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Carleson Measures on Locally Finite Trees

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Abstract
We provide a characterization of Carleson measures on locally finite trees. This characterization establishes the connection between Carleson measures and the boundedness of a suitable Poisson integral between L^p-spaces. Additionally, when the tree has bounded degree, we investigate the relationship between Carleson measures and BMO functions defined on the boundary of the tree.

Keywords Carleson measures · Trees · Poisson integral · BMO

Mathematics Subject Classification 05C05 · 05C21 · 31C05 · 43A99

1 Introduction and Preliminaries

1.1 Introduction

Carleson measures were originally introduced and characterized in the Euclidean setting by Carleson [2, 3]. In this context, it was proved that a positive measure σ is a Carleson measure if and only if the classical Poisson integral defines a bounded operator from $L^p(\mathbb{R}^n, \mathrm{d}x)$ to $L^p(\mathbb{R}^+ \times \mathbb{R}^n, \sigma)$.

Subsequently, Fefferman and Stein established a connection between Carleson measures and functions of bounded mean oscillation on $\mathbb{R}^n$ ($BMO(\mathbb{R}^n)$). Indeed, they exhibited a suitable class of operators acting on $BMO(\mathbb{R}^n)$ such that the square of the image of a function is the density of a Carleson measure with respect to the Lebesgue measure [8]. These results have been generalized to other contexts. For instance, when $\mathbb{R}^n$ is replaced by a space of homogeneous type [11], or when $\mathbb{R}^+ \times \mathbb{R}^n$ is replaced

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by a homogeneous tree (or more in general, by a radial tree), analogous results have been obtained [4, 5].

In this note, we aim to provide a characterization of Carleson measures on locally finite trees without any further restriction on the geometry of the tree. This is the content of Theorem 2.8, which is proved in Sect. 2. It is worth mentioning that a tree is not a product space; rather, it is the image of a surjective map on the product of the boundary of the tree itself and $\mathbb{Z}$. We use a natural definition of Carleson measures involving sectors instead of cylinders, which adapts to our setting the definition given in [4–6]. We point out that, because of the lack of symmetries in our settings, we cannot exploit the techniques used in the aforementioned papers. In particular, we introduce a suitable Laplacian associated with a random walk, where the probability of transitioning from a vertex to a neighbour is nonzero only if the neighbour lies below the original vertex. This hierarchical structure on the tree arises naturally from the choice of a root, which in this paper will be a point of the boundary of the tree. We then give a Poisson integral representation formula for harmonic functions associated with this distinguished Laplacian. In Sect. 2, we introduce a Hardy space H^p containing harmonic functions, which serves as the discrete counterpart of the Hardy space $H^p(\mathbb{R}^+ \times \mathbb{R})$ on the upper half-plane. It turns out that, when $p > 1$, H^p characterizes the space of harmonic functions which are Poisson integrals of L^p functions on the boundary of the tree, see Theorem 2.5. Subsequently, Carleson measures σ are characterized as measures on the tree for which the Poisson integral maps continuously the natural L^p space on the boundary of the tree to $L^p(\sigma)$. This property is shown to be equivalent to the boundedness of the identity map from H^p to $L^p(\sigma)$.

In Sect. 3, we focus on trees with bounded degree and, in Theorem 3.3, we prove a result in the spirit of [8, 11], which relates Carleson measures to *BMO* functions on the boundary of the tree. We show that a class of integral operators whose kernels satisfy suitable cancellation and decay properties maps *BMO* functions to functions that are densities of Carleson measures with respect to a suitable reference measure. Additionally, we prove that a converse statement holds true. In fact, Theorem 3.3 provides a characterization of the *BMO* space on the boundary of a tree.

Throughout the paper, C will denote a positive constant which may vary from line to line and that is independent of any involved variable but may depend on fixed parameters. Sometimes, we will stress such a dependence by adding a subscript.

1.2 Preliminaries and Notation

In this section we introduce the notation and recall some well-known results on trees.

Let T be a tree and let d denote the usual geodesic distance on T. We fix a root of the tree by choosing a point ω_* in the boundary of T, that we denote by Ω (see [9, Chapter I.1] for a detailed definition). We then introduce the *punctured boundary* $\partial T = \Omega \setminus \{\omega_*\}$. The choice of a root induces a partial order on T: given two vertices $x, y \in T$, we say that x lies below y (or equivalently, y lies above x) if $y \in [x, \omega_*)$, where $[x, \omega_*)$ denotes the infinite geodesic starting from x and ending in ω_*. Similarly, we say that $\omega \in \partial T$ lies below the vertex x if $x \in (\omega, \omega_*)$, that is x belongs to the doubly infinite geodesic with endpoints ω and ω_*. We shall write $x \leq y$ whenever

$x \in T \cup \partial T$ lies below $y \in T$ (see also [14, Section 3] for a reference about this construction).

We say that a vertex x is a neighbour of $y \in T$ if there is one edge connecting x and y or, equivalently, if $d(x, y) = 1$. In this case, we write $x \sim y$.

We fix once and for all an origin $o \in T$ and we denote by $\{x_j\}_{j=0}$ an enumeration of the geodesic starting at o and ending in ω_* such that $x_j \sim x_{j+1}$ for every $j \geq 0$. We define the level of a vertex by

$$\ell(x) = \lim_{j\to\infty} j - d(x, x_j) \quad \forall x \in T.$$

For every $x \in T$ we define the set of successors by

$$s(x) = \{y \sim x : \ell(y) = \ell(x) - 1\}$$

and the predecessor of $x \in T$ by

$$p(x) = \{y \sim x : \ell(y) = \ell(x) + 1\}.$$

We set $p^0(x) = x$ and define inductively $p^n(x) = p(p^{n-1}(x))$ for every $x \in T$ and $n \geq 1$. Similarly, we set $s_0(x) = \{x\}$ and define

$$s_n(x) = \cup_{y \in s_{n-1}(x)} s(y) \quad \forall x \in T, n \geq 1. \tag{1.1}$$

A measure on T is a positive function on T; this means that the measure of a subset of T is the sum of the values of the measure on that set. We say that a measure m on T is a *flow measure* if it satisfies the conservation property

$$m(x) = \sum_{y \in s(x)} m(y) \quad \forall x \in T. \tag{1.2}$$

Observe that by iterating (1.2), a flow measure also satisfies

$$m(x) = \sum_{y \in s_n(x)} m(y) \quad \forall x \in T, n \in \mathbb{N}.$$

Given two points $\eta, \zeta \in \overline{T} := T \cup \partial T$, their confluent is

$$\eta \wedge \zeta = \arg\min\{\ell(x) : x \in T, \eta, \zeta \leq x\},$$

and for every $x \in T$, the sector T_x and its boundary at infinity ∂T_x are

$$T_x = \{y \in T : y \leq x\},$$
$$\partial T_x = \{\omega \in \partial T : \omega \leq x\}.$$

We define the Gromov distance on $\overline{T} \times \overline{T}$, denoted ρ, by

$$\rho(\eta, \xi) = \begin{cases} 0 & \text{if } \eta = \xi, \\ e^{\ell(\eta \wedge \xi)} & \text{otherwise.} \end{cases}$$

It is straightforward that the collection of balls with positive radii in $(\partial T, \rho)$ consists of $\{\partial T_x\}_{x \in T}$. Indeed, the Gromov distance of two points only depends on the level of their confluent. Moreover, it is clear that if $\partial T_y \cap \partial T_x \neq \emptyset$, then either $\partial T_x \subset \partial T_y$ or $\partial T_y \subset \partial T_x$. Analogous considerations hold if ∂T_x and ∂T_y are replaced by T_x and T_y, respectively.

Observe that for every positive measure ν on ∂T that is positive and finite on balls, there is an associated natural flow measure given by

$$m_\nu(x) = \nu(\partial T_x) \quad \forall x \in T.$$

From now on, we shall always assume that a measure on ∂T is finite and positive on balls.

Given a measure ν on ∂T and a measure m on T, we denote by $\|\cdot\|_{L^p(\partial T, \nu)}$ and $\|\cdot\|_{L^p(T,m)}$ the corresponding L^p-norms. Sometimes, the measure will be omitted from the norm subscript if it is clear from the context.

Definition 1.1 We define the map $\Phi : \partial T \times \mathbb{Z} \to T$ such that $\Phi(\omega, j)$ is the unique vertex which lies above ω at level j. Namely, Φ is uniquely defined by $\Phi(\omega, j) \in (\omega, \omega_*)$ and $\ell(\Phi(\omega, j)) = j$.

Throughout the paper, we will make instrumental use of the Hardy–Littlewood maximal operator, whose definition we briefly recall. Given a positive measure ν on ∂T we define by $\mathcal{M}$ the associated Hardy–Littlewood maximal operator, namely,

$$\begin{aligned} \mathcal{M}f(\omega_0) &= \sup_{r>0} \frac{1}{\nu(B_\rho(\omega_0, r))} \int_{B_\rho(\omega_0,r)} |f(\omega)| \, \mathrm{d}\nu(\omega) \\ &= \sup_{x \in (\omega_0, \omega_*)} \frac{1}{m_\nu(x)} \int_{\partial T_x} |f(\omega)| \, \mathrm{d}\nu(\omega) \quad \forall \omega_0 \in \partial T, \end{aligned} \tag{1.3}$$

where $B_\rho(\omega_0, r)$ denotes the ball in $(\partial T, \rho)$ centered at ω_0 with radius r with respect to the distance ρ and f is a locally integrable function.

The next proposition establishes the weak type $(1, 1)$ of the Hardy–Littlewood maximal operator on $(\partial T, \rho, \nu)$ independently of the choice of the measure ν. This is a straightforward consequence of the covering properties of $\{\partial T_x\}_{x \in T}$ discussed above. For the sake of completeness, we provide a proof.

Proposition 1.2 *Let ν be a measure on ∂T and $\mathcal{M}$ denote the associated Hardy–Littlewood maximal operator on ∂T. Then,*

$$\|\mathcal{M}\|_{L^1(\partial T, \nu) \to L^{1,\infty}(\partial T, \nu)} \le 1,$$

where $\|\mathcal{M}\|_{L^1(\partial T,\nu)\to L^{1,\infty}(\partial T,\nu)}$ *denotes the operator norm of* $\mathcal{M}$ *between the spaces* $L^1(\partial T, \nu)$ *and* $L^{1,\infty}(\partial T, \nu)$. *Moreover,* $\mathcal{M}$ *is bounded on* $L^p(\partial T, \nu)$ *for every* $p > 1$.

Proof It suffices to prove the weak type (1, 1) boundedness since the statement about the L^p bounds follows directly by interpolating with the obvious L^∞ boundedness. Fix $\lambda > 0$ and a function $f \in L^1(\partial T, \nu)$. Define $E_\lambda = \{\mathcal{M}f > \lambda\}$ and choose a collection of balls $\{\partial T_{x_j}\}_{j\in J}$ in ∂T such that

$$E_\lambda \subset \cup_{j\in J}\partial T_{x_j},$$
$$\frac{1}{\nu(\partial T_{x_j})}\int_{\partial T_{x_j}} |f|\, d\nu > \lambda \quad \forall j \in J.$$

Since two balls with nonempty intersection are such that one contains the other, we can extract a subcollection of pairwise disjoint balls $\{\partial T_{x_j}\}_{j\in J'}$ such that

$$\cup_{j\in J}\partial T_{x_j} = \cup_{j\in J'}\partial T_{x_j}.$$

It follows that

$$\nu(E_\lambda) \le \sum_{j\in J'} \nu(\partial T_{x_j}) \le \frac{1}{\lambda}\|f\|_{L^1(\partial T,\nu)}.$$

□

2 Poisson Integral and Carleson Measures

From now on we shall assume that T is a locally finite tree, that is, every vertex in T has a finite number of neighbours. We stress that we are not assuming that the number of neighbours of a vertex is a bounded function. While the local finiteness is not essential for part of our result, it simplifies our approach by avoiding some technicalities.

We fix once and for all a positive measure ν on ∂T. We introduce a Laplace operator Δ_ν on T that will be central later. For a given function f on T we define

$$\Delta_\nu f(x) = f(x) - \sum_{y\in s(x)} f(y)\frac{m_\nu(y)}{m_\nu(x)}, \quad \forall x \in T,$$

where m_ν is the flow measure induced by ν. Δ_ν is a *probabilistic Laplacian* in the sense that

$$\Delta_\nu = I - P$$

and P acts on a function f by $Pf(x) = \sum_{y\in T} p(x, y)f(y)$ where

$$0 \le p(x, y) = \begin{cases} 0 & \text{if } x \nsim y \text{ or } y = p(x), \\ \frac{m_\nu(y)}{m_\nu(x)} & \text{if } y \in s(x), \end{cases}$$

and

$$\sum_{y \in T} p(x, y) = 1 \quad \forall x \in T.$$

Notice that, since there are no restrictions on the flow measure m_ν, $p(x, y)$ can be arbitrarily close to either 0 or 1 when $y \in s(x)$ and it is zero when $y = p(x) \sim x$.

We say that a function f is *harmonic* on $S \subset T$ if

$$\Delta_\nu f(x) = 0 \quad \forall x \in S.$$

Remark 2.1 Observe that our definition of harmonic functions coincides with that of a martingale as given in [16], in the specific case of a homogeneous isotropic tree. In our framework, harmonic functions may not satisfy certain classical properties. For instance, unlike the classical setting, there exist nontrivial, nonnegative harmonic functions on T.

Observe that a harmonic function f on T satisfies

$$f(x)m_\nu(x) = \sum_{y \in s_n(x)} f(y)m_\nu(y) \quad \forall x \in T, n \in \mathbb{N}. \tag{2.1}$$

Indeed, (2.1) holds trivially when $n = 1$. We proceed by induction: assuming (2.1) for some $n \geq 1$ and exploiting the fact that f is harmonic for all $y \in s_n(x)$,

$$\begin{aligned} f(x)m_\nu(x) &= \sum_{y \in s_n(x)} m_\nu(y) f(y) \\ &= \sum_{y \in s_n(x)} m_\nu(y) \sum_{z \in s(y)} \frac{m_\nu(z)}{m_\nu(y)} f(z) \\ &= \sum_{z \in s_{n+1}(x)} m_\nu(z) f(z) \quad \forall x \in T, \end{aligned}$$

which proves (2.1).

Therefore, the limit

$$\lim_{n \to \infty} \sum_{y \in s_n(x)} f(y) \frac{m_\nu(y)}{m_\nu(x)} \tag{2.2}$$

makes sense and it is equal to $f(x)$. Thus, since $s_n(x)$ tends to ∂T_x as $n \to \infty$ in a suitable sense, (2.2) suggests that if f is good enough, then a Poisson integral representation formula for harmonic function holds. To this purpose, we need the following notion of continuous extension of a function on $\overline{T}$.

Definition 2.2 We say that a function f on T admits a continuous extension on $\overline{T}$ if there exists a function g on ∂T such that $\lim_{x\to\omega} f(x) = g(\omega)$ for a.e. $\omega \in \partial T$. The above limit is defined as follows: for every $\varepsilon > 0$ and a.e. $\omega \in \partial T$ there exists a $\delta = \delta(\varepsilon, \omega) > 0$ such that

$$\rho(\omega, x) < \delta \implies |f(x) - g(\omega)| < \varepsilon.$$

With a slight abuse of notation we denote by f the continuous extension of f to $\overline{T}$.

The following result provides a Poisson representation formula for harmonic functions. While the proof is not hard, we present the details for the reader's convenience.

Proposition 2.3 *Let f be a bounded harmonic function on T that admits a continuous extension to $\overline{T}$. Then,*

$$f(x) = \frac{1}{m_\nu(x)} \int_{\partial T_x} f(\omega)\,\mathrm{d}\nu(\omega) \quad \forall x \in T.$$

Conversely, for every $g \in L^1_{\mathrm{loc}}(\partial T, \nu)$, the function f defined by

$$f(x) = \frac{1}{m_\nu(x)} \int_{\partial T_x} g(\omega)\,\mathrm{d}\nu(\omega), \quad \forall x \in T,$$

is harmonic on T.

Proof We first observe that $\|f\|_{L^\infty(\partial T,\nu)} \le \|f\|_{L^\infty(T,m_\nu)}$. Indeed, for a.e. $\omega \in \partial T$ and $\varepsilon > 0$ there is a $x_\varepsilon \in (\omega, \omega_*)$ such that

$$|f(\omega)| \le |f(x_\varepsilon)| + \varepsilon \le \|f\|_{L^\infty(T,m_\nu)} + \varepsilon.$$

This implies that for every $x \in T$, f is integrable on ∂T_x and thus

$$\int_{\partial T_x} f(\omega)\,\mathrm{d}\nu(\omega) \tag{2.3}$$

makes sense. Moreover, by (2.1), for every $n \in \mathbb{N}$

$$m_\nu(x) f(x) = \sum_{y\in s_n(x)} f(y) m_\nu(y) = \sum_{y\in s_n(x)} f(y)\nu(\partial T_y). \tag{2.4}$$

Define the sequence of simple functions $f_{n,x}(\omega) = f_n(\omega) = \sum_{y\in s_n(x)} f(y)\chi_{\partial T_y}(\omega)$. It is clear that

$$\int_{\partial T} f_n(\omega)\,\mathrm{d}\nu(\omega) = \sum_{y\in s_n(x)} f(y)\nu(\partial T_y) = f(x) m_\nu(x) \quad \forall n \in \mathbb{N}, \tag{2.5}$$

by (2.4). We notice that $\{\partial T_y\}_{y\in s_n(x)}$ is a partition of ∂T_x, hence for every $\omega \in \partial T_x$ we have that $f_n(\omega) = f(\omega_n)$ where ω_n is the unique vertex in $s_n(x)$ such that $\omega \le \omega_n$. Moreover,

$$\rho(\omega_n, \omega) = e^{\ell(\omega_n)} = e^{\ell(x)-n} \to 0 \quad \text{as } n \text{ tends to } \infty.$$

Since f admits a continuous extension, we deduce that

$$\lim_{n\to\infty} f_n(\omega) = f(\omega) \quad a.e. \quad \omega \in \partial T_x.$$

Observe that $|f_n(\omega)| \le \|f\|_\infty \chi_{\partial T_x}(\omega)$, so we conclude by the Lebesgue Dominated Convergence Theorem that

$$f(x)m_\nu(x) = \lim_{n\to\infty} \int_{\partial T_x} f_n(\omega) = \int_{\partial T_x} f(\omega)\, \mathrm{d}\nu(\omega).$$

It is easy to see that the converse holds: if f is such that

$$f(x) = \frac{1}{m_\nu(x)} \int_{\partial T_x} g(\omega)\, \mathrm{d}\nu(\omega),$$

for some $g \in L^1_{\mathrm{loc}}(\partial T, \nu)$, then

$$\Delta_\nu f(x) = \int_{\partial T} \Delta_\nu K(x,\omega) g(\omega)\, \mathrm{d}\nu(\omega) = 0,$$

where $K(x,\omega) := \frac{\chi_{\partial T_x}(\omega)}{m_\nu(x)}$. Indeed, it is readily seen that for every $x \in T$ and $\omega \in \partial T$

$$\begin{aligned}\Delta_\nu K(\cdot,\omega)(x) &= \frac{1}{m_\nu(x)} \chi_{\partial T_x}(\omega) - \sum_{y\in s(x)} \frac{m_\nu(y)\chi_{\partial T_y}(\omega)}{m_\nu(x)m_\nu(y)} \\ &= \begin{cases} 0 & \text{if } \omega \notin \partial T_x, \\ \frac{1}{m_\nu(x)} - \frac{1}{m_\nu(x)} \times 1 & \text{if } \omega \in \partial T_x \end{cases} \\ &= 0.\end{aligned}$$

□

For the remaining of the paper, we write $\mathcal{P}$ for the integral operator defined by

$$\mathcal{P} f(x) = \int_{\partial T} P(x,\omega) f(\omega)\, \mathrm{d}\nu(\omega), \tag{2.6}$$

where $P(x,\omega) = \frac{\chi_{\partial T_x}(\omega)}{m_\nu(x)}$. We will refer to $\mathcal{P}$ as the *Poisson integral operator*.

In Theorem 2.5, we will characterise harmonic functions that are the Poisson integral of a function in $L^p(\partial T, \nu)$. For a different approach on the Poisson representation of harmonic functions on trees, we refer the reader to [13] and the references therein. In order to state our result, we need to introduce a suitable Hardy space.

Definition 2.4 For every $p \geq 1$, we say that a harmonic function f on T belongs to H^p if

$$\begin{aligned} \|f\|_{H^p}^p &:= \sup_{k\in\mathbb{Z}} \sum_{\ell(x)=k} |f(x)|^p m_\nu(x) < \infty && \text{if } p < \infty, \\ \|f\|_{H^\infty} &:= \|f\|_{L^\infty(T)} < \infty && \text{if } p = \infty. \end{aligned}$$

H^p can be thought of as the analogue of the Hardy spaces on the upper half-plane (see for example [10, Chapter 2]).

Observe that if $g \in L^p(\partial T, \nu)$ then by Jensen's inequality

$$\|\mathcal{P}g\|_{H^p}^p = \sup_{k\in\mathbb{Z}} \sum_{\ell(x)=k} \left| \frac{1}{m_\nu(x)} \int_{\partial T_x} g(\omega)\, \mathrm{d}\nu(\omega) \right|^p m_\nu(x) \leq \|g\|_{L^p(\partial T)}^p, \tag{2.7}$$

because $\{\partial T_x\}_{\ell(x)=k}$ is a partition of ∂T.

Theorem 2.5 *Let f be a harmonic function on T and $p > 1$. Then, f is the Poisson integral of a $L^p(\partial T, \nu)$ function if and only if $f \in H^p$.*

Proof The necessary condition follows by (2.7). For the other direction, assume that $f \in H^p$ for some $p > 1$. We provide the details of the proof for $p < \infty$. The proof for $p = \infty$ is analogous with obvious modifications. We aim to show that $f = \mathcal{P}g$ for a suitable $g \in L^p(\partial T, \nu)$. We first observe that it suffices to prove that there exists $g \in L^p(\partial T, \nu)$ such that

$$\mathcal{P}g(x) = f(x) \quad \forall x : \ell(x) \leq 0. \tag{2.8}$$

Indeed, assume that (2.8) holds. If z has level > 0 then by (2.1) we see that

$$\begin{aligned} f(z) &= \frac{1}{m_\nu(z)} \sum_{y\in s_{\ell(z)}(z)} f(y) m_\nu(y) \\ &= \frac{1}{m_\nu(z)} \sum_{y\in s_{\ell(z)}(z)} \frac{m_\nu(y)}{m_\nu(y)} \int_{\partial T_y} g(\omega)\, \mathrm{d}\nu(\omega) \\ &= \int_{\partial T} P(z,\omega) g(\omega)\, \mathrm{d}\nu(\omega), \end{aligned} \tag{2.9}$$

which concludes the proof.

We prove (2.8). Let $x \in T$ be such that $\ell(x) = 0$. For every $n \in \mathbb{N}$, set

$$T_x^n = T_x \cap \{y : \ell(y) \geq -n\}$$

and consider the sequence of functions u_x^n defined by

$$u_x^n(y) = \begin{cases} f(y) & \text{if } y \in T_x^n, \\ f(x) & \text{if } y \notin T_x, \\ f(p^{-n-\ell(y)}(y)) & \text{if } y \in T_x \setminus T_x^n. \end{cases}$$

We remark that if $y \in T_x \setminus T_x^n$, $p^{-n-\ell(y)}(y)$ is the closest vertex in T_x^n to y. Note that u_x^n coincides with f on T_x^n and for every $y \in (T_x^n)^c$ one has $u_x^n(y) = u_x^n(z)$ for every $z \sim y$. We deduce that u_x^n is harmonic on T, constant on T_y for every y such that $\ell(y) \leq -n$, and

$$\sup_{y \in T} |u_x^n(y)| \leq \max_{z \in T_x^n} |f(z)|.$$

Thus, by Proposition 2.2 we conclude that for every $y \in T$

$$u_x^n(y) = \int_{\partial T} P(y, \omega) u_x^n(\omega) \, \mathrm{d}\nu(\omega),$$

where u_x^n also denotes the continuous extension of u_x^n on ∂T, which exists because u_x^n is constant on T_y for every y with level $\leq -n$. In particular, for every $y \in T_x$ and $n \geq -\ell(y)$, namely, for every n such that $y \in T_x^n$, we have that

$$f(y) = u_x^n(y) = \int_{\partial T} P(y, \omega) u_x^n(\omega) \, \mathrm{d}\nu(\omega) = \int_{\partial T_x} P(y, \omega) u_x^n(\omega) \, \mathrm{d}\nu(\omega). \quad (2.10)$$

Observe that the sequence $\{u_x^n\}_n$ is bounded on $L^p(\partial T_x, \nu)$ because

$$\int_{\partial T_x} |u_x^n(\omega)|^p \, \mathrm{d}\nu(\omega) = \sum_{z \in s_n(x)} |f(z)|^p m_\nu(z) \leq \|f\|_{H^p}^p \quad \forall n \in \mathbb{N}, \quad (2.11)$$

where we have used that for every $z \in s_n(x)$, $u_x^n = f(z)$ on ∂T_z and $\partial T_x = \cup_{z \in s_n(x)} \partial T_z$. Next, the Banach–Alaoglu Theorem implies that $\{u_x^n\}_n$ admits a subsequence $\{u_x^{n_k}\}_k$ that weakly converges in $L^p(\partial T_x, \nu)$ to a function u_x. It follows by (2.10) that for every $y \in T_x$

$$\begin{aligned} f(y) = \lim_{k \to \infty} u_x^{n_k}(y) &= \lim_{k \to \infty} \int_{\partial T_x} P(y, \omega) u_x^{n_k}(\omega) \, \mathrm{d}\nu(\omega) \\ &= \int_{\partial T_x} P(y, \omega) u_x(\omega) \, \mathrm{d}\nu(\omega), \end{aligned}$$

because $P(y, \cdot) \in L^q(\partial T_x, \nu)$ for every $y \in T_x$, where $q = p/(p-1)$. We conclude that for every $y \in T_x$, $f(y) = \mathcal{P} u_x(y)$.

We set

$$g(\omega) = u_x(\omega) \quad \forall \omega \in \partial T_x.$$

So far we have proved that $f = \mathcal{P}g$ on T_x, where $x \in T$ is a vertex of level 0. Then, by the arbitrariness of x in the set of vertices of level 0, we conclude $\mathcal{P}g = f$ on T.

It remains to prove that $g \in L^p(\partial T, \nu)$. We observe that for every $n \in \mathbb{N}$ and $x \in T$ with $\ell(x) = 0$

$$\int_{\partial T_x} |u_x^n(\omega)|^p \, \mathrm{d}\nu(\omega) = \sum_{z \in s_n(x)} |f(z)|^p m_\nu(z) \leq \sum_{z \in s_{n+1}(x)} |f(z)|^p m_\nu(z),$$

where the last inequality follows from the fact that f is harmonic for every $z \in s_n(x)$ and an application of Jensen's inequality. It follows that $n \mapsto \int_{\partial T_x} |u_x^n(\omega)|^p \, \mathrm{d}\nu(\omega)$ is increasing and bounded because $f \in H^p$. Then, on the one hand

$$\lim_{n\to\infty} \int_{\partial T_x} |u_x^n(\omega)|^p \, \mathrm{d}\nu(\omega) = \lim_{n\to\infty} \sum_{z \in s_n(x)} |f(z)|^p m_\nu(z) = c \leq \|f\|_{H^p}^p < \infty. \tag{2.12}$$

On the other hand it is known that the p-norm is weakly lower semicontinuous (see, e.g., [1, Proposition 3.5]), thus

$$\int_{\partial T_x} |u_x(\omega)|^p \, \mathrm{d}\nu(\omega) \leq \liminf_{k\to\infty} \int_{\partial T_x} |u_x^{n_k}(\omega)|^p \, \mathrm{d}\nu(\omega) = \lim_{k\to\infty} \int_{\partial T_x} |u_x^k(\omega)|^p \, \mathrm{d}\nu(\omega), \tag{2.13}$$

where in the second equality we have used (2.12). We conclude that

$$\begin{aligned}
\|g\|_{L^p(\partial T,\nu)}^p &= \sum_{x:\ell(x)=0} \int_{\partial T_x} |u_x(\omega)|^p \, \mathrm{d}\nu(\omega) \\
&= \lim_{n\to\infty} \sum_{x:\ell(x)=0, d(x,o)<n} \int_{\partial T_x} |u_x(\omega)|^p \, \mathrm{d}\nu(\omega) \\
&\leq \lim_{n\to\infty} \sum_{x:\ell(x)=0, d(x,o)<n} \lim_{k\to\infty} \int_{\partial T_x} |u_x^k(\omega)|^p \, \mathrm{d}\nu(\omega) \\
&= \lim_{n\to\infty} \lim_{k\to\infty} \sum_{x:\ell(x)=0, d(x,o)<n} \sum_{z \in s_k(x)} |f(z)|^p m_\nu(z) \\
&\leq \lim_{n\to\infty} \lim_{k\to\infty} \sum_{x:\ell(x)=-k} |f(z)|^p m_\nu(z) \\
&\leq \|f\|_{H^p}^p.
\end{aligned}$$

$\square$

We are now ready to define Carleson measures.

Definition 2.6 We say that a positive measure σ on T is a Carleson measure if there exists a constant $C > 0$ such that for every $x \in T$

$$\sigma(T_x) \leq C\nu(\partial T_x).$$

Like in the Euclidean case, we show that Carleson measures are related to the boundedness of $\mathcal{P}$ between suitable L^p spaces. To accomplish this, we first discuss some properties of the Poisson integral operator.

We define the maximal operator $\mathcal{U}$ by

$$\mathcal{U}f(\omega) = \sup_{x\in(\omega,\omega_*)} |f(x)|, \quad \forall\omega \in \partial T,$$

where f is a function on T.

Proposition 2.7 *Let Φ be as in Definition 1.1. The following hold:*

(i) *for all $f \in L^1_{\mathrm{loc}}(\partial T, \nu)$*

$$\mathcal{U}\mathcal{P}f(\omega) \leq \mathcal{M}f(\omega) \quad \forall\omega \in \partial T, \tag{2.14}$$

and thus in particular

$$\mathcal{P}f(\Phi(\omega, j)) \leq \mathcal{M}f(\omega) \quad \forall j \in \mathbb{Z}, \omega \in \partial T; \tag{2.15}$$

(ii) *for a.e. $\omega \in \partial T$, $p \in (1, \infty)$ and $f \in L^p(\partial T, \nu)$*

$$\lim_{j\to-\infty} \mathcal{P}f(\Phi(\omega, j)) = f(\omega)$$

and

$$\lim_{j\to-\infty} \int_{\partial T} |\mathcal{P}f(\Phi(\omega, j)) - f(\omega)|^p \, d\nu(\omega) = 0; \tag{2.16}$$

(iii) *for every $x \in T$*

$$\int_{\partial T} P(x, \omega)\, \mathrm{d}\nu(\omega) = 1. \tag{2.17}$$

Proof Let $f \in L^1_{\mathrm{loc}}(\partial T, \nu)$. Then, by (1.3)

$$\mathcal{U}\mathcal{P}f(\omega) \leq \sup_{x\in(\omega,\omega_*)} \frac{1}{m_\nu(x)} \int_{\partial T_x} |f(\omega)|\, \mathrm{d}\nu(\omega) = \mathcal{M}f(\omega) \quad \forall\omega \in \partial T,$$

which is (i). In order to prove (ii), assume that $f \in L^p(\partial T, \nu)$ for some $p \in (1, \infty)$. Since ∂T is a locally compact space on which $\mathcal{M}$ is of weak type $(1, 1)$, the Lebesgue differentiation Theorem holds. Thus for a.e. $\omega_0 \in \partial T$,

$$\lim_{j\to-\infty} \mathcal{P}f(\Phi(\omega_0, j)) = \lim_{j\to-\infty} \frac{1}{\nu(\partial T_{\Phi(\omega_0,j)})} \int_{\partial T_{\Phi(\omega_0,j)}} f(\omega)\, d\nu(\omega) = f(\omega_0). \tag{2.18}$$

By (i), the fact that $\mathcal{M}f \in L^p(\partial T, \nu)$, (2.18) and the Dominated Converge Theorem, we have that (2.16) holds. Finally, (iii) follows from a straightforward computation. □

We remark that since $P(x, \omega) \geq 0$ for every $(x, \omega) \in T \times \partial T$, (iii) in the above proposition implies that for every positive measure σ on T,

$$\|\mathcal{P}f\|_{L^\infty(T,\sigma)} \leq \|f\|_{L^\infty(\partial T,\nu)}, \quad \forall f \in L^\infty(\partial T, \nu). \tag{2.19}$$

Theorem 2.8 *Let ν be a positive measure on ∂T. The following facts are equivalent*

(i) *σ is a Carleson measure;*

(ii) *there exists a $C > 0$ such that for every $p > 1$ and $f \in L^p(\partial T, \nu)$*

$$\|\mathcal{P}f\|_{L^p(T,\sigma)} \leq C\|f\|_{L^p(\partial T,\nu)},$$

and

$$\|\mathcal{P}f\|_{L^{1,\infty}(T,\sigma)} \leq C\|f\|_{L^1(\partial T,\nu)};$$

(iii) *there exists $C > 0$ such that for every $p > 1$ and $f \in L^p(\partial T, \nu)$*

$$\|\mathcal{P}f\|_{L^p(T,\sigma)} \leq C\|\mathcal{P}f\|_{H^p}.$$

Proof Assume that σ is a Carleson measure. To prove (ii), by (2.19) it suffices to prove the weak type $(1, 1)$ boundedness of $\mathcal{P}$ and interpolate. For $\lambda > 0$ and $f \in L^1(\partial T, \nu)$, set $F_\lambda = \{x \in T\ |\mathcal{P}f(x)| > \lambda\}$. The fact that $x \in T_x$ for every $x \in T$ readily implies that

$$F_\lambda \subset \cup_{x\in F_\lambda} T_x.$$

Since when $T_x \cap T_y \neq \emptyset$ we have that $T_x \subset T_y$ or $T_y \subset T_x$, there exists a nonempty set $F'_\lambda \subset F_\lambda$ such that for every $x, y \in F'_\lambda$ we have that $T_x \cap T_y \neq \emptyset$ implies that $x = y$ and

$$\cup_{x\in F'_\lambda} T_x = \cup_{x\in F_\lambda} T_x.$$

It follows that

$$F_\lambda \subset \cup_{x\in F'_\lambda} T_x,$$

where $\{T_x\}_{x\in F'_\lambda}$ are pairwise disjoint. We conclude that

$$\sigma(F_\lambda) \le \sum_{x\in F'_\lambda} \sigma(T_x) \le C \sum_{x\in F'_\lambda} \nu(\partial T_x). \tag{2.20}$$

Next, observe that if $\mathcal{P}f(x) > \lambda$ then $\mathcal{U}\mathcal{P}f(\omega) > \lambda$ for every $\omega \in \partial T_x$. This means that $\partial T_x \subset \{\mathcal{U}(\mathcal{P}f) > \lambda\}$ for every $x \in F'_\lambda$. Observing that $\{\partial T_x\}_{x\in F'_\lambda}$ are pairwise disjoint, (2.20) implies

$$\sigma(F_\lambda) \le C \sum_{x\in F'_\lambda} \nu(\partial T_x \cap \{\mathcal{U}(\mathcal{P}f) > \lambda\}) \le C\nu(\{\mathcal{U}(\mathcal{P}f) > \lambda\}).$$

We conclude by (2.14) that

$$\sigma(F_\lambda) \le C\nu(\{\mathcal{M}f > \lambda\}),$$

and now the result follows by Proposition 1.2.

Next, we show that (ii) implies (iii). Indeed, assume that for $p > 1$

$$\|\mathcal{P}f\|_{L^p(T,\sigma)} \le C\|f\|_{L^p(\partial T,\nu)} \quad \forall f \in L^p(\partial T,\nu). \tag{2.21}$$

Then by (2.16) and Fatou's Lemma, for every $f \in L^p(\partial T,\nu)$ and $k \in \mathbb{Z}$

$$\begin{aligned}
\int_{\partial T} |f(\omega)|^p \,\mathrm{d}\nu(\omega) &= \sum_{x:\ell(x)=k} \int_{\partial T_x} |f(\omega)|^p \,\mathrm{d}\nu(\omega) \\
&= \sum_{x:\ell(x)=k} \lim_{j\to-\infty} \|\mathcal{P}f(\Phi(\cdot,j))\|^p_{L^p(\partial T_x)} \\
&\le \liminf_{j\to-\infty} \sum_{x:\ell(x)=k} \int_{\partial T_x} |\mathcal{P}f(\Phi(\omega,j))|^p \,\mathrm{d}\nu(\omega) \\
&\le \sup_{j\le k} \sum_{x:\ell(x)=k} \int_{\partial T_x} |\mathcal{P}f(\Phi(\omega,j))|^p \,\mathrm{d}\nu(\omega) &(2.22)\\
&= \sup_{j\le k} \sum_{x:\ell(x)=k} \sum_{\substack{y\le x\\ \ell(y)=j}} |\mathcal{P}f(y)|^p m_\nu(y) \\
&= \sup_{j\le k} \sum_{y:\ell(y)=j} |\mathcal{P}f(y)|^p m_\nu(y) \\
&\le \|\mathcal{P}f\|^p_{H^p}, &(2.23)
\end{aligned}$$

where in (2.22) we have used that $\omega \mapsto \mathcal{P}f(\Phi(\omega,j))$ is constant on $\partial T_{\Phi(\omega,j)}$ for every j fixed. Next, (2.21) together with (2.23) imply (iii).

Now assume (iii) and consider $f = \chi_{\partial T_v}$ for some $v \in T$. By (2.17), on the one hand, we have that

$$\begin{aligned}
\|\mathcal{P}f\|^p_{L^p(T,\sigma)} &= \sum_{x\in T}\left|\int_{\partial T} P(x,\omega)f(\omega)\,\mathrm{d}\nu(\omega)\right|^p \sigma(x)\\
&\geq \sum_{x\leq v}\left|\int_{\partial T} P(x,\omega)f(\omega)\,\mathrm{d}\nu(\omega)\right|^p \sigma(x)\\
&= \sum_{x\leq v}\left|\frac{1}{m_\nu(x)}\int_{\partial T_v} \chi_{\partial T_x}(\omega)\,\mathrm{d}\nu(\omega)\right|^p \sigma(x)\\
&= \sum_{x\leq v}\sigma(x)\\
&= \sigma(T_v).
\end{aligned} \tag{2.24}$$

In the second last equality, we have used that $\partial T_x \subset \partial T_v$ for every $x \leq v$ and $\nu(\partial T_x) = m_\nu(x)$. On the other hand, we claim that

$$\|\mathcal{P}f\|^p_{H^p} = m_\nu(v). \tag{2.25}$$

This would conclude the proof since (i) follows by combining (2.25), (2.24), and (iii).

To prove (2.25), we observe that

$$\mathcal{P}f(x) = \frac{1}{m_\nu(x)}\nu(\partial T_x \cap \partial T_v),$$

which is non-zero if and only if $x \leq v$ or $v \leq x$. In the first cases $\mathcal{P}f(x) = 1$ and in the second case $\mathcal{P}f(x) = \frac{m_\nu(v)}{m_\nu(x)}$.

Thus, for every $k \leq \ell(v)$ the above considerations and the definition of flow measure yield

$$\sum_{\ell(x)=k}|\mathcal{P}f(x)|^p m_\nu(x) = \sum_{x\leq v,\ell(x)=k} m_\nu(x) = m_\nu(v),$$

while for every $k > \ell(v)$, the unique vertex with level k that lies above v is $p^{k-\ell(v)}(v)$ and

$$\begin{aligned}
\sum_{\ell(x)=k}|\mathcal{P}f(x)|^p m_\nu(x) &= \left(\frac{m_\nu(v)}{m_\nu(p^{k-\ell(v)}(v))}\right)^p m_\nu(p^{k-\ell(v)}(v))\\
&= \frac{m_\nu(v)^p}{m_\nu(p^{k-\ell(v)}(v))^{p-1}}.
\end{aligned}$$

We conclude by observing that

$$\frac{m_\nu(v)^p}{m_\nu(p^{k-\ell(v)}(v))^{p-1}} \leq m_\nu(v)$$

since the above is equivalent to

$$m_\nu(v) \le m_\nu(p^{k-\ell(v)}(v)),$$

which is always true for $k \ge \ell(v)$ since m_ν is a flow measure. This proves (2.25) and concludes the proof. □

3 Carleson Measures and BMO

In this section, we assume that the number of successors of every vertex is bigger than or equal to two.

Let (X, d, μ) be a metric measure space. We say that μ is *doubling* if there exists a constant $C > 0$ such that

$$\mu(B_{2r}(x)) \le C\mu(B_r(x)) \quad \forall x \in X, \forall r > 0.$$

Similarly, μ is *locally doubling* if for every $R > 0$ there exists a constant C_R such that

$$\mu(B_{2r}(x)) \le C_R\mu(B_r(x)) \quad \forall x \in X, \forall r < R.$$

In [12, Proposition 2.2] it is proved that m is a locally doubling flow measure on T equipped with the geodesic distance if and only if there are two positive constants $c_1, c_2 > 1$ such that

$$c_2 m(y) \le m(x) \le c_1 m(y), \quad \forall x \in T, y \in s(x). \tag{3.1}$$

In the same proposition it is also proved that the inequality $m(x) \le c_1 m(y)$ for every $x \in T$ and $y \in s(x)$ implies $m(x) \ge c_1/(c_1 - 1)m(y)$ for every $x \in T$. This is where we need the assumption regarding the minimum number of successors of a vertex: the latter inequality holds as long as each vertex has at least two successors (see [12, Proposition 2.2 (ii)]). Moreover, it is known that if m is locally doubling, then the number of neighbours of a vertex is bounded on T, see [12, Corollary 2.3].

Lemma 3.1 *Let ν be a positive measure on ∂T. Then, $(\partial T, \rho, \nu)$ is doubling if and only if (T, d, m_ν) is locally doubling.*

Proof Fix $\omega_0 \in \partial T$ and observe that for a given $r > 0$

$$B_\rho(\omega_0, r) = \{\omega \in \partial T : \ell(\omega \wedge \omega_0) \le \log r\} = \partial T_{\Psi(\omega_0, r)},$$

where $\Psi(\omega_0, r)$ is the unique vertex such that $\ell(\Psi(\omega_0, r)) = \lfloor \log r \rfloor$ and $\Psi(\omega_0, r) \in (\omega_0, \omega_*)$. Then, $\nu(B_\rho(\omega, r)) = m_\nu(\Psi(\omega_0, r))$. Similarly,

$$B_\rho(\omega_0, 2r) = \partial T_{\Psi(\omega_0, 2r)},$$

and $\ell(\Psi(\omega_0, 2r)) = \lfloor \log 2r \rfloor \leq \lfloor \log r \rfloor + 1 = \ell(\Psi(\omega_0, r)) + 1 = \ell(p(\Psi(\omega_0, r)))$. Thus $(\partial T, \rho, \nu)$ is doubling if and only if there exists a constant $C > 0$ such that for every $\omega_0 \in \partial T$ and $r > 0$

$$\frac{m_\nu(\Psi(\omega_0, 2r))}{m_\nu(\Psi(\omega_0, r))} \leq C. \tag{3.2}$$

For every $x \in T$ and $\omega \in \partial T_x$, we choose $r = e^{\ell(x)+1-\log 2}$ and we get that $\Psi(\omega, r) = x$ and $\Psi(\omega, 2r) = p(x)$. Therefore, we deduce that (3.2) is equivalent to

$$\frac{m_\nu(p(x))}{m_\nu(x)} \leq C \quad \forall x \in T,$$

so we conclude by invoking (3.1) and the discussion thereafter. □

From now on we shall assume that ν is a doubling measure on ∂T. We shall introduce the *BMO* space on ∂T defined in terms of balls. More precisely, we set

$$BMO = \{b \in L^1_{\mathrm{loc}}(\partial T, \nu) : \|b\|_{BMO} < \infty\},$$

where

$$\|b\|_{BMO} := \sup_{x \in T} \frac{1}{\nu(\partial T_x)} \int_{\partial T_x} |b(\omega) - b_{\partial T_x}| \, \mathrm{d}\nu(\omega)$$

and for every $E \subset \partial T$ we set $b_E = \frac{1}{\nu(E)} \int_E b \, \mathrm{d}\nu$. Notice that this is a particular case of the *BMO* spaces considered in [7].

Definition 3.2 We define a space of integral operators that we denote by $\mathcal{O}$ such that every $\mathcal{K}$ in $\mathcal{O}$ acts on a suitable complex-valued function f on ∂T by

$$\mathcal{K} f(x) = \int_{\partial T} K(x, \omega) f(\omega) \, \mathrm{d}\nu(\omega) \quad \forall x \in T,$$

where $K(\cdot, \cdot) : T \times \partial T \to \mathbb{C}$ is an integral kernel that satisfies the following cancellation, integrability and decay properties:

(1) for every fixed $x_0 \in T$ the map $\partial T \ni \omega \mapsto K(x_0, \omega)$ is integrable and

$$\int_{\partial T} K(x_0, \omega) \, \mathrm{d}\nu(\omega) = 0;$$

(2)

$$C_K := \operatorname*{ess\,sup}_{\omega \in \partial T} \sum_{x \in T} |K(x, \omega)| m_\nu(x) < \infty;$$

(3) there exists $\alpha > 0$ such that $|K(x, \omega)| \leq \frac{m_\nu(x)^\alpha}{m_\nu(x \wedge \omega)^{\alpha+1}}$, for every $x \in T$ and $\omega \in \partial T$.

The following is the main result of this section and may be viewed as the analogous of [15, Theorem 3, p. 159].

Theorem 3.3 *Let b be a locally integrable function. Then, the following are equivalent facts:*

(i) *$b \in BMO$;*
(ii) *there exists a function $f : [0,\infty) \times (0,\infty) \to [0,\infty)$ depending on b that is increasing in the first variable and for every $\mathcal{K} \in \mathcal{O}$ the measure σ defined by $\sigma = |\mathcal{K}b| m_\nu$ is a Carleson measure satisfying*

$$\sigma(T_v) \leq f(C_K, \alpha) m_\nu(v) \quad \forall v \in T,$$

where C_K is as in (2) and α as in (3).

Proof We first prove that (i) implies (ii).
Fix $v \in T$ and assume that $b \in BMO$. We have that

$$\sigma(T_v) = \sum_{x \in T_v} |\mathcal{K}b(x)| m_\nu(x).$$

Observe that, by (1),

$$|\mathcal{K}b(x)| \leq |\mathcal{K}[(b - b_{\partial T_{p(v)}})\chi_{\partial T_{p(v)}}](x)| + |\mathcal{K}[(b - b_{\partial T_{p(v)}})\chi_{\partial T^c_{p(v)}}](x)|.$$

For notational convenience, we set $g_v = b - b_{\partial T_{p(v)}}$. Thus,

$$\sigma(T_v) \leq \sum_{x \in T_v} |\mathcal{K}(g_v \chi_{\partial T_{p(v)}})(x)| m_\nu(x) + \sum_{x \in T_v} |\mathcal{K}(g_v \chi_{\partial T^c_{p(v)}})(x)| m_\nu(x) =: I_1 + I_2.$$

Observe that by (2)

$$\begin{aligned} I_1 &\leq \sum_{x \in T_v} \int_{\partial T_{p(v)}} |K(x,\omega)||g_v(\omega)| \,\mathrm{d}\nu(\omega) m_\nu(x) \\ &= \int_{\partial T_{p(v)}} |g_v(\omega)| \sum_{x \in T_v} |K(x,\omega)| m_\nu(x) \,\mathrm{d}\nu(\omega) \\ &\leq c_1 C_K \|b\|_{BMO} m_\nu(v), \end{aligned}$$

because $m_\nu(p(v)) \leq c_1 m_\nu(v)$ by (3.1). Similarly, since

$$\partial T_v^c = \cup_{k=0}^{\infty} \partial T_{p^{k+1}(v)} \setminus \partial T_{p^k(v)}$$

and $\{T_{p^{k+1}(v)} \setminus \partial T_{p^k(v)}\}_{k=0}^{\infty}$ are pairwise disjoint,

$$I_2 \leq \sum_{x \in T_v} \sum_{k=0}^{\infty} \int_{\partial T_{p^{k+1}(v)} \setminus \partial T_{p^k(v)}} |K(x,\omega)||g_v(\omega)| \,\mathrm{d}\nu(\omega) m_\nu(x) =: \sum_{x \in T_v} \sum_{k=0}^{\infty} J_k m_\nu(x).$$

Observe that for every $x \in T_v$ and $\omega \in \partial T_{p^{k+1}(v)} \setminus \partial T_{p^k(v)}$ we have that $x \wedge \omega = p^{k+1}(v)$. Hence, by (3)

$$
\begin{aligned}
J_k &= \int_{\partial T_{p^{k+1}(v)} \setminus \partial T_{p^k(v)}} |K(x,\omega)||g_v(\omega)| \, \mathrm{d}\nu(\omega) \\
&\leq \int_{\partial T_{p^{k+1}(v)} \setminus \partial T_{p^k(v)}} \frac{m_\nu(x)^\alpha}{m_\nu(p^{k+1}(v))^{\alpha+1}} |g_v(\omega)| \, \mathrm{d}\nu(\omega) \\
&\leq \frac{m_\nu(x)^\alpha}{m_\nu(p^{k+1}(v))^{\alpha+1}} \int_{\partial T_{p^{k+1}(v)}} |g_v(\omega)| \, \mathrm{d}\nu(\omega) =: \frac{m_\nu(x)^\alpha}{m_\nu(p^{k+1}(v))^{\alpha+1}} S_k.
\end{aligned}
$$

Moreover, we set $v_k = p^k(v)$ for every $k \geq 0$ and observe that

$$
\begin{aligned}
S_k &= \int_{\partial T_{v_{k+1}}} |b(\omega) - b_{\partial T_{v_1}}| \, \mathrm{d}\nu(\omega) \\
&\leq \int_{\partial T_{v_{k+1}}} |b(\omega) - b_{\partial T_{v_{k+1}}}| \, \mathrm{d}\nu(\omega) + \int_{\partial T_{v_{k+1}}} |b_{\partial T_{v_{k+1}}} - b_{\partial T_{v_1}}| \, \mathrm{d}\nu(\omega) \\
&\leq m_\nu(v_{k+1}) \|b\|_{BMO} + m_\nu(v_{k+1}) \sum_{j=1}^{k} |b_{\partial T_{v_{j+1}}} - b_{\partial T_{v_j}}| \\
&\leq c_1 m_\nu(v_{k+1})(k+1) \|b\|_{BMO},
\end{aligned}
$$

because

$$
\begin{aligned}
|b_{\partial T_{v_{j+1}}} - b_{\partial T_{v_j}}| &\leq \frac{1}{m_\nu(v_j)} \int_{\partial T_{v_j}} |b(\omega) - b_{\partial T_{v_{j+1}}}| \, \mathrm{d}\nu(\omega) \\
&\leq \frac{m_\nu(v_{j+1})}{m_\nu(v_j)} \frac{1}{m_\nu(v_{j+1})} \int_{\partial T_{v_{j+1}}} |b(\omega) - b_{\partial T_{v_{j+1}}}| \, \mathrm{d}\nu(\omega) \\
&\leq c_1 \|b\|_{BMO}, \quad \forall j \in \mathbb{N},
\end{aligned}
$$

where the last inequality follows by (3.1). We conclude that

$$
I_2 \leq c_1 \|b\|_{BMO} \sum_{x \in T_v} m_\nu(x)^{\alpha+1} \sum_{k=0}^{\infty} (k+1) m_\nu(p^{k+1}(v))^{-\alpha}. \tag{3.3}
$$

Since $\alpha > 0$ we claim that there exists a constant $C_\alpha > 0$ such that

$$
\sum_{k=0}^{\infty} (k+1) m_\nu(p^{k+1}(v))^{-\alpha} \leq C_\alpha m_\nu(v)^{-\alpha} \tag{3.4}
$$

and

$$\sum_{x\in T_v} m_\nu(x)^{1+\alpha} \leq C_\alpha m_\nu(v)^{1+\alpha}. \tag{3.5}$$

The above claim, together with (3.3), implies that

$$I_2 \leq c_1 C_\alpha \|b\|_{BMO} \sum_{x\in T_v} m_\nu(x)^{\alpha+1} m_\nu(v)^{-\alpha} \leq c_1 C_\alpha^2 \|b\|_{BMO} m_\nu(v).$$

This, combined with the estimates involving I_1, yields

$$\sigma(T_v) \leq f(C_K, \alpha) m_\nu(v)$$

where $f(C_K, \alpha) := \|b\|_{BMO} c_1 (C_K + C_\alpha^2)$, concluding the first part of the proof. It remains to prove (3.4) and (3.5). By (3.1),

$$m_\nu(p^{k+1}(v)) \geq c_2^k m_\nu(v),$$

thus

$$\sum_{k=0}^{\infty} (k+1) m_\nu(p^{k+1}(v))^{-\alpha} \leq \sum_{k=0}^{\infty} (k+1) c_2^{-k\alpha} m_\nu(v)^{-\alpha} \leq C_\alpha m_\nu(v)^{-\alpha},$$

because $c_2 > 1$, where $C_\alpha := \sum_{k=0}^{\infty} (k+1) c_2^{-k\alpha} < \infty$. This proves (3.4). Similarly, by (3.1) again, we have that $m_\nu(x) \leq m_\nu(v) c_2^{-k}$ for every $x \in s_k(v)$ and $k \in \mathbb{N}$. Thus

$$\begin{aligned}
\sum_{x\in T_v} m_\nu(x)^{\alpha+1} &= \sum_{k=0}^{\infty} \sum_{x\in s_k(v)} m_\nu(x)^\alpha m_\nu(x) \\
&\leq \sum_{k=0}^{\infty} m_\nu(v)^\alpha c_2^{-\alpha k} \sum_{x\in s_k(v)} m_\nu(x) \\
&= m_\nu(v)^{\alpha+1} \sum_{k=0}^{\infty} \frac{1}{c_2^{\alpha k}} \\
&\leq C_\alpha m_\nu(v)^{\alpha+1},
\end{aligned} \tag{3.6}$$

where in (3.6) we have used that m_ν is a flow measure. This proves (3.5) and concludes the proof of the first part of the theorem.

We now prove that (ii) implies (i). Assume (ii). For every $y \in T$ let a_y be a function to be constructed explicitly later, supported in ∂T_y, with zero integral average, and such that

$$\|a_y\|_{L^\infty} \leq \frac{1}{m_\nu(y)}. \tag{3.7}$$

Define

$$K_{a_y}(x,\omega)=\begin{cases}a_y(\omega) & \text{if } x=y,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly K_{a_y} satisfies (1) and (3) with $\alpha=1$ because

$$|K_{a_y}(y,\omega)|=|a_y(\omega)|\le\|a_y\|_{L^\infty}\le\frac{1}{m_\nu(y)}=\frac{m_\nu(y)}{m_\nu(\omega\wedge y)^2}\quad\forall\omega\in\partial T_y,$$

by (3.7) and the fact that supp $a_y\subset\partial T_y$. Moreover, K_{a_y} fulfills (2) because

$$\sum_{x\in T}|K_{a_y}(x,\omega)|m_\nu(x)=|a_y(\omega)|m_\nu(y)\le 1\quad\forall\omega\in\partial T,$$

again by invoking (3.7). In particular, $C_{K_{a_y}}\le 1$. By (ii) and the definition of K_{a_y},

$$m_\nu(y)f(1,1)\ge\sum_{x\le y}|\mathcal{K}_{a_y}b(x)|m_\nu(x)=m_\nu(y)\left|\int_{\partial T_y}a_y(\omega)b(\omega)\,\mathrm{d}\nu(\omega)\right|,$$

from which it follows

$$\sup_{y\in T}\left|\int_{\partial T}a_y(\omega)b(\omega)\,\mathrm{d}\nu(\omega)\right|\le f(1,1),\tag{3.8}$$

where f is as in (ii). It is easy to deduce that (3.8) implies that $b\in BMO$. Indeed, for every $x\in T$ we have to prove that

$$\frac{1}{m_\nu(x)}\int_{\partial T_x}|b(\omega)-b_{\partial T_x}|\,\mathrm{d}\nu(\omega)\le C\tag{3.9}$$

for some absolute constant $C>0$. We shall give a suitable definition of a_y to get (3.9). We will deal with the real case; if b is complex-valued the argument can be slightly modified to obtain the same result. Define on ∂T_y the function a'_y by

$$a'_y(\omega)=\begin{cases}1 & \text{if } b(\omega)\ge b_{\partial T_y},\\ -1 & \text{if } b(\omega)<b_{\partial T_y}.\end{cases}$$

Set $a_y(\omega)=\frac{1}{2m_\nu(y)}[a'_y(\omega)-(a'_y)_{\partial T_y}]$ for every $\omega\in\partial T_y$. Observe that a_y has zero average on ∂T_y and satisfies (3.7) for every $x\in T$. Moreover,

$$(a'_y)_{\partial T_y}=\frac{1}{m_\nu(y)}\int_{\partial T_y}a'_y(\omega)\in[-1,1].$$

Using that a_y has zero integral average on ∂T_y and (3.8), it follows that

$$\begin{aligned} f(1,1) &\geq \left| \int_{\partial T_y} a_y(\omega) b(\omega)\, \mathrm{d}\nu(\omega) \right| \\ &= \left| \int_{\partial T_y} a_y(\omega) \big(b(\omega) - b_{\partial T_y} \big)\, \mathrm{d}\nu(\omega) \right| \\ &= \frac{1}{2m_\nu(y)} \left| \int_{\partial T_y} |b(\omega) - b_{\partial T_y}|\, \mathrm{d}\nu(\omega) - (a'_y)_{\partial T_y} \int_{\partial T_y} b(\omega) - b_{\partial T_y}\, \mathrm{d}\nu(\omega) \right| \\ &= \frac{1}{2m_\nu(y)} \int_{\partial T_y} |b(\omega) - b_{\partial T_y}|\, \mathrm{d}\nu(\omega), \end{aligned}$$

because

$$\int_{\partial T_y} b(\omega) - b_{\partial T_y}\, \mathrm{d}\nu(\omega) = 0.$$

This implies (3.9) and thus $b \in BMO$. □

Remark 3.4 Note that (3) does not imply (2). Indeed, if $K(x,\omega) = \dfrac{m_\nu(x)^\alpha}{m_\nu(\omega \wedge x)^{\alpha+1}}$ for some $\alpha > 0$ then K clearly satisfies (3) but

$$\sum_{x \in T} \frac{m_\nu(x)^\alpha}{m_\nu(\omega \wedge x)^{\alpha+1}} m_\nu(x) \geq \sum_{x \in (\omega, \omega_*)} 1 = \infty \quad \forall \omega \in \partial T,$$

so K does not satisfy (2).

Example 3.5 We provide some examples of operators in $\mathcal{O}$. Given $\delta > 0$ and $\alpha > 0$, assume that a function K_δ satisfies

$$|K_\delta(x,\omega)| \leq \frac{m_\nu(x)^\alpha}{m_\nu(\omega \wedge x)^{\alpha+1}} \min \left\{ \frac{1}{m_\nu(\omega \wedge x)}, m_\nu(\omega \wedge x) \right\}^{1+\delta}. \tag{3.10}$$

It is clear that

$$|K_\delta(x,\omega)| \leq \frac{m_\nu(x)^\alpha}{m_\nu(\omega \wedge x)^{\alpha+1}}.$$

Moreover, recalling the map Φ in Definition 1.1 and that $\omega \wedge x = \Phi(\omega, k)$ for every $x \in T_{\Phi(\omega,k)} \backslash T_{\Phi(\omega,k-1)}$ and $k \in \mathbb{N}$,

$$\sum_{x \in T} |K_\delta(x,\omega)| m_\nu(x)$$

$$
\begin{aligned}
&\le \sum_{k=-\infty}^{\infty} \sum_{\substack{x\in T_{\Phi(\omega,k)},\\ x\notin T_{\Phi(\omega,k-1)}}} \frac{m_\nu(x)^{\alpha+1}}{m_\nu(\Phi(\omega,k))^{\alpha+1}} \min\left\{\frac{1}{m_\nu(\Phi(\omega,k))}, m_\nu(\Phi(\omega,k))\right\}^{1+\delta}\\
&\le \sum_{k=-\infty}^{\infty} \min\left\{\frac{1}{m_\nu(\Phi(\omega,k))^{\alpha+2+\delta}}, m_\nu(\Phi(\omega,k))^{\delta-\alpha}\right\} \sum_{x\in T_{\Phi(\omega,k)}} m_\nu(x)^{\alpha+1}\\
&\le C \sum_{k=-\infty}^{\infty} \min\left\{\frac{1}{m_\nu(\Phi(\omega,k))^{2+\delta+\alpha}}, m_\nu(\Phi(\omega,k))^{\delta-\alpha}\right\} m_\nu(\Phi(\omega,k))^{\alpha+1}\\
&= C \sum_{k=-\infty}^{\infty} \min\left\{\frac{1}{m_\nu(\Phi(\omega,k))^{\delta+1}}, m_\nu(\Phi(\omega,k))^{\delta+1}\right\}\\
&= C\left(\sum_{k\,:\,m_\nu(\Phi(\omega,k))\ge 1} \frac{1}{m_\nu(\Phi(\omega,k))^{\delta+1}} + \sum_{k:m_\nu(\Phi(\omega,k))<1} m_\nu(\Phi(\omega,k))^{\delta+1}\right)
\end{aligned} \tag{3.11}
$$

where (3.11) is proved as in (3.5). We next claim that

$$
\sum_{k:m_\nu(\Phi(\omega,k))\ge 1} \frac{1}{m_\nu(\Phi(\omega,k))^{\delta+1}} + \sum_{k:m_\nu(\Phi(\omega,k))<1} m_\nu(\Phi(\omega,k))^{\delta+1} \le C_\delta.
$$

Indeed, set k_0 as the biggest integer such that $m_\nu(\Phi(\omega,k_0)) < 1$, which exists by (3.1). By (3.1) we deduce that

$$
\begin{aligned}
\sum_{k:m_\nu(\Phi(\omega,k))\ge 1} \frac{1}{m_\nu(\Phi(\omega,k))^{\delta+1}} &\le \frac{1}{m_\nu(\Phi(\omega,k_0+1))^{\delta+1}} \sum_{k\ge k_0+1} \frac{1}{c_2^{(k-k_0-1)(\delta+1)}}\\
&\le C\frac{1}{m_\nu(\Phi(\omega,k_0+1))^{\delta+1}} \le C_\delta,\\
\sum_{k:m_\nu(\Phi(\omega,k))<1} m_\nu(\Phi(\omega,k))^{\delta+1} &\le m_\nu(\Phi(\omega,k_0))^{\delta+1} \sum_{k\ge k_0} \frac{1}{c_2^{(k-k_0)(\delta+1)}}\\
&\le C m_\nu(\Phi(\omega,k_0))^{\delta+1} \le C_\delta.
\end{aligned}
$$

In summary, we showed that if K_δ satisfies (3.10) then it also satisfies (2). We conclude by showing that it is possible to construct a K_δ satisfying (3.10) that also fulfills (1). Indeed, for every fixed $x \in T$, the map $\partial T \ni \omega \mapsto K_\delta(x,\omega)$ is integrable because

$$
\begin{aligned}
\int_{\partial T} |K_\delta(x,\omega)|\,\mathrm{d}\nu(\omega) = &\int_{\partial T_x} |K_\delta(x,\omega)|\,\mathrm{d}\nu(\omega)\\
&+ \sum_{k=1}^{\infty} \int_{\partial T_{p^k(x)}\setminus T_{p^{k-1}(x)}} |K_\delta(x,\omega)|\,\mathrm{d}\nu(\omega)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{m_\nu(x)} m_\nu(x)^{2+\delta} \\
&\quad + \sum_{k=1}^{\infty} \frac{m_\nu(x)^\alpha}{m_\nu(p^k(x))^\alpha} \min\left\{\frac{1}{m_\nu(p^k(x))}, m_\nu(p^k(x))\right\}^{1+\delta} \\
&\leq C_{x,\delta}.
\end{aligned}
$$

Next, we have to construct a kernel that also satisfies the zero integral condition (1). For every $x \in T$ we set $c_x(\omega) = c_x(k)$ if $\omega \in \partial T_{p^k(x)} \setminus \partial T_{p^{k-1}(x)}$ if $k \geq 1$ and $c_x(\omega) = c_x(0)$ if $\omega \in \partial T_x$ where $\{c_x(k)\}_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$ is to be chosen. We also assume that $\{c_x(k)\}_{k\in\mathbb{N}}$ belongs to the unit ball of $\ell^\infty(\mathbb{N})$. Then we set

$$
K_\delta(x,\omega) = c_x(\omega) \frac{m_\nu(x)^\alpha}{m_\nu(\omega \wedge x)^{\alpha+1}} \min\left\{\frac{1}{m_\nu(\omega \wedge x)}, m_\nu(\omega \wedge x)\right\}^{1+\delta}.
$$

For notational convenience, for every $k \geq 1$ we set

$$
d_x(k) = \int_{\partial T_{p^k(x)} \setminus \partial T_{p^{k-1}(x)}} \frac{m_\nu(x)^\alpha}{m_\nu(\omega \wedge x)^{\alpha+1}} \min\left\{\frac{1}{m_\nu(\omega \wedge x)}, m_\nu(\omega \wedge x)\right\}^{1+\delta} \, \mathrm{d}\nu(\omega),
$$

and similarly

$$
d_x(0) = \int_{\partial T_x} \frac{m_\nu(x)^\alpha}{m_\nu(\omega \wedge x)^{\alpha+1}} \min\left\{\frac{1}{m_\nu(\omega \wedge x)}, m_\nu(\omega \wedge x)\right\}^{1+\delta} \, \mathrm{d}\nu(\omega).
$$

Then

$$
\int_{\partial T} K_\delta(x,\omega) \, \mathrm{d}\nu(\omega) = \sum_{k\geq 0} c_x(k) d_x(k) = \langle c_x, d_x \rangle_{\ell^2(\mathbb{N})}.
$$

Since $\{d_x(k)\}_{k\in\mathbb{N}} \in \ell^1(\mathbb{N}) \subset \ell^2(\mathbb{N})$, we can choose any $c_x \in \{d_x(k)\}_{k\in\mathbb{N}}^\perp \subset \ell^\infty(\mathbb{N})$ where the orthogonal complement is taken with respect to the $\ell^2(\mathbb{N})$ inner product. Since such a $c_x \in \ell^\infty(\mathbb{N})$ we can assume without loss of generality that $\|c_x\|_{\ell^\infty(\mathbb{N})} = 1$. We finally conclude that for such a choice

$$
\int_{\partial T} K_\delta(x,\omega) \, \mathrm{d}\nu(\omega) = 0,
$$

and thus K_δ fulfills (1), (2) and (3) of Definition 3.2.

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