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Hardy–Littlewood fractional maximal operators on homogeneous trees

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Abstract

We study the mapping properties of the Hardy–Littlewood fractional maximal operator between Lorentz spaces of the homogeneous tree and discuss the optimality of all the results.

Keywords Homogeneous trees · Hardy–Littlewood maximal function · Fractional maximal operator · Interpolation

Mathematics Subject Classification 05C05 · 42B25 · 43A85

1 Introduction

In the generality of any metric measure space (X, μ) one can define the (centered) *fractional maximal operator* of parameter $\gamma > 0$ as the operator \mathcal{M}^γ acting on locally integrable functions $f \in \mathbb{C}^X$ as follows,

$$\mathcal{M}^\gamma f(x) = \sup_{r>0} \frac{1}{\mu(B_r(x))^\gamma} \int_{B_r(x)} |f| d\mu, \quad x \in X. \quad (1.1)$$

We shall simply write \mathcal{M} for the classical Hardy–Littlewood maximal operator, which corresponds to the choice of parameter $\gamma = 1$.

If the measure μ is doubling, the space X is of homogeneous type in the sense of Coifman and Weiss and \mathcal{M} is bounded on $L^p(\mu)$ for every $p > 1$ and is of weak type $(1, 1)$ [9]. This “maximal theorem” has a fundamental role in harmonic analysis, in particular in the theory

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of singular integrals, and many attempts have been made to extend it beyond the setting of spaces of homogeneous type. While on \mathbb{R}^n , thanks to the Besicovitch covering theorem, the doubling condition can be dropped [14], the same is not true in general, even under the hypothesis that the measure of balls grows at most as a polynomial of the radius [2]. In [5, 6] Carbonaro, Mauceri and Meda developed a theory of singular integrals for metric measure spaces satisfying the isoperimetric and the so-called middle point property. In their setting, the maximal theorem holds for the dyadic maximal operator, but there is no equivalent result available for \mathcal{M} .

Despite these difficulties, there are some interesting classes of metric measure spaces where the theory seems to behave well even in the (drastic) absence of the doubling property. This is the case for non-compact symmetric spaces, where \mathcal{M} is bounded on L^p for every $p > 1$ [8] and is of weak type $(1, 1)$ [30], and for homogeneous trees, where the maximal theorem for \mathcal{M} was proved independently by Cowling, Meda, and Setti [12] and by Naor and Tao [22]. As observed in [12], the result on the homogeneous tree can be also deduced from an older theorem by Rochberg and Taibleson [26]. In these settings it is also known that the fractional maximal operator $\mathcal{M}^{1/2}$ is of restricted weak type $(2, 2)$; this was proved by Ionescu on non-compact symmetric spaces of real rank one [18] and by Veca on homogeneous trees [31].

The aim of this note is to complement the works [12, 22] and [31] extending the study of the mapping properties of \mathcal{M}^γ on Lebesgue and Lorentz spaces of the homogeneous tree T also to values of $\gamma \neq 1, 1/2$. Here, as in the aforementioned works, T is endowed with the standard graph distance and the counting measure $|\cdot|$. Observe that if T is homogeneous of order k , i.e., $|B_1(x)| = k + 2$ for every $x \in T$, then $|B_r(x)| \approx k^r$, for every $x \in T, r \in \mathbb{N}$. Hence, if x belongs to a ball B of radius r (without the requirement of being the center) then $B \subseteq B_{2r}(x)$, from which follows

$$\frac{1}{\mu(B)^\gamma} \int_B |f| d\mu \leq \frac{1}{\mu(B)^\gamma} \int_{B_{2r}(x)} |f| d\mu \approx \frac{1}{\mu(B_{2r}(x))^{\gamma/2}} \int_{B_{2r}(x)} |f| d\mu. \tag{1.2}$$

This simple observation allows to transfer boundedness results for $\mathcal{M}^{\gamma/2}$ to the *uncentered* (fractional) maximal operator $\widetilde{\mathcal{M}}^\gamma$, i.e., the operator obtained by taking the supremum in (1.1) over *all* the balls containing x , and not only on those centered at it. The same phenomenon applies to symmetric spaces, and hence for instance, the results by Ionescu and Veca on $\mathcal{M}^{1/2}$ also give that the uncentered maximal operator $\widetilde{\mathcal{M}}$ is of weak type $(1, 1)$ in the respective settings. To have such a control on the uncentered maximal operator is particularly good, since while in spaces of homogeneous type \mathcal{M}^γ and $\widetilde{\mathcal{M}}^\gamma$ are comparable, in the nondoubling setting they may be very distinct objects. For instance, as we said, \mathcal{M} is of weak type $(1, 1)$ with respect to any Borel measure in $\mathbb{R}^n, n \geq 1$, but the uncentered operator $\widetilde{\mathcal{M}}$ associated to the gaussian measure is not of weak type $(1, 1)$ on \mathbb{R}^2 [27].

That said, the interest in fractional maximal operators cannot be reduced to the study of uncentered maximal operators in nondoubling settings. Weighted L^p norm inequalities for fractional maximal operators on \mathbb{R}^n are a classical object of study [1, 21, 32] and are intimately related to boundedness properties of Riesz potentials [21, Theorem 1]. More recently, also the mapping properties of \mathcal{M}^γ on Sobolev, Hölder, Campanato, Morrey and variable exponent L^p spaces have been studied, in \mathbb{R}^n and in metric spaces [4, 16, 17, 20]. Also in the discrete setting mapping properties of maximal operators are an active area of research. Weak type estimates for \mathcal{M} have been obtained on \mathbb{Z}^d [7] and on graphs fulfilling some geometric properties [28]. On the homogeneous tree also a weighted theory for strong and weak type estimates is developing, both for \mathcal{M} [23, 24] and for \mathcal{M}^γ with $\gamma \in (0, 1)$ [15].

The paper is organized as follows. In Sect. 3 we prove endpoint results for \mathcal{M}^γ analogous to the aforementioned ones for values of $\gamma \neq 1, 1/2$. In particular, in Theorems 3.1 and 3.2 we show that if $\gamma \in (1/2, 1)$, then \mathcal{M}^γ is of restricted weak-type $(1/\gamma, 1/\gamma)$ and is bounded from $L^{1/(1-\gamma), 1/[2(1-\gamma)]}$ to $L^{1/(1-\gamma), \infty}$, and if $\gamma \in (0, 1/2]$, then it is of restricted weak-type $(1/(1-\gamma), 1/\gamma)$. While Theorem 3.2 follows from a rather simple application of a sharpened version of the Kunze–Stein phenomenon for Lorentz spaces on the homogeneous tree [10], for Theorem 3.1 a different approach based on a complex interpolation argument is needed. We also provide strong type results. In Theorem 3.4 we prove that, for any $\gamma \in (0, 1]$, \mathcal{M}^γ maps L^p boundedly to L^q when $1 \leq p \leq q \leq \infty$ and (i) $q > 1/\gamma$ and $p < 1/(1-\gamma)$ or (ii) $p = 1/(1-\gamma)$, $q = \infty$.

In Sect. 4 we discuss the optimality of the results of Sect. 3. In Theorem 4.1 we prove that Theorems 3.1 and 3.2 are optimal in the sense that, if $t \in [1, \infty)$, then \mathcal{M}^γ is unbounded from $L^{1/(1-\gamma), s}$ to $L^{1/(1-\gamma), t}$ and from $L^{p, s}$ to $L^{1/\gamma, t}$, for every $p \in [1, \infty)$ and $s \in [1, \infty]$. Proposition 4.2 is a partial converse to Theorem 3.4: it states that \mathcal{M}^γ does not map L^p boundedly to L^q for all the values of p and q not satisfying (i) and (ii) and not lying on the open segment $(1-\gamma, 1/q)$ with $0 < 1/q < \min\{\gamma, 1-\gamma\}$; to prove or disprove the L^p to L^q boundedness for points on such a critical segment seems to be a difficult problem, which we leave open for the time being. Theorem 4.5 shows that $\mathcal{M}^{1/2}$ is unbounded from $L^{2, s}$ to $L^{2, \infty}$ for every $s > 1$. This is an optimality result for the aforementioned theorem by Veca and should be considered as the tree analogue of [19, Sect. 4]. If Theorem 4.5 extends to values of $\gamma \neq 1/2$ remains an open problem. We discuss it and provide evidence that if such an extension is possible it is far from being straightforward, see Proposition 4.6.

We end the paper by comparing our strong type results with those in [15]. In particular, we show that the sufficient condition for boundedness given in [15] is not strong enough to provide a positive answer to our open question on strong boundedness on the critical segment.

It is natural to ask whether analogous results may be proved on nonhomogeneous trees. Some of them extend quite naturally to certain classes of trees, for some others the proofs are very specific to the homogeneous case and genuinely new approaches seem to be needed. This is work in progress.

In the previous pages as well as in those to come, we adopt the convention of writing $A \lesssim B$ if there exists a positive constant c , not depending on variables but possibly depending on parameters (which are the variables and which the parameters should be clear from time to time by the context) such that $A \leq cB$, and $A \approx B$ if it is both $A \lesssim B$ and $B \lesssim A$.

2 Preliminaries

Let T be a homogeneous tree, i.e., a connected graph with no cycles where each vertex has exactly $k+1$ neighbors for some $k \geq 2$. Nothing will ever depend on the specific value of k , which we assume fixed once for all. We identify T with its set of vertices and endow it with the standard graph distance d , counting the number of edges along the shortest path connecting two vertices. We fix an (arbitrary) distinguished point $o \in X$ and we abbreviate $d(o, x)$ with $\|x\|$. For every $x \in T$ and $r \in \mathbb{N}$ we denote by $B_r(x)$ the ball centered at x of radius r , i.e., $B_r(x) = \{y \in T : d(y, x) \leq r\}$ and by $S_r(x)$ the sphere centered at x of radius r , i.e., $S_r(x) = \{y \in T : d(y, x) = r\}$. We endow T with the counting measure and for every subset E of T we denote by $|E|$ its cardinality. Observe that $|B_r(x)| = |B_r(o)| \approx k^r$ for every $x \in T$ and $r \in \mathbb{N}$. It follows that the fraction maximal operator of parameter $\gamma > 0$ (1.1) on $(T, |\cdot|)$ takes the form

$$\mathcal{M}^\gamma f(x) = \sup_{r \in \mathbb{N}} \frac{1}{|B_r(x)|^\gamma} \sum_{y \in B_r(x)} |f(y)| = \sup_{r \in \mathbb{N}} \frac{1}{|B_r(o)|^\gamma} \sum_{y \in B_r(x)} |f(y)|, \quad f \in \mathbb{C}^T, x \in T.$$

We are studying the mapping properties of \mathcal{M}^γ between Lebesgue and Lorentz spaces on T . For every $p \in [1, \infty)$, we denote by L^p the space of functions $f \in \mathbb{C}^T$ such that $\|f\|_p^p = \sum_{x \in T} |f(x)|^p < \infty$ and by L^∞ the space of functions $f \in \mathbb{C}^T$ such that $\|f\|_\infty = \sup_{x \in T} |f(x)| < \infty$. For every $p \in [1, \infty]$, we denote by p' its conjugate exponent, i.e., $1/p + 1/p' = 1$. We also introduce the Lorentz spaces $L^{p,s}$ and $L^{p,\infty}$ on T , which for $p \in [1, \infty)$ and $s \in [1, \infty)$ are defined by

$$L^{p,s} = \left\{ f \in \mathbb{C}^T : \|f\|_{p,s} = \left(p \int_0^{+\infty} \lambda^s |\{x \in T : |f(x)| > \lambda\}|^{\frac{s}{p}} \frac{d\lambda}{\lambda} \right)^{1/s} < \infty \right\},$$

and

$$L^{p,\infty} = \{f \in \mathbb{C}^T : \|f\|_{p,\infty} = \sup_{\lambda > 0} \lambda |\{x \in T : |f(x)| > \lambda\}|^{1/p} < \infty\}.$$

By convention, we set $L^{\infty,\infty} = L^\infty$. For any $p, q \in [1, \infty]$, we say that an operator is of strong (weak) type (p, q) if it is bounded from L^p to L^q ($L^{q,\infty}$ respectively), and for any $p, q \in [1, \infty)$ we say that an operator is of *restricted* weak type (p, q) if it satisfies the weak type (p, q) condition when it is restricted to characteristic functions of sets of finite measure. When $q > 1$, this is equivalent to the boundedness from $L^{p,1}$ to $L^{q,\infty}$ (see [29, Th. 3.13]).

Let us recall that discrete Lorentz spaces enjoy the following embedding property, which will be of good use in the next section.

Lemma 2.1 *If an operator acting between discrete Lorentz spaces is of restricted weak type (p_0, q_0) for some $p_0, q_0 \in [1, \infty)$, then it is of restricted weak type (strong type) (p, q) for every $1 \leq p \leq p_0$ and $q_0 \leq q \leq \infty$ (respectively, for every $1 \leq p < p_0$ and $q_0 < q \leq \infty$).*

Proof For what concerns the restricted weak type boundedness it suffices to recall that since we are dealing with Lorentz spaces on a discrete setting, for any $p, p_0 \in [1, \infty)$ and $s \in [1, \infty]$, the continuous inclusion $L^{p,s} \hookrightarrow L^{p_0,s}$ holds if $p \leq p_0$. The statement regarding the strong type boundedness then follows from the general Marcinkiewicz interpolation theorem [3, Theorem 5.3.2]. □

Another important auxiliary result is the following formula for the Lorentz space norm of a radial function on T , which follows from a result of Pytlik [25] (see also [10, Lemma A3]). Here, with harmless abuse of notation, we denote by $f(n)$ the value that a radial function f takes on $S_n(o)$.

Lemma 2.2 *Let f be a radial function on T . Then, for every $p \in [1, \infty)$ and $s \in [1, \infty]$,*

$$\|f\|_{p,s} \approx \|g\|_{L^s(\mathbb{N})},$$

where $g(n) = f(n)k^{n/p}$.

When studying the mapping properties of \mathcal{M}^γ , two simple but crucial observations are in order.

Remark 2.3 For every $\gamma > 0$, \mathcal{M}^γ is unbounded from L^p to L^q when $p > q$, since the identity is not bounded on the same spaces and $\mathcal{M}^\gamma f \geq |f|$ pointwise, for every $\gamma > 0$.

Remark 2.4 For every $\gamma > 0$, $r \in \mathbb{N}$, let $a_{r,\gamma}$ denote the radial function $|B_r(o)|^{-\gamma} \chi_{B_r(o)}$. Then, for every $f \in \mathbb{C}^T$ and every $r \in \mathbb{N}$ and x in T ,

$$\begin{aligned} \frac{1}{|B_r(o)|^\gamma} \sum_{y \in B_r(x)} f(y) &= \frac{1}{|B_r(o)|^\gamma} \sum_{n=0}^r \sum_{d(x,y)=n} f(y) \\ &= \sum_{n=0}^\infty a_{r,\gamma}(n) \sum_{d(x,y)=n} f(y) = f * a_{r,\gamma}(x), \end{aligned}$$

where the convolution of f with a radial function is defined in [10, Formula (2.5)]. Hence,

$$\mathcal{M}^\gamma f(x) = \sup_{r \in \mathbb{N}} |f| * a_{r,\gamma}(x) \lesssim |f| * a_\gamma(x) =: \mathcal{A}^\gamma |f|(x), \quad x \in T, \tag{2.1}$$

where $a_\gamma(x) = k^{-\gamma \|x\|}$.

The two remarks alone are sufficient to give a full picture of the boundedness properties of \mathcal{M}^γ when $\gamma > 1$. Indeed, for $\gamma > 1$ the kernel $a_\gamma \in L^p$ for every $p \in [1, \infty]$, and a simple application of Young’s inequality gives us that the convolution operator \mathcal{A}^γ is of strong type (p, q) if $p \leq q$. Hence, it follows by the two remarks that, for $\gamma > 1$, \mathcal{M}^γ is of strong type (p, q) if and only if $p \leq q$.

The case $\gamma = 1$ is much less trivial, but by now well understood. In this case, the L^1 to L^1 boundedness fails, since $\mathcal{M}\delta_o$ does not belong to L^1 . Nevertheless, it was proved in [12, 22] that \mathcal{M} is of weak type $(1, 1)$. Since \mathcal{M} is trivially bounded from L^∞ to itself, it follows by interpolation, discrete L^p spaces inclusions, and by Remark 2.3 that it is of strong type (p, q) if and only if $1 \leq p \leq q \neq 1$.

The rest of the paper is devoted to the study of boundedness properties of fractional Hardy–Littlewood maximal operators \mathcal{M}^γ for the remaining values $\gamma \in (0, 1)$.

3 Boundedness properties of \mathcal{M}^γ

The first two theorems we present should be considered as analogues of the endpoint results in [12, 22, 31] for values of γ not necessarily equal to 1 or 1/2.

Theorem 3.1 *The operator \mathcal{M}^γ is of restricted weak type $(1/(1-\gamma), 1/\gamma)$, if $\gamma \in (0, 1/2]$, and is bounded from $L^{1/(1-\gamma), 1/[2(1-\gamma)]}$ to $L^{1/(1-\gamma), \infty}$, if $\gamma \in (1/2, 1]$.*

Proof The case $\gamma = 1$ is trivial, while the case $\gamma = 1/2$ was proved in [31, Theorem 5.1]. Assume now $\gamma \neq 1, 1/2$. For any couple of functions $\phi : T \rightarrow (0, \infty)$, $\psi : T \rightarrow \{z \in \mathbb{C} \mid |z| = 1\}$, and any $z \in \mathbb{C}$, we define a linear operator $\mathcal{T}_{z,\phi,\psi}$ which acts on $f \in \mathbb{C}^T$ as

$$\mathcal{T}_{z,\phi,\psi} f(x) = \frac{1}{|B_{\phi(x)}(o)|^{z/2}} \sum_{y \in B_{\phi(x)}(x)} f(y)\psi(y).$$

It is easy to see that for any $z \in \mathbb{C}$ and $x \in T$,

$$\sup_{\phi,\psi} |\mathcal{T}_{z,\phi,\psi} f(x)| = \mathcal{M}^{\operatorname{Re}z/2} f(x), \tag{3.1}$$

where the supremum is taken over all functions ϕ, ψ as above. Hence, if $\mathcal{T}_{z,\phi,\psi}$ is bounded from $L^{p,s}$ to $L^{q,t}$ with operator norm which does not depend on ϕ and ψ , then for every

$\varepsilon > 0$ we may find functions ϕ and ψ , and a constant C not depending on them, such that

$$\|\mathcal{M}^\gamma f\|_{q,t} - \varepsilon \leq \|\mathcal{T}_{2\gamma,\phi,\psi} f\|_{q,t} \leq C\|f\|_{p,s},$$

and by letting $\varepsilon \rightarrow 0^+$, we obtain that also \mathcal{M}^γ is bounded from $L^{p,s}$ to $L^{q,t}$. Thus, it is enough to prove that $\mathcal{T}_{2\gamma,\phi,\psi}$ is of restricted weak type $(1/(1-\gamma), 1/\gamma)$, if $\gamma \in (0, 1/2]$, and is bounded from $L^{1/(1-\gamma), 1/[2(1-\gamma)]}$ to $L^{1/(1-\gamma), \infty}$, if $\gamma \in (1/2, 1)$, with operator norm that does not depend on ϕ and ψ .

To prove this, fix two functions $\phi : T \rightarrow (0, \infty)$, $\psi : T \rightarrow \{z \in \mathbb{C} \mid |z| = 1\}$, set $\mathcal{T}_z = \mathcal{T}_{z,\phi,\psi}$ and $\tilde{\mathcal{T}}_z = \mathcal{T}_{z+1}$, and consider the families of linear operators $\{\mathcal{T}_z\}_{z \in \bar{S}}$ and $\{\tilde{\mathcal{T}}_z\}_{z \in \bar{S}}$, where \bar{S} denotes the closure of the strip $S = \{0 < \operatorname{Re} z < 1\}$. We aim to apply Cwikel and Janson’s complex interpolation result [13, Theorem 2] to these families of operators and, respectively, to the spaces, $A_0 = L^1$, $A_1 = L^{2,1}$, $B_0 = L^\infty$, $B_1 = L^{2,\infty}$ and $\tilde{A}_0 = L^{2,1}$, $\tilde{A}_1 = L^\infty$, $\tilde{B}_0 = L^{2,\infty}$, $\tilde{B}_1 = L^\infty$.

Observe that $A := A_0 \cap A_1 = A_0$ and $\tilde{A} := \tilde{A}_0 \cap \tilde{A}_1 = \tilde{A}_0$. We also set $B^+ = L^1$. It is easy to see that $\|\mathcal{M}^0 f\|_\infty = \|f\|_1$ for every $f \in L^1$. Moreover we know that \mathcal{M} is of strong type (∞, ∞) and, by [31, Theorem 5.1], that $\mathcal{M}^{1/2}$ is of restricted weak type $(2, 2)$. By means of (3.1), it follows that \mathcal{T}_z is bounded from A_0 to B_0 when $\operatorname{Re} z = 0$, and from A_1 to B_1 when $\operatorname{Re} z = 1$, and $\tilde{\mathcal{T}}_z$ is bounded from \tilde{A}_0 to \tilde{B}_0 when $\operatorname{Re} z = 0$ and from \tilde{A}_1 to \tilde{B}_1 when $\operatorname{Re} z = 1$.

We now observe that for every $b^+ \in B^+$, $a \in A$, $\tilde{a} \in \tilde{A}$, the two functions

$$z \mapsto \langle b^+, \mathcal{T}_z a \rangle, \quad \text{and} \quad z \mapsto \langle b^+, \tilde{\mathcal{T}}_z \tilde{a} \rangle,$$

both belong to $H^\infty(\{\operatorname{Re} z \geq 0\})$, the space of bounded analytic functions on $\{\operatorname{Re} z > 0\}$ which are continuous on $\{\operatorname{Re} z \geq 0\}$. Indeed, a straightforward application of Morera’s Theorem shows that both functions are entire. Moreover, for $\operatorname{Re} z \geq 0$ we have

$$|\langle b^+, \mathcal{T}_z a \rangle| \leq \sum_{x \in T} |b^+(x)| \mathcal{M}^0 a(x) = \|b^+\|_1 \|a\|_1,$$

and

$$\begin{aligned} |\langle b^+, \tilde{\mathcal{T}}_z \tilde{a} \rangle| &\leq \sum_{x \in T} |b^+(x)| \mathcal{M}^{1/2} \tilde{a}(x) \leq \|\mathcal{M}^{1/2} \tilde{a}\|_{2,\infty} \|b^+\|_{2,1} \\ &\leq \|\mathcal{M}^{1/2}\|_{L^{2,1} \rightarrow L^{2,\infty}} \|\tilde{a}\|_{2,1} \|b^+\|_1, \end{aligned}$$

where in the last two steps we used Hölder’s inequality, Veca’s theorem and the continuous inclusion of $L^{2,1}$ into L^1 . It follows, in particular, that $z \mapsto \langle b^+, \mathcal{T}_z a \rangle$ and $z \mapsto \langle b^+, \tilde{\mathcal{T}}_z \tilde{a} \rangle$ belong to $H^\infty(\bar{S})$. Therefore, [13, Theorem 2] applies to both the families $\{\mathcal{T}_z\}_{z \in \bar{S}}$ and $\{\tilde{\mathcal{T}}_z\}_{z \in \bar{S}}$, and gives that for every $\theta \in (0, 1)$

$$\mathcal{T}_\theta : [A_0, A_1]_\theta \rightarrow [B_0, B_1]^\theta, \quad \tilde{\mathcal{T}}_\theta : [\tilde{A}_0, \tilde{A}_1]_\theta \rightarrow [\tilde{B}_0, \tilde{B}_1]^\theta,$$

or equivalently (see for instance [3])

$$\mathcal{T}_\theta : L^{2/(2-\theta), 1} \rightarrow L^{2/\theta, \infty}, \quad \tilde{\mathcal{T}}_\theta : L^{2/(1-\theta), 1/(1-\theta)} \rightarrow L^{2/(1-\theta), \infty},$$

with operator norm not depending on ϕ, ψ . Recalling that $\tilde{\mathcal{T}}_\theta = \mathcal{T}_{\theta+1}$, and setting $\gamma = \theta/2$ then one has

$$\mathcal{T}_{2\gamma} : \begin{cases} L^{1/(1-\gamma), 1} \rightarrow L^{1/\gamma, \infty}, & \gamma \in (0, 1/2) \\ L^{1/(1-\gamma), 1/[2(1-\gamma)]} \rightarrow L^{1/(1-\gamma), \infty}, & \gamma \in (1/2, 1), \end{cases}$$

with operator norm not depending on ϕ, ψ . This completes the proof. □

Next theorem gives us a supplementary endpoint result which is not possible to obtain by means of the above complex interpolation argument. To prove it we exploit a sharpened version of the Kunze–Stein phenomenon for Lorentz spaces on the homogeneous tree [10, Theorem 1].

Theorem 3.2 *If $\gamma \in [1/2, 1]$ the maximal operator \mathcal{M}^γ is of restricted weak type $(1/\gamma, 1/\gamma)$.*

Proof The case $\gamma = 1$ follows by [12, 22], and the case $\gamma = 1/2$ was proved in [31, Theorem 5.1]. Assume now $\gamma \in (1/2, 1)$. By Remark 2.4, it is enough to prove the result for the linear convolution operator

$$\mathcal{A}^\gamma f(x) = f * a_\gamma(x), \quad x \in T.$$

By Lemma 2.2, $\|a_\gamma\|_{L^{1/\gamma, \infty}} \approx \sup_{n \in \mathbb{N}} k^{-\gamma n} k^{\gamma n} = 1$.

Hence, by [10, Theorem 1] we deduce that

$$\|\mathcal{A}^\gamma f\|_{1/\gamma, \infty} \lesssim \|f\|_{1/\gamma, 1} \|a_\gamma\|_{1/\gamma, \infty} \lesssim \|f\|_{1/\gamma, 1},$$

i.e., that \mathcal{A}^γ is of restricted weak type $(1/\gamma, 1/\gamma)$. □

Remark 3.3 Observe that from Theorem 3.2 one can easily deduce, arguing by duality, that \mathcal{M}^γ is of restricted weak type $(1/(1 - \gamma), 1/(1 - \gamma))$ for $\gamma \in (1/2, 1)$. Indeed, it is easily seen that the linear operator \mathcal{A}^γ is self-adjoint. Hence, by Hölder’s inequality for Lorentz spaces,

$$|\langle \mathcal{A}^\gamma f, g \rangle| = |\langle f, \mathcal{A}^\gamma g \rangle| \leq \|f\|_{1/(1-\gamma), 1} \|\mathcal{A}^\gamma g\|_{1/\gamma, \infty} \lesssim \|f\|_{1/(1-\gamma), 1} \|g\|_{1/\gamma, 1}.$$

Passing to the supremum over all functions g with $\|g\|_{1/\gamma, 1} \leq 1$, we obtain

$$\|\mathcal{A}^\gamma f\|_{1/(1-\gamma), \infty} \lesssim \|f\|_{1/(1-\gamma), 1}.$$

While this endpoint result would be sufficient for interpolation purposes, we remark that it is weaker than the result for $\gamma \in (1/2, 1)$ obtained in Theorem 3.1, since $L^{1/(1-\gamma), 1} \not\subset L^{1/(1-\gamma), 1/[2(1-\gamma)]}$ when $\gamma > 1/2$.

Thanks to the endpoint results obtained so far we can deduce strong type estimates.

Theorem 3.4 *Let $\gamma \in (0, 1]$. Then, \mathcal{M}^γ is of strong type (p, q) if one of the following conditions hold:*

- (i) $1 \leq p \leq q \leq \infty$ and $q > 1/\gamma$ and $p < 1/(1 - \gamma)$.
- (ii) $p = 1/(1 - \gamma)$ and $q = \infty$.

Proof For $\gamma = 1$ the result boils down to the (p, q) strong boundedness of \mathcal{M} for $1 \leq p \leq q \neq 1$ discussed in Sect. 2.

If $\gamma \in (0, 1/2]$ it follows from Theorem 3.1 and Lemma 2.1 that \mathcal{M}^γ is of strong type (p, q) when $p < 1/(1 - \gamma)$ and $q > 1/\gamma$. On the other hand, if $\gamma \in (1/2, 1)$, combining Theorems 3.1 and 3.2, by interpolation we have that \mathcal{M}^γ is of strong type (t, t) (and hence of strong type (p, q) with $p \leq t \leq q$), for $1/\gamma < t < 1/(1 - \gamma)$. It follows that \mathcal{M}^γ is of strong type (p, q) whenever $1 \leq p \leq q \leq \infty$ with $p < 1/(1 - \gamma)$ and $q > 1/\gamma$. This proves (i).

To prove (ii), observe that for every $x \in T$ and $r \in \mathbb{N}$, by Hölder’s inequality with exponents $p = 1/(1 - \gamma)$ and $p' = 1/\gamma$, we obtain

$$\frac{1}{|B_r(o)|^\gamma} \sum_{y \in B_r(x)} |f(y)| \leq \|f\|_{1/(1-\gamma)}.$$

Passing to the supremum on $r \in \mathbb{N}$, we get that $\mathcal{M}^\gamma f(x) \leq \|f\|_{1/(1-\gamma)}$ for every $x \in T$, hence \mathcal{M}^γ is of strong type $(1/(1-\gamma), \infty)$. \square

Let us remark that (i) above, which we deduce as a straightforward consequence of our endpoint results, for values of $\gamma \in [1/2, 1]$ also follows from a more general theorem by Cowling, Meda and Setti [11, Theorem 3.4].

4 Optimality results

In this section we discuss the optimality of the results obtained in Sect. 3 under different points of view. Recall that Theorems 3.1 and 3.2 give values of (p, q) such that \mathcal{M}^γ is bounded from $L^{p,s}$ to $L^{q,t}$ when $s = 1$ and $t = \infty$. Our first optimality result shows that these results fail if $t \neq \infty$, for any fixed $s \in [1, \infty]$.

Proposition 4.1 *Let $\gamma \in (0, 1)$. If $t \in [1, \infty)$, then \mathcal{M}^γ is unbounded from $L^{1/(1-\gamma),s}$ to $L^{1/(1-\gamma),t}$ and from $L^{p,s}$ to $L^{1/\gamma,t}$, for every $p \in [1, \infty)$ and $s \in [1, \infty]$.*

Proof For every $n \in \mathbb{N}$ set $f_n = \chi_{B_n(o)}$. It is easy to see that

$$\|f_n\|_{p,s} = |B_n(o)|^{1/p}, \quad s \in [1, \infty].$$

For any $x \in B_n(o)$,

$$\mathcal{M}^\gamma f_n(x) = \sup_{r \in \mathbb{N}} \frac{|B_r(x) \cap B_n(o)|}{|B_r(o)|^\gamma} \geq \frac{|B_{n-\|x\|}(x)|}{|B_{n-\|x\|}(o)|^\gamma} \approx k^{(n-\|x\|)(1-\gamma)}.$$

Since $\varphi_n(x) := f_n(x)k^{(n-\|x\|)(1-\gamma)}$ is radial, we may define $g_n(j) = \varphi_n(j)k^{j/p}$ and by Lemma 2.2, for any $t \in [1, \infty)$ we have

$$\|\mathcal{M}^\gamma f_n\|_{1/(1-\gamma),t} \gtrsim \|g_n\|_{L^t(\mathbb{N})} = \left(\sum_{j=0}^n k^{t(n-j)(1-\gamma)} k^{jt(1-\gamma)} \right)^{1/t} = n^{1/t} k^{n(1-\gamma)}.$$

It follows that for $t \in [1, \infty)$

$$\frac{\|\mathcal{M}^\gamma f_n\|_{1/(1-\gamma),t}}{\|f_n\|_{1/(1-\gamma),s}} \gtrsim n^{1/t} \longrightarrow \infty, \quad \text{as } n \rightarrow \infty,$$

which proves the first assertion. To prove the second, observe that if we apply \mathcal{M}^γ to a Dirac delta centered at o we get

$$\mathcal{M}^\gamma \delta_o(x) = \frac{1}{|B_{\|x\|}(o)|^\gamma}, \quad x \in T.$$

It follows by Lemma 2.2 that

$$\|\mathcal{M}^\gamma \delta_o\|_{1/\gamma,t} \approx \left(\sum_{j=0}^\infty \frac{k^{j\gamma t}}{|B_j(o)|^{\gamma t}} \right)^{1/t} \approx \left(\sum_{j=0}^\infty 1 \right)^{1/t} = \infty,$$

and this concludes the proof. \square

Now, we discuss the optimality of the values of (p, q) for which we have established strong and restricted weak type boundedness of \mathcal{M}^γ .

Proposition 4.2 *Let $\gamma \in (0, 1)$. The fractional maximal operator \mathcal{M}^γ is not of strong type (p, q) if either $q < p$, or $p = q = 1/(1 - \gamma)$, or $q \leq 1/\gamma$, or $p > 1/(1 - \gamma)$.*

Proof The first assertion is Remark 2.3, while the second and the third follow directly from Proposition 4.1. To prove the last assertion, leveraging the discrete L^p spaces inclusions, it is enough to show that \mathcal{M}^γ is unbounded from L^p to L^∞ when $p > 1/(1 - \gamma)$.

To prove this, consider the sequence of functions defined by $f_n = \chi_{B_n(o)}$, $n \in \mathbb{N}$. We know from the proof of Proposition 4.1 that $\|f_n\|_p = |B_n(o)|^{1/p}$ and that $\mathcal{M}^\gamma f_n(o) \geq |B_n(o)|^{1-\gamma}$. Thus, if $p > 1/(1 - \gamma)$,

$$\frac{\|\mathcal{M}^\gamma f_n\|_\infty}{\|f_n\|_p} \geq |B_n(o)|^{1-\gamma-1/p} \rightarrow \infty, \quad \text{as } n \rightarrow \infty,$$

which implies (ii). □

At this point, we have a complete description of the values of p, q for which \mathcal{M}^γ is of restricted weak type (p, q) and for which is not.

Corollary 4.3 *Let $\gamma \in (0, 1)$. Then, \mathcal{M}^γ is of restricted weak type (p, q) if and only if $1 \leq p \leq q \leq \infty$ and $q \geq 1/\gamma$ and $p \leq 1/(1 - \gamma)$.*

Proof The if part follows from Theorems 3.1 and 3.2, interpolation and Lemma 2.1. The only if part must hold true since otherwise, by Lemma 2.1, Proposition 4.2 would be contradicted. □

Remark 4.4 The description of the strong type boundedness region for \mathcal{M}^γ is not complete yet for $\gamma \in (0, 1)$. Indeed, the strong $(1/(1 - \gamma), q)$ boundedness for $0 < 1/q < \min\{1 - \gamma, \gamma\}$ has not been established nor disproved. This remains an open question. The best we can say at present is what we obtain interpolating the result in Theorem 3.1 with (ii) in Theorem 3.4, i.e., that \mathcal{M}^γ is bounded from $L^{1/(1-\gamma),t}$ to $L^{q,\infty}$, where $t = q/(1 + q - \gamma q)$ for $\gamma \in (1/2, 1)$ and $t = q'$ for $\gamma \in (0, 1/2]$.

Next picture summarizes all the information that we obtained on the strong type boundedness of \mathcal{M}^γ .

We end the section with a last natural question concerning optimality, that is, if the endpoint results obtained in Theorems 3.1 and 3.2 can be improved to $L^{p,s}$ to $L^{q,\infty}$ boundedness results for some $s > 1$. In the particular case $\gamma = 1/2$, the two theorems reduce to Veca’s result [31], i.e., that $\mathcal{M}^{1/2}$ is of restricted weak type $(2, 2)$, and in this case we are able to show that the answer to the above question is negative. This represents a discrete counterpart of the result by Ionescu [19] obtained in the setting of non-compact symmetric spaces mentioned in the introduction.

In order to prove this result, as well as the one that follows, it is useful to highlight the following formula, which holds true for any radial nonnegative function $f \in \mathbb{C}^T$ and can be obtained by a straightforward computation,

$$\sum_{y \in S_n(x)} f(y) \approx \begin{cases} \sum_{j=0}^n f(\|x\| + n - 2j)k^{n-j} & \text{if } n \leq \|x\|, \\ \sum_{j=0}^{\|x\|} f(\|x\| + n - 2j)k^{n-j} & \text{otherwise.} \end{cases} \tag{4.1}$$

Theorem 4.5 *For every $s > 1$, $\mathcal{M}^{1/2}$ is unbounded from $L^{2,s}$ to $L^{2,\infty}$.*

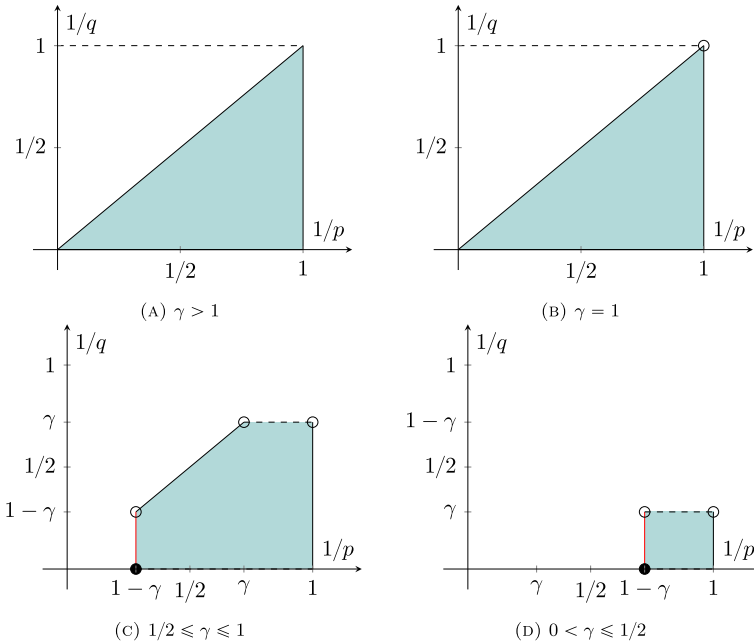


Fig. 1 \mathcal{M}^γ is of strong type (p, q) if the point $(1/p, 1/q)$ lies in the colored region or along the portion of its boundary made of black continuous lines and the black point. It is unbounded outside the colored region, at circled points and along the dashed line. We don't know whether it is bounded or not along the red segment

Proof Fix $s > 1$ and $1/s < \beta < 1$. Define $g \in \mathbb{C}^T$ by

$$g(x) = \frac{k^{-\|x\|/2}}{(1 + \|x\|)^\beta}, \quad x \in T.$$

Since g is radial, by Lemma 2.2, $\|g\|_{2,s}^s \approx \sum_{n=0}^\infty (1+n)^{-\beta s}$, thus g belongs to $L^{2,s}$. Moreover by formula (4.1), for any $x \in T$,

$$\sum_{y \in \mathcal{S}_{\|x\|}(x)} g(y) \approx \sum_{j=0}^{\|x\|} \frac{k^{-2(\|x\|-j)/2} k^{\|x\|-j}}{(1 + 2(\|x\| - j))^\beta} = \sum_{j=0}^{\|x\|} \frac{1}{(1 + 2(\|x\| - j))^\beta} \approx (1 + \|x\|)^{1-\beta}.$$

It follows,

$$\mathcal{M}^{1/2} g(x) \geq \frac{1}{|B_{\|x\|}(o)|^{1/2}} \sum_{y \in \mathcal{S}_{\|x\|}(x)} g(y) \gtrsim k^{-\|x\|/2} (1 + \|x\|)^{1-\beta} =: m(x).$$

Since m is radial and $\beta < 1$, again by Lemma 2.2,

$$\|m\|_{2,\infty} \approx \|m(\cdot)k^{1/2}\|_{L^\infty(\mathbb{N})} = +\infty.$$

Hence $\mathcal{M}^{1/2}$ does not map $L^{2,s}$ in $L^{2,\infty}$. □

More in general, one may ask whether a similar strategy can be applied also to values of $\gamma \neq 1/2$. Next proposition makes it clear that this is not the case, by showing that no radial function can serve as a counterexample to prove the optimality of Theorems 3.1 and 3.2 with respect to the parameter s .

Proposition 4.6 *Let $(L^{p,s})^\#$ denote the space of radial functions in $L^{p,s}$. Then, \mathcal{M}^γ maps boundedly $(L^{1/\gamma,s})^\#$ to $L^{1/\gamma,\infty}$ and $(L^{1/(1-\gamma),s})^\#$ to $L^{1/(1-\gamma),\infty}$ when $\gamma > 1/2$ and $(L^{1/(1-\gamma),s})^\#$ to $L^{1/\gamma,\infty}$ when $\gamma < 1/2$, for every $s \in [1, \infty]$.*

Proof Let f be a nonnegative function in $(L^{p,s})^\#$, so that by Lemma 2.2 $g(\cdot) := f(\cdot)k^{(\cdot)/p}$ belongs to $L^s(\mathbb{N})$ and $\|g\|_{L^s(\mathbb{N})} \approx \|f\|_{p,s}$. Rewriting (4.1) in terms of g and then applying Hölder’s inequality we get

$$\begin{aligned} \frac{1}{k^{n\gamma}} \sum_{y \in S_n(x)} f(y) &\approx \begin{cases} k^{-n\gamma} \sum_{j=0}^n g(\|x\| + n - 2j)k^{j(2/p-1)}k^{n/p'-\|x\|/p} & \text{if } n \leq \|x\|, \\ k^{-n\gamma} \sum_{j=0}^{\|x\|} g(\|x\| + n - 2j)k^{j(2/p-1)}k^{n/p'-\|x\|/p} & \text{otherwise,} \end{cases} \\ &\lesssim \begin{cases} k^{n(1/p'-\gamma)}k^{-\|x\|/p}\|g\|_{L^s(\mathbb{N})} & \text{if } p > 2, \\ k^{n(1/p'-\gamma)}k^{-\|x\|/p'}\|g\|_{L^s(\mathbb{N})} & \text{if } p < 2, \\ k^{n(1/p-\gamma)}k^{-\|x\|/p}\|g\|_{L^s(\mathbb{N})} & \text{if } p < 2 \text{ and } n \leq \|x\|. \end{cases} \end{aligned} \tag{4.2}$$

It follows,

$$\mathcal{M}^\gamma f(x) \lesssim \begin{cases} k^{-\|x\|/p}\|g\|_{L^s(\mathbb{N})} & \text{if } 2 < p \leq 1/(1-\gamma), \\ k^{-\|x\|/p'}\|g\|_{L^s(\mathbb{N})} & \text{if } p < 2 \text{ and } p \leq 1/(1-\gamma), \\ k^{-\|x\|/p}\|g\|_{L^s(\mathbb{N})} & \text{if } 1/\gamma \leq p < 2 \text{ and } n \leq \|x\|, \\ k^{-\gamma\|x\|}\|g\|_{L^s(\mathbb{N})} & \text{if } 1/\gamma \leq p < 2 \text{ and } n > \|x\|, \end{cases}$$

and since $x \mapsto k^{-\|x\|/t}$ belongs to $L^{p,\infty}$ if and only if $t \leq p$ and $\|g\|_{L^s(\mathbb{N})} \approx \|f\|_{p,s}$, we have

$$\mathcal{M}^\gamma : \begin{cases} (L^{p,s})^\# \rightarrow L^{p,\infty} & \text{if } 1/\gamma \leq p \leq 1/(1-\gamma) \text{ and } p \neq 2, \\ (L^{p,s})^\# \rightarrow L^{p',\infty} & \text{if } p < 2 \text{ and } p \leq 1/(1-\gamma), \end{cases}$$

The result follows by choosing $p = 1/(1-\gamma)$. □

5 Final remarks

Let $\omega : T \rightarrow \mathbb{R}_+$ and $w(A) = \sum_{x \in T} \omega(x)$, $A \subseteq T$, the associated measure. In a recent paper [15] Ghosh and Rela proved that \mathcal{M}^γ is bounded from $L^p(\omega)$ to $L^q(\omega)$, with $1 < p \leq q < \infty$ and $\gamma \in (0, 1)$, if there exists $\varepsilon \in (0, 1)$ such that $\omega \in \mathcal{Z}_{p,q}^{\varepsilon,1-\gamma}$, i.e., if

$$\sum_{x \in E} \omega(F \cap S_r(x)) \leq Ck^{\varepsilon r\gamma} \omega(E)^{1/p} \omega(F)^{1-1/q}, \quad \text{for some } C > 0.$$

It is natural to check if this sufficient condition can provide a positive answer to the only question we left open regarding strong boundedness (see Remark 4.4). We are showing here that this is not the case, since $|\cdot| \notin \mathcal{Z}_{1/(1-\gamma),q}^{\varepsilon,1-\gamma}$, for any $\varepsilon \in (0, 1)$, and any $1 < q < \infty$. To see this, observe that for any $\gamma, \gamma' \in (0, 1)$, $\mathcal{Z}_{p,q}^{\varepsilon,1-\gamma} = \mathcal{Z}_{p,q}^{\varepsilon',1-\gamma'}$, where $\varepsilon' = \varepsilon\gamma/\gamma'$. Now, choose $\gamma' \in (\varepsilon\gamma, \gamma)$ and $p = 1/(1-\gamma)$, so that $p > 1/(1-\gamma')$. If $|\cdot| \in \mathcal{Z}_{p,q}^{\varepsilon,1-\gamma}$, then we would have that $\mathcal{M}^{\gamma'}$ is bounded from L^p to L^q , with $p > 1/(1-\gamma')$, which contradicts Theorem 4.2. This shows that the result in [15] cannot improve our Theorem 3.4. However, it is not hard to prove that by using the aforementioned result combined with [22, Lemma 5.1], one can directly establish the boundedness of \mathcal{M}^γ on L^p for every $p \in \left(\frac{1}{\gamma}, \frac{1}{1-\gamma}\right)$ in the case $\gamma > 1/2$.

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