On Estimation of Angles of Arrival in Monostatic ISAC Without Instantaneous Transmit CSI

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***Abstract*—This paper explores the fundamental limits of Integrated Sensing and Communication (ISAC) in a more realistic setting compared to previous literature. We analyze a monostatic setting where the Base Station (BS) performs multi-target Angle of Arrival (AoA) estimation while simultaneously communicating with one of the targets. We assume that the BS has statistical CSI about all AoAs, with less uncertainty in the AoA of the communication receiver. The communication receiver is assumed to have perfect CSI. Utilizing a Bayesian Cramér-Rao Bound (BCRB) framework to characterize the fundamental limits of sensing under minimum mean square error (MMSE) criteria, we derive achievable BCRB-rate trade-off regions. Our approach introduces a number of transmission strategies that share power across sensing and communication beams over a coherence time. Our analysis reveals that beam allocation strategies leveraging the principal eigenvectors of the target-specific sensing matrices minimize individual AoA estimation errors, while strategies balancing sensing and communication directions optimize joint estimation performance at the cost of individual accuracy. We demonstrate that leveraging updated BCRB-based sensing information for the communication receiver, due to its lower channel uncertainty, enables significantly improved communication rates.**

I. Introduction

The conceptualization of Integrated Sensing and Communication (ISAC) can be traced back to coexistence and coordination between radar and communication systems within the spectrum. Early research explored the potential synergies between these functionalities, leading to a rapid expansion of work in this field, as highlighted in surveys such as [1], [2]. Here, we focus on the problem of the exploration of optimal trade-offs between sensing and communication capabilities presented in [2]–[4], where there are some targets to sense and other separate ones to simultaneously communicate with, under various assumption regarding channel state knowledge at various terminals. In particular, we consider a monostatic scenario where the transmitter is also the sensing receiver that only has statistical knowledge about the tragets to estimate.

The work in [3], which is the foundation for our work, addresses the fundamental trade-off in ISAC from both information-theoretic and estimation-theoretic perspectives for vector AWGN channels and the MMSE sensing metric. The authors in [3] define the BCRB-rate region as the set of all possible achievable pairs of ergodic communication rate and sensing MMSE. An inner bound can be formed through a time-sharing strategy between two optimal operating points: one where sensing is optimized and one where communication rate is optimized. This simple baseline strategy already captures the key trade-offs in ISAC. At the sensing-optimized point, the transmit signal should "align" with the sensing channel, with the input being "relatively deterministic" to improve sensing accuracy. Conversely, for the rate-optimized point, the transmit signal should "align" with the communication channel, with the input being "as random as possible" to maximize communication rate. Improved achievable regions were obtained by shaping the input covariance matrix and achieving an improved takeoff compared to time-sharing.

In this paper, we build upon the AoA estimation framework introduced in [3], considering a scenario where the transmitter has access only to statistical channel state information (CSI) for both the sensing targets and the communication receivers. Although the work in [5] operates within a comparable context, they focus primarily on optimizing the required number of dedicated sensing beams subject to a communication rate requirement for the user. In our work, the level of channel uncertainty differs between these two types of targets. Specifically, the uncertainty associated with the communication receivers is lower than that of the sensing targets. This difference arises because the transmitter can leverage the sensing BCRB for the communication target, leading to more precise estimations when transmitting communication data. Furthermore, communication channels are typically acquired earlier, allowing the transmitter to have a more accurate, though still imperfect, representation of these channels. In contrast, sensing relies on partially outdated CSI, which introduces greater uncertainty. This distinction mirrors practical ISAC scenarios, where a transmitter operates under different levels of CSI accuracy depending on the function being performed. The system must thus account for these varying degrees of uncertainty when optimizing its performance.

To address this challenge, we propose transmission strategies tailored for the two-target case, which result in achievable ISAC regions, which include sensing-only, and communication-only scenarios as special cases. Furthermore, we shed light into the trade-offs between sensing and communication performance of the proposed schemes by analyzing how the available power is allocated among the different tasks. By exploring various power allocation strategies, we examine how resource distribution impacts the achievable ISAC regions. Our analysis highlights the fundamental trade-offs between sensing accuracy and communication efficiency,

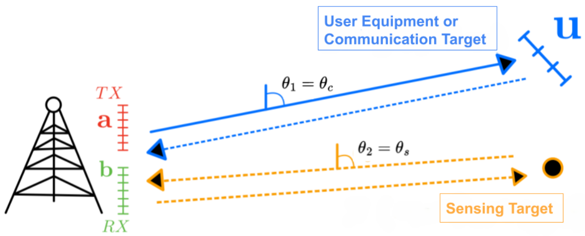

Fig. 1. Representation of the considered setting: one communication target (UE) that is also a sensing target and a separate sensing-only target.

demonstrating the crucial role of power allocation and beam design in optimizing overall system performance.

II. System Model

We consider a scenario where a Base Station (BS) is simultaneously a transmitter for the communication receiver and a monostatic radar for the sensing targets, as shown in Fig. 1 for the case of two targets. The BS features a full-duplex uniform linear array (ULA) with M_{TX} transmitting and M_{RX} receiving elements, characterized by steering vectors $\mathbf{a} \in \mathbb{C}^{M_{\text{TX}} \times 1}$ and $\mathbf{b} \in \mathbb{C}^{M_{\text{RX}} \times 1}$, respectively. We assume ideal full-duplex operation without self-interference at the BS. The communication receiver (or User Equipment, UE) has a ULA with M_{UE} elements and steering vector $\mathbf{u} \in \mathbb{C}^{M_{\text{UE}} \times 1}$. Both the UE and the sensing targets reflect the BS-transmitted signal back to the BS receiver. The sensing goal is to estimate the AoAs of the UE and the purely sensing targets.

Let T be the channel coherence time, i.e., the channel parameters undergo synchronous and i.i.d. variations every T channel uses. The BS transmits signal $\mathbf{X} \in \mathbb{C}^{M_{\text{TX}} \times T}$, and receives reflected signals $\mathbf{Y}_s \in \mathbb{C}^{M_{\text{RX}} \times T}$ from sensing targets, while the UE receives $\mathbf{Y}_c \in \mathbb{C}^{M_{\text{UE}} \times T}$, where

$$\mathbf{Y}_c = \mathbf{H}_c \mathbf{X} + \mathbf{Z}_c, \quad \mathbf{H}_c = \alpha_1 \, \mathbf{u}(\theta_1) \mathbf{a}^{\text{T}}(\theta_1), \tag{1}$$

$$\mathbf{Y}_s = \mathbf{H}_s \mathbf{X} + \mathbf{Z}_s, \quad \mathbf{H}_s = \sum_{\ell=1}^{N_s} \beta_\ell \, \mathbf{b}(\theta_\ell) \mathbf{a}^{\text{T}}(\theta_\ell), \tag{2}$$

where $\mathbf{X}$ satisfies the average power constraint

$$\text{Trace}\{\mathbb{E}\{\mathbf{R}_\mathbf{X}\}\} \leq P_{\max}, \quad \mathbf{R}_\mathbf{X} := \frac{1}{T} \mathbf{X}\mathbf{X}^{\text{H}}. \tag{3}$$

The noise terms $\mathbf{Z}_c$ and $\mathbf{Z}_s$ are i.i.d., circularly symmetric Gaussian with zero mean and variances σ_c^2 and σ_s^2, respectively. The downlink channel matrix $\mathbf{H}_c \in \mathbb{C}^{M_{\text{UE}} \times M_{\text{TX}}}$ represents the communication link, where $\theta_1 \in [0, 2\pi]$ is the AoA of the UE, and $\alpha_1 \in \mathbb{C}$ is the attenuation; θ_1 are known at the UE. The sensing channel matrix $\mathbf{H}_s \in \mathbb{C}^{M_{\text{RX}} \times M_{\text{TX}}}$ models reflections from N_s targets, with AoAs $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_{N_s})$ and gains $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_{N_s})$. The BS estimates $\boldsymbol{\theta}$, while the attenuation vector $\boldsymbol{\beta}$ is considered as nuisance parameters. We assume the channel parameters to be mutually independent, i.e.,

$$P_{\theta_1,\ldots,\theta_{N_s},\beta_1,\ldots,\beta_{N_s}} = \prod_{\ell=1}^{N_s} P_{\theta_\ell} P_{\Re\{\beta_\ell\}} P_{\Im\{\beta_\ell\}}. \tag{4}$$

We also assume that N_s is known at the BS.

Sensing Task. The sensing task consists of estimating the angle of arrivals in $\mathbf{H}_s$ in (2). As a metric for this sensing task, we use the BCRB [3], a lower bound for the Mean Squared Error (MSE) of weakly unbiased estimators. The BCRB is defined as $\epsilon := \mathbb{E}_\mathbf{X} \left\{ \text{Trace} \left\{ \mathbf{J}_{\boldsymbol{\theta}|\mathbf{X}}^{-1} \right\} \right\}$, where $\mathbf{J}_{\boldsymbol{\theta}|\mathbf{X}}$ is the Bayesian Fisher Information Matrix (BFIM) of the parameters $\boldsymbol{\theta}$ we wish to estimate. The conditioning over $\mathbf{X}$ is because, in a monostatic setting, the sensing transmitter is also the sensing receiver so the transmit signal $\mathbf{X}$ is known. The BFIM $\mathbf{J}_{\boldsymbol{\theta}|\mathbf{X}}$ of the parameters $\boldsymbol{\theta}$ is given by [3, Eq. 6]. For the AWGN model, $\mathbf{J}_{\boldsymbol{\theta}|\mathbf{X}}$ depends on the input $\mathbf{X}$ through the (random in general) sample covariance matrix $\mathbf{R}_\mathbf{X}$ in (3).

Communication Task. The communication task consists of reliably transmitting information to the UE with received signal $\mathbf{Y}_c$ in (1). As a metric for this task, we use the ergodic achievable rate [3], given by $R := \frac{1}{T} I(\mathbf{X}; \mathbf{Y}_c | \mathbf{H}_c)$, where the mutual information is averaged over the distribution of $\mathbf{H}_c$, assumed known at the UE, but not the BS.

ISAC region. The ISAC region is defined as

$$\mathcal{R}_{\text{ISAC}} := \bigcup_{\mathcal{X}(P_{\max})} \left\{ (\epsilon, R) \;\middle|\; \epsilon = \mathbb{E}_\mathbf{X} \left\{ \text{Trace} \left\{ \mathbf{J}_{\boldsymbol{\theta}|\mathbf{X}}^{-1} \right\} \right\}, \right. \tag{5}$$

$$\left. R = \frac{1}{T} I(\mathbf{X}; \mathbf{Y}_c | \mathbf{H}_c) \right\}, \tag{6}$$

where $\mathcal{X}(P_{\max})$ is the set of all possible distributions on the transmit signal $\mathbf{X}$ that satisfy the average power constraint $P_{\max}$ in (3). We assume only statistical CSI at the BS transmitter, so $\mathbf{X}$ cannot depend on the realization of $\mathbf{H}_c$ or $\mathbf{H}_s$ in each coherence time.

III. Bayesian Cramér-Rao Bound (BCRB)

Key for our study is the evaluation the BCRB for the joint estimation of the unknown parameters in the sensing channel matrix: the angles and the complex gains. A general solution to this problem was derived in [6], here we explicitly report the result applied to our setting. The BFIM of the parameter vector $\boldsymbol{\eta} := [\boldsymbol{\theta}, \Re\{\boldsymbol{\beta}\}, \Im\{\boldsymbol{\beta}\}] \in \mathbb{R}^K, K = 3N_s$, can be written as [3], [7]

$$\mathbf{J}_{\boldsymbol{\eta}|\mathbf{X}} = \mathbf{J}^P + \mathbf{F} = \begin{bmatrix} \mathbf{J}_{\boldsymbol{\theta}}^P & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_{\Re\{\boldsymbol{\beta}\}}^P & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J}_{\Im\{\boldsymbol{\beta}\}}^P \end{bmatrix} \tag{7}$$

$$+ 2 \begin{bmatrix} \Re(\mathbf{F}_{11}) & \Re(\mathbf{F}_{12}) & -\Im(\mathbf{F}_{11}) \\ \Re(\mathbf{F}_{12}) & \Re(\mathbf{F}_{22}) & -\Im(\mathbf{F}_{22}) \\ -\Im(\mathbf{F}_{11}) & -\Im^{\top}(\mathbf{F}_{22}) & \Re(\mathbf{F}_{22}) \end{bmatrix}, \tag{8}$$

where the expressions of $\mathbf{F}_{11}$, $\mathbf{F}_{12}$ and $\mathbf{F}_{22}$ can be found in [6, Eq.(11)-(13)] and are not reported here for sake of space, and where the prior Fisher information matrix $\mathbf{J}^P$ is diagonal because of the assumption in (4). Assuming the complex amplitudes are circularly symmetric, i.e., $\mathbb{E}\{\beta_i\} = 0$, implies $\mathbb{E}_{\boldsymbol{\beta}}\{\mathbf{F}_{12}\} = \mathbf{0}$. By choosing the phase reference point of the ULAs such that $\mathbf{a}_\ell^{\text{H}}(\theta) \dot{\mathbf{a}}_\ell(\theta) = \mathbf{b}_\ell^{\text{H}}(\theta) \dot{\mathbf{b}}_\ell(\theta) = 0, \forall \ell \in [N_s]$, and by the independence of the complex amplitudes, following [3], [7] we get

$$\mathbb{E}_{\boldsymbol{\beta}}\left\{\mathbf{F}_{11}\right\} = \frac{T}{\sigma_s^2}\,\mathrm{Diag}\left[\mathbb{E}\left\{|\beta_i|^2\right\} f_i(\mathbf{R_X}),\ \forall i\in[N_s]\right], \quad (9)$$

where $\forall i \in [N_s]$ we have,

$$f_i(\mathbf{R_X}) := \mathrm{Trace}\left[\overline{\mathbf{M}}_i\,\mathbf{R_X}\right], \quad \overline{\mathbf{M}}_i := \mathbb{E}_{\theta_i}\left\{\mathbf{M}_i^*\right\}, \quad (10)$$

$$\mathbf{M}_i := \|\dot{\mathbf{b}}(\theta_i)\|^2\,\mathbf{a}(\theta_i)\mathbf{a}^{\mathrm{H}}(\theta_i) + \|\mathbf{b}(\theta_i)\|^2\,\dot{\mathbf{a}}(\theta_i)\dot{\mathbf{a}}^{\mathrm{H}}(\theta_i), \quad (11)$$

where $\dot{\mathbf{a}}(\theta) := \partial\mathbf{a}/\partial\theta$ and $\dot{\mathbf{b}}(\theta) := \partial\mathbf{b}/\partial\theta$. We do not report that expression for $\mathbb{E}_{\boldsymbol{\beta}}\{\mathbf{F}_{22}\}$ as it will not be needed next.

Since we are only interested in estimating the angles of arrival, the equivalent BFIM by treating the complex amplitudes as nuisance parameters [3], [7] is given by

$$\mathbf{J}_{\boldsymbol{\theta}|\boldsymbol{X}}^{(\mathrm{equiv})} = 2\mathbb{E}_{\boldsymbol{\beta}}\left\{\mathbf{F}_{11}\right\} + \mathbf{J}_{\boldsymbol{\theta}}^{P} \quad (12)$$

$$= \mathrm{Diag}\left[\frac{2T}{\sigma_s^2}\mathbb{E}\{|\beta_i|^2\} f_i(\mathbf{R_X}) + J_{\theta_i}^P, \forall i\in[N_s]\right], \quad (13)$$

where $J_{\theta_i}^P$ denotes the prior FIM for angle θ_i. Finally, by defining BCRB for the i-th target as $\epsilon_i(\mathbf{R_X})$, where,

$$\epsilon_i\left(\mathbf{R_X}\right) := \left(\frac{2T}{\sigma_s^2}\mathbb{E}\left\{|\beta_i|^2\right\} f_i(\mathbf{R_X}) + J_{\theta_i}^P\right)^{-1}, \quad (14)$$

we can express the joint BCRB for the AoAs as

$$\epsilon = \mathbb{E}_{\mathbf{X}}\left\{\mathrm{Trace}\left[(\mathbf{J}_{\boldsymbol{\theta}|\boldsymbol{X}}^{(\mathrm{equiv})})^{-1}\right]\right\} = \sum_{i\in[N_s]}\mathbb{E}_{\mathbf{X}}\left\{\epsilon_i(\mathbf{R_X})\right\} \quad (15)$$

IV. ISAC Region Outer Bound

An easily computable outer bound is as follows

$$\mathcal{R}_{\text{ISAC}} \subseteq \left\{(\epsilon, R)\ \middle|\ \epsilon \geq \sum_{i\in[N_s]}\epsilon'_{\mathrm{min},i},\ R\leq C'\right\}. \quad (16)$$

A bound on the sensing performance is $\epsilon \geq \sum_{i\in[N_s]}\epsilon'_{\mathrm{min},i}$:

$$\epsilon_{\mathrm{min},i} := \min_{\mathbf{R_X}:\ \mathrm{Trace}\{\mathbb{E}\{\mathbf{R_X}\}\}\leq P_{\max}} \mathbb{E}\left\{\epsilon_i(\mathbf{R_X})\right\}, \quad (17)$$

$$= \min\mathbb{E}_{\mathbf{X}}\left\{\frac{1}{\frac{2T}{\sigma_s^2}\mathbb{E}\left\{|\beta_i|^2\right\}\mathrm{Trace}[\overline{\mathbf{M}}_i\mathbf{R_X}] + J_{\theta_i}^P}\right\} \quad (18)$$

$$\geq \min\frac{1}{\frac{2T}{\sigma_s^2}\mathbb{E}\left\{|\beta_i|^2\right\}\mathbb{E}_{\mathbf{X}}\left\{\mathrm{Trace}[\overline{\mathbf{M}}_i\mathbf{R_X}]\right\} + J_{\theta_i}^P} \quad (19)$$

$$= \frac{1}{\frac{2T}{\sigma_s^2}\mathbb{E}\left\{|\beta_i|^2\right\}\max\mathbb{E}_{\mathbf{X}}\left\{\mathrm{Trace}[\overline{\mathbf{M}}_i\mathbf{R_X}]\right\} + J_{\theta_i}^P} \quad (20)$$

$$= \frac{1}{\frac{2T}{\sigma_s^2}\mathbb{E}\left\{|\beta_i|^2\right\}P_{\max}\lambda_{\max}[\overline{\mathbf{M}}_i] + J_{\theta_i}^P} =: \epsilon'_{\mathrm{min},i}. \quad (21)$$

Note that the strategy that attains $\epsilon'_{\mathrm{min},i}$ is $\mathbf{X} = \sqrt{P_{\max}}\mathbf{v}_i^{(1)}\otimes[1,1,...,1]$ where $\mathbf{v}_i^{(1)}$ is the eigenvector of $\overline{\mathbf{M}}_i$ corresponding to the largest eigenvalue $\lambda_{\max}[\overline{\mathbf{M}}_i]$, and $[1,1,...,1]$ is the all-one vector of length T.

The optimal communications strategy is obtained by solving

$$C := \max_{\mathbf{R_X}:\ \mathrm{Trace}\{\mathbb{E}\{\mathbf{R_X}\}\}\leq P_{\max}} R, \quad (22)$$

$$R := \mathbb{E}_{\alpha_1,\theta_1}\left[\log\left(1 + \frac{|\alpha_1|^2}{\sigma_c^2}\mathbf{a}^H(\theta_1)\mathbb{E}\{\mathbf{R_X}\}\mathbf{a}(\theta_1)\right)\right], \quad (23)$$

attained by a zero-mean circularly symmetric Gaussian input with covariance matrix $\mathbb{E}\{\mathbf{R_X}\}$, which can be found by using a convex optimizer such as CVX, or upper bounded as follows, for $\mathbf{K_X} := \mathbb{E}\{\mathbf{R_X}\}$ and $\overline{\mathbf{L}}_1 := \mathbb{E}_{\theta_1}[\mathbf{a}(\theta_1)\mathbf{a}^{\mathrm{H}}(\theta_1)]$:

$$\max\mathbb{E}_{\alpha_1,\theta_1}\left[\log\left(1 + \frac{|\alpha_1|^2}{\sigma_c^2}\mathbf{a}^{\mathrm{H}}(\theta_1)\mathbf{K_X}\mathbf{a}(\theta_1)\right)\right] \quad (24)$$

$$\leq \max\log\left(1 + \frac{\mathbb{E}\{|\alpha_1|^2\}}{\sigma_c^2}\mathbb{E}_{\theta_1}\{\mathbf{a}^{\mathrm{H}}(\theta_1)\mathbf{K_X}\mathbf{a}(\theta_1)\}\right) \quad (25)$$

$$= \log\left(1 + \frac{\mathbb{E}\{|\alpha_1|^2\}}{\sigma_c^2}\max\mathrm{Trace}[\overline{\mathbf{L}}_1\mathbf{K_X}]\right) \quad (26)$$

$$= \log\left(1 + \frac{\mathbb{E}\{|\alpha_1|^2\}}{\sigma_c^2}P_{\max}\lambda_{\max}[\overline{\mathbf{L}}_1]\right) =: C'. \quad (27)$$

Note that the strategy that attains C' is $\mathbf{X} = \sqrt{P_{\max}}\ell_1\otimes[G_1, G_2, ..., G_T]$ where ℓ_1 is the eigenvector of $\overline{\mathbf{L}}_1$ corresponding to the largest eigenvalue $\lambda_{\max}[\overline{\mathbf{L}}_1]$, and $[G_1, G_2, ..., G_T]$ has T i.i.d. $\mathcal{N}(0,1)$ components.

V. Achievable Strategies

In an attempt to 'span' between the (deterministic) sensing-optimal strategy and the (Gaussian) communication-optimal strategy, in this work we considered

$$\mathbf{X} = [\mathbf{X}_1, \ldots, \mathbf{X}_T] : \mathbf{X}_t = \sqrt{P_{s;t}}\,\mathbf{s}_t + \sqrt{P_{c;t}}\,\mathbf{c}_t G_t, \quad (28)$$

where $\mathbf{s}_t$ is the unit-length (deterministic) sensing vector at time t, while $\mathbf{c}_t$ is the unit-length (deterministic) communication vector at time t modulated by i.i.d. $G_t\sim\mathcal{N}(0,1)$, and is subject to the power constraint

$$\frac{1}{T}\sum_{t\in[T]}(P_{s;t} + P_{c;t}) \leq P_{\max}. \quad (29)$$

Here, we consider the case of $N_s = 2$. To compute the BCRB for the strategy in (28), we take the transmitted signal $\mathbf{X}$ and compute $\epsilon = \epsilon_1 + \epsilon_2$, where

$$\epsilon_i = \mathbb{E}_{\mathbf{X}}\left\{\left(\frac{2T}{\sigma^2}\mathbb{E}\left\{|\beta_i|^2\right\}\mathrm{Trace}[\overline{\mathbf{M}}_i\mathbf{R_X}] + J_{\theta_i}^{\mathrm{P}}\right)^{-1}\right\}. \quad (30)$$

For our strategies, the achievable rate attainable is [3]

$$R_{\mathrm{in}} = \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}_{\theta_1}\left[\log_2\left(1 + \mathrm{SNR}_c\cdot\frac{\left|\mathbf{c}_t^{\mathrm{H}}\mathbf{a}(\theta_1)\right|^2}{\mathbb{E}_{\theta_1}\left[\|\mathbf{a}(\theta_1)\|^2\right]}\right)\right], \quad (31)$$

$$\mathrm{SNR}_c := P_c\,|\alpha_1|^2\,\mathbb{E}_{\theta_1}\left[\|\mathbf{a}(\theta_1)\|^2\right]/\sigma_c^2. \quad (32)$$

TABLE I
DEFINITION AND COMPUTATION OF DIRECTION VECTORS

Vector(s)	Definition and Computation
$\mathbf{v}_1^{(1)}, \mathbf{v}_2^{(1)}$	Principal eigenvectors of $\overline{\mathbf{M}}_1$ via eigen-decomposition; represent dominant sensing directions for target 1.
$\mathbf{v}_1^{(2)}, \mathbf{v}_2^{(2)}$	Principal eigenvectors of $\overline{\mathbf{M}}_2$ via eigen-decomposition; represent dominant sensing directions for target 2.
$\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2$	Principal eigenvectors of $\mathbf{R}_{\mathbf{X}}$ from numerical solution to (17); represent optimal joint sensing directions.
$\hat{\mathbf{r}}_c$	Principle eigenvector from numerical solution to (22); represent optimal direction for communication

A. Optimal Communication and Sensing directions

To construct effective transmit strategies, we focus on two key operating points: one optimized for sensing and the other for communication. For the sensing-optimal strategy, minimizing the BCRB for a single target is achieved by transmitting along the eigenvector associated with the largest eigenvalue of the matrix $\overline{\mathbf{M}}_i$. However, when jointly estimating multiple angles, an analytical solution of (17) is not straightforward. Instead, we numerically compute an achievable sensing-optimal solution by solving a joint BCRB minimization problem using CVX, yielding two dominant directions $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$. This structure is justified by the fact that, as shown in [3, Corollary 2], the optimal solution has rank N_s. On the other hand, the communication-optimal direction $\hat{\mathbf{r}}_c$ is obtained by numerically solving the rate-maximization problem (22) under a transmit power constraint. Specifically, it corresponds to the dominant eigenvector of the optimal covariance matrix that maximizes the achievable rate over the communication channel, subject to the Gaussian signaling model. These vectors- defined in TABLE I- form the basis for constructing our transmit direction vectors $\mathbf{s}_t$ and $\mathbf{c}_t$ in (28), which are used to explore different sensing-communication trade-offs in Section VI.

B. Special case 1: only deterministic

For this strategy, we transmit only deterministic signals ($P_{c;t} = 0$). Because of lack of Gaussian, rate will be zero.

$$\mathbf{X} = [\mathbf{X}_1, \ldots, \mathbf{X}_T] : \mathbf{X}_t = \sqrt{P_{\max}}\ \mathbf{s}_t. \tag{33}$$

This strategy provides insights into minimum bounds on estimation accuracy for joint AoA estimation. For this case, closed form solution of (15) is [3],

$$\epsilon_{det,i} = \left(2T \cdot \mathrm{SNR}_{s,i} \cdot \mathrm{Trace}\left[\overline{\mathbf{M}}_i \mathbf{R}_{\mathbf{X}}\right] + J^{\mathrm{P}}_{\theta_i}\right)^{-1}, \tag{34}$$

$$\mathrm{SNR}_{s,i} := P_s\, \mathbb{E}\left\{|\beta_i|^2\right\} \sigma_s^{-2}, \tag{35}$$

where $\overline{\mathbf{M}}_i$ is defined in (10).

C. Special case 2: only Gaussian

We examine scenarios where power is allocated exclusively to the direction we send Gaussian random variables, with no power allocated to deterministic signal ($P_{s;t} = 0$). This setup corresponds to a typical communication signal that may also be repurposed for ISAC for both communication and sensing. The transmit signal takes the form,

$$\mathbf{X} = [\mathbf{X}_1, \ldots, \mathbf{X}_T] : \mathbf{X}_t = \sqrt{P_{\max}}\ \mathbf{c}_t G_t, \tag{36}$$

Here, the closed-form solution of (15) is [3, Eq. 90b]

$$\epsilon_{\text{gauss,i}} = \left(2(T-1) \cdot \mathrm{SNR}_{s,i} \cdot \mathrm{Trace}\left[\overline{\mathbf{M}}_i \mathbf{R}_{\mathbf{X}}\right]\right)^{-1} (1 + r_\zeta) \tag{37}$$

where $\mathrm{SNR}_{s,i}$ was defined in (35). The correction term r_ζ is given by [3],

$$r_\zeta = \sum_{n=1}^{T-2} \frac{(-1)^n \zeta^n}{\prod_{i=1}^{n}(T-i-1)} + \underbrace{(-1)^{T-1} \cdot \frac{e^\zeta \zeta^{T-1} \Gamma(0,\zeta)}{\Gamma(T-1)}}_{O(\zeta^{T-1}\log\zeta)} \tag{38}$$

where $\Gamma(a,x) = \int_x^\infty t^{a-1} e^{-t}\ dt$, denotes the incomplete Gamma function, and the variable ζ is computed as $\zeta = J^P_{\theta_i}\left(2 \cdot \mathrm{SNR}_{s,i} \cdot \mathrm{Trace}\left[\overline{\mathbf{M}}_i\, \mathbf{R}_{\mathbf{X}}\right]\right)^{-1}$.

VI. NUMERICAL ANALYSIS

In this section, we evaluate the BCRB in (30) and the communication rate in (31) for our defined strategies. These evaluations are performed under the chosen achievable transmission strategies and based on the following assumptions. We consider a system with $M_{\text{Tx}} = M_{\text{Rx}} = 10$ antennas, spaced at half-wavelength intervals. The maximum SNR per antenna is set to 10 dB for sensing and 15 dB for communication. The coherence time is assumed to be $T = 2$. For notational simplicity, we define the AoA of the communication user as $\theta_1 = \theta_c$, and the AoA of the sensing target as $\theta_2 = \theta_s$, as illustrated in Fig. 1. Both angles are modeled as random variables following Von Mises distributions, with means $\bar{\theta}_s = 30^\circ$ and $\bar{\theta}_c = 100^\circ$, and standard deviations of 30°. The simulations are performed using 10,000 realizations of sensing angles, 1,000 realizations of communication angles, and 10,000 Gaussian samples.

The expectation over $\mathbf{a}(\theta_1)$ in (31) is taken with respect to Von Mises distribution for θ, parameterized by $(\bar{\theta}, \kappa)$, where $\kappa > 0$ controls the concentration around the mean direction $\bar{\theta}$. The parameter κ is analogous to the inverse of the variance: a larger κ indicates tighter concentration, while a smaller κ corresponds to greater spread. Here, $I_n(\kappa)$ denotes the modified Bessel function of order n. No closed-form expression exists for $\overline{\mathbf{M}}_i$ in (13) for the Von Mises distribution. The prior Fisher information can worked out to be $\frac{\kappa^2}{2}\left(1 - \frac{I_2(\kappa)}{I_0(\kappa)}\right)$. For the expectation over θ_1 (or θ_c), we use a post-sensing variance rather than the prior variance. To determine this, we first specify the initial standard deviation through a parameter κ_{init}, which defines the initial angular uncertainty for both the sensing and communication target. Subsequently, the sensing uncertainty for communication target, denoted as $\kappa_{c,\text{prior}}$, is obtained by solving the equation $1 - \frac{I_1(\kappa_{c,\text{prior}})}{I_0(\kappa_{c,\text{prior}})} = \kappa_{\text{init}}^2$, which maps the initial variance to the corresponding concentration parameter. Using the obtained $\kappa_{c,\text{prior}}$, the initial sensing BCRB, $\epsilon_{\text{initial}}$, is computed as

$\epsilon_{\text{initial}} := \epsilon_{\min}(P_{\max}, \kappa_{c,\text{prior}})$, where $\epsilon_{\min}$ corresponds to the BCRB in the sensing-optimal case in (34). $\kappa_{c,\text{post}}$, is determined by solving $1 - \frac{I_1(\kappa_{c,\text{post}})}{I_0(\kappa_{c,\text{post}})} = \epsilon_{\text{initial}}$. Here, $\kappa_{c,\text{post}}$ is a deterministic function of $P_{\max}$ and $\kappa_{c,\text{prior}}$, representing the updated uncertainty for the communication target.

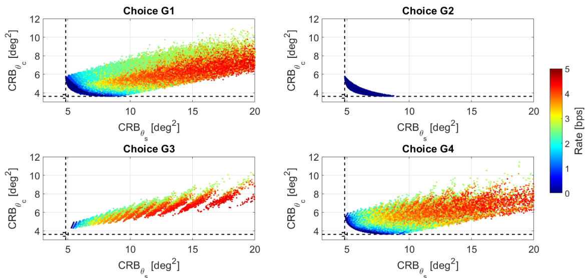

Fig. 2. Achievable BCRB-rate regions under Choices G1-G4 for sensing and communication angles $\bar{\theta}_s = 30°$, $\bar{\theta}_c = 100°$. The black lines indicate the minimum achievable BCRB for each target.

Both Deterministic and Gaussian. For the two-target scenario, the sensing optimal transmit covariance matrix has rank two [3, Corollary 2]. We define the beamforming directions $\mathbf{s}_1, \mathbf{c}_1$ and $\mathbf{s}_2, \mathbf{c}_2$ as the sensing and communication directions used in the first and second halves of the coherence time, respectively. This structure allows us to transmit in one direction during the first half and another during the second for $T = 2$, capturing a wide range of directional approaches within the general strategy in (28),

$$\mathbf{s}_1 = \sqrt{\lambda_1}\hat{\mathbf{r}}_1 + \sqrt{\lambda_2}\hat{\mathbf{r}}_2 + \sqrt{\lambda_3}\mathbf{v}_1^{(1)} + \sqrt{\lambda_4}\mathbf{v}_1^{(2)}, \tag{39}$$

$$\mathbf{c}_1 = \sqrt{\lambda_5}\hat{\mathbf{r}}_c. \tag{40}$$

In the second half, the system reuses the same directions with possibly different power allocation:

$$\mathbf{s}_2 = \sqrt{\lambda_6}\hat{\mathbf{r}}_1 + \sqrt{\lambda_7}\hat{\mathbf{r}}_2 + \sqrt{\lambda_8}\mathbf{v}_1^{(1)} + \sqrt{\lambda_9}\mathbf{v}_1^{(2)}, \tag{41}$$

$$\mathbf{c}_2 = \sqrt{\lambda_{10}}\hat{\mathbf{r}}_c. \tag{42}$$

The λ_i's are the non-negative power allocation parameters constrained as $\sum_i \lambda_i \leq 1, \lambda_i \in [0,1]$, where each λ_i represents the power fraction allocated to a specific direction vector. We analyze the impact of different spanning choices by selectively nullifying vectors through their corresponding λ values (G for General):

- Choice G1: Complete span with all vectors.
- Choice G2: Exclude communication-optimal vectors $[\lambda_5 = \lambda_{10} = 0]$.
- Choice G3: Exclude principal eigenvectors of $\overline{\mathbf{M}}_i$ $[\lambda_3 = \lambda_4 = \lambda_8 = \lambda_9 = 0]$.
- Choice G4: Exclude optimal sensing vectors $[\lambda_1 = \lambda_2 = \lambda_6 = \lambda_7 = 0]$.

For the ISAC scenario analysis, our results as depicted in Fig. 2 demonstrate several key findings across the choices above. Choice G2, which excludes information-carrying beams, achieves the minimum BCRB. Both Choice G2 and Choice G4 span along the principal eigenvectors of $\overline{\mathbf{M}}_1$ and $\overline{\mathbf{M}}_2$, resulting in minimal estimation error for the individual parameters θ_s (for sensing target) and θ_c (for communication user). When prioritizing communication while preserving single-angle estimation accuracy, Choice G4 proves effective for ISAC. Conversely, Choice G3 spans optimal sensing and communication directions, reducing joint angle estimation error at the cost of individual accuracy. Choice G1 captures the complete solution space, showing how communication rate scales with power allocation between sensing and communication vectors.

Only Deterministic. For the 'Special case 1', we span over the previously mentioned directions and principal eigenvectors for sensing, as well as include the second largest eigenvectors of $\overline{\mathbf{M}}_1$ and $\overline{\mathbf{M}}_2$ to investigate their potential influence on system performance. We now define $\mathbf{s}_1$ and $\mathbf{s}_2$ as,

$$\begin{aligned}\mathbf{s}_1 = &\sqrt{\lambda_1}\hat{\mathbf{r}}_1 + \sqrt{\lambda_2}\hat{\mathbf{r}}_2 + \sqrt{\lambda_3}\mathbf{v}_1^{(1)} + \sqrt{\lambda_4}\mathbf{v}_2^{(1)} \\ &+ \sqrt{\lambda_5}\mathbf{v}_1^{(2)} + \sqrt{\lambda_6}\mathbf{v}_2^{(2)},\end{aligned} \tag{43}$$

$$\begin{aligned}\mathbf{s}_2 = &\sqrt{\lambda_7}\hat{\mathbf{r}}_1 + \sqrt{\lambda_8}\hat{\mathbf{r}}_2 + \sqrt{\lambda_9}\mathbf{v}_1^{(1)} + \sqrt{\lambda_{10}}\mathbf{v}_2^{(1)} \\ &+ \sqrt{\lambda_{11}}\mathbf{v}_1^{(2)} + \sqrt{\lambda_{12}}\mathbf{v}_2^{(2)}.\end{aligned} \tag{44}$$

To evaluate the impact of different spanning choices, we examine scenarios where specific vectors are nullified by choosing (S for Sensing):

- Choice S1: Complete span with all vectors
- Choice S2: Exclude optimal sensing vectors $[\lambda_1 = \lambda_2 = \lambda_7 = \lambda_8 = 0]$
- Choice S3: Exclude all principal eigenvectors of $\overline{\mathbf{M}}_i$ $[\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_9 = \lambda_{10} = \lambda_{11} = \lambda_{12} = 0]$
- Choice S4: Exclude second main eigenvectors of $\overline{\mathbf{M}}_i$ $[\lambda_4 = \lambda_6 = \lambda_{10} = \lambda_{12} = 0]$
- Choice S5: Exclude sensing vectors aligned with the communication target in the first half and with the sensing target in the second half $[\lambda_3 = \lambda_4 = \lambda_{11} = \lambda_{12} = 0]$
- Choice S6: Rank-1 beam configuration with the same signals across time slots $[\lambda_1 = \lambda_7, \lambda_2 = \lambda_8, \lambda_3 = \lambda_9, \lambda_4 = \lambda_{10}, \lambda_5 = \lambda_{11}, \lambda_6 = \lambda_{12}]$

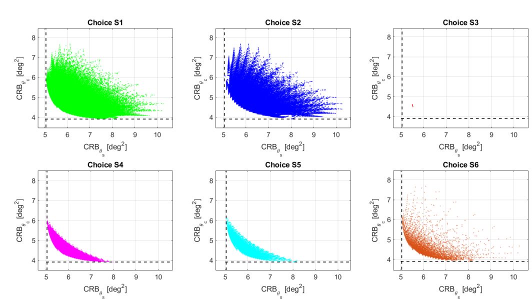

Fig. 3. BCRB-rate regions for deterministic or information-less beam scenario under Choices S1–S6 (Special Case 1).

Fig. 3 reveals how the sensing region evolves across different vector configurations. For this case with only deterministic or information-less signals, we focus exclusively on sensing performance as here $R = 0$. Configurations that utilize principal eigenvectors—i.e., Choices S2, S4, S5, and S6—attain optimal estimation performance for individual angles. However, to optimize joint estimation performance, corresponding to the lower-left corner of the BCRB region, it is essential to span across both optimal sensing directions, represented by vectors $\hat{\mathbf{r}}_1$ and $\hat{\mathbf{r}}_2$, as done in Choices S3, S4, and S6. However, achieving the two individual minimum CRB points simultaneously remains infeasible, highlighting a fundamental trade-off in joint estimation. We also observe that the rank-2 configuration (Choice S1) yields a richer set of solution points in the achievable region compared to the rank-1 configuration (Choice S6).

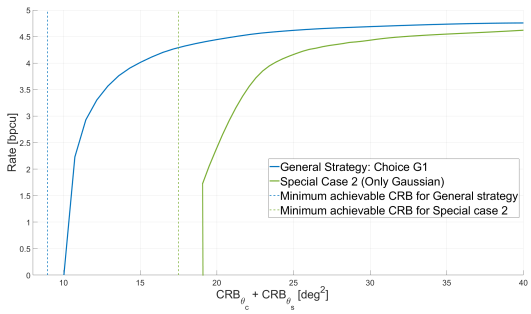

Fig. 4. Achievable BCRB-rate regions comparison for main strategy where we transmit both deterministic and Gaussian (Main Strategy in (28)), and only Gaussian (Special case 2 in (36)).

Only Gaussian. For the Gaussian case (Special Case 2), although the transmitted signal is purely Gaussian, our goal is to support both communication and sensing. Therefore, the beam directions are constructed as linear combinations of vectors optimized for both sensing and communication:

$$\mathbf{c}_1 = \sqrt{\lambda_1}\hat{\mathbf{r}}_1 + \sqrt{\lambda_2}\hat{\mathbf{r}}_2 + \sqrt{\lambda_3}\mathbf{v}_1^{(1)} + \sqrt{\lambda_4}\mathbf{v}_2^{(1)} + \sqrt{\lambda_5}\hat{\mathbf{r}}_c \quad (45)$$

$$\mathbf{c}_2 = \sqrt{\lambda_6}\hat{\mathbf{r}}_1 + \sqrt{\lambda_7}\hat{\mathbf{r}}_2 + \sqrt{\lambda_8}\mathbf{v}_1^{(1)} + \sqrt{\lambda_9}\mathbf{v}_2^{(1)} + \sqrt{\lambda_{10}}\hat{\mathbf{r}}_c \quad (46)$$

In Fig. 4, we present the inner bounds of the BCRB-rate region under the choice of (28) and (36), spanning over all possible vectors. Utilizing the updated sensing BCRB for the communication target as described previously in this section to reflect the sensing uncertainty's BCRB when taking the expectation enables substantially higher communication rates. Transmitting only information-carrying signals (special case 2) degrades sensing performance, leading to lower communication rates. This effect is more visible for realistic (lower) sensing SNRs. Again achieving the theoretical optimal performance (combined single best estimation) remains unattainable.

Fig. 5 shows the influence of key system parameters, specifically prior variance and angular separation between targets, on the ISAC tradeoff. Regarding angular separation, for a fixed $\bar{\theta}_s = 30°$, the results demonstrate that smaller separations between targets ($\bar{\theta}_c = 60°$) yield improved BCRB performance compared to widely separated targets ($\bar{\theta}_c = 100°$). This finding suggests that the system's estimation capabilities are enhanced when targets are in closer proximity, potentially

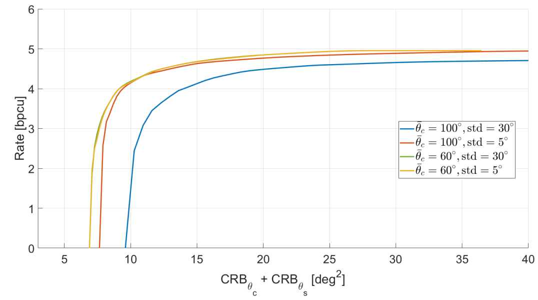

Fig. 5. Comparison of achievable ISAC regions under different prior standard deviation, communication target angle.

due to more efficient beamforming advantages. For targets with greater angular separation ($\bar{\theta}_s = 30°$, $\bar{\theta}_c = 100°$), higher prior variance significantly degrades sensing, as evidenced by the rightward shift of the blue curve. Conversely, when targets are more closely spaced ($\bar{\theta}_s = 30°$, $\bar{\theta}_c = 60°$), the impact of variance becomes less pronounced, with the performance boundaries nearly coinciding regardless of the variance.

VII. Conclusion

In this paper, we analyzed the ISAC tradeoff in a practical scenario with multiple targets, where one target serves is a communication receiver. Under specific assumptions, we showed that the BCRB reduces to a sum of single-target estimation bounds and derived the optimal sensing and communication solution using convex optimization. Through various transmission strategies, we explored achievable BCRB-rate regions and identified effective strategies through power allocation. Future work could explore numerical generalization to higher number of targets as well as considering a broadcast scenario more communication receivers.

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