

Matched pairs and Yetter-Drinfeld braces

Original

Matched pairs and Yetter-Drinfeld braces / Ferri, Davide; Sciandra, Andrea. - In: CANADIAN JOURNAL OF MATHEMATICS. - ISSN 0008-414X. - ELETTRONICO. - (2025), pp. 1-37. [10.4153/S0008414X25000239]

Availability:

This version is available at: 11583/3003087 since: 2025-09-16T17:12:15Z

Publisher:

Cambridge University Press

Published

DOI:10.4153/S0008414X25000239

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

Cambridge University Press postprint/Author's Accepted Manuscript

This article has been published in a revised form in CANADIAN JOURNAL OF MATHEMATICS <http://doi.org/10.4153/S0008414X25000239>. This version is free to view and download for private research and study only. Not for re-distribution or re-use. © copyright holder.

(Article begins on next page)

MATCHED PAIRS AND YETTER–DRINFELD BRACES

DAVIDE FERRI, ANDREA SCIANDRA

ABSTRACT. It is proven that a matched pair of actions on a Hopf algebra H is equivalent to the datum of a Yetter–Drinfeld brace, which is a novel structure generalising Hopf braces. This improves a theorem by Angiono, Galindo and Vendramin, originally stated for cocommutative Hopf braces. These Yetter–Drinfeld braces produce Hopf algebras in the category of Yetter–Drinfeld modules over H , through an operation that generalises Majid’s transmutation. A characterisation of Yetter–Drinfeld braces via 1-cocycles, in analogy to the one for Hopf braces, is given.

Every coquasitriangular Hopf algebra H will be seen to yield a Yetter–Drinfeld brace, where the additional structure on H is given by the transmutation. We compute explicit examples of Yetter–Drinfeld braces on the Sweedler’s Hopf algebra, on the algebras $E(n)$, on $SL_q(2)$, and an example in the class of Suzuki algebras.

CONTENTS

1. Introduction	1
2. Preliminaries	3
2.1. Matched pairs of actions	3
2.2. Hopf braces	6
3. Matched pairs of actions and Yetter–Drinfeld braces	6
3.1. Yetter–Drinfeld braces	6
3.2. A general example	14
3.3. A converse connection	15
4. Yetter–Drinfeld braces as 1-cocycles	16
5. Coquasitriangular Hopf algebras and Yetter–Drinfeld braces	19
5.1. Coquasitriangular bialgebras	19
5.2. Coquasitriangular Hopf algebras and matched pairs of actions	20
6. Examples	23
6.1. The Sweedler’s Hopf algebra	23
6.2. The Hopf algebras $E(n)$	25
6.3. The Hopf algebra $SL_q(2)$	25
6.4. An example in the class of Suzuki Hopf algebras	27
References	28

1. INTRODUCTION

The notion of braiding is ubiquitous in Mathematics, encompassing low-dimension topology, group theory, category theory, and the theory of the Yang–Baxter equation (YBE). In particular, notions of braided categories [20], braided groups [24, 26, 29], and (co)quasitriangular Hopf algebras [10, 11, 23, 27, 36] have been defined around the 1990s, and extensively used ever since.

On the other hand, set-theoretic solutions to the YBE have been studied on suggestion of Drinfeld [12], growing into a vibrant field of research. It is known that braided groups provide solutions to the set-theoretic YBE, and solutions to the YBE are associated with a structure group

2020 *Mathematics Subject Classification.* Primary 16T25; Secondary 18M15.

Key words and phrases. Yetter–Drinfeld Braces, Matched Pairs, Hopf Braces, Quasitriangular Hopf Algebras, Yetter–Drinfeld Modules, Braiding Operators.

[13] which turns out to be a braided group. Lu, Yan and Zhu [24] clarified that a braided group is essentially the same as a matched pair of actions. This is equivalent to bijective 1-cocycles. Afterwards, Guarnieri and Vendramin [18] proved that this is in turn equivalent to a skew brace structure: a notion that has become of capital importance, in its several generalisations (a non-exhaustive list of them being provided by [7, 31, 33]). Skew braces generalise Rump's braces [35], and the interplay between skew braces and Yang–Baxter maps generalises the correspondence between braces and involutive Yang–Baxter maps [16].

A skew brace is the datum of two group structures (G, \cdot) and (G, \bullet) on the same set G , satisfying a compatibility. An immediate Hopf-theoretic generalisation is the notion of Hopf brace, which is the datum of two Hopf algebra structures $(H, \cdot, 1, \Delta, \epsilon, S)$ and $(H, \bullet, 1, \Delta, \epsilon, T)$ on the same coalgebra (H, Δ, ϵ) , satisfying an analogous compatibility [5]. The correspondence between skew braces, matched pairs of actions, and bijective 1-cocycles generalises naturally to cocommutative Hopf braces, cocommutative matched pairs on a Hopf algebra, and bijective 1-cocycles of cocommutative Hopf algebras. The correspondence between Hopf braces and bijective 1-cocycles is true without the assumption of cocommutativity—in fact, the two structures are strongly related, and the same equivalence holds true in any braided monoidal category [15]. However, the assumption of cocommutativity is necessary for the equivalence with matched pairs; which is, in turn, crucial in order to bridge Hopf braces with the world of braiding operators. Thus, cocommutativity seemed hard to relax, without sacrificing the backbone of the correspondence theorem: this is the premise to our investigation.

In this paper, we get rid of the cocommutativity hypothesis. As in [5], we consider matched pairs of Hopf algebras, satisfying a braided-commutativity condition: we call them *matched pairs of actions on a Hopf algebra* (the name being reminiscent of matched pairs of actions on a group, extensively used by Lu, Yan and Zhu [24] with the name *compatible actions*). Then, in §3, we establish a correspondence between such matched pairs of actions, and Yetter–Drinfeld braces. A Yetter–Drinfeld brace is the datum of a Hopf algebra $H^\bullet = (H, \bullet, 1, \Delta, \epsilon, T)$, and a Hopf algebra $(H, \cdot, 1, \Delta, \epsilon, S)$ in ${}_{H^\bullet}^H \mathcal{YD}$ (which we call the transmutation of H^\bullet); where the two structures satisfy some technical hypotheses, and the same compatibility as Hopf braces. It comes out that the notions of Yetter–Drinfeld brace and of Hopf brace coincide in the cocommutative case.

It is natural to ask whether Yetter–Drinfeld braces can be interpreted as 1-cocycles as well. This is positively answered in §4.

In §5, we present a vast class of Yetter–Drinfeld braces that are, generally, not Hopf braces. Given any coquasitriangular Hopf algebra $(H, \bullet, 1, \Delta, \epsilon, T, \mathcal{R})$, we shall be able to construct a second operation \cdot and a linear map $S: H \rightarrow H$, yielding a Yetter–Drinfeld brace. In this case, the additional structure on H coincides with the transmutation of H^\bullet , introduced by Majid [27, 28]. Therefore, we obtain in particular an interpretation of the theory of transmutation, within the frame of Yetter–Drinfeld braces. The map $\sigma: H \otimes H \rightarrow H \otimes H$, $\sigma(a \otimes b) := \mathcal{R}^{-1}(a_1 \otimes b_1)b_2 \otimes a_2 \mathcal{R}(a_3 \otimes b_3)$ is a braiding operator on H . When (H^\bullet, \mathcal{R}) is cotriangular, the braiding operator σ is involutive, and the corresponding algebra $(H, \cdot, 1, \Delta, \epsilon, S)$ in $\text{Hopf}({}_{H^\bullet}^H \mathcal{YD})$ is braided-commutative with respect to the braiding of ${}_{H^\bullet}^H \mathcal{YD}$.

This entire class of Yetter–Drinfeld braces descending from coquasitriangular Hopf algebras is, in some way, “orthogonal” to the class of cocommutative Hopf braces. When (H^\bullet, \mathcal{R}) is a cocommutative coquasitriangular Hopf algebra, the corresponding Yetter–Drinfeld brace is indeed the trivial Hopf brace (H, \bullet, \bullet) .

In conclusion to this paper, §6 presents several examples of coquasitriangular Hopf algebras, and describes their associated Yetter–Drinfeld braces.

Notations and conventions. We shall denote by $(\mathcal{M}, \otimes, \mathbb{1}, \sigma)$ a braided monoidal category, with monoidal product \otimes , unit object $\mathbb{1}$, and braiding σ . Comforted by the MacLane Coherence Theorem, we shall consistently be sloppy on associativity and unit constraints. In the rest of this paper, we adopt the notation fg for the composition of maps $f \circ g$. The categories of algebras, coalgebras, bialgebras and Hopf algebras in \mathcal{M} will be denoted by $\text{Mon}(\mathcal{M})$, $\text{Comon}(\mathcal{M})$, $\text{Bimon}(\mathcal{M})$ and $\text{Hopf}(\mathcal{M})$, respectively. We shall usually denote by H a bialgebra in \mathcal{M} , by ${}_H \mathcal{M}$, \mathcal{M}_H , ${}^H \mathcal{M}$, \mathcal{M}^H ,

${}^H\mathcal{M}_H$ and ${}^H\mathcal{M}^H$ its category of left and right modules, left and right comodules, bimodules, and bicomodules, respectively. Unless otherwise specified, our setting will be the category $\mathfrak{M} = \text{Vec}_{\mathbb{k}}$ of vector spaces over \mathbb{k} , with braiding given by the canonical flip τ .

Symbols such as \bullet, \cdot will usually signify the multiplication of an algebra A . Equivalently, the multiplication will be denoted by a map m, m_\bullet or $m_\bullet: A \otimes A \rightarrow A$. The unit is denoted by 1 or, equivalently, by a map $u: \mathbb{k} \rightarrow A$. The comultiplication and the counit of a coalgebra will be denoted by Δ and ϵ , respectively. Subscripts will be added for clarity, whenever needed.

In our computations with coalgebras, we shall consistently employ *Sweedler's notation* $\Delta(a) = \sum_i a_1^i \otimes a_2^i = a_1 \otimes a_2$, omitting the summation. A similar notation is adopted for left coactions $a \mapsto a_{-1} \otimes a_0$, and for right coactions $a \mapsto a_0 \otimes a_1$.

2. PRELIMINARIES

In this section, we introduce the main notions that we are going to use throughout the paper. Namely, we describe *matched pairs of actions* on a bialgebra, and we survey the fundamentals on Hopf braces.

2.1. Matched pairs of actions. Matched pairs of groups, enabling to define a Zappa–Szép product, have been known for a long time. On the other hand, the notion of a matched pair of Hopf algebras was introduced by Singer [37] in the graded case, and the current notion was introduced by Majid [25]. We consider *matched pairs of actions* on Hopf algebras, which will turn out to be a subclass of matched pairs of Hopf algebras [30, Definition 7.2.1]. Matched pairs of actions on groups have been already used, e.g. in the seminal work of Lu, Yan and Zhu [24], and then mutated in several different contexts, such as for groupoids by Andruskiewitsch [3].

Definition 2.1. Let H be a bialgebra, \bullet be its algebra product. A *matched pair of actions* $(H, \rightharpoonup, \leftharpoonup)$ on H is the datum of a left action \rightharpoonup and a right action \leftharpoonup of H on itself, such that H is a left H -module coalgebra and a right H -module coalgebra with the respective actions, and the following conditions hold for all $a, b, c \in H$:

$$(MP.1) \quad a \rightharpoonup 1 = \epsilon(a)1, \text{ i.e. } u_H \text{ is a morphism in } {}^H\mathfrak{M};$$

$$(MP.2) \quad 1 \leftharpoonup a = \epsilon(a)1, \text{ i.e. } u_H \text{ is a morphism in } \mathfrak{M}_H;$$

$$(MP.3) \quad a \rightharpoonup (b \bullet c) = (a_1 \rightharpoonup b_1) \bullet ((a_2 \leftharpoonup b_2) \rightharpoonup c);$$

$$(MP.4) \quad (a \bullet b) \leftharpoonup c = (a \leftharpoonup (b_1 \rightharpoonup c_1)) \bullet (b_2 \leftharpoonup c_2);$$

$$(\star) \quad a \bullet b = (a_1 \rightharpoonup b_1) \bullet (a_2 \leftharpoonup b_2).$$

A *morphism of matched pairs of actions* between $(H, \rightharpoonup_H, \leftharpoonup_H)$ and $(K, \rightharpoonup_K, \leftharpoonup_K)$ is a morphism of bialgebras $H \rightarrow K$ that intertwines the two left actions and the two right actions, respectively, in the following sense:

$$f(a \rightharpoonup_H b) = f(a) \rightharpoonup_K f(b), \quad f(a \leftharpoonup_H b) = f(a) \leftharpoonup_K f(b).$$

The category of matched pairs of actions (in $\text{Vec}_{\mathbb{k}}$) will be denoted by $\text{MP}(\text{Vec}_{\mathbb{k}})$.

A pair of actions $(\rightharpoonup, \leftharpoonup)$ on a bialgebra H satisfying (MP.1)–(MP.4) allows one to construct the *double cross product* $H \bowtie H$, as described in Majid [30], which is the vector space $H \otimes H$ endowed with the algebra structure given by the product

$$(a \otimes h)(b \otimes g) := a \bullet (h_1 \rightharpoonup b_1) \otimes (h_2 \leftharpoonup b_2) \bullet g,$$

with unit $1_H \otimes 1_H$, and the usual tensor product coalgebra structure. Then $H \bowtie H$ becomes a bialgebra if and only if the additional condition

$$(MP.5) \quad (a_1 \rightharpoonup b_1) \otimes (a_2 \leftharpoonup b_2) = (a_2 \rightharpoonup b_2) \otimes (a_1 \leftharpoonup b_1)$$

holds; and in this case, (H, H) is a *matched pair of bialgebras* in the sense of Majid [30, Definition 7.2.1]. Notice that (MP.5) is trivially satisfied in case H is cocommutative.

When H is a Hopf algebra, a matched pair of bialgebras (H, H) will be usually called a *matched pair* on H .

Remark 2.2. For any bialgebra H , the actions $a \rightharpoonup b := \epsilon(a)b$, $a \leftarrow b := a\epsilon(b)$ always satisfy (MP.1)–(MP.5). They moreover satisfy (\star) if and only if \bullet is commutative. Notice that (\star) is equivalent to asking that $m_\bullet: H \bowtie H \rightarrow H$ is a morphism of algebras.

The following result simply follows by an argument of naturality.

Lemma 2.3. *Let \mathcal{M} be a (strict) braided monoidal category with braiding σ , let H be an object in \mathcal{M} , and $m: H \otimes H \rightarrow H$, $u: \mathbb{1} \rightarrow H$ any morphisms in \mathcal{M} . Then, the following hold:*

$$\begin{aligned} \text{(BR.1)} \quad & \sigma_{H,H}(\text{Id} \otimes u) = u \otimes \text{Id}; \\ \text{(BR.2)} \quad & \sigma_{H,H}(u \otimes \text{Id}) = \text{Id} \otimes u; \\ \text{(BR.3)} \quad & \sigma_{H,H}m_{23} = m_{12}(\sigma_{H,H})_{23}(\sigma_{H,H})_{12}; \\ \text{(BR.4)} \quad & \sigma_{H,H}m_{12} = m_{23}(\sigma_{H,H})_{12}(\sigma_{H,H})_{23}. \end{aligned}$$

Moreover, if m is a monomorphism in \mathcal{M} and $\varsigma: H \otimes H \rightarrow H \otimes H$ is a morphism satisfying (BR.1)–(BR.4), then ς satisfies the hexagonal axioms, and hence (H, ς) is a braided object in \mathcal{M} .

Lemma 2.4 (cf. Tambara [40, Proposition 2.2]). *Let H be a bialgebra. Suppose given a left action $\rightharpoonup: H \otimes H \rightarrow H$ and a right action $\leftarrow: H \otimes H \rightarrow H$, such that H is a left H -module coalgebra and a right H -module coalgebra with the respective actions. Define $\sigma_{H,H}: H \otimes H \rightarrow H \otimes H$ by*

$$\sigma_{H,H}(a \otimes b) := (a_1 \rightharpoonup b_1) \otimes (a_2 \leftarrow b_2).$$

Then, $\sigma_{H,H}$ satisfies (BR.1)–(BR.4) if and only if \rightharpoonup and \leftarrow satisfy (MP.1)–(MP.4).

Proof. One has

$$\sigma_{H,H}m_{12}(a \otimes b \otimes c) = \sigma_{H,H}(ab \otimes c) = (a_1b_1 \rightharpoonup c_1) \otimes (a_2b_2 \leftarrow c_2),$$

while on the other hand

$$\begin{aligned} & m_{23}(\sigma_{H,H})_{12}(\sigma_{H,H})_{23}(a \otimes b \otimes c) \\ &= m_{23}(\sigma_{H,H})_{12}(a \otimes (b_1 \rightharpoonup c_1) \otimes (b_2 \leftarrow c_2)) \\ &= m_{23}((a_1 \rightharpoonup (b_1 \rightharpoonup c_1)_1) \otimes (a_2 \leftarrow (b_1 \rightharpoonup c_1)_2) \otimes (b_2 \leftarrow c_2)) \\ &= (a_1 \rightharpoonup (b_1 \rightharpoonup c_1)) \otimes (a_2 \leftarrow (b_2 \rightharpoonup c_2))(b_3 \leftarrow c_3) \\ &= (a_1b_1 \rightharpoonup c_1) \otimes (a_2 \leftarrow (b_2 \rightharpoonup c_2))(b_3 \leftarrow c_3). \end{aligned}$$

If (BR.4) holds true then, by applying $\epsilon \otimes \text{Id}$ on both sides, we obtain $(ab) \leftarrow c = (a \leftarrow (b_1 \rightharpoonup c_1))(b_2 \leftarrow c_2)$, which is (MP.4). If conversely (MP.4) holds true, then $\sigma_{H,H}$ clearly satisfies (BR.4). One analogously proves that (BR.3) is equivalent to (MP.3). Observe that

$$\sigma_{H,H}(1 \otimes a) = (1 \rightharpoonup a_1) \otimes (1 \leftarrow a_2) = a_1 \otimes (1 \leftarrow a_2).$$

If (BR.2) holds true, then

$$a_1 \otimes (1 \leftarrow a_2) = a \otimes 1.$$

Hence, by applying $\epsilon \otimes \text{Id}$, we obtain $1 \leftarrow a = \epsilon(a)1$, which is (MP.2). Conversely, if (MP.2) holds true, then $\sigma_{H,H}$ clearly satisfies (BR.2). The equivalence between (BR.1) and (MP.1) is proven analogously. \square

Remark 2.5. Let H be a bialgebra, and suppose given

$$\sigma_{H,H}(x \otimes y) = (x_1 \rightharpoonup y_1) \otimes (x_2 \leftarrow y_2)$$

where $\rightharpoonup: H \otimes H \rightarrow H$ and $\leftarrow: H \otimes H \rightarrow H$ are such that $\epsilon(a \rightharpoonup b) = \epsilon(a)\epsilon(b) = \epsilon(a \leftarrow b)$ for all $a, b \in H$. Then, \rightharpoonup is retrieved as $(\text{Id} \otimes \epsilon)\sigma_{H,H}$, and \leftarrow as $(\epsilon \otimes \text{Id})\sigma_{H,H}$ (see [30, p. 300]).

Although there are examples of braidings that are obtained from a pair of actions satisfying (MP.1)–(MP.4) (see § 5.2), it is not true that all braidings are obtained from a pair of maps as in Remark 2.5:

Counterexample 2.6. Consider, for instance, a coquasitriangular bialgebra (H, \mathcal{R}) (see § 5.1) and the usual braiding (24) on \mathfrak{M}^H

$$\sigma_{H,H}: H \otimes H \rightarrow H \otimes H, \quad a \otimes b \mapsto b_1 \otimes a_1 \mathcal{R}(a_2 \otimes b_2).$$

If we suppose by contradiction that $\sigma_{H,H}(a \otimes b) = (a_1 \rightharpoonup b_1) \otimes (a_2 \leftarrow b_2)$ for some maps $\rightharpoonup, \leftarrow$, then Remark 2.5 yields

$$a \rightharpoonup b = \mathcal{R}(a \otimes b_2)b_1, \quad a \leftarrow b = a_1 \mathcal{R}(a_2 \otimes b),$$

but then

$$\begin{aligned} b_1 \otimes a_1 \mathcal{R}(a_2 \otimes b_2) &= \sigma_{H,H}(a \otimes b) \\ &= (a_1 \rightharpoonup b_1) \otimes (a_2 \leftarrow b_2) \\ &= \mathcal{R}(a_1 \otimes b_2)b_1 \otimes a_2 \mathcal{R}(a_3 \otimes b_3), \end{aligned}$$

and this is generally false. As an example, consider the coquasitriangular structure on the Sweedler's Hopf algebra described in § 6.1, with $a = b = g$: one has $\mathcal{R}(g \otimes g) = -1$, whence $b_1 \otimes a_1 \mathcal{R}(a_2 \otimes b_2) = -g \otimes g$ and $\mathcal{R}(a_1 \otimes b_2)b_1 \otimes a_2 \mathcal{R}(a_3 \otimes b_3) = g \otimes g$, which differ if $\text{char}(\mathbb{k}) \neq 2$.

Lemma 2.7 (cf. Majid [30, proof of Theorem 7.2.3]). *Given a bialgebra H and a morphism $\sigma_{H,H}: H \otimes H \rightarrow H \otimes H$ satisfying (BR.1)–(BR.4), the morphisms $\rightharpoonup := (\text{Id} \otimes \epsilon)\sigma_{H,H}$ and $\leftarrow := (\epsilon \otimes \text{Id})\sigma_{H,H}$ are left and right H -actions, respectively.*

Proof. We compute

$$1 \rightharpoonup c = (\text{Id} \otimes \epsilon)\sigma_{H,H}(1 \otimes c) = (\text{Id} \otimes \epsilon)(c \otimes 1) = c,$$

and also

$$\begin{aligned} ab \rightharpoonup c &= (\text{Id} \otimes \epsilon)\sigma_{H,H}m_{12}(a \otimes b \otimes c) \\ &= (\text{Id} \otimes \epsilon)m_{23}(\sigma_{H,H})_{12}(\sigma_{H,H})_{23}(a \otimes b \otimes c) \\ &= (\text{Id} \otimes \epsilon \otimes \epsilon)(\sigma_{H,H})_{12}(\sigma_{H,H})_{23}(a \otimes b \otimes c) \\ &= (\text{Id} \otimes \epsilon)\sigma_{H,H}(\text{Id} \otimes \text{Id} \otimes \epsilon)(\text{Id} \otimes \sigma_{H,H})(a \otimes b \otimes c) \\ &= (\text{Id} \otimes \epsilon)\sigma_{H,H}(a \otimes (b \rightharpoonup c)) \\ &= a \rightharpoonup (b \rightharpoonup c). \end{aligned}$$

The proof is analogous for \leftarrow . □

In general, when $\sigma_{H,H}$ does not have the form $\sigma_{H,H}(x \otimes y) = (x_1 \rightharpoonup y_1) \otimes (x_2 \leftarrow y_2)$, the actions \rightharpoonup and \leftarrow defined as in the previous lemma need not form a matched pair, as the following counterexample shows.

Counterexample 2.8. Let us consider again

$$\sigma_{H,H}: H \otimes H \rightarrow H \otimes H, \quad a \otimes b \mapsto b_1 \otimes a_1 \mathcal{R}(a_2 \otimes b_2)$$

and the left and right H -actions

$$a \rightharpoonup b = \mathcal{R}(a \otimes b_2)b_1, \quad a \leftarrow b = a_1 \mathcal{R}(a_2 \otimes b).$$

Then, we have $a \rightharpoonup bc = \mathcal{R}(a \otimes b_2 c_2)b_1 c_1$, while

$$(a_1 \rightharpoonup b_1)((a_2 \leftarrow b_2) \rightharpoonup c) = \mathcal{R}(a_1 \otimes b_2)\mathcal{R}(a_2 \otimes c_2)\mathcal{R}(a_3 \otimes b_3)b_1 c_1 = \mathcal{R}(a \otimes b_3 c_2 b_2)b_1 c_1.$$

These are generally distinct (consider for instance the Sweedler's Hopf algebra in § 6.1, $a = b = c = g$, $\text{char}(\mathbb{k}) \neq 2$).

2.2. Hopf braces. A *skew brace*, defined by Guarnieri and Vendramin [18] generalising Rump [35], is the datum of two group operations on the same set, satisfying a compatibility. Skew braces are ubiquitous in the theory of braidings and the Yang–Baxter equation, and hence yearned for a Hopf-theoretic version. This was defined by Angiono, Galindo and Vendramin [5]:

Definition 2.9. A *Hopf brace* $(H, \cdot, \bullet, 1, \Delta, \epsilon, S, T)$ is the datum of two Hopf algebra structures $(H, \cdot, 1, \Delta, \epsilon, S)$ and $(H, \bullet, 1', \Delta, \epsilon, T)$ on the same coalgebra (H, Δ, ϵ) , satisfying the following compatibility:

$$(HBC) \quad a \bullet (b \cdot c) = (a_1 \bullet b) \cdot S(a_2) \cdot (a_3 \bullet c).$$

Notice that the compatibility condition forces $1 = 1'$ (see [5, Remark 1.3]).

When no confusion arises, we shall simply indicate a Hopf brace by (H, \cdot, \bullet) . We recall the following characterisation:

Proposition 2.10 (Angiono, Galindo and Vendramin [5, Theorems 1.12, 3.3, and Corollary 2.4]). *Given a Hopf algebra H , the following data are equivalent:*

- i. a Hopf brace structure (H, \cdot, \bullet) on H ;
- ii. a Hopf algebra A , an action \rightarrow of A on H such that H is a left A -module algebra, and an isomorphism of coalgebras $\pi: A \rightarrow H$ which is a 1-cocycle of bialgebras; i.e., $\pi(ab) = \pi(a_1)(a_2 \rightarrow \pi(b))$.

In case H is cocommutative, the previous two are also equivalent to

- iii. a matched pair on H , satisfying the additional condition (\star) .

Moreover, in the cocommutative case,

$$\sigma_{H,H}: a \otimes b \mapsto (a_1 \rightarrow b_1) \otimes (a_2 \leftarrow b_2)$$

is a coalgebra isomorphism and a solution to the braid equation on H .

The additional condition (\star) is the *braided commutativity* of \bullet with respect to the braiding operator $\sigma_{H,H}$.

Remark 2.11. In the cocommutative case, the matched pair obtained from a Hopf brace (H, \cdot, \bullet) is given by $h \rightarrow k = S(h_1) \cdot (h_2 \bullet k)$ and $h \leftarrow k = T(h_1 \rightarrow k_1) \bullet h_2 \bullet k_2$. Then, the corresponding solution of the braid equation is

$$\sigma_{H,H}(x \otimes y) = (S(x_1) \cdot (x_2 \bullet y_1)) \otimes (T(S(x_3) \cdot (x_4 \bullet y_2)) \bullet x_5 \bullet y_3).$$

3. MATCHED PAIRS OF ACTIONS AND YETTER–DRINFELD BRACES

In this section, we generalise the correspondence of Proposition 2.10 by dropping the hypothesis of cocommutativity. As a result, we shall obtain a correspondence between matched pairs of actions on a Hopf algebra, and novel structures which we call *Yetter–Drinfeld braces*.

3.1. Yetter–Drinfeld braces. Let $(H, \bullet, 1, \Delta, \epsilon, T)$ be a Hopf algebra, $(H, \rightarrow, \leftarrow)$ be a matched pair of actions on H . We begin by defining a second operation \cdot and a map $S: H \rightarrow H$, mimicking [5, Proposition 3.2]:

$$(1) \quad a \cdot b := a_1 \bullet (T(a_2) \rightarrow b);$$

$$(2) \quad S(a) := a_1 \rightarrow T(a_2).$$

Notice that \bullet is retrieved from \cdot as

$$(3) \quad a \bullet b = a_1 \bullet (T(a_2) \rightarrow (a_3 \rightarrow b)) = a_1 \cdot (a_2 \rightarrow b).$$

Recall that the linearity of Δ and ϵ with respect to \rightarrow read as:

$$\Delta(a \rightarrow b) = (a_1 \rightarrow b_1) \otimes (a_2 \rightarrow b_2), \quad \epsilon(a \rightarrow b) = \epsilon(a)\epsilon(b),$$

similarly for the linearity of Δ and ϵ with respect to \leftarrow .

Remark 3.1. The condition (\star) allows one to retrieve the right action from the left action, and vice versa. Indeed, by the linearity of Δ and ϵ with respect to \rightarrow , (\star) is equivalent to

$$(4) \quad a \leftarrow b = T(a_1 \rightarrow b_1) \bullet a_2 \bullet b_2.$$

Moreover, (4) is clearly equivalent to

$$(5) \quad T(a \rightarrow b) = (a_1 \leftarrow b_1) \bullet T(b_2) \bullet T(a_2).$$

From (5) one can easily deduce

$$(6) \quad T(a_1 \rightarrow T(a_2)) = a_1 \leftarrow T(a_2).$$

Remark 3.2. Let us observe that S^2 is given on an element $a \in H$ by

$$\begin{aligned} S^2(a) &= S(a_1 \rightarrow T(a_2)) \\ &= (a_1 \rightarrow T(a_4)) \rightarrow T(a_2 \rightarrow T(a_3)) \\ &\stackrel{(6)}{=} (a_1 \rightarrow T(a_4)) \rightarrow (a_2 \leftarrow T(a_3)). \end{aligned}$$

Lemma 3.3. *One has $\Delta(a \cdot b) = (a_1 \bullet (T(a_3) \rightarrow b_1)) \otimes (a_2 \cdot b_2)$.*

Proof. We compute

$$\begin{aligned} \Delta(a \cdot b) &= \Delta(a_1 \bullet (T(a_2) \rightarrow b)) \\ &= (a_1 \bullet (T(a_3) \rightarrow b)_1) \otimes (a_2 \bullet (T(a_3) \rightarrow b)_2) \\ &= (a_1 \bullet (T(a_4) \rightarrow b_1)) \otimes (a_2 \bullet (T(a_3) \rightarrow b_2)) \\ &= (a_1 \bullet (T(a_3) \rightarrow b_1)) \otimes (a_2 \cdot b_2). \end{aligned}$$

□

Although \cdot is not a morphism of coalgebras in Vec_k , this will be sorted out by setting ourselves in a suitable category.

An immediate computation shows the form of the twice iterated Δ , which is going to come in handy later:

$$\begin{aligned} (\Delta \otimes \text{Id})\Delta(a \cdot b) &= \Delta(a_1 \bullet (T(a_3) \rightarrow b_1)) \otimes (a_2 \cdot b_2) \\ (7) \quad &= (a_1 \bullet (T(a_5) \rightarrow b_1)) \otimes (a_2 \bullet (T(a_4) \rightarrow b_2)) \otimes (a_3 \cdot b_3) \\ &= (a_1 \bullet (T(a_6) \rightarrow b_1)) \otimes (a_2 \bullet (T(a_5) \rightarrow b_2)) \otimes (a_3 \bullet (T(a_4) \rightarrow b_3)). \end{aligned}$$

Proposition 3.4. *Let $(H, \bullet, 1, \Delta, \epsilon, T)$ be a Hopf algebra, $(H, \rightarrow, \leftarrow)$ a matched pair of actions. Define \cdot and S as in (1), (2). Then, $(H, \cdot, 1)$ becomes an algebra, ϵ is a morphism of algebras with respect to this structure, and S satisfies $a_1 \cdot S(a_2) = \epsilon(a)1 = S(a_1) \cdot a_2$. Moreover, one has*

$$(8) \quad a \rightarrow b = S(a_1) \cdot (a_2 \bullet b)$$

and the two operations \bullet and \cdot satisfy the Hopf brace compatibility (HBC).

Proof. We first observe that

$$a \cdot 1 = a_1 \bullet (T(a_2) \rightarrow 1) = a_1 \bullet \epsilon(T(a_2))1 = a_1 \epsilon(a_2) = a, \quad 1 \cdot a = 1 \bullet (T(1) \rightarrow a) = a,$$

for all $a \in H$. We now prove that \cdot is associative. Given $a, b, c \in H$ we have:

$$\begin{aligned} a \cdot (b \cdot c) &= a \cdot (b_1 \bullet (T(b_2) \rightarrow c)) \\ &= a_1 \bullet \left(T(a_2) \rightarrow (b_1 \bullet (T(b_2) \rightarrow c)) \right) \\ &\stackrel{(\text{MP.3})}{=} a_1 \bullet \left(T(a_3) \rightarrow b_1 \right) \bullet \left((T(a_2) \leftarrow b_2) \rightarrow (T(b_3) \rightarrow c) \right) \\ &\stackrel{(4)}{=} a_1 \bullet (T(a_4) \rightarrow b_1) \bullet \left(T(T(a_3) \rightarrow b_2) \rightarrow (T(a_2) \rightarrow (b_3 \rightarrow (T(b_4) \rightarrow c))) \right) \end{aligned}$$

$$\begin{aligned}
&= a_1 \bullet (T(a_4) \rightharpoonup b_1) \bullet \left(T(T(a_3) \rightharpoonup b_2) \rightharpoonup (T(a_2) \rightharpoonup c) \right) \\
&= a_1 \bullet (T(a_4) \rightharpoonup b_1) \bullet \left(T(a_2 \bullet (T(a_3) \rightharpoonup b_2)) \rightharpoonup c \right) \\
&= a_1 \bullet (T(a_3) \rightharpoonup b_1) \bullet \left(T(a_2 \cdot b_2) \rightharpoonup c \right) \\
&= (a \cdot b)_1 \bullet \left(T((a \cdot b)_2) \rightharpoonup c \right) \\
&= (a \cdot b) \cdot c,
\end{aligned}$$

as desired. We then compute

$$\epsilon(a \cdot b) = \epsilon(a_1 \bullet (T(a_2) \rightharpoonup b)) = \epsilon(a_1)\epsilon(T(a_2) \rightharpoonup b) = \epsilon(T(a))\epsilon(b) = \epsilon(a)\epsilon(b).$$

Moreover, one has

$$\begin{aligned}
a_1 \cdot S(a_2) &= a_1 \bullet (T(a_2) \rightharpoonup S(a_3)) \\
&= a_1 \bullet (T(a_2) \rightharpoonup (a_3 \rightharpoonup T(a_4))) \\
&= a_1 \bullet (T(a_2) \bullet a_3 \rightharpoonup T(a_4)) \\
&= a_1 \bullet T(a_2) \\
&= \epsilon(a)1
\end{aligned}$$

and

$$\begin{aligned}
S(a_1) \cdot a_2 &= S(a_1)_1 \bullet (T(S(a_1)_2) \rightharpoonup a_2) \\
&= (a_{1_1} \rightharpoonup T(a_2)_1) \bullet (T(a_{1_2} \rightharpoonup T(a_2)_2) \rightharpoonup a_3) \\
&= (a_1 \rightharpoonup T(a_4)) \bullet (T(a_2 \rightharpoonup T(a_3)) \rightharpoonup a_5) \\
&\stackrel{(6)}{=} (a_1 \rightharpoonup T(a_4)) \bullet \left((a_2 \leftarrow T(a_3)) \rightharpoonup a_5 \right) \\
&= (a_{1_1} \rightharpoonup T(a_2)_1) \bullet \left((a_{1_2} \leftarrow T(a_2)_2) \rightharpoonup a_3 \right) \\
&\stackrel{(\text{MP.3})}{=} a_1 \rightharpoonup (T(a_2) \bullet a_3) \\
&= \epsilon(a)1.
\end{aligned}$$

As for (8), observe that

$$a \rightharpoonup b = S(a_1) \cdot a_2 \cdot (a_3 \rightharpoonup b) \stackrel{(3)}{=} S(a_1) \cdot (a_2 \bullet b).$$

We finally prove the Hopf brace compatibility (HBC):

$$\begin{aligned}
a \bullet (b \cdot c) &= a \bullet b_1 \bullet (T(b_2) \rightharpoonup c) \\
&= a_1 \bullet b_1 \bullet (T(b_2) \bullet T(a_2) \bullet a_3 \rightharpoonup c) \\
&= a_1 \bullet b_1 \bullet (T(a_2 \bullet b_2) \rightharpoonup (a_3 \rightharpoonup c)) \\
&= (a_1 \bullet b) \cdot (a_2 \rightharpoonup c) \\
&\stackrel{(8)}{=} (a_1 \bullet b) \cdot S(a_2) \cdot (a_3 \bullet c).
\end{aligned}$$

□

As promised, we now reinterpret this additional operation \cdot in a suitable category, which is going to be the category of *Yetter–Drinfeld modules* over H .

Definition 3.5. Let $(H, \bullet, 1, \Delta, \epsilon, T)$ be a Hopf algebra. A (*left-left*) *Yetter–Drinfeld module* on H is the datum of a left H -module X with an action $\rightharpoonup : H \otimes X \rightarrow X$, which is also a left H -comodule with a coaction $\rho : X \rightarrow H \otimes X$, satisfying the following compatibility for all $x \in X$, $a \in H$:

$$(\text{YD}) \quad \rho(a \rightharpoonup x) = a_1 \bullet x_{-1} \bullet T(a_3) \otimes (a_2 \rightharpoonup x_0).$$

A morphism of Yetter–Drinfeld modules is a morphism of both left H -modules and left H -comodules. We denote by ${}^H_H\mathcal{YD}$ the category of Yetter–Drinfeld modules over H .

Lemma 3.6. *Let $(H, \bullet, 1, \Delta, \epsilon, T)$ be a Hopf algebra. Suppose that $\dashv: H \otimes H \rightarrow H$ is a morphism of coalgebras, $\leftarrow: H \otimes H \rightarrow H$ is a linear map satisfying $\epsilon(a \leftarrow b) = \epsilon(a)\epsilon(b)$, and (\star) is satisfied. The following conditions are equivalent, and they are all equivalent to (MP.5):*

- (9) $(a \leftarrow b)_1 \otimes (a \leftarrow b)_2 = (a_1 \leftarrow b_1) \otimes (a_2 \leftarrow b_2)$, i.e. \leftarrow is a morphism of coalgebras;
(10) $((a_1 \dashv b_1) \bullet (a_3 \leftarrow b_3)) \otimes ((a_2 \dashv b_2) \bullet (a_4 \leftarrow b_4))$
 $= ((a_1 \dashv b_1) \bullet (a_3 \leftarrow b_3)_1) \otimes ((a_2 \dashv b_2) \bullet (a_3 \leftarrow b_3)_2)$;
(11) $(a_2 \leftarrow b_2) \otimes ((a_1 \dashv b_1) \bullet (a_3 \leftarrow b_3)) = (a_1 \leftarrow b_1) \otimes (a_2 \bullet b_2)$;
(12) $((a_1 \dashv b_1) \bullet (a_3 \leftarrow b_3)) \otimes (a_2 \dashv b_2) = (a_1 \bullet b_1) \otimes (a_2 \dashv b_2)$;

In particular, having a matched pair of actions (H, \dashv, \leftarrow) all the previous equalities hold true.

Proof. First, we notice that

$$\begin{aligned} & ((a_1 \dashv b_1) \bullet (a_3 \leftarrow b_3)_1) \otimes ((a_2 \dashv b_2) \bullet (a_3 \leftarrow b_3)_2) \\ &= ((a_1 \dashv b_1)_1 \bullet (a_2 \leftarrow b_2)_1) \otimes ((a_1 \dashv b_1)_2 \bullet (a_2 \leftarrow b_2)_2) \\ &= ((a_1 \dashv b_1) \bullet (a_2 \leftarrow b_2))_1 \otimes ((a_1 \dashv b_1) \bullet (a_2 \leftarrow b_2))_2 \\ &\stackrel{(\star)}{=} (a \bullet b)_1 \otimes (a \bullet b)_2 \\ &= (a_1 \bullet b_1) \otimes (a_2 \bullet b_2). \end{aligned}$$

Now, we check that (10) and (9) are equivalent:

$$\begin{aligned} (10) &\Leftrightarrow (T(a_1 \dashv b_1) \bullet (a_2 \dashv b_2) \bullet (a_4 \leftarrow b_4)) \otimes ((a_3 \dashv b_3) \bullet (a_5 \leftarrow b_5)) \\ &= (T(a_1 \dashv b_1) \bullet (a_2 \dashv b_2) \bullet (a_4 \leftarrow b_4)_1) \otimes ((a_3 \dashv b_3) \bullet (a_4 \leftarrow b_4)_2) \\ &\Leftrightarrow (a_2 \leftarrow b_2) \otimes ((a_1 \dashv b_1) \bullet (a_3 \leftarrow b_3)) = (a_2 \leftarrow b_2)_1 \otimes ((a_1 \dashv b_1) \bullet (a_2 \leftarrow b_2)_2) \\ &\Leftrightarrow (a_3 \leftarrow b_3) \otimes (T(a_1 \dashv b_1) \bullet (a_2 \dashv b_2) \bullet (a_4 \leftarrow b_4)) \\ &= (a_3 \leftarrow b_3)_1 \otimes (T(a_1 \dashv b_1) \bullet (a_2 \dashv b_2) \bullet (a_3 \leftarrow b_3)_2) \\ &\Leftrightarrow (a_1 \leftarrow b_1) \otimes (a_2 \leftarrow b_2) = (a \leftarrow b)_1 \otimes (a \leftarrow b)_2. \end{aligned}$$

We prove that (11) and (12) are both equivalent to (10). One has

$$\begin{aligned} (a_2 \leftarrow b_2) \otimes ((a_1 \dashv b_1) \bullet (a_3 \leftarrow b_3)) &= (a_1 \leftarrow b_1) \otimes (a_2 \bullet b_2) \\ &\Leftrightarrow ((a_1 \dashv b_1) \bullet (a_3 \leftarrow b_3)) \otimes ((a_2 \dashv b_2) \bullet (a_4 \leftarrow b_4)) = ((a_1 \dashv b_1) \bullet (a_2 \leftarrow b_2)) \otimes (a_3 \bullet b_3) \\ &\Leftrightarrow ((a_1 \dashv b_1) \bullet (a_3 \leftarrow b_3)) \otimes ((a_2 \dashv b_2) \bullet (a_4 \leftarrow b_4)) = (a_1 \bullet b_1) \otimes (a_2 \bullet b_2) \\ &\Leftrightarrow (10). \end{aligned}$$

and

$$\begin{aligned} (10) &\Leftrightarrow ((a_1 \dashv b_1) \bullet (a_3 \leftarrow b_3)) \otimes ((a_2 \dashv b_2) \bullet (a_4 \leftarrow b_4)) = (a_1 \bullet b_1) \otimes (a_2 \bullet b_2) \\ &\Leftrightarrow ((a_1 \dashv b_1) \bullet (a_3 \leftarrow b_3)) \otimes ((a_2 \dashv b_2) \bullet (a_4 \leftarrow b_4)) = (a_1 \bullet b_1) \otimes ((a_2 \dashv b_2) \bullet (a_3 \leftarrow b_3)) \\ &\Leftrightarrow ((a_1 \dashv b_1) \bullet (a_3 \leftarrow b_3)) \otimes (a_2 \dashv b_2) = (a_1 \bullet b_1) \otimes (a_2 \dashv b_2), \end{aligned}$$

where we use the fact that (10) implies (9) and $\epsilon(a \leftarrow b) = \epsilon(a)\epsilon(b)$ in the latter equivalence. This completes the proof that (9)–(12) are equivalent. Finally, using the assumption (\star) , (MP.5) clearly implies (11); and, conversely, assuming (11) (and thus (9)) we immediately retrieve (MP.5). \square

Corollary 3.7. *Let $(H, \bullet, 1, \Delta, \epsilon, T)$ be a Hopf algebra. Every matched pair of actions (H, \dashv, \leftarrow) is a matched pair (H, H) on H .*

Lemma 3.8. *Let $(H, \rightharpoonup, \leftarrow)$ be as in the setting of Proposition 3.4. Then, $(H, \rightharpoonup, \text{Ad}_L)$ is in ${}^H_H\mathcal{YD}$, where $\text{Ad}_L : H \rightarrow H \otimes H$, $a \mapsto a_1 \bullet T(a_3) \otimes a_2$ is the left H -adjoint coaction. As a consequence, denoted by $\sigma^{\mathcal{YD}}$ the braiding of ${}^H_H\mathcal{YD}$, one has that $\sigma_{H,H}^{\mathcal{YD}} : H \otimes H \rightarrow H \otimes H$ is explicitly described by $\sigma_{H,H}^{\mathcal{YD}} : a \otimes b \mapsto (a_1 \bullet T(a_3) \rightharpoonup b) \otimes a_2$.*

Proof. We already know that (H, \rightharpoonup) is a left H -module and (H, Ad_L) is a left H -comodule. Since \leftarrow is a morphism of coalgebras, the equivalent conditions of Lemma 3.6 hold true. We check the compatibility of Yetter–Drinfeld modules (YD):

$$\begin{aligned} \text{Ad}_L(a \rightharpoonup b) &= \left((a_1 \rightharpoonup b_1) \bullet T(a_3 \rightharpoonup b_3) \right) \otimes (a_2 \rightharpoonup b_2) \\ &\stackrel{(5)}{=} (a_1 \rightharpoonup b_1) \bullet (a_3 \leftarrow b_3) \bullet T(b_4) \bullet T(a_4) \otimes (a_2 \rightharpoonup b_2) \\ &\stackrel{(12)}{=} a_1 \bullet b_1 \bullet T(b_3) \bullet T(a_3) \otimes (a_2 \rightharpoonup b_2) \\ &= a_1 \bullet b_{-1} \bullet T(a_3) \otimes (a_2 \rightharpoonup b_0). \end{aligned}$$

□

Remark 3.9. Notice that the braiding $\sigma_{H,H}^{\mathcal{YD}}$ coincides with the flip map τ in case the adjoint coaction Ad_L is trivial, i.e. $a_1 \bullet T(a_3) \otimes a_2 = 1 \otimes a$, which clearly happens in the cocommutative case. Conversely, if $a_1 \bullet T(a_3) \otimes a_2 = 1 \otimes a$, then

$$a_1 \bullet T(a_3) \otimes a_2 = 1 \otimes a = 1 \otimes a_1 \epsilon(a_2) = a_2 \bullet T(a_3) \otimes a_1$$

and so $a_1 \otimes a_2 = a_2 \otimes a_1$, i.e. H is cocommutative. Furthermore, let us also observe that $\sigma_{H,H}^{\mathcal{YD}}$ coincides with τ also in case \rightharpoonup is the trivial action.

Proposition 3.10. *Let H be a Hopf algebra, $(B, \cdot, 1_B, \Delta_B, \epsilon_B)$ a bialgebra in ${}^H_H\mathcal{YD}$. Suppose that $S : B \rightarrow B$ is linear map, and a convolution inverse of Id_B : then, B is a Hopf algebra in ${}^H_H\mathcal{YD}$, with antipode S .*

*Proof.*¹ Define $\text{can} : B \otimes B \rightarrow B \otimes B$ as $\text{can} := (m_B \otimes m_B)(\text{Id}_B \otimes \sigma_{B,B}^{\mathcal{YD}} \otimes \text{Id}_B)(\text{Id}_B \otimes u_B \otimes \Delta_B)$ which is a composition of morphisms in ${}^H_H\mathcal{YD}$, and hence a morphism in ${}^H_H\mathcal{YD}$. The map can is explicitly given by $\text{can}(a \otimes b) = (a \cdot b_1) \otimes b_2$. Since S is a convolution inverse of Id_B , can is invertible with inverse given by $\text{can}^{-1} : a \otimes b \mapsto (a \cdot S(b_1)) \otimes b_2$. Since can is invertible and it is a morphism in ${}^H_H\mathcal{YD}$, the inverse is also a morphism in ${}^H_H\mathcal{YD}$. Now observe that $S = (\text{Id}_B \otimes \epsilon_B)\text{can}^{-1}(u_B \otimes \text{Id}_B)$, thus S is a composition of morphisms in ${}^H_H\mathcal{YD}$: thus S is a morphism in ${}^H_H\mathcal{YD}$, and hence B is in $\text{Hopf}({}^H_H\mathcal{YD})$. □

We merge the previous results in the following theorem:

Theorem 3.11. *Let $(H, \bullet, 1, \Delta, \epsilon, T)$ be a Hopf algebra, $(H, \rightharpoonup, \leftarrow)$ a matched pair of actions. Define \cdot and S as in (1),(2). Then:*

- i. $(H, \cdot, 1, \Delta, \epsilon, S)$ is in $\text{Hopf}({}^H_H\mathcal{YD})$ with the action \rightharpoonup and the adjoint coaction Ad_L ;
- ii. the two operations \bullet and \cdot satisfy the compatibility condition (HBC).

Proof. The compatibility (HBC) is satisfied by Proposition 3.4.

We first check that \cdot is a morphism of coalgebras in ${}^H_H\mathcal{YD}$. Using Lemma 3.3 we obtain:

$$\begin{aligned} \Delta(a \cdot b) &= (a_1 \bullet (T(a_3) \rightharpoonup b_1)) \otimes (a_2 \cdot b_2) \\ &\stackrel{(3)}{=} (a_1 \cdot (a_2 \rightharpoonup (T(a_4) \rightharpoonup b_1))) \otimes (a_3 \cdot b_2) \\ &= a_1 \cdot (a_2 \bullet T(a_4) \rightharpoonup b_1) \otimes (a_3 \cdot b_2), \end{aligned}$$

i.e., $\Delta m_\cdot = (m_\cdot \otimes m_\cdot)(\text{Id} \otimes \sigma_{H,H}^{\mathcal{YD}} \otimes \text{Id})(\Delta \otimes \Delta)$. By Proposition 3.4 we also know that $\epsilon(a \cdot b) = \epsilon(a)\epsilon(b)$. Together, they are the compatibility conditions of a bialgebra in ${}^H_H\mathcal{YD}$.

¹We are grateful to A. Ardizzone for suggesting this proof.

Our next step is proving that (H, Δ, ϵ) is a coalgebra in ${}^H_H\mathcal{YD}$. We know that Δ and ϵ are left H -linear with respect to \rightarrow . We thereby prove that they are also left H -colinear. We compute

$$\begin{aligned} (\text{Id} \otimes \Delta)\text{Ad}_L(a) &= (\text{Id} \otimes \Delta)(a_1 \bullet T(a_3) \otimes a_2) \\ &= a_1 \bullet T(a_4) \otimes a_2 \otimes a_3 \\ &= (a_1 \bullet T(a_3) \bullet a_4 \bullet T(a_6)) \otimes a_2 \otimes a_5 \\ &= (a_{1_1} \bullet T(a_{1_3})) \bullet (a_{2_1} \bullet T(a_{2_3})) \otimes a_{1_2} \otimes a_{2_2}, \end{aligned}$$

and

$$(\text{Id} \otimes \epsilon)\text{Ad}_L(a) = (a_1 \bullet T(a_3)) \otimes \epsilon(a_2) = (a_1 \bullet T(a_2)) \otimes 1_{\mathbf{k}} = \epsilon(a)1_H \otimes 1_{\mathbf{k}}.$$

This concludes the proof that (H, Δ, ϵ) is a coalgebra in ${}^H_H\mathcal{YD}$.

Clearly, the unit $u: 1_{\mathbf{k}} \mapsto 1_H$ is left H -linear and left H -colinear, so in order to conclude that $(H, \cdot, 1)$ is in $\text{Mon}({}^H_H\mathcal{YD})$ we only need to verify that \cdot is left H -linear and left H -colinear.

First we show that \cdot is left H -linear with respect to \rightarrow . Therefore, we compute

$$\begin{aligned} a \rightarrow (b \cdot c) &\stackrel{(1)}{=} a \rightarrow (b_1 \bullet (T(b_2) \rightarrow c)) \\ &\stackrel{(\text{MP.3})}{=} (a_1 \rightarrow b_1) \bullet ((a_2 \leftarrow b_2) \rightarrow (T(b_3) \rightarrow c)) \\ &\stackrel{(3)}{=} (a_1 \rightarrow b_1) \bullet ((a_2 \rightarrow b_2) \rightarrow ((a_3 \leftarrow b_3) \rightarrow (T(b_4) \rightarrow c))) \\ &= (a_1 \rightarrow b_1) \bullet (((a_2 \rightarrow b_2) \bullet (a_3 \leftarrow b_3)) \rightarrow (T(b_4) \rightarrow c)) \\ &\stackrel{(*)}{=} (a_1 \rightarrow b_1) \bullet (a_2 \bullet b_2 \rightarrow (T(b_3) \rightarrow c)) \\ &= (a_1 \rightarrow b_1) \bullet (a_2 \bullet b_2 \bullet T(b_3) \rightarrow c) \\ &= (a_1 \rightarrow b) \bullet (a_2 \rightarrow c). \end{aligned}$$

We now check that \cdot is left H -colinear with respect to Ad_L . We already know that (7) holds, i.e.

$$(\Delta \otimes \text{Id})\Delta(a \cdot b) = (a_1 \bullet (T(a_6) \rightarrow b_1)) \otimes (a_2 \bullet (T(a_5) \rightarrow b_2)) \otimes (a_3 \bullet (T(b_4) \rightarrow b_3)).$$

The request that \cdot is colinear with respect to the adjoint coaction Ad_L reads as follows:

$$(13) \quad (a \cdot b)_1 \bullet T((a \cdot b)_3) \otimes (a \cdot b)_2 = a_1 \bullet T(a_3) \bullet b_1 \bullet T(b_3) \otimes (a_2 \cdot b_2).$$

We now rephrase the condition (13). One has

$$\begin{aligned} (a \cdot b)_1 \bullet T((a \cdot b)_3) \otimes (a \cdot b)_2 &= \left(a_1 \bullet (T(a_6) \rightarrow b_1) \bullet T(a_3 \bullet (T(a_4) \rightarrow b_3)) \right) \otimes \left(a_2 \bullet (T(a_5) \rightarrow b_2) \right) \\ &= \left(a_1 \bullet (T(a_6) \rightarrow b_1) \bullet T(T(a_4) \rightarrow b_3) \bullet T(a_3) \right) \otimes \left(a_2 \bullet (T(a_5) \rightarrow b_2) \right) \\ &= \left(a_1 \bullet (T(a_6) \rightarrow b_1) \bullet T(T(a_4) \rightarrow b_3) \bullet T(a_3) \bullet b_4 \bullet T(b_5) \right) \otimes \left(a_2 \bullet (T(a_5) \rightarrow b_2) \right) \\ &\stackrel{(4)}{=} \left(a_1 \bullet (T(a_5) \rightarrow b_1) \bullet (T(a_3) \leftarrow b_3) \bullet T(b_4) \right) \otimes \left(a_2 \bullet (T(a_4) \rightarrow b_2) \right). \end{aligned}$$

Therefore, (13) becomes

$$(14) \quad \begin{aligned} &\left(a_1 \bullet (T(a_5) \rightarrow b_1) \bullet (T(a_3) \leftarrow b_3) \bullet T(b_4) \right) \otimes \left(a_2 \bullet (T(a_4) \rightarrow b_2) \right) \\ &= \left(a_1 \bullet T(a_4) \bullet b_1 \bullet T(b_3) \right) \otimes \left(a_2 \bullet (T(a_3) \rightarrow b_2) \right). \end{aligned}$$

But now, using the antipode, we obtain

$$\begin{aligned} (14) &\Leftrightarrow \left((T(a_4) \rightarrow b_1) \bullet (T(a_2) \leftarrow b_3) \right) \otimes \left(a_1 \bullet (T(a_3) \rightarrow b_2) \right) \\ &= (T(a_3) \bullet b_1) \otimes \left(a_1 \bullet (T(a_2) \rightarrow b_2) \right) \end{aligned}$$

$$\Leftrightarrow \left((T(a_3) \rightharpoonup b_1) \bullet (T(a_1) \leftarrow b_3) \right) \otimes \left(T(a_2) \rightharpoonup b_2 \right) = (T(a_2) \bullet b_1) \otimes (T(a_1) \rightharpoonup b_2).$$

Thus, we obtain that (13) is equivalent to

$$(15) \quad (T(a_3) \rightharpoonup b_1) \bullet (T(a_1) \leftarrow b_3) \otimes (T(a_2) \rightharpoonup b_2) = (T(a_2) \bullet b_1) \otimes (T(a_1) \rightharpoonup b_2).$$

The latter is implied by (12), which holds true since \leftarrow is a morphism of coalgebras (and is in fact equivalent to (12) when T is bijective). This concludes the proof that $(H, \cdot, 1)$ is an algebra in ${}^H_H\mathcal{YD}$, and hence $(H, \cdot, 1, \Delta, \epsilon)$ is a bialgebra in ${}^H_H\mathcal{YD}$ for what already observed.

We already know from Proposition 3.4 that S satisfies $S(a_1) \cdot a_2 = \epsilon(a)1_H = a_1 \cdot S(a_2)$, thus we only need to prove that S is a morphism of Yetter–Drinfeld modules. This follows from Proposition 3.10. \square

Definition 3.12. Given a matched pair $(H, \rightharpoonup, \leftarrow)$ on a Hopf algebra $(H, \bullet, 1, \Delta, \epsilon, T)$, the second structure H^\cdot from Theorem 3.11 *i.* will be called the *transmutation* of H^\bullet with respect to the matched pair.

In Remark 5.7, we shall see that this is indeed a generalisation of Majid’s transmutation [28].

Remark 3.13. By [4, Remark 1.8] the Hopf algebra $(H, \cdot, 1, \Delta, \epsilon, S)$ is, in particular, a braided Hopf algebra in the sense of Kharchenko [22, § 2.2]. If moreover T is bijective, then the braiding $\sigma^{\mathcal{YD}}$ is bijective, hence $(H, \cdot, 1, \Delta, \epsilon, S)$ is a braided Hopf algebra in the sense of Majid [29]; see also [39, Definition 5.1].

Remark 3.14. In case H is cocommutative, $\sigma_{H,H}^{\mathcal{YD}}$ coincides with the flip τ , hence $(H, \cdot, 1, \Delta, \epsilon, S)$ becomes a standard Hopf algebra and we recover the definition of Hopf brace [5].

Corollary 3.15. *Let $(H, \bullet, 1, \Delta, \epsilon, T)$ be a Hopf algebra and $(H, \rightharpoonup, \leftarrow)$ a matched pair of actions. Define \cdot as in (1), and S as in (2). Then, the following is a bialgebra structure on $H \otimes H$:*

$$\begin{aligned} (a \otimes h) \cdot_{\#} (a' \otimes h') &:= a \cdot S(h_1) \cdot (h_2 \bullet a') \otimes h_3 \bullet h', \\ \Delta_{\#}(a \otimes h) &:= a_1 \otimes a_2 \bullet T(a_4) \bullet h_1 \otimes a_3 \otimes h_2, \\ S_{\#}(a \otimes h) &:= S(T(T(a_3) \bullet h_3)) \cdot (T(T(a_4) \bullet h_2) \bullet T(a_2)) \otimes T(a_1 \bullet T(a_5) \bullet h_1). \end{aligned}$$

Proof. Let $H^\bullet := (H, \bullet)$ and $H^\cdot := (H, \cdot)$ denote the two multiplicative structures. By Theorem 3.11 we have that $(H, \cdot, 1, \Delta, \epsilon)$ is in $\text{Bimon}({}^H_H\mathcal{YD})$, thus the bosonisation $(H^\cdot \# H^\bullet, \cdot_{\#}, 1 \otimes 1, \Delta_{\#}, \epsilon_H \otimes \epsilon_H)$ is a bialgebra, where $(a \otimes h) \cdot_{\#} (a' \otimes h') := a \cdot (h_1 \rightharpoonup a') \otimes h_2 \bullet h'$ and $\Delta_{\#}(a \otimes h) := a_1 \otimes a_{2_{-1}} \bullet h_1 \otimes a_{2_0} \otimes h_2$ denote the smash product and the smash coproduct, respectively, see e.g. Heckenberger and Schneider [19, Proposition 3.8.4]. Moreover, since H^\bullet and H^\cdot are both Hopf algebras, we obtain an antipode on $H^\cdot \# H^\bullet$ given by $S_{\#}(a \otimes h) := (T(a_{-1} \bullet h_2) \rightharpoonup S(a_0)) \otimes T(a_{-2} \bullet h_1)$; see e.g. [19, Theorem 3.8.10]. Thus, we compute

$$\begin{aligned} S_{\#}(a \otimes h) &= (T(a_{-1} \bullet h_2) \rightharpoonup S(a_0)) \otimes T(a_{-2} \bullet h_1) \\ &= (T(a_2 \bullet T(a_4) \bullet h_2) \rightharpoonup S(a_3)) \otimes T(a_1 \bullet T(a_5) \bullet h_1) \\ &= (T(T(a_5) \bullet h_2) \bullet T(a_2) \rightharpoonup (a_3 \rightharpoonup T(a_4))) \otimes T(a_1 \bullet T(a_6) \bullet h_1) \\ &= (T(T(a_3) \bullet h_2) \rightharpoonup T(a_2)) \otimes T(a_1 \bullet T(a_4) \bullet h_1) \\ &= S(T(T(a_3) \bullet h_3)) \cdot (T(T(a_4) \bullet h_2) \bullet T(a_2)) \otimes T(a_1 \bullet T(a_5) \bullet h_1). \end{aligned}$$

\square

Theorem 3.11 suggests us the following definition:

Definition 3.16. A *Yetter–Drinfeld brace* (or \mathcal{YD} -brace) $(H, \cdot, \bullet, 1, \Delta, \epsilon, S, T)$ is the datum of a Hopf algebra $H^\bullet = (H, \bullet, 1, \Delta, \epsilon, T)$, a second operation \cdot on H , and a linear map $S: H \rightarrow H$ such that:

- i.* $(H, \cdot, 1, \Delta, \epsilon, S)$ is in $\text{Hopf}(\mathcal{H}_H^\bullet \mathcal{YD})$ with the action \rightarrow defined by $a \rightarrow b := S(a_1) \cdot (a_2 \bullet b)$, and the coaction given by Ad_L ;
- ii.* if we define $a \leftarrow b := T(a_1 \rightarrow b_1) \bullet a_2 \bullet b_2$, the two maps \rightarrow, \leftarrow satisfy (MP.5);
- iii.* the two operations \bullet and \cdot satisfy the Hopf brace compatibility (HBC).

Given two Yetter–Drinfeld braces H and K , a *morphism of Yetter–Drinfeld braces* is a map $f: H \rightarrow K$ that is a morphism in $\text{Hopf}(\text{Vec}_{\mathbb{k}})$ between the respective Hopf algebras in $\text{Vec}_{\mathbb{k}}$, and satisfies

$$f(a \cdot_H b) = f(a) \cdot_K f(b).$$

In particular, this implies $S_K f = f S_H$, $f \rightarrow_H = \rightarrow_K (f \otimes f)$ and $(\text{Ad}_L)_K f = (f \otimes f)(\text{Ad}_L)_H$. We denote the category of Yetter–Drinfeld braces by $\mathcal{YDBr}(\text{Vec}_{\mathbb{k}})$.

Remark 3.17. Let $f: H \rightarrow K$ be a morphism of Yetter–Drinfeld braces. Since $f: H^\bullet \rightarrow K^\bullet$ is a morphism of bialgebras, one can regard K^\bullet as a left H^\bullet -module through f , so that the condition $f(a \rightarrow_H b) = f(a) \rightarrow_K f(b)$ means that f is a morphism in ${}_{H^\bullet} \mathfrak{M}$; and H^\bullet as a left K^\bullet -comodule through f , so that the condition $(\text{Ad}_L)_K f = (f \otimes f)(\text{Ad}_L)_H$ means that f is a morphism in ${}^{K^\bullet} \mathfrak{M}$. One can consider the category ${}_{H^\bullet}^{K^\bullet} \mathcal{YD}$ of *relative* Yetter–Drinfeld modules; see [8, § 4.4, p. 183]. Clearly H^\bullet and K^\bullet are in ${}_{H^\bullet}^{K^\bullet} \mathcal{YD}$. Indeed, given $a, b \in H$, we compute:

$$\begin{aligned} (a_1 \rightarrow_H b)_-^K \bullet_K f(a_2) \otimes (a_1 \rightarrow_H b)_0^K &= f((a_1 \rightarrow_H b)_{-1} \bullet_H a_2) \otimes (a_1 \rightarrow_H b)_0 \\ &= f(a_1 \bullet_H b_{-1}) \otimes (a_2 \rightarrow_H b_0) \\ &= f(a_1) \bullet_K b_{-1}^K \otimes (a_2 \rightarrow_H b_0^K), \end{aligned}$$

where $a_{-1} \otimes a_0 = (\text{Ad}_L)_H(a)$ is the usual adjoint H^\bullet -coaction on H , while $a_{-1}^K \otimes a_0^K = (f \otimes \text{Id})(\text{Ad}_L)_H(a)$ denotes the K^\bullet -coaction on H^\bullet induced by Ad_L through f . The Yetter–Drinfeld compatibility for K^\bullet is deduced analogously. Hence we obtain that f is a morphism in ${}_{H^\bullet}^{K^\bullet} \mathcal{YD}$. Moreover, one clearly has that H^\bullet and K^\bullet are in $\text{Mon}({}_{H^\bullet}^{K^\bullet} \mathcal{YD})$ and $\text{Comon}({}_{H^\bullet}^{K^\bullet} \mathcal{YD})$; thus the condition $f(a \cdot_H b) = f(a) \cdot_K f(b)$ means that $f: H^\bullet \rightarrow K^\bullet$ is a morphism in $\text{Mon}({}_{H^\bullet}^{K^\bullet} \mathcal{YD})$. Notice that f is automatically a morphism in $\text{Comon}({}_{H^\bullet}^{K^\bullet} \mathcal{YD})$.

From Theorem 3.11, we obtain:

Corollary 3.18. *Let $(H, \bullet, 1, \Delta, \epsilon, T)$ be a Hopf algebra, $(H, \rightarrow, \leftarrow)$ a matched pair of actions. Define \cdot and S as in (1), (2). Then, $(H, \cdot, \bullet, 1, \Delta, \epsilon, S, T)$ is a Yetter–Drinfeld brace.*

Remark 3.19. Define \leftarrow as in Definition 3.16 *ii.* Since \rightarrow is a morphism of coalgebras, it is clear that \rightarrow, \leftarrow satisfy (\star) :

$$(a_1 \rightarrow b_1) \bullet (a_2 \leftarrow b_2) = (a_1 \rightarrow b_1) \bullet T(a_2 \rightarrow b_2) \bullet a_3 \bullet b_3 = \epsilon(a_1 \rightarrow b_1) a_2 \bullet b_2 = a \bullet b.$$

Observe that $\epsilon(a \leftarrow b) = \epsilon(a)\epsilon(b)$, thus we are in the hypotheses of Lemma 3.6. Consequently, Definition 3.16 *ii.* is equivalent to the request that \leftarrow is a morphism of coalgebras.

Remark 3.20. Notice that Definition 3.16 makes as much sense when we replace $\text{Vec}_{\mathbb{k}}$ with a braided monoidal category. Although this exceeds the scope of this work, we may expect several of our results in $\text{Vec}_{\mathbb{k}}$ to hold more in general. Furthermore, Definition 3.16 can be dualised to define *Yetter–Drinfeld cobraces*, leading to a generalisation of commutative Hopf cobraces [5].

Clearly, in a Yetter–Drinfeld brace, the map \rightarrow defined as in Definition 3.16 *i.* is a left H^\bullet -action, and $(H, \cdot, 1)$ is a left H^\bullet -module algebra. In fact, this is true under weaker assumptions:

Lemma 3.21. *Let $H^\bullet = (H, \bullet, 1, \Delta, \epsilon)$ be a bialgebra and $(H, \cdot, 1, \Delta, \epsilon)$ be an algebra and a coalgebra, endowed with a map $S: H \rightarrow H$ satisfying $S(a_1) \cdot a_2 = \epsilon(a)1 = a_1 \cdot S(a_2)$. Suppose that the Hopf brace compatibility (HBC) between \bullet and \cdot is satisfied. Then*

$$a \rightarrow b := S(a_1) \cdot (a_2 \bullet b)$$

is a left H^\bullet -action on H . Moreover, $(H, \cdot, 1)$ is a left H^\bullet -module algebra and, if ϵ is a morphism of algebras with respect to \cdot , then ϵ is left H^\bullet -linear with respect to \rightarrow .

Proof. First we compute $1 \rightharpoonup b = S(1) \cdot (1 \bullet b) = b$. Moreover, we have

$$\begin{aligned} a \rightharpoonup (b \rightharpoonup c) &= a \rightharpoonup (S(b_1) \cdot (b_2 \bullet c)) \\ &= S(a_1) \cdot \left(a_2 \bullet (S(b_1) \cdot (b_2 \bullet c)) \right) \\ &= S(a_1) \cdot (a_2 \bullet S(b_1)) \cdot S(a_3) \cdot (a_4 \bullet b_2 \bullet c) \\ &\stackrel{(\dagger)}{=} S(a_1 \bullet b_1) \cdot a_2 \cdot S(a_3) \cdot (a_4 \bullet b_2 \bullet c) \\ &= S(a_1 \bullet b_1) \cdot (a_2 \bullet b_2 \bullet c) \\ &= (a \bullet b) \rightharpoonup c, \end{aligned}$$

where the equality marked with (\dagger) follows from the relation $S(a_1 \bullet b) \cdot a_2 = S(a_1) \cdot (a_2 \bullet S(b))$, whose proof is the same as in [5, Lemma 1.7] and simply follows from (HBC). Moreover, we have

$$a \rightharpoonup (b \cdot c) = S(a_1) \cdot (a_2 \bullet (b \cdot c)) = S(a_1) \cdot (a_2 \bullet b) \cdot S(a_3) \cdot (a_4 \bullet c) = (a_1 \rightharpoonup b) \cdot (a_2 \rightharpoonup c)$$

and

$$(16) \quad a \rightharpoonup 1 = S(a_1) \cdot (a_2 \bullet 1) = S(a_1) \cdot a_2 = \epsilon(a)1,$$

hence $(H, \cdot, 1)$ is a left H^\bullet -module algebra. Finally, we have

$$\epsilon(a \rightharpoonup b) = \epsilon(S(a_1) \cdot (a_2 \bullet b)) \stackrel{(\ddagger)}{=} \epsilon(a_1)\epsilon(a_2)\epsilon(b) = \epsilon(a)\epsilon(b),$$

where (\ddagger) follows from the fact that ϵ is a morphism of algebras with respect to \cdot . \square

3.2. A general example. We now construct an easy example of a Yetter–Drinfeld brace on a Hopf algebra H . This construction depends on the assumption that the adjoint action on H is compatible with the comultiplication.

Lemma 3.22. *Let $H^\bullet := (H, \bullet, 1, \Delta, \epsilon, T)$ be a Hopf algebra, \rightharpoonup the left adjoint action $a \rightharpoonup b := a_1 \bullet b \bullet T(a_2)$, and \leftarrow the trivial action. Then, $(H, \rightharpoonup, \leftarrow)$ is a matched pair of actions if and only if Δ is left linear with respect to \rightharpoonup .*

Proof. Notice that H^\bullet is automatically a right H -module coalgebra. We verify (MP.1)–(MP.4) and (\star) . Clearly $a \rightharpoonup 1 = \epsilon(a)1$ holds, while (MP.2) is automatically true. Moreover, we compute

$$(a_1 \rightharpoonup b_1) \bullet ((a_2 \leftarrow b_2) \rightharpoonup c) = a_1 \bullet b \bullet T(a_2) \bullet (a_3 \rightharpoonup c) = a_1 \bullet b \bullet T(a_2) \bullet a_3 \bullet c \bullet T(a_4) = a \rightharpoonup (b \bullet c),$$

and

$$(a \leftarrow (b_1 \rightharpoonup c_1)) \bullet (b_2 \leftarrow c_2) = a \epsilon(b_1 \rightharpoonup c_1) \bullet b_2 \epsilon(c_2) = (a \bullet b) \leftarrow c.$$

Finally,

$$(a_1 \rightharpoonup b_1) \bullet (a_2 \leftarrow b_2) = a_1 \bullet b_1 \bullet T(a_2) \bullet a_3 \epsilon(b_2) = a \bullet b.$$

Thus, $(H, \rightharpoonup, \leftarrow)$ is a matched pair of actions if and only if H^\bullet is a left H -module coalgebra. Since $\epsilon(a \rightharpoonup b) = \epsilon(a)\epsilon(b)$ is automatically satisfied, H^\bullet is a left H -module coalgebra if and only if Δ is left linear with respect to \rightharpoonup ; as desired. \square

Remark 3.23. The linearity condition of Δ with respect to \rightharpoonup , i.e. $\Delta(a \rightharpoonup b) = (a_1 \rightharpoonup b_1) \otimes (a_2 \rightharpoonup b_2)$, explicitly reads:

$$a_1 \bullet b_1 \bullet T(a_4) \otimes a_2 \bullet b_2 \bullet T(a_3) = a_1 \bullet b_1 \bullet T(a_2) \otimes a_3 \bullet b_2 \bullet T(a_4).$$

The latter is equivalent to $T(a_3) \otimes a_1 \bullet b \bullet T(a_2) = T(a_1) \otimes a_2 \bullet b \bullet T(a_3)$, and then to

$$(17) \quad a_1 \bullet T(a_4) \otimes a_2 \bullet b \bullet T(a_3) = 1 \otimes a_1 \bullet b \bullet T(a_2),$$

i.e., $(\text{Id} \otimes \rightharpoonup)(\text{Ad}_L \otimes \text{Id}) = (u_H \otimes \rightharpoonup)$, where Ad_L denotes again the left adjoint coaction on H . From the matched pair of actions $(H, \rightharpoonup, \leftarrow)$ of Lemma 3.22, we can define \cdot as in (1), and S as in (2), obtaining

$$(18) \quad a \cdot b = a_1 \bullet T(a_3) \bullet b \bullet T(T(a_2)), \quad S(a) = a_1 \bullet T(a_3) \bullet T(a_2).$$

Notice that (17) clearly holds in the commutative and cocommutative cases (where the latter was already considered, e.g., in Masuoka [32, Remark p. 302]). If H^\bullet is commutative, then (H, \bullet, \bullet) is an example of a Yetter–Drinfeld brace. On the other hand, if H^\bullet is cocommutative, then the previous construction reduces to the *almost trivial* Hopf brace $(H, \bullet^{\text{op}}, \bullet)$ (the terminology is adapted from [18]).

In §5, we shall see other explicit examples coming from coquasitriangular structures. In particular, the first example will be the Sweedler’s Hopf algebra H_4 . Generalisations of H_4 are given by the Taft algebras H_{n^2} , which do not admit any (co)quasitriangular structure for $n > 2$ (see Gelaki [17]). Matched pairs (H_{n^2}, H_{n^2}) on H_{n^2} are classified by Agore [1, Theorem 2.1], and it is not difficult to see that none of these matched pairs satisfy (\star) . This explicitly shows that matched pairs of actions $(H, \rightharpoonup, \leftarrow)$ correspond to a proper subclass of the matched pairs (H, H) on H .

3.3. A converse connection. In order to show that our definition of Yetter–Drinfeld braces is optimal, we now prove that the correspondence of Theorem 3.11 has a converse. Therefore, a Yetter–Drinfeld brace and a matched pair of actions are essentially the same thing. This is in fact, as we shall point out, an isomorphism of categories.

Theorem 3.24. *Let $(H, \cdot, \bullet, 1, \Delta, \epsilon, S, T)$ be a Yetter–Drinfeld brace. Define*

$$a \leftarrow b := T(a_1 \rightharpoonup b_1) \bullet a_2 \bullet b_2.$$

Then, $(H, \rightharpoonup, \leftarrow)$ is a matched pair of actions.

Proof. Since H is in $\text{Comon}(\mathcal{H}^\bullet\text{-}\mathcal{YD})$, one has that Δ and ϵ are left H^\bullet -linear with respect to \rightharpoonup , and hence H is a left H^\bullet -module coalgebra. From the definition of \rightharpoonup we immediately have $a \bullet b = a_1 \cdot S(a_2) \cdot (a_3 \bullet b) = a_1 \cdot (a_2 \rightharpoonup b)$, whence also

$$a \cdot b = a_1 \cdot (a_2 \rightharpoonup (T(a_3) \rightharpoonup b)) = a_1 \bullet (T(a_2) \rightharpoonup b).$$

We therefore compute

$$\begin{aligned} a \leftarrow (b \bullet c) &= T(a_1 \rightharpoonup (b_1 \bullet c_1)) \bullet a_2 \bullet (b_2 \bullet c_2) \\ &= T(S(a_1) \cdot (a_2 \bullet b_1 \bullet c_1)) \bullet a_3 \bullet b_2 \bullet c_2 \\ &= T(S(a_1) \cdot (a_2 \bullet b_1) \cdot S(a_3 \bullet b_2) \cdot (a_4 \bullet b_3 \bullet c_1)) \bullet a_5 \bullet b_4 \bullet c_2 \\ &= T((a_1 \rightharpoonup b_1) \cdot ((a_2 \bullet b_2) \rightharpoonup c_1)) \bullet a_3 \bullet b_3 \bullet c_2 \\ &= T\left((a_1 \rightharpoonup b_1) \bullet (T(a_2 \rightharpoonup b_2) \rightharpoonup ((a_3 \bullet b_3) \rightharpoonup c_1))\right) \bullet a_4 \bullet b_4 \bullet c_2 \\ &= T\left((T(a_2 \rightharpoonup b_2) \bullet a_3 \bullet b_3) \rightharpoonup c_1\right) \bullet T(a_1 \rightharpoonup b_1) \bullet a_4 \bullet b_4 \bullet c_2 \\ &= (T(a_1 \rightharpoonup b_1) \bullet a_2 \bullet b_2) \leftarrow c \\ &= (a \leftarrow b) \leftarrow c. \end{aligned}$$

Moreover, we have $a \leftarrow 1 = T(a_1 \rightharpoonup 1) \bullet a_2 \bullet 1 = T(\epsilon(a_1)1) \bullet a_2 = a$, thus \leftarrow is a right H^\bullet -action. By Remark 3.19 we already know that Δ is right H^\bullet -linear with respect to \leftarrow , i.e. $\Delta(a \leftarrow b) = (a_1 \leftarrow b_1) \otimes (a_2 \leftarrow b_2)$; that H is a right H^\bullet -module coalgebra, and that (\star) is satisfied. We already know that (MP.1) is satisfied. Moreover, (MP.2) also holds:

$$(19) \quad 1 \leftarrow a = T(1 \rightharpoonup a_1) \bullet 1 \bullet a_2 = T(a_1) \bullet a_2 = \epsilon(a)1.$$

In order to show that $(H, \rightharpoonup, \leftarrow)$ is a matched pair of actions on H , it remains to prove that (MP.3) and (MP.4) are satisfied. We thereby compute

$$\begin{aligned} (a_1 \rightharpoonup b_1) \bullet \left((a_2 \leftarrow b_2) \rightharpoonup c\right) &= (a_1 \rightharpoonup b_1) \bullet \left((T(a_2 \rightharpoonup b_2) \bullet a_3 \bullet b_3) \rightharpoonup c\right) \\ &= (a_1 \rightharpoonup b_1) \bullet \left(T(a_2 \rightharpoonup b_2) \rightharpoonup (a_3 \bullet b_3 \rightharpoonup c)\right) \\ &= (a_1 \rightharpoonup b_1) \cdot (a_2 \bullet b_2 \rightharpoonup c) \\ &= (a_1 \rightharpoonup b_1) \cdot (a_2 \rightharpoonup (b_2 \rightharpoonup c)) \end{aligned}$$

$$\begin{aligned} &\stackrel{(\dagger)}{=} a \rightarrow (b_1 \cdot (b_2 \rightarrow c)) \\ &= a \rightarrow (b \bullet c), \end{aligned}$$

where (\dagger) follows since $(H, \cdot, 1)$ is a left H^\bullet -module algebra. Finally, one has

$$\begin{aligned} (a \leftarrow (b_1 \rightarrow c_1)) \bullet (b_2 \leftarrow c_2) &= T(a_1 \rightarrow (b_1 \rightarrow c_1)) \bullet a_2 \bullet (b_2 \rightarrow c_2) \bullet (b_3 \leftarrow c_3) \\ &\stackrel{(\star)}{=} T(a_1 \rightarrow (b_1 \rightarrow c_1)) \bullet a_2 \bullet b_2 \bullet c_2 \\ &= T(a_1 \bullet b_1 \rightarrow c_1) \bullet a_2 \bullet b_2 \bullet c_2 \\ &= (a \bullet b) \leftarrow c. \end{aligned}$$

□

The previous construction is converse to the one of Theorem 3.11.

Theorem 3.25. *The correspondence established in Theorem 3.11 and Theorem 3.24 is an isomorphism between $\mathcal{YDBr}(\mathbf{Vec}_{\mathbb{k}})$ and $\mathbf{MP}(\mathbf{Vec}_{\mathbb{k}})$.*

Proof. Since the constructions of Theorem 3.11 and of Theorem 3.24 are converse to each other, if we prove that these constructions are functorial then they automatically yield an isomorphism of categories.

Let $F: \mathbf{MP}(\mathbf{Vec}_{\mathbb{k}}) \rightarrow \mathcal{YDBr}(\mathbf{Vec}_{\mathbb{k}})$ be the assignment defined in Theorem 3.11. We define the assignment on morphisms as follows: if $f: (H, \rightarrow_H, \leftarrow_H) \rightarrow (K, \rightarrow_K, \leftarrow_K)$ is a morphism in $\mathbf{MP}(\mathbf{Vec}_{\mathbb{k}})$, the corresponding morphism in $\mathcal{YDBr}(\mathbf{Vec}_{\mathbb{k}})$ is $f: H \rightarrow K$ read as a morphism of Hopf algebras. Since $m_{\cdot_H} = m_{\bullet_H}(\mathrm{Id}_H \otimes \rightarrow_H)(\mathrm{Id}_H \otimes T_H \otimes \mathrm{Id}_H)(\Delta_H \otimes \mathrm{Id}_H)$ (and a similar formula holds for m_{\cdot_K}), one has

$$\begin{aligned} m_{\cdot_K}(f \otimes f) &= m_{\bullet_K}(\mathrm{Id}_K \otimes \rightarrow_K)(\mathrm{Id}_K \otimes T_K \otimes \mathrm{Id}_K)(\Delta_K \otimes \mathrm{Id}_K)(f \otimes f) \\ &= m_{\bullet_K}(\mathrm{Id}_K \otimes \rightarrow_K)(\mathrm{Id}_K \otimes T_K \otimes \mathrm{Id}_K)(f \otimes f \otimes f)(\Delta_H \otimes \mathrm{Id}_H) \\ &= m_{\bullet_K}(\mathrm{Id}_K \otimes \rightarrow_K)(f \otimes f \otimes f)(\mathrm{Id}_H \otimes T_H \otimes \mathrm{Id}_H)(\Delta_H \otimes \mathrm{Id}_H) \\ &= m_{\bullet_K}(f \otimes f)(\mathrm{Id}_H \otimes \rightarrow_H)(\mathrm{Id}_H \otimes T_H \otimes \mathrm{Id}_H)(\Delta_H \otimes \mathrm{Id}_H) \\ &= f m_{\bullet_H}(\mathrm{Id}_H \otimes \rightarrow_H)(\mathrm{Id}_H \otimes T_H \otimes \mathrm{Id}_H)(\Delta_H \otimes \mathrm{Id}_H) \\ &= f m_{\cdot_H}, \end{aligned}$$

thus f is indeed a morphism in $\mathcal{YDBr}(\mathbf{Vec}_{\mathbb{k}})$. Moreover, it is clear that F sends the identity morphism into the identity morphism. Finally, let $f: (H, \rightarrow_H, \leftarrow_H) \rightarrow (K, \rightarrow_K, \leftarrow_K)$ and $g: (K, \rightarrow_K, \leftarrow_K) \rightarrow (L, \rightarrow_L, \leftarrow_L)$ be morphisms in $\mathbf{MP}(\mathbf{Vec}_{\mathbb{k}})$, then F sends the composition gf in $\mathbf{MP}(\mathbf{Vec}_{\mathbb{k}})$ into the set-theoretic composition of maps gf , which is the composition of morphisms in $\mathcal{YDBr}(\mathbf{Vec}_{\mathbb{k}})$.

Analogously, one proves that the assignment $G: \mathcal{YDBr}(\mathbf{Vec}_{\mathbb{k}}) \rightarrow \mathbf{MP}(\mathbf{Vec}_{\mathbb{k}})$ defined in Theorem 3.24 is a functor. □

4. YETTER–DRINFELD BRACES AS 1-COCYCLES

It is known (see Angiono, Galindo and Vendramin [5, Theorem 1.12], and Fernández Vilaboa and González Rodríguez [15, Theorem 2.7]) that a Hopf brace in a braided monoidal category \mathcal{M} is tantamount to a 1-cocycle of bialgebras in \mathcal{M} . We are going to recover an analogous characterisation for Yetter–Drinfeld braces.

Definition 4.1. Let $(H, \bullet, 1, \Delta, \epsilon, T)$ be a Hopf algebra, $(A, \cdot_A, 1, \Delta, \epsilon)$ be a bialgebra in ${}^H_H\mathcal{YD}$, and let the action of H on A be denoted by \rightarrow_A . A *Yetter–Drinfeld 1-cocycle* is an isomorphism of coalgebras $\pi: H \rightarrow A$ satisfying the *1-cocycle condition*

$$(1C) \quad \pi(a \bullet b) = \pi(a_1) \cdot_A (a_2 \rightarrow_A \pi(b))$$

and satisfying the following conditions for all $a, b \in H$:

$$(20) \quad (a_1 \rightarrow_A \pi T(a_2)) \cdot_A \pi(a_3) = \epsilon(a)1_A;$$

$$(21) \quad \begin{aligned} & T\pi^{-1}(a_2 \rightarrow_A \pi(b_2)) \bullet a_3 \bullet b_3 \otimes \pi^{-1}(a_1 \rightarrow_A \pi(b_1)) \\ & = T\pi^{-1}(a_1 \rightarrow_A \pi(b_1)) \bullet a_2 \bullet b_2 \otimes \pi^{-1}(a_3 \rightarrow_A \pi(b_3)). \end{aligned}$$

Given two Yetter–Drinfeld 1-cocycles $\pi: H \rightarrow A$ and $\varpi: K \rightarrow B$, a *morphism of Yetter–Drinfeld 1-cocycles* $\pi \rightarrow \varpi$ is a pair (f, g) , where $f: H \rightarrow K$ is a morphism of Hopf algebras, and $g: A \rightarrow B$ is a morphism of algebras and coalgebras satisfying

$$g \rightarrow_A = \rightarrow_B(f \otimes g), \quad g\pi = \varpi f.$$

We denote by $\mathcal{YD1C}(\text{Vec}_{\mathbb{k}})$ the category of Yetter–Drinfeld 1-cocycles thus obtained.

Remark 4.2. Observe that, choosing $a = b = 1_H$ in (1C), we obtain $\pi(1_H) = \pi(1_H) \cdot_A \pi(1_H)$. Moreover, taking $a = 1_H$ in (20), we obtain $\pi(1_H) \cdot_A \pi(1_H) = 1_A$. Hence

$$(22) \quad \pi(1_H) = 1_A$$

is automatically satisfied. The same is true for 1-cocycles of Hopf algebras [5].

Theorem 4.3. *Let $H^\bullet := (H, \bullet, 1, \Delta, \epsilon, T)$ be a Hopf algebra. The following data are equivalent:*

- i. a coalgebra A and an isomorphism of coalgebras $\pi: H \rightarrow A$, such that A with the coaction $\rho := (\text{Id} \otimes \pi)\text{Ad}_L\pi^{-1}$ is in $\text{Bimon}(\overset{H^\bullet}{H}, \mathcal{YD})$, and π is a Yetter–Drinfeld 1-cocycle;*
- ii. a second operation \cdot and a map S , providing a Yetter–Drinfeld brace structure on H .*

This defines an equivalence of categories between the subcategory of $\mathcal{YD1C}(\text{Vec}_{\mathbb{k}})$ given by Yetter–Drinfeld 1-cocycles $\pi: H \rightarrow A$ where A has coaction $(\text{Id} \otimes \pi)\text{Ad}_L\pi^{-1}$ and $\mathcal{YDBr}(\text{Vec}_{\mathbb{k}})$.

Proof. The proof is fundamentally the same as for Hopf braces [5, Theorem 1.12].

i. to ii. Suppose given a Yetter–Drinfeld 1-cocycle $\pi: H \rightarrow A$. Define the second structure H^\cdot on H by pulling back the multiplication \cdot_A of A :

$$a \cdot b := \pi^{-1}(\pi(a) \cdot_A \pi(b)).$$

This is clearly associative and, from (22), it is immediate that H^\cdot is unitary with unit 1_H . Define $S(a) := \pi^{-1}(a_1 \rightarrow_A \pi T(a_2))$, and consequently let

$$\begin{aligned} a \rightarrow b &:= S(a_1) \cdot (a_2 \bullet b) \\ &= \pi^{-1}(\pi S(a_1) \cdot_A \pi(a_2 \bullet b)) \\ &\stackrel{(1C)}{=} \pi^{-1}((a_1 \rightarrow_A \pi T(a_2)) \cdot_A \pi(a_3) \cdot_A (a_4 \rightarrow_A \pi(b))) \\ &\stackrel{(20)}{=} \pi^{-1}(a \rightarrow_A \pi(b)). \end{aligned}$$

In particular S satisfies $S(a) = a_1 \rightarrow T(a_2)$, which is (2). Since \rightarrow is just the action \rightarrow_A pulled back through π , this is clearly a left action of H^\bullet on itself. By assumption, A is in $\text{Bimon}(\overset{H^\bullet}{H}, \mathcal{YD})$ with coaction $\rho(a) = (\text{Id} \otimes \pi)\text{Ad}_L\pi^{-1}(a) = \pi^{-1}(a_1) \bullet T(\pi^{-1}(a_3)) \otimes a_2$, hence ρ and \rightarrow satisfy the compatibility condition of a Yetter–Drinfeld module:

$$\rho(a \rightarrow_A b) = a_1 \bullet \pi^{-1}(b_1) \bullet T(\pi^{-1}(b_3)) \bullet T(a_3) \otimes (a_2 \rightarrow_A b_2),$$

moreover \cdot_A and Δ satisfy the bialgebra compatibility in $\overset{H^\bullet}{H}, \mathcal{YD}$:

$$(23) \quad \Delta(a \cdot_A b) = a_1 \cdot_A (\pi^{-1}(a_2) \bullet T(\pi^{-1}(a_4)) \rightarrow_A b_1) \otimes (a_3 \cdot_A b_2).$$

It easily follows that $(H^\cdot, \rightarrow, \text{Ad}_L)$ is in $\overset{H^\bullet}{H}, \mathcal{YD}$, and that m_\cdot, u_\cdot, Δ and ϵ are left linear with respect to \rightarrow .

We show that S is a convolution inverse of Id_H with respect to the algebra structure H^\cdot :

$$\begin{aligned} a_1 \cdot S(a_2) &= a_1 \cdot \pi^{-1}(a_2 \rightarrow_A \pi T(a_3)) \\ &= \pi^{-1}(\pi(a_1) \cdot_A (a_2 \rightarrow_A \pi T(a_3))) \\ &= \pi^{-1}\pi(a_1 \bullet T(a_2)) \\ &= \epsilon(a)1_H, \end{aligned}$$

and

$$\begin{aligned}
S(a_1) \cdot a_2 &= \pi^{-1}(a_1 \rightarrow_A \pi T(a_2)) \cdot a_3 \\
&= \pi^{-1}((a_1 \rightarrow_A \pi T(a_2)) \cdot_A \pi(a_3)) \\
&\stackrel{(20)}{=} \pi^{-1}(\epsilon(a)1_A) \\
&= \epsilon(a)1_H.
\end{aligned}$$

Let $a \leftarrow b := T(a_1 \rightarrow b_1) \bullet a_2 \bullet b_2 = T\pi^{-1}(a_1 \rightarrow_A \pi(b_1)) \bullet a_2 \bullet b_2$. Notice that (5) is automatically satisfied. We verify that \leftarrow satisfies (9):

$$\begin{aligned}
(a_1 \leftarrow b_1) \otimes (a_2 \leftarrow b_2) &= T\pi^{-1}(a_1 \rightarrow_A \pi(b_1)) \bullet a_2 \bullet b_2 \otimes T\pi^{-1}(a_3 \rightarrow_A \pi(b_3)) \bullet a_4 \bullet b_4 \\
&\stackrel{(21)}{=} T\pi^{-1}(a_2 \rightarrow_A \pi(b_2)) \bullet a_3 \bullet b_3 \otimes T\pi^{-1}(a_1 \rightarrow_A \pi(b_1)) \bullet a_4 \bullet b_4 \\
&= (a \leftarrow b)_1 \otimes (a \leftarrow b)_2.
\end{aligned}$$

Moreover, since $\epsilon(a \rightarrow b) = \epsilon(a)\epsilon(b)$, one clearly has $\epsilon(a \leftarrow b) = \epsilon(a)\epsilon(b)$, hence \leftarrow is a morphism of coalgebras: by Remark 3.19, this is equivalent to requiring (MP.5). Furthermore, $(a_1 \rightarrow b_1) \bullet (a_2 \leftarrow b_2) = a \bullet b$ holds by definition. By Lemma 3.6, we have that (12) is satisfied. Moreover, one has

$$\begin{aligned}
\pi((a_1 \bullet b) \cdot S(a_2) \cdot (a_3 \bullet c)) &= \pi(a_1 \bullet b) \cdot_A \pi(S(a_2)) \cdot_A \pi(a_3 \bullet c) \\
&\stackrel{(1c)}{=} \pi(a_1) \cdot_A (a_2 \rightarrow \pi(b)) \cdot_A \pi(S(a_3)) \cdot_A \pi(a_4) \cdot_A (a_5 \rightarrow \pi(c)) \\
&= \pi(a_1) \cdot_A (a_2 \rightarrow \pi(b)) \cdot_A (a_3 \rightarrow \pi(c)) \\
&= \pi(a_1) \cdot_A (a_2 \rightarrow (\pi(b) \cdot_A \pi(c))) \\
&= \pi(a_1) \cdot_A (a_2 \rightarrow \pi(b \cdot c)) \\
&\stackrel{(1c)}{=} \pi(a \bullet (b \cdot c)),
\end{aligned}$$

and this implies (HBC) because π is bijective.

In order to obtain that $(H \cdot, \rightarrow, \text{Ad}_L)$ is in $\text{Bimon}(\frac{H}{H} \bullet \mathcal{YD})$, it remains to prove that Δ is a morphism of algebras with respect to \cdot in $\frac{H}{H} \bullet \mathcal{YD}$, and that m is left colinear with respect to Ad_L . Both easily follow from the analogous properties of A .

Finally, by Proposition 3.10, we obtain that S is an antipode for $H \cdot$. Thus, $(H, \cdot, \bullet, 1, \Delta, \epsilon, S, T)$ is a Yetter–Drinfeld brace.

ii. to i. Suppose given a Yetter–Drinfeld brace $(H, \cdot, \bullet, 1, \Delta, \epsilon, S, T)$. Define $A := (H, \cdot, 1, \Delta, \epsilon)$. This is in $\text{Bimon}(\frac{H}{H} \bullet \mathcal{YD})$ with action $a \rightarrow b = S(a_1) \cdot (a_2 \bullet b)$, and the coaction Ad_L . Set $\pi := \text{Id}_H: H \bullet \rightarrow H \cdot$, which is clearly a coalgebra isomorphism. Moreover, we compute

$$\pi(a \bullet b) = a \bullet b = a_1 \cdot S(a_2) \cdot (a_3 \bullet b) = a_1 \cdot (a_2 \rightarrow b) = \pi(a_1) \cdot (a_2 \rightarrow \pi(b)),$$

while (20) and (21) are automatically satisfied. Thus $\pi: H \bullet \rightarrow H \cdot$ is a Yetter–Drinfeld 1-cocycle.

Functoriality. We verify that the above correspondence is functorial. Let $\pi: H \rightarrow A$, $\varpi: K \rightarrow B$ be Yetter–Drinfeld 1-cocycles, with a morphism $\pi \rightarrow \varpi$ in $\mathcal{YD1C}(\text{Vec}_{\mathbb{k}})$ given by maps $f: H \rightarrow K$, $g: A \rightarrow B$. Then $f: H \bullet \rightarrow K \bullet$ is a morphism of Hopf algebras. Moreover,

$$\begin{aligned}
f(a \cdot_H b) &= f\pi^{-1}(\pi(a) \cdot_A \pi(b)) \\
&= \varpi^{-1}g(\pi(a) \cdot_A \pi(b)) \\
&= \varpi^{-1}(g\pi(a) \cdot_B g\pi(b)) \\
&= \varpi^{-1}(\varpi f(a) \cdot_B \varpi f(b)) \\
&= f(a) \cdot_K f(b)
\end{aligned}$$

and

$$\begin{aligned}
S_K f(a) &= \varpi^{-1}(f(a)_1 \rightarrow_B \varpi T_K(f(a)_2)) \\
&= \varpi^{-1}(f(a_1) \rightarrow_B \varpi T_K(f(a_2)))
\end{aligned}$$

$$\begin{aligned}
 &= \varpi^{-1}(f(a_1) \rightarrow_B \varpi f T_H(a_2)) \\
 &= \varpi^{-1}(f(a_1) \rightarrow_B g \pi T_H(a_2)) \\
 &= \varpi^{-1}g(a_1 \rightarrow_A \pi T_H(a_2)) \\
 &= f \pi^{-1}(a_1 \rightarrow_A \pi T_H(a_2)) \\
 &= f(a_1 \rightarrow_H T_H(a_2)) \\
 &= f S_H(a),
 \end{aligned}$$

thus f is a morphism in $\mathcal{YDBr}(\mathbf{Vec}_{\mathbb{k}})$. The above construction clearly sends identity morphisms into identity morphisms, and respects the composition of maps. We denote by F the functor $\mathcal{YD1C}(\mathbf{Vec}_{\mathbb{k}}) \rightarrow \mathcal{YDBr}(\mathbf{Vec}_{\mathbb{k}})$ thus obtained.

Conversely, a functor $G: \mathcal{YDBr}(\mathbf{Vec}_{\mathbb{k}}) \rightarrow \mathcal{YD1C}(\mathbf{Vec}_{\mathbb{k}})$ is defined by the following action on the maps: if $f: H \rightarrow K$ is a morphism of Yetter–Drinfeld braces, let Gf be the pair of maps (f, f) , where the former is interpreted as a map $f: H^\bullet \rightarrow K^\bullet$, and the latter as a map $f: H^\cdot \rightarrow K^\cdot$. We check that this is a morphism in $\mathcal{YD1C}(\mathbf{Vec}_{\mathbb{k}})$ between $\text{Id}_H: H^\bullet \rightarrow H^\cdot$ and $\text{Id}_K: K^\bullet \rightarrow K^\cdot$.

We already know that $f: H^\bullet \rightarrow K^\bullet$ is a morphism of Hopf algebras and $f: H^\cdot \rightarrow K^\cdot$ is a morphism of algebras and coalgebras. Moreover, $f(a \rightarrow_H b) = f(a) \rightarrow_H f(b)$ holds true because f is a morphism in $\mathcal{YDBr}(\mathbf{Vec}_{\mathbb{k}})$. The assignment G clearly respects identities and compositions.

Equivalence. We finally check that this is an equivalence of categories. It is clear that FG is the identity functor. Given a Yetter–Drinfeld 1-cocycle $\pi: H^\bullet \rightarrow A$, one has that $GF(\pi)$ is the Yetter–Drinfeld 1-cocycle $\text{Id}: H^\bullet \rightarrow H^\cdot$, which is isomorphic to $\pi: H^\bullet \rightarrow A$ via the morphism (Id_H, π^{-1}) . It is easy to verify that this is indeed a morphism in $\mathcal{YD1C}(\mathbf{Vec}_{\mathbb{k}})$, and its inverse is clearly (Id_H, π) . \square

5. COQUASITRIANGULAR HOPF ALGEBRAS AND YETTER–DRINFELD BRACES

In this section, we shall prove that coquasitriangular bialgebras H yield matched pairs of actions $(H, \rightarrow, \leftarrow)$ satisfying the hypotheses of Theorem 3.11, and hence enable us to construct concrete examples of Yetter–Drinfeld braces. The additional structure, here, coincides with Majid’s transmutation of H [28].

5.1. Coquasitriangular bialgebras. The notion of *quasitriangular bialgebra* was introduced by Drinfeld [11], to relax the hypothesis of cocommutativity. Here we shall operate with the dual notion, which appeared in several works in between 1990 and 1993, such as in Schauenburg [36], Doi [10], Larson and Towber [23], and Majid [27].

Definition 5.1. A *coquasitriangular bialgebra*, denoted by $(H, m, u, \Delta, \epsilon, \mathcal{R})$ or simply by (H, \mathcal{R}) , is the datum of a bialgebra $(H, m, u, \Delta, \epsilon)$ and of a convolution-invertible morphism $\mathcal{R}: H \otimes H \rightarrow \mathbb{k}$, satisfying:

$$\begin{aligned}
 \text{(COQT.1)} \quad & \mathcal{R}(a_1 \otimes b_1) a_2 b_2 = b_1 a_1 \mathcal{R}(a_2 \otimes b_2); \\
 \text{(COQT.2)} \quad & \mathcal{R}(a \otimes bc) = \mathcal{R}(a_1 \otimes c) \mathcal{R}(a_2 \otimes b); \\
 \text{(COQT.3)} \quad & \mathcal{R}(ab \otimes c) = \mathcal{R}(a \otimes c_1) \mathcal{R}(b \otimes c_2).
 \end{aligned}$$

Moreover, (H, \mathcal{R}) is called *cotriangular* if $\mathcal{R}^{-1} = \mathcal{R}^{\text{op}}$, where \mathcal{R}^{-1} denotes the convolution inverse of \mathcal{R} .

If \mathcal{R} provides a coquasitriangular structure, then the convolution inverse \mathcal{R}^{-1} satisfies:

$$\begin{aligned}
 \text{(COQT.1')} \quad & a_1 b_1 \mathcal{R}^{-1}(a_2 \otimes b_2) = \mathcal{R}^{-1}(a_1 \otimes b_1) b_2 a_2; \\
 \text{(COQT.2')} \quad & \mathcal{R}^{-1}(a \otimes bc) = \mathcal{R}^{-1}(a_1 \otimes b) \mathcal{R}^{-1}(a_2 \otimes c); \\
 \text{(COQT.3')} \quad & \mathcal{R}^{-1}(ab \otimes c) = \mathcal{R}^{-1}(b \otimes c_1) \mathcal{R}^{-1}(a \otimes c_2).
 \end{aligned}$$

Let \mathcal{R} be a coquasitriangular structure on a bialgebra H . Then, it is known that \mathcal{R} and its convolution inverse \mathcal{R}^{-1} satisfy the *quantum Yang–Baxter equation* (see [30, Lemma 2.2.3]):

$$(YBE) \quad \mathcal{R}^{\pm 1}(a_1 \otimes b_1)\mathcal{R}^{\pm 1}(a_2 \otimes c_1)\mathcal{R}^{\pm 1}(b_2 \otimes c_2) = \mathcal{R}^{\pm 1}(b_1 \otimes c_1)\mathcal{R}^{\pm 1}(a_1 \otimes c_2)\mathcal{R}^{\pm 1}(a_2 \otimes b_2).$$

Moreover, if H is a Hopf algebra with antipode S , then the following relations hold true:

$$\mathcal{R}(S(a) \otimes b) = \mathcal{R}^{-1}(a \otimes b), \quad \mathcal{R}^{-1}(a \otimes S(b)) = \mathcal{R}(a \otimes b), \quad \mathcal{R}^{\pm 1}(S(a) \otimes S(b)) = \mathcal{R}(a \otimes b),$$

see [30, Lemma 2.2.2]. It is well known that a bialgebra H is coquasitriangular if and only if the category \mathfrak{M}^H of right H -comodules is braided (see [30, Exercise 9.2.9] and the considerations following [30, Theorem 9.2.4]). Indeed, given a coquasitriangular structure \mathcal{R} , the bialgebra H becomes a braided object with the braiding operator

$$\sigma_{H,H}^{\mathcal{R}}(a \otimes b) := b_1 \otimes a_1 \mathcal{R}(a_2 \otimes b_2),$$

and a braiding on \mathfrak{M}^H is defined for all X, Y in \mathfrak{M}^H by setting

$$(24) \quad \sigma_{X,Y}^{\mathcal{R}} : X \otimes Y \rightarrow Y \otimes X, \quad x \otimes y \mapsto y_0 \otimes x_0 \mathcal{R}(x_1 \otimes y_1).$$

Moreover, H is cotriangular if and only if the category \mathfrak{M}^H is symmetric, i.e. $(\sigma_{X,Y}^{\mathcal{R}})^{-1} = \sigma_{Y,X}^{\mathcal{R}}$ for all X, Y in \mathfrak{M}^H .

In the rest of this section, however, we shall use another braiding operator on H . Let (H, \mathcal{R}) be a coquasitriangular bialgebra, then the map $\sigma_{H,H} : H \otimes H \rightarrow H \otimes H$ defined by

$$(25) \quad \sigma_{H,H} : H \otimes H \rightarrow H \otimes H, \quad a \otimes b \mapsto \mathcal{R}^{-1}(a_1 \otimes b_1)b_2 \otimes a_2 \mathcal{R}(a_3 \otimes b_3),$$

is known to be a braiding operator on H , with inverse given by

$$\sigma_{H,H}^{-1} : H \otimes H \rightarrow H \otimes H, \quad a \otimes b \mapsto \mathcal{R}(b_1 \otimes a_1)b_2 \otimes a_2 \mathcal{R}^{-1}(b_3 \otimes a_3).$$

Moreover, $m_H \sigma_{H,H} = m_H$ holds.

Remark 5.2. The braiding operator (25) extends to a braiding

$$x \otimes y \mapsto \mathcal{R}^{-1}(x_{-1} \otimes y_{-1})y_0 \otimes x_0 \mathcal{R}(x_1 \otimes y_1)$$

on the category of *bicomodules* ${}^H\mathfrak{M}^H$. However, the braidings on $({}^H\mathfrak{M}^H, \otimes, \mathbb{k})$ are not yet classified, and the problem is seemingly hard. When the category of *bimodules* ${}_H\mathfrak{M}_H$ is endowed with the tensor product \otimes_H instead of $\otimes_{\mathbb{k}}$, however, a classification was found by Agore, Caenepeel and Militaru [2], which could be dualised for the category ${}^H\mathfrak{M}^H$.

Remark 5.3. Let $\sigma_{H,H}$ be the braiding operator defined in (25). Notice that

$$\sigma_{H,H}^2 = \mathcal{R}^{-1} * (\mathcal{R}^{-1})^{\text{op}} * \text{Id}_{H \otimes H} * \mathcal{R}^{\text{op}} * \mathcal{R} = (\mathcal{R}^{\text{op}} * \mathcal{R})^{-1} * \text{Id}_{H \otimes H} * (\mathcal{R}^{\text{op}} * \mathcal{R}),$$

hence we have that $\sigma_{H,H}^2 = \text{Id}_{H \otimes H}$ if and only if $\text{Id}_{H \otimes H} * (\mathcal{R}^{\text{op}} * \mathcal{R}) = (\mathcal{R}^{\text{op}} * \mathcal{R}) * \text{Id}_{H \otimes H}$. The element $\mathcal{R}^{\text{op}} * \mathcal{R}$ is also known as the *quantum Killing form*. In particular, if $\mathcal{R}^{\text{op}} = \mathcal{R}^{-1}$, one clearly has $\sigma_{H,H}^{-1} = \sigma_{H,H}$.

5.2. Coquasitriangular Hopf algebras and matched pairs of actions. Suppose given a coquasitriangular Hopf algebra $(H, \bullet, 1, \Delta, \epsilon, T, \mathcal{R})$. We shall see that the braiding operator (25) yields a matched pair satisfying (\star) , and hence a Yetter–Drinfeld brace. The product and the antipode of the corresponding Hopf algebra structure in ${}^H_H\mathcal{YD}$ coincide with those obtained by *transmutation* of H , introduced by Majid [28]; thus providing a Yetter–Drinfeld brace-theoretic interpretation of the theory of transmutation. At the end of this section, we shall also observe that the same construction cannot be carried out with the braiding (24).

Let $\sigma_{H,H}$ be defined as in (25), and define

$$(26) \quad a \rightharpoonup b = (\text{Id} \otimes \epsilon)\sigma_{H,H}(a \otimes b) = \mathcal{R}^{-1}(a_1 \otimes b_1)b_2 \mathcal{R}(a_2 \otimes b_3);$$

$$(27) \quad a \leftharpoonup b = (\epsilon \otimes \text{Id})\sigma_{H,H}(a \otimes b) = \mathcal{R}^{-1}(a_1 \otimes b_1)a_2 \mathcal{R}(a_3 \otimes b_2).$$

These are a left and a right action respectively, by Lemma 2.7, and H becomes a left and right module coalgebra with the respective actions. Moreover, these actions provide a matched pair (H, H) , see [30, Example 7.2.7]. Observe that the braiding (25) is retrieved as

$$\sigma_{H,H}(a \otimes b) = (a_1 \rightharpoonup b_1) \otimes (a_2 \leftarrow b_2).$$

Hence the fact that $(H, \rightharpoonup, \leftarrow)$ satisfies (MP.1)–(MP.4) can also be obtained using Lemma 2.3 and Lemma 2.4, while (\star) is immediate.

Remark 5.4. The bialgebra $H \bowtie H$ described in §2, when the matched pair is derived as in (26) and (27), is denoted in Majid [30] by $H \bowtie_{\mathcal{R}} H$. This has product given by

$$(a \otimes h)(b \otimes g) = \mathcal{R}^{-1}(h_1 \otimes b_1)a \bullet b_2 \otimes h_2 \bullet g\mathcal{R}(h_3 \otimes b_3).$$

Moreover, we have

$$a \leftarrow b = \mathcal{R}^{-1}(a_1 \otimes b_1)a_2\mathcal{R}(a_3 \otimes b_2) = \mathcal{R}(T(a_1) \otimes b_1)a_2\mathcal{R}(a_3 \otimes b_2) = a_2\mathcal{R}(T(a_1) \bullet a_3 \otimes b),$$

so this situation falls under the hypotheses of [30, Lemma 7.4.8] where, in our case, the coaction is given by $\text{Ad}_R: a \mapsto a_2 \otimes T(a_1) \bullet a_3$. Thus [30, Lemma 7.4.8] ensures us that H is also a right H -module with respect to \leftarrow . Hence, using \leftarrow and Ad_R one can construct the bialgebra $H \bowtie \underline{H}$ (we borrow the notation from [30]), where \underline{H} is the (right) transmutation of H [30, Theorem 7.4.1] (see also [30, Example 9.4.10]), and the symbol \bowtie denotes the bialgebra with smash product algebra structure, and smash coproduct coalgebra structure. So $H \bowtie \underline{H}$ is exactly the bosonisation $H \# \underline{H}$, built using the right action and the right coaction. By [30, Theorem 7.4.10], one also obtains an isomorphism $H \bowtie_{\mathcal{R}} H \cong H \bowtie \underline{H}$. In Remark 5.7, we shall relate H with a *left* version \overline{H} of Majid's transmutation.

Notice that

$$a \rightharpoonup b = \mathcal{R}^{-1}(a_1 \otimes b_1)b_2\mathcal{R}(a_2 \otimes b_3) = \mathcal{R}^{-1}(a_1 \otimes b_1)b_2\mathcal{R}^{-1}(a_2 \otimes T(b_3)) = b_2\mathcal{R}^{-1}(a \otimes b_1 \bullet T(b_3)).$$

In this case, (1) takes the following form:

$$(28) \quad a \cdot b = a_1 \bullet b_2\mathcal{R}^{-1}(T(a_2) \otimes (b_1 \bullet T(b_3))),$$

and also

$$(29) \quad \begin{aligned} a \cdot b &= \mathcal{R}^{-1}(T(a_3) \otimes b_1)a_1 \bullet b_2\mathcal{R}^{-1}(T(a_2) \otimes T(b_3)) \\ &= \mathcal{R}^{-1}(T(a_3) \otimes b_1)a_1 \bullet b_2\mathcal{R}^{-1}(a_2 \otimes b_3) \\ &\stackrel{(\text{coqr.1}')}{=} \mathcal{R}^{-1}(T(a_3) \otimes b_1)\mathcal{R}^{-1}(a_1 \otimes b_2)b_3 \bullet a_2 \\ &= \mathcal{R}^{-1}((a_1 \bullet T(a_3)) \otimes b_1)b_2 \bullet a_2, \end{aligned}$$

while (2) has the following form:

$$(30) \quad \begin{aligned} S(a) &= a_1 \rightharpoonup T(a_2) = T(a_3)\mathcal{R}^{-1}(a_1 \otimes T(T(a_2) \bullet a_4)) \\ &= T(a_3)\mathcal{R}(a_1 \otimes T(a_2) \bullet a_4) \\ &= T(a_4)\mathcal{R}(a_1 \otimes a_5)\mathcal{R}(a_2 \otimes T(a_3)). \end{aligned}$$

Remark 5.5. When H is cocommutative, \rightharpoonup is trivial, and hence \cdot coincides with \bullet . Thus the corresponding Yetter–Drinfeld brace is just the *trivial Hopf brace* (H, \bullet, \bullet) (the terminology is again adapted from [18]).

Theorem 5.6. *Let $(H, \bullet, 1, \Delta, \epsilon, T, \mathcal{R})$ be a coquasitriangular Hopf algebra. Define \rightharpoonup and \leftarrow as above, and define \cdot and S accordingly. Then, $(H, \cdot, \bullet, 1, \Delta, \epsilon, S, T)$ is a Yetter–Drinfeld brace. Furthermore, if \mathcal{R} is cotriangular, then \cdot is braided-commutative with respect to the braiding of ${}^H_H\mathcal{YD}$, whence also $S^2 = \text{Id}_H$.*

Proof. We already know that $(H, \rightharpoonup, \leftarrow)$ satisfies (MP.1)–(MP.4) [30, Example 7.2.7]. Moreover, it follows from (COQT.1) that H is braided-commutative, i.e. $m_\bullet \sigma_{H,H} = m_\bullet$, which translates into condition (\star) . Corollary 3.18 yields that $(H, \cdot, \bullet, 1, \Delta, \epsilon, S, T)$ is a Yetter–Drinfeld brace.

Suppose now that \mathcal{R} is cotriangular. The braided commutativity of \cdot with respect to the braiding of ${}^H_H\mathcal{YD}$ is expressed by the condition $a \cdot b = (a_1 \bullet T(a_3) \rightharpoonup b) \cdot a_2$, which in view of (29) is rewritten as

$$b_3 \bullet a_2 \mathcal{R}^{-1}(T(a_3) \otimes b_1) \mathcal{R}^{-1}(a_1 \otimes b_2) = b_2 \cdot a_3 \mathcal{R}^{-1}((a_1 \bullet T(a_5)) \otimes b_1) \mathcal{R}((a_2 \bullet T(a_4)) \otimes b_3).$$

Suppose $\mathcal{R}^{-1} = \mathcal{R}^{\text{op}}$. Then

$$\begin{aligned} & b_2 \cdot a_3 \mathcal{R}^{-1}((a_1 \bullet T(a_5)) \otimes b_1) \mathcal{R}((a_2 \bullet T(a_4)) \otimes b_3) \\ &= b_3 \bullet a_4 \mathcal{R}^{-1}(T(a_7) \otimes b_1) \mathcal{R}^{-1}(a_1 \otimes b_2) \mathcal{R}^{-1}(T(b_5) \otimes a_3) \\ & \quad \mathcal{R}^{-1}(b_4 \otimes a_5) \mathcal{R}(a_2 \otimes b_6) \mathcal{R}(T(a_6) \otimes b_7) \\ &= b_3 \bullet a_4 \mathcal{R}^{-1}(T(a_7) \otimes b_1) \mathcal{R}^{-1}(a_1 \otimes b_2) \mathcal{R}(a_3 \otimes T(b_5)) \\ & \quad \mathcal{R}(a_5 \otimes b_4) \mathcal{R}(a_2 \otimes b_6) \mathcal{R}(T(a_6) \otimes b_7) \\ &= b_3 \bullet a_2 \mathcal{R}^{-1}(T(a_5) \otimes b_1) \mathcal{R}^{-1}(a_1 \otimes b_2) \mathcal{R}(a_3 \otimes b_4) \mathcal{R}(T(a_4) \otimes b_5) \\ &= b_3 \bullet a_2 \mathcal{R}^{-1}(T(a_3) \otimes b_1) \mathcal{R}^{-1}(a_1 \otimes b_2), \end{aligned}$$

as desired. Given a Hopf algebra $(H, \cdot, 1, \Delta, \epsilon, S)$ in a braided monoidal category \mathcal{M} , it is known that, if \cdot is braided-commutative in \mathcal{M} , then S is involutive. \square

The Hopf algebra $\text{SL}_q(2)$ with $q = -1$ will be an example of a coquasitriangular Hopf algebra with involutive antipode S , which is not cotriangular.

Remark 5.7. Let us show how the Hopf algebra $(H, \cdot, 1, \Delta, \epsilon, S)$ is related to the transmutation of the Hopf algebra $(H, \bullet, 1, \Delta, \epsilon, T)$. For more details about transmutation theory we redirect the reader to Majid [28, 30]. Let $(H, \bullet, 1, \Delta, \epsilon, T, \mathcal{R})$ be a coquasitriangular Hopf algebra, so that \mathcal{R}^{-1} induces a braiding on the category ${}^H\mathfrak{M}$. It is possible to define a second product operation $\bar{\bullet}$ on H , and a second antipode \bar{T} , such that they provide, on H , a Hopf algebra structure \bar{H} in ${}^H\mathfrak{M}$ (with same comultiplication Δ and counit ϵ), where the comodule structure is given by Ad_L . This is called the *transmutation* of H ; see [30, dual of Example 9.4.10]. The transmuted product and the transmuted antipode are, respectively,

$$\begin{aligned} a \bar{\bullet} b &:= \mathcal{R}^{-1}(T(a_2) \otimes (b_1 \bullet T(b_3))) a_1 \bullet b_2, \\ \bar{T}(a) &:= \mathcal{R}^{-1}(a_1 \otimes T(T(a_2) \bullet a_4)) T(a_3). \end{aligned}$$

By (28) we have $a \bar{\bullet} b = a \cdot b$ and by (30) we have $\bar{T}(a) = S(a)$, so $(H, \cdot, 1, \Delta, \epsilon, S)$ coincides with the transmutation of $(H, \bullet, 1, \Delta, \epsilon, T)$. Hence the transmutation of coquasitriangular Hopf algebras gives us examples of Yetter–Drinfeld braces. Theorem 3.11 implies, in particular, that \bar{H} lies in $\text{Hopf}({}^H_{\bar{H}}\mathcal{YD})$, with action given by \rightharpoonup . Observe from [41, duals of Theorem 4.8 and Corollary 4.9] (with $f = \text{Id}_H \bullet$) that there is a braided monoidal equivalence between ${}_{H \cdot}({}^{H \bullet}\mathfrak{M})$ and ${}^H_{\bar{H} \bullet}\mathcal{YD}$, hence one can also recover an object in $\text{Hopf}({}_{H \cdot}({}^{H \bullet}\mathfrak{M}))$.

Remark 5.8. We know from Counterexample 2.8 that the following two actions do not form a matched pair:

$$a \rightharpoonup b = \mathcal{R}(a \otimes b_2) b_1, \quad a \leftarrow b = a_1 \mathcal{R}(a_2 \otimes b).$$

We may define \cdot as in (1):

$$a \cdot b := a_1 \bullet (T(a_2) \rightharpoonup b) = a_1 \bullet \mathcal{R}(T(a_2) \otimes b_2) b_1 = a_1 \bullet b_1 \mathcal{R}^{-1}(a_2 \otimes b_2).$$

With this product, $(H, \cdot, 1)$ is an algebra. But now observe that, since $\epsilon(a \cdot b) = \mathcal{R}^{-1}(a \otimes b)$, ϵ is not a morphism of algebras with respect to \cdot , unless $\mathcal{R}^{-1} = \epsilon \otimes \epsilon$ and hence $\mathcal{R} = \epsilon \otimes \epsilon$: in this case, by (COQT.1), H is commutative.

In the case of cocommutative Hopf braces, it is known (see [5, Corollary 2.5]) that the additional operation (1) is commutative if and only if the braiding operator $\sigma(a \otimes b) := (a_1 \rightharpoonup b_1) \otimes (a_2 \leftarrow b_2)$ is involutive. An analogous result for Yetter–Drinfeld braces may possibly hold, substituting commutativity with *braided commutativity* in ${}^H_H\mathcal{YD}$. However, we attempted mimicking the proof of [5, Theorem 2.3] in several straightforward ways, and all of them failed; thus leaving the following problem open:

Problem 5.9. Let $(H, \rightharpoonup, \leftarrow)$ be a matched pair of actions on a Hopf algebra, and let \cdot be defined as in (1). Is it true that σ is involutive if and only if \cdot is braided-commutative?

Remark 5.10. The well-known FRT construction [14] allows one to construct a coquasitriangular bialgebra $A(c)$ from a solution $c \in \text{End}_{\mathbb{k}}(V \otimes V)$ of the *braid equation* $(c \otimes \text{Id})(\text{Id} \otimes c)(c \otimes \text{Id}) = (\text{Id} \otimes c)(c \otimes \text{Id})(\text{Id} \otimes c)$. The bialgebra $A(c)$ is spanned by indeterminates t_j^i (see the construction as presented by Kassel [21]). If we adjoin to $A(c)$ inverses \bar{t} to the variables, one can canonically extend the relations, the comultiplication and the counit to \bar{t} , and define $St := \bar{t}$, obtaining a Hopf algebra $GL(c)$. Suppose now that \mathcal{R} extends so that $GL(c)$ is coquasitriangular (this happens in many concrete examples, see [30, § 4.2]). Using the coquasitriangular Hopf algebra $(GL(c), \mathcal{R})$ we can obtain a Yetter–Drinfeld brace, where \cdot is defined as in (29), and S as in (30).

6. EXAMPLES

In this section, we apply our results to known examples of coquasitriangular Hopf algebras; namely the Sweedler’s Hopf algebra, the algebras $E(n)$, the Hopf algebra $\text{SL}_q(2)$, and an example in the class of Suzuki algebras. For each example, we describe the corresponding Yetter–Drinfeld brace. Throughout this section, our notation changes from “brace-theoretic” $(H, \cdot, \bullet, 1, \Delta, \epsilon, S, T)$ to “transmutation-theoretic” $(H, \overline{\cdot}, \cdot, 1, \Delta, \epsilon, \overline{S}, S)$, in order to match the way in which the objects in the examples are usually presented.

6.1. The Sweedler’s Hopf algebra. Let us suppose $\text{char}(\mathbb{k}) \neq 2$. The Sweedler’s Hopf algebra $H_4 = E(1)$ is the free \mathbb{k} -algebra generated by two elements g and x modulo the relations

$$g^2 = 1, \quad x^2 = 0, \quad \text{and } xg = -gx,$$

where 1 denotes the unit of H_4 . It becomes a Hopf algebra with comultiplication, counit and antipode given by $\Delta(g) = g \otimes g$, $\Delta(x) = x \otimes 1 + g \otimes x$, $\epsilon(g) = 1$, $\epsilon(x) = 0$, $S(g) = g$ and $S(x) = xg$, respectively. In particular, H_4 is neither commutative nor cocommutative. We recall that H_4 has a one-parameter family of cotriangular structures given by

$$\mathcal{R} \begin{pmatrix} 1 \otimes 1 & 1 \otimes g & 1 \otimes x & 1 \otimes xg \\ g \otimes 1 & g \otimes g & g \otimes x & g \otimes xg \\ x \otimes 1 & x \otimes g & x \otimes x & x \otimes xg \\ xg \otimes 1 & xg \otimes g & xg \otimes x & xg \otimes xg \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & k & -k \\ 0 & 0 & k & k \end{pmatrix},$$

where k is a parameter in \mathbb{k} (see [30, Example 2.2.6]). The convolution inverse of \mathcal{R} is $\mathcal{R}^{-1} = \mathcal{R}^{\text{op}}$ given by the transpose matrix

$$\mathcal{R}^{-1} \begin{pmatrix} 1 \otimes 1 & 1 \otimes g & 1 \otimes x & 1 \otimes xg \\ g \otimes 1 & g \otimes g & g \otimes x & g \otimes xg \\ x \otimes 1 & x \otimes g & x \otimes x & x \otimes xg \\ xg \otimes 1 & xg \otimes g & xg \otimes x & xg \otimes xg \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & k & k \\ 0 & 0 & -k & k \end{pmatrix}.$$

Given $\alpha := a_1 1 + a_2 g + a_3 x + a_4 xg$ in H_4 , one has

$$\begin{aligned} (\Delta \otimes \text{Id})\Delta(\alpha) &= a_1 1 \otimes 1 \otimes 1 + a_2 g \otimes g \otimes g + a_3 x \otimes 1 \otimes 1 + a_3 g \otimes x \otimes 1 \\ &\quad + a_3 g \otimes g \otimes x + a_4 xg \otimes g \otimes g + a_4 1 \otimes xg \otimes g + a_4 1 \otimes 1 \otimes xg. \end{aligned}$$

Hence, given $\beta := b_1 1 + b_2 g + b_3 x + b_4 xg$ in H_4 , one has

$$\begin{aligned} \alpha \rightharpoonup \beta &= (a_1 b_1 + a_2 b_1 + k(a_3 b_3 - a_3 b_4 - a_4 b_3 + a_4 b_4)) 1 \\ &\quad + (a_1 b_2 + a_2 b_2 + k(a_3 b_4 - a_3 b_3 - a_4 b_4 + a_4 b_3)) g \end{aligned}$$

\rightarrow	1	g	x	xg
1	1	g	x	xg
g	1	g	$-x$	$-xg$
x	0	0	$k1 - kg$	$kg - k1$
xg	0	0	$kg - k1$	$k1 - kg$

\leftarrow	1	g	x	xg
1	1	1	0	0
g	g	g	0	0
x	x	$-x$	$k1 - kg$	$k1 - kg$
xg	xg	$-xg$	$k1 - kg$	$k1 - kg$

(A) Left action.
(B) Right action.

TABLE 1. Left and right actions for H_4 .

$\bar{\cdot}$	1	g	x	xg
1	1	g	x	xg
g	g	1	xg	x
x	x	xg	$k1 - kg$	$kg - k1$
xg	xg	x	$kg - k1$	$k1 - kg$

TABLE 2. Multiplication table for $\bar{\cdot}$ in H_4 .

$$+ (a_1b_3 - a_2b_3)x + (a_1b_4 - a_2b_4)xg.$$

Analogously, for \leftarrow one has:

$$\begin{aligned} \alpha \leftarrow \beta &= (a_1b_1 + a_1b_2 + a_3b_3k + a_3b_4k + a_4b_3k + a_4b_4k)1 \\ &+ (a_2b_1 + a_2b_2 - a_3b_3k - a_3b_4k - a_4b_3k - a_4b_4k)g \\ &+ (a_3b_1 - a_3b_2)x + (a_4b_1 - a_4b_2)xg. \end{aligned}$$

The two actions are summarised in Table 1. We denote by $\bar{\cdot}$ the transmuted multiplicative structure on H_4 , defined according to (29). This is explicitly given by

$$\begin{aligned} \alpha \bar{\cdot} \beta &= \alpha_1(S(\alpha_2) \rightarrow \beta) \\ &= a_1(S(1) \rightarrow \beta) + a_2g(S(g) \rightarrow \beta) + a_3x(S(1) \rightarrow \beta) \\ &\quad + a_3g(S(x) \rightarrow \beta) + a_4xg(S(g) \rightarrow \beta) + a_4(S(xg) \rightarrow \beta) \\ &= a_1\beta + a_2g(b_11 + b_2g - b_3x - b_4xg) + a_3x\beta + a_3g((kb_4 - kb_3)1 + (kb_3 - kb_4)g) \\ &\quad + a_4xg(b_11 + b_2g - b_3x - b_4xg) + a_4((kb_4 - kb_3)1 + (kb_3 - kb_4)g) \\ &= (a_1b_1 + a_2b_2 + ka_3b_3 - ka_3b_4 + ka_4b_4 - ka_4b_3)1 \\ &\quad + (a_1b_2 + a_2b_1 + ka_3b_4 - ka_3b_3 + ka_4b_3 - ka_4b_4)g \\ &\quad + (a_1b_3 + a_2b_4 + a_3b_1 + a_4b_2)x \\ &\quad + (a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1)xg. \end{aligned}$$

The multiplication $\bar{\cdot}$ is described in Table 2. The algebra $(H_4, \bar{\cdot}, 1)$ is generated by g, x modulo the relations $g \bar{\cdot} g = 1$, $x \bar{\cdot} x = k1 - kg$, $x \bar{\cdot} g = g \bar{\cdot} x$. The new antipode \bar{S} is defined by $\bar{S}(\alpha) := \alpha_1 \rightarrow S(\alpha_2)$ as in (30), whence $\bar{S}(1) = 1$, $\bar{S}(g) = g$, $\bar{S}(x) = -xg = -x \bar{\cdot} g$, and $\bar{S}(xg) = \bar{S}(x \bar{\cdot} g) = \bar{S}(g) \bar{\cdot} \bar{S}(x) = -x$. Notice that $\bar{\cdot}$ is braided-commutative and \bar{S} is involutive, as we know from Theorem 5.6. Remark 5.3 implies that the braiding $\sigma_{H_4, H_4}: \alpha \otimes \beta \mapsto (\alpha_1 \rightarrow \beta_1) \otimes (\alpha_2 \leftarrow \beta_2)$ is involutive.

Observe that $\bar{\cdot}$ is also commutative, even if $\sigma_{H_4, H_4}^{\mathcal{YD}}$ does not coincide with the flip map τ (for instance, it is easy to see that $\sigma_{H_4, H_4}^{\mathcal{YD}}$ and τ differ on $x \otimes x$). Notice moreover that $\bar{\cdot}$ is compatible with Δ in the braided sense, but not in the classical sense: indeed $(x \bar{\cdot} x)_1 \otimes (x \bar{\cdot} x)_2 = (k1 - kg) \otimes (k1 - kg) \neq (k1 - kg) \otimes 1 + 2xg \otimes g + 1 \otimes (k1 - kg) = (x_1 \bar{\cdot} x_1) \otimes (x_2 \bar{\cdot} x_2)$. In particular, this Yetter–Drinfeld brace is not a Hopf brace.

Remark 6.1. Matched pairs of bialgebras (H_4, H_4) on the Sweedler’s Hopf algebra are classified by Bontea [6, Theorem 2.1]. Other than the matched pairs arising from cotriangular structures,

$\bar{\cdot}$	1	g	x_j	$x_j g$
1	1	g	x_j	$x_j g$
g	g	1	$x_j g$	x_j
x_i	x_i	$x_i g$	$A_{ij}1 - A_{ij}g + x_i x_j$	$A_{ij}g - A_{ij}1 + x_i x_j g$
$x_i g$	$x_i g$	x_i	$A_{ij}g - A_{ij}1 + x_i x_j g$	$A_{ij}1 - A_{ij}g + x_i x_j$

 TABLE 3. Multiplication table for $\bar{\cdot}$ in $E(n)$.

\rightarrow	1	g	x_j	$x_j g$
1	1	g	x_j	$x_j g$
g	1	g	$-x_j$	$-x_j g$
x_i	0	0	$A_{ij}1 - A_{ij}g$	$A_{ij}g - A_{ij}1$
$x_i g$	0	0	$A_{ij}g - A_{ij}1$	$A_{ij}1 - A_{ij}g$

 TABLE 4. Table for the action \rightarrow in $E(n)$.

already considered above, there is only one more pair of actions on H_4 satisfying (MP.1)–(MP.5): this is the pair provided by the trivial actions $a \otimes b \mapsto \epsilon(a)b$, $a \otimes b \mapsto a\epsilon(b)$. Since H_4 is not commutative, the pair with the trivial actions does not satisfy (\star) , and hence is not a matched pair of actions.

We have, as a byproduct of Remark 6.1, the following result:

Proposition 6.2. *The only matched pair of actions $(H_4, \rightarrow, \leftarrow)$ on the Sweedler’s Hopf algebra H_4 is the matched pair defined in Table 1.*

6.2. The Hopf algebras $E(n)$. Let us suppose $\text{char}(\mathbb{k}) \neq 2$. A generalisation of the Sweedler’s Hopf algebra is provided by the Hopf algebras $E(n)$ generated as algebras by g, x_i for $i = 1, \dots, n$, with relations

$$g^2 = 1, \quad x_i^2 = 0, \quad x_i g = -g x_i, \quad x_i x_j = -x_j x_i, \quad \text{for all } i = 1, \dots, n,$$

endowed with the comultiplication $\Delta(1) = 1 \otimes 1$, $\Delta(g) = g \otimes g$, $\Delta(x_i) = x_i \otimes 1 + g \otimes x_i$, the counit $\epsilon(g) = 1$, $\epsilon(x_i) = 0$, and the antipode $S(g) = g$, $S(x_i) = x_i g$. The quasitriangular structures on $E(n)$ are classified by Panaite and Van Oystaeyen [34], and the coquasitriangular structures are obtained by duality (see e.g. Carnovale and Cuadra [9, p. 256]):

$$\mathcal{R}_A \begin{pmatrix} 1 \otimes 1 & 1 \otimes g & 1 \otimes x_j & 1 \otimes x_j g \\ g \otimes 1 & g \otimes g & g \otimes x_j & g \otimes x_j g \\ x_i \otimes 1 & x_i \otimes g & x_i \otimes x_j & x_i \otimes x_j g \\ x_i g \otimes 1 & x_i g \otimes g & x_i g \otimes x_j & x_i g \otimes x_j g \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & A_{ij} & -A_{ij} \\ 0 & 0 & A_{ij} & A_{ij} \end{pmatrix},$$

where $A = (A_{ij})_{ij}$ is an $n \times n$ matrix with entries in \mathbb{k} . It is easy to see that \mathcal{R}_A is cotriangular if and only if A is symmetric. Panaite and Van Oystaeyen [34] conjectured that \mathcal{R}_A is triangular if and only if A is symmetric; which was later proven by Carnovale and Cuadra [9, Proposition 2.1].

Assume $A_{ij} = A_{ji}$, thus $\mathcal{R}^{-1} = \mathcal{R}^{\text{op}}$. We define a second multiplicative structure $\bar{\cdot}$ as in (29). This is described in Table 3 (in particular, it is commutative). In terms of generators and relation, $(E(n), \bar{\cdot}, 1)$ is the algebra generated by g, x_i with relations $g \bar{\cdot} g = 1$, $x_i \bar{\cdot} x_j + x_j \bar{\cdot} x_i = 2A_{ij}(1 - g)$, $x_i \bar{\cdot} g = g \bar{\cdot} x_i$. The action \rightarrow is described in Table 4. The antipode \bar{S} , defined as in (30), acts as follows: $\bar{S}(1) = 1$, $\bar{S}(g) = g$, $\bar{S}(x_i) = -x_i g = -x_i \bar{\cdot} g$.

6.3. The Hopf algebra $\text{SL}_q(2)$. Let $q \in \mathbb{C}^\times$, and consider, following Kassel [21, § IV.6], the \mathbb{C} -algebra $\text{SL}_q(2)$ generated by a, b, c, d modulo the following relations:

$$\begin{aligned} ba &= qab, & ca &= qac, & db &= qbd, \\ dc &= qcd, & bc &= cb, & ad - da &= (q^{-1} - q)bc, \end{aligned}$$

$\bar{\cdot}$	a	b	c	d
a	a^2	qab	$q^{-1}ac$	ad
b	$q^{-1}ab$	$q^{-1}b^2$	$(q^2 - 1)a^2 + (1 - q^2)ad + qbc$	$(q^3 - q)ab + bd$
c	qac	qbc	$q^{-1}c^2$	cd
d	ad	bd	$(q^3 - q)ac + cd$	d^2

TABLE 5. Multiplication table for $\bar{\cdot}$ in $\text{SL}_q(2)$.

\rightarrow	a	b	c	d
a	a	$q^{-1}b$	qc	d
b	$(1 - q^{-2})b$	0	$(q - q^{-1})d + (q^{-1} - q)a$	$(1 - q^2)b$
c	0	0	0	0
d	a	qb	$q^{-1}c$	d

TABLE 6. Table for the action \rightarrow in $\text{SL}_q(2)$.

$$da - qbc = 1.$$

This is known to be a Hopf algebra with

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix},$$

and, when $q \neq 1$, it is neither commutative nor cocommutative. The Hopf algebra $\text{SL}_q(2)$ is coquasitriangular [21, Corollary VIII.7.2], with

$$\mathcal{R} \begin{pmatrix} a \otimes a & b \otimes b & a \otimes b & b \otimes a \\ c \otimes c & d \otimes d & c \otimes d & d \otimes c \\ a \otimes c & b \otimes d & a \otimes d & b \otimes c \\ c \otimes a & d \otimes b & c \otimes b & d \otimes a \end{pmatrix} = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 1 & q - q^{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{R}^{-1} \begin{pmatrix} a \otimes a & b \otimes b & a \otimes b & b \otimes a \\ c \otimes c & d \otimes d & c \otimes d & d \otimes c \\ a \otimes c & b \otimes d & a \otimes d & b \otimes c \\ c \otimes a & d \otimes b & c \otimes b & d \otimes a \end{pmatrix} = q^{1/2} \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 1 & q^{-1} - q \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Notice that $\mathcal{R}^{-1} = \mathcal{R}^{\text{op}}$ if and only if $q^{1/2} = q^{-1/2}$, if and only if $q = 1$. In this case $\mathcal{R} = \mathcal{R}^{-1} = \mathcal{R}^{\text{op}}$, and the previous matrices are the identities. The explicit form of the iterated Δ is here reported:

$$\begin{aligned} (\Delta \otimes \text{Id})\Delta(a) &= a \otimes a \otimes a + a \otimes b \otimes c + b \otimes c \otimes a + b \otimes d \otimes c; \\ (\Delta \otimes \text{Id})\Delta(b) &= a \otimes a \otimes b + a \otimes b \otimes d + b \otimes c \otimes b + b \otimes d \otimes d; \\ (\Delta \otimes \text{Id})\Delta(c) &= c \otimes a \otimes a + c \otimes b \otimes c + d \otimes c \otimes a + d \otimes d \otimes c; \\ (\Delta \otimes \text{Id})\Delta(d) &= c \otimes a \otimes b + c \otimes b \otimes d + d \otimes c \otimes b + d \otimes d \otimes d. \end{aligned}$$

We define an additional operation $\bar{\cdot}$ on $\text{SL}_q(2)$ as in (29). Using the iterated Δ , we compute Table 5 for the multiplication $\bar{\cdot}$, and Table 6 for the left action \rightarrow . Hence we have that $\bar{\cdot}$ is commutative if and only if $q = q^{-1}$, i.e., if and only if $q = \pm 1$. In particular, if $q = -1$, $\bar{\cdot}$ is commutative but \mathcal{R} is not cotriangular. The algebra $(H, \bar{\cdot}, 1)$ is generated by a, b, c, d modulo the relations

$$\begin{aligned} c\bar{\cdot}a &= q^2a\bar{\cdot}c, & b\bar{\cdot}d - d\bar{\cdot}b &= (q^2 - 1)a\bar{\cdot}b, \\ a\bar{\cdot}b &= q^2b\bar{\cdot}a, & b\bar{\cdot}c - c\bar{\cdot}b &= (q^2 - 1)a\bar{\cdot}a - (q^2 - 1)a\bar{\cdot}d, \\ a\bar{\cdot}d &= d\bar{\cdot}a, & d\bar{\cdot}c - c\bar{\cdot}d &= (q^2 - 1)c\bar{\cdot}a, \\ & & a\bar{\cdot}d - q^{-2}c\bar{\cdot}b &= 1. \end{aligned}$$

We now define the transmuted antipode $\bar{S}(x) := x_1 \rightarrow S(x_2)$, as in (30). One has

$$\bar{S}(a) = q^{-2}d + (1 - q^{-2})a, \quad \bar{S}(b) = -q^{-2}b,$$

$\bar{\cdot}$	a	b	c	d
a	$\alpha^{-2}\beta^{-2}ad$	0	0	$\alpha^{-2}\beta^{-2}a^2$
b	0	$\alpha^{-2}\beta^{-2}bc$	$\alpha^{-3}\beta^{-1}b^2$	0
c	0	$\alpha^{-3}\beta^{-1}b^2$	$\alpha^{-1}\beta^{-3}\lambda bc$	0
d	$\alpha^{-2}\beta^{-2}a^2$	0	0	$\alpha^{-2}\beta^{-2}ad$

 TABLE 7. Multiplication table for $\bar{\cdot}$ in $A_{1,2}^{\nu,\lambda}$.

\rightarrow	a	b	c	d
a	d	$\alpha^{-1}\beta c$	$\alpha\beta^{-1}b$	a
b	0	0	0	0
c	0	0	0	0
d	d	$\alpha\beta^{-1}c$	$\alpha^{-1}\beta b$	a

 TABLE 8. Table for the action \rightarrow in $A_{1,2}^{\nu,\lambda}$.

$$\overline{S}(c) = -q^{-2}c, \quad \overline{S}(d) = a.$$

Notice that \overline{S} is involutive if and only if $q = \pm 1$. Hence, if $q = -1$, \overline{S} is involutive but \mathcal{R} is not cotriangular.

6.4. An example in the class of Suzuki Hopf algebras. Let \mathbb{k} be an algebraically closed field of characteristic $\text{char}(\mathbb{k}) \neq 2$, as in Suzuki [38]. The *Suzuki Hopf algebra* $A_{1,2}^{\nu,\lambda}$, with parameters $\nu, \lambda \in \mathbb{k}$, is generated by a, b, c, d subject to the relations

$$a^2 = d^2, \quad b^2 = c^2, \quad a^2 + \nu b^2 = 1, \quad ab = ba = ac = ca = bd = db = cd = dc = 0, \quad cb = \lambda bc, \quad ad = da.$$

Let

$$\mathbf{t} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with \mathbf{t}_{ij} denoting the entry on the i -th row and j -th column. The comultiplication, counit and antipode are respectively given by:

$$\Delta \mathbf{t} = \mathbf{t} \otimes \mathbf{t}, \quad \epsilon(\mathbf{t}_{ij}) = \delta_{ij}, \quad S(\mathbf{t}_{ij}) = (\mathbf{t}_{ji})^3.$$

The map

$$\mathcal{R}_{\alpha,\beta} \begin{pmatrix} a \otimes a & a \otimes b & a \otimes c & a \otimes d \\ b \otimes a & b \otimes b & b \otimes c & b \otimes d \\ c \otimes a & c \otimes b & c \otimes c & c \otimes d \\ d \otimes a & d \otimes b & d \otimes c & d \otimes d \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \alpha & \beta & 0 \\ 0 & \beta & \alpha & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

defines a coquasitriangular structure whose convolution inverse is $\mathcal{R}_{(\alpha,\beta)}^{-1} = \mathcal{R}_{(\alpha^{-1},\beta^{-1})}$ (see [38, proof of Proposition 2.1]), which is cotriangular if $(\alpha, \beta) = (-1, -1)$ or $(\alpha, \beta) = (1, 1)$, and $(\nu, \lambda) = (1, 1)$ (see [38, Proposition 3.10]).

We define an algebra operation $\bar{\cdot}$ as in (29), whose multiplication table is reported in Table 7. In particular, $\bar{\cdot}$ is commutative. The algebra $(H, \bar{\cdot}, 1)$ is generated by a, b, c, d modulo the relations

$$\begin{aligned} a\bar{\cdot}a &= d\bar{\cdot}d, & c\bar{\cdot}c &= \alpha\beta^{-1}\lambda b\bar{\cdot}b, & c\bar{\cdot}b &= b\bar{\cdot}c, & a\bar{\cdot}d &= d\bar{\cdot}a, \\ a\bar{\cdot}b &= b\bar{\cdot}a = a\bar{\cdot}c = c\bar{\cdot}a = b\bar{\cdot}d = d\bar{\cdot}b = c\bar{\cdot}d = d\bar{\cdot}c = 0. \end{aligned}$$

The action \rightarrow is reported in Table 8. The corresponding antipode \overline{S} , defined as in (30), is identified by $\overline{S}(a) = \alpha^2\beta^2d^3$, $\overline{S}(b) = \alpha^3\beta b^3$, $\overline{S}(c) = \alpha^3\beta c^3$, $\overline{S}(d) = \alpha^2\beta^2a^3$.

We know by Theorem 5.6 that the additional operation (1) is braided commutative if \mathcal{R} is cotriangular. In all the previous examples, when \mathcal{R} is cotriangular (1) is also commutative. We could neither prove nor disprove that this is true in general, and thus the following problem is open:

Problem 6.3. Let (H, \mathcal{R}) be a cotriangular Hopf algebra. Is it true that the additional operation (29) is commutative?

Notice that the converse is false (see the example $SL_q(2)$ with $q = -1$).

Acknowledgements. The authors would like to thank A. Ardizzoni for meaningful suggestions throughout all phases of this work, and L. Vendramin and T. Weber for valuable comments. This paper was written while the authors were members of the “National Group for Algebraic and Geometric Structures and their Applications” (GNSAGA-INdAM). They were partially supported by the Ministry for University and Research (MUR) within the National Research Project (PRIN 2022) “Structures for Quivers, Algebras and Representations” (SQUARE). The first-named author was partially funded by the Vrije Universiteit Brussel Bench Fee for a Joint Doctoral Project, grant number OZR4257, and partially supported through the FWO Senior Research Project G004124N.

REFERENCES

- [1] AGORE, A.-L. Classifying bicrossed products of two Taft algebras. *J. Pure Appl. Algebra* 222, 4 (2018), 914–930.
- [2] AGORE, A.-L., CAENEPEEL, S., AND MILITARU, G. Braidings on the category of bimodules, Azumaya algebras and epimorphisms of rings. *Applied Categorical Structures* 22, 1 (2014), 29–42.
- [3] ANDRUSKIEWITSCH, N. On the quiver-theoretical quantum Yang–Baxter equation. *Selecta Mathematica* 11, 2 (2005), 203–246.
- [4] ANDRUSKIEWITSCH, N., AND SCHNEIDER, H.-J. Pointed Hopf algebras. In *New directions in Hopf algebras*, vol. 43 of *Math. Sci. Res. Inst. Publ.* Cambridge Univ. Press, Cambridge, 2002, pp. 1–68.
- [5] ANGIOLO, I., GALINDO, C., AND VENDRAMIN, L. Hopf braces and Yang–Baxter operators. *Proceedings of the American Mathematical Society* 145, 5 (2017), 1981–1995.
- [6] BONTEA, C.-G. Classifying bicrossed products of two Sweedler’s Hopf algebras. *Czechoslovak Math. J.* 64(139), 2 (2014), 419–431.
- [7] BRZEZIŃSKI, T. Trusses: Between braces and rings. *Transactions of the American Mathematical Society* 372, 6 (2019), 4149–4176.
- [8] CAENEPEEL, S., MILITARU, G., AND ZHU, S. *Frobenius and separable functors for generalized module categories and nonlinear equations*, vol. 1787 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002.
- [9] CARNOVALE, G., AND CUADRA, J. Cocycle Twisting of $E(n)$ -Module Algebras and Applications to the Brauer Group. *K-Theory* 33 (04 2004).
- [10] DOI, Y. Braided bialgebras and quadratic blgebras. *Communications in Algebra* 21, 5 (1993), 1731–1749.
- [11] DRINFELD, V. G. Quasi-Hopf algebras and Knizhnik-Zamolodchikov equations. In *Problems of Modern Quantum Field Theory: Invited Lectures of the Spring School held in Alushta USSR, April 24–May 5, 1989*. Springer, 1989, pp. 1–13.
- [12] DRINFELD, V. G. On some unsolved problems in quantum group theory. In *Quantum Groups: Proceedings of Workshops held in the Euler International Mathematical Institute, Leningrad, Fall 1990* (2006), Springer, pp. 1–8.
- [13] ETINGOF, P., SCHEDLER, T., AND SOLOVIEV, A. Set-theoretical solutions to the quantum Yang–Baxter equation. *Duke Mathematical Journal* 100, 2 (1999), 169.
- [14] FADDEEV, L. D., RESHETIKHIN, N. Y., AND TAKHTAJAN, L. A. Quantization of Lie groups and Lie algebras. *Algebraic analysis, Vol. I* (1988), 129–139.
- [15] FERNÁNDEZ VILABOA, J. M., GONZÁLEZ RODRÍGUEZ, R., RAMOS PÉREZ, B., AND RODRÍGUEZ RAPOSO, A. B. Modules over invertible 1-cocycles. *arXiv preprint arXiv:2311.05233* (2023).
- [16] GATEVA-IVANOVA, T. Set-theoretic solutions of the Yang–Baxter equation, braces and symmetric groups. *Advances in Mathematics* 338 (2018), 649–701.
- [17] GELAKI, S. *Topics on quasitriangular Hopf algebras*. Master’s Thesis. Ben Gurion University of the Negev, Israel, 1992.
- [18] GUARNIERI, L., AND VENDRAMIN, L. Skew braces and the Yang–Baxter equation. *Mathematics of Computation* 86, 307 (2017), 2519–2534.
- [19] HECKENBERGER, I., AND SCHNEIDER, H.-J. *Hopf algebras and root systems*, vol. 247. American Mathematical Soc., 2020.
- [20] JOYAL, A., AND STREET, R. Braided tensor categories. *Advances in Mathematics* 102, 1 (1993), 20–78.
- [21] KASSEL, C. *Quantum groups*, vol. 155. Springer Science & Business Media, 2012.
- [22] KHARCHENKO, V. K. Connected braided Hopf algebras. *Journal of Algebra* 307, 1 (2007), 24–48.
- [23] LARSON, R. G., AND TOWBER, J. Two dual classes of bialgebras related to the concepts of “quantum group” and “quantum Lie algebra”. *Communications in Algebra* 19, 12 (1991), 3295–3345.

- [24] LU, J.-H., YAN, M., AND ZHU, Y.-C. On the set-theoretical Yang–Baxter equation. *Duke Mathematical Journal* 104, 1 (2000).
- [25] MAJID, S. Physics for algebraists: Non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction. *Journal of Algebra* 130, 1 (1990), 17–64.
- [26] MAJID, S. Braided groups and algebraic quantum field theories. *Letters in Mathematical Physics* 22 (07 1991), 167–175.
- [27] MAJID, S. Braided groups. *Journal of Pure and Applied Algebra* 86, 2 (1993), 187–221.
- [28] MAJID, S. Transmutation theory and rank for quantum braided groups. In *Mathematical proceedings of the Cambridge Philosophical Society* (1993), vol. 113, Cambridge University Press, pp. 45–70.
- [29] MAJID, S. Algebras and Hopf algebras in braided categories. In *Advances in Hopf algebras*. CRC Press, 10 1995, pp. 55–105.
- [30] MAJID, S. *Foundations of quantum group theory*. Cambridge university press, 2000.
- [31] MARTIN-LYONS, I., AND TRUMAN, P. J. Skew bracoids. *Journal of Algebra* 638 (2024), 751–787.
- [32] MASUOKA, A. Cohomology and coquasi-bialgebra extensions associated to a matched pair of bialgebras. *Advances in Mathematics* 173, 2 (2003), 262–315.
- [33] MATSUMOTO, D. K. Dynamical braces and dynamical Yang–Baxter maps. *Journal of Pure and Applied Algebra* 217, 2 (2013), 195–206.
- [34] PANAITIE, F., AND VAN OYSTAEYEN, F. Quasitriangular structures for some pointed Hopf algebras of dimension $2n$. *Communications in Algebra* 27 (01 1999), 4929–4942.
- [35] RUMP, W. Braces, radical rings, and the quantum Yang–Baxter equation. *Journal of Algebra* 307, 1 (2007), 153–170.
- [36] SCHAUBENBURG, P. *On coquasitriangular Hopf algebras and the quantum Yang-Baxter equation*, vol. 67 of *Algebra Berichte [Algebra Reports]*. Verlag Reinhard Fischer, Munich, 1992.
- [37] SINGER, W. M. Extension theory for connected Hopf algebras. *Journal of Algebra* 21, 1 (1972), 1–16.
- [38] SUZUKI, S. A family of braided cosemisimple Hopf algebras of finite dimension. *Tsukuba Journal of Mathematics* 22 (06 1998).
- [39] TAKEUCHI, M. Survey of braided Hopf algebras. *Contemporary Mathematics* 267, 8 (2000), 301–323.
- [40] TAMBARA, D. The coendomorphism bialgebra of an algebra. *Journal of the Faculty of Science of the University of Tokyo, Section IA, Mathematics* 37, 2 (1990), 425–456.
- [41] ZHU, H. Relative Yetter–Drinfeld modules and comodules over braided groups. *Journal of Mathematical Physics* 56, 4 (2015), 041706, 11.

UNIVERSITY OF TURIN, DEPARTMENT OF MATHEMATICS “G. PEANO”,
 VIA CARLO ALBERTO 10, 10123 TORINO, ITALY.
 VRIJE UNIVERSITEIT BRUSSEL, DEPARTMENT OF MATHEMATICS AND DATA SCIENCE,
 PLEINLAAN 2, 1050, BRUSSELS, BELGIUM.
Email address: `d.ferri@unito.it`, `Davide.Ferri@vub.be`

UNIVERSITY OF TURIN, DEPARTMENT OF MATHEMATICS “G. PEANO”,
 VIA CARLO ALBERTO 10, 10123 TORINO, ITALY.
Email address: `andrea.sciandra@unito.it`