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THE DIRICHLET PROBLEM ON LOWER DIMENSIONAL BOUNDARIES: SCHAUDER ESTIMATES VIA PERFORATED DOMAINS

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Abstract. In this paper, we investigate the Dirichlet problem on lower dimensional manifolds for a class of weighted elliptic equations with coefficients that are singular on such sets. Specifically, we study the problem

$$\begin{cases} -\operatorname{div}(|y|^a A(x,y)\nabla u) = |y|^a f + \operatorname{div}(|y|^a F), \\ u = \psi, \quad \text{on } \Sigma_0, \end{cases}$$

where $(x,y) \in \mathbb{R}^{d-n} \times \mathbb{R}^n$, $2 \le n \le d$, $a+n \in (0,2)$, and $\Sigma_0 = \{|y| = 0\}$ is the lower dimensional manifold where the equation loses uniform ellipticity.

Our primary objective is to establish $C^{0,\alpha}$ and $C^{1,\alpha}$ regularity estimates up to Σ_0, under suitable assumptions on the coefficients and the data. Our approach combines perforated domain approximations, Liouville-type theorems and a blow-up argument.

1. Introduction

Let $2 \le n \le d$ be two integers and $z = (x,y) \in \mathbb{R}^{d-n} \times \mathbb{R}^n$. Let us define the lower dimensional manifold

$$\Sigma_0 := \{(x,y) \in \mathbb{R}^d : |y| = 0\},$$

which has dimension $d-n$, and the weight $|y|^a = \operatorname{dist}^a_{\Sigma_0}(z)$, where the real parameter a satisfies $a+n \in (0,2)$. We study the following equation

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla u) = |y|^a f + \operatorname{div}(|y|^a F), & \text{in } B_1 \setminus \Sigma_0, \\ u = \psi, & \text{on } \Sigma_0 \cap B_1, \end{cases} \tag{1.1}$$

where $B_1 \subset \mathbb{R}^d$ denotes the unit ball with center at 0, $A : B_1 \to \mathbb{R}^{d,d}$ is a symmetric d-dimensional matrix satisfying the following ellipticity condition

$$\lambda|\xi|^2 \le A(z)\xi\cdot\xi \le \Lambda|\xi|^2, \tag{1.2}$$

for all $\xi \in \mathbb{R}^d$ and a.e. $z \in B_1$, where $0 < \lambda \le \Lambda < \infty$ are fixed constants. The terms $f : B_1 \to \mathbb{R}$, $F : B_1 \to \mathbb{R}^d$ and $\psi : \Sigma_0 \cap B_1 \to \mathbb{R}$ belong to suitable spaces, which will be introduced later. We notice that, when $n = d$ (so that $z = y$), the lower-dimensional boundary $\Sigma_0 = \{0\}$ reduces to a single point. In this case, we assume the boundary condition is simply $u(0) = 0$. The operators ∇ and div denote the gradient and the divergence with respect to the variable z, respectively. Weak solutions to this equation are naturally defined within the framework of weighted Sobolev spaces, which will be introduced in Section 2.1. Thus, we say that u is a weak solution to (1.1) if $u \in H^{1,a}(B_1) := H^1(B_1, |y|^a dz)$, satisfies

$$\int_{B_1} |y|^a A\nabla u \cdot \nabla\phi \, dz = \int_{B_1} |y|^a (f\phi - F\cdot\nabla\phi)\, dz,$$

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for every $\phi \in C_c^\infty(B_1 \setminus \Sigma_0)$ and $u = \psi$ in the sense of the trace (see Definition 2.5).

Our primary goal is to establish local regularity estimates up to Σ_0 for weak solutions to (1.1). Specifically, we show that, under suitable assumptions on the data, solutions are $C^{0,\alpha}(B_{1/2})$ and, in some cases, may be $C^{1,\alpha}(B_{1/2})$. As we will see later, our results are *sharp* with respect to the assumptions on the data. Specifically, the function $|y|^{2-a-n}$ is a solution to (1.1), when $A = I$, $f = 0$, $F = 0$ and $\psi = 0$ and it is $C^{0,2-a-n}$ if $a+n \in [1,2)$ and $C^{1,1-a-n}$ if $a+n \in (0,1)$. The idea behind our theorems is that this solution is the *worst regular* solution to (1.1) when $A = I$, $f = 0$, $F = 0$ and $\psi = 0$.

The Dirichlet boundary condition $u = \psi$ on Σ_0 requires some clarifications. For general values of $a \in \mathbb{R}$, the trace of functions which belongs to $H^{1,a}(B_1)$ on Σ_0 might not be well defined. In the paper [28], the author shows the existence of a trace operator for a large class of weighted Sobolev spaces on lower dimensional boundaries. Specifically, the following result holds.

Theorem 1.1. [28, Theorem 2.3] *Let Γ be a $(d-n)$-dimensional C^1-manifold, dist_Γ be the distance from Γ, $a+n \in (0,2)$. Then, there exits a unique bounded linear operator*

$$T : H^1(B_1, \mathrm{dist}_\Gamma^a) \to L^2(\Gamma \cap B_1)$$

such that

$$Tu = u_{|\Gamma},$$

for every $u \in C^\infty(\overline{B_1})$.

Hence, the restriction we impose on the parameter $a+n \in (0,2)$ ensures that the boundary condition $u = \psi$ on Σ_0 makes sense. In other words, introducing the weight as a power of the distance to the boundary provides a natural framework for studying the Dirichlet problem for linear elliptic operators on lower-dimensional boundaries. Without such a weight, the solutions do not "see" the lower-dimensional sets due to capacitary reasons: for instance, a harmonic function in $B_1 \setminus \Sigma_0$ is the same as a harmonic function in the whole B_1.

We also emphasize that in the works [8,9], Cora, Vita and the author systematically study the same equation in the broader case $a+n > 0$, with a focus on solutions which satisfies an homogeneous *conormal boundary condition* on Σ_0 when $a+n \in (0,2)$.

The study of such equations falls within the theory of non uniformly elliptic operators, as the presence of the singular weight causes the operator's coefficients to blow up on the manifold Σ_0. In the seminal paper [18], the authors extended the De Giorgi-Nash-Moser theory to weighted elliptic equations, where the weight arises from quasi-conformal mappings or belongs to the Muckenhoupt A_2 class. In particular, under suitable assumptions, they proved the validity of the Harnack inequality, ensuring Hölder regularity of solutions with a non explicit Hölder exponent. Along this line, we also refer to the work [21]. Our weight $|y|^a$ belongs to the Muckenhoupt A_2 class when $a+n \in (0,2n)$, so our assumption $a+n \in (0,2)$ ensures that the known results for this class of weights apply, guaranteeing that the solutions to our problem satisfy some Hölder estimates. However, the peculiar geometry of the singular set of our weight $|y|^a$, combined with its homogeneity property, allows us to obtain more refined results compared to the general theory mentioned above.

In recent years, there have been significant contributions to the study weighted elliptic equations, where the weight behaves like the power of the distance from the boundary of a set. The most notable case is $n = 1$, which is closely related to the extension theory for fractional operators developed by Caffarelli and Silvestre in their seminal paper [5] (we refer also to [10], where the authors study fractional operators in conformal geometry). This setting is now well understood, and there is a rich literature on the regularity properties of such equations. Concerning Schauder type estimates, a notable reference can be found in [6]. Moreover, we highlight the works [31, 32, 33, 14], where the authors establish a complete Schauder regularity theory for degenerate/singular elliptic equations, and its parabolic counterpart [1, 2]. Additionally, we mention [15, 16], where elliptic and parabolic weighted equations are studied, yielding alternative regularity results.

In the paper [12], David, Feneuil and Mayboroda developed an elliptic theory for equations which are degenerate/singular on lower dimensional boundaries. In particular, in [11], the authors extensively studied our operator in the case $a+n=1$, under weaker assumptions on the coefficients A. They proved the solvability of the Dirichlet problem in this setting. We also refer to [13] and the references therein for a broader overview of this topic. Recently, in [17], the authors investigated the boundary behaviour of solutions in this high-codimensional setting and established estimates on the singular set near the boundary. Moreover, our class of operators also arises in the context of singular harmonic maps to study equations related to black holes (see [35, 36, 22, 23, 24]). In particular, in [29], the author highlights the connection between these singular harmonic maps and differential operators like ours, in the case where $a<0$. Additionally, our work is related to Mazzeo's theory of edge operators [25, 26], which emerges in boundary problems with higher codimensional boundaries and provides essential insights into solution regularity by establishing Fredholmness in degenerate Hölder or Sobolev spaces.

Finally, it seems that our operator could be a good model for free boundary problems of the obstacle type (see, for example, the classical papers [3, 4]), where the obstacle is very thin (with dimension less than $d-2$), see also [19]. As already noted above, classical elliptic operators do not allow for such problems, as they cannot "see" small sets like these due to capacitary reasons.

Main results. The main goal of this paper is to establish local $C^{0,\alpha}$ and $C^{1,\alpha}$ regularity estimates up to the singular set Σ_0 for weak solutions to (1.1). These results are presented in two main theorems: the first provides Hölder estimates for the solutions, while the second establishes Hölder estimates for the gradient. In the following, we use the notation $L^{p,a}(B_1):=L^p(B_1,|y|^a)$ to denote weighted Lebesgue spaces.

Theorem 1.2. *Let $2\le n\le d$, $a+n\in(0,2)$, $p>d/2$, $q>d$ and*

$$\alpha\in(0,2-a-n)\cap(0,2-d/p]\cap(0,1-d/q]\cap(0,1). \tag{1.3}$$

Let A be a continuous symmetric matrix satisfying (1.2) and ω be a modulus of continuity such that

$$\|A\|_{C^{0,\omega}(B_1)}:=\|A\|_{L^\infty(B_1)}+\sup_{z,z'\in B_1,z\neq z'}\frac{|A(z)-A(z')|}{\omega(|z-z'|)}\le L.$$

Let $f\in L^{p,a}(B_1)$, $F\in L^{q,a}(B_1)^d$ and $\psi\in C^{0,1}(\Sigma_0\cap B_1)$. Let u be a weak solution to (1.1).

Then, $u\in C^{0,\alpha}(B_{1/2})$ and there exists a constant $c>0$, depending only on d, n, a, λ, Λ, p, q, α and L, such that

$$\|u\|_{C^{0,\alpha}(B_{1/2})}\le c\big(\|u\|_{L^{2,a}(B_1)}+\|f\|_{L^{p,a}(B_1)}+\|F\|_{L^{q,a}(B_1)}+\|\psi\|_{C^{0,1}(\Sigma_0\cap B_1)}\big). \tag{1.4}$$

Let us point out that the assumption $\psi\in C^{0,1}(B_1)$ might not be optimal, and it is reasonable to expect that $\psi\in C^{0,\alpha}(B_1)$ would suffice.

Theorem 1.3. *Let $2\le n\le d$, $a+n\in(0,1)$, $p>d$ and*

$$\alpha\in(0,1-a-n)\cap(0,1-d/p]. \tag{1.5}$$

Let A be a α-Hölder continuous symmetric matrix satisfying (1.2) and $\|A\|_{C^{0,\alpha}(B_1)}\le L$, $f\in L^{p,a}(B_1)$, $F\in C^{0,\alpha}(B_1)$ and $\psi\in C^{1,\alpha}(\Sigma_0\cap B_1)$. Let u be a weak solution to (1.1).

Then, $u \in C^{1,\alpha}(B_{1/2})$ and there exists a constant $c > 0$, depending only on d, n, a, λ, Λ, p, α and L such that

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq c\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)} + \|\psi\|_{C^{1,\alpha}(\Sigma_0 \cap B_1)}\big). \tag{1.6}$$

In addition, u satisfies the following boundary condition

$$\begin{cases} \nabla_x u(x,0) = \nabla_x \psi(x,0), \\ (A\nabla u + F)(x,0) \cdot e_{y_i} = 0, \end{cases} \quad \text{for every } (x,0) \in \Sigma_0 \cap B_{1/2}, \text{ and } i = 1, \dots, n. \tag{1.7}$$

Before presenting the idea of the proof, we recall a method developed in [31] to establish regularity estimates for weighted elliptic equations that are degenerate or singular on a set of codimension one, that is, when $n = 1$. This case, where the singular set is an hyperplane of dimension $d-1$, does not fall under the scope of our study, which focuses on $n \geq 2$. The authors develop a regularity theory for such equations by regularizing the weight and using an approximation argument: they first establish ε-stable regularity results for solutions with the regularized weight $(\varepsilon^2 + y^2)^{a/2}$, and then take the limit as $\varepsilon \to 0$. However, this approach seems not to work in our case, where $n \geq 2$. Indeed, by regularizing the weight, that is, considering $\rho_\varepsilon^a(y) := (\varepsilon^2 + |y|^2)^{a/2}$ and solutions to uniformly elliptic problems with this type of weight, the $H^1(\rho_\varepsilon^a)$-capacity of Σ_0 will be zero. As we discussed earlier, in this situation, the *classical* Sobolev spaces will fail to capture information about the boundary condition on the lower dimensional set Σ_0. Consequently, the approximation problem with this type of weight loses critical information about the boundary condition, making it impossible to recover uniform regularity estimates for the weighted problem.

We adopt a different approach based on perforated domains, aiming to establish uniform estimates in this framework. For small $0 < \varepsilon \ll 1$, we define

$$\Sigma_\varepsilon := \{|y| \leq \varepsilon\},$$

as the ε-neighbourhood of Σ_0, noting that its boundary $\partial\Sigma_\varepsilon = \{|y| = \varepsilon\}$ has dimension $d-1$. The central idea is to approximate the lower dimensional boundary, which has dimension $d-n$, with a *classical* boundary of dimension $d-1$ and impose the Dirichlet condition on this set. Hence, we consider solutions to the problem

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla u) = |y|^a f + \operatorname{div}(|y|^a F), & \text{in } B_1 \setminus \Sigma_\varepsilon, \\ u = 0, & \text{on } \partial\Sigma_\varepsilon \cap B_1. \end{cases}$$

In this context, since the singular set Σ_0 is sufficiently far from $B_1 \setminus \Sigma_\varepsilon$, we can apply classical regularity theory to obtain $C^{0,\alpha}(B_{1/2} \setminus \Sigma_\varepsilon)$ and $C^{1,\alpha}(B_{1/2} \setminus \Sigma_\varepsilon)$ regularity estimates, with constants that may depend on ε. The crucial step is proving that these estimates are uniform as $\varepsilon \to 0$, as shown in Theorems 4.1 and 5.1. Once we have these uniform estimates, we employ an approximation argument (see Section 2.5) to pass to the limit as $\varepsilon \to 0$, thereby recovering the desired regularity results for the original problem.

We emphasize that proving $C^{1,\alpha}$ regularity for solutions to (1.1) requires a refined approach. Specifically, the uniform estimates in perforated domains require an additional assumption on the field F, as highlighted in Remark 5.2. Consequently, proving the main theorem necessitates a double approximation strategy: the first via perforated domains and the second through a standard mollification argument. This is combined with some *a priori* estimates for solutions to (1.1) under an additional boundary condition on Σ_0, as detailed in Proposition 5.3.

The strategy for establishing the ε-uniform estimates is based on a contradiction argument combined with a blow-up procedure, drawing inspiration from Simon's work [30]. A key component of this approach is the following Liouville-type theorem, which applies to entire solutions satisfying a specific growth-control condition at infinity.

Theorem 1.4. *Let $2 \le n \le d$, $a+n \in (0,2)$, $\varepsilon \ge 0$. Let A be a constant symmetric matrix satisfying* (1.2) *and u be an entire solution to*

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla u) = 0, & \text{in } \mathbb{R}^d \setminus \Sigma_\varepsilon, \\ u = 0, & \text{on } \partial\Sigma_\varepsilon, \end{cases} \tag{1.8}$$

(see Definition 2.3). Assume that there exist constants $c > 0$, $\gamma \in (0, 2-a-n)$ such that

$$|u(z)| \le c(1+|z|^\gamma), \quad \text{for a.e. } z \in \mathbb{R}^d \setminus \Sigma_\varepsilon, \tag{1.9}$$

Then, u is identically zero.

Finally, as a consequence of our main theorems, we extend our results to more general weighted equations, where the weight δ behaves like a distance function from a regular manifold Γ of dimension $d-n$ (see Definition 6.1). Specifically, in Corollaries 6.3 and 6.4, we are able to prove local $C^{0,\alpha}(B_{1/2})$ and $C^{1,\alpha}(B_{1/2})$ regularity estimates for weak solutions to the following equation

$$\begin{cases} -\operatorname{div}(\delta^a A\nabla u) = \delta^a f + \operatorname{div}(\delta^a F), & \text{in } B_1 \setminus \Gamma, \\ u = \psi, & \text{on } \Gamma \cap B_1. \end{cases} \tag{1.10}$$

The precise definition of solutions to (1.10) will be given later in Section 6.

Structure of the paper. The paper is organized as follows. In Section 2, we set up the problem by introducing weighted Sobolev spaces, discussing their basic properties, and providing the definition of weak solutions to our equation along with some preliminary results, including the approximation Lemma 2.9. Section 3 is devoted to the proof of the Liouville-type Theorem 1.4. In Sections 4 and 5, we establish the main results, Theorems 1.2 and 1.3, which concern $C^{0,\alpha}$ and $C^{1,\alpha}$ regularity, respectively. Lastly, in Section 6, we extend these results to solutions of the more general equation (1.10).

2. FUNCTIONAL SETTING AND PRELIMINARY RESULTS

2.1. **Weighted Sobolev Spaces.** Let $a+n \in (0,2)$, $R > 0$ and $B_R := \{z \in \mathbb{R}^d : |z| < R\}$ be the ball centered in 0 and radius R. We define the weighted Lebesgue spaces

$$L^{p,a}(B_R) := L^p(B_R, |y|^a dz),$$

and for vector field

$$L^{p,a}(B_R)^d := L^p(B_R, |y|^a dz)^d.$$

The Sobolev space $H^{1,a}(B_R)$ is defined as the completion of $C^\infty(\overline{B_R})$ with respect to the norm

$$\|u\|_{H^{1,a}(B_R)} = \Big(\int_{B_R} |y|^a u^2 dz + \int_{B_R} |y|^a |\nabla u|^2 dz\Big)^{1/2},$$

where $C^\infty(\overline{B_R}) = \{u_{|_{B_R}} : u \in C_c^\infty(\mathbb{R}^d)\}$.

Since we are interested in functions which vanish on $\Sigma_0 = \{|y| = 0\}$, we define the Sobolev space $\tilde{H}^{1,a}(B_R)$ as the completion of $C_c^\infty(\overline{B_R} \setminus \Sigma_0)$ with respect to the norm $\|\cdot\|_{H^{1,a}(B_R)}$.

Additionally, we define the Sobolev space $H_0^{1,a}(B_R)$ as as the completion of $C_c^\infty(B_R \setminus \Sigma_0)$ with respect to the norm $\|\cdot\|_{H^{1,a}(B_R)}$, which contains functions having zero trace on $\partial(B_R \setminus \Sigma_0)$.

2.2. **Weighted Sobolev Spaces in perforated domains.** In the rest of the paper, we use the notation $0<\varepsilon\ll 1$ to indicate that ε is a small positive number. Let us define

$$\Sigma_\varepsilon := \{(x,y) : |y|\le\varepsilon\}, \quad \partial\Sigma_\varepsilon := \{(x,y) : |y|=\varepsilon\}.$$

We set

$$L^{p,a}(B_R\setminus\Sigma_\varepsilon) := L^p(B_R\setminus\Sigma_\varepsilon, |y|^a dz),$$

and

$$L^{p,a}(B_R\setminus\Sigma_\varepsilon)^d := L^p(B_R\setminus\Sigma_\varepsilon, |y|^a dz)^d.$$

Let us define the norm

$$\|u\|_{H^{1,a}(B_R\setminus\Sigma_\varepsilon)} = \Big(\int_{B_R\setminus\Sigma_\varepsilon}|y|^a u^2 dz + \int_{B_R\setminus\Sigma_\varepsilon}|y|^a|\nabla u|^2 dz\Big)^{1/2}.$$

We define Sobolev spaces in perforated domains as follows.

- $H^{1,a}(B_R\setminus\Sigma_\varepsilon)$ as the completion of $C^\infty(\overline{B_R\setminus\Sigma_\varepsilon})$ w.r.t. the norm $\|\cdot\|_{H^{1,a}(B_R\setminus\Sigma_\varepsilon)}$,
- $\tilde H^{1,a}(B_R\setminus\Sigma_\varepsilon)$ as the completion of $C^\infty_c(\overline{B_R}\setminus\Sigma_\varepsilon)$ w.r.t. the norm $\|\cdot\|_{H^{1,a}(B_R\setminus\Sigma_\varepsilon)}$,
- $H^{1,a}_0(B_R\setminus\Sigma_\varepsilon)$ as the completion of $C^\infty_c(B_R\setminus\Sigma_\varepsilon)$ w.r.t. the norm $\|\cdot\|_{H^{1,a}(B_R\setminus\Sigma_\varepsilon)}$.

When $\varepsilon=0$, we identify the spaces

$$L^{p,a}(B_R\setminus\Sigma_0)=L^{p,a}(B_R), \quad \tilde H^{1,a}(B_R\setminus\Sigma_0)=\tilde H^{1,a}(B_R), \quad H^{1,a}_0(B_R\setminus\Sigma_0)=H^{1,a}_0(B_R).$$

Moreover, since the Poincaré inequality holds true (see Proposition 2.2) in $\tilde H^{1,a}(B_R\setminus\Sigma_\varepsilon)$ for every $0\le\varepsilon\ll 1$, we have that

$$\|u\|_{H^{1,a}_0(B_R\setminus\Sigma_\varepsilon)} = \Big(\int_{B_R\setminus\Sigma_\varepsilon}|y|^a|\nabla u|^2 dz\Big)^{1/2},$$

defines an equivalent norm to $\|\cdot\|_{H^{1,a}(B_R\setminus\Sigma_\varepsilon)}$ in $\tilde H^{1,a}(B_R\setminus\Sigma_\varepsilon)$.

Remark 2.1. For $\varepsilon>0$, functions in $\tilde H^{1,a}(B_R\setminus\Sigma_\varepsilon)$ can be identified with their trivial extensions in the whole B_R. Consequently, we have the following inclusion of spaces:

$$\tilde H^{1,a}(B_R\setminus\Sigma_\varepsilon)\subset\tilde H^{1,a}(B_R).$$

We note that this result is frequently used throughout the paper, particularly in the analysis of approximations on perforated domains.

The following proposition establishes several fundamental inequalities in the space $\tilde H^{1,a}(B_R\setminus\Sigma_\varepsilon)$: namely, the Hardy inequality, the Poincaré inequality, the Poincaré trace inequality, and a Sobolev-type inequality, all of which hold uniformly with respect to the parameter ε. These results ultimately rely on the Hardy inequality, which is by now a classical tool. We also refer to [7, 27] for other functional-type inequalities along these lines.

Proposition 2.2. *Let $2\le n\le d$, $a+n\in(0,2)$, $R>0$ and $0\le\varepsilon\ll 1$. Then, there exist a constant $c_R>0$, depending only on d, n, a and R such that*

$$\int_{B_R}|y|^a\frac{u^2}{|y|^2}dz \le c_R\int_{B_R}|y|^a|\nabla u|^2 dz, \tag{2.1}$$

$$\int_{B_R}|y|^a u^2 dz \le c_R\int_{B_R}|y|^a|\nabla u|^2 dz, \tag{2.2}$$

$$\int_{\partial B_R}|y|^a u^2 d\sigma \le c_R\int_{B_R}|y|^a|\nabla u|^2 dz, \tag{2.3}$$

$$\Big(\int_{B_R}|y|^a|u|^{2^*}dz\Big)^{2/2^*}dz\leq c_R\int_{B_R}|y|^a|\nabla u|^2dz, \tag{2.4}$$

for every $u\in C^\infty_c(\overline{B_R}\setminus\Sigma_\varepsilon)$. In the last inequality $2^*:=2d/(d-2)$ if $d>2$ and 2^* can be replaced by any $p\in[1,\infty)$ if $d=2$, and in this case, the constant $c_R>0$ also depends on p.

Proof. For the proof of the Hardy inequality (2.1), see [8, Proposition 3.4].

The Poincaré inequality (2.2) immediately follows by the validity of the Hardy inequality (2.1), in fact

$$\int_{B_R}|y|^au^2dz\leq c\int_{B_R}|y|^a\frac{u^2}{|y|^2}dz,$$

for some $c>0$ depending only on R.

By using the classical embedding $H^1(B_R)\hookrightarrow L^2(\partial B_R)$ to the function $|y|^{a/2}u\in C^\infty_c(\overline{B_R}\setminus\Sigma_\varepsilon)$, the Hardy inequality (2.1), combined with Hölder and Young inequalities, we have that

$$\begin{aligned}\int_{\partial B_R}|y|^au^2d\sigma&\leq c\int_{B_R}|\nabla(|y|^{a/2}u)|^2dz\leq c\int_{B_R}\Big(|y|^a|\nabla u|^2+\frac{a^2}{4}|y|^a\frac{u^2}{|y|^2}+a|y|^a\frac{|u|}{|y|}|\nabla u|\Big)dz\\&\leq c\int_{B_R}|y|^a|\nabla u|^2dz,\end{aligned}$$

and (2.3) holds.

Finally, let's prove the Sobolev embedding (2.4). By using $a<0$, the classical Sobolev embedding $H^1(B_R)\hookrightarrow L^{2^*}(B_R)$ to the function $|y|^{a/2}u\in C^\infty_c(\overline{B_R}\setminus\Sigma_\varepsilon)$, the Hardy inequality (2.1), the Poincaré inequality (2.2), combined with Hölder and Young inequalities, we obtain

$$\begin{aligned}\Big(\int_{B_R}|y|^a|u|^{2^*}dz\Big)^{2/2^*}&\leq c\Big(\int_{B_R}\big(|y|^{a/2}|u|\big)^{2^*}dz\Big)^{2/2^*}\leq c\int_{B_R}\Big(|y|^au^2+|\nabla(|y|^{a/2}u)|^2\Big)dz\\&\leq c\int_{B_R}\Big(|y|^au^2+|y|^a|\nabla u|^2+\frac{a^2}{4}|y|^a\frac{u^2}{|y|^2}+a|y|^a\frac{|u|}{|y|}|\nabla u|\Big)dz\leq c\int_{B_R}|y|^a|\nabla u|^2dz.\end{aligned}$$

Hence, the proof is complete. □

2.3. Weak solutions.

In this section we give the definition of weak solutions.

Definition 2.3. Let $2\leq n\leq d$, $a+n\in(0,2)$, $R>0$ and $0\leq\varepsilon\ll1$. Let A be matrix satisfying (1.2), $f\in L^{2,a}(B_R\setminus\Sigma_\varepsilon)$ and $F\in L^{2,a}(B_R\setminus\Sigma_\varepsilon)^d$. We say that u is a weak solution to

$$\begin{cases}-\operatorname{div}(|y|^aA\nabla u)=|y|^af+\operatorname{div}(|y|^aF), & \text{in } B_R\setminus\Sigma_\varepsilon,\\ u=0, & \text{on } \partial\Sigma_\varepsilon\cap B_R,\end{cases} \tag{2.5}$$

if $u\in\tilde H^{1,a}(B_R\setminus\Sigma_\varepsilon)$ and satisfies

$$\int_{B_R}|y|^aA\nabla u\cdot\nabla\phi dz=\int_{B_R}|y|^a(f\phi-F\cdot\nabla\phi)dz, \tag{2.6}$$

for every $\phi\in C^\infty_c(B_R\setminus\Sigma_\varepsilon)$.

We say that u is an entire solution to

$$\begin{cases}-\operatorname{div}(|y|^aA\nabla u)=|y|^af+\operatorname{div}(|y|^aF), & \text{in } \mathbb{R}^d\setminus\Sigma_\varepsilon,\\ u=0, & \text{on } \partial\Sigma_\varepsilon,\end{cases}$$

if u is a weak solution to (2.5) for every $R>0$.

Remark 2.4. Using the validity of the Poincaré inequality (2.2), we have existence and uniqueness for solutions to (2.5) which also satisfy a boundary condition on $\partial B_R \setminus \Sigma_\varepsilon$. In fact, if u is a weak solution to (2.5) satisfying $u - \bar{u} \in H_0^{1,a}(B_R \setminus \Sigma_\varepsilon)$, for some $\bar{u} \in \tilde{H}^{1,a}(B_R \setminus \Sigma_\varepsilon)$, then u is a minimizer to the functional

$$J(v) := \int_{B_R\setminus\Sigma_\varepsilon} |y|^a \Big(\frac{A\nabla v \cdot \nabla v}{2} - fv + F \cdot \nabla v \Big) dz,$$

over

$$X := \{ v \in H^{1,a}(B_R \setminus \Sigma_\varepsilon) : v - \bar{u} \in H_0^{1,a}(B_R \setminus \Sigma_\varepsilon) \},$$

and J is coercive. By a standard application of the Weierstrass Theorem, we have existence and uniqueness of solutions to (2.5) with prescribed trace on $\partial B_R \setminus \Sigma_\varepsilon$.

When $\varepsilon = 0$, recalling the trace Theorem 1.1, we also give a definition of weak solutions with prescribed trace on the lower dimensional boundary Σ_0.

Definition 2.5. Let $2 \le n \le d$, $a + n \in (0, 2)$, $R > 0$ and A satisfies (1.2). Let $f \in L^{2,a}(B_R)$, $F \in L^{2,a}(B_R)^d$ and $\psi \in L^2(\Sigma_0 \cap B_R)$. We say that u is a weak solution to

$$\begin{cases} -\operatorname{div}(|y|^a A \nabla u) = |y|^a f + \operatorname{div}(|y|^a F), & \text{in } B_R \setminus \Sigma_0, \\ u = \psi, & \text{on } \Sigma_0 \cap B_R, \end{cases}$$

if $u \in H^{1,a}(B_R)$, satisfies (2.6) for every $\phi \in C_c^\infty(B_R \setminus \Sigma_0)$ and $u = \psi$ on $\Sigma_0 \cap B_R$, in the sense of the trace.

2.4. **Local boundedness of solutions.**

The goal of this section is to prove $L^2 \to L^\infty$ estimates for weak solutions to (2.5). The proof is fairly standard and employs an iterative technique based on a Caccioppoli-type inequality and the Sobolev embeddings (2.4) (for example, see [34]). We include the proof for completeness. We start with the following Caccioppoli-type inequality.

Lemma 2.6. *Let $2 \le n \le d$, $a + n \in (0, 2)$, $R > 0$, $0 \le \varepsilon \ll 1$, $p \ge (2^*)'$ and $q \ge 2$. Let A be a matrix satisfying (1.2), $f \in L^{p,a}(B_R \setminus \Sigma_\varepsilon)$, $F \in L^{q,a}(B_R \setminus \Sigma_\varepsilon)^d$ and u be a weak solution to (2.5). Then, there exists $c > 0$ depending only on d, λ and Λ such that for every $0 < R_1 < R_2 < R$ there holds*

$$\int_{B_{R_1}\setminus\Sigma_\varepsilon} |y|^a |\nabla u|^2 dz \le c \Big[\frac{1}{(R_2 - R_1)^2} \int_{B_{R_2}\setminus\Sigma_\varepsilon} |y|^a |u|^2 dz + \|f\|^2_{L^{p,a}(B_{B_{R_2}}\setminus\Sigma_\varepsilon)} + \|F\|^2_{L^{q,a}(B_{B_{R_2}}\setminus\Sigma_\varepsilon)} \Big]. \tag{2.7}$$

Proof. Fix $0 < R_1 < R_2 < R$ and consider a smooth cut-off function $\eta \in C_c^\infty(B_R)$ such that

$$\operatorname{spt}(\eta) \subset B_{R_2}, \quad \eta = 1 \text{ on } B_{R_1}, \quad 0 \le \eta \le 1, \quad |\nabla \eta| \le \frac{c}{|R_2 - R_1|},$$

for some constant $c > 0$ depending only on d. Let us test the equation satisfied by u with $\eta^2 u$ (which is an admissible test function). Then, we obtain

$$\int_{B_R} |y|^a \eta^2 A \nabla u \cdot \nabla u dz = \int_{B_R} |y|^a \big(-2\eta u A \nabla u \cdot \nabla \eta + f \eta^2 u - F \cdot (\eta^2 \nabla u + 2\eta u \cdot \nabla \eta) \big) dz.$$

Let us fix $\mu > 0$ to be chosen later. By Hölder and Young inequalities, it follows that

$$\begin{aligned} \Big| \int_{B_R} |y|^a 2\eta u A \nabla u \cdot \nabla \eta dz \Big| &\le 2\Lambda \Big(\int_{B_R} |y|^a \eta^2 |\nabla u|^2 dz \Big)^{1/2} \Big(\int_{B_R} |y|^a |u|^2 |\nabla \eta|^2 dz \Big)^{1/2} \\ &\le \mu \int_{B_R} |y|^a \eta^2 |\nabla u|^2 dz + c_\mu \int_{B_R} |y|^a |u|^2 |\nabla \eta|^2 dz, \end{aligned}$$

where $c_\mu > 0$ depends on Λ and μ. Next, using the Sobolev embedding (2.4)

$$\begin{aligned}\Big|\int_{B_R} f\eta^2 u dz\Big| &\le c\|f\|_{L^{p,a}(B_{R_2}\setminus\Sigma_\varepsilon)}\|\eta u\|_{L^{2^*,a}(B_R\setminus\Sigma_\varepsilon)} \le c\|f\|_{L^{p,a}(B_{R_2}\setminus\Sigma_\varepsilon)}\|\nabla(\eta u)\|_{L^{2,a}(B_R\setminus\Sigma_\varepsilon)}\\ &\le c_\mu\|f\|^2_{L^{p,a}(B_{R_2}\setminus\Sigma_\varepsilon)} + \mu\int_{B_R}|y|^a\eta^2|\nabla u|^2dz + \mu\int_{B_R}|y|^a|u|^2|\nabla\eta|^2dz,\end{aligned}$$

where $c_\mu > 0$ depends on R and μ. Furthermore,

$$\Big|\int_{B_R} F\cdot(\eta^2\nabla u + 2\eta u\nabla\eta)dz\Big| \le c_\mu\|F\|^2_{L^{p,a}(B_{R_2}\setminus\Sigma_\varepsilon)} + \mu\int_{B_R}|y|^a\eta^2|\nabla u|^2dz + \mu\int_{B_R}|y|^a|u|^2|\nabla\eta|^2dz,$$

where $c_\mu > 0$ depends on R and μ. Hence, combining this estimates we obtain

$$\begin{aligned}&\lambda\int_{B_R}|y|^a\eta^2|\nabla u|^2dz \le \int_{B_R}|y|^a\eta^2A\nabla u\cdot\nabla u dz\\ &\le \Big|\int_{B_R}|y|^a 2\eta u A\nabla u\cdot\nabla\eta dz\Big| + \Big|\int_{B_R} f\eta^2 u dz\Big| + \Big|\int_{B_R} F\cdot(\eta^2\nabla u + 2\eta u\nabla\eta)dz\Big|\\ &\le c_\mu\Big(\int_{B_R}|y|^a|\nabla\eta|^2u^2dz + \|f\|^2_{L^{p,a}(B_{R_2}\setminus\Sigma_\varepsilon)} + \|F\|^2_{L^{p,a}(B_{R_2}\setminus\Sigma_\varepsilon)}\Big) + 3\mu\int_{B_R}|y|^a\eta^2|\nabla u|^2dz.\end{aligned}$$

Hence, choosing $\mu > 0$ small enough, we get that (2.7) holds true. □

The next lemma is to establish a no-spike estimate type.

Lemma 2.7. *Let $2 \le n \le d$, $a + n \in (0, 2)$, $0 < r < R$, $0 \le \varepsilon \ll 1$, $p > d/2$, $q > d$. Let A be a matrix satisfying (1.2), $f \in L^{p,a}(B_R \setminus \Sigma_\varepsilon)$, $F \in L^{q,a}(B_R \setminus \Sigma_\varepsilon)^d$ satisfying*

$$\|f\|_{L^{p,a}(B_R\setminus\Sigma_\varepsilon)} + \|F\|_{L^{q,a}(B_R\setminus\Sigma_\varepsilon)} \le 1.$$

Then, there exists a constant $\delta \in (0, 1)$, depending only on d, n, a, λ, Λ, p, q, r and R, such that if u is a weak solution to (2.5) in $B_R \setminus \Sigma_\varepsilon$ and it satisfies

$$\int_{B_R\setminus\Sigma_\varepsilon}|y|^a|u_+|^2dz \le \delta,$$

then

$$u \le 1 \qquad a.e.\ in\ B_r \setminus \Sigma_\varepsilon.$$

Conversely, if

$$\int_{B_R\setminus\Sigma_\varepsilon}|y|^a|u_-|^2dz \le \delta,$$

then

$$u \ge -1 \qquad a.e.\ in\ B_r \setminus \Sigma_\varepsilon.$$

Proof. We give the proof for the positive part u_+, since the other one follows the same argument.

For $\rho \in (0, R)$, fix a smooth cut-off function $\eta \in C_c^\infty(B_\rho)$ and a real number $b \in \mathbb{R}$. Let us consider $v := \eta^2(u - b)_+$ as test function in (2.5). Then,

$$\int_{B_\rho}|y|^aA\nabla u\cdot\nabla(\eta^2 v)dz = \int_{B_\rho}|y|^a\big(f\eta^2 v - F\cdot\nabla(\eta^2 v)\big)dz.$$

Using the same computations as in Lemma 2.6, combining with the fact that whenever $v > 0$, we have $\nabla u = \nabla v$, we obtain

$$\int_{B_\rho \setminus \Sigma_\varepsilon} |y|^a |\nabla(\eta v)|^2 dz \leq c\Big[\int_{B_\rho \setminus \Sigma_\varepsilon} |y|^a |\nabla \eta|^2 v^2 dz + \Big|\int_{B_\rho \setminus \Sigma_\varepsilon} |y|^a f \eta^2 v\Big| + \int_{B_\rho \setminus \Sigma_\varepsilon} |y|^a |F|^2 \chi_{\{v>0\}} dz\Big]. \tag{2.8}$$

For every $j \in \mathbb{N}$, set

$$C_j := 1 - 2^{-j}, \quad r_j := (R-r)2^{-j} + r, \quad D_j := B_{r_j} \setminus \Sigma_\varepsilon,$$

noting that $C_0 = 0$, $r_0 = R$, $C_j \uparrow 1$, $r_j \downarrow r$, $D_j \supset D_{j+1}$ and $r_j - r_{j+1} = (R-r)2^{-(j+1)}$. We define

$$V_j := (u - C_j)_+, \quad E_j := \int_{D_j} |y|^a V_j^2 dz,$$

which satisfy, for every $j \in \mathbb{N}$, $E_{j+1} \leq E_j \leq E_0 \leq \delta$ by assumption.

Let us consider a sequence of smooth function $\eta_j \in C_c^\infty(B_{r_j})$ satisfying $\eta_j = 1$ in $B_{r_{j+1}}$, $0 \leq \eta_j \leq 1$ and $|\nabla \eta_j| \leq c_d |r_j - r_{j+1}| \leq c_d 2^{j+1}$. Using (2.8) with $v = V_{j+1}$, $\rho = r_j$ and $\eta = \eta_j$ we get

$$\begin{aligned}\int_{D_j} |y|^a |\nabla(\eta_j V_{j+1})|^2 dz \leq c\Big[&\int_{D_j} |y|^a |\nabla \eta_j|^2 V_{j+1}^2 dz \\ &+ \Big|\int_{D_j} |y|^a f \eta_j^2 V_{j+1} dz\Big| + \int_{D_j} |y|^a |F|^2 \chi_{\{V_{j+1}>0\}} dz\Big].\end{aligned}$$

We estimate the first term in the right hand side as follows

$$\int_{D_j} |y|^a |\nabla \eta_j|^2 V_{j+1}^2 dz \leq c 2^{2(j+1)} E_{j+1}.$$

Next, let us fix $\tau = 2^*$ if $d \geq 3$ or $\tau > p'$ if $d = 2$. Since $p > d/2$, we can consider $\gamma > 1$ such that $p^{-1} + \tau^{-1} + \gamma^{-1} = 1$. By applying the Hölder inequality with exponent p, τ, γ, $\|f\|_{L^{p,a}(B_R \setminus \Sigma_\varepsilon)} \leq 1$ and the Sobolev embedding (2.4), it follows

$$\begin{aligned}\Big|\int_{D_j} |y|^a f \eta_j^2 V_{j+1}\Big| &\leq \|f\|_{L^{p,a}(D_j)} \Big(\int_{D_j} |y|^a |\eta_j V_{j+1}|^\tau dz\Big)^{1/\tau} \Big(\int_{D_j} |y|^a \chi_{\{V_{j+1}>0\}} dz\Big)^{1/\gamma} \\ &\leq c\Big(\int_{D_j} |y|^a |\nabla(\eta_j V_{j+1})|^2 dz\Big)^{1/2} \Big(\int_{D_j} |y|^a \chi_{\{V_{j+1}>0\}} dz\Big)^{1/\gamma} \\ &\leq \mu \int_{D_j} |y|^a |\nabla(\eta_j V_{j+1})|^2 dz + c_\mu \Big(\int_{D_j} |y|^a \chi_{\{V_{j+1}>0\}} dz\Big)^{2/\gamma},\end{aligned}$$

where in the last inequality we used the Young inequality and $\mu > 0$ is a small number to be chosen later. Since

$$\{V_{j+1} > 0\} = \{u - C_{j+1} > 0\} = \{u - C_j > 2^{-(j+1)}\} = \{V_j > 2^{-(j+1)}\},$$

it follows

$$\int_{D_j} |y|^a \chi_{\{V_{j+1}>0\}} dz = \int_{D_j} |y|^a \chi_{\{V_j^2 > 2^{-2(j+1)}\}} dz \leq 2^{2(j+1)} \int_{D_j} |y|^a V_j^2 dz = 2^{2(j+1)} E_j.$$

Using $\|F\|_{L^{q,a}(B_R\setminus\Sigma_\varepsilon)} \leq 1$, we obtain

$$\begin{aligned}\int_{D_j} |y|^a|F|^2\chi_{\{V_{j+1}>0\}}dz &\leq \Big(\int_{D_j} |y|^a|F|^q dz\Big)^{2/q}\Big(\int_{D_j} |y|^a\chi_{\{V_{j+1}>0\}}dz\Big)^{(q-2)/q}\\ &\leq \Big(\int_{D_j} |y|^a\chi_{\{V_{j+1}>0\}}dz\Big)^{q-2/q} \leq c^{j+1}E_j^{1-2/q}.\end{aligned}$$

Combining together the previous estimates, and choosing $\mu > 0$ small enough it follows

$$\int_{D_j} |y|^a|\nabla(\eta_j V_{j+1})|^2dz \leq c^{j+1}\big(E_j + E_j^{2/\gamma} + E_j^{1-2/q}\big).$$

Next, using the Hölder inequality in the definition of E_{j+1} and the Sobolev embedding (2.4), it follows that

$$\begin{aligned}E_{j+1} = \int_{D_{j+1}} |y|^aV_{j+1}^2dz &\leq \Big(\int_{D_{j+1}} |y|^a|V_{j+1}|^\tau dz\Big)^{2/\tau}\Big(\int_{D_{j+1}} |y|^a\chi_{\{V_{j+1}>0\}}dz\Big)^{(2^*-2)/2^*}\\ &\leq \int_{D_j} |y|^a|\eta_j V_{j+1}|^\tau dz\Big)^{2/\tau}c^{j+1}E_j^{1-2/\tau} \leq \Big(\int_{D_j} |y|^a|\nabla(\eta_j V_{j+1})|^2dz\Big)c^{j+1}E_j^{1-2/\tau}\\ &\leq c^{j+1}\big(E_j^{2-2/\tau} + E_j^{1+2/\gamma-2/\tau} + E_j^{2-2/q-2/\tau}\big) \leq c^{j+1}E_j^{1+\overline{\gamma}},\end{aligned}$$

where

$$\overline{\gamma} := \min\Big\{1-\frac{2}{\tau}, \frac{2}{\gamma}-\frac{2}{\tau}, 1-\frac{2}{q}-\frac{2}{\tau}\Big\} > 0.$$

The positivity of $\overline{\gamma}$ follows by the fact that for $d = 3$ we set $\tau = 2^*$, and for $d = 2$ we choose $\tau > p'$. Then,

$$\begin{cases} E_{j+1} \leq c^{j+1}E_j^{1+\overline{\gamma}},\\ E_0 \leq \delta,\end{cases}$$

which implies

$$E_j \leq c^{\sum_{i=0}^j i(1+\overline{\gamma})^{j-i}}E_0^{(1+\overline{\gamma})^j} \leq (C\delta)^{(1+\overline{\gamma})^j}.$$

Finally, by choosing δ such that $c\delta < 1$, and taking the limit as $j \to \infty$ we obtain that $E_j \to 0$, that is, $\int_{B_r\setminus\Sigma_\varepsilon} |y|^a(u-1)_+^2 = 0$, which yields $u \leq 1$ a.e. in $B_r \setminus \Sigma_\varepsilon$. □

Finally, the next lemma states the L^∞_{loc} boundedness of weak solutions to (2.5).

Lemma 2.8. *Let* $2 \leq n \leq d$, $a + n \in (0, 2)$, $R > 0$, $0 \leq \varepsilon \ll 1$, $p > d/2$, $q > d$. *Let* A *be a matrix satisfying* (1.2), $f \in L^{p,a}(B_R \setminus \Sigma_\varepsilon)$, $F \in L^{q,a}(B_R \setminus \Sigma_\varepsilon)^d$ *and let* u *be a weak solution to* (2.5). *Then, for every* $r \in (0, R)$, *there exists* $c > 0$ *depending only on* d, n, a, λ, Λ, p, q *and* r *such that*

$$\|u\|_{L^\infty(B_r\setminus\Sigma_\varepsilon)} \leq c\big(\|u\|_{L^{2,a}(B_R\setminus\Sigma_\varepsilon)} + \|f\|_{L^{p,a}(B_R\setminus\Sigma_\varepsilon)} + \|F\|_{L^{q,a}(B_R\setminus\Sigma_\varepsilon)}\big). \tag{2.9}$$

Proof. Let us define

$$v := \theta u, \quad \theta := \frac{\sqrt{\delta}}{\|u\|_{L^{2,a}(B_R\setminus\Sigma_\varepsilon)} + \|f\|_{L^{p,a}(B_R\setminus\Sigma_\varepsilon)} + \|F\|_{L^{q,a}(B_R\setminus\Sigma_\varepsilon)}},$$

where $\delta > 0$ is as in Lemma 2.7. One has that v satisfies the hypothesis of Lemma 2.7, hence, $v_+ \leq 1$ in $B_r \setminus \Sigma_\varepsilon$, which implies

$$\|u_+\|_{L^\infty(B_r\setminus\Sigma_\varepsilon)} \leq \frac{1}{\sqrt{\delta}}(\|u\|_{L^{2,a}(B_R\setminus\Sigma_\varepsilon)} + \|f\|_{L^{p,a}(B_R\setminus\Sigma_\varepsilon)} + \|F\|_{L^{q,a}(B_R\setminus\Sigma_\varepsilon)}).$$

Repeating the same argument with v_- one has that

$$\|u_-\|_{L^\infty(B_r\setminus\Sigma_\varepsilon)} \leq \frac{1}{\sqrt{\delta}}(\|u\|_{L^{2,a}(B_R\setminus\Sigma_\varepsilon)} + \|f\|_{L^{p,a}(B_R\setminus\Sigma_\varepsilon)} + \|F\|_{L^{q,a}(B_R\setminus\Sigma_\varepsilon)}).$$

Hence, we have that (2.9) holds true, by choosing $C = 2/\sqrt{\delta}$. The proof is complete. □

2.5. **Approximation result.** In the spirit of [31, Lemma 2.12, Lemma 2.15] and [1, Lemma 4.2], the goal of this section is to provide an approximation result, which allows us to construct a family of solutions to (2.5) in perforated domains ($0 < \varepsilon \ll 1$), which converges in a suitable sense to weak solutions of (2.5), when $\varepsilon = 0$.

Lemma 2.9. *Let $2 \leq n \leq d$, $a + n \in (0, 2)$, $R > 0$. Let A be a matrix satisfying (1.2), $f \in L^{2,a}(B_R)$, $F \in L^{2,a}(B_R)^d$ and let u be a weak solution to (2.5) with $\varepsilon = 0$.*

Then, for every $r \in (0, R)$, there exists a family $\{u_\varepsilon\}_{0<\varepsilon\ll 1}$, such that u_ε are weak solutions to

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla u_\varepsilon) = |y|^a f + \operatorname{div}(|y|^a F), & \text{in } B_r \setminus \Sigma_\varepsilon, \\ u = 0, & \text{on } \partial\Sigma_\varepsilon \cap B_r, \end{cases} \tag{2.10}$$

satisfying

$$\|u_\varepsilon\|_{H^{1,a}(B_r\setminus\Sigma_\varepsilon)} \leq c\big(\|u\|_{H^{1,a}(B_R)} + \|f\|_{L^{2,a}(B_R)} + \|F\|_{L^{2,a}(B_R)}\big), \tag{2.11}$$

for some constant $c > 0$ depending only on d, n, a, λ, Λ, R, r and, up to consider the trivial extension of u_ε in the whole B_r (see Remark 2.1), there exists a sequence $\varepsilon_k \to 0$ such that

$$u_{\varepsilon_k} \to u \text{ in } H^{1,a}(B_r).$$

Proof. Let us fix $r \in (0, R)$ and consider a cut-off function $\xi \in C_c^\infty(B_R)$ such that

$$\xi = 1 \text{ in } B_r, \quad \operatorname{spt}(\xi) \subset B_{\frac{R+r}{2}}, \quad 0 \leq \xi \leq 1, \quad |\nabla\xi| \leq c_0,$$

for some $c_0 > 0$ depending only on d, R and r, and define $\tilde{u} = \xi u \in H_0^{1,a}(B_R)$.

Fixed $\phi \in C_c^\infty(B_R \setminus \Sigma_0)$, and using the equation (2.6) satisfied by u, we get

$$\begin{aligned} \int_{B_R} |y|^a A\nabla\tilde{u} \cdot \nabla\phi dz &= \int_{B_R} |y|^a \Big(\xi A\nabla u \cdot \nabla\phi u A\nabla\xi \cdot \nabla\phi\Big) dz \\ &= \int_{B_R} |y|^a \Big(A\nabla u \cdot \nabla(\phi\xi) - \phi A\nabla u \cdot \nabla\xi + uA\nabla\xi \cdot \nabla\phi\Big) dz \\ &= \int_{B_R} |y|^a \Big(f\phi\xi - F \cdot \nabla(\phi\xi) - \phi A\nabla u \cdot \nabla\xi + uA\nabla\xi \cdot \nabla\phi\Big) dz \\ &= \int_{B_R} |y|^a \Big(f\phi\xi - \xi F \cdot \nabla\phi - \phi F \cdot \nabla\xi - \phi A\nabla u \cdot \nabla\xi + uA\nabla\xi \cdot \nabla\phi\Big) dz, \end{aligned}$$

that is, $\tilde{u}$ is a weak solution to

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla\tilde{u}) = |y|^a \tilde{f} + \operatorname{div}(|y|^a \tilde{F}), & \text{in } B_R \setminus \Sigma_0, \\ u = 0, & \text{on } \partial B_R \setminus \Sigma_0, \\ u = 0, & \text{on } \Sigma_0 \cap B_R, \end{cases} \tag{2.12}$$

where we have set

$$\tilde{f} = f\xi - F \cdot \nabla\xi - A\nabla u \cdot \nabla\xi, \qquad \tilde{F} = F\xi - uA\nabla\xi.$$

We estimate the right hand side in (2.12) in the following way

$$\begin{aligned}\int_{B_R}|y|^a\tilde{f}^2dz &\le 2\int_{B_R}|y|^a\Big((\xi f)^2+(F\cdot\nabla\xi)^2+(A\nabla u\cdot\nabla\xi)^2\Big)dz\\ &\le 2\int_{B_R}|y|^a\Big(f^2+c_0^2|F|^2+|A\nabla\xi|^2|\nabla u|^2\Big)dz\\ &\le c\int_{B_R}|y|^a\Big(f^2+|F|^2+|\nabla u|^2\Big)dz,\end{aligned}\tag{2.13}$$

for some $c>0$ depending only on d, Λ, R and r. By performing similar computations, we get

$$\int_{B_R}|y|^a|\tilde{F}|^2dz \quad\le c\int_{B_R}|y|^a\big(|F|^2+u^2\Big)dz,\tag{2.14}$$

for some $c>0$ depending only on d, Λ, R and r

Fixed $0<\varepsilon_0\ll 1$, for every $0<\varepsilon<\varepsilon_0$, recalling Remark 2.4, let u_ε be the unique weak solution to

$$\begin{cases}-\operatorname{div}(|y|^aA\nabla u_\varepsilon)=|y|^a\tilde{f}+\operatorname{div}(|y|^a\tilde{F}), & \text{in } B_R\setminus\Sigma_\varepsilon,\\ u_\varepsilon=0, & \text{on } \partial B_R\setminus\Sigma_\varepsilon,\\ u_\varepsilon=0, & \text{on } \Sigma_\varepsilon\cap B_R.\end{cases}\tag{2.15}$$

By using $u_\varepsilon\in H^{1,a}_0(B_R\setminus\Sigma_\varepsilon)$ as test function in (2.15), up to consider the trivial extension in the whole B_r (see Remark 2.1), combined with (1.2), Poincaré inequality (2.2) and Hölder inequality, we get

$$\begin{aligned}\lambda\int_{B_R}|y|^a|\nabla u_\varepsilon|^2dz &\le \int_{B_R}|y|^aA\nabla u_\varepsilon\cdot\nabla u_\varepsilon dz=\int_{B_R}|y|^a\Big(\tilde{f}u_\varepsilon+\tilde{F}\cdot\nabla u_\varepsilon\Big)dz\\ &\le \Big(\int_{B_R}|y|^a|\tilde{f}|^2dz\Big)^{1/2}\Big(\int_{B_R}|y|^a|u_\varepsilon|^2dz\Big)^{1/2}+\Big(\int_{B_R}|y|^a|\tilde{F}|^2dz\Big)^{1/2}\Big(\int_{B_R}|y|^a|\nabla u_\varepsilon|^2dz\Big)^{1/2}\\ &\le c\Big(\int_{B_R}|y|^a|\nabla u_\varepsilon|^2dz\Big)^{1/2}\Big(\|\tilde{f}\|_{L^{2,a}(B_R)}+\|\tilde{F}\|_{L^{2,a}(B_R)}\Big),\end{aligned}$$

and then, by using (2.13) and (2.14), we have that there exists a constant $c>0$ depending only on d, n, a, λ and Λ such that

$$\|u_\varepsilon\|_{H^{1,a}(B_R\setminus\Sigma_\varepsilon)}\le c\big(\|f\|_{L^{2,a}(B_R)}+\|F\|_{L^{2,a}(B_R)}+\|u\|_{H^{1,a}(B_R)}\big).\tag{2.16}$$

So, we get that $\{u_\varepsilon\}\subset H^{1,a}_0(B_R\setminus\Sigma_\varepsilon)\subset H^{1,a}_0(B_R)$ is uniformly bounded. Hence, there exists $\bar{u}\in H^{1,a}_0(B_R)$ and a sequence $\varepsilon_k\to 0$, such that

$$u_{\varepsilon_k}\rightharpoonup\bar{u},\quad \text{weakly in } H^{1,a}(B_R).\tag{2.17}$$

Next, we prove that $\bar{u}=\tilde{u}$. Let $\phi\in C^\infty_c(B_R\setminus\Sigma_0)$ be a test function in the equation (2.15) satisfied by u_ε. Then, $\mathrm{spt}(\phi)\subset B_R\setminus\Sigma_\varepsilon$ for every ε small enough. By using (2.17), we have

$$\int_{\mathrm{spt}(\phi)}|y|^a\tilde{f}\phi-\tilde{F}\cdot\nabla\phi=\int_{\mathrm{spt}(\phi)}|y|^aA\nabla u_{\varepsilon_k}\cdot\nabla\phi\to\int_{\mathrm{spt}(\phi)}|y|^aA\nabla\bar{u}\cdot\nabla\phi,\quad \text{as } \varepsilon_k\to 0,$$

so, $\bar{u}$ is a weak solution to

$$\begin{cases}-\operatorname{div}(|y|^aA\nabla\bar{u})=|y|^a\tilde{f}+\operatorname{div}(|y|^a\tilde{F}), & \text{in } B_R\setminus\Sigma_0,\\ \bar{u}=0, & \text{on } \partial B_R\setminus\Sigma_0,\\ \bar{u}=0, & \text{on } \Sigma_0\cap B_R.\end{cases}$$

By uniqueness of weak solution to (2.12) (see Remark 2.4), we get that $\bar{u} = \tilde{u}$ in $H^{1,a}_0(B_R)$.

Finally, we prove that $u_{\varepsilon_k} \to \tilde{u}$ strongly in $H^{1,a}(B_R)$. By testing (2.12) with $\tilde{u}$, we get

$$\int_{B_R} |y|^a A\nabla\tilde{u}\cdot\nabla\tilde{u}dz = \int_{B_R} |y|^a\Big(\tilde{f}\tilde{u} - \tilde{F}\cdot\nabla\tilde{u}\Big)dz, \tag{2.18}$$

and, by testing (2.15) with u_ε combined with (2.17), we have

$$\int_{B_R} |y|^a A\nabla u_\varepsilon\cdot\nabla u_\varepsilon dz = \int_{B_R} |y|^a\Big(\tilde{f}\tilde{u}_\varepsilon - \tilde{F}\cdot\nabla\tilde{u}_\varepsilon\Big)dz \to \int_{B_R} |y|^a\Big(\tilde{f}\tilde{u} - \tilde{F}\cdot\nabla\tilde{u}\Big)dz, \tag{2.19}$$

along a subsequence $\varepsilon_k \to 0$. Putting together (2.18) and (2.19) we obtain

$$\lim_{\varepsilon_k\to 0}\int_{B_R} |y|^a A\nabla u_{\varepsilon_k}\cdot\nabla u_{\varepsilon_k} dz = \int_{B_R} |y|^a A\nabla\tilde{u}\cdot\nabla\tilde{u}dz.$$

Since A satisfies (1.2), one has that $\|\nabla u_{\varepsilon_k}\|_{L^{2,a}(B_R)} \to \|\nabla\tilde{u}\|_{L^{2,a}(B_R)}$. This, combined with (2.17), allows us to assert that

$$u_{\varepsilon_k} \to \tilde{u}, \quad \text{strongly in } H^{1,a}(B_R). \tag{2.20}$$

Finally, since $\tilde{u} = u$, $\tilde{f} = f$, $\tilde{F} = F$ in B_r, we have that u_ε is a weak solution to (2.10) in B_r and, by using (2.16) and (2.20), our statement follows. □

3. LIOUVILLE THEOREMS

The goal of this section is to prove the Liouville type Theorem 1.4 for homogeneous entire solutions to (1.8) with constant coefficients. The proof is based on a spectral trace inequality which is stable with respect to ε, using an argument similar to [31, Theorem 3.4]. We start with a couple of results which are crucial to treat the case $n = d$.

Lemma 3.1. *Let $n = d$, $a + n \in (0, 2)$, $0 \le \varepsilon \ll 1$ and $r > 0$. Let $A \in \mathbb{R}^{n,n}$ be a positive definite symmetric matrix and define $\Omega_r := \{A^{-1}y\cdot y < r^2\}$. Then,*

$$\int_{\Omega_r} |y|^a A\nabla v\cdot\nabla v dz \ge (2 - a - n)\int_{\partial\Omega_r} |y|^a v^2 d\sigma,$$

for every $v \in C^\infty_c(\overline{\Omega}_r \setminus \Sigma_\varepsilon)$.

Proof. We provide the result for $r = 1$. The case for generic $r > 0$ follows by a scaling argument.

Since A is a positive definite symmetric matrix, it is well defined the square root $A^{1/2}$, which is a positive definite symmetric matrix too. The homogeneous function

$$\bar{u}(y) := |A^{-1/2}y|^{2-a-n},$$

is solution (in a point-wise sense) to

$$-\operatorname{div}(|y|^a A\nabla\bar{u}) = 0, \quad \text{in } \mathbb{R}^n \setminus \Sigma_0, \tag{3.1}$$

and satisfies

$$\nabla\bar{u}(y) = (2 - a - n)|A^{-1/2}y|^{-a-n}A^{-1}y, \qquad \nabla\bar{u}(y)\cdot y = (2 - a - n)\bar{u}(y). \tag{3.2}$$

Indeed, equations (3.1) and (3.2) are verified by a straightforward computation.

Fix $v \in C^\infty_c(\overline{\Omega}_1 \setminus \Sigma_\varepsilon)$. Then,

$$\int_{\Omega_1} |y|^a A\nabla\bar{u}\cdot\nabla\Big(\frac{v^2}{\bar{u}}\Big)dy = \int_{\Omega_1} |y|^a\Big(A\nabla v\cdot\nabla v - \Big|A^{1/2}\nabla v - \frac{v}{\bar{u}}A^{1/2}\nabla\bar{u}\Big|^2\Big)dy \le \int_{\Omega_1} |y|^a A\nabla v\cdot\nabla v dy, \tag{3.3}$$

On the other hand, by using the divergence theorem, (3.1) and (3.2), we have

$$\int_{\Omega_1} |y|^a A\nabla\bar u\cdot\nabla\Big(\frac{v^2}{\bar u}\Big)dy=\int_{\partial\Omega_1}|y|^a\frac{v^2}{\bar u}A\nabla\bar u\cdot\nu d\sigma=(2-a-n)\int_{\partial\Omega_1}|y|^a v^2 d\sigma, \tag{3.4}$$

so, putting together (3.3) and (3.4), our statement follows. □

Lemma 3.2. *Let $n=d$, $a+n\in(0,2)$, $R>0$ and $0\le\varepsilon\ll 1$. Let $A\in\mathbb{R}^{n,n}$ be a positive definite diagonal matrix and define $\Omega_r:=\{A^{-1}y\cdot y<r^2\}$, for every $r>0$ such that $\partial\Omega_r\subset B_R\setminus\Sigma_\varepsilon$. Let u be a weak solution to*

$$\begin{cases}-\operatorname{div}(|y|^a A\nabla u)=0, & \text{in } B_R\setminus\Sigma_\varepsilon,\\ u=0, & \text{on } \partial\Sigma_\varepsilon\cap B_R,\end{cases} \tag{3.5}$$

Up to consider the trivial extension of u in the whole B_R (see Remark 2.1), let us define

$$E(u,r):=\frac{1}{r^{n+a-2}}\int_{\Omega_r}|y|^a A\nabla u\cdot\nabla u dy,$$
$$H(u,r):=\frac{1}{r^{n+a-1}}\int_{\partial\Omega_r}|y|^a u^2 d\sigma.$$

Then,

$$\partial_r H(v,r)=\frac{2}{r}E(v,r),\qquad \text{for every } r\in(0,R).$$

Proof. When $\varepsilon>0$, classical regularity theory ensures that the function u is smooth in $\overline{\Omega_r\setminus\Sigma_\varepsilon}$. Consequently, the result immediately follows trough explicit computations.

When $\varepsilon=0$, we proceed by an approximation argument. Fixed $0<\delta\ll 1$, by using the approximation Lemma 2.9, we find a family $\{u_\varepsilon\}_{0<\varepsilon\ll 1}$ of solutions to (3.5) in $B_{R-\delta}\setminus\Sigma_\varepsilon$ such that $u_{\varepsilon_k}\to u$ in $H^{1,a}(B_{R-\delta})$ along a sequence $\varepsilon_k\to 0$ and, by applying the trace Poincaré inequality (2.3) (which also holds in Ω_r) we get that $v_{\varepsilon_k}\to v$ in $L^{2,a}(\partial\Omega_r)$. Hence, we have that

$$\begin{aligned}&\int_{\Omega_r}|y|^a A\nabla u_{\varepsilon_k}\cdot\nabla u_{\varepsilon_k}dy\to\int_{\Omega_r}|y|^a A\nabla u\cdot\nabla u dy,\\ &\int_{\partial\Omega_r}|y|^a u_{\varepsilon_k}^2 d\sigma\to\int_{\partial\Omega_r}|y|^a u^2 d\sigma.\end{aligned} \tag{3.6}$$

By utilizing the result obtained in the case $\varepsilon>0$ one finds that

$$\partial_r H(u_{\varepsilon_k},r)=\frac{2}{r}E(u_{\varepsilon_k},r). \tag{3.7}$$

By applying (3.6), we can take the limit as $\varepsilon_k\to 0$ in (3.7) to obtain $\partial_r H(u,r)=\frac{2}{r}E(u,r)$. □

The following lemma allows us to handle the unweighted variables x. Its proof relies on the method of difference quotients and an iterative application of the Caccioppoli-type inequality (2.7). For a detailed proof, see [33, Corollary 4.2, Lemma 4.3] in a quite similar context.

Lemma 3.3. *Let $2\le n<d$, $a+n\in(0,2)$, $\varepsilon\ge 0$. Let A be a constant symmetric matrix satisfying (1.2) and u be an entire solution to (1.8). Then, the following holds true.*

- *i) For every $i=1,\dots,d-n$, the function $\partial_{x_i}u$ is an entire solution to the same problem.*
- *i) If u satisfies the growth condition (1.9) for some $\gamma>0$, then u must be a polynomial in the variable x of degree almost $\lfloor\gamma\rfloor$.*

Proof of Theorem 1.4. Let u be an entire solution to (1.8) and let us suppose that $n = d$ and so $z = y \in \mathbb{R}^n$.

By contradiction let us suppose that $u \not\equiv 0$. Let $r_0 > 0$ such that $\Sigma_\varepsilon \subset \Omega_r := \{A^{-1}y \cdot y \leq r^2\}$ for every $r \geq r_0$ and define

$$E(u,r) = \frac{1}{r^{n+a-2}} \int_{\Omega_r} |y|^a A\nabla u \cdot \nabla u dy,$$

$$H(u,r) = \frac{1}{r^{n+a-1}} \int_{\partial\Omega_r} |y|^a u^2 d\sigma.$$

By applying Lemma 3.1 and Lemma 3.2, we get

$$\partial_r H(u,r) = \frac{2}{r} E(u,r) \geq \frac{2(2-a-n)}{r} H(u,r),$$

which implies

$$H(u,r) \geq H(u,r_0) r^{2(2-a-n)}, \qquad \text{for every } r > r_0,$$

by Gronwall's inequality. On the other hand, since A satisfies (1.2), the growth condition (1.9) implies

$$H(u,r) \leq c(1 + r^{2\gamma}).$$

Combining these two inequalities we get

$$H(u,r_0) \leq c r^{2(\gamma-(2-a-n))}.$$

Taking the limit as $r \to \infty$ and using $\gamma < 2 - a - n$ we get $H(u, r_0) = 0$. Since $r_0 > 0$ is arbitrary, we deduce that $u \equiv 0$ in $\mathbb{R}^d \setminus \Omega_{r_0}$. Moreover, since u is a solution to (1.8) and satisfies $u = 0$ on $\partial(\Omega_{r_0} \setminus \Sigma_\varepsilon)$, we apply the existence and uniqueness result (see Remark 2.4) to conclude that $u \equiv 0$ in $\Omega_{r_0} \setminus \Sigma_\varepsilon$. Therefore, $u \equiv 0$ in $\mathbb{R}^d \setminus \Sigma_\varepsilon$, which leads to a contradiction.

Let us consider the case $n < d$. By Lemma 3.3, one has that u is polynomial in the variable x. Hence, if $\gamma \in (0,1)$, the function u must be constant in x, so $u(x,y) = u(y)$ and our statement follows by using the result obtained in the case $n = d$. If $\gamma \in [1,2)$, we have that u must be linear in x, that is

$$u(x,y) = u_i(y) + \sum_{i=1}^{d-n} x_i u_i(y),$$

for some unknown functions $u_i(y)$. First,

$$|u_0(y)| = |u(0,y)| \leq c(1 + |y|^\gamma).$$

On the other hand,

$$|u(e_{x_i}, y)| = |u_i(y) + u_0(y)| \leq c(1 + |y|^\gamma),$$

and so

$$|u_i(y)| \leq |u_0(y)| + c(1 + |y|^\gamma) \leq c(1 + |y|^\gamma).$$

Hence, every u_i satisfies the growth condition (1.9) for every $i = 0, \dots, d - n$.

Next, for every $i = 1, \dots, d - n$, by applying Lemma 3.3 we have that $\partial_{x_i} u(x,y) = u_i(y)$ is an entire solution to (1.8) and satisfies (1.9). Then, the result obtained in the case $d = n$ allows us to conclude that $u_i = 0$ for every $i = 1, \dots, d - n$ and so $u(x,y) = u_0(y)$. Hence, using again the case $d = n$, we have that u must be zero and our statement follows. □

4. Hölder estimates for weak solutions

The goal of this section is to prove Theorem 1.2, which we obtain as a by-product of ε-uniform Hölder estimates for solutions in perforated domains and the approximation Lemma 2.9.

Theorem 4.1. *Let $2 \le n \le d$, $a+n \in (0,2)$, $p > d/2$, $q > d$ and α satisfying (1.3). Let A be a continuous symmetric matrix satisfying (1.2) and ω be a modulus of continuity such that*

$$\|A\|_{L^\infty(B_1)} + \sup_{z,z' \in B_1, z \neq z'} \frac{|A(z)-A(z')|}{\omega(|z-z'|)} \le L. \tag{4.1}$$

Let $f \in L^{p,a}(B_1)$ and $F \in L^{q,a}(B_1)^d$. For $0 < \varepsilon \ll 1$, let $\{u_\varepsilon\}$ be a family of solutions to

$$\begin{cases} -\operatorname{div}(|y|^a A \nabla u_\varepsilon) = |y|^a f + \operatorname{div}(|y|^a F), & \text{in } B_1 \setminus \Sigma_\varepsilon, \\ u_\varepsilon = 0, & \text{on } \partial\Sigma_\varepsilon \cap B_1. \end{cases} \tag{4.2}$$

Then, there exists a constant $c > 0$, depending only on d, n, a, λ, Λ, p, q, α and L such that

$$\|u_\varepsilon\|_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_\varepsilon)} \le c\big(\|u_\varepsilon\|_{L^{2,a}(B_1\setminus\Sigma_\varepsilon)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{L^{q,a}(B_1)}\big). \tag{4.3}$$

Proof. By classical regularity theory, we know that solutions to (4.2) are $C^{0,\alpha}(B_{1/2} \setminus \Sigma_\varepsilon)$ and that (4.3) holds with a constant $c > 0$ that may also depend on ε. Our goal is to show that it is possible to provide a constant $c > 0$ that is uniform in ε.

Without loss of generality, we can assume that

$$\|u_\varepsilon\|_{L^{2,a}(B_1\setminus\Sigma_\varepsilon)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{L^{q,a}(B_1)} \le c,$$

for some $c > 0$, which not depends on ε. Moreover, by using the local uniform bound of weak solutions in (see Lemma 2.8), it follows that

$$\|u_\varepsilon\|_{L^\infty(B_{3/4}\setminus\Sigma_\varepsilon)} \le c, \tag{4.4}$$

for some $c > 0$, which not depends on ε.

Step 1. Contradiction argument and blow-up sequences. By contradiction let us suppose that there exist $p > d/2$, $q > d$, α satisfying (1.3), $\{u_k\}_k := \{u_{\varepsilon_k}\}_k$ as $\varepsilon_k \to 0$ such that

$$\begin{cases} -\operatorname{div}(|y|^a A \nabla u_k) = |y|^a f + \operatorname{div}(|y|^a F), & \text{in } B_1 \setminus \Sigma_{\varepsilon_k}, \\ u_k = 0, & \text{on } \partial\Sigma_{\varepsilon_k} \cap B_1, \end{cases} \tag{4.5}$$

and

$$\|u_k\|_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_{\varepsilon_k})} \to \infty.$$

Let us fix a smooth cut-off function $\eta \in C^\infty_c(B_1)$ such that

$$\operatorname{spt}(\phi) \subset B_{3/4}, \quad \eta = 1 \text{ in } B_{1/2}, \quad 0 \le \eta \le 1.$$

By (4.4), one has that

$$L_k := [\eta u_k]_{C^{0,\alpha}(B_1\setminus\Sigma_{\varepsilon_k})} \to \infty.$$

By definition of Hölder seminorm, take two sequences of points $z_k = (x_k, y_k), \hat{z}_k = (\hat{x}_k, \hat{y}_k) \in B_1 \setminus \Sigma_{\varepsilon_k}$ such that

$$\frac{|(\eta u_k)(z_k) - (\eta u_k)(\hat{z}_k)|}{|z_k - \hat{z}_k|^\alpha} \ge \frac{L_k}{2}, \tag{4.6}$$

define $r_k := |z_k - \hat{z}_k|$ and observe that at least one of z_k or $\hat{z}_k$ belongs to $B_{3/4} \setminus \Sigma_{\varepsilon_k}$. First, by using the local uniform bound of weak solutions (4.4), we have that $r_k \to 0$, in fact

$$L_k \leq \frac{4\|\eta u_k\|_{L^\infty(B_{3/4}\setminus\Sigma_{\varepsilon_k})}}{r_k^\alpha} \leq \frac{c}{r_k^\alpha},$$

which implies

$$r_k \leq \frac{c^{1/\alpha}}{L_k^{1/\alpha}} \to 0, \qquad \text{as } k \to \infty.$$

From now on we distinguish three cases.

- **Case 1:** $\dfrac{|y_k|}{r_k} \to \infty, \quad \dfrac{|y_k| - \varepsilon_k}{r_k} \to \infty,$
- **Case 2:** $\dfrac{|y_k|}{r_k} \to \infty, \quad \dfrac{|y_k| - \varepsilon_k}{r_k} \leq c,$
- **Case 3:** $\dfrac{|y_k|}{r_k} \leq c,$

for some constant $c > 0$ which not depends on k. Let $z_k^0 := (x_k, y_k^0)$ be the projection of z_k on $\partial\Sigma_{\varepsilon_k}$ and define

$$\tilde{z}_k = (\tilde{x}_k, \tilde{y}_k) := \begin{cases} (x_k, y_k), & \text{in } \textbf{Case 1}, \\ (x_k, y_k^0), & \text{in } \textbf{Case 2}, \\ (x_k, 0), & \text{in } \textbf{Case 3}. \end{cases}$$

Define the sequence of domains

$$\Omega_k := \frac{B_1 \setminus \Sigma_{\varepsilon_k} - \tilde{z}_k}{r_k} = \Big\{ z = (x,y) : |\tilde{z}_k + r_k z| < 1, \text{ and } |\tilde{y}_k + r_k y| > \varepsilon_k \Big\},$$

and, for every $z \in \Omega_k$, let us define the sequence of functions

$$v_k(z) := \frac{(\eta u_k)(\tilde{z}_k + r_k z) - (\eta u_k)(\tilde{z}_k)}{r_k^\alpha L_k}, \qquad w_k(z) := \frac{\eta(\tilde{z}_k)\big(u_k(\tilde{z}_k + r_k z) - u_k(\tilde{z}_k)\big)}{r_k^\alpha L_k},$$

in **Case 1** and **Case 2**, and

$$v_k(z) := \frac{(\eta u_k)(\tilde{z}_k + r_k z)}{r_k^\alpha L_k}, \qquad w_k(z) := \frac{\eta(\tilde{z}_k) u_k(\tilde{z}_k + r_k z)}{r_k^\alpha L_k},$$

in **Case 3**.

Step 2. Blow-up domains. Let us define

$$\Omega_\infty := \big\{ z = (x,y) \in \mathbb{R}^d : \text{exists } \hat{k} \text{ such that } z \in \Omega_k \text{ for every } k \geq \hat{k} \big\}. \tag{4.7}$$

In this section we show who is the limit domain $\Omega_\infty := \lim_{k\to\infty} \Omega_k$, along a suitable subsequence. First, in every case, for every $z \in \mathbb{R}^d$, one has that

$$|\tilde{z}_k + r_k z| < |z_k| + r_k|z| \leq 3/4 + o(1) < 1.$$

Hence, to prove that $z \in \Omega_\infty$, we only need to show that $\tilde{z}_k + r_k z \notin \Sigma_{\varepsilon_k}$, that is,

$$|\tilde{y}_k + r_k y| > \varepsilon_k. \tag{4.8}$$

Let us start with **Case 1**, recalling that $\tilde{z}_k = z_k$. Fix $z \in \mathbb{R}^d$ and by contradiction let us suppose that (4.8) does not hold. Then, since $|\cdot|$ is a Lipschitz function, one has that

$$\frac{|y_k| - \varepsilon_k}{r_k} \leq \frac{|y_k| - |y_k + r_k y|}{r_k} \leq c|y|,$$

and taking the limit as $k \to \infty$, it follows

$$|y| \geq \infty,$$

which is a contradiction. Hence, $\Omega_\infty = \mathbb{R}^d$.

Next, let us consider the **Case 2** and recall that $\tilde{z}_k = (x_k, y_k^0)$, $|y_k^0| = \varepsilon_k$, $r_k/\varepsilon_k \to 0$. Defining

$$\bar{e} := \lim_{k\to\infty} \frac{y_k^0}{|y_k^0|},$$

we claim that $\Omega_\infty = \Pi := \{(x,y) : y \cdot \bar{e} > 0\}$, which is an half-space. We observe that, for every $y \in \mathbb{R}^n$,

$$\frac{|y_k^0 + r_k y| - |y_k^0|}{r_k} - \frac{y_k^0}{|y_k^0|} \cdot y \leq c\frac{r_k}{\varepsilon_k} \to 0, \tag{4.9}$$

as $k \to \infty$. Indeed, by using Lagrange's Theorem to the function $|\cdot|$, there exists y^* (which could depend on k) such that $|y^*| \leq |y|$ and denoting $y_k^* := y_k^0 + r_k y^*$, we have

$$\begin{aligned}\Big|\frac{|y_k^0 + r_k y| - |y_k^0|}{r_k} - \frac{y_k^0 \cdot y}{|y_k^0|}\Big| &= \Big|\frac{y_k^* \cdot y}{|y_k^*|} - \frac{y_k^0 \cdot y}{|y_k^0|}\Big| \leq \Big|\frac{y_k^* \cdot y}{|y_k^*|} - \frac{y_k^* \cdot y}{|y_k^0|}\Big| + \Big|\frac{y_k^* \cdot y}{|y_k^0|} - \frac{y_k^0 \cdot y}{|y_k^0|}\Big| \\ &\leq c\Big|\frac{|y_k^0| - |y_k^*|}{|y_k^0|}\Big| + c\Big|\frac{|y_k^* - y_k^0|}{|y_k^0|}\Big| \leq c\frac{r_k}{|y_k^0|} = c\frac{r_k}{\varepsilon_k} \to 0, \quad \text{as } k \to \infty,\end{aligned}$$

so (4.9) holds true. Let us fix $z = (x,y)$ such that $\bar{e} \cdot y = \delta > 0$ and suppose by contradiction that (4.8) doesn't hold, that is

$$\frac{|y_k^0 + r_k y| - |y_k^0|}{r_k} - \frac{y_k^0}{|y_k^0|} \cdot y + \frac{y_k^0}{|y_k^0|} \cdot y \leq 0.$$

So, by taking the limit as $k \to \infty$ and using (4.9), we obtain

$$\bar{e} \cdot y \leq 0,$$

which is a contradiction. In analogous way, we have that every $z = (x,y)$ such $|\bar{e} \cdot y| = -\delta < 0$ satisfies $z \notin \Omega_k$. Hence, since $\delta > 0$ is arbitrary, the claim follows, that is, $\Omega_\infty = \Pi$.

Finally, let us consider the **Case 3**, recall that $\varepsilon_k < |y_k| \leq c r_k$ and $\tilde{z}_k = (x_k, 0)$. First, up to consider a subsequence, the following limit is well defined

$$\bar{\varepsilon} := \lim_{k\to\infty} \frac{\varepsilon_k}{r_k} \in [0, c].$$

Let us fix $z = (x,y)$ such that $|y| = \bar{\varepsilon} + \delta$ for some $\delta > 0$ and suppose by contradiction that (4.8) doesn't hold. Then,

$$\bar{\varepsilon} + \delta = |y| \geq \frac{\varepsilon_k}{r_k} \to \bar{\varepsilon},$$

which is a contradiction, so $z \in \Omega_k$. Instead, fixed $z = (x,y)$ such that $|y| = \bar{\varepsilon} - \delta$ for some $\delta > 0$, one has $z \notin \Omega_k$. Since $\delta > 0$ is arbitrary, we have that

$$\Omega_\infty = \mathbb{R}^d \setminus \Sigma_{\bar{\varepsilon}} = \{(x,y) : |y| > \bar{\varepsilon}\}.$$

Resuming, we have shown that the limit domain is

$$\Omega_\infty = \begin{cases} \mathbb{R}^d, & \text{in } \textbf{Case 1}, \\ \Pi, & \text{in } \textbf{Case 2}, \\ \mathbb{R}^d \setminus \Sigma_{\bar\varepsilon}, & \text{in } \textbf{Case 3}, \end{cases} \tag{4.10}$$

where $\Pi := \{(x,y) \in \mathbb{R}^d : \bar e \cdot y \geq 0\}$ is an half-space.

Step 3. Hölder estimates and convergence of the blow-up sequences. Let us fix a compact set $K \subset \Omega_\infty$. For every $z, z' \in K$ such that $z \neq z'$, we have

$$|v_k(z) - v_k(z')| = \frac{|(\eta u_k)(\tilde z_k + r_k z) - (\eta u_k)(\tilde z_k + r_k z')|}{r_k^\alpha L_k} \leq |z - z'|^\alpha,$$

that is,

$$[v_k]_{C^{0,\alpha}(K)} \leq 1. \tag{4.11}$$

In **Case 1** and **Case 2**, by using $v_k(0) = 0$, we get the uniform bound $\|v_k\|_{C^{0,\alpha}(K)} \leq c$, for every compact subset $K \subset \Omega_\infty$. Instead, in **Case 3**, since $\varepsilon_k / r_k \leq c$, one has that

$$|v_k(x,y)| = |v_k(x,y) - v_k(0, y_k^0/r_k)| \leq [v_k]_{C^{0,\alpha}(K)} (|x| + |y - y_k^0/r_k|)^\alpha \leq c,$$

for some $c > 0$ which depends only on K, where we have used the boundary condition $\eta u_k = 0$ on $\partial\Sigma_{\varepsilon_k}$. Hence, we have that $\|v_k\|_{L^\infty(K)} \leq c$, which implies that $\|v_k\|_{C^{0,\alpha}(K)} \leq c$ also in this case.

By applying the Arzelà-Ascoli Theorem we conclude that $v_k \to \bar v$ uniformly in K. By a standard diagonal argument, we can take the the limit as $k \to \infty$ in (4.11) to obtain

$$[\bar v]_{C^{0,\alpha}(\Omega_\infty)} \leq 1,$$

which implies that $\bar v$ satisfies the growth condition

$$|\bar v| \leq c(1 + |z|^\alpha), \quad \text{a.e. in } \Omega_\infty. \tag{4.12}$$

Moreover, since $u_{\varepsilon_k} = 0$ on $\partial\Sigma_{\varepsilon_k}$, and then $v_k = 0$ on $(\partial\Sigma_{\varepsilon_k} - \tilde z_k)/r_k$, by employing the local uniform convergence we have that $\bar v = 0$ on $\partial\Pi$ in **Case 2** and $\bar v = 0$ on $\partial\Sigma_{\bar\varepsilon}$ in **Case 3**.

Furthermore, the sequences v_k and w_k converge to the same limit function. Let us fix a compact set $K \subset \Omega_\infty$. For every $z \in K$, by using (4.4), we have

$$|v_k(z) - w_k(z)| \leq \frac{(\eta(\tilde z_k + r_k z) - \eta(\tilde z_k)) \|u_k\|_{L^\infty(B_{3/4} \setminus \Sigma_{\varepsilon_k})}}{r_k^\alpha L_k} \leq \frac{c r_k^{1-\alpha}}{L_k} \to 0, \quad \text{as } k \to \infty.$$

Hence, the sequences v_k and w_k have the same asymptotic behaviour as $k \to \infty$ on every $K \subset \Omega_\infty$, which implies that $w_k \to \bar v$ uniformly on K.

Step 4. The limit function $\bar v$ is not constant. Let us consider the sequences of points

$$\xi_k^1 := \frac{z_k - \tilde z_k}{r_k}, \quad \xi_k^2 := \frac{\hat z_k - \tilde z_k}{r_k}.$$

By (4.6), we have

$$|v_k(\xi_k^1) - v_k(\xi_k^2)| = \frac{|(\eta u_k)(\hat z_k) - (\eta u_k)(z_k)|}{r_k^\alpha L_k} \geq \frac{1}{2}.$$

In **Case 1**, we have that

$$\xi_k^1 = 0, \quad \xi_k^2 = \frac{\hat z_k - z_k}{r_k} \in \partial B_1,$$

then, $\xi_k^1 \to 0$, $\xi_k^2 \to \xi^2 \neq 0$.

In **Case 2**,

$$\xi_k^1 = \frac{(0, y_k - y_k^0)}{r_k}, \quad \xi_k^2 = \frac{(\hat{x}_k - x_k, \hat{y}_k - y_k^0)}{r_k}.$$

Since $(|y_k| - \varepsilon_k)/r_k \leq c$ uniformly in k, $|\xi_k^1 - \xi_k^2| = 1$, we have that $\xi_k^1 \to \xi^1$, $\xi_k^2 \to \xi^2$ and $\xi^1 \neq \xi^2$.

In **Case 3**,

$$\xi_k^1 = \frac{(0, y_k)}{r_k}, \quad \xi_k^2 = \frac{(\hat{x}_k - x_k, \hat{y}_k)}{r_k}.$$

Since $|y_k|/r_k \leq c$ uniformly in k, $|\xi_k^1 - \xi_k^2| = 1$, we have that $\xi_k^1 \to \xi^1$, $\xi_k^2 \to \xi^2$ and $\xi^1 \neq \xi^2$.

Hence, by the local uniform convergence $v_k \to \bar{v}$ we get that $|\bar{v}(\xi^1) - \bar{v}(\xi^2)| \geq 1/2$, so $\bar{v}$ is not constant.

Step 5. $\bar{v}$ is an entire solution to a homogeneous equation with constant coefficients. Let us define $A_k(x,y) := A(\tilde{z}_k + r_k z)$ and $(\bar{x}, \bar{y}) = \lim_{k\to\infty}(\tilde{x}_k, \tilde{y}_k)$. By (4.1), we have that

$$|A_k(z) - A_k(z')| = |A_k(\tilde{z}_k + r_k z) - A_k(\tilde{z}_k + r_k z')| \leq L\omega(r_k|z - z'|) \to 0, \qquad \text{as } k \to \infty,$$

where ω is a modulus of continuity. Hence, the Arzelá-Ascoli theorem yiels that the matrix $\bar{A} := \lim_{k\to\infty} A_k(z) = A(\bar{x}, \bar{y})$ is well defined and it is a constant coefficients symmetric matrix satisfying (1.2). Next, set

$$\rho_k(y) := \begin{cases} \dfrac{|\tilde{y}_k + r_k y|}{|\tilde{y}_k|}, & \text{in } \textbf{Case 1} \text{ and } \textbf{Case 2}, \\ |y|, & \text{in } \textbf{Case 3}, \end{cases}$$

observing that in **Case 1** and **Case 2**

$$|\rho_k(y)| = 1 + o(1), \tag{4.13}$$

as $k \to \infty$ on every compact subset $K \subset \mathbb{R}^d$.

Fix $\phi \in C_c^\infty(\Omega_\infty)$. since u_k is a weak solution to (4.5), a straightforward computation shows us that

$$\begin{aligned} \int_{\mathrm{spt}(\phi)} \rho_k^a(y) A_k(z) \nabla w_k(z) \cdot \nabla\phi(z) dz &= \frac{r_k^{2-\alpha}\eta(\tilde{z}_k)}{L_k} \int_{\mathrm{spt}(\phi)} \rho_k^a(y) f(\tilde{z}_k + r_k z)\phi(z) dz \\ &\quad - \frac{r_k^{1-\alpha}\eta(\tilde{z}_k)}{L_k} \int_{\mathrm{spt}(\phi)} \rho_k^a(y) F(\tilde{z}_k + r_k z) \cdot \nabla\phi(z) dz. \end{aligned} \tag{4.14}$$

Now, we aim to prove that the right-hand side of (4.14) vanishes as $k \to \infty$. First, we focus on the term involving the function f. Let us consider the **Case 1** and **Case 2** together. By using the Hölder inequality, (4.13) and $a < 0$, we get

$$\begin{aligned} &\Big| \int_{\mathrm{spt}(\phi)} \frac{|\tilde{y}_k + r_k y|^a}{|\tilde{y}_k|^a} f(\tilde{z}_k + r_k z)\phi(z) dz \Big| \\ &\leq c\|\phi\|_{L^\infty} \Big(r_k^{-d} |\tilde{y}_k|^{-a} \int_{B_1} |y|^a |f|^p dz \Big)^{1/p} \Big(\int_{\mathrm{spt}(\phi)} \frac{|\tilde{y}_k + r_k y|^a}{|\tilde{y}_k|^a} dz \Big)^{1/p'} \leq c r_k^{d/p}. \end{aligned}$$

Hence,

$$\frac{r_k^{2-\alpha}\eta(\tilde{z}_k)}{L_k} \Big| \int_{\mathrm{spt}(\phi)} \rho_k^a(y) f(\tilde{z}_k + r_k z)\phi(z) dz \Big| \leq c r_k^{2-\alpha-d/p} L_k^{-1} \to 0,$$

as $k \to \infty$, by the hypothesis (1.3), which implies $\alpha \leq 2 - d/p$.

Let us consider now the **Case 3**. By using the Hölder inequality and $a<0$, we get

$$\begin{aligned}&\Big|\int_{\mathrm{spt}(\phi)}|y|^a f(\tilde z_k+r_kz)\phi(z)dz\Big|\\&\le c\|\phi\|_{L^\infty}\Big(r_k^{-d-a}\int_{B_1}|y|^a|f|^p dz\Big)^{1/p}\Big(\int_{\mathrm{spt}(\phi)}|y|^a dz\Big)^{1/p'}\le cr_k^{-d/p},\end{aligned}$$

and then

$$\frac{r_k^{2-\alpha}\eta(\tilde z_k)}{L_k}\Big|\int_{\mathrm{spt}(\phi)}|y|^a f(\tilde z_k+r_kz)\phi(z)dz\Big|\le cr_k^{2-\alpha-d/p}L_k^{-1}\to 0,$$

as before.

The second member of the the right hand side of (4.14) vanishes as $k\to\infty$ by using similar computations. In fact, in every cases, we have

$$\frac{r_k^{1-\alpha}\eta(\tilde z_k)}{L_k}\Big|\int_{\mathrm{spt}(\phi)}\rho_k^a(y)F(\tilde z_k+r_kz)\cdot\nabla\phi(z)dz\Big|\le cr_k^{1-\alpha-d/q}L_k^{-1}\to 0,$$

by the assumption (1.3), which implies that $\alpha\le 1-d/q$.

Finally, we prove that the left hand side of (4.14) converges in the following sense

$$\int_{\mathrm{spt}(\phi)}\rho_k^a(y)A_k(z)\nabla w_k(z)\cdot\nabla\phi(z)dz\to\int_{\mathrm{spt}(\phi)}\bar\rho^a(y)\bar A\nabla\bar v(z)\cdot\nabla\phi(z)dz, \tag{4.15}$$

where

$$\bar\rho(y):=\begin{cases}1, & \text{in } \textbf{Case 1} \text{ and } \textbf{Case 2},\\ |y|, & \text{in } \textbf{Case 3}.\end{cases}$$

Let us fix $R>0$ such that $\mathrm{spt}(\phi)\subset B_{2R}\cap\Omega_k$ for every k large enough. Since w_k is uniformly bounded in $L^\infty(B_{2R}\cap\Omega_\infty)$, one has that w_k is uniformly bounded in $L^2(B_{2R}\cap\Omega_\infty,\rho_k^a(y)dz)$ and, by applying the Caccioppoli-type inequality (2.7), we get that w_k is uniformly bounded in $H^1(B_{2R}\cap\Omega_\infty,\rho_k^a(y)dz)$. Using $\rho_k^a\to\bar\rho$ and $A_k\to\bar A$ a.e. in Ω_∞ and arguing as in the proof of Lemma 2.9, we can conclude that (4.15) holds true and $\bar v\in H^1(B_R\cap\Omega_\infty,\bar\rho^a)$. A similar approach is carried out in [8, Theorem 1.3], where analogous computations are performed.

Finally, recalling the definition of the limit domain Ω_∞, see (4.10), and using *Step 3* for the boundary condition, we conclude that

- in **Case 1**, $\bar v$ is an entire solution to
$$-\operatorname{div}(\bar A\nabla\bar v)=0,\quad\text{in }\mathbb{R}^d,$$
- in **Case 2**, $\bar v$ is an entire solution to
$$\begin{cases}-\operatorname{div}(\bar A\nabla\bar v)=0, & \text{in }\Pi,\\ \bar v=0, & \text{on }\partial\Pi,\end{cases}$$
- in **Case 3**, $\bar v$ is an entire solution to
$$\begin{cases}-\operatorname{div}(|y|^a\bar A\nabla\bar v)=0, & \text{in }\mathbb{R}^d\setminus\Sigma_{\bar\varepsilon},\\ \bar v=0, & \text{on }\partial\Sigma_{\bar\varepsilon}.\end{cases}$$

Step 6. Liouville Theorems and conclusion. By (4.12) we have that $\bar{v}$ satisfies the growth condition

$$|\bar{v}(z)| \leq c(1+|z|^\alpha),$$

for every $z \in \Omega_\infty$, where $\alpha < \min\{1, 2-a-n\}$ by hypothesis (1.3). In **Case 1** and **Case 2**, by invoking the classical Liouville Theorem, we can conclude that $\bar{v}$ must be constant and this is a contradiction since $\bar{v}$ is not constant by *Step 4*. In **Case 3**, we have that $\bar{v}$ satisfies the hypothesis of the Liouville Theorem 1.4 and so $\bar{v}$ must be identically zero, which is a contradiction. Then, $L_k \leq c$ uniformly in k, which implies that $[u_\varepsilon]_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_\varepsilon)} \leq c$. The proof is complete. ☐

Proof of the Theorems 1.2. Let u be a weak solution to (1.1) and consider the trivial extension of ψ in B_1, that is, $\psi(x,y) = \psi(x)$, for every $(x,y) \in B_1$. Let us define

$$v := u - \psi.$$

Since ψ is a Lipschitz function, we have that $v \in \tilde{H}^{1,a}(B_1)$ and v is a weak solution to

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla v) = |y|^a f + \operatorname{div}(|y|^a(F - A\nabla\psi)), & \text{in } B_1 \setminus \Sigma_0, \\ v = 0, & \text{on } \Sigma_0 \cap B_1. \end{cases}$$

By applying Lemma 2.9 we find a sequence $\{v_{\varepsilon_k}\}$ as $\varepsilon_k \to 0$, such that every v_{ε_k} is solution to

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla v_{\varepsilon_k}) = |y|^a f + \operatorname{div}(|y|^a(F - A\nabla\psi)), & \text{in } B_{3/4} \setminus \Sigma_{\varepsilon_k}, \\ v_{\varepsilon_k} = 0, & \text{on } \partial\Sigma_{\varepsilon_k} \cap B_{3/4}, \end{cases}$$

and $v_{\varepsilon_k} \to v$ in $H^{1,a}(B_{3/4})$ as $\varepsilon_k \to 0$. By applying the Theorem 4.1 to the sequences $\{v_{\varepsilon_k}\}$, combined with (2.11), we get that

$$\|v_{\varepsilon_k}\|_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_{\varepsilon_k})} \leq c\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{L^{q,a}(B_1)} + \|\psi\|_{C^{0,1}(\Sigma_0\cap B_1)}\big),$$

for some $c > 0$ depending only on d, n, a, λ, Λ, p, q, α and L.

By applying Arzelà-Ascoli Theorem, we get that $v_{\varepsilon_k} \to w$ in $C^{0,\gamma}_{\mathrm{loc}}(B_{1/2} \setminus \Sigma_0)$, for every $\gamma \in (0,\alpha)$, and by the a.e. convergence $v_{\varepsilon_k} \to v$ it follows that $v = w$. Furthermore, by taking $z, z' \in B_{1/2} \setminus \Sigma_0$ such that $z \neq z'$, we have that

$$\begin{aligned} \frac{|v(z) - v(z')|}{|z - z'|^\alpha} &= \lim_{\varepsilon_k \to 0} \frac{|v_{\varepsilon_k}(z) - v_{\varepsilon_k}(z')|}{|z - z'|^\alpha} \\ &\leq c\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{L^{q,a}(B_1)} + \|\psi\|_{C^{0,1}(\Sigma_0\cap B_1)}\big), \end{aligned}$$

which implies that

$$\begin{aligned} [v]_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_0)} &= \sup_{\substack{z,z' \in B_{1/2}\setminus\Sigma_0 \\ z\neq z'}} \frac{|v(z) - v(z')|}{|z - z'|^\alpha} \\ &\leq c\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{L^{q,a}(B_1)} + \|\psi\|_{C^{0,1}(\Sigma_0\cap B_1)}\big). \end{aligned}$$

By continuity, we can extend v to the entire $B_{1/2}$ in such a way that $[v]_{C^{0,\alpha}(B_{1/2})} = [v]_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_0)}$. Finally, combining the previous estimates with the L^∞_{loc} bounds of solutions (see Lemma 2.8), we obtain

$$\begin{aligned} \|u\|_{C^{0,\alpha}(B_{1/2})} &\leq \|v\|_{C^{0,\alpha}(B_{1/2})} + \|\psi\|_{C^{0,\alpha}(\Sigma_0\cap B_{1/2})} \\ &\leq c\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{L^{q,a}(B_1)} + \|\psi\|_{C^{0,1}(\Sigma_0\cap B_1)}\big). \end{aligned}$$

that is, $u \in C^{0,\alpha}(B_{1/2})$ and (1.4) holds true. ☐

5. SCHAUDER ESTIMATES FOR WEAK SOLUTIONS

This section is devoted to the proof of the Theorem 1.3, which establishes the $C^{1,\alpha}_{\rm loc}$ regularity for weak solutions. To achieve this result, we proceed as follows: first, as in Theorem 4.1, we prove ε-uniform estimates for solutions in perforated domains, with an additional assumption on the field F. As we will see in Remark 5.2, this condition cannot be removed. Next, we show *a priori* estimates for solutions, which also satisfy an additional boundary condition on Σ_0. Afterwards, we establish the main result through a double approximation process: the first involves perforated domains and using Lemma 2.9, while the second one is a standard approximation via convolution with a family of mollifiers.

Theorem 5.1. *Let $2 \le n \le d$, $a + n \in (0, 1)$, $p > d$ and α satisfying (1.5). Let A be a α-Hölder continuous symmetric matrix satisfying (1.2) and $\|A\|_{C^{0,\alpha}(B_1)} \le L$, $f \in L^{p,a}(B_1)$, $F \in C^{0,\alpha}(B_1)$ be a field such that $F(x, 0)\cdot e_{y_i} = 0$ for every $(x, 0) \in B_1$ and for every $i = 1, \dots , n$. For $0 < \varepsilon \ll 1$, let $\{u_\varepsilon\}$ be a family of solutions to (4.2).*

Then, there exists a constant $c > 0$, depending only on d, n, a, λ, Λ, p, α and L such that

$$\|u_\varepsilon\|_{C^{1,\alpha}(B_{1/2}\setminus\Sigma_\varepsilon)} \le c\big(\|u_\varepsilon\|_{L^{2,a}(B_1\setminus\Sigma_\varepsilon)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)}\big). \tag{5.1}$$

In addition, u_ε satisfies

$$|\nabla u_\varepsilon(z)| \le c\big(\|u_\varepsilon\|_{L^{2,a}(B_1\setminus\Sigma_\varepsilon)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)}\big)\varepsilon^\alpha, \quad \text{for every } z \in \partial\Sigma_\varepsilon \cap B_{1/2}. \tag{5.2}$$

Proof. Under the assumptions of the theorem, classical Schauder theory ensures that solutions to (4.2) are $C^{1,\alpha}(B_{1/2} \setminus \Sigma_\varepsilon)$ and that (5.1) holds with a constant $c > 0$ that may also depend on ε. Our goal is to show that it is possible to provide a constant $c > 0$ that not depends on ε.

Without loss of generality, we can assume that

$$\|u_\varepsilon\|_{L^{2,a}(B_1\setminus\Sigma_\varepsilon)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{L^{q,a}(B_1)} \le c,$$

for some $c > 0$, which not depends on ε. Moreover, for every $\beta \in (0, 1)$, the assumptions of Theorem 4.1 are satisfied, so

$$\|u_\varepsilon\|_{C^{0,\beta}(B_{3/4}\setminus\Sigma_\varepsilon)} \le c, \tag{5.3}$$

for some $c > 0$, which not depends on ε.

Step 1. Contradiction argument and blow-up sequences. By contradiction let us suppose that there exists $p > d$, α satisfying (1.5), $\{u_k\}_k := \{u_{\varepsilon_k}\}_k$, as $\varepsilon_k \to 0$ such that u_k is solution to (4.5) and

$$\|\nabla u_k\|_{C^{0,\alpha}(B_{1/2}\setminus\Sigma_{\varepsilon_k})} \to \infty.$$

Let us fix a smooth cut-off function $\eta \in C^\infty_c(B_1)$ such that

$$\mathrm{spt}(\phi) \subset B_{3/4}, \quad \eta = 1 \text{ in } B_{1/2}, \quad 0 \le \eta \le 1.$$

One has that

$$\|\eta u_k\|_{C^{1,\alpha}(B_1\setminus\Sigma_{\varepsilon_k})} \to \infty.$$

Let us define

$$L_k := [\nabla(\eta u_k)]_{C^{0,\alpha}(B_1\setminus\Sigma_{\varepsilon_k})},$$

and notice that it cannot be possible that $[\nabla(\eta u_k)]_{C^{0,\alpha}(B_1)} \le c$ and $\|\nabla(\eta u_k)\|_{L^\infty(B_1)} \to \infty$, since $\nabla(\eta u_k) = 0$ outside of $B_{3/4} \setminus \Sigma_{\varepsilon_k}$. Hence, $L_k \to \infty$.

By definition of Hölder seminorm, take two sequences of points $z_k = (x_k, y_k), \hat{z}_k = (\hat{x}_k, \hat{y}_k) \in B_1 \setminus \Sigma_{\varepsilon_k}$ such that

$$\frac{|\nabla(\eta u_k)(z_k) - \nabla(\eta u_k)(\hat{z}_k)|}{|z_k - \hat{z}_k|^\alpha} \geq \frac{L_k}{2}, \tag{5.4}$$

define $r_k := |z_k - \hat{z}_k|$ and observe that at least one of z_k or $\hat{z}_k$ belongs to $B_{3/4} \setminus \Sigma_{\varepsilon_k}$. We distinguish three cases.

- **Case 1:** $\dfrac{|y_k|}{r_k} \to \infty, \quad \dfrac{|y_k| - \varepsilon_k}{r_k} \to \infty,$
- **Case 2:** $\dfrac{|y_k|}{r_k} \to \infty, \quad \dfrac{|y_k| - \varepsilon_k}{r_k} \leq c,$
- **Case 3:** $\dfrac{|y_k|}{r_k} \leq c,$

for some constant $c > 0$ which not depends on k. We notice that $r_k \to 0$ in **Case 1** and **Case 2** and we show later that $r_k \to 0$ also in **Case 3**.

Let $z_k^0 := (x_k, y_k^0)$ be the projection of z_k on $\partial\Sigma_{\varepsilon_k}$, define

$$\tilde{z}_k = (\tilde{x}_k, \tilde{y}_k) := \begin{cases} (x_k, y_k), & \text{in } \textbf{Case 1}, \\ (x_k, y_k^0), & \text{in } \textbf{Case 2}, \\ (x_k, 0), & \text{in } \textbf{Case 3}, \end{cases}$$

and the sequence of domains

$$\Omega_k := \frac{B_1 \setminus \Sigma_{\varepsilon_k} - \tilde{z}_k}{r_k}.$$

For every $z \in \Omega_k$, let us define the sequence of functions

$$v_k(z) := \frac{(\eta u_k)(\tilde{z}_k + r_k z) - (\eta u_k)(\tilde{z}_k) - \nabla(\eta u_k)(\tilde{z}_k) \cdot r_k z}{L_k r_k^{1+\alpha}},$$
$$w_k(z) := \frac{\eta(\tilde{z}_k) u_k(\tilde{z}_k + r_k z) - (\eta u_k)(\tilde{z}_k) - (\eta \nabla u_k)(\tilde{z}_k) \cdot r_k z}{L_k r_k^{1+\alpha}},$$

in **Case 1** and **Case 2**, and

$$v_k(z) := \frac{(\eta u_k)(\tilde{z}_k + r_k z)}{L_k r_k^{1+\alpha}}, \quad w_k(z) := \frac{\eta(\tilde{z}_k) u_k(\tilde{z}_k + r_k z)}{L_k r_k^{1+\alpha}},$$

in **Case 3**. Furthermore, let us define the limit domain Ω_∞ as in (4.7).

Step 2. Gradient Hölder estimates and convergence of the blow-up sequences. Let us fix a compact set $K \subset \Omega_\infty$. For every $z, z' \in K$ such that $z \neq z'$ we have

$$|\nabla v_k(z) - \nabla v_k(z')| = \frac{|\nabla(\eta u_k)(\tilde{z}_k + r_k z) - \nabla(\eta u_k)(\tilde{z}_k + r_k z')|}{r_k^\alpha L_k} \leq |z - z'|^\alpha,$$

that is

$$[\nabla v_k]_{C^{0,\alpha}(K)} \leq 1. \tag{5.5}$$

In **Case 1** and **Case 2**, since $v_k(0) = 0$ and $|\nabla v_k(0)| = 0$, we get the uniform bound of the norm

$$\|v_k\|_{C^{1,\alpha}(K)} \leq c.$$

In **Case 3** we use a different argument to show a uniform bound of $\|v_k\|_{C^{1,\alpha}(K)}$. First, for every point $z'_k=(x'_k,y'_k)\in\partial\Sigma_{\varepsilon_k}\cap B_{3/4}$, one has that one has that $(\eta u_k)(z'_k)=0$ and, recalling that the normal vector to $\partial\Sigma_{\varepsilon_k}$ at the point z'_k is the vector $(0,y'_k)/|y'_k|\in\mathbb{S}^{d-1}$, it follows that

$$\nabla(\eta u_k)(z'_k)\cdot\vec{e}=0,\quad \text{for every vector } \vec{e}\perp(0,y'_k).$$

Let us fix $i,h\in\{1,\dots,n\}$ such that $i\neq h$. Then,

$$\begin{aligned}\big|\nabla(\eta u_k)(x'_k,y'_k)\cdot e_{y_i}\big| &= \big|\big(\nabla(\eta u_k)(x'_k,y'_k)-\nabla(\eta u_k)(x'_k,\varepsilon_k e_{y_h})\big)\cdot e_{y_i}\big|\\ &\le[\nabla(\eta u_k)]_{C^{0,\alpha}(B_1\setminus\Sigma_{\varepsilon_k})}|y'_k-\varepsilon_k e_{y_h}|^\alpha\le 2^\alpha L_k\varepsilon_k^\alpha,\end{aligned}\tag{5.6}$$

and

$$|\nabla(\eta u_k)(x'_k,y'_k)\cdot e_{x_j}|=0,\quad \text{for every } j=1,\dots,d-n.\tag{5.7}$$

Hence, by using (5.6) and (5.7), it follows that

$$|\nabla(\eta u_k)(x'_k,y'_k)|\le cL_k\varepsilon_k^\alpha.\tag{5.8}$$

By using interpolation inequality in Hölder spaces [20, Lemma 6.35], one has

$$\|v_k\|_{C^{1,\alpha}(K)}\le c(\|v_k\|_{L^\infty(K)}+[v_k]_{C^{1,\alpha}(K)}),$$

and, by using the first order expansion of v_k at the point $(0,y_k^0/r_k)$ we obtain

$$\begin{aligned}|v_k(x,y)|&\le|v_k(x,y)-v_k(0,y_k^0/r_k)-\nabla v_k(0,y_k^0/r_k)\cdot z|+|\nabla v_k(0,y_k^0/r_k)|\\ &\le c(|x|+|y-y_k^0/r_k|)^{1+\alpha}+|\nabla v_k(0,y_k^0/r_k)|\le c+|\nabla v_k(0,y_k^0/r_k)|,\end{aligned}\tag{5.9}$$

for some $c>0$ which depends only on K, noticing that $|y_k^0/r_k|=\varepsilon_k/r_k\le c$ uniformly in k in **Case 3**. So, if

$$|\nabla v_k(0,y_k^0/r_k)|\le c,\tag{5.10}$$

uniformly in k, we can take the $\sup_K$ in (5.9) to obtain a uniform of $\|v_k\|_{L^\infty(K)}$ which implies $\|v_k\|_{C^{1,\alpha}(K)}\le c$. Then, by using (5.8) and recalling that $(x_k,y_k^0)\in\partial\Sigma_{\varepsilon_k}\cap B_{3/4}$, we get that

$$|\nabla v_k(0,y_k^0/r_k)|=\frac{|\nabla(\eta u_k)(x_k,y_k^0)|}{r_k^\alpha L_k}\le c\frac{\varepsilon_k^\alpha}{r_k^\alpha}\le c.$$

So, (5.10) holds true and $\|v_k\|_{C^{1,\alpha}(K)}\le c$ uniformly in k.

Then, we may apply the Arzelà-Ascoli Theorem to infer that $v_k\to\bar{v}$ in $C^{1,\gamma}(K)$ for any $\gamma\in(0,\alpha)$. By a standard diagonal argument, we can take the the limit as $k\to\infty$ in (5.5) to obtain

$$[\nabla\bar{v}]_{C^{1,\alpha}(\Omega_\infty)}\le 1,$$

which implies that

$$|\bar{v}(z)|\le c(1+|z|^{1+\alpha}),\quad \text{a.e. in } \Omega_\infty.\tag{5.11}$$

Moreover, v_k and w_k converge to the same limit function $\bar{w}$. Let us fix a compact set $K\subset\Omega_\infty$. In **Case 1** and **Case 2**, for every $z\in K$, by using (5.3) and exploiting the first order expansion of η, we have

$$\begin{aligned}|v_k(z)-w_k(z)|&=\frac{|(\eta u_k)(\tilde{z}_k+r_kz)-\eta(\tilde{z}_k)u_k(\tilde{z}_k+r_kz)-(u_k\nabla\eta)(\tilde{z}_k)\cdot r_kz|}{r_k^{1+\alpha}L_k}\\ &\le\frac{|u_k(\tilde{z}_k+r_kz)(\eta(\tilde{z}_k+r_kz)-\eta(\tilde{z}_k)-\nabla\eta(\tilde{z}_k)\cdot r_kz)|}{r_k^{1+\alpha}L_k}+c\frac{|\nabla\eta(\tilde{z}_k)||u_k(\tilde{z}_k+r_kz)-u_k(\tilde{z}_k)|}{r_k^\alpha L_k}\end{aligned}$$

$$\leq c\frac{\|u_k\|_{L^\infty(B_{3/4}\setminus\Sigma_{\varepsilon_k})}r_k^{1-\alpha}}{L_k}+c\frac{\|u_k\|_{C^{0,\beta}(B_{3/4}\setminus\Sigma_{\varepsilon_k})}r_k^{\beta-\alpha}}{L_k}\leq\frac{cr_k^{\beta-\alpha}}{L_k}\to 0,$$

as $k\to\infty$, since we can choose $\beta\in(\alpha,1)$. In **Case 3**, one has that

$$|v_k(z)-w_k(z)|=\frac{|\eta(\tilde{z}_k+r_kz)-\eta(\tilde{z}_k)|\cdot|u_k(\tilde{z}_k+r_kz)|}{r_k^{1+\alpha}L_k}$$
$$\leq\frac{c|u_k(\tilde{z}_k+r_kz)-u_k(z_k^0)|}{r_k^\alpha L_k}\leq\frac{c\|u_k\|_{C^{0,\beta}(B_{3/4}\setminus\Sigma_{\varepsilon_k})}|\tilde{z}_k+r_kz-z_k^0|^\beta}{r_k^\alpha L_k}\leq\frac{r_k^{\beta-\alpha}}{L_k}\to 0,$$

where we have used the facts that $u_k(z_k^0)=0$ and $|\tilde{z}_k+r_kz-z_k^0|\leq r_k|x|+r_k|y|+|y_k^0|\leq cr_k+\varepsilon_k\leq cr_k$ in **Case 3**. Hence, the sequences v_k and w_k have the same asymptotic behaviour as $k\to\infty$ on every $K\subset\Omega_\infty$, which implies that $w_k\to\bar{v}$ uniformly on K.

Step 3. $\nabla\bar{v}$ is not constant. Let us define the sequence of points

$$\xi_k^1:=\frac{z_k-\tilde{z}_k}{r_k},\quad \xi_k^2:=\frac{\hat{z}_k-\tilde{z}_k}{r_k}.$$

By using (5.4) we get

$$|\nabla v_k(\xi_k^1)-\nabla v_k(\xi_k^2)|=\frac{|\nabla(\eta u_k)(z_k)-(\nabla\eta u_k)(\hat{z}_k)|}{r_k^\alpha L_k}\geq\frac{1}{2}>0. \tag{5.12}$$

Arguing as Theorem 4.1, *Step 4*, we have that $\xi_k^1\to\xi_1$ and $\xi_k^2\to\xi_2$ and $\xi^1\neq\xi^2$. Since $\nabla v_k\to\nabla\bar{v}$ uniformly on compact set by *Step 2*, we can take the limit in (5.12) to obtain $|\nabla\bar{v}(\xi^1)-\nabla\bar{v}(\xi^2)|>\delta_0/2$.

Step 4. $r_k\to 0$ in **Case 3**. By contradiction let us suppose that $r_k\to\bar{r}>0$. Fixed $z\in\Omega_\infty$, we have that

$$|\bar{v}(z)|=\Big|\lim_{k\to\infty}v_k(z)\Big|=\Big|\lim_{k\to\infty}\frac{(\eta u_k)(\tilde{z}_k+r_kz)}{L_kr_k^{1+\alpha}}\Big|\leq\frac{2\|u_k\|_{L^\infty(B_{3/4}\setminus\Sigma_{\varepsilon_k})}}{r_k^{1+\alpha}L_k}\leq\frac{c}{L_k}\to 0,$$

hence, $\bar{v}=0$, which is a contradiction with *Step 3*, where we have proved that $\nabla\bar{v}$ is not constant. Then, $r_k\to 0$. By arguing as in Theorem 4.1, *Step 3*, we have that the limit domain Ω_∞ is defined by (4.10).

Step 5. $\bar{v}$ satisfies a homogeneous Dirichlet boundary condition in **Case 2** *and* **Case 3**. First, in **Case 3**, since $v_k=0$ on $\partial\Sigma_{\varepsilon_k/r_k}$ and $v_k\to\bar{v}$ uniformly on every compact set $K\subset\Omega_\infty$, one can conclude that $\bar{v}=0$ on $\partial\Sigma_{\bar{\varepsilon}}$.

In **Case 2**, recalling that $\tilde{z}_k=z_k^0\in\partial\Sigma_{\varepsilon_k}$, let us fix a boundary point $z\in(\partial\Sigma_{\varepsilon_k}-y_k^0)/r_k$ and denote by $z^\perp=(x,y^\perp)$ the projection of z on the hyperplane $\Pi_k=\{y\cdot y_k^0=0\}$. Recalling (4.9) and observing that $|y_k^0+r_ky|=|y_k^0|=\varepsilon_k$, one has that

$$|y-y^\perp|=\Big|y\cdot\frac{y_k^0}{|y_k^0|}\Big|=\Big|\frac{|y_k^0+r_ky|-|y_k^0|}{r_k}-y\cdot\frac{y_k^0}{|y_k^0|}\Big|\leq c\frac{r_k}{\varepsilon_k}. \tag{5.13}$$

Then, by using (5.13), (5.8), and noting that $\nabla(\eta u_k)(z_k^0)\cdot y=\nabla(\eta u_k)(z_k^0)\cdot(y-y^\perp)$, we obtain

$$|v_k(z)|=\frac{|\nabla(\eta u_k)(z_k^0)\cdot y|}{r_k^\alpha L_k}\leq\frac{|\nabla(\eta u_k)(z_k^0)|}{r_k^\alpha L_k}|y-y^\perp|\leq c\Big(\frac{\varepsilon_k}{r_k}\Big)^\alpha\frac{r_k}{\varepsilon_k}\to 0,$$

as $k\to\infty$. Therefore, since $v_k\to\bar{v}$ uniformly on every compact set $K\subset\Omega_\infty$, we conclude that $\bar{v}=0$ on $\partial\Pi$.

Step 6. $\bar{v}$ is an entire solution to a homogeneous equation with constant coefficients. Let $A_k(z) := A(\tilde{z}_k + r_k z)$ and $\bar{z} := \lim_{k\to\infty} \tilde{z}_k$. By using the α-Hölder continuity of A, we can define $\bar{A} := A(\bar{z}) = \lim_{k\to\infty} A_k(z)$, which is a constant coefficients symmetric matrix satisfying (1.2). Let us define

$$\rho_k(y) := \begin{cases} \dfrac{|y_k + r_k y|}{|y_k|}, & \text{in } \textbf{Case 1} \text{ and } \textbf{Case 2}, \\ |y|, & \text{in } \textbf{Case 3}, \end{cases}$$

and fix $\phi \in C_c^\infty(\Omega_\infty)$. Since u_k is a solution to (4.5), we have that

$$\begin{aligned} &\int_{\mathrm{spt}(\phi)} \rho_k^a(y) A_k(z) \nabla w_k(z) \cdot \nabla \phi(z) dz = \frac{\eta(\tilde{z}_k) r_k^{1-\alpha}}{L_k} \int_{\mathrm{spt}(\phi)} \rho_k^a(y) f(\tilde{z}_k + r_k z) \phi(z) dz \\ &- \frac{\eta(\tilde{z}_k) r_k^{-\alpha}}{L_k} \int_{\mathrm{spt}(\phi)} \rho_k^a(y) F(\tilde{z}_k + r_k z) \cdot \nabla \phi(z) dz - \frac{r_k^{-\alpha}}{L_k} \int_{\mathrm{spt}(\phi)} \rho_k^a(y) A_k(z) P_k \cdot \nabla \phi(z) dz \\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}, \end{aligned} \tag{5.14}$$

where we have set

$$P_k := \begin{cases} (\eta \nabla u_k)(\tilde{z}_k), & \text{in } \textbf{Case 1} \text{ and } \textbf{Case 2}, \\ 0, & \text{in } \textbf{Case 3}. \end{cases}$$

We want to show that the right hand side vanishes as $k \to \infty$. The term I vanishes exactly as in the Theorem 4.1, *Step 5*, by using the integrability assumption $f \in L^{p,a}(B_1)$, with $p > d$ and $\alpha \in (0, 1 - d/p]$ by (1.5).

Next, by using the divergence theorem, we have

$$\begin{aligned} &\Big| \int_{\mathrm{spt}(\phi)} \rho_k^a(y) F(\tilde{z}_k + r_k z) \cdot \nabla \phi(z) dz \Big| \\ &\le \int_{\mathrm{spt}(\phi)} \rho_k^a(y) |F(\tilde{z}_k + r_k z) - F(\tilde{z}_k)| |\nabla \phi(z)| dz + \Big| \int_{\mathrm{spt}(\phi)} \rho_k^a(y) F(\tilde{z}_k) \cdot \nabla \phi(z) dz \Big| \\ &\le c \|F\|_{C^{0,\alpha}(B_1)} r_k^\alpha + \Big| \int_{\mathrm{spt}(\phi)} \nabla \rho_k^a(y) \cdot F(\tilde{z}_k) \phi(z) dz \Big| \\ &\le c r_k^\alpha + \Big| \int_{\mathrm{spt}(\phi)} \nabla \rho_k^a(y) \cdot F(\tilde{z}_k) \phi(z) dz \Big|. \end{aligned} \tag{5.15}$$

In **Case 3**, since $F(x, 0) \cdot e_{y_i} = 0$, one has that

$$\nabla |y|^a \cdot F(\tilde{z}_k) = a |y|^{a-2} F(x_k, 0) \cdot y = 0,$$

for every $z \in \mathrm{spt}(\phi)$, so the second term in (5.15) is zero and

$$|\mathrm{II}| \le \frac{c}{L_k} \to 0, \quad \text{as } k \to \infty.$$

Instead, in **Case 1** and **Case 2**, since $|\tilde{y}_k + r_k y| \ge |\tilde{y}_k|/2$ for $y \in \mathrm{spt}(\phi)$, we have that

$$\begin{aligned} \big| \nabla \rho_k^a(y) \cdot F(\tilde{z}_k) \big| &= \Big| \frac{a r_k \rho_k^a(y) (\tilde{y}_k + r_k y)}{|\tilde{y}_k + r_k y|^2} \cdot \Big(F(\tilde{x}_k, \tilde{y}_k) - F(\tilde{x}_k, 0) \Big) \Big| \\ &\le \frac{c r_k [F]_{C^{0,\alpha}(B_1)} |\tilde{y}_k|^\alpha}{|\tilde{y}_k|} \le c r_k |\tilde{y}_k|^{\alpha - 1}. \end{aligned} \tag{5.16}$$

Thus, by using (5.15) and (5.16), one has that

$$|\mathrm{II}| \leq \frac{c}{L_k} + \frac{cr_k^{-\alpha}}{L_k} r_k |\tilde{y}_k|^{\alpha-1} \leq \frac{c}{L_k}\Big(1 + \Big(\frac{r_k}{|\tilde{y}_k|}\Big)^{1-\alpha}\Big) \leq cL_k^{-1} \to 0, \quad \text{as } k \to \infty,$$

in **Case 1** and **Case 2**, since $r_k/|\tilde{y}_k| \leq c$.

Finally, we prove that the third member of (5.14) goes to zero as $k \to \infty$. In **Case 3**, $P_k = 0$ so $\mathrm{III} = 0$. Instead, in **Case 1** and **Case 2**, one has

$$\begin{aligned} |\mathrm{III}| \leq &\Big|\frac{r_k^{-\alpha}}{L_k}\int_{\mathrm{spt}(\phi)} \rho_k^a(y) A(\tilde{z}_k)(\eta\nabla u_k)(\tilde{z}_k) \cdot \nabla\phi(z)dz\Big| \\ &+ \Big|\frac{r_k^{-\alpha}}{L_k}\int_{\mathrm{spt}(\phi)} \rho_k^a(y)(A(\tilde{z}_k + r_k z) - A(\tilde{z}_k))(\eta\nabla u_k)(\tilde{z}_k) \cdot \nabla\phi(z)dz\Big|. \end{aligned} \tag{5.17}$$

First, we show that the first member in (5.17) vanishes. Recall that $r_k/|y_k| \to 0$, $\rho_k^a \to 1$ and (x_k, y_k^0) is the projection of z_k on $\partial\Sigma_{\varepsilon_k}$. By using (5.3) and (5.8) we get

$$\begin{aligned} |(\eta\nabla u_k)(\tilde{z}_k))| &\leq |\nabla(\eta u_k)(\tilde{x}_k, \tilde{y}_k)| + |(u_k\nabla\eta)(\tilde{x}_k, \tilde{y}_k)| \\ &\leq |\nabla(\eta u_k)(\tilde{x}_k, \tilde{y}_k) - \nabla(\eta u_k)(x_k, y_k^0)| + |\nabla(\eta u_k)(x_k, y_k^0)| \\ &+ |\nabla\eta(\tilde{x}_k, \tilde{y}_k)(u_k(\tilde{x}_k, \tilde{y}_k) - u_k(x_k, y_k^0))| \\ &\leq cL_k|\tilde{y}_k - y_k^0|^\alpha + cL_k\varepsilon_k^\alpha + c|\tilde{y}_k - y_k^0|^\alpha \leq cL_k|\tilde{y}_k|^\alpha. \end{aligned} \tag{5.18}$$

On the other hand, since $|\tilde{y}_k + r_k y| \geq |\tilde{y}_k|/2$ for $y \in \mathrm{spt}(\phi)$, it follows

$$|\nabla\rho_k^a(y)| = \Big|\frac{ar_k\rho_k^a(y)(\tilde{y}_k + r_k y)}{|\tilde{y}_k + r_k y|^2}\Big| \leq c\frac{r_k}{|\tilde{y}_k|}\rho_k^a(y). \tag{5.19}$$

hence, by combining (5.18) and (5.19), and using the divergence theorem, it follows that

$$\Big|\frac{r_k^{-\alpha}}{L_k}\int_{\mathrm{spt}(\phi)} \nabla\rho_k^a(y) \cdot A(\tilde{z}_k)(\eta\nabla u_k)(\tilde{z}_k)\phi(z)dz\Big| \leq c\Big(\frac{r_k}{|\tilde{y}_k|}\Big)^{1-\alpha} \to 0,$$

that is, the first member in (5.17) vanishes. Next, we show that the second member vanishes as $k \to \infty$. In this case, we need to reason in two steps (as done in [31, Remark 5.3] and [1, Theorem 7.1]): first, we prove uniform estimates in $C^{1,\alpha'}$ space for some suboptimal $\alpha' \in (0, \alpha)$; then, by using these estimates, we conclude the optimal regularity with exponent α. Let us fix $\alpha' \in (0, \alpha)$. By using interpolation inequality in Hölder spaces [20, Lemma 6.35], we can estimate the second term of (5.17) as follows

$$\begin{aligned} &\Big|\frac{r_k^{-\alpha'}}{L_k}\int_{\mathrm{spt}(\phi)} \rho_k^a(y)(A(\tilde{z}_k + r_k z) - A(\tilde{z}_k))(\eta\nabla u_k)(\tilde{z}_k) \cdot \nabla\phi(z)dz\Big| \\ &\leq \frac{cr_k^{\alpha-\alpha'}\|(\eta\nabla u_k)\|_{L^\infty(B_{3/4}\setminus\Sigma_{\varepsilon_k})}}{L_k} \leq \frac{cr_k^{\alpha-\alpha'}}{L_k}\big(\|u_k\nabla\eta\|_{L^\infty(B_{3/4}\setminus\Sigma_{\varepsilon_k})} + \|\nabla(\eta u_k)\|_{L^\infty(B_{3/4}\setminus\Sigma_{\varepsilon_k})}\big) \\ &\leq \frac{cr_k^{\alpha-\alpha'}(\|u_k\|_{L^\infty(B_{3/4}\setminus\Sigma_{\varepsilon_k})} + [\nabla(\eta u_k)]_{C^{0,\alpha}(B_{3/4}\setminus\Sigma_{\varepsilon_k})})}{L_k} \leq cr_k^{\alpha-\alpha'} \to 0, \end{aligned}$$

since $\alpha' < \alpha$. If we have uniform estimates in $C^{1,\alpha'}$ space, then $\|\eta\nabla u_k\|_{L^\infty(B_{3/4}\setminus\Sigma_{\varepsilon_k})} \leq c$. Hence, restarting the proof with the optimal α and the additional information above, in the previous computation we get

$$\begin{aligned}&\left|\frac{r_k^{-\alpha}}{L_k}\int_{\mathrm{spt}(\phi)}\rho_k^a(y)(A(\tilde{z}_k+r_kz)-A(\tilde{z}_k))(\eta\nabla u_k)(\tilde{z}_k)\cdot\nabla\phi(z)dz\right|\\&\leq\frac{c\|(\eta\nabla u_k)\|_{L^\infty(B_{3/4}\setminus\Sigma_{\varepsilon_k})}}{L_k}\leq\frac{c}{L_k}\to 0.\end{aligned}$$

Combining all the previous results, we conclude that the right-hand side of (5.14) vanishes as $k\to\infty$.

Finally, by the same considerations of Theorem 4.1, we obtain that the left hand side of (5.14) converges in the following sense

$$\int_{\mathrm{spt}(\phi)}\rho_k^aA_k\nabla w_k\cdot\nabla\phi dz\to\int_{\mathrm{spt}(\phi)}\bar{\rho}^a\bar{A}\nabla\bar{v}\cdot\nabla\phi dz,$$

where

$$\bar{\rho}(y):=\begin{cases}1, & \text{in } \textbf{Case 1} \text{ and } \textbf{Case 2},\\ |y|, & \text{in } \textbf{Case 3},\end{cases}$$

and $\bar{v}\in H^1_{\mathrm{loc}}(\Omega_\infty,\bar{\rho}^a(y)dz)$.

Then, recalling who is Ω_∞, see (4.10), and using the *Step 5* for the Dirichlet boundary condition, we conclude that

- in **Case 1**, $\bar{v}$ is an entire solution to

$$-\operatorname{div}(\bar{A}\nabla\bar{v})=0,\quad \text{in } \mathbb{R}^d,$$

- in **Case 2**, $\bar{v}$ is an entire solution to

$$\begin{cases}-\operatorname{div}(\bar{A}\nabla\bar{v})=0, & \text{in } \Pi,\\ \bar{v}=0, & \text{on } \partial\Pi,,\end{cases}$$

- in **Case 3**, $\bar{v}$ is an entire solution to

$$\begin{cases}-\operatorname{div}(|y|^a\bar{A}\nabla\bar{v})=0, & \text{in } \mathbb{R}^d\setminus\Sigma_{\bar{\varepsilon}},\\ \bar{v}=0, & \text{on } \partial\Sigma_{\bar{\varepsilon}},\end{cases}$$

Step 7. Liouville Theorems and conclusion. Since $\bar{v}$ satisfies the growth condition (5.11) with $1+\alpha\in(0,2)\cap(0,2-a-n)$, invoking the classical Liouville Theorem in **Case 1** and **Case 2** shows that $\bar{v}$ must be a linear function. In contrast, applying the Liouville Theorem 1.4 in **Case 3** implies that $\bar{v}$ must be zero.

This contradicts *Step 3*, as $\nabla\bar{v}$ is not constant. Consequently, $L_k=[\nabla(\eta u_k)]_{C^{0,\alpha}(B_1\setminus\Sigma_{\varepsilon_k})}$ must be bounded, which implies that $\|\eta u_\varepsilon\|_{C^{1,\alpha}(B_1\setminus\Sigma_\varepsilon)}\leq c$. Thus, (5.1) holds true.

Furthermore, recalling (5.8) and using (5.1), we conclude that (5.2) follows. This completes the proof. □

Remark 5.2. The homogeneous Dirichlet condition on $\partial\Sigma_\varepsilon$ is too restrictive to handle all possible fields F, thus preventing the validity of ε-uniform $C^{1,\alpha}$ estimates in the general case.

Let us suppose that $n = d = 2$, $-2 < a < -1$ and consider the function $u(y) = u(y_1, y_2) := y_1 + y_2$ which is a weak solution to

$$\begin{cases} -\operatorname{div}(|y|^a \nabla u) = \operatorname{div}(|y|^a F), & \text{in } B_1 \setminus \Sigma_0, \\ u = 0, & \text{on } \Sigma_0 \cap B_1. \end{cases}$$

where $F := -\nabla u = (-1, -1)$. By applying Lemma 2.10 we find that there exists a family u_ε which are weak solutions to

$$\begin{cases} -\operatorname{div}(|y|^a \nabla u_\varepsilon) = \operatorname{div}(|y|^a F), & \text{in } B_{3/4} \setminus \Sigma_\varepsilon, \\ u = 0, & \text{on } \Sigma_\varepsilon \cap B_{3/4}, \end{cases}$$

and satisfies $\|u_\varepsilon\|_{H^{1,a}(B_{3/4} \setminus \Sigma_\varepsilon)} \leq c$ and $u_\varepsilon \to u$ in $H^{1,a}(B_{3/4})$. If Theorem 5.1 holds true for this equation, we obtain that (5.2) works (since it depends only on the Dirichlet boundary condition satisfied by u_ε), that is

$$|\nabla u_\varepsilon| \leq c\varepsilon^\alpha, \qquad \text{on } \partial\Sigma_\varepsilon \cap B_{1/2},$$

and then, taking the limit as $\varepsilon \to 0$ in a suitable sense (see the proof of the Theorem 1.3), we deduce that $\nabla u = 0$ on $\Sigma_0 \cap B_{1/2}$, which contradicts $\nabla u = (1, 1)$.

The next proposition provides *a priori* estimates for solutions that additionally satisfy an extra boundary condition on the lower dimensional boundary Σ_0.

Proposition 5.3. *Let $2 \leq n \leq d$, $a + n \in (0, 1)$, $p > d$ and α satisfying (1.5). Let A be a α-Hölder continuous symmetric matrix satisfying (1.2) and $\|A\|_{C^{0,\alpha}(B_1)} \leq L$, $f \in L^{p,a}(B_1)$ and $F \in C^{0,\alpha}(B_1)$. Let $u \in C^{1,\alpha}(B_1)$ be a weak solution to*

$$\begin{cases} -\operatorname{div}(|y|^a A \nabla u) = |y|^a f + \operatorname{div}(|y|^a F), & \textit{in } B_1 \setminus \Sigma_0, \\ u = 0, & \textit{on } \Sigma_0 \cap B_1, \end{cases}$$

such that u satisfies the boundary condition

$$\begin{cases} \nabla_x u(x, 0) = 0, \\ (A\nabla u + F)(x, 0) \cdot e_{y_i} = 0, \end{cases} \qquad \textit{for every } (x, 0) \in \Sigma_0 \cap B_1, \ \textit{and } i = 1, \dots, n. \tag{5.20}$$

Then, there exists a constant $c > 0$, depending only on d, n, a, λ, Λ, p, α and L such that

$$\|u\|_{C^{1,\alpha}(B_{1/2})} \leq c\big(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)}\big). \tag{5.21}$$

Proof. The proof is quite similar to the one of Theorem 5.1, so we avoid some details. Without loss of generality, let us suppose that

$$\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F\|_{C^{0,\alpha}(B_1)} \leq c.$$

By contradiction let us suppose that (5.21) doesn't holds, hence, there exist $p > d$, α satisfying (1.5), $\{A_k\}_k$, $\{f_k\}_k$, $\{F_k\}_k$, $\{u_k\}_k$, such that u_k is solution to

$$\begin{cases} -\operatorname{div}(|y|^a A_k \nabla u_k) = |y|^a f_k + \operatorname{div}(|y|^a F_k), & \text{in } B_1 \setminus \Sigma_0, \\ u_k = 0, & \text{on } \Sigma_0 \cap B_1, \end{cases}$$

satisfies the boundary condition (5.20) in $\Sigma_0 \cap B_1$ and

$$\|u_k\|_{C^{1,\alpha}(B_{1/2})} \to \infty.$$

Let us fix a smooth cut-off function $\eta \in C_c^\infty(B_1)$ such that

$$\mathrm{spt}(\phi) \subset B_{3/4}, \quad \eta = 1 \text{ in } B_{1/2}, \quad 0 \le \eta \le 1,$$

hence, we have

$$L_k := [\nabla(\eta u_k)]_{C^{0,\alpha}(B_1)} \to \infty.$$

By definition of Hölder seminorm, take two sequences of points $z_k = (x_k, y_k), \hat z_k = (\hat x_k, \hat y_k) \in B_1$ such that

$$\frac{|\nabla(\eta u_k)(z_k) - \nabla(\eta u_k)(\hat z_k)|}{|z_k - \hat z_k|^\alpha} \ge \frac{L_k}{2},$$

and define $r_k := |z_k - \hat z_k|$. Now we distinguish two cases.

- **Case 1:** $\dfrac{|y_k|}{r_k} \to \infty$,
- **Case 2:** $\dfrac{|y_k|}{r_k} \le c$,

for some $c > 0$ which not depends on k. Let us define

$$\tilde z_k = (\tilde x_k, \tilde y_k) := \begin{cases} (x_k, y_k), & \text{in } \textbf{Case 1}, \\ (x_k, 0), & \text{in } \textbf{Case 2}, \end{cases}$$

the sequence of domains

$$\Omega_k := \frac{B_1 \setminus \Sigma_0 - \tilde z_k}{r_k},$$

the sequences of functions

$$v_k(z) := \frac{(\eta u_k)(\tilde z_k + r_k z) - (\eta u_k)(\tilde z_k) - \nabla(\eta u_k)(\tilde z_k) \cdot r_k z}{L_k r_k^{1+\alpha}},$$
$$w_k(z) := \frac{\eta(\tilde z_k) u_k(\tilde z_k + r_k z) - (\eta u_k)(\tilde z_k) - (\eta \nabla u_k)(\tilde z_k) \cdot r_k z}{L_k r_k^{1+\alpha}},$$

for $z \in \Omega_k$ and set Ω_∞ as in (4.7).

By following the argument in Theorem 5.1, *Step 2*, we have that $\|v_k\|_{C^{1,\alpha}(K)} \le 1$ for every compact subset $K \subset \Omega_\infty$. Therefore, we can apply the Arzelà-Ascoli Theorem to conclude that $v_k \to \bar v$ in $C^{1,\gamma}(K)$ for every $\gamma \in (0, \alpha)$, and that $|\bar v(z)| \le c(1 + |z|^{1+\alpha})$, for a.e. $z \in \Omega_\infty$. Moreover, it follows that $w_k \to \bar v$ as well.

Next, by the same reasoning used in Theorem 5.1, *Step 3*, we obtain that $\nabla \bar v$ is not constant. Continuing as in *Step 4* of Theorem 5.1, we conclude that $r_k \to 0$, which implies that the limit domain Ω_∞ is given by

$$\Omega_\infty := \begin{cases} \mathbb{R}^d, & \text{in } \textbf{Case 1}, \\ \mathbb{R}^d \setminus \Sigma_0, & \text{in } \textbf{Case 2}. \end{cases}$$

Eventually, we prove that $\bar v$ is an entire solution to a homogeneous equation with constant coefficients and we reach a contradiction by invoking a Liouville type Theorem. Since $\|A_k\|_{C^{0,\alpha}(B_1)} \le L$, we have that $A_k(\tilde z_k + r_k z) \to A(\bar z) := \bar A$, which is a constant matrix satisfying (1.2). Let us define

$$\rho_k(y) := \begin{cases} \dfrac{|y_k + r_k y|}{|y_k|}, & \text{in } \textbf{Case 1}, \\ |y|, & \text{in } \textbf{Case 2}. \end{cases}$$

and fix $\phi \in C_c^\infty(\Omega_\infty)$. A straightforward computation show us that

$$\begin{aligned}\int_{\mathrm{spt}(\phi)} \rho_k^a(y) A_k(\tilde z_k + r_k z)\nabla w_k(z)\cdot\nabla\phi(z)dz &= \frac{\eta(\tilde z_k) r_k^{1-\alpha}}{L_k}\int_{\mathrm{spt}(\phi)} \rho_k^a(y) f_k(\tilde z_k + r_k z)\phi(z)dz\\ &-\frac{\eta(\tilde z_k) r_k^{-\alpha}}{L_k}\int_{\mathrm{spt}(\phi)} \rho_k^a(y)\big(F_k(\tilde z_k + r_k z) - F_k(\tilde z_k)\big)\cdot\nabla\phi(z)dz\\ &-\frac{r_k^{-\alpha}}{L_k}\int_{\mathrm{spt}(\phi)} \rho_k^a(y)\big(A_k(\tilde z_k + r_k z) - A_k(\tilde z_k)\big)(\eta\nabla u_k)(\tilde z_k)\cdot\nabla\phi(z)dz\\ &-\frac{r_k^{-\alpha}}{L_k}\rho_k^a(y)\big(\eta A_k\nabla u_k + \eta F_k\big)(\tilde z_k)\cdot\nabla\phi(z)dz = \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}.\end{aligned}$$

The terms I, II, III vanishes as $k\to\infty$ exactly as in Theorem 5.1, *Step 6*.

Next, we show that the fourth member goes to zero. By using the divergence theorem, we get

$$\int_{\mathrm{spt}(\phi)} \rho_k^a(y)\big(\eta A_k\nabla u_k + \eta F_k\big)(\tilde z_k)\cdot\nabla\phi(z)dz = \int_{\mathrm{spt}(\phi)} \nabla\rho_k^a(y)\cdot\big(\eta A_k\nabla u_k + \eta F_k\big)(\tilde z_k)\phi(z)dz.$$

In **Case 2**, since, $\tilde z_k = (x_k, 0)\in\Sigma_0$, u_k satisfies the boundary condition (5.20) we have that

$$\eta(\tilde z_k)(A_k\nabla u_k + F_k)(\tilde z_k)\cdot e_{y_i} = 0, \quad \text{for every } i = 1,\dots,n,$$

so IV$=0$.

Let us consider the **Case 1**, recalling that $\tilde z_k = z_k$, $r_k/|y_k|\to 0$ and $\rho_k^a\to 1$. Arguing as Theorem 5.1, *Step 6*, we have that

$$|\nabla\rho_k^a(y)| \le c\frac{r_k}{|y_k|}\rho_k^a(y),$$

and by using the boundary condition (5.20)

$$\begin{aligned}&\left|(\eta A_k\nabla u_k + \eta F_k)(z_k)\cdot\frac{y_k + r_k y}{|y_k + r_k y|}\right|\\ &\le \left|\big((\eta A_k\nabla u_k + \eta F_k)(z_k) - (\eta A_k\nabla u_k + \eta F_k)(x_k,0)\big)\cdot\frac{y_k + r_k y}{|y_k + r_k y|}\right|\\ &\le [\eta A_k\nabla u_k + \eta F_k]_{C^{0,\alpha}(B_{3/4})}|y_k|^\alpha \le cL_k|y_k|^\alpha.\end{aligned}$$

Hence, combining these two inequalities we obtain

$$\left|\nabla\rho_k^a(y)\cdot(\eta A_k\nabla u_k + \eta F_k)(z_k)\right| \le c\frac{r_k}{|y_k|^{1-\alpha}}L_k,$$

so

$$|\mathrm{IV}| = \frac{r_k^{-\alpha}}{L_k}\left|\int_{\mathrm{spt}(\phi)}\nabla\rho_k^a(y)\cdot\big(\eta A_k\nabla u_k + \eta F_k\big)(z_k)\phi(z)dz\right| \le c\Big(\frac{r_k}{|y_k|}\Big)^{1-\alpha}\to 0,$$

as $k\to\infty$.

On the other hand, arguing as in the *Step 5* of the Theorem 4.1, we have that

$$\int_{\mathrm{spt}(\phi)} \rho_k^a(y) A_k(\tilde z_k + r_k z)\nabla w_k(z)\cdot\nabla\phi(z)dz \to \int_{\mathrm{spt}(\phi)} \bar\rho^a\bar A\nabla\bar v\cdot\nabla\phi dz,$$

where

$$\bar\rho(y) := \begin{cases}1, & \text{in } \textbf{Case 1},\\ |y|, & \text{in } \textbf{Case 2}.\end{cases}$$

Then, we have that

- in **Case 1**, $\bar{v}$ is an entire solution to

$$-\operatorname{div}(\bar{A}\nabla\bar{v}) = 0, \quad \text{in } \mathbb{R}^d,$$

- in **Case 2**, $\bar{v}$ is an entire solution to

$$\begin{cases} -\operatorname{div}(|y|^a\bar{A}\nabla\bar{v}) = 0, & \text{in } \mathbb{R}^d \setminus \Sigma_0, \\ \bar{v} = 0, & \text{on } \Sigma_0, \end{cases}.$$

where $\bar{A}$ is symmetric constant matrix satisfying (1.2). From this point on, as in Theorem 5.1, *Step 7*, by invoking appropriate Liouville type Theorems we get a contradiction and the thesis follows. □

Proof of the Theorems 1.3. As in the proof of Theorem 1.2, without loss of generality, we may consider the function $u - \psi$ and provide the proof of the theorem for solutions to (1.1) with homogeneous Dirichlet boundary conditions. Once this is established, the general case follows directly. We divide the proof in two steps.

Step 1. First, we claim that if $A \in C^{1,\alpha}(B_1)$ and $F \in C^{1,\alpha}(B_1)$ then $u \in C^{1,\alpha}(B_{1/2})$ and u satisfies the boundary condition (5.20) in $\Sigma_0 \cap B_{1/2}$.

Let us split the matrix A in blocks as follows

$$A = \begin{pmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{pmatrix},$$

where $A_1 : B_R \to \mathbb{R}^{d-n,d-n}$, $A_2 : B_R \to \mathbb{R}^{d-n,n}$, $A_3 : B_R \to \mathbb{R}^{n,n}$. Let us consider the decomposition $F = (F_x, F_y)$, where $F_x = (F_{x_1}, \ldots, F_{x_{d-n}})$ and $F_y = (F_{y_1}, \ldots, F_{y_n})$ and define the scalar function

$$g(x,y) = A_3^{-1}(x,0)F_2(x,0) \cdot y,$$

which belongs to $C^{1,\alpha}(B_1)$, since the block A_3 satisfies the uniformly elliptic condition (1.2), and

$$g(x,0) = 0, \qquad \nabla g(x,0) = \big(0, (A_3^{-1}F_2)(x,0)\big).$$

The function $v := u - g \in \tilde{H}^{1,a}(B_1)$ is a weak solution to

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla v) = |y|^a f + \operatorname{div}(|y|^a(F - A\nabla g)), & \text{in } B_1 \setminus \Sigma_0, \\ v = 0, & \text{on } \Sigma_0 \cap B_1, \end{cases}$$

where the field $F - A\nabla g \in C^\infty(B_1)$ satisfies

$$(F - A\nabla g)(x,0) \cdot e_{y_i} = 0.$$

Arguing as in the proof of Theorem 1.2, by applying Lemma 2.9, we can find a sequence $\{v_{\varepsilon_k}\}$ as $\varepsilon_k \to 0$, such that every v_{ε_k} is solution to

$$\begin{cases} -\operatorname{div}(|y|^a A\nabla v_{\varepsilon_k}) = |y|^a f + \operatorname{div}(|y|^a(F - A\nabla g)), & \text{in } B_{3/4} \setminus \Sigma_{\varepsilon_k}, \\ v_{\varepsilon_k} = 0, & \text{on } \partial\Sigma_{\varepsilon_k} \cap B_{3/4}, \end{cases}$$

and $v_{\varepsilon_k} \to v$ in $H^{1,a}(B_{3/4} \setminus \Sigma_0)$ as $\varepsilon_k \to 0$. By applying the Theorem 5.1 to the sequences $\{v_{\varepsilon_k}\}$, combined with the estimate (2.11), we get that

$$\|v_{\varepsilon_k}\|_{C^{1,\alpha}(B_{1/2}\setminus\Sigma_{\varepsilon_k})} \le c\big(\|v\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F - A\nabla g\|_{C^{0,\alpha}(B_1)}\big),$$

for some $c > 0$ which not depends on ε_k.

By applying Arzelà-Ascoli Theorem we get that $v_{\varepsilon_k} \to w$ in $C^{1,\gamma}_{\mathrm{loc}}(B_{1/2} \setminus \Sigma_0)$, for every $\gamma \in (0,\alpha)$, and by the a.e. convergences $v_{\varepsilon_k} \to v$ it follows that $w = v$. Moreover, by taking $z, z' \in B_{1/2} \setminus \Sigma_0$ such that $z \neq z'$, we have that

$$\begin{aligned}\frac{|\nabla v(z) - \nabla v(z')|}{|z-z'|^\alpha} &= \lim_{\varepsilon_k \to 0} \frac{|\nabla v_{\varepsilon_k}(z) - \nabla v_{\varepsilon_k}(z')|}{|z-z'|^\alpha} \\ &\leq c\big(\|v\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F - A\nabla g\|_{C^{0,\alpha}(B_1)}\big),\end{aligned}$$

which implies that

$$\begin{aligned}[v]_{C^{1,\alpha}(B_{1/2}\setminus\Sigma_0)} &= \sup_{\substack{z,z' \in B_{1/2}\setminus\Sigma_0 \\ z \neq z'}} \frac{|\nabla v(z) - \nabla v(z')|}{|z-z'|^\alpha} \\ &\leq c\big(\|v\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F - A\nabla g\|_{C^{0,\alpha}(B_1)}\big),\end{aligned}$$

By continuity, we can extend v to the whole $B_{1/2}$ in such a way that $[v]_{C^{1,\alpha}(B_{1/2})} = [v]_{C^{1,\alpha}(B_{1/2}\setminus\Sigma_0)}$. Combining the previous inequality with the L^∞_{loc} bound of solutions (see Lemma 2.8) and interpolation inequality in Hölder spaces [20, Lemma 6.35], we get

$$\|v\|_{C^{1,\alpha}(B_{1/2})} \leq c(\|v\|_{L^\infty(B_{1/2})} + [v]_{C^{1,\alpha}(B_{1/2})}) \leq c\big(\|v\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F - A\nabla g\|_{C^{0,\alpha}(B_1)}\big).$$

Hence, $v \in C^{1,\alpha}(B_{1/2})$, which immediately implies $u \in C^{1,\alpha}(B_{1/2})$.

Next, let us prove that u satisfies the boundary condition (5.20). Let us fix $z = (x,y) \in B_{1/2} \setminus \Sigma_0$ and let z_k^0 be the projection of z on $\partial\Sigma_{\varepsilon_k} \cap B_{1/2}$. By using Theorem 5.1 and (5.2), we get

$$\begin{aligned}|\nabla v(z)| &\leq |\nabla v(z) - \nabla v_{\varepsilon_k}(z)| + |\nabla v_{\varepsilon_k}(z) - \nabla v_{\varepsilon_k}(z_k^0)| + |\nabla v_{\varepsilon_k}(z_k^0)| \\ &\leq |\nabla v(z) - \nabla v_{\varepsilon_k}(z)| + [v_{\varepsilon_k}]_{C^{1,\alpha}(B_{1/2}\setminus\Sigma_{\varepsilon_k})}|z - z_k^0|^\alpha + c[v_{\varepsilon_k}]_{C^{1,\alpha}(B_{1/2}\setminus\Sigma_{\varepsilon_k})}\varepsilon_k^\alpha \\ &\leq o(1) + c\big(\|v\|_{L^{2,a}(B_1)} + \|f\|_{L^{p,a}(B_1)} + \|F - A\nabla g\|_{C^{0,\alpha}(B_1)}\big)|y|^\alpha, \quad \text{as } \varepsilon_k \to 0.\end{aligned}$$

By taking the limit as $|y| \to 0$ in the previous inequality, we conclude that $\nabla v = 0$ on $\Sigma_0 \cap B_{1/2}$ and, by construction, it follows that

$$0 = \nabla_x v(x,0) = \nabla_x u(x,0) - \nabla_x g(x,0) = \nabla_x u(x,0),$$

and

$$0 = (A\nabla v)(x,0)\cdot e_{y_i} = (A\nabla u - A\nabla g)(x,0)\cdot e_{y_i} = (A\nabla u + F)(x,0)\cdot e_{y_i},$$

that is, (5.20) holds true.

Step 2. Finally, we prove that if $A, F \in C^{0,\alpha}(B_1)$, then $u \in C^{1,\alpha}(B_{1/2})$ and satisfies (1.6), (1.7). Up to consider the function $u - \psi \in \tilde{H}^{1,a}(B_1)$, we can suppose that u satisfies a Dirichlet homogeneous boundary condition $u = 0$ on $\Sigma_0 \cap B_1$.

Let $\{\rho_\delta\}_{\delta>0}$ be a family of smooth mollifiers and define $A_\delta := A * \rho_\delta$ and $F_\delta := F * \rho_\delta$, which satisfies $\|A_\delta\|_{C^{0,\alpha}(B_{3/4})} \leq \|A\|_{C^{0,\alpha}(B_1)}$ and $\|F_\delta\|_{C^{0,\alpha}(B_{3/4})} \leq \|F\|_{C^{0,\alpha}(B_1)}$. Using an approximation argument similar to the one in Lemma 2.9, but with a more standard approach, and following a method analogous to [1, Theorem 4.1] in a similar context, we construct a family $\{u_\delta\}_{\delta>0}$ of solutions to

$$\begin{cases} -\operatorname{div}(|y|^a A_\delta \nabla u_\delta) = |y|^a f + \operatorname{div}(|y|^a F_\delta), & \text{in } B_{4/5} \setminus \Sigma_0, \\ u = 0, & \text{on } \Sigma_0 \cap B_{4/5}, \end{cases}$$

satisfies the following properties

$$\|u_\delta\|_{H^{1,a}(B_{4/5})} \leq c(\|u\|_{H^{1,a}(B_1)} + \|f\|_{L^{2,a}(B_1)} + \|F\|_{L^{2,a}(B_1)}),$$

$$u_\delta \to u \text{ strongly in } H^{1,a}(B_{4/5}),$$

for some $c > 0$ depending only on d, n, a, λ and Λ.

Since $A_\delta, F_\delta \in C^\infty(B_{4/5})$, by using *Step 1*, we get that $u \in C^{1,\alpha}(B_{3/4})$ and u satisfies the boundary condition (5.20) on $\Sigma_0 \cap B_{3/4}$. Hence, the assumptions of the Proposition 5.3 are satisfies, which implies that u_δ satisfies (5.21), which immediately implies that

$$\|u_\delta\|_{C^{1,\alpha}(B_{1/2})} \leq c(\|u\|_{L^{2,a}(B_1)} + \|f\|_{L^{2,a}(B_1)} + \|F\|_{L^{2,a}(B_1)}),$$

and the Arzelá-Ascoli Theorem allows us to taking the limit as $\delta \to 0$ to infer that $u \in C^{1,\alpha}(B_{1/2})$, satisfies (1.6) and the boundary condition (1.7). The proof is complete. ☐

6. Regularity on manifolds

In this section, we prove how to extend Theorems 1.2 and 1.3, namely the local $C^{0,\alpha}$ and $C^{1,\alpha}$ regularity of weak solutions to (1.1), to a class of equations with weights that are singular on $(d-n)$-dimensional manifolds, that is, for weak solutions to (1.10).

For $2 \leq n < d$, let $\Gamma \subset \mathbb{R}^d$ be a $(d-n)$-dimensional C^1-manifold such that $0 \in \Gamma$ and there exists a parametrization $\varphi \in C^1(\Sigma_0 \cap B_1; \mathbb{R}^n)$ such that, up to perform a dilation, we have

$$B_1 \cap \Gamma = \{(x,y) : y = \varphi(x)\} \cap B_1, \quad \varphi(0) = 0. \tag{6.1}$$

The diffeomorphism

$$\Phi(x,y) := (x, y + \varphi(x)), \tag{6.2}$$

whose inverse straightens the lower dimensional boundary Γ to Σ_0, satisfies $\Phi(\Sigma_0 \cap B_1) \subset \Gamma \cap B_1$ and the Jacobian associated to Φ is

$$J_\Phi(x,y) = \begin{pmatrix} I_{d-n} & 0 \\ J_\varphi(x) & I_n \end{pmatrix}, \quad \text{with } |\det J_\Phi| \equiv 1.$$

Given a $(d-n)$-dimensional manifold Γ parametrized by φ, we give the definition of admissible weights with respect to φ.

Definition 6.1. Let $2 \leq n < d$ and $\alpha \in [0,1)$. Let Γ be a $(d-n)$-dimensional $C^{1,\alpha}$-manifold and $\varphi \in C^{1,\alpha}(\Sigma_0 \cap B_1; \mathbb{R}^n)$ be a parametrization of Γ in the sense of (6.1) and define the diffeomorphism Φ as in (6.2).

We say that δ is an α-admissible weight with respect to the parametrization φ if $\delta \in C^{0,1}(B_1)$ and the two following condition holds true.

i) There exist constants $0 < c_0 \leq c_1$ such that

$$c_0 \leq \frac{\delta}{\mathrm{dist}_\Gamma} \leq c_1. \tag{6.3}$$

ii)

$$\tilde{\delta}(x,y) := \frac{\delta(\Phi(x,y))}{|y|} \in C^{0,\alpha}(B_1). \tag{6.4}$$

We point out that the condition i) in the Definition 6.1 implies that δ in a equivalent distance from Γ to the standard distance function dist_Γ. So, given δ, which satisfies condition i), similarly to what was done in Section 2.1, we can define the functional spaces $H^1(B_1,\delta^a)$, which is equivalent to $H^1(B_1,\mathrm{dist}_\Gamma^a)$. Then, by Theorem 1.1, it follows that it is well defined the bounded linear trace operator $T: H^1(B_1,\delta^a)\to L^2(\Gamma\cap B_1)$ such that $Tu=u_{|_{\Gamma\cap B_1}}$, for every $u\in C^\infty(\overline{B_1})$. In light of this, we define a notion of weak solutions for the equation (1.10), which satisfy a Dirichlet boundary on Γ.

Definition 6.2. Let $2\le n<d$, $a+n\in(0,2)$, A be a matrix satisfying (1.2). Let Γ be a $(d-n)$-dimensional C^1-manifold and $\varphi\in C^1(\Sigma_0\cap B_1;\mathbb{R}^n)$ be a parametrization of Γ in the sense of (6.1), $\delta\in C^{0,1}(B_1)$ satisfying i) of the Definition 6.1. Let $f\in L^2(B_1,\delta^a)$, $F\in L^2(B_1,\delta^a)^d$ and $\psi\in L^2(\Gamma\cap B_1)$.

We say that u is a weak solution to (1.10) if $u\in H^1(B_1,\delta^a)$, satisfies

$$\int_{B_1}\delta^a A\nabla u\cdot\nabla\phi dz=\int_{B_1}\delta^a(f\phi-F\cdot\nabla\phi)dz, \tag{6.5}$$

for every $\phi\in C_c^\infty(B_1\setminus\Gamma)$ and $u=\psi$ in $L^2(\Gamma\cap B_1)$, in the sense of the trace.

Finally, we can state the main results of this section, namely $C^{0,\alpha}_{\mathrm{loc}}$ and $C^{1,\alpha}_{\mathrm{loc}}$ regularity for weak solutions to (1.10).

Corollary 6.3. *Let $2\le n<d$, $a+n\in(0,2)$, $p>d/2$, $q>d$ and α satisfying (1.3). Let $\varphi\in C^1(\Sigma_0\cap B_1;\mathbb{R}^n)$ be the parametrization defined in (6.1). Let δ be a 0-admissible weight with respect to φ in the sense of Definition 6.1 and define $\tilde\delta\in C^0(B_1)$ as in (6.4). Let A be a continuous symmetric matrix satisfying (1.2), $f\in L^p(B_1,\delta^a)$, $F\in L^q(B_1,\delta^a)^d$, $\psi\in C^{0,1}(\Gamma\cap B_1)$ and u be a weak solution to (1.10), in the sense of Definition 6.2. Let us suppose that there exists a modulus of continuity ω such that*

$$\|A\|_{C^{0,\omega}(B_1)}+\|\tilde\delta\|_{C^{0,\omega}(B_1)}+\|\varphi\|_{C^{1,\omega}(\Sigma_0\cap B_1;\mathbb{R}^n)}\le L$$

Then, $u\in C^{0,\alpha}(B_{1/2})$ and there exists a constant $c>0$, depending only on d, n, a, λ, Λ, p, q, α and L such that

$$\|u\|_{C^{0,\alpha}(B_{1/2})}\le c\big(\|u\|_{L^2(B_1,\delta^a)}+\|f\|_{L^p(B_1,\delta^a)}+\|F\|_{L^q(B_1,\delta^a)}+\|\psi\|_{C^{0,1}(\Gamma\cap B_1)}\big).$$

Corollary 6.4. *Let $2\le n<d$, $a+n\in(0,1)$, $p>d$ and α satisfying (1.5). Let $\varphi\in C^{1.\alpha}(\Sigma_0\cap B_1;\mathbb{R}^n)$ be the parametrization defined in (6.1). Let δ be a α-admissible weight with respect to φ in the sense of Definition 6.1 and define $\tilde\delta\in C^{0,\alpha}(B_1)$, as in (6.4). Let A be a α-Hölder continuous symmetric matrix satisfying (1.2), $f\in L^p(B_1,\delta^a)$, $F\in C^{0,\alpha}(B_1)$, $\psi\in C^{1,\alpha}(\Gamma\cap B_1)$ and u be a weak solution to (1.10), in the sense of Definition 6.2. Let us suppose that*

$$\|A\|_{C^{0,\alpha}(B_1)}+\|\tilde\delta\|_{C^{0,\alpha}(B_1)}+\|\varphi\|_{C^{1,\alpha}(\Sigma_0\cap B_1;\mathbb{R}^n)}\le L.$$

Then, $u\in C^{1,\alpha}(B_{1/2})$ and there exists a constant $c>0$, depending only on d, n, a, λ, Λ, p, α and L such that

$$\|u\|_{C^{1,\alpha}(B_{1/2})}\le c\big(\|u\|_{L^2(B_1,\delta^a)}+\|f\|_{L^p(B_1,\delta^a)}+\|F\|_{C^{0,\alpha}(B_1)}+\|\psi\|_{C^{1,\alpha}(\Gamma\cap B_1)}\big). \tag{6.6}$$

Moreover, denoting by $T_z\Gamma$ the tangent space to Γ at the point $z\in\Gamma$, we have that u satisfies the following boundary condition for every $z\in\Gamma\cap B_{1/2}$,

$$\begin{cases}\nabla u(z)\cdot\tau(z)=\nabla\psi(z)\cdot\tau(z), & \textit{for every } \tau(z)\in T_z\Gamma,\\ (A\nabla u+F)(z)\cdot\nu(z)=0, & \textit{for every } \nu(z)\perp T_z\Gamma.\end{cases} \tag{6.7}$$

Since the proofs of Corollaries 6.3 and 6.4 are quite similar, we will only provide the proof of the second one, as it is more involved.

Proof of Corollary 6.4. Let us define the diffeomorphism Φ as in (6.2), which is of class $C^{1,\alpha}(B_1)$, by the assumption $\varphi \in C^{1,\alpha}(\Sigma_0 \cap B_1)$. Defining

$$\tilde{u}(x,y) := u \circ \Phi(x,y), \qquad \tilde{\psi}(x) = \psi \circ \Phi(x,0) \in C^{1,\alpha}(\Sigma_0 \cap B_1).$$

we have that $\tilde{u} \in H^{1,a}(B_1)$ and $\tilde{u} = \tilde{\psi}$ on $\Sigma_0 \cap B_1$ in the sense of the traces. Since δ is an α-admissible weight in the sense of Definition 6.1, the conditions (6.3) and (6.4) implies that

$$\tilde{\delta}^a(x,y) = \frac{\delta(\Phi(x,y))^a}{|y|^a} \in C^{0,\alpha}(B_1), \quad \tilde{\delta}^a \geq \tilde{c}_0 > 0, \tag{6.8}$$

for some constant $\tilde{c}_0 > 0$, where $\tilde{\delta} = \delta \circ \Phi$.

Let $\phi \in C_c^\infty(B_1 \setminus \Gamma)$ be a test function in (6.5). By taking the change of variables $z = \Phi(x,y)$ it follows that

$$0 = \int_{B_1} \delta^a \big(A\nabla u \cdot \nabla\phi - f\phi + F \cdot \nabla\phi\big) dz = \int_{B_1} |y|^a \big(\tilde{A}\nabla\tilde{u} \cdot \nabla\tilde{\phi} - \tilde{f}\tilde{\phi} + \tilde{F} \cdot \nabla\tilde{\phi}\big) dz,$$

where

$$\tilde{A} := \tilde{\delta}^a (J_\Phi^{-1})(A \circ \Phi)(J_\Phi^{-1})^T, \quad \tilde{f} := \tilde{\delta}^a f \circ \Phi, \quad \tilde{F} := \tilde{\delta}^a (J_\Phi^{-1}) F \circ \Phi, \quad \tilde{\phi} := \phi \circ \Phi. \tag{6.9}$$

By using (6.8), we have that

$$\tilde{A} \in C^{0,\alpha}(B_1) \text{ and satisfies (1.2)}, \quad \tilde{f} \in L^{p,a}(B_1), \quad \tilde{F} \in C^{0,\alpha}(B_1).‘$$

Hence, we have proved that $\tilde{u}$ is a weak solution to

$$\begin{cases} -\operatorname{div}(|y|^a \tilde{A}\nabla\tilde{u}) = |y|^a \tilde{f} + \operatorname{div}(|y|^a \tilde{F}), & \text{in } B_1 \setminus \Sigma_0, \\ \tilde{u} = \tilde{\psi}, & \text{on } \Sigma_0 \cap B_1, \end{cases}$$

and $\tilde{u}$ satisfies the hypothesis of the Theorem 1.3. Hence, $\tilde{u}$ satisfies (1.6) and composing back with the diffeomorphism Φ^{-1} we get that u satisfies (6.6).

Eventually, let us prove that u satisfies the boundary condition (6.7). By Theorem 1.3, we have that $\tilde{u}$ satisfies the boundary condition

$$\nabla_x \tilde{u} = \nabla_x \tilde{\psi}, \quad (\tilde{A}\nabla\tilde{u} + \tilde{F}) \cdot e_{y_i} = 0, \quad \text{on } \Sigma_0 \cap B_{1/2}, \text{ for every } i = 1, \ldots, n. \tag{6.10}$$

Since $\tilde{u}(z) = u(\Phi(z))$, one has that

$$\nabla\tilde{u}(z) = J_\Phi^T(z) \nabla u(\Phi(z)).$$

and, noting $\Phi(x,0) = (x, \varphi(x)) \in \Gamma$, we have that

$$\nabla\tilde{u}(x,0) \cdot e_{x_j} = J_\Phi^T(x,0) \nabla u(\Phi(x,0)) \cdot e_{x_j} = \nabla u(x,\varphi(x)) \cdot J_\Phi(x,0) e_{x_j}.$$

Thus, for every $j = 1, \ldots, d-n$, and $(x,0) \in \Sigma_0 \cap B_{1/2}$, we have

$$\nabla u(x,\varphi(x)) \cdot J_\Phi(x,0) e_{x_j} = \nabla\psi(x,\varphi(x)) \cdot J_\Phi(x,0) e_{x_j}. \tag{6.11}$$

Next, recalling (6.8), (6.9) and (6.10), we find

$$\begin{aligned} 0 = \tilde{\delta}^{-a}(\tilde{A}\nabla\tilde{u} + \tilde{F})(x,0) \cdot e_{y_i} &= J_\Phi^{-1}(x,0)(A\nabla u + F)(x,\varphi(x)) \cdot e_{y_i} \\ &= (A\nabla u + F)(x,\varphi(x)) \cdot (J_\Phi^{-1})^T(x,0) e_{y_i}. \end{aligned} \tag{6.12}$$

Finally, observing that the tangent space to Γ at the point $z = (x, \varphi(x))$ is given by

$$T_{(x,\varphi(x))}\Gamma := \{(\xi, J_\varphi(x)\xi) : \xi \in \mathbb{R}^{d-n}\},$$

and noting that

$$J_\Phi(x,0)e_{x_j} \cdot (J_\Phi^{-1})^T(x,0)e_{y_i} = 0, \quad \text{for every } j = 1, \dots, d-n, \quad i = 1, \dots, n,$$

it follows that (6.11) and (6.12) implies that (6.7) holds true. This complete the proof. $\square$

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