^d X_j^{(\boldsymbol{E}_k)}$ where*

$$\boldsymbol{X}^{(\boldsymbol{E}_k)} = (\mathbf{1} - \boldsymbol{E}_k)\boldsymbol{Z}_0 + \boldsymbol{E}_k\boldsymbol{Z}_1, \tag{4.5}$$

where $\boldsymbol{E}_k$ is an extremal point of $\mathcal{E}_d(p)$, for $k \in \{1, \ldots, n_p^{\mathcal{D}}\}$.

Proof The proof is similar to the one of Theorem 4.1. □

Theorem 4.3 provides the following relationship:

$$\mathcal{S}_d^p(F) \leftrightarrow \mathcal{S}_d^p \leftrightarrow \mathcal{E}_d(p) \leftrightarrow \mathcal{D}_d(dp).$$

From this relationship, it follows that the number of extremal points in $\mathcal{S}_d^p(F)$ is $n_p^{\mathcal{D}}$, that is significantly lower than the number of extremal points in $\mathcal{G}_d^p(F)$ and they are analytical. Therefore we can also find them in high dimensions.

Finally, we have the following generalization of Theorem 4.2.

Theorem 4.4 *Let $\boldsymbol{I}, \boldsymbol{I}' \in \mathcal{B}_d(p)$ and let $\boldsymbol{X} \in \mathcal{G}_d^p(F)$ and $\boldsymbol{X}' \in \mathcal{G}_d^p(F)$, for any cdf F, be respectively built from $\boldsymbol{I}$ and $\boldsymbol{I}'$, as in (3.1). Then,*

$$\sum_{j=1}^d I_j \preceq_{cx} \sum_{j=1}^d I'_j \implies \sum_{j=1}^d X_j \preceq_{cx} \sum_{j=1}^d X'_j.$$

Proof The proof of this Theorem is along the same lines at the one of Theorem 4.2. □

We conclude this section by considering the two examples with exponential and discrete margins previously discussed in Example 1 and Example 2 but here in the special case of identically distributed risks.

Example 1 (Discrete margins, **continued**) For the case of identically distributed discrete margins, we obtain the following expression for the pmf of $S^{(\boldsymbol{X})}$:

$$f_{S^{(\boldsymbol{X})}}(k) = \sum_{j=0}^d f_{S^{(\boldsymbol{I})}}(j) f_{Z_0}^{*(d-j)} * f_{Z_1}^{*j}(k), \quad k \in \mathbb{N}, \tag{4.6}$$

where * denotes the convolution product and f^{*j} denotes the j-fold convolution product of the probability density function measure f with itself. It follows from (3.9) that the pgf of S is given by

$$\mathcal{P}_{S^{(\boldsymbol{X})}}(s) = \sum_{j=0}^{d} f_{S^{(\boldsymbol{I})}}(j)\mathcal{P}_{Z_0}^{d-j}(s)\mathcal{P}_{Z_1}^{j}(s), \quad s \in [0,1], \tag{4.7}$$

where $\mathcal{P}_{Z_0}$ and $\mathcal{P}_{Z_1}$ are the pgf of Z_0 and Z_1, respectively. As previously mentioned, it is more convenient to use the FFT algorithm to extract the values of the pmf of $S^{(\boldsymbol{X})}$ from the pgf in (4.7) rather than finding those values directly from (4.6). The procedure is illustrated in Example 6 provided in Sect. 7. □

Example 2 (Exponential margins, **continued**) Firstly, assume $\boldsymbol{X} \in \mathcal{G}_d^p(Exp(\lambda))$, that is $X_j \sim Exp(\lambda)$ for every $j \in \{1,\ldots,d\}$, with GFGM(p) copula. The LST of $S^{(\boldsymbol{X})}$ in (3.13) becomes

$$\mathcal{L}_{S^{(\boldsymbol{X})}}(t) = (\mathcal{L}_{W_1}(t))^d \left(\sum_{j=0}^{d} f_{S^{(\boldsymbol{I})}}(j) \left(\mathcal{L}_{W_2}(t)\right)^j\right), \quad t \geq 0,$$

where $S^{(\boldsymbol{I})} = \sum_{j=1}^{d} I_j$. By identification of the LST, we conclude

$$F_{S^{(\boldsymbol{X})}}(x) = f_{S^{(\boldsymbol{I})}}(0)F_{W_1}^{*d}(x) + \sum_{j=1}^{d} f_{S^{(\boldsymbol{I})}}(j)F_{W_1}^{*d} * F_{W_2}^{*j}(x), \quad x \geq 0,$$

where $F_{W_1}^{*d}$ corresponds to the cdf of an $Erlang(d, \frac{\lambda}{1-p})$ distribution and $F_{W_2}^{*j}$ corresponds to the cdf of an $Erlang(j,\lambda)$ distribution for $j \in \{1,\ldots,d\}$. □

4.2 Minimal convex sums

The solution to the problem of finding the distribution of the vectors with minimal convex sums is a corollary of our main theorems, Theorem 4.2 and Theorem 4.4. Before introducing it, we need some notations. Following Definition 3.4 of Puccetti and Wang (2015), we say that $\boldsymbol{X} \in \mathcal{F}$ is a Σ_{cx}-smallest element in a class of distributions $\mathcal{F}$ if, for all $\boldsymbol{X}' \in \mathcal{F}$, $\sum_{j=1}^{d} X_j \preceq_{cx} \sum_{j=1}^{d} X'_j$. A Σ_{cx}-smallest element in a Fréchet class does not always exist, see Example 3.1 of Bernard et al. (2014). However, the authors of Hu and Wu (1999) found the distribution of the exchangeable Bernoulli random vector that is the Σ_{cx}-smallest element in the class of exchangeable Bernoulli pmfs $\mathcal{E}_d(p)$. Since there is a one-to-one correspondence between $\mathcal{E}_d(p)$ and $\mathcal{D}_d(dp)$, see (2.2), and the sums of the components of a random vector with pmf in $\mathcal{B}_d(p)$ are rvs with distribution in $\mathcal{D}_d(dp)$, a Σ_{cx}-smallest element always exists in the class $\mathcal{B}_d(p)$. Actually, for each $p \in (0,1)$, we can also build non-exchangeable Σ_{cx}-smallest elements of $\mathcal{B}_d(p)$ following Theorem 5.2 of Fontana and Semeraro

(2024). In the proof of Lemma 3.1 of Bernard et al. (2017), the authors provide a way to construct a random variable with Σ_{cx}-smallest pmf. Let $\boldsymbol{I} \in \mathcal{B}_d(p)$ and let $\boldsymbol{U}$ be the corresponding uniform random vector with GFGM(p) copula. The following corollary is a straightforward but important consequence of Theorem 4.2 and Theorem 4.4.

Corollary 4.1 *Let $\boldsymbol{I} \in \mathcal{B}_d(p)$ be a Σ_{cx}-smallest element.*

1. *Let $\boldsymbol{U}$ be a uniform random vector whose joint cdf is the GFGM(p) copula corresponding to $\boldsymbol{I}$. Then, $\boldsymbol{U}$ is a Σ_{cx}-smallest element in $\mathcal{C}_d^p$.*
2. *Let $\boldsymbol{X} \in \mathcal{G}_d^p(F)$ with joint cdf defined with the GFGM(p) copula corresponding to $\boldsymbol{I}$. Then, $\boldsymbol{X}$ is a Σ_{cx}-smallest element in $\mathcal{G}_d^p(F)$.*

Consequently, distributions, for which the lower bounds of a convex functional are reached, are built using a Σ_{cx}-smallest element of $\mathcal{B}_d(p)$. Obviously, using the upper Fréchet bound of $\mathcal{B}_d(p)$, we build the distributions of vectors with maximal convex sums.

5 Sharp bounds for risk measures

This section finds sharp bounds for functions of aggregated risks represented by random variables with cdf in $\mathcal{S}_d^{\boldsymbol{p}}(F_1, \ldots, F_d)$. As a consequence of Corollary 2.1, to derive sharp bounds for risk measures in the class $\mathcal{S}_d^{\boldsymbol{p}}(F_1, \ldots, F_d)$, we can proceed by enumerating their values on the extremal points. This is computationally expensive because the number of extremal points $n_p^{\mathcal{B}}$ explodes and becomes larger, as highlighted by the authors of Fontana and Semeraro (2024). In this section, we illustrate how this problem can be solved by finding sharp bounds for risk measures in the classes $\mathcal{C}_d^p$ and $\mathcal{G}_d^p(F_X)$, for any $p \in (0, 1)$.

As a motivation, we start with an example showing that the assumption of common margins in Lemma 4.2 and Theorem 4.4 is necessary. Note that there is no restriction on the chosen discrete distributions.

Example 3 Consider the class $\mathcal{G}_3^{2/5}(F_1, F_2, F_3)$, where F_1, F_2, and F_3 are the discrete cdfs provided in Table 2.

Let $\boldsymbol{X}$, $\boldsymbol{X}'$, and $\boldsymbol{X}'' \in \mathcal{G}_3^{2/5}(F_1, F_2, F_3)$ with copulas respectively defined by the Bernoulli rvs $\boldsymbol{I}$, $\boldsymbol{I}'$, and $\boldsymbol{I}''$ respectively distributed according to $\boldsymbol{f}$, $\boldsymbol{f}'$, and $\boldsymbol{f}'' \in \mathcal{B}_3(\frac{2}{5})$ given in Table 3.

Table 2 Marginal cdfs

k	$F_1(k)$	$F_2(k)$	$F_3(k)$
0	0.1	0.1	0.8
1	0.2	0.4	1.0
2	0.3	0.7	1.0
3	1.0	1.0	1.0

Table 3 Bernoulli pmfs

$\boldsymbol{i}$	(0,0,0)	(1,0,0)	(0,1,0)	(1,1,0)	(0,0,1)	(1,0,1)	(0,1,1)	(1,1,1)
$\boldsymbol{f}$	0	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	0	0	0
$\boldsymbol{f}'$	$\frac{1}{5}$	0	$\frac{2}{5}$	0	0	$\frac{2}{5}$	0	0
$\boldsymbol{f}''$	0	$\frac{2}{5}$	$\frac{1}{5}$	0	$\frac{1}{5}$	0	$\frac{1}{5}$	0

Table 4 Pmfs of $S^{(\boldsymbol{X})}$ and $S^{(\boldsymbol{X}')}$

k	0	1	2	3	4	5	6	7	8	9
$f_{S^{(\boldsymbol{X})}}(k)$	0.0080	0.0338	0.0640	0.1328	0.2467	0.2592	0.2312	0.0242	0	0
$f_{S^{(\boldsymbol{X}')}}(k)$	0.0032	0.0249	0.0602	0.1556	0.2636	0.2569	0.2004	0.0352	0	0
$f_{S^{(\boldsymbol{X}'')}}(k)$	0.0029	0.0214	0.0549	0.1588	0.2798	0.2521	0.1976	0.0324	0	0

Table 4 exhibits the values of pmfs of the discrete random variables $S^{(\boldsymbol{X})} = \sum_{j=1}^{3} X_j$, $S^{(\boldsymbol{X}')} = \sum_{j=1}^{3} X'_j$, and $S^{(\boldsymbol{X}'')} = \sum_{j=1}^{3} X''_j$; those values are computed using the pgf and the `FFT` algorithm as explained in Example 1. As one can see by looking at the support of the Bernoulli distributions in Table 3, although $\boldsymbol{I}$ and $\boldsymbol{I}''$ have the same distribution of the sum, $S^{(\boldsymbol{X})}$ and $S^{(\boldsymbol{X}'')}$ are not equal in distribution. Moreover, it is easy to show that $S^{(\boldsymbol{I})} = \sum_{j=1}^{3} I_j \preceq_{cx} S^{(\boldsymbol{I}')} = \sum_{j=1}^{3} I'_j$, but $S^{(\boldsymbol{X})} \npreceq_{cx} S^{(\boldsymbol{X}')}$, since $Var(S^{(\boldsymbol{X})}) = 2.0633 \geq 1.8865 = Var(S^{(\boldsymbol{X}')})$. □

We consider two convex risk measures, the widely used expected shortfall (ES) and the entropic risk measure. Then we consider, consistently with financial regulations such as Basel III or Solvency II, the value-at-risk (VaR), which is not a convex measure. Below, we recall the definition of these three measures of risk.

Definition 5.1 Let Y be a random variable representing a loss with finite mean. Then, the value-at-risk at level $\alpha \in (0, 1)$ is given by

$$\mathrm{VaR}_\alpha(Y) = \inf\{y \in \mathbb{R} : \Pr(Y \leq y) \geq \alpha\}.$$

Definition 5.2 Let Y be a random variable representing a loss with finite mean. The expected shortfall at level $\alpha \in (0, 1)$ is defined as

$$\mathrm{ES}_\alpha(Y) = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_u(Y) \mathrm{d}u. \tag{5.1}$$

The expected shortfall defined in (5.1) also admits the following representation

$$\mathrm{ES}_\alpha(Y) = \mathrm{VaR}_\alpha(Y) + \frac{1}{1-\alpha} E[\max(Y - \mathrm{VaR}_\alpha(Y), 0)], \; for \;\; \alpha \in (0, 1).$$

Definition 5.3 The entropic risk measure is defined by

$$\Psi_\gamma(Y) = \frac{1}{\gamma}\log(E[e^{\gamma Y}]),$$

assuming that there exists a real number $\gamma_0 > 0$ such that $E[e^{\gamma Y}]$ is finite for $\gamma \in (0, \gamma_0)$.
Using the results from Corollary 4.1, we can analytically find the lower bounds of the convex risk measures considered for exponential margins and discrete margins, respectively. Although the VaR_α is not convex we can find its bounds in $\mathcal{S}_d^p(F)$. The authors of Fontana et al. (2021) prove that the bounds of the VaR_α in a class of univariate distributions that has a convex polytope structure are reached at the extremal points. We consider a random vector $\boldsymbol{X}$ with distribution in the class $\mathcal{G}_d^p(F)$, whose sum $S^{(\boldsymbol{X})} = X_1 + \cdots + X_d$ have cdf in $\mathcal{S}_d^p(F)$. Despite the number of extremal points of $\mathcal{G}_d^p(F)$ being $n_p^{\mathcal{B}}$, as a consequence of Theorem 4.3 we can restrict our attention to the $n_p^{\mathcal{D}}$ extremal points of $\mathcal{S}_d^p(F)$. By computing the VaR_α of the random variable $S^{(\boldsymbol{X},\boldsymbol{E}_k)}$, for every $k \in \{1, \ldots, n_p^{\mathcal{D}}\}$, it is possible to find maximum and minimum values that $\mathrm{VaR}_\alpha(X_1 + \cdots + X_d)$ can reach.

Remark 1 In some cases, the minimum VaR_α is reached on the minimal Σ_{cx}-element of the polytope. In fact, from the proof of Theorem 3.A.4 in Shaked and Shanthikumar (2007), it follows that, if $S \preceq_{cx} S'$, then there exists $\tilde{\alpha} \in (0, 1)$ such that, $\mathrm{VaR}_\alpha(S) \leq \mathrm{VaR}_\alpha(S')$, for every $\alpha \in (\tilde{\alpha}, 1)$. Therefore, if $\boldsymbol{X} = (X_1, \ldots, X_d)$ is a Σ_{cx}-smallest element of its Fréchet class, then there exists $\tilde{\alpha} \in (0, 1)$ such that, $\mathrm{VaR}_\alpha(\sum_{j=1}^d X_j) \leq \mathrm{VaR}_\alpha(\sum_{j=1}^d X'_j)$, for every $\alpha \in (\tilde{\alpha}, 1)$, for every random vector $\boldsymbol{X}' = (X'_1, \ldots, X'_d)$ of the same Fréchet class of $\boldsymbol{X}$.

We present below two numerical illustrations of our results.

Example 4 Consider the case $d = 5$, $p = \frac{1}{2}$. Then, $dp = \frac{5}{2}$ and $j_1^{\vee} = 2$, $j_2^{\wedge} = 3$. The class $\mathcal{D}_5(\frac{5}{2})$ has $n_{1/2}^{\mathcal{D}} = 9$ extremal points provided in Table 5. Let us compute value-at-risk, expected shortfall, and entropic risk measures of the extremal pmfs for $\alpha = 0.8$ and $\gamma = 0.1$. Results are reported in Table 6. The choice of α has been made to exhibit the case where the minimum VaR_α is not the Σ_{cx}-smallest element of the class. Table 7 reports instead the same risk measures evaluated on the corresponding FGM copulas. Notice that the bounds for the sums are at the extremal copulas corresponding to the upper Fréchet bound and to the Σ_{cx}-smallest Bernoulli pmfs for the convex measures, as proved in Corollary 4.1. The minimum VaR_α in $\mathcal{C}_5^{1/2}$ is reached at $C_{\boldsymbol{r}_7^D}$ while for the Bernoulli case it is at $\boldsymbol{r}_9^D$. This proves that the minimum VaR_α of the sum of the components of U in $\mathcal{C}_5^{1/2}$ is not inherited from the underlying Bernoulli pmf. □

The following Example 5 is the main example of application of our results. We consider a high dimensional portfolio of risks for six scenarios: three different GFGM(p) dependence structures for two Fréchet classes, with exponential and discrete margins, discussed in a theoretical setting in Examples 1 and 2.

Table 5 Extremal pmfs of the class $\mathcal{D}_5\left(\frac{5}{2}\right)$

y	$\boldsymbol{r}_1^D$	$\boldsymbol{r}_2^D$	$\boldsymbol{r}_3^D$	$\boldsymbol{r}_4^D$	$\boldsymbol{r}_5^D$	$\boldsymbol{r}_6^D$	$\boldsymbol{r}_7^D$	$\boldsymbol{r}_8^D$	$\boldsymbol{r}_9^D$
0	$\frac{1}{6}$	$\frac{3}{8}$	$\frac{1}{2}$	0	0	0	0	0	0
1	0	0	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{5}{8}$	0	0	0
2	0	0	0	0	0	0	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{6}$
3	$\frac{5}{6}$	0	0	$\frac{3}{4}$	0	0	$\frac{1}{2}$	0	0
4	0	$\frac{5}{8}$	0	0	$\frac{1}{2}$	0	0	$\frac{1}{4}$	0
5	0	0	$\frac{1}{2}$	0	0	$\frac{3}{8}$	0	0	$\frac{1}{6}$

Table 6 Values of risk measures of the sum of the components of Bernoulli random vectors with the extremal probability mass functions of the class $\mathcal{D}_5\left(\frac{5}{2}\right)$

	$\boldsymbol{r}_1^D$	$\boldsymbol{r}_2^D$	$\boldsymbol{r}_3^D$	$\boldsymbol{r}_4^D$	$\boldsymbol{r}_5^D$	$\boldsymbol{r}_6^D$	$\boldsymbol{r}_7^D$	$\boldsymbol{r}_8^D$	$\boldsymbol{r}_9^D$
$\mathrm{VaR}_{0.8}(S^{(\boldsymbol{I})})$	3	4	$\underline{5}$	3	4	$\underline{5}$	3	4	$\boxed{2}$
$\mathrm{ES}_{0.8}(S^{(\boldsymbol{I})})$	$\boxed{3}$	4	$\underline{5}$	$\boxed{3}$	4	$\underline{5}$	$\boxed{3}$	4	4.5
$\Psi_{0.1}(S^{(\boldsymbol{I})})$	2.5584	2.6803	$\underline{2.8093}$	2.5362	2.6121	2.6927	$\boxed{2.5125}$	2.5387	2.5667

The minimum values are squared and the maximum ones are underlined

Table 7 Values of risk measures of the sum of components of uniform vectors with joint cdf defined by FGM copulas corresponding to the extremal probability mass functions of the class $\mathcal{D}_5\left(\frac{5}{2}\right)$

	$C_{\boldsymbol{r}_1^D}$	$C_{\boldsymbol{r}_2^D}$	$C_{\boldsymbol{r}_3^D}$	$C_{\boldsymbol{r}_4^D}$	$C_{\boldsymbol{r}_5^D}$	$C_{\boldsymbol{r}_6^D}$	$C_{\boldsymbol{r}_7^D}$	$C_{\boldsymbol{r}_8^D}$	$C_{\boldsymbol{r}_9^D}$
$\mathrm{VaR}_{0.8}(S^{(\boldsymbol{U})})$	3.0308	3.3281	$\underline{3.4928}$	3.0158	3.1710	3.2636	$\boxed{2.9729}$	3.0180	3.0476
$\mathrm{ES}_{0.8}(S^{(\boldsymbol{U})})$	3.4627	3.7345	$\underline{3.8401}$	3.3641	3.5161	3.5846	$\boxed{3.2753}$	3.3228	3.3477
$\Psi_{0.1}(S^{(\boldsymbol{U})})$	2.5210	2.5350	$\underline{2.5486}$	2.5181	2.5264	2.5345	$\boxed{2.5153}$	2.5180	2.5207

The minimum values are squared and the maximum ones are underlined

Example 5 Consider the classes $\mathcal{G}_{100}^p\left(Exp\left(\frac{1}{10}\right)\right)$ and $\mathcal{G}_{100}^p(F)$, where F is the discrete cdf whose pmf is given by

$$f(y) = \begin{cases} 0.8 & y = 0, \\ 0.2\left[\left(\frac{y}{100}\right)^3 - \left(\frac{y-1}{100}\right)^3\right] & y \in \{1, \ldots, 100\}. \end{cases}$$

We consider three different cases of GFGM(p) dependencies for each class, that is, we consider $\mathcal{G}_{100}^p\left(Exp\left(\frac{1}{10}\right)\right)$ and $\mathcal{G}_{100}^p(F)$, where each case is associated to a common $p \in \left\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right\}$.

The bounds for the convex measures are reached at the distributions of the two classes $\mathcal{G}_{100}^p\left(Exp\left(\frac{1}{10}\right)\right)$ and $\mathcal{G}_{100}^p(F)$ corresponding to the minimal and maximal convex sums in $\mathcal{D}_{100}(100p)$, for $p = \frac{1}{3}, p = \frac{1}{2}$ and $p = \frac{2}{3}$. The minimal convex sum is the pmf in $\mathcal{D}_{100}(100p)$ with support on the pair (33, 34) when $p = \frac{1}{3}$, with support on the point 50 when $p = \frac{1}{2}$, and support on the pair (66, 67) when $p = \frac{2}{3}$. Table 8

Table 8 Bounds of convex risk measures in the classes $\mathcal{G}^p_{100}\left(Exp\left(\frac{1}{10}\right)\right)$ and $\mathcal{G}^p_{100}(F)$

	$\mathcal{G}^p_{100}\left(Exp\left(\frac{1}{10}\right)\right)$			$\mathcal{G}^p_{100}(F)$		
	$p=\frac{1}{3}$	$p=\frac{1}{2}$	$p=\frac{2}{3}$	$p=\frac{1}{3}$	$p=\frac{1}{2}$	$p=\frac{2}{3}$
min $ES_{0.95}$	1191.2742	1189.2721	1192.3324	2152.595	2122.718	2019.207
max $ES_{0.95}$	1858.1846	1702.8444	1540.6192	2858.955	3448.241	4440.057
min $\Psi_{0.001}$	1003.9212	1003.8215	1003.9237	1555.710	1551.957	1546.627
max $\Psi_{0.001}$	1124.6343	1125.0510	1101.5259	1888.303	2216.540	2843.312

Table 9 $\mathrm{VaR}_{0.95}$ Bounds: $\mathcal{F}_{100}(G)$ is the Fréchet class with 100 identically distributed risks, $X_j \sim G$, for every $j \in \{1,\ldots,100\}$, and $\mathcal{G}^p_{100}(G)$, where $G = Exp\left(\frac{1}{10}\right)$ or $G = F$

	$Exp\left(\frac{1}{10}\right)$			F		
	$p=\frac{1}{3}$	$p=\frac{1}{2}$	$p=\frac{2}{3}$	$p=\frac{1}{3}$	$p=\frac{1}{2}$	$p=\frac{2}{3}$
Lower bound $\mathcal{F}_{100}(G)$	842.3299	842.3299	842.3299	1045.963	1045.963	1045.963
Min $\mathcal{G}^p_{100}(G)$	1149.7294	1147.0118	1150.2229	2016	1994	1960
Max $\mathcal{G}^p_{100}(G)$	1791.3283	1645.0538	1488.2312	2688	3258	4225
Upper bound $\mathcal{F}_{100}(G)$	3995.7323	3995.7323	3995.7323	9606.61	9606.61	9606.61

provides the sharp bounds for the convex risk measures with exponential and discrete margins, respectively.

The VaR_α is bounded by its evaluations on the extremal pmfs of the two classes $\mathcal{S}^p_{100}\left(Exp\left(\frac{1}{10}\right)\right)$ and $\mathcal{S}^p_{100}(F)$. When $d = 100$, the number of extremal points $n^{\mathcal{D}}_p$ is lower than or equal to 2501, see Corollary 4.6 of Fontana et al. (2021), and we find bounds by enumeration. Table 9 provides the bounds for the VaR_α in the above-mentioned classes and also the analytical bounds for the whole Fréchet classes given in Equation (4) of Bernard et al. (2017). We mention that the minimum $\mathrm{VaR}_{0.95}$ in the class $\mathcal{G}^p_{100}(F)$ is not reached at the distribution corresponding to the minimal convex pmf in $\mathcal{D}_{100}\left(\frac{200}{3}\right)$, whose VaR_α is 1961.

We conclude this example by considering Pearson's correlation of the exchangeable Σ_{cx}-smallest element $\boldsymbol{X}^e_{cx}$ in $\mathcal{G}^{1/3}_{100}\left(Exp\left(\frac{1}{10}\right)\right)$ and Pearson's correlation matrix of a vector $\boldsymbol{X}_{cx}$ corresponding to the Bernoulli Σ_{cx}-smallest element provided by the authors of Fontana and Semeraro (2024) in Theorem 5.2. Using (3.11), Pearson's correlation $\rho_P(X_{j_1}, X_{j_2})$ (denoted by $\rho_{j_1 j_2}$) is equal to $Cov(I_{j_1}, I_{j_1})$, for $1 \le j_1 < j_2 \le d$.

We therefore have to find the covariance of the exchangeable Σ_{cx}-smallest element and the covariance matrix of the Σ_{cx}-smallest element f_{cx} in $\mathcal{B}_{100}(\frac{1}{3})$, obtained following Theorem 5.2 of Fontana and Semeraro (2024), and given by

$$f_{cx}(\boldsymbol{x}) = \begin{cases} \frac{1}{3} & \boldsymbol{x} = (\underbrace{1,\ldots,1}_{33},0,\ldots,0,0,\ldots,0), \\ \frac{1}{3} & \boldsymbol{x} = (0,\ldots,0,\underbrace{1,\ldots,1}_{34},0,\ldots,0), \\ \frac{1}{3} & \boldsymbol{x} = (0,\ldots,0,0,\ldots,0,\underbrace{1,\ldots,1}_{33}). \end{cases}$$

The equicorrelation of the exchangeable Σ_{cx}-smallest element in $\mathcal{G}_{100}^{1/3}\left(Exp\left(\frac{1}{10}\right)\right)$ is $\rho_e = -0.0022$, that is the minimal correlation in the subclass of exchangeable distributions in $\mathcal{G}_{100}^{1/3}\left(Exp\left(\frac{1}{10}\right)\right)$. Let $A = \{1,\ldots,33\}^2 \cup \{34,\ldots,67\}^2 \cup \{68,\ldots,100\}^2$. The entries of Pearson's correlation matrix of $\boldsymbol{X}_{cx}$ are given by

$$\rho_P(X_{j_1}, X_{j_2}) = \begin{cases} 1 & j_1 = j_2, \\ \frac{2}{9} & j_1 \neq j_2, (j_1, j_2) \in A, \\ -\frac{1}{9} & j_1 \neq j_2, (j_1, j_2) \in \{1,\ldots,100\}^2 \setminus A. \end{cases}$$

The mean ρ_m of Pearson's correlations of the random vector $\boldsymbol{X}_{cx}$ is given by

$$\rho_m = \frac{2}{99 \times 100} \sum_{j_1=1}^{99} \sum_{j_2=j_1+1}^{100} \rho_P(X_{j_1}, X_{j_2}) = -0.0022. \tag{5.2}$$

From (5.2) we notice that $\rho_m = \rho_e$, the equicorrelation of the exchangeable vector $\boldsymbol{X}_{cx}^e$. This result is a consequence of Corollary 5.2 in Fontana and Semeraro (2024) and of the fact that the Pearson's correlations in the class with the exponential margins is equal to the covariance of the corresponding Bernoulli pair. □

6 Applications and estimation procedures

The FGM family of copulas is used to build dependent models in different fields of application. To mention a few, we find applications in reliability, e.g., Sha (2021), survival analysis, e.g., Louzada et al. (2013)), as well as insurance. For example, in healthcare insurance context, the authors of Shi and Zhang (2015) study whether individuals' private information on health risks affects their medical care utilization. Pervaz et al. (2024) also uses a copula-based approach to examine the severity of driver injury in motor vehicle crashes in a developing country. For an application with health data, see e.g. Susam (2025).

In the bidimensional case, the simple analytical formulation and the consequent statistical tractability of FGM copulas are some of the reasons for their success. In dimensions higher than two, although the analytical formula of the FGM copula remains simple, the parameter space becomes more complex. As discussed at the end of Sect. 2, the geometrical representation allows us to prove that the parameter space is a convex polytope, whose generators are the parameters of the extremal copulas. Indeed, the geometrical representation in (2.6) is a parametric representation of FGM (or GFGM) copulas in terms of the extremal copulas $C_{\boldsymbol{R}_k}$, for $k \in \{1, \ldots, n_{\boldsymbol{p}}^{\mathcal{B}}\}$. For any $C \in \mathcal{C}_d^{\boldsymbol{p}}$ and for $\boldsymbol{u} \in [0,1]^d$, we have

$$C(\boldsymbol{u}) = C(\boldsymbol{u}; \boldsymbol{\lambda}) = \sum_{k=1}^{n_{\boldsymbol{p}}^{\mathcal{B}}} \lambda_k C_{\boldsymbol{R}_k}(\boldsymbol{u}), \quad \text{for } \lambda_1, \ldots, \lambda_{n_{\boldsymbol{p}}^{\mathcal{B}}} \geq 0, \text{ and } \sum_{k=1}^{n_{\boldsymbol{p}}^{\mathcal{B}}} \lambda_k = 1. \quad (6.1)$$

From (6.1), it follows that, in any dimension, the space of the parameters $\boldsymbol{\lambda}$ is the standard simplex. For the bivariate FGM copula, Susam (2025) reviews several estimation approaches. Although the investigation of estimation procedures is outside the scope of this work, we mention some directions that are of interest for our future research development. Given a random sample of size *n*, and observations denoted by $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, we can implement, beyond the traditional maximum likelihood estimation, the inference functions for margins (IFM) method (see Joe and Xu (1996)) or the semiparametric pseudo maximum likelihood (PML) technique, as proposed by Genest et al. (1995). The IFM and PML methods are both two-stage approaches in which the marginal distributions are estimated separately from the dependence structure. They differ in the estimation of the marginals, which is done parametrically in the IFM method and nonparametrically in the PML technique. Denote by $\hat{F}_j$ the estimated marginal cdfs obtained by one of the two methods. The second step of both methods consists in maximizing the pseudo-log-likelihood function

$$L(\boldsymbol{\lambda}) = \sum_{h=1}^{n} \log\left(c\big(\hat{F}_1(x_{h,1}), \ldots, \hat{F}_d(x_{h,d}); \boldsymbol{\lambda}\big)\right), \quad (6.2)$$

under the constraints $\lambda_1 \geq 0, \ldots, \lambda_{n_{\boldsymbol{p}}^{\mathcal{B}}} \geq 0$ and $\lambda_1 + \cdots + \lambda_{n_{\boldsymbol{p}}^{\mathcal{B}}} = 1$, where $c(\boldsymbol{u}; \lambda)$ is the density of $C(\boldsymbol{u}; \lambda)$. As in Susam (2025), we can consider minimum-distance methods and perform goodness-of-fit tests. The geometrical representation of the FGM copulas facilitates the estimation and, consequently, their use in dimensions higher than two. However, some computational issues remain in high dimensions, where finding all the generators is still an open issue.

7 Remarks and conclusion

We conclude with one example in low dimension of the general class $\mathcal{G}_d(\boldsymbol{p})$, with $\boldsymbol{p} = (p_1, \ldots, p_d)$, and we leave its theoretical investigation to further research. We find the risk measures' sharp bounds for the sum $S = X_1 + X_2 + X_3$, where X_i have discrete distributions and $\boldsymbol{X}$ has a GFGM copula with vector parameter $\boldsymbol{p}$. In fact, for $d = 3$ we are able to find the extremal points of $\mathcal{B}_d(\boldsymbol{p})$, to construct the corresponding copulas, and to find the generators of the convex polytope $\mathcal{G}_d^{\boldsymbol{p}}(F_1, \ldots, F_d)$. We evaluate the risk measures on the extremal point and we find sharp bounds by enumeration. Furthermore, we find the expected allocation and the expected contribution of X_i for each risk $\boldsymbol{X}$ with extremal pmf, following Blier-Wong et al. (2025).

Example 6 We consider the class $\mathcal{B}_d(\boldsymbol{p})$ with $d = 3$ and $\boldsymbol{p} = (\frac{1}{2}, \frac{1}{3}, \frac{2}{3})$. In this case there are $n_{\boldsymbol{p}}^{\mathcal{B}} = 12$ extremal points that can be found by using `4ti2`, 4ti2 team (2018). The extremal points $\boldsymbol{r}_k$, $k \in \{1, \ldots, 12\}$, of the class $\mathcal{B}_3(\boldsymbol{p})$ are reported in Table 10. Note that there are three extremal pmfs whose sum is minimal under the convex order: r_1, r_2, and r_4. These three vectors have two couples of Bernoulli rvs with minimal covariance and the remaining pair — (I_1, I_2) for r_1, (I_1, I_3) for r_2, and (I_2, I_3) for r_4 — has maximal covariance.

We now consider the class $\mathcal{G}_3^{\boldsymbol{p}}(F_1, F_2, F_3)$, where F_i, $i \in \{1, 2, 3\}$, is the cdf whose pmf f_i is defined by

$$f_i(y) = \begin{cases} 1 - a_i & y = 0, \\ a_i[\left(\frac{y}{n}\right)^{c_i} - \left(\frac{y-1}{n}\right)^{c_i}] & y \in \{1, \ldots, n\}, \end{cases}$$

and we choose $n = 1000$, $a_1 = 0.2$, $a_2 = 0.1$, $a_3 = 0.3$ and $c_1 = 3$, $c_2 = 4$, $c_3 = 2$. Also, we have $E[X_1] = 150.09995$, $E[X_2] = 80.04997$, $E[X_3] = 200.14995$ and $E[S] = 430.29987$.

The lower and upper bounds of convex risk measures are reached at the random vectors corresponding to the extremal points $\boldsymbol{r}_1$ and $\boldsymbol{r}_{11}$, respectively. In Table 11, we provide Pearson's correlation coefficients $\rho(X_1, X_2)$, $\rho(X_1, X_3)$, and $\rho(X_2, X_3)$ for all of the twelve extremal dependence structures. Notice that the correlation matrix of $\boldsymbol{X}_1$ also has negative entries, i.e., $(X_{1,1}, X_{1,3})$ and $(X_{1,2}, X_{1,3})$ are negatively correlated, while $(X_{1,1}, X_{1,2})$ has the same correlation as $(X_{11,1}, X_{11,2})$. This last equality follows observing that the correlation $\rho_1(1, 2)$ between $(r_{1,1}, r_{1,2})$ and the correlation $\rho_{11}(1, 2)$ between $(r_{11,1}, r_{11,2})$ are equal, in fact

$$\rho_1(1, 2) = \frac{r_1((1, 1, 0)) + r_1((1, 1, 1)) - \frac{1}{2}\frac{1}{3}}{\sqrt{\frac{1}{2}(1 - \frac{1}{2})\frac{1}{3}(1 - \frac{1}{3})}} = 0.0605$$

and

$$\rho_{11}(1, 2) = \frac{r_{11}((1, 1, 0)) + r_{11}((1, 1, 1)) - \frac{1}{2}\frac{1}{3}}{\sqrt{\frac{1}{2}(1 - \frac{1}{2})\frac{1}{3}(1 - \frac{1}{3})}} = 0.0605$$

coincide since $r_1((1, 1, 0)) + r_1((1, 1, 1)) = r_{11}((1, 1, 0)) + r_{11}((1, 1, 1)) = \frac{1}{3}$. Table [12] provides the values of risk measures on the extremal points.

We conclude this example by finding the contribution of risk X_j, $j \in \{1, 2, 3\}$, to the standard deviation of the sum $S = X_1 + X_2 + X_3$, to the VaR_α and to the ES_α, for all the extremal dependence structures. We recall the definitions of expected allocation and of expected contribution of the risk X_j to a total outcome $S = y$, for $y \in \{0, 1, \ldots, 3000\}$. The expected allocation of each risk X_j in Definition 1.1 of Blier-Wong et al. (2025) is given by

Table 10 Extremal points $\boldsymbol{r}_k, k \in \{1, \ldots, 12\}$, of the class $\mathcal{B}_3(\frac{1}{2}, \frac{1}{3}, \frac{2}{3})$

$\boldsymbol{x}$	$\boldsymbol{r}_1$	$\boldsymbol{r}_2$	$\boldsymbol{r}_3$	$\boldsymbol{r}_4$	$\boldsymbol{r}_5$	$\boldsymbol{r}_6$	$\boldsymbol{r}_7$	$\boldsymbol{r}_8$	$\boldsymbol{r}_9$	$\boldsymbol{r}_{10}$	$\boldsymbol{r}_{11}$	$\boldsymbol{r}_{12}$
(0,0,0)	0	0	0	0	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$
(1,0,0)	0	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	0	$\frac{1}{6}$	$\frac{1}{6}$	0	0	0
(0,1,0)	0	$\frac{1}{3}$	0	0	0	$\frac{1}{12}$	$\frac{1}{6}$	0	0	0	0	0
(1,1,0)	$\frac{1}{3}$	0	$\frac{1}{6}$	0	0	0	0	0	0	0	0	$\frac{1}{12}$
(0,0,1)	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{5}{12}$	0	0	$\frac{1}{3}$	0	$\frac{1}{6}$	0
(1,0,1)	$\frac{1}{6}$	$\frac{1}{2}$	0	$\frac{1}{6}$	0	0	$\frac{1}{2}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{5}{12}$
(0,1,1)	0	0	0	$\frac{1}{3}$	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{3}$	0	$\frac{1}{6}$	0	$\frac{1}{4}$
(1,1,1)	0	0	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{4}$	0	0	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	0

$$E[X_j \mathbf{1}\{S = y\}], \quad y \in \mathbb{N}_+,$$

where $\mathbf{1}$ is the indicator function, such that $\mathbf{1}\{A\} = 1$, if A is verified, and $\mathbf{1}\{A\} = 0$, otherwise. The expected contribution of each risk X_j, $j \in \{1,2,3\}$, is provided in Equation (6) of Blier-Wong et al. (2025), and is defined by

$$E[X_j | S = y] = \frac{E[X_j \mathbf{1}\{S = y\}]}{\Pr(S = y)}, \quad y \in \mathbb{N}_+,$$

assuming that $\Pr(S = y) > 0$. The expected contribution of X_j to the VaR_α is given by $E[X_j | S = \mathrm{VaR}_\alpha(S)]$. We now recall the expression for the contribution to the ES_α based on the Euler-based allocation rule provided in Tasche (1999):

$$CES_\alpha(X_j, S) = \frac{E[X_j] - E[X_j \mathbf{1}\{S \leq \mathrm{VaR}_\alpha(S)\}] + \beta_S E[X_j \mathbf{1}\{S = \mathrm{VaR}_\alpha(S)\}]}{1-\alpha},$$

where

$$\beta_S = \begin{cases} \frac{\Pr(S \leq \mathrm{VaR}_\alpha(S)) - \alpha}{\Pr(S = \mathrm{VaR}_\alpha(S))} & \text{if } \Pr(S = \mathrm{VaR}_\alpha(S)) > 0, \\ 0 & \text{if } \Pr(S = \mathrm{VaR}_\alpha(S)) = 0, \end{cases}$$

and

$$E[X_j \mathbf{1}\{S \leq k\}] = \sum_{y=1}^{k} E[X_j \mathbf{1}\{S = y\}], \quad k \in \mathbb{N}_+.$$

Finally, the contribution of X_j to the standard deviation of S based on Euler's rule is given by

$$CStd(X_j, S) = \frac{Cov(X_j, S)}{\sqrt{Var(S)}} = \frac{Var(X_j) + \sum_{j' \neq j} Cov(X_j, X_{j'})}{\sqrt{Var(S)}}, \quad j = 1,2,3. \tag{7.1}$$

Fig. 1 reports the contributions to the $\mathrm{VaR}_{0.95}$, the $\mathrm{ES}_{0.95}$ and the standard deviation of S. □

The geometrical structure of GFGM copulas inherited from the geometrical structure of multivariate Bernoulli distributions has proven to be a powerful tool for studying the properties of random vectors with GFGM dependence.

The last Example 6 finds the bounds by enumeration of their values in the extremal points, which becomes computationally challenging in high dimensions. Under the assumption of identically distributed risks with GFGM(p) dependence structure, we show the effectiveness of our theoretical results in studying the risk of high dimensional — $d = 100$ — portfolios. The extension of these theoretical results to the whole GFGM copulas relies on extending corresponding results in the class of multivariate Bernoulli distributions, and this is part of our ongoing research. Another, more applicative, part is to use this novel geometrical representation to investigate

Table 11 Pearson's coefficients of $\boldsymbol{X}$

	$\boldsymbol{r}_1$	$\boldsymbol{r}_2$	$\boldsymbol{r}_3$	$\boldsymbol{r}_4$	$\boldsymbol{r}_5$	$\boldsymbol{r}_6$
$\rho(X_1, X_2)$	0.0605	− 0.0605	0.0605	− 0.0605	0.0000	0.0302
$\rho(X_1, X_3)$	− 0.1610	0.1610	− 0.1610	− 0.1610	− 0.1610	−0.0805
$\rho(X_2, X_3)$	− 0.1229	− 0.1229	− 0.0307	0.0615	0.0615	0.0154
	$\boldsymbol{r}_7$	$\boldsymbol{r}_8$	$\boldsymbol{r}_9$	$\boldsymbol{r}_{10}$	$\boldsymbol{r}_{11}$	$\boldsymbol{r}_{12}$
$\rho(X_1, X_2)$	− 0.0605	− 0.0605	0.0605	0.0000	0.0605	−0.0302
$\rho(X_1, X_3)$	0.1610	0.0000	0.0000	0.1610	0.1610	0.0805
$\rho(X_2, X_3)$	− 0.0307	0.0615	0.0615	0.0615	0.0615	0.0154

Table 12 Values of risk measures of the sum of the components of vectors with joint cdf defined by the GFGM copulas corresponding to the extremal probability mass functions ($\boldsymbol{r}_i, i \in \{1, \ldots, 12\}$) of the class $\mathcal{B}_3(\frac{1}{2}, \frac{1}{3}, \frac{2}{3})$

	$\boldsymbol{r}_1$	$\boldsymbol{r}_2$	$\boldsymbol{r}_3$	$\boldsymbol{r}_4$	$\boldsymbol{r}_5$	$\boldsymbol{r}_6$
$VaR_{0.95}(S)$	1219.00	1532.00	1360.00	1342.00	1403.00	1479.00
$ES_{0.95}(S)$	1590.08	1733.70	1665.46	1641.07	1683.14	1724.32
$\Psi_{0.001}(S)$	555.98	587.74	566.80	563.46	570.51	580.07
Std(*S*)	473.23	521.70	488.85	485.22	494.70	508.47
	$\boldsymbol{r}_7$	$\boldsymbol{r}_8$	$\boldsymbol{r}_9$	$\boldsymbol{r}_{10}$	$\boldsymbol{r}_{11}$	$\boldsymbol{r}_{12}$
$VaR_{0.95}(S)$	1561.00	1493.00	1567.00	1618.00	1643.00	1535.00
$ES_{0.95}(S)$	1802.17	1771.05	1824.07	1888.55	1906.84	1818.89
$\Psi_{0.001}(S)$	602.12	590.22	603.90	622.97	629.61	601.55
Std(*S*)	535.91	518.49	536.10	558.13	566.39	531.66

The minimum values are squared and the maximum ones are underlined

the dependence structure of the class and of their extremal points, which are good candidates for representing extremal dependence also in high dimension.

Appendix: GFGM copulas

In this section, we illustrate the effect of the parameter p on the dependence flexibility of 3-dimensional GFGM(p) copulas. We consider three values of p: $p = 1/2$, corresponding to the FGM copulas, a lower value $p = 1/3$, and a higher value $p = 2/3$. For each value of p, we consider three copulas corresponding to three different Bernoulli distributions. In particular, for each p, we have the comonotonic case, the exchangeable Σ_{cx}-smallest element, and an extremal point with a comonotonic pair and two countermonotonic pairs.

Table 13 provides Spearman's ρ_S for each case, while Figs. 2, 3, and 4 show the corresponding scatter plots. From Table 13, we deduce that the range of dependence spanned is increasing with the value of the parameter p. Thus, with the higher value $p = 2/3$, we expand the range of dependence modeled by the traditional FGM copula. Specifically, we obtain higher positive dependence in the comonotonic case (for

Table 13 Spearman's correlation coefficients for each pair of uniform random variables for the three values of $p = 1/3, 1/2, 2/3$

"CM" stands for the comonotonic copula, "exch. Σ_{cx}" for the exchangeable minimal convex copula, and " *2CMT-CM*" for the copula with a comonotonic pair and two countermonotonic pairs

		$\rho_s(U_1, U_2)$	$\rho_s(U_1, U_3)$	$\rho_s(U_2, U_3)$
$p = \frac{1}{3}$	CM	0.24	0.24	0.24
	exch. Σ_{cx}	− 0.12	− 0.12	− 0.12
	2CMT-CM	− 0.12	− 0.12	0.24
$p = \frac{1}{2}$	CM	0.33	0.33	0.33
	exch. Σ_{cx}	− 0.11	− 0.11	− 0.11
	2CMT-CM	− 0.33	− 0.33	0.33
$p = \frac{2}{3}$	CM	0.375	0.375	0.375
	exch. Σ_{cx}	− 0.1875	− 0.1875	−0.1875
	2CMT-CM	− 0.1875	− 0.1875	0.375

$p = 2/3$ we have $\rho_S = 0.38$, while for $p = 1/2$ we have $\rho_S = 0.34$) as well as lower negative dependence in the Σ_{cx}-smallest case (for $p = 2/3$ we have $\rho_S = -0.1875$, while for $p = 1/2$ we have $\rho_S = -0.12$).

The scatter plots in Figs. 2 and 3 illustrate how the dependence becomes slightly higher as the parameter p increases. In particular, Fig. 2 shows the scatter plots of the copulas with comonotonic components for the three values $p = 1/3$ (first row), 1/2 (second row), and 2/3 (third row). We observe that in the scatter plots in the first row, corresponding to the case $p = 1/3$, the data points are widely scattered, showing only weak positive dependence. As p increases, the points are more concentrated along the main diagonal and the positive dependence becomes stronger. Figure 3, instead, presents the scatter plots obtained from the exchangeable Σ_{cx}-smallest copulas for the three values of p. We observe that, for all the values of p, the dependence is slightly negative for all pairs. The last row corresponding to $p = 2/3$ shows a stronger negative dependence, as shown by the major concentration of points along the anti-diagonal. Finally, Fig. 4 illustrates the copulas obtained by considering the Bernoulli distributions with two countermonotonic pairs, (I_1, I_2) (first column) and (I_1, I_3) (second column), and one comonotonic pair, (I_2, I_3) (third column). In each row, the scatter plots on the left and in the middle show negative dependence (points concentrate along the anti-diagonal), while the scatterplot on the right exhibits positive dependence (points concentrate along the main diagonal). We note that the dependence for $p = 1/3$ is weaker compared to the other two cases. Indeed, the dispersion of the scatter points decreases as p increases.

We conclude this appendix by reporting Figs. 3 and 4 of Blier-Wong et al. (2024a) in Figs. 5 and 6. They show the heatmaps of the copula density function in the bivariate case for different dependence parameters.

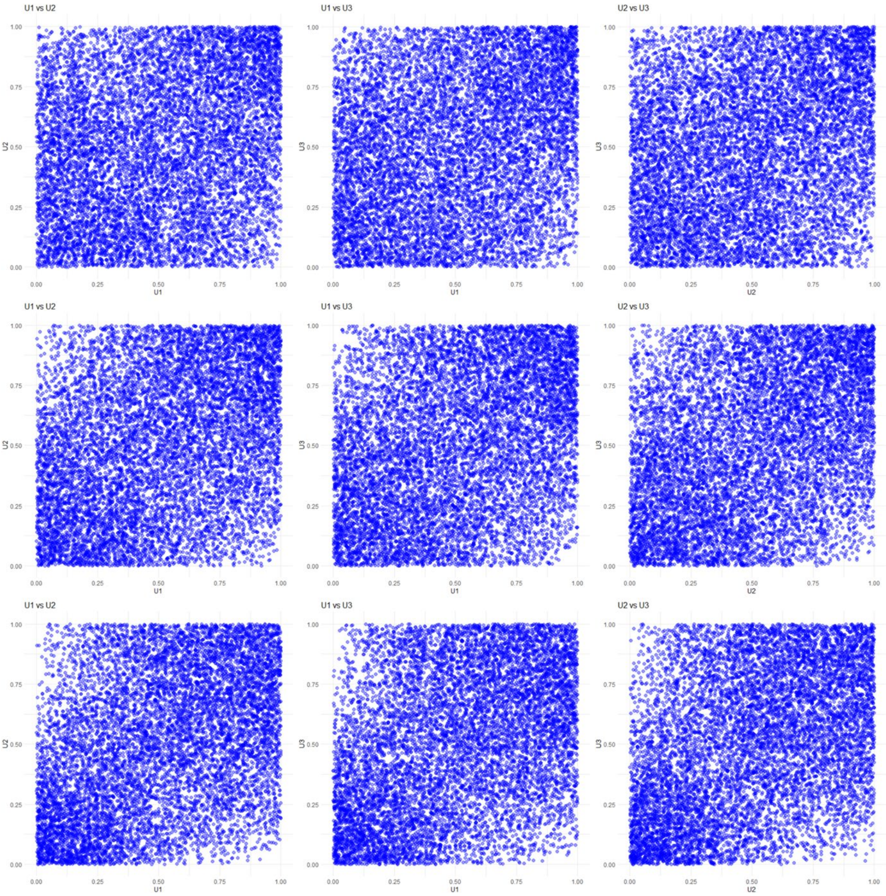


Fig. 2 Scatterplots of the three copulas built from the comonotonic Bernoulli random vectors. Each row is a different copula: The first row refers to $p = \frac{1}{3}$, the second row to $p = \frac{1}{2}$, and the last row to $p = \frac{2}{3}$. On the left there are the scatterplots between the components (U_1, U_2), between (U_1, U_3) in the middle, and between (U_2, U_3) on the right

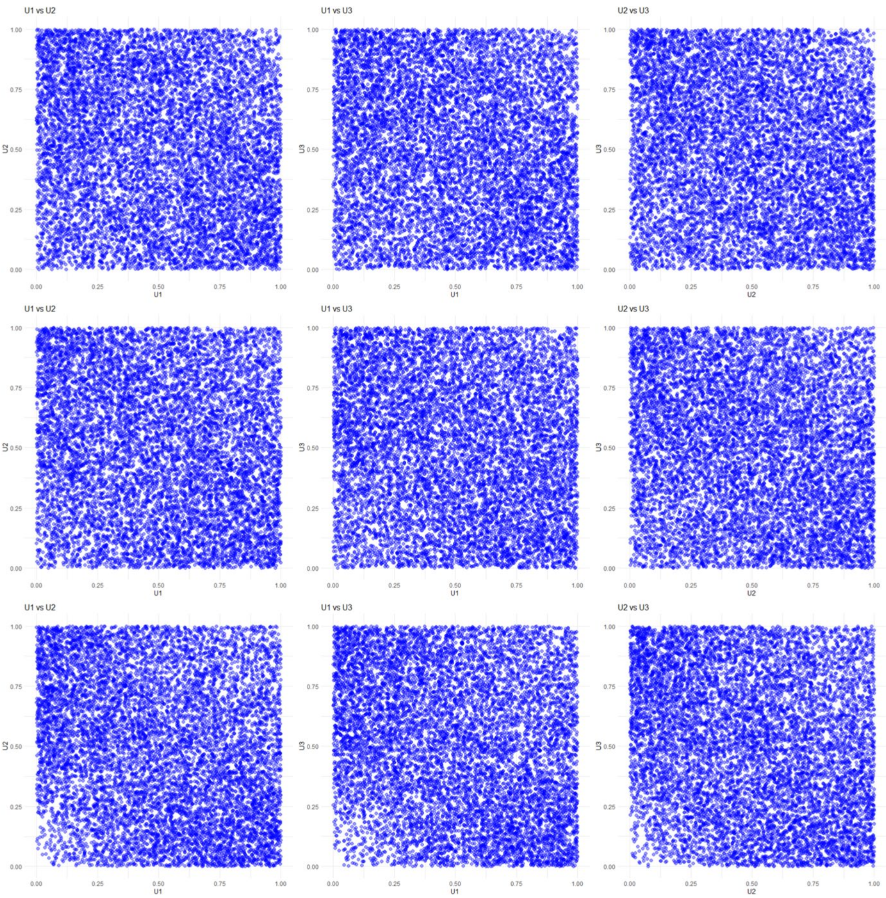


Fig. 3 Scatterplots of the copulas built from the exchangeable Σ_{cx}-smallest Bernoulli random vectors. Each row is a different copula: The first row refers to $p = \frac{1}{3}$, the second row to $p = \frac{1}{2}$, and the last row to $p = \frac{2}{3}$. On the left there are the scatterplots between the components (U_1, U_2), between (U_1, U_3) in the middle, and between (U_2, U_3) on the right

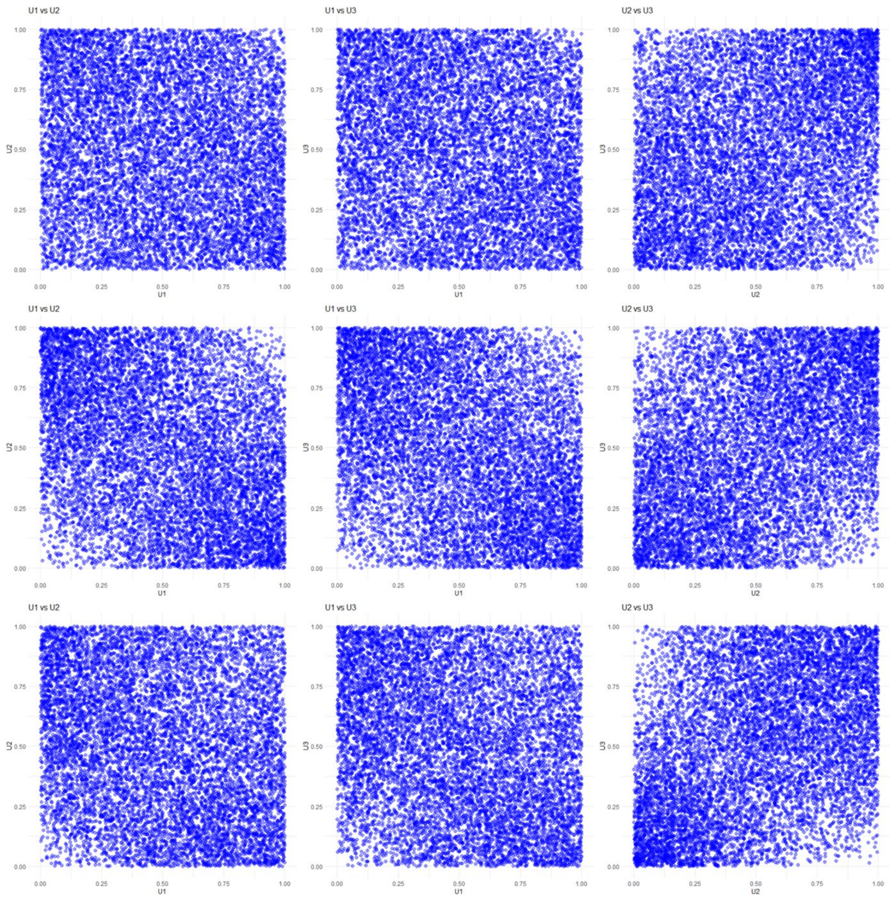
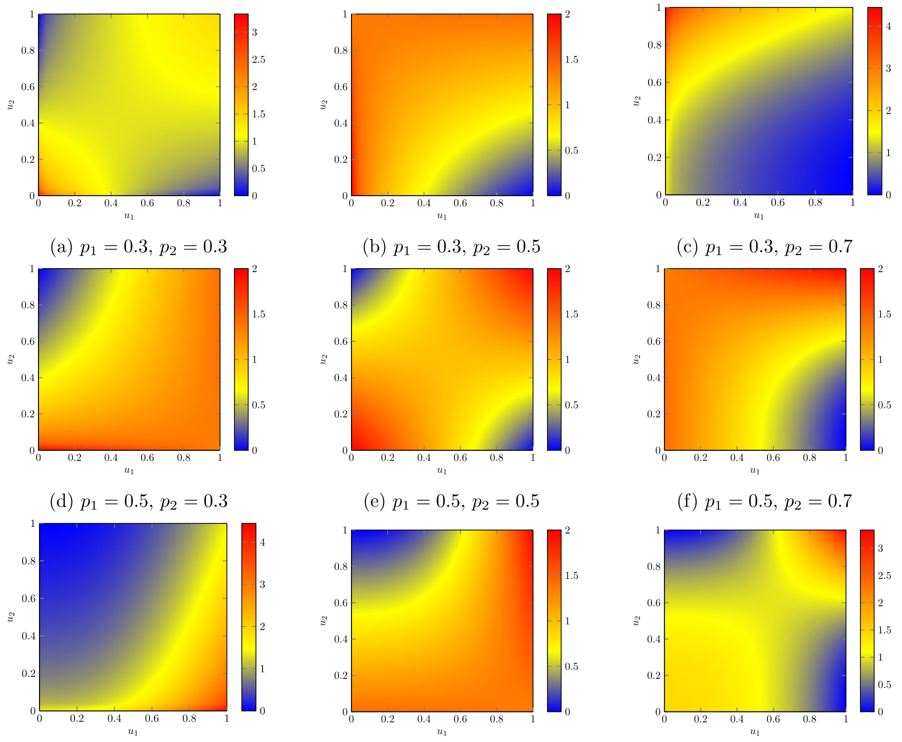


Fig. 4 Scatterplots of the copulas built from the extremal Bernoulli random vectors with comonotonic pair and two countermonotonic pairs. Each row is a different copula: The first row refers to $p = 1/3$, the second row to $p = 1/2$, and the last row to $p = 2/3$. On the left there are the scatterplots between the components (U_1, U_2), between (U_1, U_3) in the middle, and between (U_2, U_3) on the right

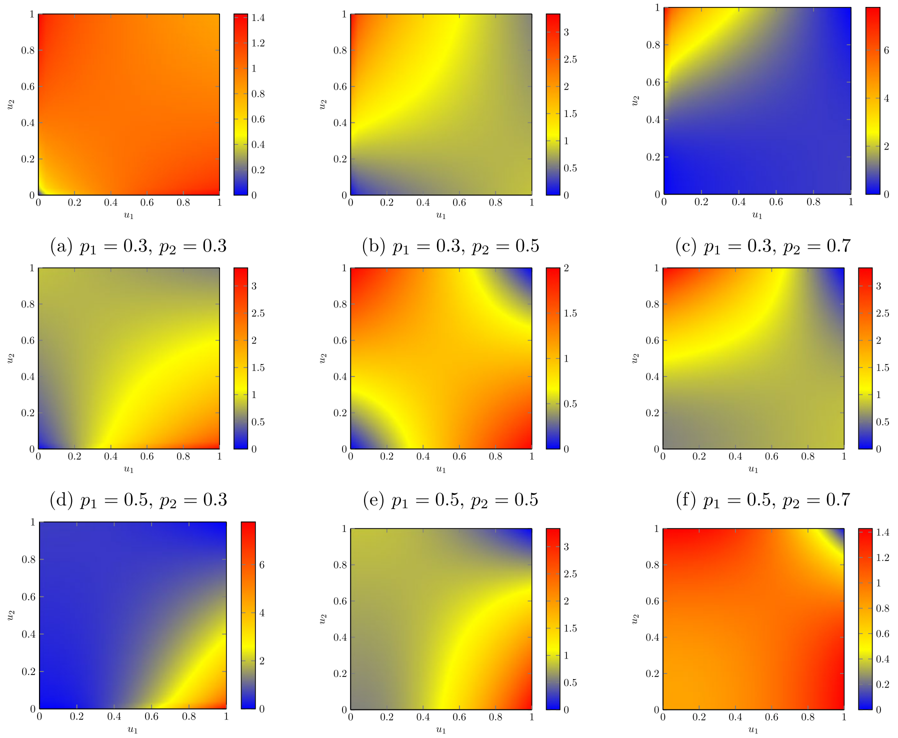


(a) $p_1 = 0.3$, $p_2 = 0.3$ (b) $p_1 = 0.3$, $p_2 = 0.5$ (c) $p_1 = 0.3$, $p_2 = 0.7$

(d) $p_1 = 0.5$, $p_2 = 0.3$ (e) $p_1 = 0.5$, $p_2 = 0.5$ (f) $p_1 = 0.5$, $p_2 = 0.7$

(g) $p_1 = 0.7$, $p_2 = 0.3$ (h) $p_1 = 0.7$, $p_2 = 0.5$ (i) $p_1 = 0.7$, $p_2 = 0.7$

Fig. 5 Heatmaps for density functions associated to the maximal dependence structure (Fig. 3 in Blier-Wong et al. (2024a))



(a) $p_1 = 0.3$, $p_2 = 0.3$ (b) $p_1 = 0.3$, $p_2 = 0.5$ (c) $p_1 = 0.3$, $p_2 = 0.7$

(d) $p_1 = 0.5$, $p_2 = 0.3$ (e) $p_1 = 0.5$, $p_2 = 0.5$ (f) $p_1 = 0.5$, $p_2 = 0.7$

(g) $p_1 = 0.7$, $p_2 = 0.3$ (h) $p_1 = 0.7$, $p_2 = 0.5$ (i) $p_1 = 0.7$, $p_2 = 0.7$

Fig. 6 Heatmaps for density functions associated to the minimal dependence structure (Fig. 4 in Blier-Wong et al. (2024a))

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