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Abstract

The Friedkin-Johnsen (FJ) model has been extensively validated across social science, systems and control, game theory, and algorithmic research. We introduce an advanced generalization — termed the FJ-MM model — that incorporates memory effects and multi-hop influence. This extension allows agents to naturally integrate both current and past opinions at each iteration. We analyze the stability and equilibrium properties of the FJ-MM model, demonstrating that they can be derived from those of a standard FJ model with an appropriately modified influence matrix. We examine the convergence rate of the FJ-MM model and demonstrate that, as can be expected, the time lags introduced by memory and higher-order neighbor influences result in slower convergence. Numerical results illustrate that memory and multi-hop influence reshape the final opinion landscape, e.g., by reducing polarization.

Keywords: Opinion dynamics, Social dynamics, Higher-Order Neighbors

1. Introduction

Agent-based opinion dynamics modeling is rapidly advancing, drawing researchers from social sciences, economics, physics, engineering, and other fields [1, 2, 3, 4, 5]. The most widely studied models in engineering and mathematics

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rely on iterative *opinion averaging* as the primary mechanism driving opinion formation, a concept originating from early work on social power [6]. Recent experiments confirm a tendency toward opinion averaging in small groups [7] and large-scale online communities [8]. A central element of averaging-based models is a weighted graph of social influence, which may be static or evolve. Each individual (a node in the graph) updates their opinion by computing the weighted average of the opinions of their adjacent nodes.

The Friedkin-Johnsen (FJ) model [1] is a seminal and widely studied framework for opinion formation that extends iterative averaging, yielding a range of final outcomes from consensus to multimodal, polarized states. Besides incorporating a weighted influence digraph, the FJ model assigns each agent a fixed *innate* opinion that factors into every opinion update. An extra parameter, interpreted either as social influence susceptibility [1] or as conformity under group pressure [9], regulates how strongly an agent adheres to their innate opinion. The FJ model has been studied from systems and control [10, 11], game-theoretic [12] and algorithmic [13, 14] perspectives; a number of experiments have been conducted to validate it [7, 15, 16].

In this paper, we analyze a generalization of the FJ model that incorporates hereditary effects. At each update, an agent is influenced not only by current opinions (both their own and those of others) but also by past opinions. It is important to note that our primary motivation is not to account for communication delays — a topic that has been extensively addressed in the opinion dynamics literature [17, 18]. Rather, the motivation stems from intertwined effects of *higher-order neighbors* [19], or the multi-hop influence[1] and individual memory. We refer to this model as the **FJ-MM**: the FJ model with Memory and Multi-hop influence.

In [19], the FJ model is extended to assume that agents average opinions from *long-range* connections (via walks of a given length) along with those of

[1]The term "multi-hop" [20, 21] refers to indirect communication occurring through multiple intermediary nodes connecting the source and target.

adjacent nodes in each update[2]. Though the exact mechanism of this multi-hop influence is unspecified in [19], a plausible explanation is that agents share not only their own opinions but also information from their nearest neighbors, thereby disseminating those opinions to individuals who would otherwise lack access to them. This explanation implies that secondary neighbor influence involves a time lag, which is neglected in [19]. For instance, if opinions from agent k reach agent i via the path $i \to j \to k$ at step t, then agent j must have received information about k at one of the earlier steps. Agent i at time t relies on agent j's *memory* of past interactions. Similarly, social media recommender algorithms may surface j's comment on k's old post to i, thereby inducing a time-lagged multi-hop influence [22]. Besides this, many social media platforms (such as Facebook) provide personalized prompts that direct users back to previous discussion threads, posts, and images [23].

It is thus plausible to assume that agents assign positive weights not only to current opinions but also to past opinions. Although memory effects have been explored in consensus algorithms [24], models addressing memory's impact on opinion formation are less common [25, 26, 27].

The paper is organized as follows. Section 2 introduces the FJ-MM model and notation. Section 3 presents the stability analysis of the model and illustrates the impact of memory and multi-hop influence on opinion formation outcomes. Section 4 examines the convergence rate of the FJ-MM model. Section 5 concludes the paper.

2. Model Definition and Framework

2.1. Notation

We use $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{m\times n}$ to denote real numbers, real n-dimensional (column) vectors, and real $m \times n$ matrices, respectively. The all-0 and all-1 column

[2]While the original FJ model implies chains of influence (i.e., walks in the influence graph) [1], this influence remains indirect — agents do not directly incorporate the higher-order neighbors' opinions into each iteration.

vectors are denoted by $\mathbf{0}_n$ and $\mathbf{1}_n \in \mathbb{R}^n$ respectively, and the $n \times n$ identity matrix by I_n, with dimensions omitted when clear from context. Capital letters denote matrices, and their entries are denoted by lowercase letters, e.g., $W = (w_{ij})$. For vectors and matrices, the relations $\geq, >, \leq, <$ are understood entry-wise. Given a vector $v \in \mathbb{R}^n$, the symbol $[v]$ denotes the diagonal matrix $D \in \mathbb{R}^{n\times n}$ with entries $d_{ii} = v_i$. Given an arbitrary matrix M, we denote by $\mathrm{diag}(M)$ the diagonal matrix whose diagonal entries are the same as those of M.

A matrix M is Schur stable if $\rho(M) < 1$, where $\rho(M)$ is the spectral radius, i.e., the maximum absolute value of the eigenvalues of M. The well-known Perron–Frobenius theorem states that for any nonnegative matrix $M \geq 0$, the spectral radius $\rho(M)$ is a (real) eigenvalue of M. A matrix $M \geq 0$ is (row) stochastic if $M\mathbf{1} = \mathbf{1}$, and (row) substochastic if $M\mathbf{1} \leq \mathbf{1}$. The Geršgorin disk theorem implies that $\rho(M) \leq 1$ for all substochastic matrices, and $\rho(M) = 1$ if M is stochastic.

A (weighted directed) graph is defined by $\mathcal{G}[W] = (\mathcal{V}, \mathcal{E}, W)$, where the set of nodes $\mathcal{V} = \{1, \dots, n\}$ represents the individuals (agents), the set of directed edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ indicates the presence of social influence, and $W = (w_{ij}) \geq 0$ is the weighted $n \times n$ adjacency matrix such that $w_{ij} > 0$ iff $(i, j) \in \mathcal{E}$. For matrices $W^1, W^2 \geq 0$, the graph $\mathcal{G}[W^1 + W^2]$ corresponds to the union of graphs $\mathcal{G}[W^1]$ and $\mathcal{G}[W^2]$.

2.2. The FJ and FJ-MM Models

Consider a social group where each agent $i = 1, \dots, n$ holds a state $x_i(t) \in \mathbb{R}$ at discrete time t, representing their opinion on some topic; denote the group's opinion vector by $x(t) := [x_1(t), \dots, x_n(t)]^\top$.

The FJ model [1] is defined by (a) the stochastic *influence matrix* $W = (w_{ij}) \in \mathbb{R}^{n\times n}$; (b) the diagonal *susceptibility matrix* $\Lambda \in \mathbb{R}^{n\times n}$ with entries $\lambda_{ii} \in [0, 1]$, and (c) the vector of *innate opinions* $s \in \mathbb{R}^n$. At each step, an agent's new opinion is determined by their innate opinion s_i and the weighted average of their own and others' opinions, $\bar{x}_i(t)$:

$$\begin{aligned} x_i(t+1) &= \lambda_{ii}\bar{x}_i(t) + (1-\lambda_{ii})s_i, \\ \bar{x}_i(t) &:= \sum_{j\in\mathcal{V}} w_{ij}x_j(t). \end{aligned} \tag{1}$$

The weight $w_{ij} > 0$, assigned by agent i to agent j, reflects i's appraisal of j — such as recognition of expertise or trust. The coefficient λ_{ii} represents an agent's openness to assimilating others' opinions, or their *susceptibility* to social influence: an agent with $\lambda_{ii} = 0$ is fully stubborn, anchored to their innate opinion s_i, while $\lambda_{ii} = 1$ corresponds to the classical French–DeGroot iterative averaging. In the matrix form, the state vector $x(t)$ dynamics are as follows

$$x(t+1) = \Lambda W x(t) + (I-\Lambda)s, \quad \forall t = 0, 1, \dots \tag{2}$$

In this paper, we focus on an extended version of the FJ model, where the average of opinions $\bar{x}_i(t)$, computed by each agent i, is expanded to incorporate some opinions from previous steps as follows:

$$\bar{x}_i(t) = \sum\nolimits_{j\in\mathcal{V}} \sum\nolimits_{\ell=1}^{L} w_{ij}^{(\ell)} x_j(t-\ell+1), \tag{3}$$

which leads to the matrix equation

$$x(t+1) = \Lambda \sum\nolimits_{\ell=1}^{L} W^{(\ell)} x(t-\ell+1) + (I-\Lambda)s. \tag{4}$$

Definition 1 (FJ-MM)**.** *The system* (4)*, defined by the diagonal matrix* $\mathbf{0} \leq \Lambda \leq I_n$ *and matrices* $W^{(\ell)} \geq 0$*,* $\ell = 1, \dots, L$*, whose sum* $W^{(1)} + \dots + W^{(L)}$ *is a* stochastic *matrix, is referred to as the FJ-MM model with the depth of memory* L *(the case* $L = 1$ *corresponds to the original FJ model).*

Note that the case $\Lambda = I_n$ corresponds to the French–DeGroot dynamics with memory — a model that has been studied in the context of delay robustness in consensus algorithms [28]. In this paper, we are primarily interested in the generic case where the FJ-MM dynamics is Schur stable, which is only possible when $\lambda_{ii} < 1$ for some agent i.

Remark 1 (Initial Condition vs. Innate Opinions)**.** *The initial condition of the FJ-MM model is given by the sequence* $x(-L+1), \dots, x(0)$*. In the original FJ model (*$L = 1$*), it is often assumed that* $x(0) = s$*, as the innate opinions, according to [15], retain information about the agents' past experiences and thus serve a role similar to the initial state vector* $x(0)$*. Thus, a natural choice for the initial condition is* $x(-L+1) = \ldots = x(0) = s$*. This choice, however, is not crucial as we are primarily interested in the asymptotic stability of the equilibrium, which does not depend on the initial conditions.*

It can be proven, using the induction on t, that the opinions never leave the convex hull spanned by all initial opinions $x_i(-L+1), \dots, x_i(0)$ and the innate opinions s_i of all agents $i \in \mathcal{V}$, which implies the following simple proposition.

Proposition 1. *All solutions of the FJ-MM system* (4) *are bounded.*

2.3. Main Use Cases

We illustrate the flexibility of the multiple influence weight matrices $W^{(\ell)}$ by considering several scenarios (Use Cases 1–3) that generalize the standard FJ model [1]. From now on, we adopt the following assumption for brevity and simplicity.

Assumption 1 (One-Step Memory)**.** *The FJ-MM model* (4) *has the depth of memory* $L = 2$*, being thus*

$$x(t+1) = \Lambda\big(W^{(1)}x(t) + W^{(2)}x(t-1)\big) + (I - \Lambda)s. \tag{5}$$

Eq. (5) can be rewritten in the state-space form as

$$y(t+1) = \bar{A}_d y(t) + \begin{pmatrix} 0 \\ (I-\Lambda)s \end{pmatrix}, \tag{6}$$

where, by definition, one has

$$y(t) := \begin{pmatrix} x(t-1) \\ x(t) \end{pmatrix}, \quad \bar{A}_d := \begin{pmatrix} 0 & I \\ \Lambda W^{(2)} & \Lambda W^{(1)} \end{pmatrix}.$$

We primarily deal with the influence matrices

$$W^{(1)} = (I - [\beta])W, \quad W^{(2)} = [\beta]\tilde{W}, \tag{7}$$

where W and $\tilde{W}$ are stochastic matrices, and $\beta \in [0,1]^n$ is some vector. Thus, the weighted average of the neighbors' opinions in (3) at each time t can be expressed as:

$$\bar{x}_i(t) = (1 - \beta_i) \sum_{j \in \mathcal{V}} w_{ij} x_j(t) + \beta_i \sum_{j \in \mathcal{V}} \tilde{w}_{ij} x_j(t-1),$$

where $\sum_{j \in \mathcal{V}} w_{ij} = \sum_{j \in \mathcal{V}} \tilde{w}_{ij} = 1$. The parameter $\beta_i \in [0,1]$ admits a simple interpretation: it represents the total influence weight that agent i allocates to the *past* opinions of herself and others, i.e., $\beta_i = \sum_{j \in \mathcal{V}} (\beta_i \tilde{w}_{ij})$, while the remaining weight, $1 - \beta_i$, is distributed across the *current* opinions. As in the original FJ model (corresponding to $\beta = \mathbf{0}$), the weights w_{ij} and $\tilde{w}_{ij}$ reflect the level of trust that agent i places in the current and past opinions of agent j, respectively. However, as already noted, the mechanisms by which agent i receives the current and past opinions of agent j can differ fundamentally: while current opinions are directly communicated by other individuals, past opinions may be accessible only through "rumors" spread by them or may rely on their memory. For these reasons, W and $\tilde{W}$ may correspond to entirely different graphs.

Use Case 1 (Secondary Neighbors)**.** *Our first use case is inspired by the model in [19], where agents receive opinions from both direct and secondary neighbors in the influence graph $\mathcal{G}[W]$, defined by stochastic matrix W, whereas $\tilde{W} = W^2$ in* (7)*, meaning that if agent i receives agent's k opinion via an intermediary j, the weight is proportional to $w_{ij}w_{jk}$. Summing over all such two-step paths, the total weight corresponds to (i,k) entry of the weighted walk matrix W^2.*

Unlike [19], which assumes availability of the up-to-date secondary neighbors' opinions, we assume that an opinion reaching agent i via the path $i \to j \to k$ is delayed by one step, as it reflects agent j's memory of earlier interactions with agent k.

Use Case 2 (Social Inertia)**.** *A possible explanation for the inclusion of the previous opinion vector* $x(t-1)$ *is social inertia and status quo bias [29], which leads agents to be reluctant to change their beliefs and behaviors. To accommodate this effect, one can choose* $\tilde{W} = I$ *in* (7)*.*

Use Case 3 (Recent Memory Influence)**.** *From the social psychology literature it is known that agents do not immediately forget their neighbors' previous opinion (i.e. recent memory) [30]. Consequently, we may consider* $\tilde{W} = W$*.*

Finally, we note that the general FJ-MM model can accommodate various other scenarios; for example, $\tilde{W}$ in (4) could be a convex combination of the matrices from use cases 1–3, capturing the combined effects of multi-hop influence, inertia, and memory, or either referring to Use Case 1 we may not know the weight assigned by our neighbors to their ones and thus it could be natural to assign equal weights to all secondary neighbors accessible through j. It should also be noticed that those effects do not exhaust possible explanations for the general FJ-MM dynamics. For instance, the FJ-MM model (5) also applies to cases where memory is caused by communication lag: agents receive messages from others with time lag, while their own opinions remain up-to-date. In this case, $W^{(1)} = \mathrm{diag}(W)$ and $W^{(2)} = W - \mathrm{diag}(W)$.

3. Asymptotic Stability Criterion

In this section, we focus on the system (5) and examine its asymptotic behavior, establishing conditions for the convergence to a unique equilibrium. We will use the notion of a *comparison system.*

Definition 2. *Given the FJ-MM model* (5)*, its **comparison FJ system** is defined as*

$$\begin{aligned} x(t+1) &= \bar{A}x(t) + (I-\Lambda)s, \\ \bar{A} &:= \Lambda(W^{(1)} + W^{(2)}). \end{aligned} \tag{8}$$

Remark 2. *The concept of the comparison system extends naturally to the case* $L > 2$ *by denoting* $\bar{A} := \Lambda \sum_{\ell=1}^{L} W^{(\ell)}$*. In this setting, the comparison model* (8) *generalizes the dynamical model introduced in [19]. Here the matrices $W^{(\ell)}$ can be arbitrary substochastic matrices whose sum is stochastic, and need not correspond to weighted walk matrices as in [19].*

3.1. Stability Criterion

The next theorem shows that the comparison system (8) characterizes the asymptotic stability of the FJ-MM and its unique equilibrium.

Theorem 1. *The three statements are equivalent:*

(i) the FJ-MM (5) *is exponentially stable, i.e., $\rho(\bar{A}_d) < 1$;*

(ii) the comparison FJ system (8) *is exponentially stable, that is, $\rho(\bar{A}) < 1$;*

(iii) the subset of nodes $\hat{\mathcal{V}} := \{i \in \mathcal{V} : \lambda_{ii} < 1\} \subset \mathcal{V}$ is non-empty and globally reachable in the union of graphs $\mathcal{G}[W^{(1)}]$ and $\mathcal{G}[W^{(2)}]$. This holds, in particular, if $\Lambda < I$.

If (i)-(iii) hold, then all solutions of the FJ-MM model (5) *and of the system* (8) *converge to the point*

$$\bar{x} = \left(I - \bar{A}\right)^{-1}(I - \Lambda)s. \tag{9}$$

Proof. We first rewrite the dynamical system (5) as

$$y(t) = \bar{A}_d y(t-1) + \bar{c}. \tag{10}$$

where, by definition $y(t-1) := \begin{bmatrix} x(t-1) \\ x(t) \end{bmatrix}$ and

$$\bar{A}_d := \begin{bmatrix} 0 & I \\ \Lambda W^{(2)} & \Lambda W^{(1)} \end{bmatrix}, \ \bar{c} := \begin{bmatrix} 0 \\ (I-\Lambda)s \end{bmatrix}.$$

The system (10) (equivalent to the FJ-MM) is exponentially stable if and only if the matrix $\bar{A}_d$ is Schur stable, i.e. $\rho(\bar{A}_d) < 1$. The latter spectral radius, in accordance with the Perron-Frobenius theorem, serves as the maximum real eigenvalue $\lambda \geq 0$ of $\bar{A}_d$, that is, maximum of λ such that $\bar{A}_d v = \lambda v$ with some non-zero vector $v = [v_1^\top; v_2^\top]^\top \neq 0$.

First notice that since $\bar{A}_d$ is a substochastic matrix, $\rho(\bar{A}_d) \leq 1$. Solving the equation $\bar{A}_d v = \lambda v$, we observe that $v_2 = \lambda v_1$, one arrives at the following:

$$(\Lambda W^{(2)} + \Lambda W^{(1)}\lambda)v_1 = \lambda^2 v_1. \tag{11}$$

Recall that $\bar{A}_d$ is Schur stable if and only if 1 is not an eigenvalue of $\bar{A}_d$, or in other words that (11) has no non-trivial solution $v_1 \neq 0$ when $\lambda = 1$. This last requirement (see Eq. (11)) is equivalent to the nonexistence of any nonzero vector v_1 satisfying

$$\Lambda(W^{(1)} + W^{(2)})v_1 = v_1,$$

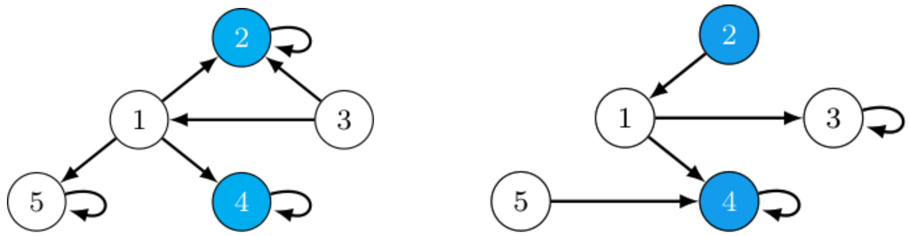
i.e. $\lambda = 1$ is not an eigenvalue of $\bar{A} = \Lambda\big(W^{(1)} + W^{(2)}\big)$. Finally, as $\bar{A}$ is also substochastic, this implies that $\rho(\bar{A}) < 1$. We have thus proven the **equivalence of (i) and (ii):** the matrices $\bar{A}_d$, which determines the stability of the FJ-MM, and $\bar{A}$, which determines the stability of the comparison model, are either both Schur stable or both have eigenvalue 1.

The **equivalence of (ii) and (iii)** is immediate from the criterion for the FJ models stability [31, 3], stating that $\rho(\Lambda\hat{W}) < 1$ for a stochastic matrix $\hat{W} = W^{(1)} + W^{(2)}$ if and only if the subset of nodes $\hat{\mathcal{V}}$ is globally reachable in the graph $\mathcal{G}[\hat{W}]$. Finally, note that a state vector x is an equilibrium of the system (5) or (8) if and only if $x = \bar{A}x + (I - \Lambda)s$, which equation has a unique solution (9) for every s whenever $(I - \bar{A})$ is an invertible matrix. □

For instance, the FJ-MM with matrices (7) is, obviously, stable if $\beta < \mathbf{1}$ and $\hat{\mathcal{V}}$ is reachable in the graph the $\mathcal{G}[W]$ or, symmetrically, $\beta > \mathbf{0}$ and the $\hat{\mathcal{V}}$ is reachable in $\mathcal{G}[\tilde{W}]$. It is easy to construct an example where neither ΛW nor $\Lambda\tilde{W}$ are Schur stable, yet the FJ-MM model with matrices in (7) is exponentially stable for $\mathbf{0} < \beta < \mathbf{1}$ as the set $\hat{\mathcal{V}}$ becomes reachable in the union of two graphs (see Fig. 1).

3.2. Numerical Examples

Next, we analyze the equilibrium achieved by the FJ-MM model with matrices in (7) and compare it with that of the original FJ model in (2) via numerical



(a) Graph associated to W (b) Graph associated to $\tilde{W}$

Figure 1: Example of two matrices $W, \tilde{W}$ with $\rho(\Lambda W) = \rho(\Lambda \tilde{W}) = 1$, whereas the FJ-MM model is Schur stable $\rho(\Lambda([\mathbf{1} - \beta]W + [\beta]\tilde{W})) < 1$ or $\mathbf{0} < \beta < \mathbf{1}$. The colored nodes correspond to $\hat{\mathcal{V}} = \{i : \lambda_{ii} < 1\}$.

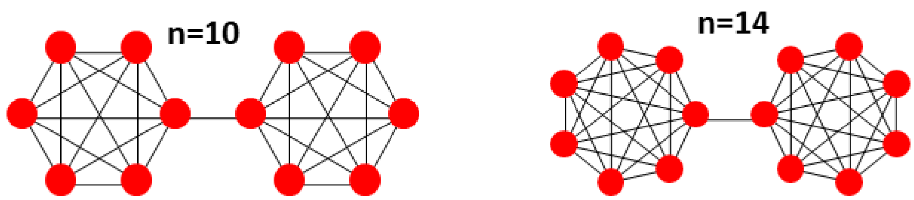


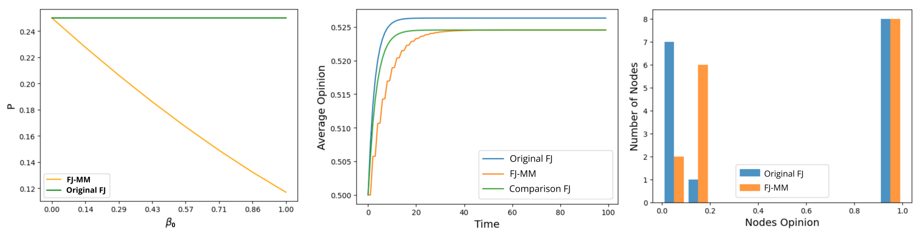
Figure 2: Examples of barbell graphs with different numbers of nodes.

simulations. In our examples the influence graph[3] is a Barbell graph, obtained by connecting two identical cliques by an edge (see Fig. 2).

As initial condition, we let the two cliques be two polarized communities, with opinion fixed to 0 and 1, respectively. All the nodes are assumed to be fully open to social influence (i.e., $\lambda_{ii} = 1$), except for several totally stubborn ($\lambda_{ii} = 0$) nodes to be explicitly specified. We consider the scenario of Use Case 1, that is, $\tilde{W} = W^2$ in (7).

A formal way to measure polarization is to measure how far we are from the state of complete neutrality. One such measure is the polarization index [32],

[3]Henceforth, for each graph specified below, we construct the matrix W under the assumption that every agent assigns equal weights to all of its neighbors. Consequently, all nonzero entries in each row of W are identical and correspond one-to-one with the arcs emanating from that node.



(a) Polarization Index (b) Average Opinion Dynamics (c) Final Opinions

Figure 3: Illustrations to (a) Example 1 and (b,c) Example 2.

defined as $P = \frac{(\bar{x}-x^*)^\top(\bar{x}-x^*)}{n}$, where $\bar{x}$ is the equilibrium state and $x^* = \bar{x}^\top \mathbf{1}/n$ is its mean value. In the next example, we compute the polarization index as a function of the rescaling parameter β. For simplicity, we assume that $[\beta] = \beta_0 I$ (the agents assign equal total weights to past opinions).

Example 1. *Consider a barbell graph with 10 nodes, initialized as previously described. Let the agents at the endpoints of the connecting edge be completely stubborn (i.e., $\lambda_{ii} = 0$) and $[\beta] = \beta_0 I$, where β_0 is changing. Figure 3a displays P as a function of β_0. It can be seen that as the influence of secondary neighbors (β_0) increases, the polarization becomes weaker.*

Example 2. *Consider a barbell graph with 16 nodes, initialized as previously described. Set $[\beta] = 0.8I$, and assign $\lambda_{ii} = 0$ to the two nodes at the endpoints of the central edge and to two randomly selected nodes (one from each clique). Fig. 3b compares the average opinion dynamics for the FJ-MM, the comparison FJ, and the original FJ model with matrix W, showing that the FJ-MM model converges more slowly than both the original FJ and the comparison models. Figure 3c shows the final opinion distribution.*

4. Convergence Rate

Here, we focus on the analysis of the convergence rate. Writing the FJ-MM model as in (10) and introducing its comparison FJ model (8), the convergence rates are of these systems are determined by the spectral radii $\rho(\bar{A}_d)$ and $\bar{\rho}(A)$

respectively. For the previously introduced Use Cases, it is also interesting to compare these convergence rates with the one of the original FJ model (2).

First, we will show that the FJ-MM system cannot converge faster than the comparison FJ model associated to it.

Proposition 2. *The convergence rate of the FJ-MM system in* (5) *does not exceed the one of the comparison model* (8)*:* $\rho(\bar{A}_d) \geq \rho(\bar{A})$.

Proof. Consider the eigenvalue-eigenvector system associated to the dynamical system in (8). Given $\rho(\bar{A})$ the spectral radius of $\bar{A}$, it exists $v \in \mathbb{R}^n$ such that $\bar{A}v = \rho(\bar{A})v$, with v eigenvector associated to maximum eigenvalue $\rho(\bar{A})$. Let us now define $x = [v; v]$. Then, it holds:

$$\bar{A}_d x = \begin{bmatrix} 0 & I \\ \Lambda W^{(2)} & \Lambda W^{(1)} \end{bmatrix} x = \begin{bmatrix} v \\ \rho(\bar{A})v \end{bmatrix} \geq \rho(\bar{A})x \tag{12}$$

From Corollary 3.2 in [33], one has $\rho(\bar{A}_d) \geq \rho(\bar{A})$. □

The result is in line with what intuitively expected as the memory term drives the node opinion back to the past, influenced by opinions at previous steps.

4.1. A Special Case: Homogeneous Susceptibility

We now focus on the homogeneous susceptibility case, where the spectral radius can be directly computed from previous results, gaining further insight into the convergence rate, especially in comparison with the original FJ model.

Assumption 2. *Assume that* $\Lambda = \sigma I$ *with* $\sigma \in \mathbb{R}$ *with*[4] $\sigma \in (0, 1)$*. Suppose also that all elements of the vector* β *are equal* $\beta_i = \beta_0 \in (0, 1)$ *for every* $i \in \mathcal{V}$.

Lemma 1. *If* $\Lambda = \sigma I$*, then for every stochastic matrix* $\hat{W}$ *one has* $\rho(\Lambda\hat{W}) = \sigma$*. In particular,* $\rho(\bar{A}) = \sigma \leq \rho(\bar{A}_d)$.

[4]The case of $\Lambda = 0$ is, obviously, degenerate, as all agents are totally stubborn and the dynamics terminate in a single step. The case $\Lambda = I$ corresponds to the DeGroot model, where there is no asymptotic stability.

Proof. The first statement is straightforward since $\rho(\sigma\hat{W}) = \sigma$. The second statement follows from Corollary 2. □

By virtue this lemma, it can be easily shown that the inequality in Proposition 2 holds strictly in several cases.

Proposition 3. *Let Assumption 2 hold and consider the FJ-MM defined such that W is some stochastic matrix and $\tilde{W}$ characterizes the joint impact of social inertia and recent memory influence, i.e. $\tilde{W} = \alpha_1 W + \alpha_2 I$ where $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$. Then the FJ-MM model converges strictly slower than the comparison FJ and the original FJ model with matrix W. More precisely,*

$$\rho(\bar{A}_d) = \frac{\sigma(1-\beta_0) + \sqrt{\sigma(1-\beta_0)^2 + 4\beta_0}}{2} > \sigma. \tag{13}$$

Proof. We first compute the spectral radius for the FJ-MM model. Recalling that the eigenvalues of $\bar{A}_d$ are such scalars λ such that $\bar{A}_d v = \lambda v$, with some vector $v = [v_1^\top; v_2^\top]^\top$, then observing that $v_2 = \lambda v_1$, one proves that

$$(\Lambda\beta W + \Lambda(1-\beta_0)(\alpha_1 W + \alpha_2 I)\lambda)v_1 = \lambda^2 v_1. \tag{14}$$

Substituting $\Lambda = \sigma I$ and using the Perron-Frobenius theorem, the spectral radius $\rho(\bar{A}_d)$ is the maximum of numbers $\lambda \geq 0$ such that

$$\begin{aligned} 0 &= \det(\lambda^2 I - \lambda\sigma(1-\beta_0)W - \sigma\beta_0(\alpha_1 W + \alpha_2 I)) \\ &= \det((\lambda^2 - \sigma\beta_0\alpha_2)I - [\lambda\sigma(1-\beta_0) + \sigma\beta_0\alpha_1]) \end{aligned} \tag{15}$$

Notice that $\lambda\sigma(1-\beta_0) + \sigma\beta_0\alpha_1 > 0$, because if it were not, then $\lambda = 0$ and $\alpha_1 = 0$, which would imply $\lambda^2 - \sigma\beta_0\alpha_2 = -\sigma\beta_0 < 0$. Consequently, λ could not be a root of equation (15). Hence, the real number

$$\mu := \frac{\lambda^2 - \sigma\beta_0\alpha_2}{\lambda\sigma(1-\beta_0) + \sigma\beta_0\alpha_1} \tag{16}$$

appears as an eigenvalue of W. Conversely, if μ is an eigenvalue of W and satisfies (16), then λ is an eigenvalue of $\bar{A}_d$. By treating (16) as a quadratic equation in λ and selecting its maximal (nonnegative) root, the spectral radius $\rho(\bar{A}_d)$ is given by

$$\rho(\bar{A}_d) = \max_{\mu} \frac{\sigma(1-\beta_0)}{2}\left(\mu + \sqrt{\mu^2 + \frac{4\beta_0(\alpha_1\mu + \alpha_2)}{\sigma(1-\beta_0)^2}}\right)$$

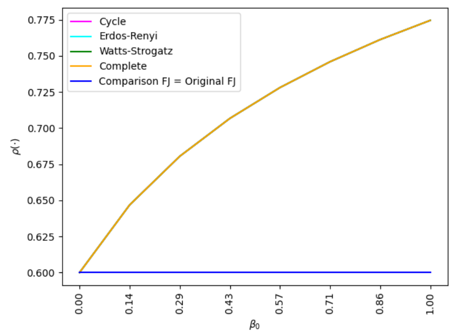


Figure 4: Spectral radii of the FJ-MM and the comparison FJ (Example 3). The convergence rate under Assumption 2 does not depend on the underlying graph.

which maximum is taken over all real μ eigenvalues of W and, obviously, is achieved at $\mu = 1$.

Recalling that $\alpha_1 + \alpha_2 = 1 > \sigma$, one arrives at

$$\begin{aligned} \rho(\bar{A}_d) &= \tfrac{\sigma(1-\beta_0)}{2}\left(1 + \sqrt{1 + \tfrac{4\beta_0(\alpha_1+\alpha_2)}{\sigma(1-\beta_0)^2}}\right) \\ &> \tfrac{\sigma(1-\beta_0)}{2}\left(1 + \sqrt{1 + \tfrac{4\beta_0}{(1-\beta_0)^2}}\right) = \sigma, \end{aligned}$$

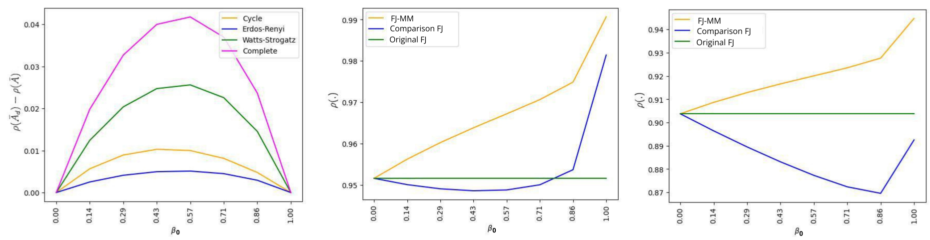
which finishes the proof of Proposition 3. □

The following numerical example illustrates that, if Assumption 2 holds, the spectral radius in Use Case 2 depends only on β_0 and σ but not on the graph.

Example 3. *Fig. 4 shows numerical simulations which compare the maximal eigenvalue of the FJ-MM model for different graph types as a function of* β_0*, assuming that* $\Lambda = 0.6I$ *and* $[\beta] = \beta_0 I$*. Technically, it considers an undirected cycle graph with* $N = 20$ *nodes, an Erdös–Rényi graph with* $N = 150$ *and probability of edge creation* 0.4*, a Watts-Strogatz graph with* $N = 200$*, degree* $0.6N$ *and rewiring probability equal to* 0.7*, and a complete graph with* $N = 50$.

It can be observed that the spectral radius is, first, independent of the graph structure and, second, strictly greater than the spectral radius of the comparison FJ model when $\beta_0 > 0$.

Similar computations allow to prove a similar estimate of the convergence rate for the Use Case 1.



(a) Example 4 (b) Example 5, $|\hat{\mathcal{V}}| = 0.15|\mathcal{V}|$ (c) Example 5, $|\hat{\mathcal{V}}| = 0.5|\mathcal{V}|$

Figure 5: Illustration to (a) Example 4 and (b,c) Example 5.

Proposition 4. *Let Assumption 2 holds and consider the FJ-MM with matrices* (7)*, where W is some stochastic matrix and $\tilde{W} = W^2$. Then, the FJ-MM converges slower both than the comparison FJ and the original FJ model. Precisely,*

$$\rho(\bar{A}_d) = \frac{\sigma(1-\beta_0) + \sqrt{\sigma^2(1-\beta_0)^2 + 4\sigma\beta_0}}{2} > \sigma. \tag{17}$$

4.2. Numerical Analysis

In general, the convergence rates of the FJ-MM and its corresponding comparison FJ system depend not only on the vector β, but also on the graphs corresponding to W and $\tilde{W}$ and on the susceptibility matrix Λ. Since an analytical characterization of these dependencies is nontrivial, we illustrate is through numerical experiments. Here, the diagonal entries of Λ are heterogeneous: nodes belonging to a randomly chosen subset $\hat{\mathcal{V}}$ are assigned $\lambda_{ii} = 0$ (indicating complete stubbornness), while the others have $\lambda_{ii} = 1$.

First, let us focus on the dependence of the convergence rate on the network choice in the setting of Use Case 2.

Example 4. *We compare the convergence rates of the FJ-MM and the comparison FJ model for different network choices under the assumption that $\tilde{W} = I$ (Social Inertia). We adopt the same graphs as in Example 3. The set of randomly chosen stubborn nodes has cardinality $|\hat{\mathcal{V}}| = 0.2|\mathcal{V}|$. In Fig. 5a we notice that the gap between the spectral radii of the FJ-MM and the comparison FJ* *model is no longer a monotonic function of* β_0 *and thus differs substantially from the homogeneous* λ_{ii} *case (Proposition 3). This outcome is expected since, when* $\beta_0 = 1$*, both models lose asymptotic stability, yielding* $\rho(\bar{A}_d) = \rho(\bar{A}) = 1$*. Unlike the situation in Example 3, the gap depends on the network's topology.*

In the case of secondary neighbors (Use Case 1), the relationship between the spectral radii and β_0 becomes even more complex and substantially depends on Λ. Specifically, while the spectral radius of the comparison model exhibits a pronounced minimum at some $\beta_0 = \beta_* \in (0, 1)$, the spectral radius of the FJ-MM model increases monotonically to 1 as $\beta_0 \to 1$. Furthermore, the gap between the two models widens as the number of stubborn agents, $|\hat{\mathcal{V}}|$, increases. This is illustrated in our final example.

Example 5 (Influence of cardinality of $\hat{\mathcal{V}}$)**.** *Consider a Watts-Strogatz graph randomly generated, with* $N = 200$ *nodes, degree* $0.6N$ *and rewiring probability* 0.7*, under the assumption that* $\tilde{W} = W^2$*. In Fig.5b-c, we compare the spectral radii of the FJ-MM, the comparison FJ and the original FJ model with matrix* W*, for the cases* 15% *of individuals are stubborn vs.* 50% *of stubborn agents .*

5. Conclusion

The generalized FJ model, called FJ-MM, integrates memory effects and higher-order neighbor influences to account for both current and past opinions through direct and secondary connections. While the convergence properties of the FJ-MM model reduce to those of a comparison model, our analysis and simulations show that including past opinions significantly impacts the convergence rate. Additionally, numerical results suggest that multi-hop influence reduces polarization in the final opinion landscape.

These findings open several avenues for future research on both the steady-state and transient properties of the FJ-MM model. The FJ model naturally gives rise to a centrality measure in influence networks [1, 3], including the PageRank as a special case. A similar centrality metric can be defined for the

FJ-MM model, raising the question of how memory effects and multi-hop interactions influence a node's centrality. Another open problem involves detecting the most informative nodes in a network — the problem addressed for the FJ model in [34]. It is worth noting that the convergence of the Friedkin–Johnsen model on time-varying networks remains underexplored; some sufficient conditions are provided in [31]. Extending this convergence analysis to the FJ–MM model poses even greater challenges. Even for classical FJ model, the relationship between the convergence rate and the influence graph is not fully explored (some results can be found in [10, 31]); a challenge that becomes even more pronounced for the FJ-MM model. While the limitation $L = 2$ can be easily relaxed, it is plausible that the effective depth of memory is time-varying or even random; e.g., social media platforms can randomly retrieve events or posts from several months or years ago. Another direction for future research is identification of the FJ-MM models, extending the results from [11].

CRediT authorship contribution statement

Roberta Raineri: Conceptualization, Methodology, Formal analysis, Investigation, Software, Validation, Data Curation, Visualization, Writing — Original Draft

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Anton Proskurnikov: Conceptualization, Methodology, Investigation, Resources, Writing — Review & Editing, Supervision, Project administration, Funding acquisition

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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