Università degli Studi di Torino
Politecnico di Torino

Department of Mathematics

Doctoral Program in Pure and Applied Mathematics
XXXVII cycle, academic years 2021-2024

Doctoral Dissertation

Thesis title :

Four-Dimensional Toric Orbifolds and Holography

Author
Alessio Fontanarossa

Advisor
Dario Martelli

The Coordinator of the PhD program
Anna Maria Fino

Turin, April 2025

ABSTRACT

In this dissertation, I summarize my work on supergravity solutions involving the spindle $\mathbb{\Sigma} \equiv \mathbb{WCP}^1_{[m_-,m_+]}$ and its generalizations, which give rise to $\mathbb{M}_4$, four-dimensional orbifolds. These solutions are relevant to holography, as they are supersymmetric and explicitly contain an Anti-de Sitter factor or are asymptotically locally AdS_4. The first part of the thesis focuses on compact four-dimensional orbifolds, which can be described by compact labeled polytopes, referred to as quadrilaterals. These take the form of $\mathbb{M}_4^{(1)} = \Sigma_{\mathrm{g}} \ltimes \mathbb{\Sigma}_2$ or $\mathbb{M}_4^{(2)} = \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$, representing a non-trivial fibration of a spindle $\mathbb{\Sigma}_2$ over either a smooth Riemann surface Σ_{g} of genus $\mathrm{g} > 1$ or another spindle $\mathbb{\Sigma}_1$, respectively. Both classes can be thought of as orbifold generalizations of Hirzebruch surfaces, and we describe $\mathbb{M}_4^{(2)}$ in terms of toric geometry. A more sophisticated generalization, where each facet of the polytope corresponds to a different spindle, is presented later. I demonstrate how the entropies of $\mathrm{AdS}_2 \times \mathbb{M}_4$ solutions and the gravitational central charges of $\mathrm{AdS}_3 \times \mathbb{M}_4$ are reproduced by extremizing an off-shell free energy, constructed by gluing simple factors known as gravitational blocks. The second part of the thesis deals with non-compact four-dimensional solutions, arising as the total space of complex line bundles over the spindle. The on-shell action, computed via holographic renormalization, reveals a rich structure linked to the twist or anti-twist realization of supersymmetry on the spindle bolt. This structure is then compared to general predictions derived from the equivariant localization of the action, finding perfect agreement.

PUBLICATIONS

This dissertation is based on PhD research projects conducted in collaboration with Mr. Matteo Kevin Crisafio, Dr. Federico Faedo and Prof. Dario Martelli and carried out at the Department of Mathematics of University of Turin between November 2021 and December 2024.

These efforts have resulted in the following publications [1, 2, 3]:

- F. Faedo, A. Fontanarossa and D. Martelli, *Branes wrapped on orbifolds and their gravitational blocks*, *Lett. Math. Phys.* **113** (2023) [2210.16128]
- F. Faedo, A. Fontanarossa and D. Martelli, *Branes wrapped on quadrilaterals*, *Lett. Math. Phys.* **115** (2025) 26 [2402.08724]
- M.K. Crisafio, A. Fontanarossa and D. Martelli, *Nuts, bolts and spindles*, *Lett. Math. Phys.* **115** (2025) 27 [2412.00428]

Contents

Contents

Chapter 1

Introduction

1.1 The AdS/CFT correspondence

The holographic principle Since Hawking's 1974 confirmation of Bekenstein's conjecture—that a black hole's entropy is proportional to the area of its horizon rather than the volume it encloses—the idea of the *holographic principle* has taken shape. First proposed by 't Hooft and later refined by Susskind, this principle found its first concrete realization in 1997 with Maldacena's groundbreaking work [4]. The modern holographic principle, now famously known as the *AdS/CFT correspondence*, establishes a deep relationship between a theory of (super)gravity in an AdS_{p+2} (Anti-de Sitter) spacetime and a strongly coupled quantum model defined on its boundary, $\partial\mathrm{AdS}_{p+2} \simeq \mathbb{R}^{p+1}$. Specifically, the boundary theory is a $(p+1)$-dimensional Conformal Field Theory (CFT), typically enhanced by supersymmetry. Due to the weak-strong nature of the duality, the correspondence cannot be rigorously *proven* from first principles and remains, for this reason, a conjecture. Its most well-known formulation concerns type IIB superstring theory in the $\mathrm{AdS}_5 \times S^5$ background, with N units of Ramond-Ramond F_5 flux threading S^5. The proposed dual theory is $\mathcal{N} = 4$, $d = 4$ supersymmetric Yang-Mills (SYM) with gauge group $SU(N)$, R-symmetry group $SU(4)_R$, and a squared coupling proportional to the string coupling, $g^2_{\mathrm{SYM}} \propto g_s$. A key insight leading to the formulation of the correspondence is the realization that Dp-branes—extended objects in superstring theory where open strings can end—are the same as (extremal) p-branes in supergravity, its low-energy limit. On the other hand, string oscillations generate fields, suggesting the existence of a field theory on the branes. This connection can be explicitly seen by computing gluon scattering amplitudes on N Dp-branes in string theory and identifying the effective low-energy field theory that reproduces them. For a stack of D3-branes, whose near-horizon geometry is $\mathrm{AdS}_5 \times S^5$, the resulting field theory is precisely four-dimensional SYM.

The AdS/CFT conjecture Beyond the foundational example of D3-branes, the catalogue of dualities has expanded by following similar reasoning, incorporating other Ricci-flat cones as well. For instance, one can probe D3-branes at the singularity of a Calabi-Yau cone, replacing the S^5 with a more general five-dimensional Sasaki-Einstein (SE) space. In this case, since a general SE_5 has fewer symmetries than S^5, the resulting dual field theory exhibits only $\mathcal{N} = 1$ supersymmetry [5]. For a general p-brane, viewed as a solution of a D-dimensional supergravity theory (which can be either type IIA, type IIB, or

M-theory), the line element in the Einstein frame (for $D = 10$) is given by

$$\mathrm{d}s_E^2 = H_p^{-\frac{D-p-3}{D-2}} \left(-\mathrm{d}t^2 + \dots \mathrm{d}x_p^2\right) + H_p^{\frac{p+1}{D-2}} \left(\mathrm{d}r^2 + r^2 \mathrm{d}s^2_{S^{D-p-2}}\right) , \quad H_p(r) = c + \left(\frac{L}{r}\right)^{D-p-3} ,$$

where S^{D-p-2} is a $(D-p-2)$-dimensional sphere, and the brane is located at $r = 0$, which corresponds to the tip of the cone over the sphere. In the near-horizon limit, the geometry generically takes the form $\mathrm{AdS}_{p+2} \times S^{D-p-2}$. For example, when $p = 3$ and $D = 10$, this describes the well-known case of D3-branes. Beyond just matching (super)symmetries—where $\mathrm{ISO}(\mathrm{AdS}_{p+2}) = SO(2, p+1)$ coincides with the conformal group in $(p, 1)$ dimensions, and the isometry group of S^5 is $SO(6) \simeq SU(4)_R$—the AdS/CFT correspondence encapsulates deep dynamical insights.Indeed, a more precise formulation of the AdS/CFT conjecture is as follows. Given an effective string-theoretic action in AdS_{p+2} (meaning that the vacuum solution is AdS_{p+2}),

$$S^{\text{eff, string}}_{\mathrm{AdS}_{p+2}}[g_{\hat{\mu}\hat{\nu}}, \hat{\Phi}] ,$$

which depends on a set of bulk fields $(g_{\hat{\mu}\hat{\nu}}, \hat{\Phi})$ and bulk coordinates $x^{\hat{\mu}} = (z, x^\mu)$, where $\mu = 1, \dots, p+1$, we can define the so-called "quantum gravity" path integral:

$$Z^{\text{string}}_{\mathrm{AdS}_{p+2}}[\hat{\Phi}(\Phi)] \equiv \int_{\Phi(x^\mu) = \hat{\Phi}(0, x^\mu)} \mathcal{D}\hat{\Phi}\, \mathrm{e}^{-S^{\text{eff, string}}_{\mathrm{AdS}_{p+2}}[g_{\hat{\mu}\hat{\nu}}, \hat{\Phi}]} .$$

This quantity depends on the values of the fields at the boundary of AdS_{p+2} $(z \to 0)$, denoted as $\Phi(x^\mu)$. On the field theory side, we can consider the generating functional for correlation functions,

$$Z^{\text{CFT}}_{\mathbb{R}^{p+1}}[\hat{\Phi}] \equiv \int \mathcal{D}\Theta \mathrm{e}^{-S_{\mathrm{CFT}_{p+1}} + \int \mathrm{d}^{p+1}x\, \Theta(x^\mu)\Phi(x^\mu)} .$$

According to the AdS/CFT conjecture, these two partition functions are equal, implying that the presence of bulk fields $\hat{\Phi}(x^{\hat{\mu}})$ modifies the boundary theory in a way such that their boundary values $\Phi(x^\mu)$ act as sources for the dual operators $\Theta(x^\mu)$ in the CFT. This equivalence is conjectured to hold for any number of coincident branes N, though in practice, computing the string-theoretic path integral is highly non-trivial due to high-energy ("stringy") corrections. To make progress, one typically considers the large-N approximation, where supergravity becomes a reliable effective description. In this limit, the string partition function simplifies to a sum over on-shell supergravity actions with fixed boundary conditions,

$$Z^{\text{string}}_{\mathrm{AdS}_{p+2}} \longrightarrow \sum \mathrm{e}^{-S^{\text{on-shell, supergravity}}_{\mathrm{AdS}_{p+2}}} .$$

Microstates counting It is striking that AdS/CFT, according to the previous discussion, gives access to the microscopic counting of microstates of a certain black hole. In particular, for a black hole with electric charges q_a and angular momenta J_i, the partition function (in the grand canonical ensemble) of the dual field theory may be written as

$$Z^{\text{GC}}_{\text{CFT}}(\Delta_a, \omega_i; p_a) = \sum_{q_a, J_i} c(q_a, J_i) \mathrm{e}^{\mathrm{i}(\Delta_a q_a + \omega_i J_i)} , \tag{1.1.1}$$

where (Δ_a, ω_i) are chemical potentials associated to (q_a, J_i), respectively, and $c(q_a, J_i)$ is the number of microstates which preserve the same supersymmetry of the dual black hole in the CFT twisted by the fluxes p_a (magnetic charges). Then, by definition, the entropy of the black hole should be counted as a measure of the disorder, namely

$$\mathrm{e}^{S_{\mathrm{BH}}(q_a,J_i)} \equiv c(q_a, J_i) = \int \frac{\mathrm{d}\Delta_a}{2\pi}\frac{\mathrm{d}\omega_i}{2\pi} Z^{\mathrm{GC}}_{\mathrm{CFT}} \mathrm{e}^{-\mathrm{i}(\Delta_a q_a + \omega_i J_i)} \,. \tag{1.1.2}$$

In the large charges limit, we can use again saddle point approximation to get

$$S_{\mathrm{BH}}(q_a, J_i) \simeq \mathcal{S}(\Delta_a, \omega_i; q_a, J_i)\Big|_{\bar{\Delta}_a,\bar{\omega}_i} \equiv \Big[\log Z^{\mathrm{GC}}_{\mathrm{CFT}}(\Delta_a, \omega_i; p_a) - \mathrm{i}(\Delta_a q_a + \omega_i J_i)\Big]\Big|_{\bar{\Delta}_a,\bar{\omega}_i} \,,$$

where we have introduced the *entropy function* $\mathcal{S}(\Delta_a, \omega_i; q_a, J_i)$, and $(\bar{\Delta}a, \bar{\omega}i)$ are its extremizing values. Finally, by the AdS/CFT correspondence, $\log Z_{\mathrm{CFT}} \simeq -S^{\text{on-shell}}_{\text{supergravity}}$, so that the black hole entropy can be obtained by extremizing the (Legendre transform of the) (renormalized) on-shell action on the gravity side. This is where both opportunities and challenges arise. On the one hand, computing the on-shell action is not always possible—for example, when the full black hole solution is unknown. On the other hand, evaluating the partition function of the dual CFT is generally a difficult task. Here it comes the sun, and the keywords are *rigid supersymmetry* and *supersymmetric localization.* Localization, introduced by Pestun [6], allows the reduction of a complicated path integral to a simpler, often finite-dimensional integral using supersymmetry. Specifically, one can deform the path integral by adding a supersymmetry-exact term; this deformation ensures that the integral collapses to a saddle-point approximation around the supersymmetric locus. Meanwhile, rigid supersymmetry provides a framework in which a (flat-space) field theory can be placed on a curved background while preserving a certain amount of supersymmetry. This is achieved by solving a generalized Killing spinor equation, as discussed firstly in [7].

More AdS/CFT dualities Before the advent of [7], it was only known how to preserve supersymmetry by using the *topological twist* trick for Riemann surfaces Σ_{g} of genus g (or the *no-twist* for two-spheres, taking the standard Killing spinors solving $2D_\mu\epsilon = \gamma_\mu\epsilon$) [8], leading to the topological-twisted index [9] (or the superconformal index [10]). Schematically, supersymmetry on $\mathbb{R}^{p-1} \times \Sigma_{\mathrm{g}}$ is preserved by adding a R-symmetry background gauge field a^R_μ equal to the local spin connection w_μ. This changes the equations for the parallel spinor, so that the variation of the gravitino reduces to $(D_\mu - \mathrm{i}a^R_\mu)\epsilon = 0 = \partial_\mu\epsilon$, namely the spinor is constant and the theory is said to be topologically twisted. As a consequence, the R-symmetry gauge line bundle is identified with the tangent bundle to Σ_{g}, and a^R_μ is identified as a connection on the tangent bundle, so that by definition

$$\frac{1}{2\pi}\int_{\Sigma_{\mathrm{g}}} \mathrm{d}a_R = \frac{1}{4\pi}\int_{\Sigma_{\mathrm{g}}} \mathrm{d}^2x\sqrt{g}R \equiv \chi_{\Sigma_{\mathrm{g}}} = 2(1-\mathrm{g})\,. \tag{1.1.3}$$

Due to this trick, new examples of AdS/CFT dualities have been proposed, starting with the the seminal paper of Maldacena-Nuñez [11], followed by many others (see *e.g.* [12, 13, 14, 15, 16]). The idea, summarized in figure 1.1, is to start with a (flat) $(p+1)$-dimensional SCFT with a known dual in terms of branes, which can be then put on $\mathbb{R}^{p-1}\times\Sigma_{\mathrm{g}}$ preserving supersymmetry with the topological twist trick. We can then compactify on the compact factor Σ_{g}, performing an RG flow across dimensions, reaching another (different!) flat SCFT on $\mathbb{R}^{p-1}$. The dual description of this flow is a black brane in AdS_{p+2}, which interpolates between a conformal boundary $\mathbb{R}^{p-1} \times \Sigma_{\mathrm{g}}$, and a near horizon geometry $\mathrm{AdS}_p \times \Sigma_{\mathrm{g}}$. The interpretation is in terms of a stack of p-branes, with two directions among the p wrapped on the Riemann surface with a topological twist to preserve supersymmetry, and the low-energy theory—which is a $(p+1-2=p-1)$-dimensional field theory living on the brane world-volume, the non-compactified dimensions—is then a natural candidate to be dual to $\mathrm{AdS}_p \times \Sigma_{\mathrm{g}}$. The first successful matching has been accomplished considering statical, magnetically charged AdS_4 black holes [17], the topologically-twisted index [9] and its large N limit [18]. In the latter reference, it has also been shown that the extremization of the large N limit of the index with respect to the fugacities is necessary to match the entropy of the dual four-dimensional black holes, according to our previous discussion. The extremization procedure has been dubbed as $\mathcal{I}$-extremization [18, 19], interpreted as necessary to select the exact R-symmetry among all the possible trial R-symmetries in the dual $d=1$ quantum mechanics. Similar extremization process have been proven to be useful in other dimensions, as the $d=2$, c-extremization [20, 15], the $d=3$, F-maximization [21, 22] and the $d=4$, a-maximization [23]. For the very essence of the AdS/CFT conjecture, everything that happens on a side should have a counterpart on the other one, and this phenomena are not an exception. Since the field theory has an R-symmetry, it is better that the dual geometry realizes the same symmetry. Indeed, dual to these various extremization principles, there is a similar extremization of the *Sasakian volume* with respect to the trial Killing vectors (Reeb vectors) and of the volume of the Gauntlett-Kim (GK) geometry [24, 25, 26], the *master volume* [27, 28, 29, 30, 31, 32, 33, 34, 35, 36]. The power of these tools lies in the fact that the explicit knowledge of the metrics is no needed, and they are purely geometrical tools. Following this stream of ideas, and the factorization in blocks of the sphere partition functions [37], it has been proposed that all the gravitational relevant objects (as entropies or gravitational central charges) can be written in terms of some *gravitational blocks* [38]; an "experimental" result (for that time!) was that all the relevant functions come as sum over fixed points. Moreover the relevant objects (or, *extremal functions*), should be topological in nature, as for the volumes of SE and GK. These (supposedly true) features are particularly useful when the dual field theory computation is not available, as happens usually for the spindle (or more complicated orbifolds), which we shall now review.

1.2 Spindles and other orbifolds

The solution presented in [39] changed drastically the perspective on the AdS/CFT correspondence, enriching the landscape of known dualities. In [39], a supersymmetric solution in minimal $D=5$ gauged supergravity of the form $\mathrm{AdS}_3 \times \Sigma_{[m_-,m_+]}$ was constructed, where $\Sigma_{[m_-,m_+]}$ is informally known as a *spindle*.

Spindle geometry The spindle, which more properly is the weighted complex projective line $\mathbb{WCP}^1_{[m_-,m_+]} \equiv \Sigma_{[m_-,m_+]}$ for some co-prime positive integers $m_\pm$, is an *orbifold* and can be described in various ways. From a point of view, the spindle can be obtained

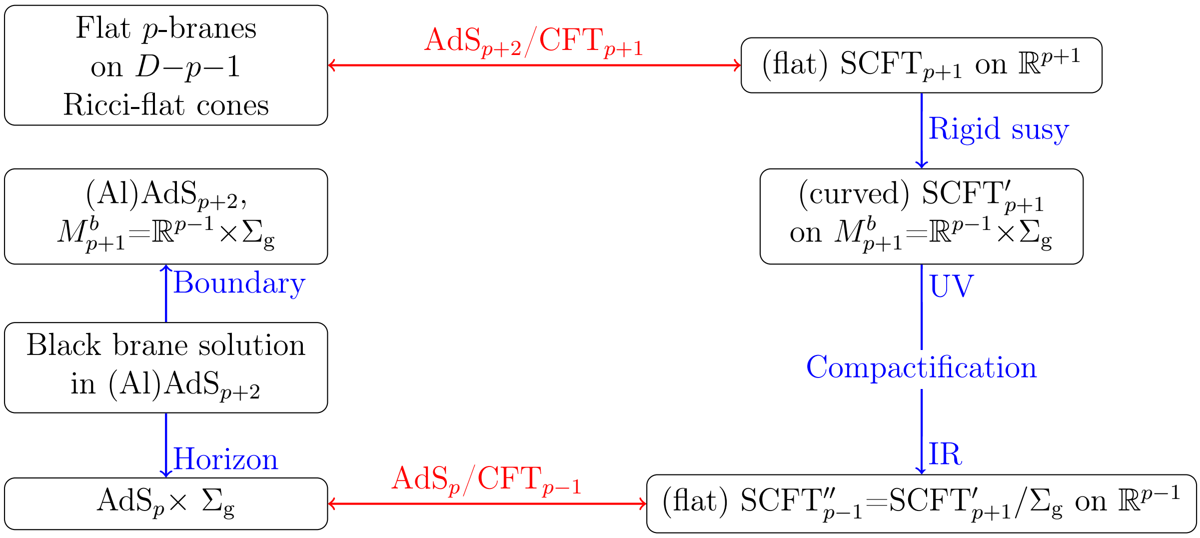


Figure 1.1: A schematic summary for holography involving p-branes wrapped on a Riemann surface (or a sphere).

as the weighted quotient of $U(1)$ on the three-sphere embedded in $\mathbb{C}^2$[1]

$$\Sigma_{[m_-,m_+]} = \frac{S^3 \subset \mathbb{C}^2}{U(1)_{m_\pm}}\,, \quad (z_1,z_2)\sim(\lambda^{m_+}z_1,\lambda^{m_-}z_2)\,, \quad \lambda\in U(1)\,, \quad \gcd(m_-,m_+)=1\,.$$

This description relies on the fact that the quotient M/G, with G a Lie group acting on M, is naturally an orbifold and then M is a principal G-orbibundle over M/G. In this case the action is not free at $z_i=0$, resulting in two orbifold points at the poles of the sphere. For $M=S^3$ and[2] $G=U(1)_{m_\pm}$, this results in $U(1)_{m_\pm}\hookrightarrow S^3\to\Sigma_{[m_-,m_+]}$. From this point of view, this is completely analogous to the very well-known Hopf fibration which realizes the three-sphere as a (smooth) bundle over S^2 as $U(1)\hookrightarrow S^3\to S^2\simeq\mathbb{CP}^1$, a fact that will be useful during this thesis. Similarly, as we can also construct $U(1)\hookrightarrow(S^3/\mathbb{Z}_t\simeq L(t,1))\to_t\mathbb{CP}^1$ (where $\to_t$ is a short-hand for the Chern number of the fibration being t), it is possible to construct an orbibundle over the spindle with the lens space $L(t,1)$ as total space. A perhaps more accurate interpretation is obtained by using the usual description in term of patches, covering spaces and gluing. Indeed, the spindle is a "bad orbifold" [40], in the sense that it is not possible to move to a covering space which is a manifold[3]. The very fact that the spindle is an orbifold, implies that near the orbifold points it can be modelled (locally) on open subsets of $\mathbb{R}^2/\Gamma$, with Γ a discrete group ($\Gamma=\mathbb{Z}_{m_\pm}$ for the spindle). It follows that there exist always an adapted coordinate system (θ,ϕ), with $\theta\in[\theta_-,\theta_+]$ and $\phi\in[0,2\pi)$, for which the metric on the

[1]The requirement that $m_\pm$ have to be mutually prime is not strictly necessary. If instead $\gcd(m_-,m_+)=m_0$, there is a sub-action for which the resulting space is $\Sigma_{[m_-/m_0,m_+/m_0]}/\mathbb{Z}_{m_0}$.

[2]The notation $G=U(1)_{m_\pm}$ should be intended as the group being $G=U(1)$, but with the weighted action on S^3.

[3]More precisely, a "good orbifold" can be realized as a global quotient of a discrete group, whilst the spindle is obtained as a quotient with $G=U(1)_{m_\pm}$. The spindle is the only (and hence, the more general!), bad orbifold in two real dimensions. The adjective "bad" comes from mathematical literature and should not be interpreted as having a negative connotation, except that it makes things more challenging (but also interesting!).

spindle takes the form

$$\mathrm{d}s^2_{\Sigma_{[m_-,m_+]}} = \mathrm{d}\theta^2 + \frac{\sin^2\theta}{f(\theta)^2}\mathrm{d}\phi^2\,, \quad f(\theta) \underset{\theta\to\theta_\pm}{\simeq} m_\pm + O(\theta^2)\,,$$
$$\chi_{\Sigma} \equiv \frac{1}{4\pi}\int_{\Sigma_{m_\pm}} \mathrm{d}^2x\sqrt{g}R = \frac{1}{f(\theta_-)} + \frac{1}{f(\theta_+)} = \frac{m_- + m_+}{m_- m_+}\,,$$

such that near the poles $\theta_\pm$ there is a conical deficit $2\pi(1-m_\pm^{-1})$. The above discussion, together with the fact that the spindle has conical singularities, implies that Σ does not admit any metric with constant scalar curvature. This is also evident from its characteristic, also called "orbifold Euler characteristic" or "orbifold Gauss-Bonnet theorem" [41]. More generally one could consider Σ^n, a two sphere with $2n$ orbifold points of order m_i, and in that case the orbifold characteristic can be obtained from the characteristic of the underlying smooth S^2 as $\chi_{\Sigma^n} = \chi_{S^2} + \sum_i^{2n}(1-m_i)/m_i$. For $n=1$, we have indeed χ_Σ. The spindle is special in the sense that it posses a $U(1)_\phi$ azimuthal symmetry, since we have factored out only one $U(1)$ among $U(1)^2 \subset SO(4) = \mathrm{ISO}(S^3)$, which is the symmetry of the original S^3. This $U(1)$ symmetry will be fundamental in many ways; one among all, it makes the spindle a toric orbifold.

Supersymmetry on the spindle Apart from being a mathematical object interesting by itself, it has introduced some novelties in the game. First of all, [39] shows that it makes sense to wrap the brane world-volumes also on a (compact) orbifold, which has in turn a non-constant curvature[4]. The consequence is that it is not possible anymore to choose a background R-symmetry gauge field identified with the local connection on the tangent (orbi) bundle; it follows that the spinors are not simply constant on the spindle. Indeed, supersymmetry on the Σ is realized in a novel way, which naturally generalizes both the topological twist and the no-twist condition. Compactly, we can write

$$\mathfrak{n}_R \equiv \frac{1}{2\pi}\int_{\Sigma_{m_\pm}} \mathrm{d}A_R = \frac{m_- + \sigma m_+}{m_- m_+}\,, \quad \sigma = \pm 1\,.$$

Later, it has been demonstrated that these are the only possible ways to preserve supersymmetry on the spindle [42]. The case $\sigma=-1$, dubbed *anti-twist*, reproduces formally the no-twist condition in the limit for which the spindle becomes a smooth S^2, *i.e.* $m_\pm \to 1$. Similarly, $\sigma=+1$ is naturally referred to as *topologically topological twist*, since the R-symmetry gauge field is again identified as a connection on the tangent bundle, but the corresponding local curvatures are different. By uplifting the solution on a five-dimensional (regular) SE_7 manifold to type IIB supergravity, [39] shows that it is possible to constrain the parameters to obtain a smooth ten-dimensional solution. Then, since the central charge computed from the gravity solution and the anomaly polynomial computation [43, 44] of the dual $d=2$, $\mathcal{N}=(0,2)$ SCFTs (understood as in figure 1.1) have been shown to match perfectly, one can conclude that it is possible to wrap branes also on orbifolds. Interestingly, in performing the c-extremization procedure, one should let the trial R-symmetry to be a mixing of the two-dimensional one and the angular momentum on

[4]It is interesting to notice that the local solutions containing spindles can give, in some degenerate limits, to solution globally describing branes wrapped on disks or Riemann surfaces with non-constant scalar curvature.

the spindle, with an "equivariant" parameter ϵ, for the above-mentioned $U(1)_\phi$ azimuthal symmetry. Thus, every time there is a spindle in the game, one should introduce also a rotational fugacity ϵ, and the extremal function will be extremized also over it. It is also worth pointing out that, as in other many cases, the local gravity solution was known before the advent of the spindle [45], but the dual $d = 2$ field theory remained obscure. The interpretation in terms of D3-branes, allows to identify the dual theory as obtained from the parent $d = 4$, $\mathcal{N} = 1$ theory (dual to $\mathrm{AdS}_5 \times \mathrm{SE}_5$) compactified on the spindle, along with a background field A_R to preserve supersymmetry (the anti-twist, in this case).

The "spindly" black hole From a physical perspective and the above considerations, it is reasonable that it should exist a more general solution, which looks like $\mathrm{AdS}_3 \times \mathbb{\Sigma}$ near the horizon (in the IR), and asymptotically locally AdS_5 in the UV. Unfortunately, such a black string solution has not been found (yet). However, there is (at least) a case in which the full solution is known. Indeed, in the second paper concerning spindles [46], it is showed that a solution $\mathrm{AdS}_2 \times \mathbb{\Sigma}$ arises from the horizon of an accelerating four dimensional Plebański-Demianski black hole in minimal $d = 4$, $\mathcal{N} = 2$ gauged supergravity. In this case it is the acceleration itself responsible for the conical singularities of the horizon, in that it turns out that it is proportional to $m_- - m_+$. Again, the uplift to M-theory can be made smooth and allows to conjecture that the solution arises from a stack of N M2-branes wrapped on the spindle. The general lesson is: lower-dimensional singular solutions can have regular uplifts. Since in odd dimensions (of field theory) there are no anomalies, one has to perform the full localization (and large N limit) computation to verify carefully this statement. This program took four years to complete, starting from the computation of the on-shell action, which is more subtle than one can expect [47], to the field theory side [48, 49, 50].

Branes wrapped on spindles From this moment on, a plethora of solutions containing spindles have been constructed in various spacetime dimensions and in various supergravity theories, starting with [51, 52], where M5- and D4-branes wrapped on the spindle have been considered, completing the "minimal" scenario of wrapped branes. A (hopefully) complete list is [39, 46, 53, 54, 51, 55, 56, 57, 52, 58, 42, 59, 60, 61, 62, 63, 64, 65, 1, 66, 67, 68, 69, 2, 70, 71, 72]. The on-shell action for the accelerating Plebański-Demianski solution of [46] has been *computed* in [47], by considering the supersymmetric black hole first, and taking the extremal limit after (as put forward in [73]). This procedure effectively defines the on-shell action for the extremal black hole, which otherwise is divergent as a consequence of the temperature being zero at the extremality. In all the other cases, when the full black string solution is not known, the on-shell action can only be *conjectured*. Indeed this has been done for example for the multi-charge spindle solution [56], which generalizes [46]. Then, it is striking that there exist again a gravitational block formula which allows to reproduce the relevant gravitational quantity from an extremization procedure. For all the branes wrapped on spindles, it has been conjectured in [52] a *spindly gravitational block* decomposition for a $U(1)^r$ gauged supergravity theory in $D = d + 1$ dimensions, where r is for the rank of the (Cartan subgroup of the) gauge group. Namely, it has been proposed that the exact superconformal R-symmetry of the $(d-2)$ SCFT obtained compactifying the d dimensional one on the spindle is determined by extremizing an[5] *off-shell free energy* with a constraint:

$$
\begin{gathered}
F^{\kappa}(\varphi_i,\epsilon;\mathfrak{n}_i,m_\pm,\sigma)\equiv\frac{1}{\epsilon}\big[\mathcal{F}_d(\varphi_i+\mathfrak{n}_i\epsilon)+\kappa\,\mathcal{F}_d(\varphi_i-\mathfrak{n}_i\epsilon)\big]\,,\\
\sum_{i=1}^{r}\mathfrak{n}_i=\frac{m_++\sigma m_-}{m_-m_+}\,,\qquad\sum_{i=1}^{r}\varphi_i-\frac{m_+-\sigma m_-}{2m_-m_+}\epsilon=2\,,\quad\kappa=\pm1\,.
\end{gathered}
$$

In this formula, as usual, the R-symmetry flux $\mathfrak{n}_R=\sum_i^r\mathfrak{n}_i$ sums up to the twist or the anti-twist through the spindle, ϵ is the equivariant parameter (or rotational fugacity) associated to the $U(1)_\phi$ of the spindle, and φ_i are the fugacities associated to the other (continuous) global symmetries of the theory. The power of this formula is that it is valid for all the solutions involving p-branes wrapped on the spindle (with near horizon geometries $\mathrm{AdS}_p\times\Sigma$) with minor modifications. In particular, what changes across the dimensions is the form of the gravitational block $\mathcal{F}_d(\Delta_i)$ and the choice of the sign κ—see table 1.1. The blocks are proportional to the objects which are expected to count the degrees of freedom of the d dimensional theory. For $d=3$, $\mathcal{F}_3$ is proportional to the (large N limit of the) S^3 *off-shell* free-energy ($F_{S^3}^{\text{off-shell}}$, as computed in [74]) of the $d=3$, $\mathcal{N}=6$ ABJM theory [75] with gauge group $U(1)_k\times U(1)_{-k}$ and $SO(6)_R\sim SU(4)$ R-symmetry, enhanced to $\mathcal{N}=8$ and $SO(8)_R$ for $k=1,2$. In $d=4$, it is proportional to the *trial* central charge a_{4d}^{trial} of the $d=4$, $\mathcal{N}=4$ SYM theory with gauge group $U(N)$ and R-symmetry group $SO(6)_R$. Similarly, they are proportional to $F_{S^5}^{\text{off-shell}}$ computed in [76] of the $d=5$, $\mathcal{N}=1$ Seiberg theory [77] with gauge group $USp(2N)$ with $SU(2)_R$ of R-symmetry. Finally, $\mathcal{F}_6$ is proportional to a_{6d}^{trial} of the $d=6$, $\mathcal{N}=(0,2)$ theory [78]. The extremal functions F^κ are expected to be reproduced by the large N limit of the (logarithm of the) corresponding localized partition functions (depending on the fugacities). These off-shell free energies contains all the previous conjectures for D3- and M2-branes wrapped on the spindle [53, 47, 56], and generalize all the existing conjectures for entropy functions (or off-shell central charges) regarding branes wrapped on two-spheres (or Riemann surfaces) [38, 79, 80, 53]. Later, some of these block decompositions have been demonstrated. From the gravity point of view, using the GK geometry master volume (for M2- and D3-branes) [81, 82], or the equivariant volume [83], or the (equivariant) localization of the action (for M5-branes) [84]. From the field theory side, the only available complete computation has been done for M2-branes wrapped on the spindle of the accelerating black hole [48, 49, 50].

	$d=3$	$d=4$	$d=5$	$d=6$
$\mathcal{F}_d$	$c_3(\Delta_1\Delta_2\Delta_3\Delta_4)^{1/2}$	$c_4(\Delta_1\Delta_2\Delta_3)$	$c_5(\Delta_1\Delta_2)^{3/2}$	$c_6(\Delta_1\Delta_2)^2$
c_d	$\frac{-2^{1/2}\pi\sqrt{k}}{3}N^{3/2}=-F_{S^3}$	$\frac{-3}{2}N^2=-6a_{4d}$	$\frac{-2^{5/2}\pi}{15}\frac{N^{5/2}}{\sqrt{8-N_f}}=\frac{4}{27}F_{S^5}$	$\frac{-9}{256}N^3=-\frac{63}{256}a_{6d}$
κ	$-\sigma$	-1	$-\sigma$	-1

Table 1.1: In this table are summarized the gravitational blocks $\mathcal{F}_d(\Delta_i)$ for various space-time dimensions d. Adapted from [52].

[5]Conceptually, an off-shell free energy (or extremal function) is the same as an entropy function, but we will deserve this name for black holes, with $d=3$.

Branes wrapped on orbifolds Once it has been established that valuable insights on holography can be gained by wrapping various branes on spindles, it seems natural to investigate if it is also possible to consider higher dimensional orbifolds in the game [54, 52, 58, 61, 64, 65, 1, 2, 72]. A first attempt in this direction involves a simple four-dimensional orbifold $\mathbb{M}_4 = \Sigma_{\rm g} \times \mathbb{\Sigma}$ in $D = 5$ gauged supergravity, arising from M5-branes wrapped on a Riemann surface first, and then further wrapped on the spindle [54]. A similar solution in $D = 6$, of the form $\mathrm{AdS}_2 \times \Sigma_{\rm g} \times \mathbb{\Sigma}$, has later been found in [52] (and generalized in [58]), and both have been generalized in [64]. As before, an extremal function has been proposed in [1], which should reproduce the entropy of the hypothetical black hole with $\mathbb{M}_4 = \Sigma_{\rm g} \times \mathbb{\Sigma}$ in the horizon. Employing the same building blocks of table 1.1, it has been considered

$$
\begin{aligned}
S(\varphi_i, \epsilon_i; \mathfrak{n}_i, \mathfrak{s}_i) = \frac{-1}{4\epsilon_1\epsilon_2}\Big[&\mathcal{F}_5(\varphi_i + \mathfrak{n}_i\epsilon_1 + \mathfrak{s}_i\epsilon_2) + \mathcal{F}_5(\varphi_i - \mathfrak{n}_i\epsilon_1 + \mathfrak{s}_i\epsilon_2) \\
&- \mathcal{F}_5(\varphi_i + \mathfrak{n}_i\epsilon_1 - \mathfrak{s}_i\epsilon_2) - \mathcal{F}_5(\varphi_i - \mathfrak{n}_i\epsilon_1 - \mathfrak{s}_i\epsilon_2)\Big] \,, \\
\mathfrak{n}_1 + \mathfrak{n}_2 = \frac{m_+ - m_-}{m_- m_+}\,, \quad \mathfrak{s}_1 + \mathfrak{s}_2 = 2(1-\mathrm{g})\,, \quad &\varphi_1 + \varphi_2 - \frac{m_- + m_+}{m_- m_+}\epsilon_1 = 2\,.
\end{aligned}
$$

Extremizing this with respect to (φ_i, ϵ_i) reproduces indeed the entropy of the system. In a sense, the extremal function is now "doubled": there are two couple of fluxes, through the spindle $(\mathfrak{n}_i)$ and through the Riemann surface $(\mathfrak{s}_i)$, the number of blocks has expanded two times, and crucially there are two rotational fugacities ϵ_i associated to the $U(1)^2$ rotational symmetry[6] of $\Sigma_{\rm g} \times \mathbb{\Sigma}$. A much more complicated solution in $D = 7$ comprising four-dimensional orbifolds has been constructed in [61], by constructing a new consistent truncation of maximal $D = 7$ gauged to minimal $D = 5$ gauged supergravity, reducing on the spindle. Then, it has been possible to uplift the known $\mathrm{AdS}_3 \times \Sigma_{\rm g}$ and $\mathrm{AdS}_3 \times \mathbb{\Sigma}_1$ solutions [85, 39] of minimal $D = 5$ supergravity to $D = 7$. Summarizing, the seven-dimensional solutions contain $\mathbb{M}_4^{(1)} = \Sigma_{\rm g} \ltimes \mathbb{\Sigma}_2$ (where $\ltimes$ means a non-trivial fibration of the second factor over the first one) and even the more complicated $\mathbb{M}_4^{(2)} = \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$. Following this, a similar solution in $D = 6$, of the form $\mathrm{AdS}_2 \times \mathbb{M}_4^{(1,2)}$ has later been constructed again from a truncation point of view [65]. Building on [61, 65], we studied in detail the toric properties of the $\mathbb{M}_4^{(1,2)}$ orbifolds and we conjectured again an off-shell free energy, now called *orbifold entropy function*, for D4- (and D8-) and M5-branes wrapped over generic (toric) $\mathbb{M}_4$. Since a toric orbifold can be described in terms of a labelled polytope [86], the extremal function turns out to be written only in terms of the geometrical data characterizing the topology of the solution. It will be matter of the chapter 2 to review all the relevant toric geometry needed to understand and formulate the extremal problem from the conjectural off-shell free energy, which will be applied then in chapters 3 and 4. The way in which the conjecture is formulated is very general and it should be (in principle) applicable also to more generic toric orbifolds then the solutions of [61, 65, 1], with a different number of fixed points, or also to toric manifolds, for which explicit solutions are not known (but in $D = 7$ there is the anomaly polynomial

[6] Clearly, the Riemann surface does not possesses an $U(1)$ symmetry, but we can think to extend the rotational symmetry of S^2 formally to $\Sigma_{\rm g}$. In the same sense, with a slightly abuse of notation, we will say (several times) also that $\Sigma_{\rm g} \times \mathbb{\Sigma}$ is toric.

computation of [44], to which our formula boils down). We tested it over all known solutions [61, 65, 52, 1]. As a further precision check of the validity of our conjecture in [1], we applied it also to a more complicated four-dimensional orbifold $\mathbb{M}_4$ which arises as a solution of both $D = 6$ and $D = 7$ gauged supergravities in the form $\mathrm{AdS}_{2,3} \times \mathbb{M}_4$. We dubbed these solutions as *quadrilaterals* [87], since the polytope describing $\mathbb{M}_4$ is a quadrilateral (as also for the simpler solutions [61, 65, 52, 1]). We will review in detail this type of solution and its toric properties in chapter 4. Later, the validity of our proposal has been (partially) demonstrated using the idea of equivariant integration of the anomaly polynomial in (for M5-branes on a generic toric $\mathbb{M}_4$) [83], the equivariant volume with higher times (for D4- and M5-branes on $\mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$) [88], or from the equivariant localization of the action (for M5-branes on a generic toric $\mathbb{M}_4$) [89]. In general equivariant localization of the volume [83, 88] and of the action [90, 83, 84, 89, 91, 88, 92, 93, 94, 95] seems a promising way to better understand extremization problems.

"Euclidean" spindles The solutions described up to this point contain, in almost all cases, a spindle appearing as a factor $\mathrm{AdS}_{D-2} \times \mathbb{\Sigma}$ in some D-dimensional gauged supergravity. More generally, it is also possible to find solutions with one or more spindle appearing together, as $\mathrm{AdS}_{2,3} \times \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$. In these cases, the standard interpretation is that there exist a black brane wrapping the spindle (or $\mathbb{M}_4$), which near-horizon geometry takes the above forms. This expectation has been confirmed in various cases, with anomaly polynomial computations or using rigid supersymmetry and localization. However, it is interesting to notice that it is possible to find spindles also in other forms. As there exist both $\mathrm{AdS}_2 \times S^2$ and its counterpart $\mathrm{AdS}_2 \times \mathbb{\Sigma}$ as solutions of minimal $d = 4$, $\mathcal{N} = 2$ gauged supergravity, one can wonder if it exists a generalization of the known (supersymmetric) Euclidean solution $\mathbb{C} \hookrightarrow \mathcal{O}(-t) \to S^2$ [96] involving the spindle. In this situation, the two-sphere appears as a zero section of the (complex) line bundle $\mathcal{O}(-t)$, and as a consequence it takes the name *bolt* [97]. Usually, the line element of Bolt solutions is the same as for the Taub-NUT ones [98, 99, 100], and the different topology ($\mathbb{R}^4$ or quotient thereof for Taub-NUT spaces) comes from the value of the parameters in the metric functions. The initial interest in Taub-NUT solutions relies on the fact that they were the first example of a *gravitational instanton* [101] in pure gravity, the analogue of a Yang-Mills instanton, with an (anti) self-dual field strength ($\star F_{\mu\nu} = \pm F_{\mu\nu}$). They are defined as spaces with (anti) self-dual Riemann tensor $R_{\mu\nu\rho\sigma} = \pm 1/2 \varepsilon_{\rho\sigma\alpha\beta} R^{\alpha\beta}{}_{\mu\nu}$, and consequently they are Ricci flat, $R_{\mu\nu} = 0$, giving a leading contribution to the path integral. The Taub-NUT solution, and also its multi-NUT generalization [101], is asymptotically locally flat (ALF), contrarily to the self-dual (bolt) Eguchi-Hanson (EH) solution with topology of a disk fibration over S^2 [102, 103], which is asymptotically flat (AF). There is also a generalization of the EH to more than one center, which is still self-dual but only ALF [104]. More generally, one can define *nuts and bolts* as sub-manifold fixed by some Killing vectors, with dimension zero (Σ_0) or two (Σ_2) [97]. Employing this definition, a new search for gravitational instantons with a cosmological constant has started [105, 106, 107], perhaps with also a graviphoton [108] and with supersymmetry [109, 110, 111, 96]. It is notable that in [3] we constructed a supersymmetric solution with topology of a *spindle bolt* ($\mathcal{M}_4 = \mathbb{C}/\mathbb{Z}_v \hookrightarrow \mathcal{O}(-t) \to \mathbb{\Sigma}_{[m_-,m_+]}$) in Einstein-Maxwell-$\Lambda$ supergravity. For this solution, the boundary $\mathcal{M}_3 = L(t, 1)$ is generically a squashed (and possibly branched [49]) lens space. Differently from the spindly accelerating black hole, for which $\mathcal{M}_3 = S^1 \times \Sigma$ and which realizes only the anti-twist for the spindle, we find that both the types of supersymmetry realizations are admitted, according to the values of the parameters. As in the context of supersymmetric black holes [73, 112, 47], and as is natural from the holographic point of view, we shall allow the solution to take complex values, finding that only in the case of the twist is the metric always real. Moreover, there exists a number of limits for which our spindle bolt comes back to the old known NUTs and Bolts solutions. It is matter of chapter 5 the study of this solution, with a particular focus on the on-shell action, which neatly characterizes the twist or the anti-twist. It should be also clear that generically our regular Euclidean solutions do not give rise to regular Lorentzian spacetimes [3]; it follows that in general the spindle bolt is not the Euclidean counterpart of a four-dimensional black hole.

1.3 Structure of the thesis

The rest of the thesis contains approximately one chapter (3, 4 and 5) for each paper [1, 2, 3]. Moreover, there is an introductory chapter (2) which collects useful facts about toric geometry, and a summary of each chapter at its end. The following is a preview of the content of the thesis, whilst a more comprehensive introduction to each chapter can be found at its beginning:

- Chapter 2. This chapter presents the notion of toric orbifolds in section 2.1, with a particular focus on four-dimensional orbifolds (section 2.1.5), which are relevant for this thesis. The main result of this chapter is the construction of the entropy functions for four-dimensional toric orbifolds in section 2.2, which heavily uses the description in terms of polytopes and Chern classes (sections 2.1.1 and 2.1.4, respectively);

- Chapter 3. After having presented the relevant supergravity theories in $D = 6, 7$ (section 3.1) and their uplift to massive type IIA and M-theory respectively (section 3.2), we start analysing the solution of [1]. The local solution, its supersymmetry and the global analysis (sections 3.3 and 3.4) are complemented with a detailed study of the toric properties of the solution in section 3.5.1. With this, the entropy function conjecture of section 2.2 is applied to the system, in section 3.6;

- Chapter 4 follows, to a large extent, the structure of the previous chapter, presenting the results of [2]. The main difference lies in a greater difficulty in the global study of the solution (section 4.2), which relies heavily on the toric geometry tools presented in chapter 2, in particular on the adjunction formula introduced in section 2.1.4. As a consequence, also the solution of the quantization conditions is more complicated, and is tackled in section 4.3. Finally, the successful application of the off-shell free energy extremization is reported in section 4.5, with a perfect match with the entropy and central charge computed in section 4.4;

- Chapter 5. Here we start considering the Maxwell-Einstein-Λ theory and its uplift to M-theory in section 5.1, on which [3] is based. This chapter is mainly divided into two: 5.3 and 5.4. The main result of the first part is the computation of the more general on-shell action for the Plebański-Demianski solution, in section 5.3.3. In the second one, which deals with the Carter-Plebański solution, there is a detailed study of the local and global properties of the solution, both in the bulk and in the boundary. The main findings are contained in section 5.4.4 and 5.4.6, where it is showed that there are two families, clearly distinguished by the value of the parameters, for which the on-shell action behaves differently. Finally, a comparison with the results of equivariant localization (section 5.2) is presented in section 5.4.6;

- Chapter 6 discusses our findings and open problems, whilst appendix A contains some conventions and equation of motions for the $D = 6, 7$ theories and appendix B describes in more details the limits which bring the spindle bolt to the older nuts and bolts solutions.

Chapter 2

Toric geometry and entropy functions

In this chapter, we briefly review (some) key aspects of toric geometry related to orbifolds (or manifolds), which will be utilized throughout this thesis. Our primary focus is on four-dimensional toric orbifolds $\mathbb{M}_4$, as all the solutions we examine exhibit a $U(1)^2$ symmetry and are in complex dimension $m = 2$. We find that the metrics of the solutions in chapters 3 and 4 are compatible with an integrable complex structure, making them complex (or Hermitian). However, they lack a closed symplectic two-form, $\omega_{(2)}$, and are therefore not Kähler. Moreover, they are not even conformally Kähler, meaning that it does not exist a one form $\eta_{(1)}$ such that $\mathrm{d}\omega_{(2)} = \eta_{(1)} \wedge \omega_{(2)}$. Similarly, the metric on the toric orbifold that we will present in chapter 5 is complex but not Kähler, but diversely they will be conformally closed. Furthermore, we will show in detail that there exist two integrable commuting complex structures and symplectic forms $\omega_{\pm}$. This means that $(g, \omega_{\pm})$ define two *ambiHermitian* structures, whilst the rescaled metric and symplectic form $(g'_{\pm}, \omega'_{\pm})$ define two *ambiKähler* structures. Possessing also the same $U(1)^2$ symmetry of the non-rescaled metric, they are *ambitoric*. Another important difference is that the toric orbifolds $\mathbb{M}_4$ of chapters 3 and 4 are *compact* and arise as the internal space of solutions of the form $\mathrm{AdS}_{2,3} \times \mathbb{M}_4$ in some gauged supergravities in dimension $D = 6, 7$. The simplest example in this class is $\Sigma \hookrightarrow S^2$, where the fibre is the spindle, which is compact. Differently, the orbifolds of the chapter 5 describe the complex line bundle over the spindle, namely $\mathbb{M}_4 = \mathbb{C} \hookrightarrow \mathcal{O}(-t) \to \Sigma$, and as such they are *non-compact*. It is interesting to notice that in turn $(g'_{\pm}, \omega'_{\pm})$ describe a (non-toric) compact orbifold, with an “artificial” boundary introduced by the “conformal compactification”[1].

Even if our metrics will not be Kähler, as highlighted in [83, 88], since our interest lies in topological quantities independent of the metric, we can confidently apply various results from standard symplectic toric geometry. The ultimate objective of this chapter is to formulate an extremal problem that enables us to compute relevant gravitational objects (such as entropies or gravitational central charges) using only the topological data of the solutions, referred to as “toric data”. These data include a standard Delzant polytope [113], accompanied by specific labels assigned to each facet of the polytope. This generalization, as studied by Lerman and Tolman [86], will be essential for properly understanding the toric properties of our solutions.

[1] As a simple example, consider $S^2 \simeq \mathbb{C} \cup \{\infty\} = \mathbb{C}^* \cup \{0\} \cup \{\infty\}$, which is clearly toric. However, its conformal compactification, the compact disk has a boundary and it is not toric.

2.1 Toric orbifolds

Let us start with a $2m$-dimensional symplectic orbifold $(\mathbb{M}^{\mathrm{s}}_{2m}, \omega_{(2)})$, which is an orbifold equipped with a non-degenerate $(\omega^m_{(2)} \propto \mathrm{vol}\mathbb{M}^{\mathrm{s}}_{2m} \neq 0)$ and closed $(\mathrm{d}\omega_{(2)} = 0)$ two-form $\omega_{(2)}$. If there is an effective Hamiltonian action of the torus $\mathbb{T}^m = U(1)^m$ on $(\mathbb{M}^{\mathrm{s}}_{2m}, \omega_{(2)})$, the symplectic orbifold is said to be toric: $(\mathbb{M}^{\mathrm{s,t}}_{2m}, \omega_{(2)})$. In particular, for a generic Lie group G we consider its action on $\mathbb{M}^{\mathrm{s}}_{2m}$, which is a map $\eta : G \times \mathbb{M}^{\mathrm{s}}_{2m} \to \mathbb{M}^{\mathrm{s}}_{2m}$. The action is effective if all the elements $g \in G$ (except the identity) move at least a point in $\mathbb{M}^{\mathrm{s}}_{2m}$. To each element ξ_I in the algebra $\mathfrak{g}$ of G, $I = 1, \ldots, \dim(\mathfrak{g})$, we associate a vector $V(\xi_I) = V_I$ such that $\mathrm{d}\eta : \mathfrak{g} \times \mathbb{M}^{\mathrm{s}}_{2m} \to \mathrm{T}_p\,\mathbb{M}^{\mathrm{s}}_{2m}$ acts as $\mathrm{d}\eta(\xi_I, p) \equiv V(\xi_I)|_p$. In this way the vector field corresponds to the orbit of the action η. The action is said to be a *symplectomorphism* if it preserves the symplectic form for each element of the group G, or

$$\mathcal{L}_{V(\xi_I)}\omega_{(2)} = 0 \implies \mathrm{d}(i_{V(\xi_I)}\omega_{(2)}) = 0\,, \quad I = 1, \ldots, \dim(\mathfrak{g})\,, \tag{2.1.1}$$

from which it follows that, at least locally, it exists a function ν_I such that $i_{V(\xi_I)}\omega_{(2)} = -\mathrm{d}(\nu_{V(\xi_I)} + c_I)$, with c_I a constant. The action is said to be *Hamiltonian* if the previous statement can be extended globally on all the orbifold, *i.e.* for each point we can find a function $\mu_I = \mu_{V(\xi_I)} : \mathbb{M}^{\mathrm{s}}_{2m} \to \mathfrak{g}^*$ such that

$$i_{V(\xi_I)}\omega_{(2)} = -\mathrm{d}(\mu_{V(\xi_I)} + c_I)\,. \tag{2.1.2}$$

The function $\vec{\mu}$ defined in this way is called a *moment map* (or momentum map). A symplectic toric orbifold is thus a pair $(\mathbb{M}^{\mathrm{s}}_{2m}, \omega_{(2)})$ with an effective Hamiltonian action of the torus $G = \mathbb{T}^m$ represented by the moment map (2.1.2). From now on, we will use $(\mathbb{M}_{2m}, \omega_{(2)})$ to indicate such an orbifold.

A convenient set of coordinates on $(\mathbb{M}_{2m}, \omega_{(2)})$ are the symplectic (or Darboux, or action-angle) coordinates (y_I, ϕ_I), where ϕ_I are angular coordinates on the torus, with standard periodicity $\Delta\phi_I = 2\pi$. In terms of these coordinates the symplectic form is given by

$$\omega_{(2)} = \sum_{I=1}^{m} \mathrm{d}y_I \wedge \mathrm{d}\phi_I\,, \tag{2.1.3}$$

and it is clear that y_I play the role of the moment map. Indeed $V_I = \partial_{\phi_I}$ and

$$i_{\partial_{\phi_I}}\omega_{(2)} = -\mathrm{d}y_I\,. \tag{2.1.4}$$

2.1.1 Polytopes

According to a generalization of the Delzant's theorem [113] by Lerman and Tolman [86], compact symplectic toric orbifolds are completely characterized by the associated *labelled polytope* $\mathcal{P}$, and the total space $\mathbb{M}_{2m}$ arises as a torus fibration over $\mathcal{P}$. In particular, from a labelled polytope it is always possible to reconstruct the toric orbifold, up to symplectomorphisms. The presence of a label $m_a \in \mathbb{Z}$ associated to a facet $\mathcal{F}_a$ of the polytope, signals the presence of a *local* normal orbifold singularity $\mathbb{Z}_{m_a}$ to the associated divisor $D_a \equiv \mu^{-1}(\mathcal{F}_a)$. We will need such a generalization, since the orbifolds $\mathbb{M}_4$ considered in this thesis does not fit in the standard picture of symplectic toric geometry due to the presence of labels. Indeed the normal vectors $\vec{v}_a$ of $\mathcal{P}$ will not be primitive (*i.e.*

$\det(\vec{v}_a, \vec{v}_{a+1}) \neq 1$). Moreover, in the context of toric varieties, the fan of $\mathbb{M}_{2m}$ generated by $\vec{v}_a$ is normal and therefore all the (orbifold) singularities are in complex co-dimension higher than one. For $m = 2$ this means that, for standard fans, orbifold singularities can happen only on points, but not on toric divisors D_a, which have co-dimension equal to one. As a consequence, we are dealing with more general objects than toric varieties, known as a "stacks" [114, 115, 116]. The fan of a stack is called *stacky fan* [117] and is dual to the labelled polytope that we now introduce.

A labelled polytope is a rational simple convex polytope $\mathcal{P}$ in $\mathbb{R}^m$, obtained as the image of $\mathbb{M}_{2m}$ under the moment maps y_I, equipped with a label attached to each of its facets. This object can be described concisely as the subset

$$\mathcal{P} = \bigcap_{a=1}^{n} \{ \vec{y} \in \mathbb{R}^m : l_a(\vec{y}) \equiv \vec{y} \cdot \vec{v}_a - \lambda_a = y_I v_a^I - \lambda_a \leq 0 \} \,, \tag{2.1.5}$$

with $\vec{v}_a \in \mathbb{Z}^m$ and $\lambda_a \in \mathbb{R}$ constants. The linear equations $l_a(y) = 0$ define the facets $\mathcal{F}_a = \{l_a(y) = 0\}$ of the polytope and we denote with n their total number. In our conventions, $\vec{v}_a$ are the outward-pointing normal vectors to the facets $\mathcal{F}_a$. A standard Delzant's polytope is *simple* (exactly m facets intersect at a vertex[2] p_a), *rational* (the vectors of the polytope can be chosen to form a $\mathbb{Z}$-basis of the lattice $\mathbb{Z}^m$) and *smooth* ($\det(\vec{v}_a, \vec{v}_{a+1}) = 1$). The situation here changes, in that the vectors $\vec{v}_a$ entering in (2.1.5) are not primitive due to the labels m_a. The labels m_a enter in the above construction as a common factor in the entries of the vectors $\vec{v}_a$, which can be written as

$$\vec{v}_a = m_a \vec{\hat{v}}_a \,, \tag{2.1.6}$$

where $\vec{\hat{v}}_a$ are primitive vectors and therefore they define an ordinary fan. The vertices p_a correspond to fixed points of the $\mathbb{T}^m$ action. Assuming that a given fixed point is obtained as the intersection of the facets $\mathcal{F}_{a_I}$, with $I = 1, \ldots, m$, the order of the orbifold singularity at this point is given by

$$d_A = |\det(\vec{v}_{a_1}, \ldots, \vec{v}_{a_m})| \,. \tag{2.1.7}$$

The toric orbifold $\mathbb{M}_{2m}$ viewed as a torus fibration over the polytope $\mathcal{P}$ is non-degenerate in the interior of $\mathcal{P}$. Differently, at each facet $\mathcal{F}_a$ a particular one-cycle $S^1 \subset \mathbb{T}^m$ collapses, thus determining a symplectic subspace of $\mathbb{M}_{2m}$ of (real) codimension two, which is the preimage of $\mathcal{F}_a$ under the moment maps and is called a toric divisor $D_a = \mu^{-1}(\mathcal{F}_a)$. The toric divisor D_a is thus a $2(m-1)$-cycle in $\mathbb{M}_{2m}$ and the intersection of q of them is a toric sub-orbifold of codimension $2q$. The fixed points p_a correspond then to the intersection of exactly $q = m$ distinct divisors, where all the cycles of $\mathbb{T}^m$ collapse. We will denote with s the number of fixed points in $\mathcal{P}$.

2.1.2 Toric orbifolds from Kähler quotients

Toric orbifolds can be reconstructed from the polytopes via symplectic reduction [86, 118], generalizing familiar results for toric manifolds. We now review the procedure, showing

[2] Notice that, with a standard abuse of notation, we will use p_a for the fixed points and for the vertices of the polytope, which are the image of the fixed points under the moment map μ.

also that the symplectic two-form $\omega_{(2)}$ is also Kähler, *i.e.* there exist a complex structure compatible with $\omega_{(2)}$.

We start by considering the standard $\mathbb{T}^n$ action on $\mathbb{C}^n$ (with coordinates $z_a = r_a \mathrm{e}^{2\pi \mathrm{i}\varphi_a}$) and its subgroup $K \in \mathbb{T}^n$ defined as

$$K = \left\{ (\mathrm{e}^{2\pi \mathrm{i} \mathcal{Q}_a}, \dots, \mathrm{e}^{2\pi \mathrm{i} \mathcal{Q}_n}) \in \mathbb{T}^n / \sum_{a=1}^{n} \mathcal{Q}_a \vec{v}_a \in \mathbb{Z}^n \right\} . \tag{2.1.8}$$

where $\vec{v}_a$ are the same vectors as in (2.1.5). The subgroup K can be written as the sum of a discrete part Γ and a continuous $\mathbb{T}^{n-m}$ one

$$K = \Gamma \oplus \left\{ (\mathrm{e}^{2\pi \mathrm{i} Q_a^\alpha c_\alpha}, \dots, \mathrm{e}^{2\pi \mathrm{i} Q_n^\alpha c_\alpha}) \in \mathbb{T}^n / \sum_{\alpha=1}^{n-m} Q_a^\alpha \vec{v}_a = 0 \right\} , \quad \mathcal{Q}_a = \sum_{\alpha=1}^{n-m} Q_a^\alpha c_\alpha \, , \quad c_\alpha \in \mathbb{R} \, , \tag{2.1.9}$$

where Q_a^α is called, in a physics context, the "GLSM charge" of the quotient and consists of a matrix with $a = 1, \dots, n$ columns and $\alpha = 1, \dots, n-m$ rows. They generate the continuous part and define a map $Q_a^\alpha : \mathbb{T}^{n-m} \to \mathbb{T}^n$ such that $Q_a^\alpha(\mathrm{e}^{2\pi \mathrm{i} c_\alpha}) = \mathrm{e}^{2\pi \mathrm{i} Q_a^\alpha c_\alpha} = \mathrm{e}^{2\pi \mathrm{i} \mathcal{Q}_a}$. One now constructs $\mathbb{M}_{2m}$ as the symplectic reduction

$$\mathbb{M}_{2m} = \mathbb{C}^n // K \equiv (\mu_K^{-1}(0) \subset \mathbb{C}^n)/K = \left\{ z_a \in \mathbb{C}^n / \mu_\alpha^K(z, \bar{z}) = 0 \, , \, \alpha = 1, \dots, n-m \right\} / K \, , \tag{2.1.10}$$

where μ_α^K is the moment map associated with the action of K on $\mathbb{C}^n$. To find its expression we can start from the standard Kähler form $\omega(\mathbb{C}^n)$ and moment map $\mu_a^{\mathbb{T}^n}$ for the $\mathbb{T}^n$ action of $\mathbb{C}^n$. In particular we have

$$\omega(\mathbb{C}^n) \equiv \frac{\mathrm{i}}{2} \sum_a^n \mathrm{d}z_a \wedge \mathrm{d}\bar{z}_a = \sum_{a=1}^n \frac{\mathrm{d}|z_a|^2}{2} \wedge \mathrm{d}\varphi_a \implies \mu_a^{\mathbb{T}^n}(z, \bar{z}) = \frac{|z_a|^2}{2} + \lambda_a \, , \tag{2.1.11}$$

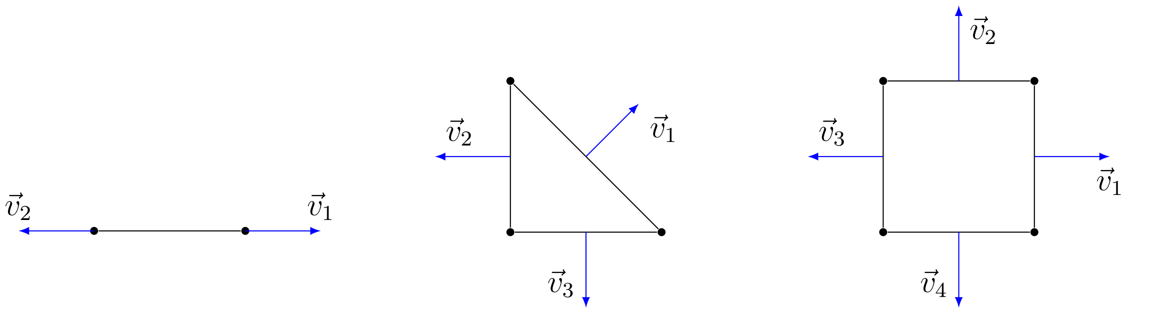
where, as usual, we used the freedom in the definition of the moment map to add a constant λ_a. These λ_a are called Kähler moduli for reasons that will be clear momentarily and are the same appearing in (2.1.5). From (2.1.11) we can deduce the moment map $\mu_\alpha^K : \mathbb{C}^n \to (\mathrm{Lie}K)^* = \mathbb{R}^{n-m}$ by composition

$$\mu_\alpha^K(z, \bar{z}) \equiv \sum_{a=1}^n Q_a^\alpha \mu_a^{\mathbb{C}^n}(z, \bar{z}) \, . \tag{2.1.12}$$

It is easy to verify that this is the correct expression. Since $Q_a^\alpha \varphi_a$ are the infinitesimal generators of K, we compute

$$i_{Q_a^\alpha \partial_{\varphi_a}} \omega(\mathbb{C}^n) = -\mathrm{d}\mu_\alpha^K(z, \bar{z}) \, , \tag{2.1.13}$$

which agrees with (2.1.2). Then it is possible to show that the Hamiltonian effective action of $\mathbb{T}^n$ on $\mathbb{C}^n$ induces an Hamiltonian effective action of $\mathbb{T}^m$ on $\mathbb{M}_{2m}$, which is then toric, and the image under the moment map $\mu_I^{\mathbb{T}^m} = y_I$ is exactly the polytope (2.1.5) with vectors given in (2.1.9) and Kähler parameters as in (2.1.11). Moreover the two-form $\omega_{(2)}$ of $\mathbb{M}_{2m}$ is inherited from $\omega(\mathbb{C}^n)$ as $p^* \omega_{(2)} = i^* \omega(\mathbb{C}^n)$, where $p : \mu_K^{-1}(0) \to \mathbb{M}_{2m}$ and $i : \mu_K^{-1}(0) \to \mathbb{C}^n$ are the projection and the inclusion map, respectively. Finally, it



(a) Polytope of $\mathbb{CP}^1 \simeq S^2$. (b) Polytope of $\mathbb{CP}^2$. (c) Polytope of $F_0 \simeq \mathbb{CP}^1 \times \mathbb{CP}^1$.

Figure 2.1: Examples of polytopes regarding the complex projective space.

is possible to show that the compatible complex structure of $\mathbb{C}^n$ furnishes a compatible complex structure also for the quotient, which acquires a natural Kähler structure. From the above discussion, one sees that given a certain polytope defined by $\vec{v}_a$, the charges which construct it as a quotient have the vectors in their kernel, *i.e.* $\sum_{\alpha=1}^{n-m} Q_a^\alpha \vec{v}_a = 0$. A final comment is that a non-compact polytope represents a Calabi-Yau (CY) space if and only if all the vectors $\vec{v}_a$ lie on a hyperplane [119], or equivalently if it exists a vector $\vec{k}$ such that $\vec{k} \cdot \vec{v}_a = 1$ for each $a = 1, \dots, n$. It follows that there must be an $\bar{I}$ for which all $v_a^{\bar{I}}$ are equal. Then, an equivalent condition is $\sum_{\alpha=1}^{n-m} Q_a^\alpha = 0$.

For completeness we now present some basic examples of polytopes and GLSM charges that will be useful (or generalized) throughout this thesis. Since $\mathbb{C}^* \simeq \mathbb{R}_+ \times S^1$ and $\mathbb{C} = \mathbb{C}^* \cup \{0\}$, we have that $\mathbb{C}$ is toric and its polytope is simply an half-line ($\mathbb{C}^*$) with the point at the origin. Similarly, the polytope of $\mathbb{C}^2$ can be taken to be the union of two half-lines, joining at the origin. It is easy to construct now the polytope of $S^2 \simeq \mathbb{CP}^1 \simeq \mathbb{C} \cup \{+\infty\} = \mathbb{C}^* \cup \{0, +\infty\}$, which is depicted in figure 2.1a. To construct the polytope of $\mathbb{CP}^2$, we recall the definition of the (complex) projective space

$$\mathbb{CP}^m = \frac{\mathbb{C}^{m+1} \setminus 0}{\mathbb{C}^*} \simeq \frac{S^{2m+1} \subset \mathbb{C}^{m+1}}{U(1)}\,, \quad (z_1, \dots, z_{m+1}) \sim (\lambda z_1, \dots, \lambda z_{m+1})\,, \quad \lambda \in U(1)\,. \tag{2.1.14}$$

In the notation of (2.1.9), we have $n = m+1$, $a = 1, \dots, m$ and $\alpha = 1$. The GLSM charge and a choice of vectors for $\mathbb{CP}^2$ satisfying $Q_a \vec{v}_a = 0$ (see figure 2.1b) are then given by

$$\mathbb{CP}^2: \quad Q_a^{\alpha=1} = (1,1,1) \implies \vec{v}_1 = (1,1)\,, \quad \vec{v}_2 = (-1,0)\,, \quad \vec{v}_3 = (0,-1)\,. \tag{2.1.15}$$

In general, a polytope with n facets in $\mathbb{R}^{m=n-1}$ represents $\mathbb{CP}^{m=n-1}$. Then, the polytope of $\mathbb{CP}^3$ is a tetrahedron and the one of $\mathbb{CP}^m$ is sometimes called an m-simplex. Finally, the polytope for $\mathbb{CP}^1 \times \mathbb{CP}^1$ is simply the product of two segments representing $\mathbb{CP}^1$, as in figure 2.1c. From $\mathbb{CP}^2$ one can generate other interesting manifolds, known as *del Pezzo surfaces* dP_l, by replacing a vertex of the polytope $\mathbb{CP}^2$ with a $\mathbb{CP}^1$ (or, by "blowing-up" l-times[3] $\mathbb{CP}^2$). So for example $dP_1 = \widehat{\mathbb{CP}^2}$ is a trapezoid (see figure 2.2a), with vectors and charge given by

$$dP_1: \quad \begin{array}{ll} \vec{v}_1 = (1,0)\,, & \vec{v}_2 = (-1,1)\,, \\ \vec{v}_3 = (-1,0)\,, & \vec{v}_4 = (0,-1)\,, \end{array} \implies Q_a^\alpha = \begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}\,, \tag{2.1.16}$$

[3] In mathematical literature, dP_l is defined as the blow-up of $\mathbb{CP}^2$ $(9-l)$-times.

and $dP_2 = \widehat{dP_1} = \widehat{\mathbb{CP}^1 \times \mathbb{CP}^1}$. The manifold $\mathbb{CP}^1 \times \mathbb{CP}^1 \simeq S^2 \times S^2$ is also the simplest example of a *Hirzebruch surface*, which in general is defined as $F_t = \mathbb{CP}^1 \hookrightarrow_t \mathbb{CP}^1$ with Chern number t. As anticipated, $F_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$. A method for constructing F_t, which is a $m = 2$-dimensional complex space, is by the Kähler quotient

$$F_t = \mathbb{C}^{n=4}//U(1)^{n-m=2}\,, \quad Q_a^\alpha = \begin{pmatrix} 0 & 1 & -t & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \implies \begin{array}{ll} \vec{v}_1 = (1,0)\,, & \vec{v}_2 = (-t,1)\,, \\ \vec{v}_3 = (-1,0)\,, & \vec{v}_4 = (0,-1)\,. \end{array} \tag{2.1.17}$$

The resulting polytope is depicted in figure 2.2b. In chapter 3 (see section 3.5.1) we will find the orbifold version of these surfaces, also called *orbifold Hirzebruch surfaces* [120], with a label m_a attached to each facet. In particular, each facet will be a spindle. We also see that $\widehat{\mathbb{CP}^2} = dP_1 \simeq F_1$, whilst more generally F_t extends $\mathrm{d}P_1$. Finally, we want to discuss $\mathbb{C}/\mathbb{Z}_t$ (which is a simple orbifold) and its blow-up. The action of the quotient is

$$\mathbb{C}^{n=2}/\mathbb{Z}_t : \quad (z_1, z_2) \sim (\mathrm{e}^{\frac{2\pi i k}{t}} z_1, \mathrm{e}^{\frac{2\pi i k q}{t}} z_2)\,, \quad z_i \in \mathbb{C}\,, \quad k = 0, \dots, t-1\,, \quad \gcd(t, |q|) = 1\,. \tag{2.1.18}$$

For $q \in \mathbb{Z}$, this is the most general $\mathbb{Z}_t$ action one can consider. In this case there is not any continuous part in the quotient, with $K = \Gamma = \mathbb{Z}_t$, hence we introduce $r_\pm$ such that $r_- + q r_+ = t$ and

$$\mathbb{C}^2/\mathbb{Z}_t : \quad \mathcal{Q}_a = \left(\frac{k}{t}, \frac{kq}{t}\right) \implies \vec{v}_1 = (r_-, -q)\,, \quad \vec{v}_2 = (r_+, 1)\,, \quad \det(\vec{v}_1, \vec{v}_2) = t\,. \tag{2.1.19}$$

Notice that when $|z_i|^2 = 1$, so that we are considering $S^3 \subset \mathbb{C}^2$, this is the definition of the lens space[4] $L(p,q)$ which is a smooth manifold since this time the action is free (the quotient does not act on the origin of $\mathbb{C}^2$, which is a fixed point). At this stage the blow-up can be made by simply replacing the vertex of the polytope of $\mathbb{C}^2/\mathbb{Z}_t$ with a vertical segment, which is the polytope for $\mathbb{CP}^1$. In this way we obtain again a non-compact polytope with two fixed points (see figure 2.2c)

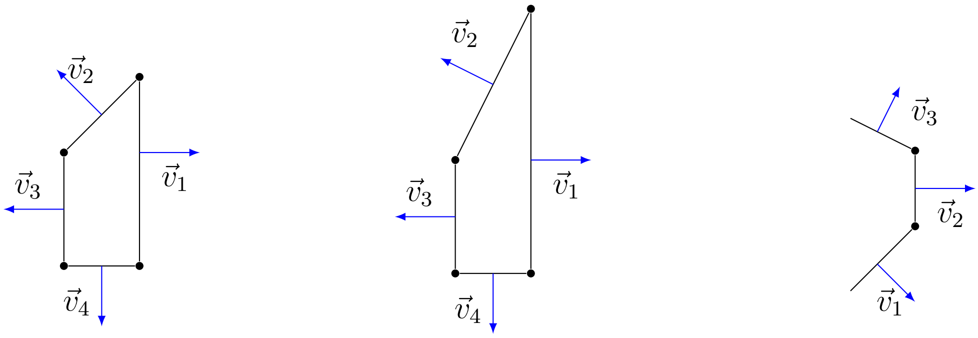
$$\widehat{\mathbb{C}^2/\mathbb{Z}_t} : \quad \vec{v}_1 = (r_-, -q)\,, \quad \vec{v}_2 = (1,0)\,, \quad \vec{v}_3 = (r_+, 1)\,. \tag{2.1.20}$$

It is also interesting to notice that $\widehat{\mathbb{C}^2/\mathbb{Z}_t} \simeq \mathbb{C} \hookrightarrow \mathcal{O}(-t) \to \mathbb{CP}^1 = \mathbb{C}^3//U(1)$, with action

$$\mathbb{C}^{n=3}//U(1)^{n-m=1}\,, \quad Q_a^1 = (1, -t, q) \implies \vec{v}_1 = (r_-, -q)\,, \quad \vec{v}_2 = (1,0)\,, \quad \vec{v}_3 = (r_+, 1)\,. \tag{2.1.21}$$

Again, we will find a generalization of this space in chapter 5 (see section 5.4.3), where the compact face of the polytope will be a spindle. In general, we can use $\mathbb{WCP}^1_{[m_-,m_+]}$ to perform the blow-ups. It is now easy to see that, for example, both $\mathbb{C}^2/\mathbb{Z}_t$ and its blow-up can be CY spaces. From the charges in (2.1.21), we have the condition $t = q + 1$ (or $r_\pm = 1$ from the vectors, which implies again the same condition). The same conclusion is deduced from (2.1.19).

[4]Sometimes the action is given with $q \mapsto -q$. However such definitions are equivalent, since one has the following chain of diffeomorphisms: $L(t,q) \simeq L(-t,q) \simeq L(t,-q) \simeq L(t, q+kt)$.



(a) $dP_1 = \widehat{\mathbb{CP}^2} \simeq F_1$. (b) F_t. (c) $\widehat{\mathbb{C}^2/\mathbb{Z}_t} \simeq \mathcal{O}(-t) \to \mathbb{CP}^1$.

Figure 2.2: Examples of polytopes of more involved manifolds.

2.1.3 Kähler structure

The most general Kähler $\mathbb{T}^m$-invariant metric, in action-angle coordinates (y_I, ϕ_I), is given by [121]

$$\mathrm{d}s^2 = G_{IJ}(\vec{y})\mathrm{d}y_I\mathrm{d}y_J + G^{IJ}(\vec{y})\mathrm{d}\phi_I\mathrm{d}\phi_J\,, \quad G^{IJ} = (G^{-1})_{IJ}\,, \quad G^{IJ}G_{JK} = \delta^I_K\,. \tag{2.1.22}$$

and the almost complex structure J compatible with $\omega_{(2)}$ is

$$J = \begin{pmatrix} 0 & -G^{IJ} \\ G_{IJ} & 0 \end{pmatrix}\,. \tag{2.1.23}$$

The condition that J is integrable is equivalent to require that G_{IJ} is an Hessian matrix

$$G_{IJ} = \partial_{y_I}\partial_{y_J}G(\vec{y})\,, \tag{2.1.24}$$

where the function $G(\vec{y})$ is called symplectic potential. It is also important to impose that G_{IJ} is positive definite and the correct behaviour on the facets $\mathcal{F}_a$ of the polytope. The result is that [118]

$$\det G_{IJ} = f(y)\prod_{a=1}^{n} l_a(\vec{y})^{-1}\,, \quad G(\vec{y}) = \frac{1}{2}\sum_{a=1}^{n} l_a(\vec{y})\log l_a(\vec{y}) + h(\vec{y}) \tag{2.1.25}$$

where $f(\vec{y}) > 0$ and $h(\vec{y})$ are smooth function on the whole polytope, facets included. The canonical choice $h(\vec{y}) = 0$ gives a canonical form of the metric

$$G^{\text{can}}_{IJ} = \frac{1}{2}\sum_{a=1}^{n}\frac{v^I_a v^J_a}{l_a(\vec{y})}\,. \tag{2.1.26}$$

The fact that the metric G_{IJ} can be expressed as derivatives of a potential reminds the theory of complex (Kähler) orbifolds. Indeed, defining holomorphic coordinates

$$z_I = x_I + \mathrm{i}\phi_I\,, \quad x_I = \partial_{y_I}G(\vec{y})\,, \tag{2.1.27}$$

the symplectic two-form acquires a standard form

$$\omega_{(2)} = 2\mathrm{i}\partial\bar{\partial}F(\vec{x})\,, \quad F(\vec{x}) \equiv \left[y_I\partial_{y_I}G(\vec{y}) - G(\vec{y})\right]\big|_{y_I=\partial_{x_I}F(\vec{x})}\,, \tag{2.1.28}$$

and $F(\vec{x}) = F(z_I, \bar{z}_I)$ is identified with a Legendre transform of $G(\vec{y})$. In these coordinates the metric reads

$$\mathrm{d}s^2 = F_{IJ}(\vec{x})(\mathrm{d}x^I \mathrm{d}x^J + \mathrm{d}\phi^I \mathrm{d}\phi^J) = F_{IJ}(\vec{x})(\mathrm{d}z^I \mathrm{d}\bar{z}^J + \mathrm{d}\bar{z}^I \mathrm{d}z^J)\,, \tag{2.1.29}$$

where, similarly as before

$$F_{IJ} = \partial_{x_I}\partial_{x_J} F(\vec{x}) = G^{IJ}(y_I = \partial_{x_I} F(\vec{x}))\,. \tag{2.1.30}$$

and $F(\vec{x})$ is the Kähler potential. In the complex coordinates z_I we can easily compute the (real) Ricci two-form[5]

$$\rho_{(2)} = -\mathrm{i}\partial\bar{\partial} \log \det F_{IJ}(\vec{x}) = -\mathrm{i}\partial\bar{\partial} \log \det G^{IJ}(\vec{y})\,, \tag{2.1.31}$$

which is also defined as the derivative of the (real) "Ricci potential" $P_{(1)}$, namely a connection one-form on the anti-canonical bundle

$$\rho_{(2)} = \mathrm{d}P_{(1)}\,. \tag{2.1.32}$$

Moreover, for complex orbifolds (or manifolds), the integrability of the almost complex structure is equivalent to the non-closedness of the holomorphic $(m, 0)$ form[6]

$$\mathrm{d}\Omega^{(m,0)} = \mathrm{i}P_{(1)} \wedge \Omega^{(m,0)}\,. \tag{2.1.33}$$

Similarly to (2.1.31), we can deduce from (2.1.33) a general expression for the holomorphic form [35]

$$\Omega^{(m,0)} = \mathrm{e}^{\mathrm{i}\alpha(z,\bar{z})}[\det F_{IJ}(z, \bar{z})]^{1/2}\mathrm{d}z^1 \wedge \ldots \wedge \mathrm{d}z^m\,, \tag{2.1.34}$$

for a complex function $\alpha(z, \bar{z})$ such that $\partial\bar{\partial}\alpha(z, \bar{z}) = 0$, *i.e.* it is harmonic. In general, equations (2.1.33) and (2.1.31) are valid for all the complex orbifolds $\mathbb{M}_4$ that we will construct.

2.1.4 Chern classes

To each facet of the polytope $\mathcal{P}$ is associated a toric divisor $D_a = \mu^{-1}(\mathcal{F}_a)$ and a line bundle L_a. Its first Chern class has been computed in [122, 118]

$$c_1(L_a) = -\frac{\mathrm{i}}{2\pi}\left[\partial\bar{\partial} \log |l_a|\right]\,, \tag{2.1.35}$$

where $[\alpha]$ denotes the cohomology class of a given differential form α. An explicit representative can be taken to be

$$c_1(L_a) = \mathrm{d}\mu_a^I \wedge \mathrm{d}\phi_I\,, \quad \mu_a^I = -\frac{1}{4\pi}\frac{G^{IJ}(\vec{y})v_a^J}{l_a(\vec{y})}\,, \tag{2.1.36}$$

where μ_a^I can be considered basically moment map for the torus $\mathbb{T}^m$ on D_a, indeed

$$i_{\partial_{\phi_I}} c_1(L_a) = -\mathrm{d}\mu_a^I\,. \tag{2.1.37}$$

[5]The following identity on torus invariant functions $\partial\bar{\partial}f(\vec{y}) = -\frac{\mathrm{i}}{2}\mathrm{d}(G^{IJ}\partial_{y_I} f(\vec{y})\mathrm{d}\phi_J)$ holds.

[6]With "holomorphic volume form" we mean a top-form on the holomorphic tangent bundle; in general $\Omega^{(m,0)}$ is not an holomorphic form, in the sense that $\bar{\partial}\Omega^{(m,0)} \neq 0$.

The relation between these moment map and the ones for $\mathbb{M}_{2m}$ ($\mu_I = y_I$) is obtained by expressing the coordinates

$$\mu_I = y_I = -2\pi \sum_{a=1}^{n} \lambda_a \mu_I^a + \frac{1}{2} \sum_{a=1}^{n} G^{IJ}_{\text{can}}(\vec{y}) v_a^J \,, \tag{2.1.38}$$

which can be verified contracting both sides with $G_{IJ}(\vec{y})$. Then

$$\omega_{(2)} = \sum_{I=1}^{m} \mathrm{d}y_I \wedge \mathrm{d}\phi_I = -2\pi \sum_a^n \lambda_a c_1(L_a) + \frac{1}{2}\mathrm{d}\Big[\sum_a^n G^{IJ} v_a^J \mathrm{d}\phi_I\Big] \,. \tag{2.1.39}$$

Since the last term is a well defined one-form, we have that in co-homology

$$\frac{[\omega_{(2)}]}{2\pi} = -\sum_a^n \lambda_a c_1(L_a) \,, \tag{2.1.40}$$

where we see that λ_a are indeed the Kähler moduli. Moreover, using

$$G^{IJ}(\vec{y}) G_{JK}(\vec{y}) = \delta^I_K \implies \sum_a^n \mu_a^I v_a^J = -\frac{\delta^{IJ}}{2\pi} \,, \tag{2.1.41}$$

and (2.1.36) it can be easily verified

$$\sum_a^n v_I^a c_1(L_a) = 0 \,. \tag{2.1.42}$$

These are m constraints and imply that there are only $n-m$ independent line bundles L_a. Similarly, in homology, we can write

$$\sum_a^n v_I^a D_a = 0 \,, \tag{2.1.43}$$

where we have used that $c_1(L_a)$ are Poincaré dual to D_a:

$$\int_{\mathbb{M}_{2m}} c_1(L_a) \wedge \alpha_{2(m-1)} = \int_{D_a} \alpha_{2(m-1)} \,, \tag{2.1.44}$$

where $\alpha_{2(m-1)}$ is any $2(m-1)$-form on the orbifold $\mathbb{M}_{2m}$. Again, this implies that the are only $n-m$ independent $2(m-1)$-cycles in homology.

Recall that we have introduced two types of vectors, the $\vec{v}_a$ which enters in the polytope (2.1.5) and in the above discussion, and the hatted primitive $\hat{\vec{v}}_a = \vec{v}_a/m_a$. Associated to this distinction, we can consider hatted line bundles $\hat{L}_a$ and first Chern classes $c_1(\hat{L}_a)$ defined as

$$c_1(\hat{L}_a) = m_a c_1(L_a) \,. \tag{2.1.45}$$

Notice that this definition is consistent with (2.1.36), supplemented with (2.1.26) and (2.1.6). Correspondingly, we can introduce hatted divisors $\hat{D}_a$, called *branch divisors*, associated to $\hat{\vec{v}}_a$ as

$$\hat{D}_a = m_a D_a \,, \tag{2.1.46}$$

where from this point of view the D_a are *ramification divisors* (see *e.g.* [123]). More precisely, the last equation can be re-written as $D_a = \hat{D}_a \times \mathrm{pt}/\mathbb{Z}_{m_a}$, where this means that at each point on D_a the structure group $\mathbb{Z}_{m_a}$ is acting on the transverse complex direction $\mathbb{C}$ normal to D_a. In other terms, we can see the action of $\mathbb{Z}_{m_a}$ on the divisor D_a from the metric by zooming in near to the locus in which a specific $U(1)$ degenerates. It should be clear, instead, that moving exactly on this locus we are able to study only $\hat{D}_a$. This also means that if we restrict to some loci $\mathcal{L}_a$ in the metric defined by a specific radial coordinate r set to 0, we will end up with $\mathcal{L}_a = \hat{D}_a$. We will see an explicit example of this phenomenon in section 3.5.1. Finally, due to the action of the structure group, we have

$$\int_{\hat{D}_a} \alpha_{2(m-1)} = m_a \int_{D_a} \alpha_{2(m-1)} \,. \tag{2.1.47}$$

It will also be useful to notice that given a toric ramification divisor D_a of the total space $\mathbb{M}_{2m}$, the first Chern class of the tangent bundle $T\mathbb{M}_{2m}$ can be decomposed by means of the *adjunction formula* as

$$c_1(T\mathbb{M}_{2m})\big|_{D_a} = c_1(TD_a) + c_1(L_a)\big|_{D_a} \,. \tag{2.1.48}$$

In this formula the first Chern class of the line bundles L_a is given by (2.1.36), while $c_1(T\mathbb{M}_{2m})$ follows from

$$c_1(T\mathbb{M}_{2m}) = \sum_{a=1}^{n} c_1(L_a) = -K_{\mathbb{M}_{2m}} \,, \tag{2.1.49}$$

where K is the canonical bundle of $\mathbb{M}_{2m}$. These two equations can be combined into

$$c_1(TD_a) = \sum_{b\neq a}^{n} c_1(L_b)\big|_{D_a} \,. \tag{2.1.50}$$

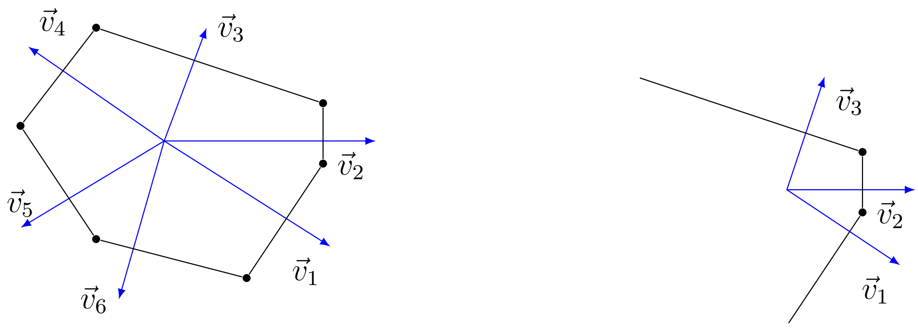
We will use the adjunction formula (2.1.48) in an integrated form. Indeed in this way we will be able to connect the informations extracted from the metrics (*i.e.* the Ricci forms of $\mathbb{M}_4$ and of D_a) and the ones coming from the topology of the solutions, encoded in the polytopes.In the next section we review this idea in details.

2.1.5 Four dimensional toric orbifolds

We now focus to four-dimensional ($m = 2$) toric orbifolds, which are central for this thesis. Notice that for $m = 2$, the simplicity condition for the polytope is automatically satisfied, in that each fixed point is at the intersection of exactly two facets $\mathcal{F}_a$. Moreover, for compact polytopes the number of facets n and f, the number of fixed points[7], coincide. Differently, for non-compact polytopes we have $f = n-1$—See figure 2.3. We now define,

[7] For clarity, we summarize here the notation

$$\begin{aligned} &I = 1,\dots,m = \dim_{\mathbb{C}}\mathbb{M}\,, \qquad && a = 1,\dots,n = \text{number of facets } \mathcal{F}\,, \\ &\alpha = 1,\dots,n-m\,, && A = 1,\dots,f = \text{number of fixed points } f\,. \end{aligned}$$



(a) Compact polytope. (b) Non-compact polytope.

Figure 2.3: Examples of four-dimensional polytopes, The vectors are ordered counter-clockwise and defined to be outward-pointing. On the left it is shown a compact polytope with $n = f = 6$ fixed points; on the right a non-compact polytope with $n = 3$ facets and $f = 2$ fixed points. The vectors $\vec{v}_a$ are *not* primitive.

for compact toric orbifolds, the intersection matrix of two toric divisors, which is computed to be[8]

$$D_{ab} \equiv D_a \cdot D_b = \int_{\mathbb{M}_4} c_1(L_a) \wedge c_1(L_b) = \begin{cases} \dfrac{1}{d_{a-1,a}} & \text{if } b = a-1\,, \\ \dfrac{1}{d_{a,a+1}} & \text{if } b = a+1\,, \\ -\dfrac{d_{a-1,a+1}}{d_{a-1,a}\, d_{a,a+1}} & \text{if } b = a\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{2.1.51}$$

and we have defined

$$d_{ab} = \det(\vec{v}_a, \vec{v}_b)\,. \tag{2.1.52}$$

Some useful identities are then given by

$$D_{aa} = \int_{D_a} c_1(L_a) = -\frac{d_{a-1,a+1}}{d_{a-1,a}\, d_{a,a+1}}\,, \qquad \sum_{a,b}^{n} D_{ab} = \int_{\mathbb{M}_4} c_1(T\mathbb{M}_4) \wedge c_1(T\mathbb{M}_4)\,, \tag{2.1.53}$$

where we have employed (2.1.44), and using also (2.1.49) we get

$$\begin{aligned} \sum_{b}^{n} D_{ab} &= \int_{\mathbb{M}_4} c_1(L_a) \wedge \sum_{b}^{n} c_1(L_b) = \int_{\mathbb{M}_4} c_1(L_a) \wedge c_1(T\mathbb{M}_4) \\ &= \int_{D_a} c_1(T\mathbb{M}_4) = \frac{d_{a-1,a} + d_{a,a+1} - d_{a-1,a+1}}{d_{a-1,a}\, d_{a,a+1}}\,. \end{aligned} \tag{2.1.54}$$

Notice that

$$d_{ab} = m_a m_b\, \hat{d}_{ab} \implies D_{ab} = \frac{1}{m_a m_b} \hat{D}_{ab}\,. \tag{2.1.55}$$

where, as always, the hatted quantities are referred to the $\vec{\hat{v}}$ defined in (2.1.6).

[8]Obviously we can compute also the intersection of an higher number of divisors, see *e.g.* [83].

We note that D_{ab} is invariant under $SL(2,\mathbb{Z})$ transformations of the basis and more generally for any transformation in $SL(2,\mathbb{R})$. This follows immediately from the fact that given any two vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$, $\det(\vec{v}_1, \vec{v}_2)$ is invariant under $SL(2,\mathbb{R})$ transformations of the two vectors. In particular, taking a generic matrix $S \in SL(2,\mathbb{R})$ and transforming the vectors as $\vec{v}'_a = S\,\vec{v}_a$, we have that S acts on the matrix $(\vec{v}_1, \vec{v}_2)$ simply by matrix multiplication, *i.e.* $(\vec{v}'_1, \vec{v}'_2) = S\,(\vec{v}_1, \vec{v}_2)$, and therefore

$$\det(\vec{v}'_1, \vec{v}'_2) = \det(S)\det(\vec{v}_1, \vec{v}_2) = \det(\vec{v}_1, \vec{v}_2)\,. \tag{2.1.56}$$

The importance of this property will become clear momentarily, since it allows us to compute the matrix D_{ab} from a generic set of vectors.

Using the adjunction formula (2.1.48), we can relate the toric data, encoded in the intersection matrix D_{ab}, and the complex formalism, embodied by $c_1(T\mathbb{M}_4)$ and $c_1(TD_a)$. We start noticing that the adjunction formula (2.1.48) can be integrated to give

$$D_{aa} = \int_{D_a} c_1(L_a) = \int_{D_a} \big[c_1(T\mathbb{M}_4) - c_1(TD_a)\big]\,. \tag{2.1.57}$$

Moreover,

$$\begin{aligned} D_{a\,a-1} + D_{a\,a+1} &= \int_{\mathbb{M}_4} c_1(L_a) \wedge \big[c_1(L_{a-1}) + c_1(L_{a+1})\big] = \int_{D_a} \big[c_1(L_{a-1}) + c_1(L_{a+1})\big] \\ &= \sum_{b\neq a}^{n} \int_{D_a} c_1(L_b) = \sum_{b\neq a}^{n} D_{ab} = \sum_{b}^{n} D_{ab} - D_{aa} = \int_{D_a} c_1(TD_a)\,, \end{aligned} \tag{2.1.58}$$

where we have used (2.1.44) in the first line, (2.1.51) from the first to the second line, and (2.1.54), (2.1.57) to get the result. From the point of view of complex geometry we can rewrite (2.1.57) and (2.1.58) as

$$\begin{aligned} D_{aa} &= -\frac{d_{a-1,a+1}}{d_{a-1,a}\,d_{a,a+1}} = \frac{1}{2\pi}\int_{D_a} [\rho_{T\mathbb{M}_4} - \rho_{TD_a}] = \frac{1}{2\pi m_a}\int_{\hat{D}_a} \Big[\rho_{T\mathbb{M}_4}\big|_{\hat{D}_a} - \rho_{T\hat{D}_a}\Big]\,, \\ D_{a\,a-1} + D_{a\,a+1} &= \frac{d_{a-1,a} + d_{a,a+1}}{d_{a-1,a}\,d_{a,a+1}} = \frac{1}{2\pi}\int_{D_a} \rho_{TD_a} = \frac{1}{2\pi m_a}\hat{\chi}_a\,, \end{aligned} \tag{2.1.59}$$

where we have used (2.1.47). Moreover we have introduced the Euler characteristic of a divisor $\hat{D}_a$

$$\hat{\chi}_a = \frac{1}{2\pi}\int_{\hat{D}_a} \rho_{T\hat{D}_a}\,, \tag{2.1.60}$$

and identified $\rho_{T\hat{D}_a} = \rho_{TD_a}$, since they are local quantities and do not know anything about the global action of the structure group $\mathbb{Z}_{m_a}$.

We now specialize to the case of four fixed points $n = s = 4$, that we refer to as quadrilaterals, as in figure 2.4. Their basis-independent information is encoded in the following 4×4 intersection matrix (2.1.51)

$$D_{ab} = \begin{pmatrix} \frac{d_{2,4}}{d_{4,1}\,d_{1,2}} & \frac{1}{d_{1,2}} & 0 & \frac{1}{d_{4,1}} \\ \frac{1}{d_{1,2}} & -\frac{d_{1,3}}{d_{1,2}\,d_{2,3}} & \frac{1}{d_{2,3}} & 0 \\ 0 & \frac{1}{d_{2,3}} & -\frac{d_{2,4}}{d_{2,3}\,d_{3,4}} & \frac{1}{d_{3,4}} \\ \frac{1}{d_{4,1}} & 0 & \frac{1}{d_{3,4}} & \frac{d_{1,3}}{d_{3,4}\,d_{4,1}} \end{pmatrix}\,, \tag{2.1.61}$$

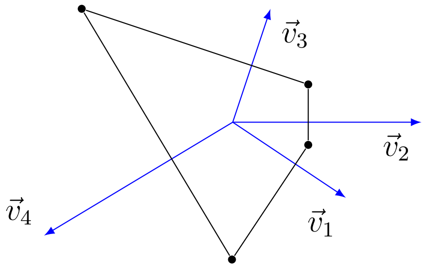


Figure 2.4: Outward-pointing fan and polytope of a generic quadrilateral.

where a vector identity implies that the $d_{a,b} \in \mathbb{Z}$ are actually not independent, but satisfy the relation

$$d_{1,2}\, d_{3,4} - d_{2,3}\, d_{4,1} = d_{1,3}\, d_{2,4}\,, \quad \hat{d}_{1,2}\, \hat{d}_{3,4} - \hat{d}_{2,3}\, \hat{d}_{4,1} = \hat{d}_{1,3}\, \hat{d}_{2,4}\,. \tag{2.1.62}$$

Thus, a generic quadrilateral is characterized by five independent integer parameters $\hat{d}_{a,b}$ (subject to (2.1.62)) and four integer labels m_a, which we refer to as the toric data of the orbifold.

2.2 Entropy functions for quadrilaterals

In this section we use part of the toric geometry reviewed up to this point to write an *off-shell free energy* for compactifications of five- and six-dimensional SCFTs on a four-dimensional toric orbifold $\mathbb{M}_4$. This, in particular, is completely characterized by the toric data of the polytope $\mathcal{P}$, which are the non-necessarily primitive vectors $\vec{v}_a$. On the other hand, the twisted compactification is realized by coupling the SCFTs to two background gauge fields A_i for the $U(1)^2$ symmetry, with field strengths $F_i = \mathrm{d}A_i$. We conjecture that the solutions that we will present, which take the form $\mathrm{AdS}_{D-4} \times \mathbb{M}_4$ with $D = 6, 7$, are holographically dual to $d = 1, 2$ SCFTs obtained compactifying on $\mathbb{M}_4$ the $d = 5$, $\mathcal{N} = 1$ Seiberg theory [77] dual to [124] and the six-dimensional $\mathcal{N} = (0, 2)$ dual to a stack of M5-branes. The underlying idea is based on the idea of gluing "gravitational blocks", introduced in [38]. This recipe extends the results of [52] to four-dimensional toric orbifolds with an arbitrary number of fixed points $n = s \geq 3$. Starting from first principles, for $D = 6$, this function should be derived from the large N limit of $Z^{d=5\ \mathrm{SCFT}}_{S^1\times\mathbb{M}_4}$, the localized partition function of the $d = 5$ SCFT placed on the background of $S^1 \times \mathbb{M}_4$. Similarly, for $D = 7$, one should integrate on $\mathbb{M}_4$ the anomaly polynomial of the six-dimensional SCFTs associated with M5-branes. Whilst for $d = 5$ a solid proof is still lacking, in $d = 6$ the (equivariant) integration of the anomaly polynomial on $\mathbb{M}_4$ has been performed in [83], finding agreement with the equivariant integration of the action results [89]. Instead, in $d = 5$, the gravitational blocks for $\mathbb{M}_4 = \Sigma_1 \ltimes \Sigma_2$ has been retrieved in [88] through the integration of the equivariant volume of the associated geometry, with the addition of higher times. Throughout this thesis we will provide supporting evidences of these statements by comparing the explicit results from the solutions (*i.e.* entropies or gravitational central charges) and the extremization of our (conjectural) off-shell free energies.

2.2.1 Twisting data

We start by considering two background gauge fields $-A_i$, which can be thought of as connections over two line orbibundles E_i. Since the $c_1(L_a)$ form a basis for $H^2(\mathbb{M}_4,\mathbb{Q})$ (see (2.1.40)), we can decompose the first Chern class of E_i as

$$c_1(E_i) = -\frac{\mathrm{d}A_i}{2\pi} = -\sum_{a=1}^{n} \mathfrak{p}_i^a c_1(L_a)\,, \quad \mathfrak{p}_i^a \in \mathbb{Q}\,. \tag{2.2.1}$$

Then we define the "physical fluxes" as[9]

$$\mathfrak{q}_i^a \equiv \frac{1}{2\pi}\int_{D_a} F_i = \frac{1}{2\pi m_a}\int_{\hat{D}_a} F_i = \frac{\hat{\mathfrak{q}}_i^a}{m_a}\,. \tag{2.2.2}$$

Notice that, as a consequence of (2.1.44) and the homological relation (2.1.43), we have

$$\mathfrak{q}_i^a = \sum_{b=1}^{n} \mathfrak{p}_i^b \int_{D_a} c_1(L_b) = \sum_{b=1}^{n} D_{ab}\,\mathfrak{p}_i^b\,, \qquad \sum_{a=1}^{n} \mathfrak{q}_i^a \vec{v}_a = 0\,, \tag{2.2.3}$$

where D_{ab} is the intersection matrix defined in (2.1.51). The latter equation is a constraint on the physical fluxes and implies that, for fixed i, only $n-2$ of the $\mathfrak{q}_i^a$ are independent. On the other hand, $H^2(\mathbb{M}_4,\mathbb{Q})$ has dimension $n-2$, thus, for fixed i, only $n-2$ of the $\mathfrak{p}_i^a$ are linearly independent. Therefore, the first equation in (2.2.3) can be inverted to obtain $\mathfrak{p}_i^a$ in terms of $\mathfrak{q}_i^a$ once the redundant equations are eliminated. The two additional degrees of freedom are the signal of a "shift symmetry"

$$\tilde{\mathfrak{p}}_i^a = \mathfrak{p}_i^a + \det(\vec{\lambda}_i, \vec{v}_a) \implies \mathfrak{q}_i^a = \sum_{b=1}^{n} D_{ab}\,\mathfrak{p}_i^b = \sum_{b=1}^{n} D_{ab}\,\tilde{\mathfrak{p}}_i^b\,, \tag{2.2.4}$$

for any two-dimensional constant vector $\vec{\lambda}_i$. A similar condition applies also to the R-symmetry fluxes

$$\mathfrak{q}_R^a \equiv \mathfrak{q}_1^a + \mathfrak{q}_2^a \implies \sum_{a=1}^{n} \mathfrak{q}_R^a \vec{v}_a = 0\,. \tag{2.2.5}$$

A possible choice to solve this constraint is

$$\mathfrak{q}_R^a = \sum_{b=1}^{n} D_{ab}\,\sigma^b\,, \tag{2.2.6}$$

where σ^a are, *a priori*, arbitrary coefficients. Comparing (2.2.6) and the first equation in (2.2.3) we get

$$\mathfrak{p}_1^a + \mathfrak{p}_2^a = \sigma^a + \det(\vec{W}, \vec{v}_a)\,, \tag{2.2.7}$$

for a generic vector $\vec{W}$. It is easy to see that we can fix $\vec{W}$ to a suitable value

$$\vec{\tilde{W}} = \vec{W} + \vec{\lambda}_1 + \vec{\lambda}_2\,, \tag{2.2.8}$$

[9]Here and in the following we shall rename the background gauge fields as $g_c A_i \mapsto A_i$, which is more natural from the field theory point of view. The nomenclature "physical fluxes" should not be confused with the fact that the natural fluxes in supergravity are computed on the branch divisors $\hat{D}_a$.

by using the gauge symmetry (2.2.4). In particular, we conjectured that σ^a are n arbitrary signs, *i.e.* $\sigma^a = \pm 1$. This assumption is supported by all the example we will provide in the thesis. Later, in [89], and recently in [94] (in a purely four-dimensional context), it has been explicitly demonstrated, within the equivariant localization approach, that these σ^a *must be* signs in general. Indeed, given a spinor ψ on a (compact) toric[10] $S^4 \hookrightarrow \mathbb{M}_4$ with $U(1)^2 \times U(1)^2$ symmetry, a supersymmetric Killing vector ξ obtained as a bilinear in ψ, and a fixed point $p_{A=a} = D_a \cap D_{a+1}$ under the toric action, it is easy to show that the spinor charge under ξ is related to the weights of the toric action. More precisely, $\mathcal{L}_\xi \psi|_{p_a} = \frac{\mathrm{i}}{2}(\Phi_1^a + \Phi_2^a + s_1^a \epsilon_1^a + s_2^a \epsilon_2^a)\psi|_{p_a}$, with $(\Phi_{1,2}^a, \epsilon_{1,2}^a)$ toric weights that we shall introduce momentarily, and $s_{1,2}^a = \pm 1$ signs. Consistency with (2.2.6),(2.2.7) implies immediately $s_1^a = \sigma^a$ and $s_2^a = \sigma^{a+1}$, showing indeed that $\sigma^a = \pm 1$ in general. Further, it can be shown that $\Gamma_\star \psi|_{p_a} = s_1^a s_2^a \psi|_{p_a} = \sigma^a \sigma^{a+1} \psi|_{p_a}$ with $\Gamma_\star$ the chirality matrix in eight-dimensions, so that the chirality of the spinor (at a fixed point p_a) is determined by the very same signs.

Denoting as E_R the R-symmetry line bundle, equation (2.2.6) can be rewritten as

$$c_1(E_R) = -\sum_{a=1}^{n} \sigma^a c_1(L_a)\,, \tag{2.2.9}$$

Notice that the standard topological twist corresponds to identifying E_R with the orbifold canonical line bundle $K_{\mathbb{M}_4}^{\text{orb}} = -\sum_a D_a$ and hence $\sigma^a_{\text{top-twist}} = (+, \dots, +)$. See *e.g.* [125] for a related discussion of orbifold line bundles. In other words, we expect that it should be possible to compactify a SCFT on a toric orbifold $\mathbb{M}_4$, turning on a background R-symmetry gauge field with magnetic fluxes given in (2.2.6), with $\sigma^a = \pm 1$ parameterizing the different supersymmetry-preserving twists. Of course, we have not proven that all these different twists preserve supersymmetry, nor that there cannot exist more general twists. It would be interesting to carry out such an analysis extending the results of [42], where it was demonstrated that, in whole generality, on a spindle the only two supersymmetry-preserving twists are the twist and the anti-twist.

2.2.2 The recipe

We conjecture that given a general class of SCFTs in $d = 5, 6$ compactified on a four-dimensional (compact) toric orbifold $\mathbb{M}_4$, with an arbitrary twist parameterized by a set of signs σ^a as in (2.2.6), the corresponding entropy/central charge, respectively, is determined by the constrained extremization of the following *off-shell free energy*

$$F(\varphi_i, \epsilon_i; \mathfrak{q}_i^a) = k_d \sum_{a=1}^{n} \frac{\eta_d^a}{d_{a,a+1}} \frac{\mathcal{F}_d(\Phi_i^a)}{\epsilon_1^a \epsilon_2^a}\,, \tag{2.2.10}$$

where the sum is over the number $n = s$ of fixed points (or facets). Here, $\mathcal{F}_d$ are the usual gravitational blocks (*cf.* table 2 of [52] or table 1.1)

$$\mathcal{F}_5(\Delta_i) = -\frac{4\sqrt{2}\pi\, N^{5/2}}{15\sqrt{8-N_f}}(\Delta_1\Delta_2)^{3/2}\,, \quad \mathcal{F}_6(\Delta_i) = -\frac{9N^3}{256}(\Delta_1\Delta_2)^2\,, \tag{2.2.11}$$

[10] This will be the internal space for the uplift of our solutions to massive type IIA supergravity and M-theory, see section 3.2.

the variables Φ_i^a are defined as

$$\Phi_i^a = \varphi_i - \mathfrak{p}_i^a \epsilon_1^a - \mathfrak{p}_i^{a+1} \epsilon_2^a \,, \tag{2.2.12}$$

and the auxiliary quantities $\epsilon_{1,2}^a$ read

$$\epsilon_1^a = -\frac{\det(\vec{v}_{a+1}, \vec{\epsilon})}{d_{a,a+1}} \,, \qquad \epsilon_2^a = \frac{\det(\vec{v}_a, \vec{\epsilon})}{d_{a,a+1}} \,. \tag{2.2.13}$$

with[11] $\vec{\epsilon} = (\epsilon_1, \epsilon_2)$. Here, ϵ_1 and ϵ_2 may be interpreted as the fugacities associated with the two $U(1)$ rotational symmetries and parameterize their mixing with the R-symmetry. As anticipated, $\epsilon_{1,2}^A$ can be seen as the toric weights for the $U(1)^2$ action existing on $\mathbb{M}_4$, whilst Φ_i^a are the analogous for the S^4 in $S^4 \hookrightarrow \mathbb{M}_4$. The relation (2.2.12) can also be obtained instead of being postulated as a definition; see [89]. $\eta_d^a = \pm$ are signs that at this stage must be tuned by hand: we speculate that in $d = 5$ they are related to the type of twist as $\eta_5^a = \sigma^a \sigma^{a+1}$ while for $d = 6$ we take all of them with the same value, which is, ultimately, related to the chirality of the SCFT in question. In particular, $\eta_6^a = \kappa$, with $\kappa = \pm 1$ so that the extremization of the off-shell free energy reproduces the central charge of the $d = 2$ SCFTs with $\mathcal{N} = (0, 2)$ or $\mathcal{N} = (2, 0)$, respectively, extracted from the anomaly polynomial of the $d = 6$ theory. We recall that $d_{a,b} = \det(\vec{v}_a, \vec{v}_b)$, whereas k_d are numerical constants which assume, *a posteriori*, the values

$$k_5 = -1 \,, \qquad k_6 = \frac{64}{9} \,. \tag{2.2.14}$$

In this construction φ_i and ϵ_i are variables with respect to which one has to extremize $F(\varphi_i, \epsilon_i)$, $\vec{v}_a$ (and subsequently m_a) are data describing the toric orbifold $\mathbb{M}_4$ and $\mathfrak{p}_i^a$ are related to the physical fluxes $\mathfrak{q}_i^a$ through (2.2.3). The variables of the off-shell free energy, namely φ_i and ϵ_i, are subject to the constraint

$$\varphi_1 + \varphi_2 - \det(\vec{W}, \vec{\epsilon}) = 2 \,, \tag{2.2.15}$$

where $\vec{W}$ is a two-dimensional constant vector parameterizing the "gauge invariance" of the problem, discussed above in (2.2.4). This condition is inherited from the R-symmetry constraint $\Delta_1 + \Delta_2 = 2$, where Δ_i are the fugacities parameterizing the R-symmetry within the Cartan subgroup of the global symmetries of the d-dimensional parent theory. The whole construction enjoys two symmetries

1. "shift symmetry": if under the redefinitions (2.2.4) and (2.2.8) we require also $\tilde{\varphi}_i = \varphi_i + \det(\vec{\lambda}_i, \vec{\epsilon})$, we have

$$\Phi_i^a(\tilde{\varphi}, \tilde{\mathfrak{p}}) = \Phi_i^a(\varphi, \mathfrak{p}) \,, \quad \tilde{\varphi}_1 + \tilde{\varphi}_2 - \det(\vec{\tilde{W}}, \vec{\epsilon}) = \varphi_1 + \varphi_2 - \det(\vec{W}, \vec{\epsilon}) = 2 \,. \tag{2.2.16}$$

   It follows that (2.2.10) remains unchanged;

2. As we already mentioned around (2.1.56), the quantity $\det(\vec{v}_1, \vec{v}_2)$ is invariant under $SL(2, \mathbb{R})$ transformations of the two vectors $\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$. This property is fundamental for the consistency of our construction. A given toric orbifold can be described

[11] The two variables $\epsilon_{1,2}$ are different from the ones adopted in [52]. Specifically, $\epsilon_i^{\text{here}} = 2\epsilon_i^{\text{there}}$.

by an infinite number of polytopes equivalent under $SL(2,\mathbb{Z})$ transformations of the vectors $\vec{v}_a$. These transformations are generated by rotations of the basis vectors (e_1, e_2) that give rise to effective torus actions. On the other hand, we expect the free energy not to be affected by these transformations and, indeed, this is the case. If we perform an $SL(2,\mathbb{Z})$ rotation of the vectors $\vec{v}_a$, the expression of the intersection matrix D_{ab} is retained, and the explicit form of the off-shell free energy is not modified provided we apply exactly the same transformation to $\vec{\epsilon}$ and $\vec{W}$.

The last step of the procedure is the constrained extremization of $F(\varphi_i, \epsilon_i)$, which can be performed defining the function

$$\mathcal{S}(\varphi_i, \epsilon_i, \Lambda; \mathfrak{q}_i^a) = F(\varphi_i, \epsilon_i; \mathfrak{q}_i^a) + \Lambda\big(\varphi_1 + \varphi_2 - \det(\vec{W}, \vec{\epsilon}) - 2\big) \tag{2.2.17}$$

and extremizing it with respect to φ_i, ϵ_i and the Lagrangian multiplier Λ. As a consequence of Euler's theorem, $F(\varphi_i^*, \epsilon_i^*) = -\frac{2}{h}\Lambda^*$, with $h = 1$ for $d = 5$, $h = 2$ for $d = 6$.

Notice that an important simplification comes in the sub-case with $A_1 = A_2$, which implies $\mathfrak{q}_1^a = \mathfrak{q}_2^a$ and, in turn, $\mathfrak{p}_1^a = \mathfrak{p}_2^a$. Indeed the off-shell free energy (2.2.17) becomes symmetric under $\varphi_1 \leftrightarrow \varphi_2$ and the necessary condition $(\partial_{\varphi_1} - \partial_{\varphi_2})\mathcal{S}(\varphi_i, \epsilon_i, \Lambda; \mathfrak{q}_1^a = \mathfrak{q}_2^a) = 0$ requires that the value at the extremum are equal: $\varphi_1^* = \varphi_2^*$. Moreover, employing the "shift symmetry" one can always fix $\vec{\lambda}_i$ such that $\vec{W} = 0$, and (2.2.7) implies $\mathfrak{p}_i^a = \sigma^a/2$. Finally, the constraint (2.2.15) now gives $\varphi_1^* = \varphi_2^* = 1$. Summarizing these facts, for $A_1 = A_2$ one can always take

$$\vec{W} = 0\,, \quad \mathfrak{p}_i^a = \frac{\sigma^a}{2}\,, \quad \varphi_1^* = \varphi_2^* = 1\,, \quad \Lambda = 0\,, \tag{2.2.18}$$

and extremize $\mathcal{S}(1, \epsilon_i, 0; \mathfrak{q}_1^a = \mathfrak{q}_2^a)$ only with respect to the remaining variables, *i.e.* the fugacities ϵ_i.

2.3 Summary of the chapter

In this chapter we have reviewed some important properties of toric orbifolds, with a particular focus on four dimensions ($m = 2$). With the tools developed and discussed here, we can extract all the toric data that characterize the underlying topology of a certain supergravity solution. These toric data consist of a set of primitive, $\mathbb{Z}^2$-valued and counter-clockwise ordered vectors $\vec{v}_a$, along with a label m_a attached to each facet $\mathcal{F}_a$ of the polytope, whose normal vectors are given by $\vec{v}_a$. All these informations can then be systematically organized into an off-shell free energy, which, upon extremization, reproduces the relevant gravitational quantities.

In particular, equations (2.1.59), that in synthesis read

$$-\frac{d_{a-1,a+1}}{d_{a-1,a}\, d_{a,a+1}} = \frac{1}{2\pi m_a}\int_{\hat{D}_a}\Big[\rho_{T\mathbb{M}_4}\big|_{\hat{D}_a} - \rho_{T\hat{D}_a}\Big]\,, \tag{2.3.1}$$

are the "master formulas" to study the $U(1)_1 \times U(1)_2$ -invariant (complex) quadrilateral orbifolds of the next chapters. We now summarize the strategy for clarity

- left hand side of (2.1.59): by studying the loci $\mathcal{L}_a = \hat{D}_a$ on which the Killing vectors $\xi_a = b_a^1 \partial_{\phi_1} + b_a^2 \partial_{\phi_2}$ (normalized to have unitary surface gravity) degenerate, we will be able to extract a "fake" set of normal vectors $\vec{V}_a$ through the relation[12]

$$\xi_a = \vec{V}_a \cdot (e_1, e_2)\,, \tag{2.3.2}$$

where $\{e_1, e_2\}$ is a "fake" basis for the torus action. "Fake" means that the vectors $\vec{V}_a$ can be $\mathbb{R}^2$-valued and the basis can be non-effective. For the purposes of computing the left hand of (2.1.59), this fact does not represent a problem. Indeed, as explained around (2.1.56), D_{ab}—and in turn also d_{ab}—are invariant under $S \in SL(2, \mathbb{R})$ rotations of the polytope. Since an effective basis $\{E_1, E_2\}$ will be obtained as $E_I = S^{-1}_{JI} e_J$ and the respective $\mathbb{Z}^2$-valued vectors as $v_a^I = S_{IJ} V_a^J$, we can continue with the "generic" $\vec{V}_a$ to compute the $SL(2, \mathbb{R})$-invariant determinants $d_{a,b}$. Notice that, being in general $V_a^I \in \mathbb{R}$, we can not extract the labels simply by the definition as $m_a = \gcd(V_a^1, V_a^2)$. However, we can obtain it using (2.1.59);

- right hand side of (2.1.59): we can consider the metrics on $(\mathbb{M}_4, \mathcal{L}_a = \hat{D}_a)$ and use $\mathrm{d}\Omega_X^{(n,0)} = \mathrm{i} P_X \wedge \Omega_X^{(n,0)}$ for $X = T\mathbb{M}_4, T\hat{D}_a$ to obtain $\rho_X = \mathrm{d}P_x$. We can then integrate them on $\hat{D}_a$, so that the master formulas (2.1.59) tell us the labels m_a.

In the next chapter we start presenting the supergravity solutions to which we will apply this extremization procedure. Locally, the metrics will take the form $\mathrm{AdS}_{D-4} \times \mathbb{M}_4$ where, after a suitable constraining of the parameters, $\mathbb{M}_4$ will be a $U(1)^2$ toric four-dimensional orbifold with four fixed-points. After having extracted all the relevant toric data from the solutions, we will be able to reproduce the relevant gravitational quantities from the extremization of (2.2.17).

[12] This relation and its use will be extensively explained in section 3.5.1. See also [126]

Chapter 3

Supergravity solutions with special quadrilaterals

In this section we review the first type of solution we will analyse, which takes the form of $\mathrm{AdS}_{D-4} \times \mathbb{M}_4$ with $D = 6, 7$. There are then two classes, where[1] $\mathbb{M}_4^{(1)} = \Sigma_\mathrm{g} \ltimes \mathbb{\Sigma}_2$ is the non-trivial fibration of a spindle over a smooth Riemann surface $\Sigma_\mathrm{g} = \mathbb{H}/\Gamma$ with $\Gamma \subset PSL(2, \mathbb{R})$ (and with genus $\mathrm{g} > 1$) or $\mathbb{M}_4^{(2)} = \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$, the fibration of a spindle over another (different) spindle. These solutions are a generalization of previous ones containing simpler four-dimensional toric orbifolds of the form $\mathbb{M}_4 = \Sigma_\mathrm{g} \times \mathbb{\Sigma}$, found in $D = 6, 7$ and in various supergravity theories [54, 52, 58]. Strictly speaking, $\mathbb{M}_4^{(1)}$ is not toric in any sense. However, taking an analytic continuation, we will obtain $S^2 \ltimes \mathbb{\Sigma}_2$, which is toric. The metric $\mathrm{AdS}_{D-4} \times S^2 \ltimes \mathbb{\Sigma}_2$ is still a solution, although non supersymmetric, of the supergravity theories we will consider. Thus we will extract the relevant toric data from this solution and use it to extremize (2.2.17). As we shall see, even if using a slightly inaccurate method, the result will match perfectly the gravitational computation. In this sense we will often say that $\mathbb{M}_4^{(1)}$ is toric. For $\mathbb{M}_4^{(2)}$ the situation is more complicated, since we will not be able to construct a symplectic closed two form $\omega_{(2)}$. Consequently we can not use the standard procedure for extracting the toric data, *i.e.* we do not rely on a moment map. However the toric data will follow from the co-dimension two loci where one $U(1) \subset U(1)^2$ degenerates, as explained at the end of the previous chapter. In particular, these can be determined by studying the Killing vectors of the explicit metrics on $\mathbb{M}_4^{(1,2)}$. We can then associate a compact polytope defined using these data as normal vectors $\vec{v}_a$ to the facets $\mathcal{F}_a$ and identify the vertices as fixed points of the torus action, as in standard symplectic toric geometry. We will focus on the supergravity constructions mainly in $D = 6$ [1], but an analogous story similarly holds for $D = 7$ [61]. Here $\mathrm{AdS}_3 \times \mathbb{M}_4^{(1,2)}$ have been constructed from consistent truncation of maximal $D = 7$ gauged supergravity on $\mathbb{\Sigma}_2$ down to minimal $D = 5$ gauged supergravity. A similar truncation from $D = 6$ to $D = 4$ is presented in [65].

[1]The symbol $\ltimes$ means a non-trivial fibration, so that for example $\mathbb{M}_4^{(1)}$ can be equivalently written as $\mathbb{\Sigma}_2 \hookrightarrow \Sigma_\mathrm{g}$.

3.1 $U(1)^2$ gauged supergravity models in $D = 6, 7$

In this section we discuss solutions of specific $D = 6, 7$ matter-coupled gauged supergravity theories with gauge group $U(1)^2$. Both of them comprise two gauge fields A_1, A_2, a $(D-4)$-form B and two real scalar fields $\vec{\varphi} = (\varphi_1, \varphi_2)$.

The $\mathcal{N} = (1,1)$, $D = 6$ model can be obtained as a sub-sector of an extension of Romans $F(4)$ gauged supergravity [127], coupled to three vector multiplets [128, 129]. The bosonic part of the action reads[2]

$$S_{\rm 6D} = \frac{1}{16\pi G_{(6)}}\int \mathrm{d}^6x\,\sqrt{-g}\Big[R - V_6 - \frac{1}{2}|\mathrm{d}\vec{\varphi}|^2 - \frac{1}{2}\sum_{i=1}^{2} X_i^{-2}|F_i|^2 - \frac{1}{8}(X_1X_2)^2|H|^2 \\ - \frac{m^2}{4}(X_1X_2)^{-1}|B|^2 - \frac{1}{16}\varepsilon^{\mu\nu\rho\sigma\tau\lambda}B_{\mu\nu}\Big(F_{1\,\rho\sigma}F_{2\,\tau\lambda} + \frac{m^2}{12}B_{\rho\sigma}B_{\tau\lambda}\Big)\Big]\,, \tag{3.1.1}$$

where $F_i = \mathrm{d}A_i$, $H = \mathrm{d}B$ and the scalar fields $\vec{\varphi}$ are parameterized as

$$X_i = \mathrm{e}^{-\vec{a}_i\cdot\vec{\varphi}}\,, \quad \vec{a}_1 = \big(2^{-1/2}, 2^{-3/2}\big)\,, \quad \vec{a}_2 = \big(-2^{-1/2}, 2^{-3/2}\big)\,. \tag{3.1.2}$$

The scalar potential is

$$V_6 = m^2X_0^2 - 4g_c^2X_1X_2 - 4mg_c\,X_0(X_1 + X_2)\,, \tag{3.1.3}$$

with g_c the gauge coupling and m the mass parameter, and where for later convenience we defined $X_0 = (X_1X_2)^{-3/2}$. In appendix A we present the equations of motion stemming from this action. Moreover, for $D = 6$, we will construct explicitly the Killing spinor. Indeed, a solution to the equations of motion of the model is supersymmetric if and only if it satisfies also the following set of Killing spinor equations [128]:

$$\mathcal{D}_\mu\epsilon^A + \frac{1}{8}\big[g_c(X_1 + X_2) + mX_0\big]\Gamma_\mu\epsilon^A \\ + \frac{1}{32}\big[m(X_1X_2)^{-1/2}B_{\nu\lambda}\Gamma_7\delta^A_B + \mathrm{i}\big(X_1^{-1}F_1 + X_2^{-1}F_2\big)_{\nu\lambda}(\sigma^3)^A{}_B\big]\big(\Gamma_\mu{}^{\nu\lambda} - 6\delta^\nu_\mu\,\Gamma^\lambda\big)\epsilon^B \\ - \frac{1}{96}(X_1X_2)H_{\nu\lambda\rho}\Gamma_7\big(\Gamma_\mu{}^{\nu\lambda\rho} - 3\delta^\nu_\mu\,\Gamma^{\lambda\rho}\big)\epsilon^A = 0\,, \tag{3.1.4}$$

$$\frac{1}{4}\big(X_1^{-1}\partial_\mu X_1 + X_2^{-1}\partial_\mu X_2\big)\Gamma^\mu\epsilon^A - \frac{1}{8}\big[g_c(X_1 + X_2) - 3mX_0\big]\epsilon^A \\ + \frac{1}{32}\big[m(X_1X_2)^{-1/2}B_{\mu\nu}\Gamma_7\delta^A_B - \mathrm{i}\big(X_1^{-1}F_1 + X_2^{-1}F_2\big)_{\mu\nu}(\sigma^3)^A{}_B\big]\Gamma^{\mu\nu}\epsilon^B \\ + \frac{1}{96}(X_1X_2)H_{\mu\nu\lambda}\Gamma_7\Gamma^{\mu\nu\lambda}\epsilon^A = 0\,, \tag{3.1.5}$$

$$\frac{1}{2}\big(X_1^{-1}\partial_\mu X_1 - X_2^{-1}\partial_\mu X_2\big)\Gamma^\mu(\sigma^3)^A{}_B\epsilon^B - g_c(X_1 - X_2)(\sigma^3)^A{}_B\epsilon^B \\ - \frac{\mathrm{i}}{4}\big(X_1^{-1}F_1 - X_2^{-1}F_2\big)_{\mu\nu}\Gamma^{\mu\nu}\epsilon^A = 0\,, \tag{3.1.6}$$

[2] Here and in what follows we define, for any p-form ω, $|\omega|^2 = \frac{1}{p!}\,\omega_{\mu_1\dots\mu_p}\omega^{\mu_1\cdots\mu_p}$.

where

$$\mathcal{D}_\mu\epsilon^A \equiv \partial_\mu\epsilon^A + \frac{1}{4}\,\omega_\mu{}^{ab}\Gamma_{ab}\epsilon^A - \frac{\mathrm{i}}{2}\,g_c(A_1+A_2)_\mu(\sigma^3)^A{}_B\epsilon^B\,, \quad (\sigma^3)^A{}_B\epsilon^B = (-1)^{A+1}\epsilon^A\,. \tag{3.1.7}$$

These follow from setting to zero the supersymmetry variations of the fermionic fields of the theory with three vector multiplets [128], that do not vanish automatically in the sub-truncation that we are considering. Here $(\sigma^3)^A{}_B$ is the usual third Pauli matrix, $\{\Gamma_a,\Gamma_b\} = 2\eta_{ab}$ and $\Gamma_7 \equiv \Gamma^0\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5$. The $SU(2)$ indices A, B are raised and lowered as $\epsilon^A = \varepsilon^{AB}\epsilon_B$ and $\epsilon_A = \epsilon^B\varepsilon_{BA}$, where $\varepsilon_{AB} = -\varepsilon_{BA}$ and its inverse matrix ε^{AB} is defined such that $\varepsilon^{AB}\varepsilon_{AC} = \delta^B_C$. The supersymmetry parameter ϵ^A is an eight-component symplectic-Majorana spinor, hence it satisfies the condition

$$\varepsilon^{AB}\epsilon^*_B = \mathcal{B}_6\epsilon_A\,, \tag{3.1.8}$$

where $\mathcal{B}_6$ is related to the six-dimensional charge conjugation matrix $\mathcal{C}_6$ by $\mathcal{B}_6 = -\mathrm{i}\,\mathcal{C}_6\Gamma^0$.

The seven-dimensional theory can be constructed as a consistent truncation [130] of the $D = 7$, maximal $SO(5)$ gauged supergravity [131]. The bosonic part of the action is [132]

$$\begin{aligned} S_{7\mathrm{D}} = \frac{1}{16\pi G_{(7)}}\int \mathrm{d}^7x\,\sqrt{-g}\Big[R - V_7 - \frac{1}{2}|\mathrm{d}\vec{\varphi}|^2 - \frac{1}{2}\sum_{i=1}^{2}X_i^{-2}|F_i|^2 - \frac{1}{2}(X_1X_2)^2|H|^2 \\ - \frac{1}{24}\varepsilon^{\mu\nu\rho\sigma\tau\lambda\eta}B_{\mu\nu\rho}\Big(F_{1\,\sigma\tau}F_{2\,\lambda\eta} - \frac{g_c}{12}H_{\sigma\tau\lambda\eta}\Big)\Big]\,. \end{aligned} \tag{3.1.9}$$

Again, the field strengths are given by $F_i = \mathrm{d}A_i$ and $H = \mathrm{d}B$, while now the scalar fields $\vec{\varphi}$ take the parameterization

$$X_i = \mathrm{e}^{-\vec{a}_i\cdot\vec{\varphi}}\,, \quad \vec{a}_1 = \big(2^{-1/2}, 10^{-1/2}\big)\,, \quad \vec{a}_2 = \big(-2^{-1/2}, 10^{-1/2}\big)\,. \tag{3.1.10}$$

The scalar potential is

$$V_7 = \frac{g_c^2}{2}\big[X_0^2 - 8X_1X_2 - 4X_0(X_1+X_2)\big]\,, \tag{3.1.11}$$

with g_c the gauge coupling and, in this context, $X_0 = (X_1X_2)^{-2}$. Additionally, the following self-duality condition must hold

$$(X_1X_2)^2\star H = -g_cB + \frac{1}{2}A_1\wedge F_2 + \frac{1}{2}A_2\wedge F_1 + \mathrm{d}\lambda_{(2)}\,, \tag{3.1.12}$$

for some two-form $\lambda_{(2)}$. In appendix A the equations of motion are presented.

3.2 Uplift to massive IIA and M-theory

Any (supersymmetric) local solution of the $D = 6, 7$ theories presented in the previous section can be uplifted (locally) to a solution of massive type IIA [1, 65] and 11d supergravity [61], respectively. In the next section we will describe solutions of the form $\mathrm{AdS}_{(D-4)} \times \mathbb{M}_4$, with $\mathbb{M}_4$ a four-dimensional toric orbifold. Thus, in the uplifted solution, it makes sense to compute the entropy (for $D = 6$) and the gravitational central charge (for $D = 7$) of the (putative) black string which has $\mathbb{M}_4$ in the horizon.

The higher-dimensional theories we consider are described by the equations of motion stemming from the massive type IIA action, written in the string frame,[3]

$$
\begin{aligned}
S_{\text{mIIA}} = \frac{1}{16\pi G_{(10)}} \bigg\{ \int \mathrm{d}^{10}x\sqrt{-g}\Big[\mathrm{e}^{-2\Phi}\big(R + 4|\mathrm{d}\Phi|^2 - \frac{1}{2}|H_{(3)}|^2\big) - \frac{1}{2}\big(F_{(0)}^2 + |F_{(2)}|^2 + |F_{(4)}|^2\big)\Big] \\
- \frac{1}{2}\int \big(B_{(2)} \wedge \mathrm{d}C_{(3)} \wedge \mathrm{d}C_{(3)} + 2F_{(0)}B_{(2)}^3 \wedge \mathrm{d}C_{(3)} + 6F_{(0)}^2 B_{(2)}^5\big)\bigg\} \,,
\end{aligned}
\tag{3.2.1}
$$

and from the 11d supergravity action

$$
S_{\text{11d}} = \frac{1}{16\pi G_{(11)}} \bigg\{ \int \mathrm{d}^{11}x\sqrt{-g}\Big[R - \frac{1}{2}|F_{(4)}|^2\Big] + \frac{1}{6}\int \big(F_{(4)} \wedge F_{(4)} \wedge C_{(3)}\big)\bigg\} \,. \tag{3.2.2}
$$

In these conventions, the ten-dimensional field strengths are

$$
H_{(3)} = \mathrm{d}B_{(2)} \,, \quad F_{(2)} = \mathrm{d}C_{(1)} + F_{(0)}B_{(2)} \,, \quad F_{(4)} = \mathrm{d}C_{(3)} - H_{(3)} \wedge C_{(1)} + \frac{1}{2}F_{(0)}B_{(2)} \wedge B_{(2)} \,, \tag{3.2.3}
$$

while the eleven-dimensional four-form flux is simply $F_{(4)} = \mathrm{d}C_{(3)}$.

The uplifted metrics, written in the string frame for massive type IIA, read

$$
\begin{aligned}
\mathrm{d}s^2_{(D+4)} =& \lambda^2 \mu_0^{(D-7)/3} X_0^{-(D-7)/6} \Delta^{1/(D-4)} \Big\{ \mathrm{d}s^2_{(D)} \\
&+ g_c^{-2}\Delta^{-1}\big[X_0^{-1}\mathrm{d}\mu_0^2 + X_1^{-1}\big(\mathrm{d}\mu_1^2 + \mu_1^2\sigma_1^2\big) + X_2^{-1}\big(\mathrm{d}\mu_2^2 + \mu_2^2\sigma_2^2\big)\big]\Big\} \,, \\
\Delta \equiv& X_0\mu_0^2 + X_1\mu_1^2 + X_2\mu_2^2 \,, \quad \sigma_i \equiv \mathrm{d}\phi_i - g_c A_i^{(D)} \,, \quad \Delta\phi_i = 2\pi \,,
\end{aligned}
\tag{3.2.4}
$$

where $\mathrm{d}s^2_{(D)}$ is the line element of the lower-dimensional solution, the one-forms σ_i are built up from the D-dimensional gauge fields and $\lambda \in \mathbb{R}_+$ is a constant which realizes the scaling symmetry of the supergravity theories (see section 3 of [1] and 3.2 of [61]) and plays a crucial role for the correct quantization of the ten-dimensional solutions, as we shall see momentarily. The coordinates μ_a, with $a = 0, 1, 2$, such that $\sum \mu_a^2 = 1$ can be taken as

$$
\mu_0 = \sin\xi \,, \quad \mu_1 = \cos\xi\sin\eta \,, \quad \mu_2 = \cos\xi\cos\eta \,, \quad \eta \in [0, \pi/2] \,, \quad
\begin{aligned}
\xi \in& (0, \tfrac{\pi}{2}] \,, \quad d = 10 \,, \\
\xi \in& [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \,, \quad d = 11 \,,
\end{aligned}
\tag{3.2.5}
$$

In $d = 10$ the line element (3.2.4) describes, at each point of M_6, a four-dimensional hemisphere, that we denote as $\mathbb{S}^4$. On the contrary, in $d = 11$ the dimensional reduction is performed on a (squashed) four-sphere, that we continue to denote as $\mathbb{S}^4$.

The four-form flux, common to both higher-dimensional supergravities, can be written

[3] Here we use the shortcut $B_{(2)}^n$ to denote the wedge product of $B_{(2)}$ with itself n times, divided by $n!$.

as

$$
\begin{aligned}
\frac{-F_{(4)}}{\lambda^{(2D-11)}} = \frac{1}{2g_c^3\Delta\mu_0^{(D-4)/3}} & \bigg\{ \frac{U}{2\Delta}\mathrm{d}\mu_1^2\wedge\mathrm{d}\mu_2^2\wedge\sigma_1\wedge\sigma_2 \\
& - \frac{X_1X_2}{\Delta}\Big[\frac{\mu_0^2X_0^2}{X_1X_2}\big(\mu_1^2\mathrm{d}\frac{X_1}{X_0}\wedge\mathrm{d}\mu_2^2-\mu_2^2\mathrm{d}\frac{X_2}{X_0}\wedge\mathrm{d}\mu_1^2\big)-\mu_1^2\mu_2^2\mathrm{d}\log\frac{X_1}{X_2}\wedge\mathrm{d}\mu_0^2\Big]\wedge\sigma_1\wedge\sigma_2 \\
& + g_c\Big[F_1\wedge\big(X_0\mu_0^2\mathrm{d}\mu_1^2-X_1\mu_1^2\mathrm{d}\mu_0^2\big)\wedge\sigma_2+F_2\wedge\big(X_0\mu_0^2\mathrm{d}\mu_2^2-X_2\mu_2^2\mathrm{d}\mu_0^2\big)\wedge\sigma_1\Big]\bigg\} \\
& + \frac{\delta_{D,6}}{4g_c\mu_0^{(D-4)/3}}\Big[(X_1X_2)^2\star_6 H\wedge\mathrm{d}\mu_0^2+\frac{3m^2}{X_1X_2}\mu_0^2\star_6 B\Big] \\
& + \frac{\delta_{D,7}}{2g_c\mu_0^{(D-4)/3}}\Big[(X_1X_2)^2\star_7 H\wedge\mathrm{d}\mu_0^2+2g_c\mu_0^2H\Big]\,,
\end{aligned}
\tag{3.2.6}
$$

where, for convenience, we defined the function

$$
U = 2\sum_a X_a^2\mu_a^2 - \Big[\frac{(10-D)}{3}X_0+2(X_1+X_2)\Big]\Delta\,. \tag{3.2.7}
$$

Finally, the dilaton, Romans mass $F_{(0)}$ and the other ten-dimensional fields read

$$
\mathrm{e}^{\Phi} = \lambda^2\mu_0^{-5/6}\Delta^{1/4}(X_1X_2)^{-5/8}\,,\quad F_{(0)}=\frac{m}{\lambda^3}=\frac{2g_c}{3\lambda^3}\,,\quad F_{(2)}=\frac{g_c}{3\lambda}\mu_0^{2/3}B\,,\quad B_{(2)}=\frac{\lambda^2\mu_0^{2/3}}{2}B\,. \tag{3.2.8}
$$

Often, it will be useful to define the reduced functions

$$
\Delta_X=(X_1X_2)^{(1-D)/2}\Delta\,,\quad U_X=(X_1X_2)^{-1}U \implies \frac{U}{\Delta^2}=(X_1X_2)^{(2-D)}\frac{U_X}{\Delta_X^2}\,. \tag{3.2.9}
$$

Entropy, central charge and flux quantization

In order to compute the entropy and the central charge of the uplifted solutions, we first need to write the higher-dimensional metric in the form

$$
\mathrm{d}s^2_{(D+4)} = \mathrm{e}^{2A}\big(\mathrm{d}s^2_{\mathrm{AdS}_{D-4}}+\mathrm{d}s^2_{M_8}\big)\,, \tag{3.2.10}
$$

where $\mathrm{d}s^2_{M_8}$ is the line element of the internal fibred space $\mathbb{S}^4\hookrightarrow M_8\to\mathbb{M}_4$. The entropy and the central charge can then be read from the $(D-4)$-dimensional effective Newton constant $G_{(D-4)}$ as[4]

$$
S=\frac{1}{4G_{(2)}}=\frac{8\pi^2}{(2\pi\ell_s)^8}\int \mathrm{e}^{8A-2\Phi}\,\mathrm{vol}M_8\,,\quad c=\frac{3}{2G_{(3)}}=\frac{48\pi^2}{(2\pi\ell_s)^9}\int\mathrm{e}^{9A}\,\mathrm{vol}M_8\,. \tag{3.2.11}
$$

Moreover, in order to ensure that the uplifted solution is globally well-defined, we will need to impose the quantization of fluxes. In particular, the four-form flux yields the following constraints

$$
\frac{1}{(2\pi\ell_s)^3}\int_{\mathbb{S}^4}F_{(4)}=N\in\mathbb{N}\,,\quad \frac{1}{(2\pi\ell_s)^3}\int_{\mathbb{M}_4}F_{(4)}=K\in\mathbb{N}\,, \tag{3.2.12}
$$

[4]We are using conventions in which $G_{(11)}=(2\pi\ell_s)G_{(10)}$, that is the same to say that the eleven-dimensional Planck length is equal to ℓ_s.

where N can be interpreted as the number of D4 and M5-branes wrapped over $\mathbb{M}_4$, in ten and eleven dimensions respectively. In massive type IIA supergravity one has to impose the additional condition

$$(2\pi\ell_s)F_{(0)} = n_0 \in \mathbb{N}\,, \tag{3.2.13}$$

closely related to the presence of D8-branes. Indeed, our background has a boundary at $\xi = 0$, where $\mu_0 = 0$ and the warp factor is singular, which corresponds to the location of an O8-plane and $N_f = 8 - n_0$ coincident D8-branes. The integration over $\mathbb{S}^4$ can be performed in generality, since only the first line of (3.2.6) contributes. Using the identity

$$\int_{\mathbb{S}^4} \frac{U}{\Delta^2} \frac{\mu_1\mu_2}{\mu_0^{(D-4)/3}}\, \mathrm{d}\mu_1 \wedge \mathrm{d}\mu_2 \wedge \mathrm{d}\phi_1 \wedge \mathrm{d}\phi_2 = -(D-3)(D-5)\pi^2\,, \tag{3.2.14}$$

the condition (3.2.12), together with (3.2.13) (in the case of massive type IIA), can be solved to give g_c and λ in terms of N (and n_0)

$$D = 6: \quad g_c^8 = \frac{1}{(2\pi\ell_s)^8}\frac{18\pi^6}{N^3 n_0}\,, \quad \lambda^8 = \frac{8\pi^2}{9Nn_0^3}\,, \tag{3.2.15}$$

$$D = 7: \quad \frac{g_c^3}{\lambda^3} = \frac{1}{(2\pi\ell_s)^3}\frac{8\pi^2}{N}\,. \tag{3.2.16}$$

Collectively, we can write

$$N = \frac{\lambda^{2D-11}\pi^2(D-3)(D-5)}{(2\pi\ell_s)^3 g_c^3}\,, \tag{3.2.17}$$

These are very general since do not depend on the details of the lower-dimensional solutions. Indeed, they are the same as for other known six- and seven-dimensional solutions (*cf.* [52] and [51, 61]). Diversely, the integration along a representative of $\mathbb{M}_4$, which we take to be at the pole of the hemisphere $\mathbb{S}^4$ (*i.e.* at $\xi = \pi/2$), takes contributions from the last two lines and as such it depends explicitly on the form of the field B and $H = \mathrm{d}B$ in $D = 6, 7$. It is worth noticing that, being N and n_0 integers, the second equation for $D = 6$ would be inconsistent with $\lambda = 1$. Such a problem arises from the fact that, without introducing λ, the constraints (3.2.12) and (3.2.13) would have to be imposed on an unique dimensionless parameter, namely $(g\ell_s)$. This makes the scaling symmetry crucial to establish the regularity of the ten-dimensional uplifted solution.

3.3 Local form of the solutions and supersymmetry

We start analysing the local form of the solutions in $D = 6$, whilst global issue will be addressed later on.

3.3.1 $\mathrm{AdS}_2 \times \Sigma_\mathrm{g} \ltimes \Sigma_2$

The first local solutions to the supergravity action (3.1.1) that we consider are of the type $\mathrm{AdS}_2 \times \Sigma_\mathrm{g} \ltimes \Sigma_2$, Σ_g Riemann surface of genus $\mathrm{g} > 1$. As anticipated, for $\mathrm{g} = 0$,

$\mathrm{AdS}_2 \times S^2 \ltimes \Sigma_2$ is still a solution to the equations of motion, but it is not supersymmetric. Specifically, we consider[5]

$$\begin{aligned}
\mathrm{d}s^2 &= (y^2 h_1 h_2)^{1/4}\left[\frac{1}{4}\,\mathrm{d}s^2_{\mathrm{AdS}_2} + \frac{1}{2}\,\mathrm{d}s^2_{\Sigma_\mathrm{g}} + \frac{y^2}{F}\,\mathrm{d}y^2 + \frac{F}{h_1 h_2}\Big(\mathrm{d}z - \frac{1}{2m}\,\omega_{\Sigma_\mathrm{g}}\Big)^2\right], \\
A_i &= -\frac{y^3}{h_i}\Big(\mathrm{d}z - \frac{1}{2m}\,\omega_{\Sigma_\mathrm{g}}\Big) + \alpha_i \mathrm{d}z\,, \quad X_i = (y^2 h_1 h_2)^{3/8} h_i^{-1}\,, \quad B = \frac{y}{2m}\,\hat{\mathrm{vol}}(\mathrm{AdS}_2)\,,
\end{aligned} \tag{3.3.1}$$

where $\mathrm{d}s^2_{\mathrm{AdS}_2}$ denotes the unit radius metric on AdS_2, $\mathrm{d}s^2_{\Sigma_\mathrm{g}}$ the unit radius metric on Σ_g and $\omega_{\Sigma_\mathrm{g}}$ is normalized such that $\mathrm{d}\omega_{\Sigma_\mathrm{g}} = \hat{\mathrm{vol}}(\Sigma_\mathrm{g})$. Explicitly, we can take

$$\mathrm{d}s^2_{\Sigma_\mathrm{g}} = \mathrm{d}x^2 + \sinh^2\!x\,\mathrm{d}\psi^2\,, \qquad \omega_{\Sigma_\mathrm{g}} = \cosh x\,\mathrm{d}\psi\,, \tag{3.3.2}$$

but all the following computations can be performed without adopting an explicit metric on Σ_g nor an explicit expression for $\omega_{\Sigma_\mathrm{g}}$. In these normalization $\mathrm{vol}(\Sigma_\mathrm{g}) = 4\pi(\mathrm{g}-1)$ and $\chi_{\Sigma_\mathrm{g}} = 2(1-\mathrm{g})$. The functions $h_i(y)$ and $F(y)$ are given by

$$h_i(y) = \frac{2g_c}{3m}\,y^3 + q_i\,, \qquad F(y) = m^2 h_1(y) h_2(y) - y^4\,, \tag{3.3.3}$$

with q_1, q_2 two real parameters. A curvature singularity lies at $y=0$, hence without loss of generality, in what follows we will restrict to $y>0$.

We now solve (3.1.4)-(3.1.6), showing that the solution is indeed supersymmetric. We employ the following orthonormal frame

$$\begin{aligned}
e^{\hat{a}} &= \frac{y^{1/4}(h_1h_2)^{1/8}}{2}\,\hat{e}^{\hat{a}}\,, \qquad & e^{\check{a}} &= \frac{y^{1/4}(h_1h_2)^{1/8}}{\sqrt{2}}\,\check{e}^{\check{a}}\,, \\
e^{4} &= \frac{y^{5/4}(h_1h_2)^{1/8}}{F^{1/2}}\,\mathrm{d}y\,, \qquad & e^{5} &= \frac{y^{1/4}F^{1/2}}{(h_1h_2)^{3/8}}\Big(\mathrm{d}z - \frac{1}{2m}\,\omega_{\Sigma_\mathrm{g}}\Big)\,,
\end{aligned} \tag{3.3.4}$$

where $\hat{e}^{\hat{a}}$, $\hat{a}=0,1$, is the zweibein on AdS_2, whose coordinates are denoted as $x^{\hat{\mu}}$, and $\check{e}^{\check{a}}$, $\check{a}=2,3$, is the zweibein on Σ_g, whose coordinates are denoted as $x^{\check{\mu}}$. Equation (3.1.4) then splits into the following system

$$\begin{aligned}
\Big(\partial_{\hat{\mu}} + \frac{1}{4}\omega_{\hat{\mu}}{}^{\hat{a}\hat{b}}\Gamma_{\hat{a}\hat{b}}\Big)\epsilon^A + \hat{e}^{\hat{a}}_{\hat{\mu}}\left[-\frac{\mathrm{i}}{4}\,\Gamma_{\hat{a}}{}^{45}(\sigma^3)^A{}_B\epsilon^B + \frac{1}{4}\,\Gamma_{\hat{a}}{}^{2345}\epsilon^A\right] &= 0\,, \\
\partial_{\check{\mu}}\epsilon^A - \frac{1}{2}\left[(\omega_{\Sigma_\mathrm{g}})_{\check{\mu}} + \frac{1}{\sqrt{2}}\,\Gamma_{\check{\mu}}{}^{45}\right]\big(\Gamma^{23}\epsilon^A + \mathrm{i}(\sigma^3)^A{}_B\epsilon^B\big) &= 0\,, \\
\partial_y\epsilon^A - \frac{1}{16y}\left[(2+y\tilde{h}')\epsilon^A - \mathrm{i}\,\frac{4y^2}{F^{1/2}}(4-y\tilde{h}')\Gamma^5(\sigma^3)^A{}_B\epsilon^B\right] &= 0\,, \\
\partial_z\epsilon^A - \mathrm{i}\,\frac{g_c}{2}\Big(\alpha_1+\alpha_2-\frac{2m}{g_c}\Big)(\sigma^3)^A{}_B\epsilon^B &= 0\,,
\end{aligned} \tag{3.3.5}$$

where $\tilde{h} \equiv \log(h_1h_2)$ and we may take

$$\alpha_1 + \alpha_2 = \frac{2m}{g_c}\,, \tag{3.3.6}$$

[5]Here we have inserted a gauge transformation $\alpha_i \mathrm{d}z$ for reasons that will be clear momentarily.

so that the Killing spinors are independent of z. Equations (3.1.5) and (3.1.6) yield the same constraint

$$F^{1/2}\Gamma^4\epsilon^A + m(h_1h_2)^{1/2}\epsilon^A + \mathrm{i}\, y^2\Gamma^{45}(\sigma^3)^A{}_B\epsilon^B = 0\,. \tag{3.3.7}$$

We note that the equations along y and z in (3.3.5) and equation (3.3.7) are the same as in the $\mathrm{AdS}_4 \times \Sigma$ system (*cf.* equations (3.3) and (3.4) of [52]), a fact that will play an important role later on. We consider the following decomposition of the gamma matrices

$$\Gamma^{\tilde{a}} = \gamma^{\tilde{a}} \otimes \rho_*\,, \qquad \Gamma^{4,5} = I_4 \otimes \rho^{1,2}\,, \tag{3.3.8}$$

where $\gamma^{\tilde{a}}$, $\tilde{a} = 0, \ldots, 3$, are the (Lorentzian) gamma matrices in $D = 4$, $\rho^{\hat{\imath}}$, $\hat{\imath} = 1, 2$, are the (Euclidean) gamma matrices in $D = 2$ and $\rho_* = -\mathrm{i}\,\rho^1\rho^2$ is the related chiral matrix. For $\rho^{\hat{\imath}}$ we choose the following representation

$$\rho^{\hat{\imath}} = \sigma^{\hat{\imath}}\,, \qquad \rho_* = \sigma^3\,, \tag{3.3.9}$$

and we take $\mathcal{B}_2^\rho = -\sigma^2$. For consistency, the six-dimensional matrix $\mathcal{B}_6$ decomposes as

$$\mathcal{B}_6 = (\mathcal{B}_4\gamma_5) \otimes (\mathcal{B}_2^\rho\rho_*)\,, \tag{3.3.10}$$

where $\gamma_5 = \mathrm{i}\,\gamma^0\gamma^1\gamma^2\gamma^3$ is the four-dimensional chiral matrix. Moreover, we decompose the four-dimensional gamma matrices $\gamma^{\tilde{a}}$ as

$$\gamma^{\hat{a}} = \beta^{\hat{a}} \otimes \tau_*\,, \qquad \gamma^{2,3} = I_2 \otimes \tau^{1,2}\,, \tag{3.3.11}$$

where $\beta^{\hat{a}}$ are the (Lorentzian) gamma matrices in $D = 2$, $\tau^{\hat{\imath}}$, $\hat{\imath} = 1, 2$, are the (Euclidean) gamma matrices in $D = 2$ and $\tau_* = -\mathrm{i}\,\tau^1\tau^2$ is the related chiral matrix. Explicitly, we adopt the same representation as in (3.3.9)

$$\tau^{\check{\imath}} = \sigma^{\check{\imath}}\,, \qquad \tau_* = \sigma^3\,, \tag{3.3.12}$$

and we take $\mathcal{B}_2^\tau = \sigma^1$. The four-dimensional matrices $\mathcal{B}_4$ and γ_5 decompose as

$$\mathcal{B}_4 = (\mathcal{B}_2^\beta\beta_*) \otimes \mathcal{B}_2^\tau\,, \qquad \gamma_5 = \beta_* \otimes \tau_*\,, \tag{3.3.13}$$

where $\beta_* = -\beta^0\beta^1$ is the (Lorentzian) chiral matrix in $D = 2$. Adopting these decompositions, the Killing spinor takes the form of a tensor product

$$\epsilon^A = \vartheta_{\mathrm{AdS}_2} \otimes \chi^A_{\Sigma_{\mathrm{g}}} \otimes \eta^A_{\Sigma_2}\,, \tag{3.3.14}$$

where $\vartheta = \vartheta(x^{\hat{\mu}})$ is a Majorana Killing spinor on AdS_2, hence $\hat{\nabla}_{\hat{\mu}}\vartheta = \frac{1}{2}\beta_{\hat{\mu}}\vartheta$ and $\vartheta^* = \mathcal{B}_2^\beta\vartheta$, $\chi^A = \chi^A(x^{\check{\mu}})$ are two two-component spinors defined on the Riemann surface Σ_{g}

$$\chi^1 = \begin{pmatrix} 0 \\ \xi_\chi \end{pmatrix}\,, \qquad \chi^2 = \begin{pmatrix} -\xi_\chi^* \\ 0 \end{pmatrix}\,, \qquad \chi^2 = -\mathrm{i}\,\sigma^2(\chi^1)^* \tag{3.3.15}$$

with $\xi_\chi \in \mathbb{C}$ a constant and $\eta^{1,2}$ are given by

$$\eta^1 = \xi_\eta\, y^{1/8}(h_1h_2)^{-3/16}\begin{pmatrix} f_+^{1/2} \\ -f_-^{1/2} \end{pmatrix}\,, \qquad \eta^2 = -\mathrm{i}\,\xi_\eta^*\, y^{1/8}(h_1h_2)^{-3/16}\begin{pmatrix} f_-^{1/2} \\ -f_+^{1/2} \end{pmatrix}\,, \qquad \eta^2 = \mathrm{i}\,\sigma^1(\eta^1)^*\,. \tag{3.3.16}$$

Here, $\xi_\eta \in \mathbb{C}$ is another complex constant and we have defined

$$f_\pm(y) \equiv m(h_1h_2)^{1/2} \pm y^2\,, \quad F(y) = f_+(y)\,f_-(y)\,. \tag{3.3.17}$$

We can now count the number of supersymmetries preserved by our $\mathrm{AdS}_2 \times \Sigma_\mathrm{g} \ltimes \Sigma_2$ solution. ϑ is a Majorana spinor, hence it has two real degrees of freedom, while the tensor product $\chi^A \otimes \eta^A$ is fully determined by the product $(\xi_\chi\xi_\eta)$, which has two real degrees of freedom. Therefore, there are four real independent Killing spinors, that is one quarter of the number of supersymmetries of the six-dimensional $\mathcal{N} = (1,1)$ theory, hence the solution is 1/4-BPS.

3.3.2 $\mathrm{AdS}_2 \times \Sigma_1 \ltimes \Sigma_2$

A second class of solutions to the model described by action (3.1.1) is the $\mathrm{AdS}_2 \times \Sigma_1 \ltimes \Sigma_2$ background, where a two-dimensional spindle Σ_2 is non-trivially fibred over a different two-dimensional spindle Σ_1. Remarkably, there exists a specific limit of the coordinates and the parameters in which the $\mathrm{AdS}_2 \times \Sigma_\mathrm{g} \ltimes \Sigma_2$ solutions are retrieved, both their bosonic content and the Killing spinors (see appendix B of [1]). The solution takes the form

$$\begin{aligned} \mathrm{d}s^2 &= (y^2h_1h_2)^{1/4}\left[\frac{x^2}{4}\mathrm{d}s^2_{\mathrm{AdS}_2} + \frac{x^2}{q}\mathrm{d}x^2 + \frac{q}{4x^2}\mathrm{d}\psi^2 + \frac{y^2}{F}\mathrm{d}y^2 + \frac{F}{h_1h_2}\Big(\mathrm{d}z - \frac{1}{2m}\Big(1-\frac{\mathtt{a}}{x}\Big)\mathrm{d}\psi\Big)^2\right], \\ A_i &= -\frac{y^3}{h_i}\Big(\mathrm{d}z - \frac{1}{2m}\Big(1-\frac{\mathtt{a}}{x}\Big)\,\mathrm{d}\psi\Big) + \alpha_i\mathrm{d}z\,, \quad X_i = (y^2h_1h_2)^{3/8}h_i^{-1}\,, \quad B = \frac{\mathtt{a}y}{2m}\,\hat{\mathrm{vol}}\mathrm{AdS}_2\,, \end{aligned} \tag{3.3.18}$$

where α_i are again gauge parameters and $\mathtt{a} \in \mathbb{R}_+$. The metric functions are given by

$$h_i(y) = \frac{2g_c}{3m}\,y^3 + q_i\,, \quad F(y) = m^2h_1(y)h_2(y) - y^4\,, \quad q(x) = x^4 - 4x^2 + 4\mathtt{a}x - \mathtt{a}^2\,. \tag{3.3.19}$$

Notice that formally taking $\mathtt{a} = 0$ we retrieve the $\mathrm{AdS}_4 \times \Sigma$ background studied in [52].

In order to construct the local form of the Killing spinors and to demonstrate that this solution is supersymmetric, we must first specialise the Killing spinor equations (3.1.4)-(3.1.6) to our system employing the orthonormal frame

$$\begin{aligned} e^{\hat{a}} &= \frac{x\,y^{1/4}(h_1h_2)^{1/8}}{2}\,\hat{e}^{\hat{a}}\,, \quad e^2 = \frac{x\,y^{1/4}(h_1h_2)^{1/8}}{q^{1/2}}\,\mathrm{d}x\,, \quad e^3 = \frac{q^{1/2}y^{1/4}(h_1h_2)^{1/8}}{2x}\,\mathrm{d}\psi\,, \\ e^4 &= \frac{y^{5/4}(h_1h_2)^{1/8}}{F^{1/2}}\,\mathrm{d}y\,, \quad e^5 = \frac{y^{1/4}F^{1/2}}{(h_1h_2)^{3/8}}\Big(\mathrm{d}z - \frac{1}{2m}\Big(1-\frac{\mathtt{a}}{x}\Big)\,\mathrm{d}\psi\Big)\,, \end{aligned} \tag{3.3.20}$$

where $\hat{e}^{\hat{a}}$, $\hat{a} = 0, 1$, is the zweibein on AdS_2, whose coordinates are denoted as $x^{\hat{\mu}}$. Equation (3.1.4) then splits into the following system

$$
\begin{aligned}
\Big(\partial_{\hat\mu}+\frac{1}{4}\omega_{\hat\mu}^{\hat a\hat b}\Gamma_{\hat a\hat b}\Big)\epsilon^A+\hat e^{\hat a}_{\hat\mu}\Big[\frac{q^{1/2}}{4x}\,\Gamma_{\hat a}{}^{2}\epsilon^A-\mathrm{i}\,\frac{x}{4}\,\Gamma_{\hat a}{}^{45}(\sigma^3)^A{}_B\epsilon^B+\frac{\mathtt{a}}{4x}\,\Gamma_{\hat a}{}^{2345}\epsilon^A\Big]&=0\,,\\
\partial_x\epsilon^A-\frac{\mathtt{a}}{2xq^{1/2}}\,\Gamma^{345}\epsilon^A-\mathrm{i}\,\frac{x}{2q^{1/2}}\,\Gamma^{245}(\sigma^3)^A{}_B\epsilon^B&=0\,,\\
\partial_\psi\epsilon^A-\frac{\mathrm{i}}{2}\Big(1-\frac{\mathtt{a}}{x}\Big)(\sigma^3)^A{}_B\epsilon^B+\frac{2q-xq'}{8x^3}\,\Gamma^{23}\epsilon^A+\frac{\mathtt{a}q^{1/2}}{4x^3}\,\Gamma^{245}\epsilon^A&\\
-\mathrm{i}\,\frac{q^{1/2}}{4x}\,\Gamma^{345}(\sigma^3)^A{}_B\epsilon^B&=0\,,\\
\partial_y\epsilon^A-\frac{1}{16y}\Big[(2+y\tilde h')\epsilon^A-\mathrm{i}\,\frac{4y^2}{F^{1/2}}(4-y\tilde h')\Gamma^5(\sigma^3)^A{}_B\epsilon^B\Big]&=0\,,\\
\partial_z\epsilon^A-\mathrm{i}\,\frac{g_c}{2}\Big(\alpha_1+\alpha_2-\frac{2m}{g_c}\Big)(\sigma^3)^A{}_B\epsilon^B&=0\,,
\end{aligned}
\tag{3.3.21}
$$

where $\tilde h\equiv\log(h_1h_2)$ is defined as before and we fix the same gauge as in (3.3.6). Equations (3.1.5) and (3.1.6) yield, again, the same constraint (3.3.7).

The explicit construction of the Killing spinors proceeds in a way similar to the $\mathrm{AdS}_2\times\Sigma_{\mathrm{g}}\ltimes\Sigma_2$ case. First of all, we employ the following decomposition of the six-dimensional gamma matrices (*cf.* equations (3.3.8) and (3.3.11)):

$$
\Gamma^{\hat a}=\beta^{\hat a}\otimes\tau_*\otimes\rho_*\,,\qquad\Gamma^{2,3}=I_2\otimes\tau^{1,2}\otimes\rho_*\,,\qquad\Gamma^{4,5}=I_2\otimes I_2\otimes\rho^{1,2}\,,\tag{3.3.22}
$$

which implies,

$$
\mathcal{B}_6=\mathcal{B}_2^\beta\otimes(\mathcal{B}_2^\tau\tau_*)\otimes(\mathcal{B}_2^\rho\rho_*)\,,\tag{3.3.23}
$$

where $\beta^{\hat a}$ are the (Lorentzian) gamma matrices in $D=2$, $\rho^{\hat\imath}$ and $\tau^{\hat\imath}$, $\hat\imath=1,2$, are the (Euclidean) gamma matrices in $D=2$ and ρ_* and τ_* are the related chiral matrices. In the Euclidean sectors we adopt the representation (3.3.9)

$$
\rho^{\hat\imath}=\tau^{\hat\imath}=\sigma^{\hat\imath}\,,\qquad\rho_*=\tau_*=\sigma^3\,,\tag{3.3.24}
$$

and we take $\mathcal{B}_2^\rho=-\sigma^2$ and $\mathcal{B}_2^\tau=\sigma^1$. The six-dimensional Killing spinors consist again in a tensor product, namely

$$
\epsilon^A=\vartheta_{\mathrm{AdS}_2}\otimes\chi^A_{\Sigma_1}\otimes\eta^A_{\Sigma_2}\,,\tag{3.3.25}
$$

with $\vartheta=\vartheta(x^{\hat\mu})$ Majorana Killing spinor on AdS_2, $\chi^A=\chi^A(x,\psi)$ two two-component spinors defined on the spindle Σ_1 and η^A given in (3.3.16). The symplectic-Majorana condition (3.1.8) and the Killing spinor equations (3.3.21)-(3.3.7) are satisfied given that

$$
\chi^1=\xi_\chi\,x^{-1/2}\begin{pmatrix}Q_+^{1/2}\\-Q_-^{1/2}\end{pmatrix}\,,\qquad\chi^2=\xi_\chi^*\,x^{-1/2}\begin{pmatrix}Q_-^{1/2}\\Q_+^{1/2}\end{pmatrix}\,,\qquad\chi^2=-\mathrm{i}\,\sigma^2(\chi^1)^*\tag{3.3.26}
$$

where $\xi_\chi\in\mathbb{C}$ is a constant and

$$
Q_\pm(x)\equiv x^2\pm(\mathtt{a}-2x)\,,\qquad q(x)=Q_+(x)\,Q_-(x)\,.\tag{3.3.27}
$$

The count of the real degrees of freedom is identical to the $\mathrm{AdS}_2\times\Sigma_{\mathrm{g}}\ltimes\Sigma_2$ case and shows that the solution is 1/4-BPS, since it preserves four real supercharges.

3.4 Global analysis

We now proceed to the global analysis of the local solutions presented in sections 3.3.1 and 3.3.2, for which we set $m = 2g_c/3$ without loss of generality[6]. First we study the conditions for which the (y, z) part of the metric describe a spindle $\Sigma_2^{[n_-,n_+]}(y, z)$ and then, analogously, the conditions for having a well-defined base $\Sigma_1^{[m_-,m_+]}(x, \psi)$. Subsequently, we will take into account the fibration of the second spindle on the base $\mathbb{B} = \Sigma_{\rm g}, \Sigma_1$. Finally, we study the regularity of the uplifted solution by computing the fluxes of $F_{(4)}$ over $\mathbb{M}_4^{(1,2)}$, computing also the entropy of the putative black-hole with horizon $\mathbb{M}_4^{(1,2)}$. We will find a number of regularity conditions, which must be studied within the domain of integers. All details are reported in appendix A of [1], whilst here we focus on the structure of the analysis.

3.4.1 Regularity in $D = 6$

$\Sigma_2^{[n_-,n_+]}(y, z)$

Fixed a generic point on the base $\mathbb{B}$, that can be $\mathbb{B} = \Sigma_{\rm g}, \Sigma_1$, the metric Σ_2 reads

$$\mathrm{d}s^2_{\Sigma_2} = \frac{y^2}{F(y)}\,\mathrm{d}y^2 + \frac{F(y)}{h_1(y)h_2(y)}\,\mathrm{d}z^2\,. \tag{3.4.1}$$

In order to have a well-defined metric and positive scalars X_i, we need to take $F(y) \geq 0$ and $h_i(y) > 0$ in a closed interval not containing the curvature singularity in $y = 0$, thus without loss of generality we restrict to $y > 0$. Moreover, denoting $[y_-, y_+]$ the range of the coordinate y (so that $F(y_\pm) = 0$), Σ_2 is a proper spindle given that [52]

$$\frac{gF'(y_-)}{3y_-^3}\,\Delta z = \frac{2\pi}{n_-}\,, \qquad \frac{gF'(y_+)}{3y_+^3}\,\Delta z = -\frac{2\pi}{n_+}\,, \tag{3.4.2}$$

where $n_\pm$ are two co-prime integers and Δz is the periodicity of the z coordinate. The signs $\pm$ in these conditions are due to $\mathrm{sign}(F'(y_\pm)) = \mp$ due to $F(y) > 0$. These two relations ensure that at the poles $y = y_\pm$ there are $\mathbb{C}/\mathbb{Z}_{n_\pm}$ orbifold singularities, respectively. An additional constraint comes from the quantization of the magnetic fluxes across Σ_2, which arises from the requirement that gA_i be well-defined connection one-forms on $U(1)$ bundles over Σ_2. Given any point of $\Sigma_{\rm g}$, we must have [42]

$$\mathfrak{t}_i \equiv \frac{g_c}{2\pi}\int_{\Sigma_2} F_i = \frac{p_i}{n_- n_+}\,, \qquad p_i \in \mathbb{Z}\,. \tag{3.4.3}$$

It can be shown that [52]

$$p_1 + p_2 = n_+ + n_- \implies \mathfrak{t}_1 + \mathfrak{t}_2 = \frac{n_- + n_+}{n_- n_+}\,, \tag{3.4.4}$$

meaning that the R-symmetry gauge field $A_R \equiv g_c(A_1 + A_2)$ is a connection one-form on the line bundle $\mathcal{O}(n_- + n_+)$ over Σ_2, and therefore the integers p_i can be conveniently

[6]Taking $g > 0$ and $m > 0$, we can apply the rescaling (2.4) of [52], together with $B \mapsto (m/lg_c)^{-1/2}B$, in order to absorb m in the coupling constant.

parameterized as

$$p_1=\frac{n_-+n_+}{2}(1+\mathtt{z})\,,\quad p_2=\frac{n_-+n_+}{2}(1-\mathtt{z})\,,\tag{3.4.5}$$

where $\mathtt{z}$ is an appropriate rational number. The value of the sum p_1+p_2 implies that A_R realizes on the bundle the type of twist that was dubbed "twist", as for the D4 [52] and M5-branes [51] wrapped on a spindle. All these conditions (3.4.2)-(3.4.5) were studied in [52], to which we refer for a detailed analysis, and it was proven that they can be satisfied if $n_-<n_+$, $p_1<0$, $p_2>0$ and if[7]

$$\begin{aligned}q_1(\mathtt{z})=&w\frac{3(1-\mathtt{x}^2)}{g_c^2(\mathtt{x}^2+3)^2(\mu-\mathtt{x})}\big[3\mu(1+\mathtt{x}^2)-2\mathtt{x}-\mathtt{x}(\mathtt{x}^2+3)\mathtt{z}\big]=q_2(-\mathtt{z})\,,\\ y_\pm=&w(1\pm\mathtt{x})=\frac{1\pm\mathtt{x}}{g_c(\mathtt{x}^2+3)}\sqrt{\frac{9\mu(\mathtt{x}^2+1)-3\mathtt{x}(\mathtt{x}^2+5)}{2(\mu-\mathtt{x})}}\,,\quad\Delta z=\chi_2\frac{3\pi(\mathtt{x}^2+3)(\mu-\mathtt{x})}{8g_c\mathtt{x}^2}\,.\end{aligned}\tag{3.4.6}$$

Here, the Euler characteristic of Σ_2 χ_2 and μ are defined as

$$\chi_2=\frac{n_++n_-}{n_-n_+}\,,\quad\mu\equiv\frac{n_+-n_-}{n_++n_-}\,,\tag{3.4.7}$$

and $\mathtt{x}$ is the only solution of the quartic equation

$$\mathtt{x}^4+\big(8\mathtt{z}^2-3-9\mu^2\big)\mathtt{x}^2+12\mu\mathtt{x}-9\mu^2=0\,,\tag{3.4.8}$$

lying inside the range $0<\mathtt{x}<1$.

$\Sigma_1^{[m_-,m_+]}(x,\psi)$

The base spindle $\Sigma_1^{[m_-,m_+]}(x,\psi)$, described by the metric

$$\mathrm{d}s^2_{\Sigma_1}=\frac{x^2}{q(x)}\,\mathrm{d}x^2+\frac{q(x)}{4x^2}\,\mathrm{d}\psi^2\,,\tag{3.4.9}$$

was thoroughly studied in [46], thus we will take advantage of the results presented therein, setting $\mathtt{j}=0$. Σ_1 is indeed a spindle, characterized by the two co-prime integers $m_\pm$, with[8] $m_-<m_+$, if the parameter $\mathtt{a}$, the periodicity of the coordinate ψ and the range of $x\in[x_-,x_+]$ (so that $q(x_\pm)=0$) are given by

$$\mathtt{a}=\frac{m_+^2-m_-^2}{m_+^2+m_-^2}\,,\quad\Delta\psi=\frac{\sqrt{m_+^2+m_-^2}}{\sqrt{2}\,m_-m_+}2\pi\,,\quad x_\pm=\pm1\mp\frac{\sqrt{2}\,m_\mp}{\sqrt{m_+^2+m_-^2}}\,.\tag{3.4.10}$$

At the north and south poles of Σ_1, namely $x=x_\mp$, are present $\mathbb{C}/\mathbb{Z}_{m_\mp}$ orbifold singularities, respectively. Moreover, defined the vector field

$$A_{\text{4d}}=\frac{1}{2}\Big(1-\frac{\mathtt{a}}{x}\Big)\,\mathrm{d}\psi\,,\tag{3.4.11}$$

its magnetic flux through the spindle Σ_1 is

$$\frac{1}{2\pi}\int_{\Sigma_1}F_{\text{4d}}=\frac{m_+-m_-}{2m_-m_+}\,.\tag{3.4.12}$$

In the four-dimensional theory [46] this implies that $2A_{\text{4d}}$ is a connection one-form on the line bundle $\mathcal{O}(m_+-m_-)$ over Σ_1 and, as a consequence, that we have the anti-twist.

[7] Notice that, with respect to [52], $x^{\text{there}}\to\mathtt{x}^{\text{here}}$.

[8] Notice that we exchanged m_- and m_+ with respect to [46], so that $n_\pm^{\text{there}}\to m_\mp^{\text{here}}$.

$\mathbb{B}(x,\psi) \ltimes \Sigma_2^{[n_-,n_+]}(y,z)$

As we already mentioned, the internal space has the structure of a spindle Σ_2 fibred over $\mathbb{B} = \Sigma_{\rm g}, \Sigma$. This fibration is well-defined in the orbifold sense if the one-form η describing the fibration is globally defined, *i.e.*

$$\Sigma_{\rm g}: \quad \frac{1}{2\pi}\int_{\Sigma_{\rm g}} \mathrm{d}\eta_{\Sigma_{\rm g}} = t\,, \quad \Sigma_1: \quad \frac{1}{2\pi}\int_{\Sigma_1} \mathrm{d}\eta_{\Sigma_1} = \frac{t}{m_-m_+}\,, \quad t\in\mathbb{Z}\,, \tag{3.4.13}$$

where $\eta_{\mathbb{B}}$ is the one-form representing the fibration

$$\Sigma_{\rm g}: \quad \eta_{\Sigma_{\rm g}} \equiv \frac{2\pi}{\Delta z}\Big(\mathrm{d}z - \frac{3}{4g_c}\,\omega_{\Sigma_{\rm g}}\Big)\,, \quad \Sigma_1: \quad \eta_{\Sigma_1} \equiv \frac{2\pi}{\Delta z}\Big(\mathrm{d}z - \frac{3}{4g_c}\Big(1-\frac{\mathtt{a}}{x}\Big)\,\mathrm{d}\psi\Big)\,. \tag{3.4.14}$$

This requirement yields a quantization condition relating the second spindle data $(n_\pm, \mathtt{z})$ and the genus $\rm g > 1$ or the integers $m_\pm$ of the base, namely

$$\Sigma_{\rm g}: \quad t = -\frac{8\mathtt{x}^2(\mathrm{g}-1)}{\chi_2(\mathtt{x}^2+3)(\mu-\mathtt{x})}\,, \quad \Sigma_1: \quad t = -\frac{4\mathtt{x}^2(m_+-m_-)}{\chi_2(\mathtt{x}^2+3)(\mu-\mathtt{x})}\,, \quad t\in\mathbb{Z}\,, \tag{3.4.15}$$

where we have used (3.4.6). The analysis of this constraint is reported in appendix A of [1]. The last conditions follow from the quantization of the fluxes F_i through two-cycles in $\mathbb{B} \ltimes \Sigma_2$. Their integration through Σ_2 were computed in (3.4.3), hence they are already appropriately quantized. Next, we define the two two-cycles $S_\pm \equiv \{y = y_\pm\}$ which correspond to two copies of the base $\mathbb{B}$ at the two poles of Σ_2, and compute

$$\mathfrak{s}_i^\pm \equiv \frac{g_c}{2\pi}\int_{S_\pm} F_i\,, \tag{3.4.16}$$

obtaining, *e.g.*,

$$\begin{aligned}
\Sigma_{\rm g}: \quad \mathfrak{s}_1^- &= \frac{3}{2}\frac{y_-^3}{h_1(y_-)}\,(\mathrm{g}-1) = t\chi_2\,\frac{\mathtt{x}^3-(2+2\mathtt{z}+3\mu)\mathtt{x}^2+(3-2\mathtt{z})\mathtt{x}-3\mu}{8\mathtt{x}^2}\,,\\
\Sigma_1: \quad \mathfrak{s}_1^- &= \frac{3}{4}\frac{y_-^3}{h_1(y_-)}\,\frac{m_+-m_-}{m_+m_-} = \frac{t\chi_2}{m_+m_-}\,\frac{\mathtt{x}^3-(2+2\mathtt{z}+3\mu)\mathtt{x}^2+(3-2\mathtt{z})\mathtt{x}-3\mu}{8\mathtt{x}^2}\,.
\end{aligned} \tag{3.4.17}$$

where we substituted t as in (3.4.15). Notice that the right hand sides of these equations formally coincide for $m_+ = m_- = 1$. The fluxes of F_2 can be obtained from the previous formulas by exchanging $A_1 \to A_2$, which is equal to exchange $q_1 \to q_2$ or $p_1 \to p_2$ and, in turn, $\mathtt{z} \to -\mathtt{z}$. As a consequence, the quantization of its fluxes automatically holds. The quantization through S_+ can be addressed easily since an explicit computation shows that

$$\Sigma_{\rm g}: \quad \mathfrak{s}_i^+ = \mathfrak{s}_i^- + t\,\frac{p_i}{n_-n_+}\,, \quad \Sigma_1: \quad \mathfrak{s}_i^+ = \mathfrak{s}_i^- + \frac{t}{m_-m_+}\,\frac{p_i}{n_-n_+}\,, \tag{3.4.18}$$

which agrees with the homology relation $S_+ - S_- = t\,\Sigma_2$ or $S_+ - S_- = (t/m_-m_+)\,\Sigma_2$. Similarly to what happened in (3.4.13), the quantization condition requires (3.4.17) to be an integer, modulo m_-m_+ if the base is Σ_1. Again, some examples for which this constraint is satisfied are reported in appendix A of [1].

Focussing on the R-symmetry gauge field $A_R \equiv g_c(A_1 + A_2)$, its fluxes read

$$\Sigma_{\rm g}: \quad \frac{1}{2\pi}\int_{S_\pm} F_R = 2(\mathrm{g}-1) \pm \frac{t}{n_\pm}\,, \qquad \Sigma_1: \quad \frac{1}{2\pi}\int_{S_\pm} F_R = \frac{m_+ - m_-}{m_- m_+} \pm \frac{t}{m_- m_+ n_\pm}\,, \tag{3.4.19}$$

and, as expected, they are correctly quantized if all the previous conditions are satisfied. Notice that in both cases, the first term is the Euler characteristic of the base $\mathbb{B}$ (or its generalization due to the anti-twist on Σ_1). Moreover the second term reflects the presence of a fibration, and $n_\pm$ is present since the normal bundle to $S_\pm$ contains orbifold singularities $\mathbb{Z}_{n_\pm}$, as we shall see momentarily.

3.4.2 Massive type IIA and entropy

For the system at hand, the reduced functions (3.2.9) reads

$$\Delta_h(y) = h_1 h_2\, \mu_0^2 + y^3 h_2\, \mu_1^2 + y^3 h_1\, \mu_2^2\,, \quad U_h(y) = 2\big[(y^3 - h_1)(y^3 - h_2)\mu_0^2 - y^6\big] - \frac{4}{3}\Delta_h(y)\,. \tag{3.4.20}$$

It follows that

$$F_{(0)} = \frac{2g_c}{3\lambda^3}\,, \qquad \mathrm{e}^{\Phi} = \lambda^2 \mu_0^{-5/6} y^{-3/2} \Delta_h^{1/4}\,. \tag{3.4.21}$$

The other fields depend explicitly on $\mathbb{B} = \Sigma_{\rm g}, \Sigma_1$. In particular, for $\Sigma_{\rm g}$ we have

$$\begin{aligned}
\mathrm{d}s^2_{\text{s.f.}} &= \frac{\lambda^2 \Delta_h^{1/2}}{\mu_0^{1/3} y}\bigg\{\frac{1}{4}\,\mathrm{d}s^2_{\mathrm{AdS}_2} + \mathrm{d}s^2_{\Sigma_{\rm g}} + \frac{y^2}{F}\,\mathrm{d}y^2 + \frac{F}{h_1 h_2}\Big(\mathrm{d}z - \frac{1}{2m}\,\omega_{\Sigma_{\rm g}}\Big)^2 \\
&\quad + \frac{y}{g_c^2 \Delta_h}\big[y^3\,\mathrm{d}\mu_0^2 + h_1\big(\mathrm{d}\mu_1^2 + \mu_1^2\sigma_1^2\big) + h_2\big(\mathrm{d}\mu_2^2 + \mu_2^2\sigma_2^2\big)\big]\bigg\}\,, \qquad B_{(2)} = \frac{\lambda^2 y \mu_0^{2/3}}{4m}\mathrm{volAdS}_2\,, \\
F_{(4)} &= -\frac{\lambda \mu_0^{1/3} h_1 h_2}{g_c^3 \Delta_h}\bigg\{\frac{U_h}{\Delta_h}\frac{\mu_1\mu_2}{\mu_0}\,\mathrm{d}\mu_1 \wedge \mathrm{d}\mu_2 \wedge \sigma_1 \wedge \sigma_2 \\
&\quad - \sum_{i\neq j}\Big[g_c\, F_i \wedge \mathrm{d}\phi_j - \frac{y^3(h_i' - 3y^{-1}h_i)h_j}{\Delta_h h_i}\mu_i^2\,\mathrm{d}y \wedge \sigma_i \wedge \sigma_j\Big] \wedge \big(\mu_0\mu_j\,\mathrm{d}\mu_j - y^3 h_j^{-1}\mu_j^2\,\mathrm{d}\mu_0\big)\bigg\} \\
&\quad + \frac{\lambda y \mu_0^{4/3}}{2}\,\mathrm{vol}\Sigma_{\rm g} \wedge \mathrm{d}y \wedge \mathrm{d}z - \frac{\lambda y^2 \mu_0^{1/3}}{3}\,\mathrm{vol}\Sigma_{\rm g} \wedge \mathrm{d}z \wedge \mathrm{d}\mu_0\,, \quad F_{(2)} = \frac{g_c y \mu_0^{2/3}}{6\lambda m}\mathrm{volAdS}_2\,.
\end{aligned}$$

Instead, when the base of the fibration is Σ_1 we have

$$\begin{aligned}
\mathrm{d}s^2_{\text{s.f.}} &= \frac{\lambda^2 \Delta_h^{1/2}}{\mu_0^{1/3} y}\bigg\{\frac{x^2}{4}\,\mathrm{d}s^2_{\mathrm{AdS}_2} + \frac{x^2}{q}\mathrm{d}x^2 + \frac{q}{4x^2}\mathrm{d}\psi^2 + \frac{y^2}{F}\,\mathrm{d}y^2 + \frac{F}{h_1 h_2}\Big(\mathrm{d}z - \frac{1}{2m}\Big(1 - \frac{\mathtt{a}}{x}\Big)\,\mathrm{d}\psi\Big)^2 \\
&\quad + \frac{y}{g_c^2 \Delta_h}\big[y^3\,\mathrm{d}\mu_0^2 + h_1\big(\mathrm{d}\mu_1^2 + \mu_1^2\sigma_1^2\big) + h_2\big(\mathrm{d}\mu_2^2 + \mu_2^2\sigma_2^2\big)\big]\bigg\}\,, \qquad B_{(2)} = \frac{\lambda^2 \mathtt{a} y \mu_0^{2/3}}{4m}\mathrm{volAdS}_2\,, \\
F_{(4)} &= -\frac{\lambda \mu_0^{1/3} h_1 h_2}{g_c^3 \Delta_h}\bigg\{\frac{U_h}{\Delta_h}\frac{\mu_1\mu_2}{\mu_0}\,\mathrm{d}\mu_1 \wedge \mathrm{d}\mu_2 \wedge \sigma_1 \wedge \sigma_2 \\
&\quad - \sum_{i\neq j}\Big[g_c\, F_i \wedge \mathrm{d}\phi_j - \frac{y^3(h_i' - 3y^{-1}h_i)h_j}{\Delta_h h_i}\mu_i^2\,\mathrm{d}y \wedge \sigma_i \wedge \sigma_j\Big] \wedge \big(\mu_0\mu_j\,\mathrm{d}\mu_j - y^3 h_j^{-1}\mu_j^2\,\mathrm{d}\mu_0\big)\bigg\} \\
&\quad + \frac{\lambda \mathtt{a} y \mu_0^{4/3}}{x^2}\mathrm{vol}\Sigma_1 \wedge \mathrm{d}y \wedge \mathrm{d}z - \frac{2\lambda \mathtt{a} y^2 \mu_0^{1/3}}{3x^2}\mathrm{vol}\Sigma_1 \wedge \mathrm{d}z \wedge \mathrm{d}\mu_0\,, \quad F_{(2)} = \frac{\mathtt{a} g_c y \mu_0^{2/3}}{6\lambda m}\mathrm{volAdS}_2\,.
\end{aligned}$$

The integration along $\mathbb{S}^4$ gives, as expected, the same result as in (3.2.17). In both cases, the integration along $\mathbb{M}_4^{(1,2)}$ receives contributions only from the first term in the last line of $F_{(4)}$, since it is proportional to $\mathrm{vol}\mathbb{M}_4^{(1,2)}$. The result can be expressed in terms of $(\mathrm{g}, n_\pm, \mathtt{z}, N)$ or $(m_\pm, n_\pm, \mathtt{z}, N)$, and reads

$$\begin{aligned} \Sigma_{\mathrm{g}}: \quad & K = N\chi_2(\mathrm{g}-1)\,\frac{3[3\mu(\mathtt{x}^2+1)-\mathtt{x}(\mathtt{x}^2+5)]}{8\mathtt{x}(\mathtt{x}^2+3)} \in \mathbb{N}\,, \\ \Sigma_1: \quad & K = N\chi_2\frac{m_+-m_-}{m_+m_-}\,\frac{3[3\mu(\mathtt{x}^2+1)-\mathtt{x}(\mathtt{x}^2+5)]}{8\mathtt{x}(\mathtt{x}^2+3)} \in \mathbb{N}\,. \end{aligned} \tag{3.4.22}$$

The analysis of these constraint is left to appendix A of [1]. With all these expressions at hand we can now compute the entropy of the systems through the formula (3.2.11). We obtain

$$\begin{aligned} \Sigma_{\mathrm{g}}: \quad & S_{\Sigma_{\mathrm{g}}\ltimes\Sigma_2} = \frac{8\pi^2}{(2\pi\ell_s)^8}\frac{3\pi^2\lambda^4}{20g^4}\,4\pi(\mathrm{g}-1)\,(y_+^3-y_-^3)\Delta z = (\mathrm{g}-1)\,F_{S^3\times\Sigma_2}\,, \\ \Sigma_1: \quad & S_{\Sigma_1\ltimes\Sigma_2} = \frac{8\pi^2}{(2\pi\ell_s)^8}\frac{3\pi^2\lambda^4}{20g^4}\,(x_+-x_-)\Delta\psi\,(y_+^3-y_-^3)\Delta z = \frac{1}{2\pi}\,A_{\Sigma_1}F_{S^3\times\Sigma_2}\,, \end{aligned} \tag{3.4.23}$$

where $F_{S^3\times\Sigma_2}$ is the free energy of $d=3$, $\mathcal{N}=2$ SCFTs that arise from a system of N D4-branes and N_f D8-branes wrapped on a spindle [52]

$$F_{S^3\times\Sigma_2} = \chi_2\,\frac{\sqrt{3}\pi N^{5/2}}{5\sqrt{8-N_f}}\frac{[3\mu(\mathtt{x}^2+1)-\mathtt{x}(\mathtt{x}^2+5)]^{3/2}}{\mathtt{x}(\mathtt{x}^2+3)(\mu-\mathtt{x})^{1/2}}\,, \tag{3.4.24}$$

and A_{Σ_1} is the area of the horizon of the (putative) four-dimensional supersymmetric black hole solutions with $\mathrm{AdS}_2\times\Sigma_1$ near-horizon studied in [46], namely

$$A_{\Sigma_1} = \frac{\sqrt{2}\sqrt{m_+^2+m_-^2}-(m_++m_-)}{m_+m_-}\,\pi\,. \tag{3.4.25}$$

3.5 Toric data

Having completed the global analysis of our solutions, we can now construct the labelled polytope and extract the toric data that we will use in the next section to reproduce the gravitational results (3.4.23). Recall from the discussion in section 2.1 that there is a distinction between $\hat{D}_a$ and D_a, the branch and ramified divisors. Consequently there are "long" vectors $\vec{v}_a$, which are not primitive, and "short" vectors $\vec{\hat{v}}_a$ for which $\det(\vec{\hat{v}}_a, \vec{\hat{v}}_{a+1}) = 1$. We will start by studying the orbifold $S^2 \ltimes \Sigma_2$ which is toric in the standard sense, *i.e.* it admits a (conformally) closed symplectic two form $\omega_{(2)}$ and a moment map $\vec{\mu}$. This, even if not supersymmetric, is a solution of the six-dimensional model (3.1.1) and can be obtained by an analytic continuation of $\mathbb{M}_4^{(1)} = \Sigma_{\mathrm{g}}\ltimes\Sigma_2$. Moreover we will provide a method for extracting the same toric data by analysing the loci $\mathcal{L}_a$ in which $U(1)\subset U(1)^2$ collapses, which is exactly the definition of a divisor $\hat{D}_a$. Then we can apply the same method to the toric orbifold $\mathbb{M}_4^{(2)} = \Sigma_1 \ltimes \Sigma_2$, even if we are not able to construct a (conformally) closed $\omega_{(2)}$. Here there is a subtlety, in that it is not clear *a priori* if the loci in which $U(1)\subset U(1)^2$ degenerates is identifiable with D_a or $\hat{D}_a$. We will provide evidences for which $\mathcal{L}_a = \hat{D}_a$ by analysing in details the metric (3.3.18). Also, if we denote with r the distance from $\mathcal{L}_a$, it is clear that for $r = \epsilon > 0$ we can inspect the surroundings of $\mathcal{L}_a$, where it is visible the action of the structure group $\mathbb{Z}_{m_a}$, so that $D_a = \hat{D}_a \times \mathrm{pt}/\mathbb{Z}_{m_a}$.

3.5.1 $\mathrm{AdS}_2 \times \mathbb{M}_4$

$S^2 \ltimes \Sigma_2$

Before studying the $\Sigma_1 \ltimes \Sigma_2$ orbifold, we start with $\mathbb{M}_4 = S^2 \ltimes \Sigma_2$. Although non-supersymmetric, the $\mathrm{AdS}_2 \times S^2 \ltimes \Sigma_2$ backgrounds form a family of solutions to the equations of motion very similar to (3.3.1), but with a different value of the radius of the Riemann surface, now a sphere. Specifically, the space we consider is the four-dimensional toric orbifold $S^2 \ltimes \Sigma_2$ with metric

$$\mathrm{d}s^2_{S^2\ltimes\Sigma_2} = \frac{1}{6}\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\psi^2\right) + \frac{y^2}{F}\,\mathrm{d}y^2 + \frac{F}{h_1h_2}\left(\mathrm{d}z + \frac{1}{2m}\cos\theta\,\mathrm{d}\psi\right)^2 . \tag{3.5.1}$$

In this case, the condition on t (3.4.15) in order to have a well-defined fibration reads

$$t = -\frac{1}{m}\frac{2\pi}{\Delta z} \in \mathbb{Z}\,, \tag{3.5.2}$$

where t results to be negative. For fixed values of $y \neq y_{\mp}$, metric (3.5.1) describes (topologically) a squashed $L(-t,1) = S^3/\mathbb{Z}_{-t}$ lens space, where S^3 is a squashed three-sphere written as a Hopf fibration. Therefore a natural basis of an effective two-torus action is[9] [126]

$$E_1 \equiv \partial_{\phi_1} = \partial_{\nu_2}\,, \quad E_2 \equiv \partial_{\phi_2} = \partial_\psi - \frac{t}{2}\,\partial_{\nu_2}\,, \tag{3.5.3}$$

where we introduced the 2π-periodic coordinate $\nu_2 = \frac{2\pi}{\Delta z}z$. In total we have four fixed points p_I, $I = 1,\ldots,4$, under the action of the two-torus, which are all the possible combinations obtained by pairing the poles of the sphere ($\theta = 0,\pi$) and the poles of the spindle ($y = y_\pm$)

$$\begin{aligned} p_1 &= \{\theta = 0,\, y = y_-\}\,, \qquad p_2 = \{\theta = 0,\, y = y_+\}\,, \\ p_3 &= \{\theta = \pi,\, y = y_+\}\,, \qquad p_4 = \{\theta = \pi,\, y = y_-\}\,. \end{aligned} \tag{3.5.4}$$

We now consider the conformally rescaled metric $\mathrm{d}s^2 = \Gamma(y)\,\mathrm{d}s^2_{S^2\ltimes\Sigma_2}$, with $\Gamma(y) > 0$, with compatible symplectic two-form

$$\omega_{(2)} = \Gamma(y)\left[\frac{1}{6}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\psi + \frac{y}{(h_1h_2)^{1/2}}\,\mathrm{d}y\wedge\left(\mathrm{d}z + \frac{1}{2m}\cos\theta\,\mathrm{d}\psi\right)\right] . \tag{3.5.5}$$

When $\Gamma'(y) = -\frac{3y}{m(h_1h_2)^{1/2}}\Gamma(y)$ [10], the two-form (3.5.5) is closed and can be written as

$$\omega_{(2)} = \mathrm{d}\psi\wedge\mathrm{d}\left[\frac{1}{6}\Gamma(y)\cos\theta\right] + \mathrm{d}\nu_2\wedge\mathrm{d}\left[-\frac{1}{3t}\Gamma(y)\right] = \mathrm{d}\left[\frac{1}{3t}\Gamma(y)\right]\wedge\mathrm{d}\phi_1 - \mathrm{d}\left[\Gamma(y)\frac{1+\cos\theta}{6}\right]\wedge\mathrm{d}\phi_2\,. \tag{3.5.6}$$

[9] We exchanged E_1 and E_2 with respect to [126] in order to have the vectors $\vec{v}_a$ ordered counter-clockwise.

[10] A real and positive solution to this differential equation exists and is unique, up to an irrelevant overall constant.

From this expression we can derive the moment maps with respect to the basis (3.5.3) according to the general prescription (2.1.4)

$$\vec{\mu}(y,\theta) = \frac{1}{3t}\Gamma(y)\left(1, -\frac{t(1+\cos\theta)}{2}\right), \tag{3.5.7}$$

and compute the image of the fixed points (3.5.4)

$$\begin{aligned} \vec{\mu}(p_1) &= \frac{1}{3t}\Gamma(y_-)\,(1,-t)\,, \qquad & \vec{\mu}(p_2) &= \frac{1}{3t}\Gamma(y_+)\,(1,-t)\,, \\ \vec{\mu}(p_3) &= \frac{1}{3t}\Gamma(y_+)\,(1,0)\,, \qquad & \vec{\mu}(p_4) &= \frac{1}{3t}\Gamma(y_-)\,(1,0)\,, \end{aligned} \tag{3.5.8}$$

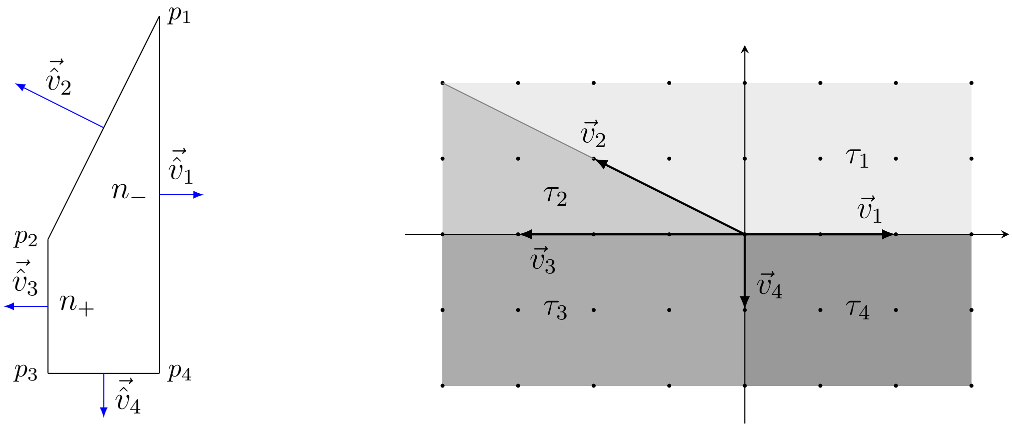
which are vertices of the moment polytope. In order to correctly draw the polytope we notice that, since $\Gamma'(y) < 0$, we have $\Gamma(y_-) > \Gamma(y_+)$. The $\mathbb{M}_4 = S^2 \ltimes \Sigma_2$ orbifold is then characterized by

$$\begin{aligned} \hat{D}_1 &= \{y = y_-\}: \quad m_1 = n_-\,, \qquad & \hat{D}_2 &= \{\theta = 0\}: \quad m_2 = 1\,, \\ \hat{D}_3 &= \{y = y_+\}: \quad m_3 = n_+\,, \qquad & \hat{D}_4 &= \{\theta = \pi\}: \quad m_4 = 1\,, \end{aligned} \tag{3.5.9}$$

and by the primitive vectors

$$\hat{\vec{v}}_1 = (1,0)\,, \quad \hat{\vec{v}}_2 = (t,1)\,, \quad \hat{\vec{v}}_3 = (-1,0)\,, \quad \hat{\vec{v}}_4 = (0,-1)\,. \tag{3.5.10}$$

The long vectors are, as usual, $\vec{v}_a = m_a \hat{\vec{v}}_a$. The resulting labelled polytope and stacky fan are depicted in figure 3.1, and if the labels are stripped off this is exactly the polytope



(a) Labelled polytope. (b) Stacky fan.

Figure 3.1: Labelled polytope and stacky fan of the $S^2 \ltimes \Sigma_2$ orbifold. The different facets $\mathcal{F}_a$ are labelled with the integers m_a. The cones generated by $\vec{v}_a$ and $\vec{v}_{a+1}$ are denoted as τ_a.

corresponding to the Hirzebruch surfaces $\mathbb{F}_{-t}$, see figure 2.2b.

The method we employed to construct the labelled polytope above is rigorous and the result fits in the theory of symplectic toric orbifolds [86]. As anticipated, we now reproduce (3.5.9) and (3.5.10) by a different analysis, *i.e.* studying the loci $\mathcal{L}_a$ in which a generic Killing vectors $\xi = a\partial_\psi + b\partial_{\nu_2}$ degenerate. Explicitly, denoting with ξ_a the Killing vector that degenerates at $\mathcal{L}_a$, we have

$$\xi_1 = n_- \,\partial_{\nu_2}\,, \quad \xi_2 = \partial_\psi + \frac{t}{2}\,\partial_{\nu_2}\,, \quad \xi_3 = n_+ \,\partial_{\nu_2}\,, \quad \xi_4 = \partial_\psi - \frac{t}{2}\,\partial_{\nu_2}\,, \tag{3.5.11}$$

normalized so to have unitary surface gravity

$$\kappa^2_{\text{grav}} = \frac{\partial_\mu|\xi_a|^2\partial^\mu|\xi_a|^2}{4|\xi_a|^2}\Big|_{\mathcal{L}_a} = 1\,. \tag{3.5.12}$$

Expanded in the basis (3.5.3), the degenerating Killing vectors read

$$\xi_1 = n_-\,E_1\,, \quad \xi_2 = t\,E_1 + E_2\,, \quad \xi_3 = n_+\,E_1\,, \quad \xi_4 = E_2\,. \tag{3.5.13}$$

We notice that the following relation

$$\xi_a = \vec{v}_a \cdot (E_1, E_2)\,, \tag{3.5.14}$$

holds, but only for long-vectors $\vec{v}_a$. As anticipated around (2.3.2), we will use the relation (3.5.14) to infer the toric data starting from the information about the degenerate Killing vectors.

$\Sigma_{\rm g} \ltimes \Sigma_2$

Let us make some comments on the orbifold $\Sigma_{\rm g} \ltimes \Sigma_2$, with ${\rm g} > 1$. This cannot be toric in any sense, therefore none of the previous considerations can be applied. Nevertheless, for our purposes, we can think about extending the results obtained for the $S^2 \ltimes \Sigma_2$ toric orbifold to this case, and adopt as "toric data" the same vectors $\vec{\hat{v}}_a$ (3.5.10) and the same integers m_a (3.5.9). We will confirm the validity of this proposal at the end of section 3.6.1.

$\Sigma_1 \ltimes \Sigma_2$

After the warm-up with the $\mathbb{M}_4 = S^2 \ltimes \Sigma_2$ case, we are now in the position to study the $\mathbb{M}_4^{(2)} = \Sigma_1 \ltimes \Sigma_2$ orbifold. As anticipated we have not been able to construct a conformal symplectic structure, therefore we do not have a moment map to derive the labelled polytope, but we will use the prescription (3.5.14). Recall that the metric on $\Sigma_1 \ltimes \Sigma_2$ is

$$\mathrm{d}s^2_{\Sigma_1\ltimes\Sigma_2} = \frac{x^2}{q(x)}\,\mathrm{d}x^2 + \frac{q(x)}{4x^2}\,\mathrm{d}\psi^2 + \frac{y^2}{F(y)}\,\mathrm{d}y^2 + \frac{F(y)}{h_1(y)h_2(y)}\Big(\mathrm{d}z - \frac{1}{2m}\Big(1 - \frac{\mathtt{a}}{x}\Big)\,\mathrm{d}\psi\Big)^2\,. \tag{3.5.15}$$

A simple direct calculation shows that it has four Killing vectors degenerating at the poles of the two spindles. Defining the four loci

$$\mathcal{L}_1 = \{y = y_-\}\,, \quad \mathcal{L}_2 = \{x = x_-\}\,, \quad \mathcal{L}_3 = \{y = y_+\}\,, \quad \mathcal{L}_4 = \{x = x_+\}\,, \tag{3.5.16}$$

intersecting at the the four fixed points

$$\begin{aligned} p_1 &= \{x = x_-,\, y = y_-\}\,, \qquad & p_2 &= \{x = x_-,\, y = y_+\}\,, \\ p_3 &= \{x = x_+,\, y = y_+\}\,, \qquad & p_4 &= \{x = x_+,\, y = y_-\}\,, \end{aligned} \tag{3.5.17}$$

and normalizing the degenerating Killing vectors so that they have unitary surface gravity as before, the Killing vectors read

$$\begin{aligned} \xi_1 &= n_-\,\partial_{\nu_2}\,, \qquad \xi_2 = m_-\Big(\partial_{\nu_1} - \frac{\mathtt{a}-x_-}{2mx_-}\frac{\Delta\psi}{\Delta z}\,\partial_{\nu_2}\Big)\,, \\ \xi_3 &= n_+\,\partial_{\nu_2}\,, \qquad \xi_4 = m_+\Big(\partial_{\nu_1} - \frac{\mathtt{a}-x_+}{2mx_+}\frac{\Delta\psi}{\Delta z}\,\partial_{\nu_2}\Big)\,, \end{aligned} \tag{3.5.18}$$

where, in addition to $\nu_2 = \frac{2\pi}{\Delta z}z$, we introduced also the 2π-periodic coordinate $\nu_1 = \frac{2\pi}{\Delta\psi}\psi$. In order to derive the normal vectors $\vec{v}_a$ we need to find an appropriate basis in which to expand the ξ_a. Inspired by (3.5.3), where ∂_{ν_2} is the first basis element and ξ_4 plays the role of E_2, we consider the following basis

$$e_1 = \partial_{\nu_2}\,, \quad e_2 = \partial_{\nu_1} - \frac{\mathtt{a}-x_+}{2mx_+}\frac{\Delta\psi}{\Delta z}\,\partial_{\nu_2}\,, \tag{3.5.19}$$

in which the degenerating Killing vectors are decomposed as

$$\begin{aligned} \xi_1 &= n_-\,e_1\,, \qquad \xi_2 = \frac{t}{m_+}\,e_1 + m_-\,e_2\,, \\ \xi_3 &= n_+\,e_1\,, \qquad \xi_4 = m_+\,e_2\,, \end{aligned} \tag{3.5.20}$$

where we made use of the quantization condition on t in (3.4.15) in the form

$$\frac{t}{m_+m_-} = -\frac{\mathtt{a}(x_+ - x_-)}{2mx_-x_+}\frac{\Delta\psi}{\Delta z}\,. \tag{3.5.21}$$

Note that, even though $x_\mp$ and $\mathtt{a}$ are real numbers, all the coefficients in (3.5.20) are rational. Extracting the normal vectors, one gets

$$\vec{V}_1 = n_-(1,0)\,, \quad \vec{V}_2 = \Big(\frac{t}{m_+}, m_-\Big)\,, \quad \vec{V}_3 = n_+(-1,0)\,, \quad \vec{V}_4 = m_+(0,-1)\,. \tag{3.5.22}$$

which are not primitive, as expected, but also they are not $\mathbb{Z}^2$-valued. The underlying reason is that the basis (3.5.19) is not effective. A better basis can be obtained rotating the vectors $\vec{v}_a$ through the following $SL(2,\mathbb{Q})$ matrix

$$S = \begin{pmatrix} 1 & -r_-/m_+ \\ 0 & 1 \end{pmatrix}\,, \tag{3.5.23}$$

where r_- is an integer prime to m_+. Defining $\vec{v}_a \equiv S\,\vec{V}_a$, we have

$$\vec{v}_1 = (n_-, 0)\,, \quad \vec{v}_2 = (r_+, m_-)\,, \quad \vec{v}_3 = (-n_+, 0)\,, \quad \vec{v}_4 = (r_-, -m_+)\,, \tag{3.5.24}$$

where we have introduced $a_\pm \in \mathbb{Z}$ such that

$$r_\pm = a_\pm t \in \mathbb{Z}\,, \quad a_+m_+ + a_-m_- = 1 \implies r_+m_+ + r_-m_- = t\,, \tag{3.5.25}$$

and always exist by Bézout's lemma for co-prime $m_\pm$. The vectors $\vec{v}_a$ now take values in $\mathbb{Z}^2$. We can now define a new basis $\{E_1, E_2\}$, obtained as $E_I = S^{-1}_{JI}e_J$, such that

$$\xi_a = \vec{v}_a \cdot (E_1, E_2)\,, \tag{3.5.26}$$

and the degenerating Killing vectors take the form

$$\xi_1 = n_- \, E_1 \,, \quad \xi_2 = m_- \, E_2 + r_+ \, E_1 \,, \quad \xi_3 = n_+ \, E_1 \,, \quad \xi_4 = m_+ \, E_2 - r_- \, E_1 \,. \tag{3.5.27}$$

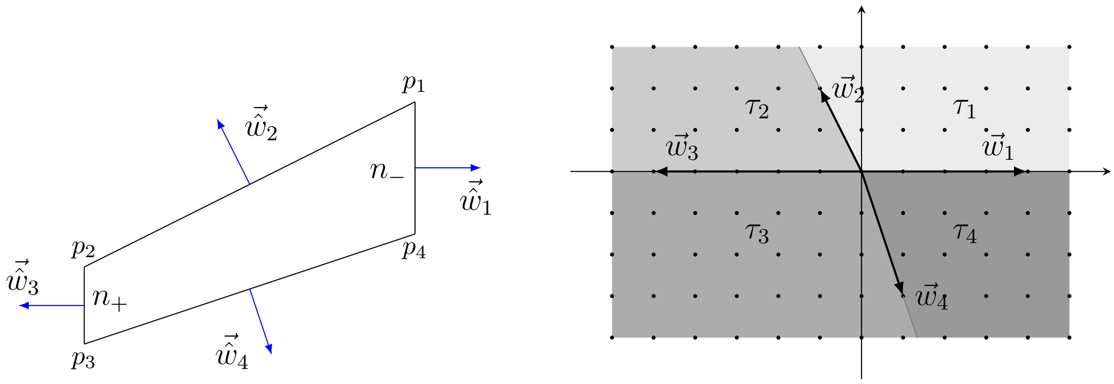
All the coefficients of this decomposition are now integers, indicating that in this basis the torus action is effective. For completeness we also write the basis vectors in terms of the Killing vectors ∂_{ν_1} and ∂_{ν_2}[11]

$$E_1 = \partial_{\nu_2} \,, \quad E_2 = \partial_{\nu_1} + \frac{r_+ + r_-}{m_+ - m_-} \partial_{\nu_2} \,. \tag{3.5.28}$$

To extract the labels we have to take now the greatest common divisors of each entries of the vectors in (3.5.24). This procedure is subtle, as depend on $\gcd(t, m_\pm)$. For a generic value of t, *i.e.* $\gcd(t, m_\pm) = 1$, the correct assignment for the labels is

$$\gcd(t, m_\pm) = 1 : \quad m_a = (n_-, 1, n_+, 1) \,. \tag{3.5.29}$$

The combinatorial data can be presented in the form of a stacky fan or labelled polytope, as depicted in figure 3.2. Notice that for $n_+ = n_- = 1$ this reduces to the "Hirzebruch orbifold" discussed in [120] in the context of toric stacks.



(a) Labelled polytope. (b) Stacky fan.

Figure 3.2: Toric data of the $\Sigma_1 \ltimes \Sigma_2$ orbifold corresponding to the vectors (3.5.24). In this picture $\gcd(t, m_\pm) = 1$.

We now examine the case $\gcd(t, m_\pm) \neq 1$ and, contemporarily, understand better the difference between the branch and the ramified divisors, getting some insights on the reason for which the loci in (3.5.16) are to be identified as $\mathcal{L}_a = \hat{D}_a$. Let us start with $\gcd(t, m_\pm) = 1$. To focus on these $\mathcal{L}_a$, we define adapted coordinates

$$\nu_1 = m_+ \phi_+ - m_- \phi_- \,, \quad \nu_2 = \frac{\Delta\psi}{\Delta z} \left[-m_+ \frac{\mathtt{a} - x_+}{2 m x_+} \phi_+ + m_- \frac{\mathtt{a} - x_-}{2 m x_-} \phi_- \right] , \quad \frac{(2\pi)^2}{-t} = \Delta\phi_+ \Delta\phi_- \,, \tag{3.5.30}$$

[11]Notice that E_2 is finite also when $m_+ = m_- = 1$. Indeed, in this limit we have $r_+ + r_- = t$, which is proportional to $m_+ - m_-$ (*cf.* (3.4.15)), thus eliminating the vanishing denominator.

where the last requirement comes from $\Delta\nu_1\Delta\nu_2 = (2\pi)^2 = |\det(\mathrm{Jac})|\Delta\phi_+\Delta\phi_-$. The total metric, restricted to $\mathcal{L}_a$, reads

$$\begin{aligned} \mathrm{d}s^2_{\Sigma_1\ltimes\Sigma_2}\big|_{y=y_\mp} &: \quad \mathrm{d}s^2_{(1)} = \mathrm{d}s^2_{(3)} = \frac{x^2}{q}\,\mathrm{d}x^2 + \frac{q}{4x^2}\,\mathrm{d}\psi^2\,, \\ \mathrm{d}s^2_{\Sigma_1\ltimes\Sigma_2}\big|_{x=x_-} &: \quad \mathrm{d}s^2_{(2)} = \frac{y^2}{F}\,\mathrm{d}y^2 + \frac{t^2}{m_-^2}\frac{F}{h_1h_2}\left(\frac{\Delta z}{2\pi}\right)^2\mathrm{d}\phi_+^2\,, \\ \mathrm{d}s^2_{\Sigma_1\ltimes\Sigma_2}\big|_{x=x_+} &: \quad \mathrm{d}s^2_{(4)} = \frac{y^2}{F}\,\mathrm{d}y^2 + \frac{t^2}{m_+^2}\frac{F}{h_1h_2}\left(\frac{\Delta z}{2\pi}\right)^2\mathrm{d}\phi_-^2\,. \end{aligned} \tag{3.5.31}$$

Looking, *e.g.*, at $\mathrm{d}s^2_{(2)}$, we see that it describes $\Sigma_2/\mathbb{Z}_{m_-}$ if we require $(\Delta\phi_+)^{(2)} = -2\pi/t$, where $(\Delta\phi_+)^{(2)}$ is the periodicity of ϕ_+ when referred to $\mathcal{L}_2$. From (3.5.30), we have then $(\Delta\phi_-)^{(2)} = 2\pi$. Similarly, the line element $\mathrm{d}s^2_{(4)}$ describes $\Sigma_2/\mathbb{Z}_{m_+}$ when $(\Delta\phi_-)^{(4)} = -2\pi/t$ and $(\Delta\phi_+)^{(4)} = 2\pi$. The reason for which the periodicities seem not to be globally defined and "jump" to different values on the loci $\mathcal{L}_2$ and $\mathcal{L}_4$, is that the quotients must be done locally in patches, and then suitably glued. Thus we should have defined local coordinates $\phi_\pm^{(2)}$ and $\phi_\pm^{(4)}$on a patch that contains $\mathcal{L}_2$ or $\mathcal{L}_4$, for which $(\Delta\phi_+)^{(2)}$ is more precisely $\Delta(\phi_+^{(2)})$. Eventually, we will write $\Delta\phi_+^{(2)}$ when it is clear what we mean. We thus deduce

$$\gcd(t, m_\pm) = 1: \quad \hat{D}_1 = \Sigma_1\,, \quad \hat{D}_2 = \frac{\Sigma_2}{\mathbb{Z}_{m_-}}\,, \quad \hat{D}_3 = \Sigma_1\,, \quad \hat{D}_4 = \frac{\Sigma_2}{\mathbb{Z}_{m_+}}\,, \tag{3.5.32}$$

where $\hat{D}_{1,3}$ are copies of the base Σ_1 at the north and south poles of the fibre Σ_2, respectively. On the other hand, at the poles of Σ_1, the four-dimensional orbifold is locally modelled by $(\mathbb{C}\times\Sigma_2)/\mathbb{Z}_{m_\mp}$, and as a consequence $\hat{D}_{2,4}$ are *global* quotients of the fibre Σ_2. Notice that, even if $\hat{D}_{2,4} = \Sigma_2/\mathbb{Z}_{m_\mp}$, there are no orbifold singularities at generic points (*i.e.* different from the north and south poles) on $\hat{D}_{2,4}$. These are then divisors associated with an ordinary fan, with primitive vectors $\vec{\hat{v}}_{2,4} = \vec{v}_{2,4}$. We still need to extract the labels and to do this, we need to move in the surroundings of the $\mathcal{L}_a$. Zooming in for example near $\mathcal{L}_1$, the metric at leading order in $R^2 = \frac{4y_-^2}{F'(y_-)}(y-y_-)$ reads[12]

$$\mathrm{d}s^2_{\Sigma_1\ltimes\Sigma_2} \underset{y\to y_-}{\simeq} \frac{x^2}{q}\,\mathrm{d}x^2 + \frac{q}{4x^2}\,\mathrm{d}\psi^2 + \mathrm{d}R^2 + \frac{R^2}{n_-^2}\left(\mathrm{d}\tilde{z} - \frac{1}{2m}\left(1-\frac{\mathtt{a}}{x}\right)\frac{1}{\Delta z}\,\mathrm{d}\psi\right)^2. \tag{3.5.33}$$

When we set $y = y_-$, the metric reduces to $\mathrm{d}s^2_{(1)}$ in (3.5.31), which is $\hat{D}_1 = \Sigma_1$. However, at each point $x_- \le x \le x_+$, from (3.5.33) we see that there is a normal conical singularity giving, *locally*, a copy of $\Sigma_1 \ltimes (\mathbb{C}/\mathbb{Z}_{n_-})$. As a consequence, n_- is a label with associated divisor $D_1 = \hat{D}_1 \times \mathrm{pt}/\mathbb{Z}_{n_-}$. The metric (3.5.33) can be compared with its counterpart when approaching the locus $\mathcal{L}_2$. In this case, we zoom in near $x = x_-$ defining the new coordinate R such that $x = x_- + \frac{q'(x_-)}{4x_-^2}R^2$. In the adapted coordinates (3.5.30), the metric on $\mathbb{M}_4^{(2)} = \Sigma_1 \ltimes \Sigma_2$ at leading order in R^2 reads

$$\mathrm{d}s^2_{\Sigma_1\ltimes\Sigma_2} \underset{x\to x_-}{\simeq} \mathrm{d}R^2 + R^2\big(\mathrm{d}\phi_-^{(2)} + c(y)\,\mathrm{d}\phi_+^{(2)}\big)^2 + \frac{y^2}{F}\,\mathrm{d}y^2 + \frac{t^2}{m_-^2}\frac{F}{h_1h_2}\left(\frac{\Delta z}{2\pi}\right)^2(\mathrm{d}\phi_+^{(2)})^2\,, \tag{3.5.34}$$

[12]See section 4.4 of [118] for an analogous computation.

with $c(y)$ a regular function on $[y_-, y_+]$ which at the endpoints of the interval takes the values $c(y_\pm) = -m_+/m_-$. Assuming $(\Delta\phi_-)^{(2)} = 2\pi$ as before, the first contribution to the total metric is the smooth complex plane in polar coordinates, therefore no singularity along the divisor D_2 is present. Setting $R = 0$, which accounts in moving exactly on the divisor, the line element reduces to the second line of (3.5.31), which, as already discussed, corresponds to the metric on $\Sigma_2/\mathbb{Z}_{m_-}$, once the correct periodicity for $\phi_+^{(2)}$ is considered. An analogous reasoning can be applied to $\mathcal{L}_3$ and $\mathcal{L}_4$, showing that $D_3 = \hat{D}_3 \times \mathrm{pt}/\mathbb{Z}_{n_+}$, but no singularity along D_4. From the above analysis we can confirm again that

$$\gcd(t, m_\pm) = 1 : \quad \begin{gathered} m_a = (n_-, 1, n_+, 1)\,, \\ \hat{D}_1 = \Sigma_1\,, \quad \hat{D}_2 = \frac{\Sigma_2}{\mathbb{Z}_{m_-}}\,, \quad \hat{D}_3 = \Sigma_1\,, \quad \hat{D}_4 = \frac{\Sigma_2}{\mathbb{Z}_{m_+}}\,. \end{gathered} \tag{3.5.35}$$

We can now come back to the case when t and $m_\pm$ have common factors, and we expect that the labels change accordingly. On general grounds, it is known that in this case the fibration does not smooth the orbifold points on the base [46]. Then, the four-dimensional orbifold will be characterized by four different labels with $m_2 = \gcd(t, m_-)$ and $m_4 = \gcd(t, m_+)$, as we shall understand in a moment. To see explicitly the difference with the previous case, we can zoom in near $y = y_\pm$ in (3.5.34), obtaining

$$\begin{aligned} \mathrm{d}s^2_{\mathbb{M}_4} \underset{x_-, y_-}{\simeq} \mathrm{d}R^2 + R^2\Big(\mathrm{d}\phi_-^{(2)} - \frac{m_+}{m_-}\,\mathrm{d}\phi_+^{(2)}\Big)^2 + \mathrm{d}P^2 + \Big(\frac{t}{m_- n_-}\Big)^2 P^2 (\mathrm{d}\phi_+^{(2)})^2\,, \\ \mathrm{d}s^2_{\mathbb{M}_4} \underset{x_-, y_+}{\simeq} \mathrm{d}R^2 + R^2\Big(\mathrm{d}\phi_-^{(2)} - \frac{m_+}{m_-}\,\mathrm{d}\phi_+^{(2)}\Big)^2 + \mathrm{d}P^2 + \Big(\frac{t}{m_- n_+}\Big)^2 P^2 (\mathrm{d}\phi_+^{(2)})^2\,. \end{aligned} \tag{3.5.36}$$

For coprime $m_\pm$, we can take $t = k_+\bar{t}_+ = k_-\bar{t}_-$ and $m_\pm = k_\pm \overline{m}_\pm$ (with $\gcd(k_+, k_-) = 1$), which implies $t = k_- k_+ \bar{t}$, such that $\gcd(t, m_\pm) = k_\pm$ and

$$\begin{aligned} \mathrm{d}s^2_{\mathbb{M}_4} \underset{x_-, y_-}{\simeq} \mathrm{d}R^2 + R^2\Big(\mathrm{d}\phi_-^{(2)} - \frac{m_+}{m_-}\,\mathrm{d}\phi_+^{(2)}\Big)^2 + \mathrm{d}P^2 + \Big(\frac{k_+\bar{t}}{\overline{m}_- n_-}\Big)^2 P^2 (\mathrm{d}\phi_+^{(2)})^2\,, \\ \mathrm{d}s^2_{\mathbb{M}_4} \underset{x_-, y_+}{\simeq} \mathrm{d}R^2 + R^2\Big(\mathrm{d}\phi_-^{(2)} - \frac{m_+}{m_-}\,\mathrm{d}\phi_+^{(2)}\Big)^2 + \mathrm{d}P^2 + \Big(\frac{k_+\bar{t}}{\overline{m}_- n_+}\Big)^2 P^2 (\mathrm{d}\phi_+^{(2)})^2\,. \end{aligned} \tag{3.5.37}$$

The requirement $\Delta\phi_+^{(2)} = 2\pi/(-k_+\bar{t})$ translates into

$$\frac{2\pi}{-k_+\bar{t}}\,\Delta\phi_-^{(2)} = \Delta\phi_+^{(2)}\Delta\phi_-^{(2)} = \frac{(2\pi)^2}{-t} \qquad \Longrightarrow \qquad \Delta\phi_-^{(2)} = \frac{2\pi}{k_-}\,. \tag{3.5.38}$$

Then the space is $(\Sigma_2/\mathbb{Z}_{\overline{m}_-}) \ltimes (\mathbb{C}/\mathbb{Z}_{k_-})$ and k_- acts as a label on the transverse space to the divisor. Subsequently, there exists an associated divisor D_2 with $\hat{D}_2 = \Sigma_2/\mathbb{Z}_{\overline{m}_-}$. This can also be seen from the vectors (3.5.24), which become

$$\vec{v}_1 = n_-(1,0)\,, \quad \vec{v}_2 = k_-(a_+ k_+ \bar{t}, \overline{m}_-)\,, \quad \vec{v}_3 = n_+(-1,0)\,, \quad \vec{v}_4 = k_+(a_- k_- \bar{t}, -\overline{m}_+)\,, \tag{3.5.39}$$

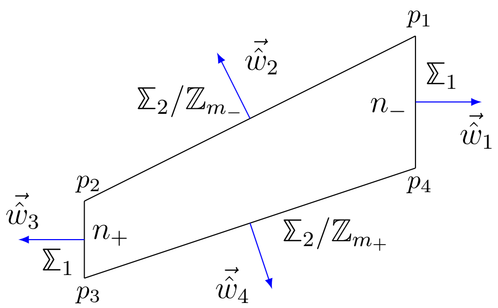
and the labels and divisors are

$$\gcd(t, m_\pm) = k_\pm : \quad \begin{gathered} m_a = (n_-, k_-, n_+, k_+)\,, \\ \hat{D}_1 = \Sigma_1\,, \quad \hat{D}_2 = \frac{\Sigma_2}{\mathbb{Z}_{\overline{m}_-}}\,, \quad \hat{D}_3 = \Sigma_1\,, \quad \hat{D}_4 = \frac{\Sigma_2}{\mathbb{Z}_{\overline{m}_+}}\,, \end{gathered} \tag{3.5.40}$$

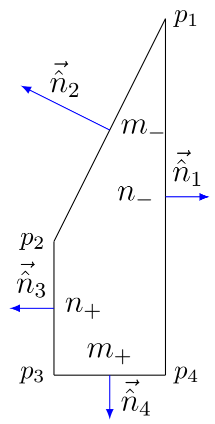
as stated before. The case $\overline{m}_\pm = 1$ (that is $t = m_- m_+ \bar{t}$) is particularly familiar, in that the connection is on $\mathcal{O}(m_- m_+ \bar{t})$ and at a fixed value of y the metric (3.5.15) describes a branched lens space $\mathbb{L}_{m_\pm}(-t, 1)$ [49]. Using an $SL(2,\mathbb{Z})$ transformation, the vectors (3.5.39) can then be rotated to

$$\vec{n}_1 = n_-(1,0)\,, \qquad \vec{n}_2 = m_-(\bar{t},1)\,, \qquad \vec{n}_3 = n_+(-1,0)\,, \qquad \vec{n}_4 = m_+(0,-1)\,, \quad (3.5.41)$$

which are in $\mathbb{Z}^2$, and the labels are simply $m_a = (n_-, m_-, n_+, m_+)$. The associated labelled polytope is sketched in figure 3.3b, and is clearly a labelled (or, stacky) version of the polytope associated with the Hirzebruch surfaces $\mathbb{F}_{-\bar{t}}$ (as in figure 2.2b).



(a) $\gcd(t, m_\pm) = 1$.



(b) $t = m_+ m_- \bar{t}$.

Figure 3.3: Polytope of the $\Sigma_1 \ltimes \Sigma_2$ orbifold corresponding to the vectors (3.5.24) and (3.5.41), respectively.

3.5.2 $\mathrm{AdS}_3 \times \mathbb{M}_4$

We consider here the $\mathrm{AdS}_3 \times \mathbb{M}_4^{(1,2)}$ geometry presented in section 4 and 3 of [61], respectively. Moreover, we present some relevant formulas for these systems which will be used later on. First we study $\mathbb{M}_4^{(1)} = \Sigma_\mathrm{g} \ltimes \Sigma_2$ subjected, as before, to the analytic continuation $\Sigma_\mathrm{g} \mapsto S^2$. We will as brief as possible, since the computations are very similar to the one presented in the previous sections. The aim is to extract the relevant toric data, and for this we will introduce the smallest quantity of informations about the solution.

The metric for the $S^2 \ltimes \Sigma_2$ is given by[13]

$$\mathrm{d}s^2 = \frac{4(yP)^{1/5}}{9}\left[\mathrm{d}s^2_{\mathrm{AdS}_3} + \frac{3}{4}\,\mathrm{d}s^2_{S^2} + \frac{9y}{16Q}\,\mathrm{d}y^2 + \frac{9Q}{4P}\left(\mathrm{d}z - \frac{2}{3}\,\omega_{S^2}\right)^2\right], \qquad (3.5.42)$$

where $\mathrm{d}\omega_{S^2} = -\mathrm{vol}S^2$ and

$$h_i(y) = y^2 + q_i\,, \quad P(y) = h_1(y)\,h_2(y)\,, \quad Q(y) = -y^3 + \frac{1}{4}P(y)\,. \qquad (3.5.43)$$

[13] In order to make the comparison with [61] easier, we relabelled ϕ and a therein as z and $\mathtt{a}$ and exchanged $n_+ \leftrightarrow n_-$.

We can take explicitly

$$\mathrm{d}s^2_{S^2} = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\psi^2 \,, \quad \omega_{S^2} = \cos\theta \mathrm{d}\psi \,. \tag{3.5.44}$$

The requirement of having a globally well-defined fibration

$$\frac{1}{2\pi}\int_{S^2} \mathrm{d}\eta_{S^2} = t \in \mathbb{Z}\,, \quad \eta_{S^2} \equiv \frac{2\pi}{\Delta z}\Big(\mathrm{d}z - \frac{2}{3}\omega_{S^2}\Big)\,, \tag{3.5.45}$$

yields the relation[14]

$$\frac{4}{3t} = \frac{\Delta z}{2\pi}\,. \tag{3.5.46}$$

Explicitly, the $\mathrm{AdS}_3 \times \Sigma_{\mathrm{g}} \ltimes \Sigma_2$ is globally regular if [61]

$$t = \frac{12(\mathrm{g}-1)n_+ n_-(n_+ - n_-)}{[\mathtt{s} - (p_1+p_2)][\mathtt{s} + 2(p_1+p_2)]}\,, \quad \mathtt{s} = \sqrt{7(p_1^2+p_2^2) + 2p_1p_2 - 6(n_+^2 + n_-^2)}\,. \tag{3.5.47}$$

with $(t, \mathtt{s}) \in \mathbb{Z}$ and p_i are two integers related to the magnetic fluxes as in (2.11) of [61].

We can define again a 2π-periodic coordinate $\nu_2 = \frac{2\pi}{\Delta z}z$ and for fixed $y \neq y_\mp$ this metric describes again a lens space $S^3/\mathbb{Z}_t$. Thus we consider as a basis of an effective torus action

$$e_1 = \partial_{\nu_2}\,, \quad e_2 = \partial_\psi + \frac{t}{2}\,\partial_{\nu_2}\,. \tag{3.5.48}$$

The four fixed points are

$$\begin{aligned} p_1 &= \{\theta = 0,\, y = y_-\}\,, \qquad & p_2 &= \{\theta = 0,\, y = y_+\}\,, \\ p_3 &= \{\theta = \pi,\, y = y_+\}\,, \qquad & p_4 &= \{\theta = \pi,\, y = y_-\}\,, \end{aligned} \tag{3.5.49}$$

where $y_\mp$ are the two middle roots of $Q(y)$, with $y_- < y_+$. These points define the four divisors

$$D_1 = \{y = y_-\}\,, \quad D_2 = \{\theta = 0\}\,, \quad D_3 = \{y = y_+\}\,, \quad D_4 = \{\theta = \pi\}\,, \tag{3.5.50}$$

and, with respect to the basis (3.5.48), the degenerate normalized Killing vectors and the orbifold labels are

$$\begin{aligned} &D_1 : \ \xi_1 = n_-\, e_1\,, \qquad m_1 = n_-\,, \qquad && D_2 : \ \xi_2 = e_2\,, \qquad && m_2 = 1\,, \\ &D_3 : \ \xi_3 = n_+\, e_1\,, \qquad m_3 = n_+\,, \qquad && D_4 : \ \xi_4 = e_2 - t\, e_1\,, \qquad && m_4 = 1\,. \end{aligned} \tag{3.5.51}$$

We can now construct the polytope in the standard fashion, by considering the four-dimensional conformally-rescaled metric $\mathrm{d}s^2 = \Gamma(y)\,\mathrm{d}s^2_{S^2\ltimes\Sigma_2}$ and the related symplectic two-form

$$\omega_{(2)} = \Gamma(y)\left[\frac{1}{3}\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\psi + \frac{y^{1/2}}{2P^{1/2}}\,\mathrm{d}y \wedge \Big(\mathrm{d}z - \frac{2}{3}\cos\theta\,\mathrm{d}\psi\Big)\right]. \tag{3.5.52}$$

When $\Gamma'(y) = \frac{y^{1/2}}{P^{1/2}}\Gamma(y)$, the two-form (3.5.52) is closed and can be written as

$$\omega_{(2)} = \mathrm{d}\psi \wedge \mathrm{d}\left[\frac{1}{3}\Gamma(y)\cos\theta\right] + \mathrm{d}\nu_2 \wedge \mathrm{d}\left[-\frac{2}{3t}\Gamma(y)\right]. \tag{3.5.53}$$

[14]Notice that here t is a positive integer, due to the different convention of [61] in defining ω_{S^2}.

From this expression we can derive the moment maps with respect to the basis (3.5.48)

$$\vec{\mu}(y,\theta) = -\frac{2}{3t}\Gamma(y)\left(1, \frac{t(1-\cos\theta)}{2}\right), \tag{3.5.54}$$

whose action on the fixed points (3.5.49) is

$$\begin{aligned} \vec{\mu}(p_1) &= \frac{2}{3t}\Gamma(y_-)\,(1,0)\,, \qquad & \vec{\mu}(p_2) &= \frac{2}{3t}\Gamma(y_+)\,(1,0)\,, \\ \vec{\mu}(p_3) &= \frac{2}{3t}\Gamma(y_+)\,(1,t)\,, \qquad & \vec{\mu}(p_4) &= \frac{2}{3t}\Gamma(y_-)\,(1,t)\,. \end{aligned} \tag{3.5.55}$$

Since we have chosen the same basis as in (3.5.3), the resultant polytope is similar to the one constructed in section 3.5.1. Indeed, it can be read off from figure 3.1a exchanging $p_1 \leftrightarrow p_3$, $p_2 \leftrightarrow p_4$ and $n_+ \leftrightarrow n_-$. The normal vectors $\vec{v}_a$ change accordingly with respect to equation (3.5.10), *i.e.* $\vec{v}_1 \leftrightarrow \vec{v}_3$ and $\vec{v}_2 \leftrightarrow \vec{v}_4$, together with $t \to -t$. The quantities we derived satisfy relation (3.5.14).

We now move to the $\mathrm{AdS}_3 \times \Sigma_1 \ltimes \Sigma_2$ solutions presented in section 3 of[15] [61]. The metric on the toric orbifold is given by

$$\mathrm{d}s^2_{\Sigma_1\ltimes\Sigma_2} = \frac{x}{f}\,\mathrm{d}x^2 + \frac{f}{36x^2}\,\mathrm{d}\psi^2 + \frac{y}{4Q}\,\mathrm{d}y^2 + \frac{Q}{P}\Big(\mathrm{d}z - \frac{1}{3}\left(1-\frac{\mathtt{a}}{x}\right)\mathrm{d}\psi\Big)^2, \tag{3.5.56}$$

where functions h_i, P and Q are the same as in (3.5.43) and

$$f(x) = 4x^3 - 9x^2 + 6\mathtt{a}x - \mathtt{a}^2\,. \tag{3.5.57}$$

The coordinate x ranges between x_- and x_+, the two smallest roots of $f(x)$ with $x_- < x_+$, and y lies between y_- and y_+ as before. The fibration is globally well-defined if

$$\frac{m_+m_-}{2\pi}\int_{\Sigma_1}\mathrm{d}\eta_{\Sigma_1} = t \in \mathbb{Z}\,, \qquad \eta_{\Sigma_1} \equiv \frac{2\pi}{\Delta z}\Big(\mathrm{d}z - \frac{1}{3}\left(1-\frac{\mathtt{a}}{x}\right)\mathrm{d}\psi\Big)\,, \tag{3.5.58}$$

which gives the relation

$$\frac{t}{m_+m_-} = -\frac{\mathtt{a}(x_+-x_-)}{3x_-x_+}\frac{\Delta\psi}{\Delta z}\,. \tag{3.5.59}$$

Explicitly the seven-dimensional solution is regular if [61]

$$t = \frac{-6(m_+-m_-)n_+n_-(n_+-n_-)}{[\mathtt{s}-(p_1+p_2)][\mathtt{s}+2(p_1+p_2)]}\,, \qquad \mathtt{s} = \sqrt{7(p_1^2+p_2^2)+2p_1p_2-6(n_+^2+n_-^2)}\,, \tag{3.5.60}$$

with again $(t,\mathtt{s}) \in \mathbb{Z}$.

Taking inspiration from section 3.5.1, we introduce

$$E_1 = \partial_{\nu_2}\,, \qquad E_2 = \partial_{\nu_1} + \frac{r_+ + r_-}{m_+ - m_-}\,\partial_{\nu_2}\,, \tag{3.5.61}$$

[15] In addition to $n_+ \leftrightarrow n_-$, we also exchanged $m_+ \leftrightarrow m_-$ with respect to [61]. Notice that again here $t < 0$.

where $\nu_1 = \frac{2\pi}{\Delta\psi}\psi$ and $r_\pm$ are two integers such that $r_+ m_+ + r_- m_- = t$, as in (3.5.25). In total there are four Killing vectors degenerating at the divisors $\hat{D}_1 = \{y = y_-\}$, $\hat{D}_2 = \{x = x_-\}$, $\hat{D}_3 = \{y = y_+\}$, $\hat{D}_4 = \{x = x_+\}$, which, in the basis (3.5.61), read

$$\xi_1 = n_- \, E_1 \,, \quad \xi_2 = m_- \, E_2 + r_+ \, E_1 \,, \quad \xi_3 = n_+ \, E_1 \,, \quad \xi_4 = m_+ \, E_2 - r_- \, E_1 \,. \tag{3.5.62}$$

These are formally the same Killing vectors given in (3.5.27). The "long" normal vectors can be obtained by means of (3.5.26)

$$\vec{v}_1 = (n_-, 0) \,, \quad \vec{v}_2 = (r_+, m_-) \,, \quad \vec{v}_3 = (-n_+, 0) \,, \quad \vec{v}_4 = (r_-, -m_+) \,. \tag{3.5.63}$$

Following the same arguments of section 3.5.1, we derive the labels $m_a = (n_-, 1, n_+, 1)$. This orbifold is characterized by the same stacky fan and labelled polytope of figure 3.3.

3.6 Entropy functions

We have now collected all the informations to put in practice the extremization procedure explained in 2.2. On a side, we have the entropy (and the gravitational central charge) computed explicitly from the solutions in section 3.4.2; on the other side we have extracted all the toric data we need in section 3.5.1 and 3.5.2.

3.6.1 Application to $\mathrm{AdS}_2 \times \Sigma_1 \ltimes \Sigma_2$ and $\mathrm{AdS}_2 \times \Sigma_g \ltimes \Sigma_2$

The first example to which we can apply the prescription presented in section 2.2 is the $\mathrm{AdS}_2 \times \Sigma_1 \ltimes \Sigma_2$ solution (3.3.18). We now briefly recollect some of the results needed in the construction. The toric orbifold $\mathbb{M}_4 = \Sigma_1 \ltimes \Sigma_2$ can be described by the toric data (see section 3.5.1)

$$\begin{aligned} &m_1 = n_- \,, && \vec{v}_1 = (1, 0) \,, && m_2 = 1 \,, && \vec{v}_2 = (r_+, m_-) \,, \\ &m_3 = n_+ \,, && \vec{v}_3 = (-1, 0) \,, && m_4 = 1 \,, && \vec{v}_4 = (r_-, -m_+) \,, \end{aligned} \tag{3.6.1}$$

with $r_+ m_+ + r_- m_- = t$ as in (3.5.25). The vectors $\vec{v}_a$ are $\mathbb{Z}^2$-valued, primitive (for $\gcd(t, m_\pm) = 1$) and ordered counter-clockwise, as required by the recipe. Keeping in mind all the observations about the divisors made in section 3.5.1, the figure 3.3a, the divisors (3.5.35) are

$$\begin{aligned} &&& m_a = (n_-, 1, n_+, 1) \,, \\ \gcd(t, m_\pm) = 1 : \quad &\hat{D}_1 = S_- = \Sigma_1 \,, && \hat{D}_2 = \frac{\Sigma_2}{\mathbb{Z}_{m_-}} \,, \quad \hat{D}_3 = S_+ = \Sigma_1 \,, \quad \hat{D}_4 = \frac{\Sigma_2}{\mathbb{Z}_{m_+}} \,. \end{aligned} \tag{3.6.2}$$

The physical fluxes defined in (2.2.2) read

$$\begin{aligned} \mathfrak{q}_i^1 &= \frac{g_c}{2\pi} \int_{D_1} F_i = \frac{g_c}{2\pi n_-} \int_{S_-} F_i = \frac{\mathfrak{s}_i^-}{n_-} \,, & \mathfrak{q}_i^2 &= \frac{g_c}{2\pi} \int_{D_2} F_i = \frac{g_c}{2\pi m_-} \int_{\Sigma_2} F_i = \frac{\mathfrak{t}_i}{m_-} \,, \\ \mathfrak{q}_i^3 &= \frac{g_c}{2\pi} \int_{D_3} F_i = \frac{g_c}{2\pi n_+} \int_{S_+} F_i = \frac{\mathfrak{s}_i^+}{n_+} \,, & \mathfrak{q}_i^4 &= \frac{g_c}{2\pi} \int_{D_4} F_i = \frac{g_c}{2\pi m_+} \int_{\Sigma_2} F_i = \frac{\mathfrak{t}_i}{m_+} \,, \end{aligned} \tag{3.6.3}$$

where $\mathfrak{s}_i^{\pm}$ and $\mathfrak{t}_i$ are given, respectively, in (3.4.17) and (3.4.3), and recall (2.1.47), namely $\int_{\hat{D}_a} \alpha_2 = m_a \int_{D_a} \alpha_2$. The intersection matrix D_{ab}, computed straightforwardly using (2.1.51), reads

$$D_{ab} = \begin{pmatrix} -\frac{t}{m_+ m_-}\frac{1}{n_-^2} & \frac{1}{m_-}\frac{1}{n_-} & 0 & \frac{1}{m_+}\frac{1}{n_-} \\ \frac{1}{m_-}\frac{1}{n_-} & 0 & \frac{1}{m_-}\frac{1}{n_+} & 0 \\ 0 & \frac{1}{m_-}\frac{1}{n_+} & \frac{t}{m_+ m_-}\frac{1}{n_+^2} & \frac{1}{m_+}\frac{1}{n_+} \\ \frac{1}{m_+}\frac{1}{n_-} & 0 & \frac{1}{m_+}\frac{1}{n_+} & 0 \end{pmatrix} = \hat{D}_{ab}\frac{1}{m_a m_b} \, . \tag{3.6.4}$$

Recalling the R-symmetry fluxes (3.4.4) and (3.4.19)

$$\mathfrak{t}_1 + \mathfrak{t}_2 = \frac{n_- + n_+}{n_- n_+} \, , \qquad \mathfrak{s}_1^{\pm} + \mathfrak{s}_2^{\pm} = \frac{m_+ - m_-}{m_- m_+} \pm \frac{t/n_{\pm}}{m_- m_+} \, , \tag{3.6.5}$$

and imposing (2.2.6) we obtain the vector of twists

$$\sigma^a = (+,+,+,-) \, . \tag{3.6.6}$$

Taking advantage of the “shift symmetry”, we impose $n_+ \mathfrak{p}_i^1 = n_- \mathfrak{p}_i^3$ and $m_+ \mathfrak{p}_i^2 = m_- \mathfrak{p}_i^4$. Therefore, the relation (2.2.3) yields the identifications

$$\frac{\mathfrak{p}_i^1}{n_-} = \frac{\mathfrak{p}_i^3}{n_+} = \frac{\mathfrak{t}_i}{2} \, , \qquad \frac{\mathfrak{p}_i^2}{m_-} = \frac{\mathfrak{p}_i^4}{m_+} = \frac{\mathfrak{s}_i^*}{2} \, , \qquad \mathfrak{s}_i^* \equiv \frac{\mathfrak{s}_i^+ + \mathfrak{s}_i^-}{2} \, . \tag{3.6.7}$$

In particular $\mathfrak{s}_1^*$ explicitly reads

$$\mathfrak{s}_1^* = \frac{t\chi_2}{m_+ m_-} \frac{\mathtt{x}^3 - 3\mu \mathtt{x}^2 + (3 - 2\mathtt{z})\mathtt{x} - 3\mu}{8\mathtt{x}^2} \, , \tag{3.6.8}$$

and $\mathfrak{s}_2^*$ can be obtained replacing of $\mathtt{z} \to -\mathtt{z}$. With this definition, $\mathfrak{s}_i^{\pm}$ can be written as

$$\mathfrak{s}_i^{\pm} = \mathfrak{s}_i^* \pm \frac{t}{2m_+ m_-} \mathfrak{t}_i \, . \tag{3.6.9}$$

System (2.2.7) can be solved to get the auxiliary vector

$$\vec{W} = \left(\frac{r_+ m_+ - r_- m_-}{2m_- m_+} \frac{n_+ - n_-}{2n_- n_+} - \frac{m_+ + m_-}{2m_- m_+} , \frac{n_+ - n_-}{2n_- n_+} \right) \, , \tag{3.6.10}$$

which gives the constraint (2.2.15)

$$\varphi_1 + \varphi_2 + \frac{n_+ - n_-}{2n_- n_+}\epsilon_1 + \left(\frac{m_+ + m_-}{2m_- m_+} - \frac{r_+ m_+ - r_- m_-}{m_- m_+} \frac{n_+ - n_-}{4n_- n_+} \right) \epsilon_2 = 2 \, . \tag{3.6.11}$$

The ingredients $\epsilon_{1,2}^a$ and Φ_i^a can be constructed by means of (2.2.13) and (2.2.12), whereas the off-shell free energy (2.2.10) reads

$$F(\varphi_i, \epsilon_i; \mathfrak{s}_i^{\pm}, \mathfrak{t}_i, m_\ell) = -\frac{\mathcal{F}_5(\Phi_i^1)}{d_{1,2}\,\epsilon_1^1 \epsilon_2^1} - \frac{\mathcal{F}_5(\Phi_i^2)}{d_{2,3}\,\epsilon_1^2 \epsilon_2^2} + \frac{\mathcal{F}_5(\Phi_i^3)}{d_{3,4}\,\epsilon_1^3 \epsilon_2^3} + \frac{\mathcal{F}_5(\Phi_i^4)}{d_{4,1}\,\epsilon_1^4 \epsilon_2^4} \, , \tag{3.6.12}$$

where the relative signs have been fixed using the prescription $\eta_5^a = \sigma^a \sigma^{a+1}$.

We can now proceed to the extremization of the off-shell free energy and, after some work, we get the critical values

$$\begin{aligned}\varphi_i^* &= \frac{2m_+m_-}{m_+-m_-}\Big(1+\eta\,\frac{m_++m_-}{2m_-m_+}\frac{2\pi}{\Delta\psi}\Big)\,\mathfrak{s}_i^*\,,\\ \epsilon_1^* &= -\frac{1}{m}\frac{2\pi}{\Delta z}+\eta\,\frac{2(r_++r_-)}{m_+-m_-}\frac{2\pi}{\Delta\psi}\,,\qquad \epsilon_2^*=-2\eta\,\frac{2\pi}{\Delta\psi}\,,\end{aligned} \tag{3.6.13}$$

where the sign ambiguity $\eta=\pm1$ arises by solving the equations over the complex numbers[16]. Inserting these values back into (3.6.12) we obtain

$$F(\varphi_i^*,\epsilon_i^*;\mathfrak{s}_i^\pm,\mathfrak{t}_i,m_a)=\frac{-\eta\sqrt{2}\sqrt{m_+^2+m_-^2}-(m_++m_-)}{2m_-m_+}\,F_{S^3\times\Sigma_2}\,, \tag{3.6.14}$$

where $F_{S^3\times\Sigma_2}$ is given in (3.4.24). In order to get a positive entropy we need to pick $\eta=-1$ and in this case the result agrees with the gravitational entropy (3.4.23).

We propose that the procedure we formulated can be extended also to non-toric four-dimensional orbifolds, for example when $\mathbb{B}=\Sigma_\mathrm{g}$. Whilst the genus g should be set to zero in all the extremization formulas (*i.e.* (2.2.7) and (2.2.6)), it has to appear explicitly in the fluxes (3.4.17). Finally, we have to set $m_\pm=1$ and we will take the same toric data extracted from S^2, equations (3.5.10) and (3.5.9). The fluxes $\mathfrak{p}_i^a$ are identified as in (3.6.7) and $\mathfrak{s}_i^*$ retains the same form, as follows from the observation made under (3.4.17). The constraint (2.2.15) and the vector of twist are

$$\varphi_1+\varphi_2+\frac{n_+-n_-}{2n_+n_-}\epsilon_1-t\,\frac{n_+-n_-}{4n_+n_-}\epsilon_2=2\,,\quad \sigma^a=(+,+,+,+)\,. \tag{3.6.15}$$

The off-shell free energy is extremized by

$$\varphi_i^*=\frac{\mathfrak{s}_i^*}{\mathrm{g}-1}\,,\qquad \epsilon_1^*=-\frac{1}{m}\frac{2\pi}{\Delta z}\,,\qquad \epsilon_2^*=0\,, \tag{3.6.16}$$

to which corresponds the critical value

$$F(\varphi_i^*,\epsilon_i^*;\mathfrak{s}_i^\pm,\mathfrak{t}_i,m_a)=(\mathrm{g}-1)\,F_{S^3\times\Sigma_2}\,. \tag{3.6.17}$$

Also in this case, the result returned by the extremization agrees with the entropy computed from the ten-dimensional supergravity solution (3.4.23).

3.6.2 Application to AdS$_3\times\Sigma_1\ltimes\Sigma_2$ and AdS$_3\times\Sigma_1\ltimes\Sigma_2$

The toric data of the $\Sigma_1\ltimes\Sigma_2$ toric orbifold are derived in section 3.5.2 and they are indeed equal to (3.6.1). The physical fluxes can be computed as in the previous example and, compactly, they read

$$\mathfrak{q}_i^a=\Big(\frac{\mathfrak{s}_i^-}{n_-},\frac{\mathfrak{t}_i}{m_-},\frac{\mathfrak{s}_i^+}{n_+},\frac{\mathfrak{t}_i}{m_+}\Big)\,, \tag{3.6.18}$$

[16]Generically, in presence of rotation we have to work with the complex numbers [47], therefore we continue to do so also in the static case.

where $\mathfrak{s}_1^-$ corresponds to the flux computed in (3.15) of [61], $\mathfrak{s}_2^-$ and $\mathfrak{s}_i^+$ can be derived from the former and $\mathfrak{t}_i = p_i/(n_- n_+)$. Using the gauge symmetry we can impose $n_+ \mathfrak{p}_i^1 = n_- \mathfrak{p}_i^3$ and $m_+ \mathfrak{p}_i^2 = m_- \mathfrak{p}_i^4$, which yields, again, to the identifications (3.6.7). In this case $\mathfrak{s}_i^*$ reads

$$\mathfrak{s}_i^* = \frac{t p_i[-6p_i + 4(n_+ + n_-) - \mathtt{s}]}{6m_+ m_- n_+ n_- (n_+ - n_-)} \,. \tag{3.6.19}$$

where $(t, \mathtt{s})$ have been introduced in (3.5.60). The fluxes $\mathfrak{s}_i^\pm$ are related to $\mathfrak{s}_i^*$ through

$$\mathfrak{s}_i^\pm = \mathfrak{s}_i^* \pm \frac{t}{2m_+ m_-} \mathfrak{t}_i \,. \tag{3.6.20}$$

The constraint (2.2.15) takes the same form of (3.6.11), whilst, as discussed in section 2.2, in $d = 6$ we take $\eta_6^a = +$. Remarkably, this implies that F is *quadratic* in the φ_i, as expected for the off-shell central charge of a two-dimensional SCFT [20]. Explicitly, this reads

$$\begin{aligned} F(\varphi_i, \epsilon_i; \mathfrak{s}_i^\pm, \mathfrak{t}_i, m_\ell) = -\Bigg[& \frac{\mathfrak{t}_2 \mathfrak{s}_1^* + \mathfrak{t}_1 \mathfrak{s}_2^*}{8} \Bigg(\mathfrak{t}_1 \mathfrak{t}_2 \epsilon_1^2 + \frac{(r_+^2 m_+^2 + r_-^2 m_-^2)\mathfrak{t}_1 \mathfrak{t}_2 + 2m_+^2 m_-^2 \mathfrak{s}_1^* \mathfrak{s}_2^*}{2m_+^2 m_-^2} \epsilon_2^2 \\ & - \frac{r_+ m_+ - r_- m_-}{m_+ m_-} \mathfrak{t}_1 \mathfrak{t}_2 \epsilon_1 \epsilon_2 \Bigg) + \frac{\mathfrak{t}_2 \mathfrak{s}_2^* \varphi_1^2 + \mathfrak{t}_1 \mathfrak{s}_1^* \varphi_2^2}{2} + (\mathfrak{t}_2 \mathfrak{s}_1^* + \mathfrak{t}_1 \mathfrak{s}_2^*)\varphi_1 \varphi_2 \\ & + \frac{2m_+ m_- \epsilon_1 - (r_+ m_+ - r_- m_-)\epsilon_2}{8m_+^2 m_-^2} \, t \, \mathfrak{t}_1 \mathfrak{t}_2 (\mathfrak{t}_2 \varphi_1 + \mathfrak{t}_1 \varphi_2) \Bigg] N^3 \,. \end{aligned} \tag{3.6.21}$$

The extremization procedure, realized by means of Lagrangian multipliers, gives the critical values

$$\begin{aligned} \varphi_i^* &= \frac{2m_+ m_-}{m_+ - m_-} \left(1 - \frac{m_- + m_+}{2m_+ m_-} \frac{2\pi}{\Delta\psi} \right) \mathfrak{s}_i^* \,, \\ \epsilon_1^* &= -\frac{4}{3} \frac{2\pi}{\Delta z} - \frac{2(r_+ + r_-)}{m_+ - m_-} \frac{2\pi}{\Delta\psi} \,, \qquad \epsilon_2^* = 2 \frac{2\pi}{\Delta\psi} \,, \end{aligned} \tag{3.6.22}$$

which, plugged into the off-shell central charge, gives

$$F(\varphi_i^*, \epsilon_i^*; \mathfrak{s}_i^\pm, \mathfrak{t}_i, m_a) = \frac{4(m_+ - m_-)^3}{3m_+ m_- (m_+^2 + m_+ m_- + m_-^2)} a_{4\mathrm{d}} \,, \tag{3.6.23}$$

where $a_{4\mathrm{d}}$ is the central charge of $d = 4$, $\mathcal{N} = 1$ SCFTs that arise from N M5-branes wrapped on a spindle [51]

$$a_{4\mathrm{d}} = \frac{3p_1^2 p_2^2 (\mathtt{s} + p_1 + p_2)}{8n_+ n_- (n_+ - p_1)(p_2 - n_+)[\mathtt{s} + 2(p_1 + p_2)]^2} N^3 \,. \tag{3.6.24}$$

This matches exactly the central charge of the $\mathrm{AdS}_3 \times \Sigma_1 \ltimes \Sigma_2$ system computed in [61]. In general, we expect that the anomaly polynomial computation of [61] and the prescription presented in this paper should be equivalent and connected by a suitable gauge choice and a possible redefinition of $\epsilon_{1,2}$, mixing the two related $U(1)$ isometries. Notice that the extremizing values are identical to the corresponding quantities for the $\mathrm{AdS}_2 \times \Sigma_1 \ltimes \Sigma_2$ solution, for $m = 3/4$.

It is now a matter of algebra to repeat the analysis for the solution $\mathrm{AdS}_3 \times \Sigma_{\mathrm{g}} \ltimes \mathbb{\Sigma}_2$, using the same toric data extracted from $\mathrm{AdS}_3 \times S^2 \ltimes \mathbb{\Sigma}_2$ in section 3.5.2

$$\begin{aligned} m_1 &= n_-\,, & \vec{v}_1 &= (-1,0)\,, & m_2 &= 1\,, & \vec{v}_2 &= (0,-1)\,, \\ m_3 &= n_+\,, & \vec{v}_3 &= (1,0)\,, & m_4 &= 1\,, & \vec{v}_4 &= (-t,1)\,. \end{aligned} \tag{3.6.25}$$

We take the physical fluxes to be as in (3.6.18), where $\mathfrak{t}_i = p_i/(n_- n_+)$ is unchanged and $\mathfrak{s}_i^{\pm}$ can be read from (4.7) of [61]. The gauge symmetry allows us to identify the $\mathfrak{p}_i^a$ as in equation (3.6.7) with also

$$\mathfrak{s}_i^{\pm} = \mathfrak{s}_i^* \pm \frac{t}{2}\mathfrak{t}_i\,, \quad \mathfrak{s}_i^* = \frac{t p_i[-6p_i + 4(n_+ + n_-) - \mathfrak{s}]}{6n_+ n_-(n_+ - n_-)}\,, \tag{3.6.26}$$

where $(t, \mathfrak{s})$ can be read from (3.5.47). The constraint (2.2.15) and the vector of twist are

$$\varphi_1 + \varphi_2 - \frac{n_+ - n_-}{2n_+ n_-}\epsilon_1 - t\,\frac{n_+ - n_-}{4n_+ n_-}\epsilon_2 = 2\,, \quad \sigma^a = (+,-,+,-)\,, \tag{3.6.27}$$

which is the same as in (3.6.15), with $t \mapsto -t$ and $\epsilon_{1,2} \mapsto -\epsilon_{1,2}$, due to the different conventions used in [61]. Taking all the $\eta_6^a = -$ and extremizing equation (2.2.17) with respect to φ_i, ϵ_i and Λ, we obtain the critical values

$$\varphi_i^* = -\frac{\mathfrak{s}_i^*}{\mathrm{g}-1}\,, \quad \epsilon_1^* = \frac{4}{3}\frac{2\pi}{\Delta z}\,, \quad \epsilon_2^* = 0\,, \tag{3.6.28}$$

as well as the off-shell central charge at the extremum

$$F(\varphi_i^*, \epsilon_i^*; \mathfrak{s}_i^{\pm}, \mathfrak{t}_i, m_a) = \frac{32}{3}(\mathrm{g}-1)\, a_{4\mathrm{d}}\,, \tag{3.6.29}$$

with $a_{4\mathrm{d}}$ given in (3.6.24). This result agrees with the central charge of the $\mathrm{AdS}_3 \times \Sigma_{\mathrm{g}} \ltimes \mathbb{\Sigma}_2$ system studied in [61]. The origin of the (different) choice of the η_6^a can be traced to the sign of the gauge potential of the five-dimensional solution that the authors of [61] uplifted to obtain the $\mathrm{AdS}_3 \times \Sigma_{\mathrm{g}} \ltimes \mathbb{\Sigma}_2$ backgrounds. The component along AdS_3 of their Killing spinors satisfy the corresponding Killing spinor equations, but with a minus, namely $\hat{\nabla}_{\hat{\mu}}\vartheta = -\frac{1}{2}\alpha_{\hat{\mu}}\vartheta$, which implies that in the two-dimensional dual SCFTs $(2,0)$ supersymmetries are preserved, instead of $(0,2)$ as in our case. This fact leads to a different sign in the extraction of the central charge from the anomaly polynomial. See the discussion under (2.2.13). Notice that, again, the extremizing values are the same as for the $\mathrm{AdS}_s \times \Sigma_{\mathrm{g}} \ltimes \mathbb{\Sigma}_2$ for $m = 3/4$.

3.7 Summary of the chapter

In this chapter, we have conducted a detailed study of a six-dimensional background comprising two gauge fields A_i, a two-form field B, two scalars X_i, and a metric $g_{\mu\nu}$, describing the topology $\mathrm{AdS}_2 \times \mathbb{M}_4^{(1,2)}$, where $\mathbb{M}_4^{(1,2)}$ represents four-dimensional toric orbifolds. In the first case, $\mathbb{M}_4^{(1)} = \Sigma_{\mathrm{g}} \ltimes \mathbb{\Sigma}_2$ is a non-trivial fibration of a spindle over a Riemann surface with genus $\mathrm{g} > 1$. In the second case, $\mathbb{M}_4^{(2)} = \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}$, the base is another spindle. After presenting the local form of the solutions and demonstrating the supersymmetry of the $\mathrm{AdS}_2 \times \mathbb{M}_4^{(1,2)}$ backgrounds in each case (section 3.3), we addressed the global regularity conditions in both $D = 6$ and in the ten-dimensional massive type IIA supergravity. This analysis enabled us, in section 3.4.2, to compute the entropies of the putative black holes whose near-horizon geometry is $\mathrm{AdS}_2 \times \mathbb{M}_4^{(1,2)}$ from a gravitational perspective. The extraction of the toric data for this system, as discussed in section 3.5, was a multifaceted process. We carefully examined the orbifold geometry of $\mathbb{M}_4^{(1,2)}$, emphasizing the role and meaning of the labels, as well as the significant distinction between the ramification divisors D_a and the branch divisors $\hat{D}_a$. Finally, in section 3.6, we tested our off-shell free energy conjecture on the $\mathrm{AdS}_2 \times \mathbb{M}_4^{(1,2)}$ system, finding perfect agreement with the gravitational computations. We also extended this conjecture to the analogous $\mathrm{AdS}_3 \times \mathbb{M}_4^{(1,2)}$ system in $D = 7$, previously studied in [61], with the toric data extracted in section 3.5.2.

In the next chapter, we will construct a more sophisticated supergravity solution in both $D = 6$ and $D = 7$, which lacks a simple fibration structure. Consequently, the analysis will be significantly more involved. Nonetheless, our off-shell free energy conjecture will again prove successful, allowing us to recover the gravitational entropy and central charge.

Chapter 4

Supergravity solutions with more general quadrilaterals

In the previous chapter we have studied in the details the toric orbifold $\mathrm{AdS}_{(D-4)} \times \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$ in $D = 6, 7$, both from the geometrical point of view and as solutions of D-dimensional gauged supergravities. Upon uplifting them to massive Type IIA supergravity in $d = 10$ or M-theory in $d = 11$, these solution can be interpreted as arising from a stack of D4 or M5-branes wrapped on $\mathbb{M}_4 = \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$. The orbifold $\mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$ is not, of course, the most general geometry with $U(1)^2$ symmetry. For example, the polytope representing its toric properties in figure 3.3a has two parallel sides, as a consequence of the fibration structure present in the metric (3.3.18).

In this chapter we consider more general four-dimensional toric orbifolds $\mathbb{M}_4$, that should come as near-horizon geometries of some black strings in $D = 6, 7$. We will refer to them as *quadrilaterals*, because they are characterized (again) by certain labelled two-dimensional polytopes with four edges [87, 133]. Remarkably, there exists a specific limit which brings the new solutions to the form $\mathrm{AdS}_{D-4} \times \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$, recovering also the fermionic contents (see appendix A of [2]). Recalling that there exists another limit which brings $\mathbb{M}_4^{(2)} = \mathbb{\Sigma}_1 \ltimes \mathbb{\Sigma}_2$ to $\mathbb{M}_4^{(1)} = \Sigma_{\mathrm{g}} \ltimes \mathbb{\Sigma}_2$ (appendix B of [1]), the new solutions we present momentarily are the more involved and complete four-dimensional (known) toric orbifolds available in literature.

Their local form can be obtained as an analytic continuation of the asymptotically AdS_D, non-extremal, charged and rotating[1] black strings constructed in [134], with the addition of a NUT parameter N_y. The solutions of our interest can then be obtained performing an analytic continuation, which results in the presence of an AdS_{D-4} factor instead of the S^{D-4}. In particular, the time and radial coordinates t and r as well as the mass parameter M of [134] are identified as

$$t = \psi \,, \quad r = \mathrm{i}\, x \,, \quad M = \mathrm{i}^{D-5} N_x \,. \tag{4.0.1}$$

with ψ an angular coordinate. Similarly, it is possible to retrieve the seven-dimensional solutions presented in [135], equivalent to the ones of [134], but written in generalized Boyer–Lindquist coordinates. Moreover, the solutions presented here and the Kerr–NUT–AdS black holes constructed in [136] overlap and agree in the intersecting region of their

[1] It is crucial that they have a single non-zero angular momentum, so that the solutions contain a round $(D-4)$-sphere.

parameters. Indeed, the uncharged subfamily of the former coincides with the $r \to 0$ limit of the latter, when the mass parameter is set to zero and only one angular momentum is retained. Lastly, in both six and seven dimensions, the AdS vacuum solution with radius $L_{\mathrm{AdS}} = 1/g$ is recovered setting $N_y = N_x = 0$. Here, the constant g is related to the gauge coupling as $g = 2g_c/3$ in $D = 6$ and as $g = g_c/2$ in $D = 7$.

4.1 Local form of the solution

The background we consider takes the form

$$\mathrm{d}s^2 = (H_1 H_2)^{1/(D-2)} \left(\frac{x^2 y^2}{a^2} \, \mathrm{d}s^2_{\mathrm{AdS}_{D-4}} + \mathrm{d}s^2_{\mathbb{M}_4} \right), \tag{4.1.1}$$

where $\mathrm{d}s^2_{\mathrm{AdS}_{D-4}}$ denotes the unit radius metric on AdS_{D-4} and we defined the four-dimensional (orbifold) metric

$$\begin{aligned}
\mathrm{d}s^2_{\mathbb{M}_4} =& \frac{(H_1 H_2)^{-1}}{\Xi^2 (x^2 - y^2)} \Big[(V_y^2 \Delta_x - V_x^2 \Delta_y) \, \mathrm{d}\psi^2 + (\widetilde{V}_y^2 \Delta_x - \widetilde{V}_x^2 \Delta_y) \, \mathrm{d}\phi^2 \\
& - \frac{4 c_1 c_2 \tilde{c}_1 \tilde{c}_2}{a^2} \left(\frac{N_y \Delta_x}{y^{D-5}} - \frac{N_x \Delta_y}{x^{D-5}} \right) \mathrm{d}\psi \, \mathrm{d}\phi \Big] + \frac{y^2 - x^2}{\Delta_y} \, \mathrm{d}y^2 + \frac{x^2 - y^2}{\Delta_x} \, \mathrm{d}x^2 \, .
\end{aligned} \tag{4.1.2}$$

We take the spacetime orientation as induced by the coordinate ordering $(\mathrm{AdS}, y, \phi, x, \psi)$. The scalar fields are

$$X_i = (H_1 H_2)^{(D-3)/2(D-2)} H_i^{-1} \, , \tag{4.1.3}$$

the gauge potentials are given by

$$\begin{aligned}
A_i =& \frac{2 N_y s_i c_i \tilde{c}_i}{\Xi (x^2 - y^2) H_i y^{D-5}} \left(\frac{\tilde{c}_1 \tilde{c}_2}{\tilde{c}_i^2} \, V_x^{(i)} \, \mathrm{d}\psi - \frac{c_1 c_2}{c_i^2} \, \widetilde{V}_x^{(i)} \, \mathrm{d}\phi \right) \\
& - \frac{2 N_x s_i c_i \tilde{c}_i}{\Xi (x^2 - y^2) H_i x^{D-5}} \left(\frac{\tilde{c}_1 \tilde{c}_2}{\tilde{c}_i^2} \, V_y^{(i)} \, \mathrm{d}\psi - \frac{c_1 c_2}{c_i^2} \, \widetilde{V}_y^{(i)} \, \mathrm{d}\phi \right),
\end{aligned} \tag{4.1.4}$$

and the $(D-4)$-form is

$$\begin{aligned}
B =& - \frac{4 s_1 s_2 (N_y x^3 - N_x y^3)}{a^2 (x^2 - y^2)} \, \mathrm{vol}\mathrm{AdS}_2 \, , && (D = 6) \, , \\
B =& - \frac{2 s_1 s_2 (N_y x^4 - N_x y^4)}{a^3 (x^2 - y^2)} \, \mathrm{vol}\mathrm{AdS}_3 - \frac{2 N_y g s_1 s_2 x}{\Xi a^2 y^2} \, \mathrm{d}\psi \wedge \mathrm{d}\phi \wedge \mathrm{d}x \\
& - \frac{2 N_x g s_1 s_2 y}{\Xi a^2 x^2} \, \mathrm{d}\psi \wedge \mathrm{d}\phi \wedge \mathrm{d}y \, , && (D = 7) \, .
\end{aligned} \tag{4.1.5}$$

Apart from the constants

$$\Xi = 1 - a^2 g^2 \, , \quad s_i = \sinh \delta_i \, , \quad c_i = \cosh \delta_i \, , \quad \tilde{c}_i = \sqrt{1 + a^2 g^2 s_i^2} \, , \tag{4.1.6}$$

we have a number of the metric functions, which can be summarized as follows

$$
\begin{aligned}
\Delta_y &= -y^2 + a^2 - \frac{2N_y}{y^{D-5}} + g^2\Big(y^2 - \frac{2N_y s_1^2}{y^{D-5}}\Big)\Big(y^2 - \frac{2N_y s_2^2}{y^{D-5}}\Big) - a^2 g^2 y^2 - \frac{2N_y a^2 g^2 s_1^2 s_2^2}{y^{D-5}}\,,\\
\Delta_x &= -x^2 + a^2 - \frac{2N_x}{x^{D-5}} + g^2\Big(x^2 - \frac{2N_x s_1^2}{x^{D-5}}\Big)\Big(x^2 - \frac{2N_x s_2^2}{x^{D-5}}\Big) - a^2 g^2 x^2 - \frac{2N_x a^2 g^2 s_1^2 s_2^2}{x^{D-5}}\,,\\
V_y^2 &= V_y^{(1)} V_y^{(2)}\,, \qquad \widetilde{V}_y^2 = \widetilde{V}_y^{(1)}\widetilde{V}_y^{(2)}\,, \qquad V_x^2 = V_x^{(1)} V_x^{(2)}\,, \qquad \widetilde{V}_x^2 = \widetilde{V}_x^{(1)}\widetilde{V}_x^{(2)}\,,\\
V_y^{(i)} &= 1 - g^2\Big(y^2 - \frac{2N_y s_i^2}{y^{D-5}}\Big)\,, \qquad \widetilde{V}_y^{(i)} = 1 - \frac{1}{a^2}\Big(y^2 - \frac{2N_y s_i^2}{y^{D-5}}\Big)\,,\\
V_x^{(i)} &= 1 - g^2\Big(x^2 - \frac{2N_x s_i^2}{x^{D-5}}\Big)\,, \qquad \widetilde{V}_x^{(i)} = 1 - \frac{1}{a^2}\Big(x^2 - \frac{2N_x s_i^2}{x^{D-5}}\Big)\,,\\
H_i &= 1 + \frac{2s_i^2}{x^2 - y^2}\Big(\frac{N_y}{y^{D-5}} - \frac{N_x}{x^{D-5}}\Big)\,, .
\end{aligned} \tag{4.1.7}
$$

In our discussion we shall assume $g_c > 0$ in both theories, therefore $g > 0$. The background is completely determined by a real parameter a, two charges $\delta_{1,2}$ and two parameters N_y and N_x (corresponding to the NUT and the mass of the Lorentzian solution). When $x = 0$ or $y = 0$ a curvature singularity is encountered, therefore the range of definition of these two coordinates will have to avoid the origin of the real axis. Moreover, in the rest of the paper we shall consider these two ranges to be disjoint, in order to prevent the occurrence of a singularity when $x = y$. This solution possesses two symmetries, under inversions

$$
x \leftrightarrow y\,, \quad N_x \leftrightarrow N_y\,, \tag{4.1.8}
$$

and under a specific rescaling

$$
\begin{aligned}
&y \to \gamma\, y\,, && x \to \gamma\, x\,, && \psi \to \gamma\, \phi\,, && \phi \to \gamma\, \psi\,,\\
&a \to \gamma^2 a\,, && N_y \to \gamma^{D-1} N_y\,, && N_x \to \gamma^{D-1} N_x\,, && s_i \to \gamma^{-1} s_i\,,
\end{aligned} \tag{4.1.9}
$$

with $\gamma = \pm 1/(ag)$, which connects solutions with positive and negative values of Ξ.

Equal charges

The complicated form of the (ψ, ϕ) part of the metric (4.1.2) makes the study of the general solution computationally involved. For this reason, we now set $\delta_1 = \delta_2 \equiv \delta$, for which the charges are equal. In this simplified case, the metric on the toric orbifold (4.1.2) acquires the following diagonal structure

$$
\begin{aligned}
\mathrm{d}s^2_{\mathbb{M}_4} =& \frac{1}{\Xi^2 H^2}\left[\frac{-\Delta_y}{x^2 - y^2}\big(V_x\,\mathrm{d}\psi - \widetilde{V}_x\,\mathrm{d}\phi\big)^2 + \frac{\Delta_x}{x^2 - y^2}\big(V_y\,\mathrm{d}\psi - \widetilde{V}_y\,\mathrm{d}\phi\big)^2\right]\\
&+ \frac{x^2 - y^2}{-\Delta_y}\,\mathrm{d}y^2 + \frac{x^2 - y^2}{\Delta_x}\,\mathrm{d}x^2\,,
\end{aligned} \tag{4.1.10}
$$

where we defined the functions $H \equiv H_1 = H_2$, $V_\bullet \equiv V_\bullet^{(1)} = V_\bullet^{(2)}$ and $\widetilde{V}_\bullet \equiv \widetilde{V}_\bullet^{(1)} = \widetilde{V}_\bullet^{(2)}$. This class of solutions is characterized by having equal scalar fields and equal gauge fields, where the latter now read

$$A_1 = A_2 = \frac{2sc\tilde{c}}{\Xi(x^2-y^2)H}\left[\frac{N_y}{y^{D-5}}\big(V_x\,\mathrm{d}\psi - \widetilde{V}_x\,\mathrm{d}\phi\big) - \frac{N_x}{x^{D-5}}\big(V_y\,\mathrm{d}\psi - \widetilde{V}_y\,\mathrm{d}\phi\big)\right] + \alpha\mathrm{d}\phi + \beta\mathrm{d}\psi\,, \tag{4.1.11}$$

with $s = \sinh\delta$, $c = \cosh\delta$ and $\tilde{c} = \sqrt{1 + a^2g^2s^2}$ and $(\alpha, \beta) \in \mathbb{R}$ two constants to be fixed later. The metric has a correct Lorentzian signature if and only if

$$\frac{\Delta_x}{x^2-y^2} > 0\,, \qquad \frac{\Delta_y}{y^2-x^2} > 0\,, \qquad H(x,y) > 0\,, \tag{4.1.12}$$

in two closed intervals $[x_-, x_+]$ and $[y_-, y_+]$ not intersecting and not containing the curvature singularities in $x = 0$ and $y = 0$. Without loss of generality we restrict to positive values of x and y. All the details of the signature analysis are reported in section 3.3 of [2] and we do not report them here, since they are straightforward but tedious.

4.1.1 Supersymmetry of the $D = 6$ solutions with equal charges

In [134], the supersymmetry conditions was obtained as $ags_1s_2 = \pm 1$, with δ_1 and δ_2 of the same sign, either positive or negative. Our solution is the analytic continuation of the black holes of [134], with the addition of NUT. Since the mass parameter of the latter does not appear explicitly in the supersymmetry condition, we expect that the introduction of a NUT parameter does not affect this constraint. Furthermore, since the BPS condition is not altered by (4.0.1), we conjecture that the solution we presented is supersymmetric when $ags_1s_2 = \pm 1$, with δ_1 and δ_2 of the same sign. We now confirm this expectation, at least in the equal charge case in $D = 6$, by explicitly constructing the six-dimensional Killing spinor.

We start defining the orthonormal frame on $\mathbb{M}_4$

$$\begin{aligned} \hat{e}^1 &= \sqrt{\frac{x^2-y^2}{-\Delta_y}}\,\mathrm{d}y\,, \qquad & \hat{e}^2 &= \frac{1}{\Xi\,H}\sqrt{\frac{-\Delta_y}{x^2-y^2}}\,\big(V_x\,\mathrm{d}\psi - \widetilde{V}_x\,\mathrm{d}\phi\big)\,, \\ \hat{e}^3 &= \sqrt{\frac{x^2-y^2}{\Delta_x}}\,\mathrm{d}x\,, \qquad & \hat{e}^4 &= \frac{1}{\Xi\,H}\sqrt{\frac{\Delta_x}{x^2-y^2}}\,\big(V_y\,\mathrm{d}\psi - \widetilde{V}_y\,\mathrm{d}\phi\big)\,, \end{aligned} \tag{4.1.13}$$

and the six-dimensional one

$$e^{\hat{a}} = H^{1/4}\,\frac{xy}{a}\,\hat{e}^{\hat{a}}\,, \qquad e^{\hat{\imath}+1} = H^{1/4}\,\hat{e}^{\hat{\imath}}\,. \tag{4.1.14}$$

The coordinates on AdS_2 and $\mathbb{M}_4$ are denoted as $x^{\hat{\mu}}$ and $x^{\hat{\alpha}}$, respectively. When the charges are equal, equation (3.1.6) automatically vanishes. Instead, equation (3.1.5) reduces to the constraint

$$\sqrt{\frac{\Delta_y}{y^2-x^2}}\,\Gamma^3\epsilon^A + \sqrt{\frac{\Delta_x}{x^2-y^2}}\,\Gamma^5\epsilon^A + \mathrm{i}\,\frac{c\tilde{c}}{s}\,(\sigma^3)^A{}_B\epsilon^B - gH\big(y\,\Gamma^{23} + x\,\Gamma^{45}\big)\epsilon^A = 0\,, \tag{4.1.15}$$

where, we recall, $g = m = 2g_c/3$. Writing this algebraic equation schematically as $M \cdot \epsilon = 0$, we need to impose $\det(M) = 0$ in order to have non-trivial solutions, *i.e.* $\epsilon \neq 0$. This necessary condition is satisfied for any value of x and y if and only if

$$(ags^2)^2 = 1 \implies ags^2 = -\kappa\,, \qquad \kappa = \pm 1\,, \tag{4.1.16}$$

which agree with [134]. Multiplying (4.1.15) by its complex conjugate we obtain

$$\frac{1}{y}\sqrt{\frac{\Delta_y}{y^2-x^2}}\,\Gamma^2\epsilon^A+\frac{1}{x}\sqrt{\frac{\Delta_x}{x^2-y^2}}\,\Gamma^4\epsilon^A+gH\epsilon^A+\Big[g\big(y\,\partial_x H+x\,\partial_y H\big)-\frac{\kappa\,a}{xy}\Big]\Gamma^{2345}\epsilon^A=0\,, \tag{4.1.17}$$

and, thanks to this equation, the AdS_2 components of (3.1.4) can be written in a simple fashion as

$$\partial_{\hat\mu}\epsilon^A+\frac{1}{4}\,\hat\omega_{\hat\mu}{}^{\hat a\hat b}\,\Gamma_{\hat a\hat b}\epsilon^A+\frac{\kappa}{2}\,\hat e_{\hat\mu}^{\hat c}\,\Gamma_{\hat c}{}^{2345}\epsilon^A=0\,, \tag{4.1.18}$$

where $\hat\omega_{\hat\mu}{}^{\hat a\hat b}$ is the spin connection on AdS_2. The components along the coordinates y and x read

$$\begin{aligned}\partial_y\epsilon^A+\frac{3}{8}H^{-1}\partial_y H\,\epsilon^A+\frac{1}{2}\sqrt{\frac{y^2-x^2}{\Delta_y}}\,\Gamma^2\bigg[&g\,\partial_y(yH)-gx\,\partial_y H\,\Gamma^{2345}\\&+\sqrt{\frac{\Delta_x}{x^2-y^2}}\,\frac{1}{x^2-y^2}\big(y\,\Gamma^{235}+x\,\Gamma^4\big)\bigg]\epsilon^A=0\,,\end{aligned} \tag{4.1.19}$$

$$\begin{aligned}\partial_x\epsilon^A+\frac{3}{8}H^{-1}\partial_x H\,\epsilon^A+\frac{1}{2}\sqrt{\frac{x^2-y^2}{\Delta_x}}\,\Gamma^4\bigg[&g\,\partial_x(xH)-gy\,\partial_x H\,\Gamma^{2345}\\&-\sqrt{\frac{\Delta_y}{y^2-x^2}}\,\frac{1}{x^2-y^2}\big(y\,\Gamma^{2}+x\,\Gamma^{345}\big)\bigg]\epsilon^A=0\,,\end{aligned} \tag{4.1.20}$$

whilst the remaining components along the angular directions ϕ and ψ are, respectively,

$$\begin{aligned}\partial_\phi\epsilon^A=&\,\frac{3ig}{2}\bigg[\alpha+\frac{2sc\tilde c(N_xy\widetilde V_y-N_yx\widetilde V_x)}{\Xi\,xy(x^2-y^2)H}\bigg](\sigma^3)^A{}_B\epsilon^B+\frac{1}{2\Xi\,H}\Bigg\{\sqrt{\frac{-\Delta_x\,\Delta_y}{(x^2-y^2)^2}}\,\frac{H}{a^2}\big(y\,\Gamma^{25}-x\,\Gamma^{34}\big)\\&+g\widetilde V_x\sqrt{\frac{\Delta_y}{y^2-x^2}}\big[\partial_y(yH)\,\Gamma^3+x\,\partial_y H\,\Gamma^{245}\big]+g\widetilde V_y\sqrt{\frac{\Delta_x}{x^2-y^2}}\big[\partial_x(xH)\,\Gamma^5+y\,\partial_x H\,\Gamma^{234}\big]\\&-\bigg[\frac{x(\Delta_y\widetilde V_x-\Delta_x\widetilde V_y)}{(x^2-y^2)^2}+\frac{\Delta_x'\widetilde V_y}{2(x^2-y^2)}\bigg]\Gamma^{45}+\bigg[\frac{y(\Delta_y\widetilde V_x-\Delta_x\widetilde V_y)}{(x^2-y^2)^2}+\frac{\Delta_y'\widetilde V_x}{2(x^2-y^2)}\bigg]\Gamma^{23}\Bigg\}\,\epsilon^A\,,\end{aligned} \tag{4.1.21}$$

$$\begin{aligned}\partial_\psi\epsilon^A=&\,\frac{3ig}{2}\bigg[\beta-\frac{2sc\tilde c(N_xyV_y-N_yxV_x)}{\Xi\,xy(x^2-y^2)H}\bigg](\sigma^3)^A{}_B\epsilon^B-\frac{1}{2\Xi\,H}\Bigg\{\sqrt{\frac{-\Delta_x\,\Delta_y}{(x^2-y^2)^2}}\,g^2H\big(y\,\Gamma^{25}-x\,\Gamma^{34}\big)\\&+gV_x\sqrt{\frac{\Delta_y}{y^2-x^2}}\big[\partial_y(yH)\,\Gamma^3+x\,\partial_y H\,\Gamma^{245}\big]+gV_y\sqrt{\frac{\Delta_x}{x^2-y^2}}\big[\partial_x(xH)\,\Gamma^5+y\,\partial_x H\,\Gamma^{234}\big]\\&-\bigg[\frac{x(\Delta_yV_x-\Delta_xV_y)}{(x^2-y^2)^2}+\frac{\Delta_x'V_y}{2(x^2-y^2)}\bigg]\Gamma^{45}+\bigg[\frac{y(\Delta_yV_x-\Delta_xV_y)}{(x^2-y^2)^2}+\frac{\Delta_y'V_x}{2(x^2-y^2)}\bigg]\Gamma^{23}\Bigg\}\,\epsilon^A\,.\end{aligned} \tag{4.1.22}$$

Assuming (4.1.16) for both $D=6,7$, we can decompose Δ_x and Δ_y as

$$\Delta_x=\Delta_x^+\,\Delta_x^-\,,\qquad\Delta_y=\Delta_y^+\Delta_y^-\,, \tag{4.1.23}$$

where

$$\begin{aligned}\Delta_x^\pm &= g\Big(x^2 - \frac{2N_x s^2}{x^{D-5}}\Big) - \kappa a \pm (1-\kappa a g)x\,,\\ \Delta_y^\pm &= g\Big(y^2 - \frac{2N_y s^2}{y^{D-5}}\Big) - \kappa a \pm (1-\kappa a g)y\,.\end{aligned} \tag{4.1.24}$$

Considering, *e.g.*, Δ_x, every root of this function is necessarily a root of either Δ_x^+ or Δ_x^-, therefore, given a generic root $x_\pm$ of Δ_x, we have

$$x_\pm^2 - \frac{2N_x s^2}{x_\pm^{D-5}} = \frac{\kappa a}{g} - \tau^{(x_\pm)}\frac{1-\kappa a g}{g}\,x_\pm\,, \tag{4.1.25}$$

with $\tau^{(x_\pm)} = \pm$ according to the fact that $x_\pm$ is a root of $\Delta_x^\pm$, respectively. Thanks to this relation and the corresponding one for $y_\pm$, we can obtain simplified expressions for many functions, when evaluated at one of the roots $x_\pm$ or $y_\pm$. In particular, the function H reads

$$(x_\pm^2 - y_\pm^2)H(x_\pm, y_\pm) = -\frac{1-\kappa a g}{g}\big(\tau^{(x_\pm)}x_\pm - \tau^{(y_\pm)}y_\pm\big)\,, \tag{4.1.26}$$

while the derivatives of Δ_x and Δ_y become, collectively,

$$\Delta_z'(z_\pm) = -2\tau^{(z_\pm)}(1-\kappa a g)\big[(D-3)g z_\pm^2 + \tau^{(z_\pm)}(D-4)(1-\kappa a g)z_\pm - \kappa(D-5)a\big]\,, \tag{4.1.27}$$

where z can be either x or y.

With all the equations made explicit, we can now proceed to decompose the six-dimensional gamma matrices. In particular, we adopt the following decomposition

$$\Gamma^{\hat a} = \beta^{\hat a}\otimes\gamma_*\,,\qquad \Gamma^{\hat\imath+1} = I_2\otimes\gamma^{\hat\imath}\,, \tag{4.1.28}$$

where $\beta^{\hat a}$ are the (Lorentzian) gamma matrices in $D=2$ and $\gamma^{\hat\imath}$ are the (Euclidean) gamma matrices in $D=4$. The related chiral matrices are $\beta_* = -\beta^0\beta^1$ and $\gamma_* = -\gamma^1\gamma^2\gamma^3\gamma^4$, respectively. As a consequence, the six-dimensional matrices $\mathcal{B}_6$ and Γ_* decompose as[2]

$$\mathcal{B}_6 = \mathcal{B}_2\otimes(\mathcal{B}_4\gamma_*)\,,\qquad \Gamma_* = \beta_*\otimes\gamma_*\,. \tag{4.1.29}$$

The ansatz for the six-dimensional Killing spinor is

$$\epsilon^A = \vartheta_{\mathrm{AdS}_2}\otimes\zeta^A_{\mathbb{M}_4}\,, \tag{4.1.30}$$

where $\vartheta = \vartheta(x^{\hat\mu})$ is a Majorana spinor on AdS_2, hence $\vartheta^* = \mathcal{B}_2\vartheta$, and $\zeta^A = \zeta^A(x^{\hat\alpha})$ are two four-component spinors defined on $\mathbb{M}_4$. In this way, equation (4.1.18) reduces to

$$\Big(\hat\nabla_{\hat\mu}\vartheta - \frac{\kappa}{2}\,\beta_{\hat\mu}\vartheta\Big)\otimes\zeta^A = 0 \implies \hat\nabla_{\hat\mu}\vartheta = \frac{\kappa}{2}\,\beta_{\hat\mu}\vartheta\,, \tag{4.1.31}$$

which implies that ϑ must be a Killing spinor on AdS_2. We see that κ, which enters in the supersymmetry constraint, reflects in the chirality of the spinor ϑ on AdS_2.

Employing the decompositions (4.1.28) and (4.1.30), equation (4.1.15) becomes

$$\sqrt{\frac{\Delta_y}{y^2-x^2}}\,\gamma^2\zeta^A + \sqrt{\frac{\Delta_x}{x^2-y^2}}\,\gamma^4\zeta^A + \mathrm{i}\,\frac{c\tilde c}{s}\,(\sigma^3)^A{}_B\zeta^B - gH\big(y\,\gamma^{12} + x\,\gamma^{34}\big)\zeta^A = 0\,. \tag{4.1.32}$$

[2] $\mathcal{B}_4$ is related to the four-dimensional charge conjugation matrix $\mathcal{C}_4$ by $\mathcal{B}_4 = (\mathcal{C}_4)^t$.

Equations (4.1.19) and (4.1.20) boil down to

$$\begin{aligned}\partial_y\big(H^{3/8}\zeta^A\big)+\frac{1}{2}\sqrt{\frac{y^2-x^2}{\Delta_y}}\,\gamma^1\bigg[g\,\partial_y(yH)+gx\,\partial_yH\,\gamma_*\\+\sqrt{\frac{\Delta_x}{x^2-y^2}}\,\frac{1}{x^2-y^2}\big(y\,\gamma^{124}+x\,\gamma^3\big)\bigg]\big(H^{3/8}\zeta^A\big)=0\,,\end{aligned}\tag{4.1.33}$$

$$\begin{aligned}\partial_x\big(H^{3/8}\zeta^A\big)+\frac{1}{2}\sqrt{\frac{x^2-y^2}{\Delta_x}}\,\gamma^3\bigg[g\,\partial_x(xH)+gy\,\partial_xH\,\gamma_*\\-\sqrt{\frac{\Delta_y}{y^2-x^2}}\,\frac{1}{x^2-y^2}\big(y\,\gamma^1+x\,\gamma^{234}\big)\bigg]\big(H^{3/8}\zeta^A\big)=0\,,\end{aligned}\tag{4.1.34}$$

while (4.1.21) and (4.1.22) give

$$\begin{aligned}\partial_\phi\zeta^A=&\,\frac{3ig}{2}\bigg[\alpha+\frac{2sc\tilde{c}(N_xy\widetilde{V}_y-N_yx\widetilde{V}_x)}{\Xi\,xy(x^2-y^2)H}\bigg](\sigma^3)^A{}_B\zeta^B+\frac{1}{2\Xi\,H}\bigg\{\sqrt{\frac{-\Delta_x\,\Delta_y}{(x^2-y^2)^2}}\,\frac{H}{a^2}\big(y\,\gamma^{14}-x\,\gamma^{23}\big)\\&+g\widetilde{V}_x\sqrt{\frac{\Delta_y}{y^2-x^2}}\big[\partial_y(yH)\,\gamma^2+x\,\partial_yH\,\gamma^{134}\big]+g\widetilde{V}_y\sqrt{\frac{\Delta_x}{x^2-y^2}}\big[\partial_x(xH)\,\gamma^4+y\,\partial_xH\,\gamma^{123}\big]\\&-\bigg[\frac{x(\Delta_y\widetilde{V}_x-\Delta_x\widetilde{V}_y)}{(x^2-y^2)^2}+\frac{\Delta_x'\widetilde{V}_y}{2(x^2-y^2)}\bigg]\gamma^{34}+\bigg[\frac{y(\Delta_y\widetilde{V}_x-\Delta_x\widetilde{V}_y)}{(x^2-y^2)^2}+\frac{\Delta_y'\widetilde{V}_x}{2(x^2-y^2)}\bigg]\gamma^{12}\bigg\}\zeta^A\,,\end{aligned}\tag{4.1.35}$$

$$\begin{aligned}\partial_\psi\zeta^A=&\,\frac{3ig}{2}\bigg[\beta-\frac{2sc\tilde{c}(N_xyV_y-N_yxV_x)}{\Xi\,xy(x^2-y^2)H}\bigg](\sigma^3)^A{}_B\zeta^B-\frac{1}{2\Xi\,H}\bigg\{\sqrt{\frac{-\Delta_x\,\Delta_y}{(x^2-y^2)^2}}\,g^2H\big(y\,\gamma^{14}-x\,\gamma^{23}\big)\\&+gV_x\sqrt{\frac{\Delta_y}{y^2-x^2}}\big[\partial_y(yH)\,\gamma^2+x\,\partial_yH\,\gamma^{134}\big]+gV_y\sqrt{\frac{\Delta_x}{x^2-y^2}}\big[\partial_x(xH)\,\gamma^4+y\,\partial_xH\,\gamma^{123}\big]\\&-\bigg[\frac{x(\Delta_yV_x-\Delta_xV_y)}{(x^2-y^2)^2}+\frac{\Delta_x'V_y}{2(x^2-y^2)}\bigg]\gamma^{34}+\bigg[\frac{y(\Delta_yV_x-\Delta_xV_y)}{(x^2-y^2)^2}+\frac{\Delta_y'V_x}{2(x^2-y^2)}\bigg]\gamma^{12}\bigg\}\zeta^A\,.\end{aligned}\tag{4.1.36}$$

From now on, in order to solve the Killing spinor equations (4.1.32)–(4.1.36) we shall employ an explicit representation for the four-dimensional gamma matrices $\gamma^{\hat{\imath}}$, namely

$$\gamma^1=\sigma^1\otimes\sigma^3\,,\quad\gamma^2=\sigma^2\otimes\sigma^3\,,\quad\gamma^3=I_2\otimes\sigma^1\,,\quad\gamma^4=I_2\otimes\sigma^2\,.\tag{4.1.37}$$

In this representation we have

$$\mathcal{B}_4=-\sigma^1\otimes\sigma^2\,,\quad\gamma_*=\sigma^3\otimes\sigma^3\,.\tag{4.1.38}$$

Since the symplectic-Majorana condition (3.1.8) relates the two Killing spinors ϵ^A as

$$\epsilon^2=(\mathcal{B}_6\epsilon^1)^*=\vartheta\otimes(\mathcal{B}_4\gamma_*\zeta^1)^*\,,\tag{4.1.39}$$

we can focus uniquely on the spinor ζ^1. Writing it as

$$\zeta^1=H^{-3/8}\big(\chi_1,\chi_2,\chi_3,\chi_4\big)\,,\tag{4.1.40}$$

with $\chi_i = \chi_i(y, \phi, x, \psi)$ complex functions of the coordinates on $\mathbb{M}_4$, equation (4.1.32) is solved by[3]

$$\chi_1 = -\sqrt{\frac{\Delta_y}{y^2 - x^2}\,\frac{y-x}{\Delta_y^-}}\,\chi_3\,, \quad \chi_2 = -\sqrt{-\frac{\Delta_y}{\Delta_x}\,\frac{\Delta_x^-}{\Delta_y^-}}\,\chi_3\,, \quad \chi_4 = -\sqrt{\frac{x^2-y^2}{\Delta_x}\,\frac{\Delta_x^-}{x+y}}\,\chi_3\,, \tag{4.1.41}$$

where the functions $\Delta_\bullet^\pm$ have been introduced previously in (4.1.24). In terms of the unique unknown function χ_3, equations (4.1.33) and (4.1.34) read

$$\partial_y \chi_3 = \frac{1}{2}\,\partial_y\left[\log\left|\frac{\Delta_y^-}{y-x}\right|\right]\chi_3\,, \quad \partial_x \chi_3 = \frac{1}{2}\,\partial_x\left[\log\left|\frac{\Delta_x^-}{x-y}\right|\right]\chi_3\,, \tag{4.1.42}$$

while (4.1.35) and (4.1.36) boil down respectively to

$$\partial_\phi \chi_3 = \frac{\mathrm{i}g}{2}\Big(3\alpha - \frac{\kappa}{ag}\Big)\chi_3\,, \quad \partial_\psi \chi_3 = \frac{\mathrm{i}g}{2}(3\beta + 1)\chi_3\,. \tag{4.1.43}$$

Tuning the gauge parameters to

$$\alpha = \frac{\kappa}{3ag} = -\frac{s^2}{3}\,, \quad \beta = -\frac{1}{3}\,, \tag{4.1.44}$$

the system can be easily solved, finding an angular-independent Killing spinor with components given by

$$\begin{aligned} \chi_1 &= -c\left(\frac{\Delta_x^+\Delta_y^+}{y+x}\right)^{1/2}, & \chi_2 &= c\left(\frac{\Delta_x^-\Delta_y^+}{x-y}\right)^{1/2}, \\ \chi_3 &= c\left(\frac{\Delta_x^+\Delta_y^-}{y-x}\right)^{1/2}, & \chi_4 &= -c\;\mathrm{sign}(x-y)\Big(-\frac{\Delta_x^-\Delta_y^-}{x+y}\Big)^{1/2}, \end{aligned} \tag{4.1.45}$$

with $c \in \mathbb{C}$ a constant. From the symplectic-Majorana condition (4.1.39), we obtain

$$\zeta^2 = (\mathcal{B}_4\gamma_*\zeta^1)^* = \mathrm{i}\,H^{-3/8}\big(-\chi_4^*, -\chi_3^*, \chi_2^*, \chi_1^*\big)\,, \tag{4.1.46}$$

which, as it can be shown explicitly, satisfies all the related Killing spinor equations.

We can now count the amount of supersymmetry preserved by our $\mathrm{AdS}_2 \times \mathbb{M}_4$ background. Since ϑ is a two-dimensional Majorana spinor, it has two real independent degrees of freedom. Similarly, ζ^A is completely determined by the complex constant c, accounting for two real degrees of freedom. Therefore, there are in total four real independent Killing spinors and thus, being sixteen the number of supersymmetries of the six-dimensional $\mathcal{N} = (1,1)$ theory, our solution is 1/4-BPS.

4.2 Global analysis

Having established the local form of the solution and its supersymmetry properties, we move to the study of the regularity. Since the four-dimensional orbifold $\mathbb{M}_4$ possesses a

[3]Here and in what follows we assume $\delta > 0$. When $\delta < 0$ it is sufficient to exchange $\Delta_\bullet^+ \leftrightarrow \Delta_\bullet^-$.

$U(1)^2$ isometry associated with the two Killing vectors ∂_ψ and ∂_ϕ of the metric (4.1.10), it is natural to study this system within the framework of toric geometry. The global analysis will be then divided in steps. As outlined in section 2.3, we will approach the problem by studying the degeneration of four Killing vectors ξ_a on some divisors $\mathcal{L}_a = \hat{D}_a$, which allows us to extract the $SL(2,\mathbb{R})$-invariant toric data $d_{a,b}$. From the determinants $d_{a,b}$ we can compute the intersection matrix D_{ab} through (2.1.61). Finally, computing the Ricci form $\rho_{(a)}$ of each divisor $\hat{D}_a$ and using (2.1.59), we will be able to extract all the labels m_a.

4.2.1 Degenerating Killing vectors

The metric (4.1.10) has in total four degenerate Killing vectors

$$\mathtt{j}_\pm = \frac{2a^2}{\Delta_x'(x_\pm)}\big[\widetilde{V}_x(x_\pm)\,\partial_\psi + V_x(x_\pm)\,\partial_\phi\big] \equiv J_\pm^{(\psi)}\partial_\psi + J_\pm^{(\phi)}\partial_\phi\,, \tag{4.2.1}$$

$$\mathtt{k}_\pm = \frac{2a^2}{\Delta_y'(y_\pm)}\big[\widetilde{V}_y(y_\pm)\,\partial_\psi + V_y(y_\pm)\,\partial_\phi\big] \equiv K_\pm^{(\psi)}\partial_\psi + K_\pm^{(\phi)}\partial_\phi\,, \tag{4.2.2}$$

whose norm vanishes at $x = x_\pm$ and $y = y_\pm$, respectively. All of them are normalized so to have unitary surface gravity. In order to present the results in a uniform way we introduce the four loci

$$\hat{D}_1 = \{x = x_-\}\,,\quad \hat{D}_2 = \{y = y_-\}\,,\quad \hat{D}_3 = \{x = x_+\}\,,\quad \hat{D}_4 = \{y = y_+\}\,, \tag{4.2.3}$$

and define the four vectors

$$\xi_1 = \mathtt{j}_-\,,\quad \xi_2 = \mathtt{k}_-\,,\quad \xi_3 = \mathtt{j}_+\,,\quad \xi_4 = \mathtt{k}_+\,, \tag{4.2.4}$$

so that ξ_a is the Killing vector that degenerates at $\hat{D}_a$.

For later convenience, we define the two alternative sets of coordinates $\psi_\pm$ and $\chi_\pm$ such that $\mathtt{j}_\pm = \varkappa_J\,\partial_{\psi_\pm}$ or $\mathtt{k}_\pm = \varkappa_K\,\partial_{\chi_\pm}$, with $\varkappa_J = \pm$ and $\varkappa_K = \pm$. These coordinates are adapted to the direction generated by the corresponding Killing vector ξ_a and, for this reason, they will play a fundamental role when zooming in on the different loci $\hat{D}_a$. They are defined by

$$\psi = \varkappa_J \sum_{\sigma=\pm} J_\sigma^{(\psi)}\psi_\sigma\,,\quad \phi = \varkappa_J \sum_{\sigma=\pm} J_\sigma^{(\phi)}\psi_\sigma\,, \tag{4.2.5}$$

or, alternatively, by

$$\psi = \varkappa_K \sum_{\sigma=\pm} K_\sigma^{(\psi)}\chi_\sigma\,,\quad \phi = \varkappa_K \sum_{\sigma=\pm} K_\sigma^{(\phi)}\chi_\sigma\,. \tag{4.2.6}$$

Here ψ_σ should be intended as well-defined coordinates on a patch which contains $\hat{D}_1$ or $\hat{D}_3$, and subsequently we will denote them as $\psi_\sigma^{(1)}$ or $\psi_\sigma^{(3)}$, with periodicities $\Delta\psi_\sigma^{(1)}$ and $\Delta\psi_\sigma^{(3)}$, as explained in section 3.5.1. The same applies also to χ_σ, which will be written as $\chi_\sigma^{(2)}$ and $\chi_\sigma^{(4)}$ on $\hat{D}_2$ and $\hat{D}_4$. The reason for the introduction of $\varkappa_J$ and $\varkappa_K$ and the prescription for their choice will be explained shortly. The Jacobian matrices of these two transformations have determinant

$$\det(\mathbf{J}) = J_+^{(\psi)} J_-^{(\phi)} - J_-^{(\psi)} J_+^{(\phi)}, \quad \det(\mathbf{K}) = K_+^{(\psi)} K_-^{(\phi)} - K_-^{(\psi)} K_+^{(\phi)}, \tag{4.2.7}$$

and the periodicities of the old and new coordinates are connected by

$$\begin{aligned} \Delta\psi\,\Delta\phi &= \big|\det(\mathbf{J})\big|\,\Delta\psi_+^{(1)}\,\Delta\psi_-^{(1)} = \big|\det(\mathbf{J})\big|\,\Delta\psi_+^{(3)}\,\Delta\psi_-^{(3)}\,, \\ \Delta\psi\,\Delta\phi &= \big|\det(\mathbf{K})\big|\,\Delta\chi_+^{(2)}\,\Delta\chi_-^{(2)} = \big|\det(\mathbf{K})\big|\,\Delta\chi_+^{(4)}\,\Delta\chi_-^{(4)}\,. \end{aligned} \tag{4.2.8}$$

The signs $\varkappa_J$ and $\varkappa_K$ are given by

$$\begin{aligned} \varkappa_J &= \operatorname{sign}\big[\Xi\,(x^2-y^2)\det(\mathbf{J})\big] = \operatorname{sign}(x^2-y^2)\,, \\ \varkappa_K &= -\operatorname{sign}\big[\Xi\,(x^2-y^2)\det(\mathbf{K})\big] = -\operatorname{sign}(x^2-y^2)\,, \end{aligned} \tag{4.2.9}$$

where, in the last steps, we restricted to $x, y > 0$ and imposed the signature conditions $N_x, N_y > 0$; in this case, both $\det(\mathbf{J})$ and $\det(\mathbf{K})$ have the same sign of Ξ.

We can now extract a set of fake vectors $\vec{V}_a$ by using the relation $\xi_a = \vec{V}_a \cdot (e_1, e_2)$ on a fake basis $\{e_1, e_2\}$, which will not be effective. Specifically, we introduce the 2π-periodic coordinates $\nu_1 = \frac{2\pi}{\Delta\psi}\psi$, $\nu_2 = \frac{2\pi}{\Delta\phi}\phi$ and consider the basis $\{e_1, e_2\} = \{\partial_{\nu_1}, \partial_{\nu_2}\}$. From the Killing vectors (4.2.1) and (4.2.2) we derive the vectors

$$\begin{aligned} \vec{V}_1 &= \Big(\frac{2\pi}{\Delta\psi}\,J_-^{(\psi)}, \frac{2\pi}{\Delta\phi}\,J_-^{(\phi)}\Big)\,, & \vec{V}_2 &= \Big(\frac{2\pi}{\Delta\psi}\,K_-^{(\psi)}, \frac{2\pi}{\Delta\phi}\,K_-^{(\phi)}\Big)\,, \\ \vec{V}_3 &= \Big(\frac{2\pi}{\Delta\psi}\,J_+^{(\psi)}, \frac{2\pi}{\Delta\phi}\,J_+^{(\phi)}\Big)\,, & \vec{V}_4 &= \Big(\frac{2\pi}{\Delta\psi}\,K_+^{(\psi)}, \frac{2\pi}{\Delta\phi}\,K_+^{(\phi)}\Big)\,. \end{aligned} \tag{4.2.10}$$

Since we will not be able to determine the periodicities $\Delta\psi$ and $\Delta\phi$ separately, but only in the combination $(\Delta\psi\Delta\phi)$, we actually do not know the explicit form of the vectors in (4.2.10), thus in particular we do not know whether they belong to $\mathbb{Z}^2$. However this is not a problem, as explained in section 2.3. In particular, we can not simply use the definition $m_a = \gcd(V_a^1, V_a^2)$ to extract the labels of the four dimensional orbifold $\mathbb{M}_4$. There is only one *caveat*: the vectors (4.2.10) must be dual to a convex polytope and, in our conventions, must be ordered counter-clockwise, which imply $\det(\vec{V}_a, \vec{V}_{a+1}) > 0$ for any a. If this condition is not met, the vectors can be ordered in the correct way by means of a reflection about a line in the $\mathbb{Z}^2$-plane, which can be realized, *e.g.*, swapping the two components of each vector. This transformation accounts in exchanging ∂_{ν_1} and ∂_{ν_2} and the two bases $\{\partial_{\nu_2}, \partial_{\nu_1}\}$ and $\{E_1, E_2\}$ will now be related through a matrix with determinant equal to -1. As a consequence, the intersection matrices computed starting from the two sets of vectors extracted from the two aforementioned bases will have opposite sign. In order to keep track of this fact we introduce the sign $\varkappa_D = \pm$, telling whether the vectors are ordered as in (4.2.10) $(+)$ or with the components swapped $(-)$. $\varkappa_D$ will multiply every determinant computed from the vectors (4.2.10) and its value will be fixed shortly.

Using the set of vectors (4.2.10) we compute

$$\begin{aligned} d_{1,2} &= \frac{-\varkappa_D(2\pi)^2}{\Delta\psi\Delta\phi}\frac{4\Xi a^2(x_-^2-y_-^2)H(x_-,y_-)}{\Delta_x'(x_-)\Delta_y'(y_-)}\,, & d_{2,3} &= \frac{\varkappa_D(2\pi)^2}{\Delta\psi\Delta\phi}\frac{4\Xi a^2(x_+^2-y_-^2)H(x_+,y_-)}{\Delta_x'(x_+)\Delta_y'(y_-)}\,, \\ d_{3,4} &= \frac{-\varkappa_D(2\pi)^2}{\Delta\psi\Delta\phi}\frac{4\Xi a^2(x_+^2-y_+^2)H(x_+,y_+)}{\Delta_x'(x_+)\Delta_y'(y_+)}\,, & d_{4,1} &= \frac{\varkappa_D(2\pi)^2}{\Delta\psi\Delta\phi}\frac{4\Xi a^2(x_-^2-y_+^2)H(x_-,y_+)}{\Delta_x'(x_-)\Delta_y'(y_+)}\,, \end{aligned} \tag{4.2.11}$$

where we made use of the identity

$$J^{(\psi)}_{\sigma_1}K^{(\phi)}_{\sigma_2} - K^{(\psi)}_{\sigma_2}J^{(\phi)}_{\sigma_1} = -\frac{4\Xi a^2 (x^2_{\sigma_1} - y^2_{\sigma_2})H(x_{\sigma_1}, y_{\sigma_2})}{\Delta'_x(x_{\sigma_1})\Delta'_y(y_{\sigma_2})}\,, \quad \text{for} \quad \sigma_1, \sigma_2 = \pm\,. \tag{4.2.12}$$

Notice that, even if we have extracted the invariant toric data $d_{a,b}$ from the fake vectors (4.2.10), we have not completed the analysis. Indeed, we have to impose a quantization condition

$$d_{a,b} \in \mathbb{Z}\,, \tag{4.2.13}$$

which is not automatically guaranteed. This constraint is necessary for the system to be well-defined, and we will study it in details momentarily. Imposing the condition $\det(\vec{v}_a, \vec{v}_{a+1}) = \det(\vec{V}_a, \vec{V}_{a+1}) > 0$ we can fix the value of $\varkappa_D$, obtaining[4]

$$\varkappa_D = \operatorname{sign}\big[\Xi(x^2 - y^2)\big]\,. \tag{4.2.14}$$

Recalling the definition of $\varkappa_J$ and $\varkappa_K$ in (4.2.9), we then have

$$\varkappa_D \varkappa_J = \operatorname{sign}\big[\det(\mathbf{J})\big]\,, \quad \varkappa_D \varkappa_K = -\operatorname{sign}\big[\det(\mathbf{K})\big]\,. \tag{4.2.15}$$

Following the notation of [1], we also define

$$t_J \equiv d_{1,3} = -\varkappa_D \frac{2\pi}{\Delta\psi}\frac{2\pi}{\Delta\phi}\det(\mathbf{J})\,, \quad t_K \equiv d_{2,4} = -\varkappa_D \frac{2\pi}{\Delta\psi}\frac{2\pi}{\Delta\phi}\det(\mathbf{K})\,. \tag{4.2.16}$$

The intersection matrix describing a given set of toric divisors was defined in (2.1.61). In our construction, with a bit of computation the diagonal terms can be cast in the form

$$D_{11} = -\varkappa_D \frac{\Delta\psi}{2\pi}\frac{\Delta\phi}{2\pi}\frac{\Delta'_x(x_-)^2}{4\Xi a^2}\left[\frac{1}{(x^2_- - y^2_+)H(x_-, y_+)} - \frac{1}{(x^2_- - y^2_-)H(x_-, y_-)}\right], \tag{4.2.17}$$

$$D_{22} = -\varkappa_D \frac{\Delta\psi}{2\pi}\frac{\Delta\phi}{2\pi}\frac{\Delta'_y(y_-)^2}{4\Xi a^2}\left[\frac{1}{(x^2_+ - y^2_-)H(x_+, y_-)} - \frac{1}{(x^2_- - y^2_-)H(x_-, y_-)}\right], \tag{4.2.18}$$

$$D_{33} = \varkappa_D \frac{\Delta\psi}{2\pi}\frac{\Delta\phi}{2\pi}\frac{\Delta'_x(x_+)^2}{4\Xi a^2}\left[\frac{1}{(x^2_+ - y^2_+)H(x_+, y_+)} - \frac{1}{(x^2_+ - y^2_-)H(x_+, y_-)}\right], \tag{4.2.19}$$

$$D_{44} = \varkappa_D \frac{\Delta\psi}{2\pi}\frac{\Delta\phi}{2\pi}\frac{\Delta'_y(y_+)^2}{4\Xi a^2}\left[\frac{1}{(x^2_+ - y^2_+)H(x_+, y_+)} - \frac{1}{(x^2_- - y^2_+)H(x_-, y_+)}\right], \tag{4.2.20}$$

whereas the off-diagonal terms are simply ($D_{ab} = D_{ba}$)

$$D_{12} = \frac{1}{d_{1,2}}\,, \quad D_{23} = \frac{1}{d_{2,3}}\,, \quad D_{34} = \frac{1}{d_{3,4}}\,, \quad D_{41} = \frac{1}{d_{4,1}}\,, \tag{4.2.21}$$

with $d_{a,a+1}$ given in (4.2.11).

[4] In order to do so we must use the relations between the signs of $\Delta'_x(x_\pm)$ and $\Delta'_y(y_\pm)$.

4.2.2 Complex structure and divisors

We now define the holomorphic $(2,0)$-form

$$\Omega = (\hat{e}^1 + \mathrm{i}\hat{e}^2)\wedge(\hat{e}^3 + \mathrm{i}\hat{e}^4) \equiv \Omega^{(12)}\wedge\Omega^{(34)}\,, \tag{4.2.22}$$

which, by construction, is compatible with the metrics (4.1.10). The fact that the associated complex structure is integrable, and thus $\mathbb{M}_4$ is Hermitian, follows from the relation $\mathrm{d}\Omega = \mathrm{i}P_\rho\wedge\Omega$, where P_ρ is the Ricci potential and reads

$$P_\rho = \frac{H\,\partial_y\big(H^{-2}\Delta_y\big)}{2\Xi(x^2-y^2)}\big(V_x\,\mathrm{d}\psi - \widetilde{V}_x\,\mathrm{d}\phi\big) - \frac{H\,\partial_x\big(H^{-2}\Delta_x\big)}{2\Xi(x^2-y^2)}\big(V_y\,\mathrm{d}\psi - \widetilde{V}_y\,\mathrm{d}\phi\big)\,. \tag{4.2.23}$$

It is then clear that the four (real) codimension two loci $\hat{D}_a$ are (complex) divisors in $\mathbb{M}_4$ and we will now describe these in more detail. We first zoom in on $\hat{D}_1$, determined by the condition $x = x_-$. To this end, we define the coordinate R_- such that $x = x_- + R_-$ and obtain

$$\mathrm{d}s^2_{(1)} = \frac{y^2-x_-^2}{\Delta_y}\,\mathrm{d}y^2 + \frac{\Delta_x'(x_-)^2\det(\mathbf{J})^2}{4\Xi^2a^4H(x_-,y)^2}\frac{\Delta_y}{y^2-x_-^2}\,(\mathrm{d}\psi_+^{(1)})^2\,. \tag{4.2.24}$$

$\hat{D}_1$ is a complex orbifold of (complex) dimension one, therefore it is natural to define its holomorphic $(1,0)$-form

$$\Omega_{(1)} = \sqrt{\frac{y^2-x_-^2}{\Delta_y}}\,\mathrm{d}y + \mathrm{i}\,\frac{|\Delta_x'(x_-)\det(\mathbf{J})|}{2|\Xi|a^2H(x_-,y)}\sqrt{\frac{\Delta_y}{y^2-x_-^2}}\,\mathrm{d}\psi_+^{(1)}\,. \tag{4.2.25}$$

Notice that, defined $\Omega^{(12)}\equiv\hat{e}^1+\mathrm{i}\hat{e}^2$, with $\hat{e}^1$ and $\hat{e}^2$ given in (4.1.13), $\Omega_{(1)}$ is simply $\Omega^{(12)}$ restricted to $\hat{D}_1$. It can be checked that $\mathrm{d}\Omega_{(1)} = \mathrm{i}P_{(1)}\wedge\Omega_{(1)}$, where

$$P_{(1)} = \frac{|\Delta_x'(x_-)\det(\mathbf{J})|}{4|\Xi|a^2}\left[\frac{\Delta_y'}{(x_-^2-y^2)H(x_-,y)} - \frac{\Delta_y\,\partial_y\big((x_-^2-y^2)H(x_-,y)^2\big)}{(x_-^2-y^2)^2H(x_-,y)^3}\right]\mathrm{d}\psi_+^{(1)}\,. \tag{4.2.26}$$

It follows

$$\hat{\chi}_{(1)} = \frac{1}{2\pi}\int_{\hat{D}_1}\rho_{(1)} = \frac{\Delta\psi_+^{(1)}}{2\pi}\frac{|\Delta_x'(x_-)\det(\mathbf{J})|}{4|\Xi|a^2}\left[\frac{\Delta_y'(y_+)}{(x_-^2-y_+^2)H(x_-,y_+)} + \frac{-\Delta_y'(y_-)}{(x_-^2-y_-^2)H(x_-,y_-)}\right]. \tag{4.2.27}$$

Endowed with metric (4.2.24), $\hat{D}_1$ is a compact surface parameterized by the periodic azimuthal coordinate $\psi_+^{(1)}$ and the compact "polar" coordinate y, with $y\in[y_-,y_+]$. As y approaches one of the endpoints of this interval, say $y_\pm$, the line element becomes

$$\mathrm{d}s^2_{(1)} \underset{y_\pm}{\simeq} \mathrm{d}\varrho_\pm^2 + \varrho_\pm^2\,\frac{\Delta_x'(x_-)^2\Delta_y'(y_\pm)^2\det(\mathbf{J})^2}{16\Xi^2a^4(x_-^2-y_\pm^2)^2H(x_-,y_\pm)^2}\,(\mathrm{d}\psi_+^{(1)})^2\,, \tag{4.2.28}$$

where we defined $\varrho_\pm^2 = |y-y_\pm|$. In order to have a smooth orbifold metric on $\hat{D}_1$ we must impose the following conditions at the north (y_-) and south (y_+) poles, respectively,

$$\begin{aligned}
\frac{|\Delta_x'(x_-)\det(\mathbf{J})|}{4|\Xi|a^2}\frac{\Delta_y'(y_-)}{(x_-^2-y_-^2)H(x_-,y_-)}\,\Delta\psi_+^{(1)} &= -\frac{2\pi}{\mathtt{m}_-^{(1)}}\,, \qquad \mathtt{m}_-^{(1)}\in\mathbb{N}\,,\\
\frac{|\Delta_x'(x_-)\det(\mathbf{J})|}{4|\Xi|a^2}\frac{\Delta_y'(y_+)}{(x_-^2-y_+^2)H(x_-,y_+)}\,\Delta\psi_+^{(1)} &= \frac{2\pi}{\mathtt{m}_+^{(1)}}\,, \qquad \mathtt{m}_+^{(1)}\in\mathbb{N}\,,
\end{aligned} \tag{4.2.29}$$

where the minus in the first relation is due to the fact that $\Delta_y'(y_-)/(x_-^2 - y_-^2) < 0$. By imposing (4.2.29), we have required that

$$\hat{D}_1 = \frac{\mathbb{WCP}^1_{[\bar{\mathtt{m}}_-^{(1)},\bar{\mathtt{m}}_+^{(1)}]}}{\mathbb{Z}_{\mathtt{m}_0^{(1)}}}\,, \quad \mathtt{m}_0^{(1)} = \gcd(\mathtt{m}_-^{(1)}, \mathtt{m}_+^{(1)})\,, \tag{4.2.30}$$

and in terms of the $\mathtt{m}_\pm^{(1)}$ its Euler characteristic (4.2.27) can be written as

$$\hat{\chi}_{(1)} = \frac{1}{\mathtt{m}_-^{(1)}} + \frac{1}{\mathtt{m}_+^{(1)}}\,. \tag{4.2.31}$$

A similar analysis can be repeated the remaining divisors, obtaining

$$\hat{\chi}_{(2)} = \frac{\Delta\chi_+^{(2)}}{2\pi}\frac{|\Delta_y'(y_-)\det(\mathbf{K})|}{4|\Xi|a^2}\left[\frac{-\Delta_x'(x_+)}{(x_+^2-y_-^2)H(x_+,y_-)} + \frac{\Delta_x'(x_-)}{(x_-^2-y_-^2)H(x_-,y_-)}\right], \tag{4.2.32}$$

$$\hat{\chi}_{(3)} = \frac{\Delta\psi_-^{(3)}}{2\pi}\frac{|\Delta_x'(x_+)\det(\mathbf{J})|}{4|\Xi|a^2}\left[\frac{\Delta_y'(y_+)}{(x_+^2-y_+^2)H(x_+,y_+)} + \frac{-\Delta_y'(y_-)}{(x_+^2-y_-^2)H(x_+,y_-)}\right], \tag{4.2.33}$$

$$\hat{\chi}_{(4)} = \frac{\Delta\chi_-^{(4)}}{2\pi}\frac{|\Delta_y'(y_+)\det(\mathbf{K})|}{4|\Xi|a^2}\left[\frac{-\Delta_x'(x_+)}{(x_+^2-y_+^2)H(x_+,y_+)} + \frac{\Delta_x'(x_-)}{(x_-^2-y_+^2)H(x_-,y_+)}\right], \tag{4.2.34}$$

and

$$\hat{\chi}_{(a)} = \frac{1}{\mathtt{m}_-^{(a)}} + \frac{1}{\mathtt{m}_+^{(a)}}\,, \tag{4.2.35}$$

where $\mathtt{m}_\pm^{(a)}$ are eight integer parameters, labelling a spindle $\Sigma_{[\mathtt{m}_-^{(a)},\mathtt{m}_+^{(a)}]}$ for each divisor $\hat{D}_a$.

4.2.3 Extracting the labels

We can now use the master formulas (2.1.59) to extract the labels m_a of the system. An explicit computation gives

$$\begin{aligned}\rho\big|_{\hat{D}_1} - \rho_{(1)} =& \varkappa_J \frac{\Delta_x'(x_-)}{\Delta_x'(x_+)}\,\partial_y\left[\frac{(x_+^2-y^2)H(x_+,y)}{(x_-^2-y^2)H(x_-,y)}\right]\mathrm{d}y\wedge\mathrm{d}\psi_+^{(1)} \\ &- \frac{|\Delta_x'(x_-)\det(\mathbf{J})|}{2|\Xi|a^2}\,\partial_y\left[\frac{y\,\Delta_y}{(x_-^2-y^2)^2H(x_-,y)}\right]\mathrm{d}y\wedge\mathrm{d}\psi_+^{(1)}\,,\end{aligned} \tag{4.2.36}$$

hence, performing the integration,

$$\begin{aligned}D_{11} &= \varkappa_J\frac{1}{m_1}\frac{\Delta\psi_+^{(1)}}{2\pi}\frac{\Delta_x'(x_-)}{\Delta_x'(x_+)}\left[\frac{(x_+^2-y_+^2)H(x_+,y_+)}{(x_-^2-y_+^2)H(x_-,y_+)} - \frac{(x_+^2-y_-^2)H(x_+,y_-)}{(x_-^2-y_-^2)H(x_-,y_-)}\right] \\ &= -\varkappa_J\frac{1}{m_1}\frac{\Delta\psi_+^{(1)}}{2\pi}\frac{\Delta_x'(x_-)^2\det(\mathbf{J})}{4\Xi a^2}\left[\frac{1}{(x_-^2-y_+^2)H(x_-,y_+)} - \frac{1}{(x_-^2-y_-^2)H(x_-,y_-)}\right],\end{aligned} \tag{4.2.37}$$

where, in the last line, we used the identity

$$\begin{aligned}&\frac{(x_\pm^2-y_+^2)H(x_\pm,y_+)}{(x_\mp^2-y_+^2)H(x_\mp,y_+)} - \frac{(x_\pm^2-y_-^2)H(x_\pm,y_-)}{(x_\mp^2-y_-^2)H(x_\mp,y_-)} = \\ &\quad = \mp\frac{\det(\mathbf{J})}{4\Xi a^2}\left[\frac{\Delta_x'(x_-)\Delta_x'(x_+)}{(x_\mp^2-y_+^2)H(x_\mp,y_+)} - \frac{\Delta_x'(x_-)\Delta_x'(x_+)}{(x_\mp^2-y_-^2)H(x_\mp,y_-)}\right].\end{aligned} \tag{4.2.38}$$

We can now compare the expressions for D_{11} presented in (4.2.17) and (4.2.37), thus obtaining

$$\varkappa_D \frac{\Delta\psi}{2\pi}\frac{\Delta\phi}{2\pi} = \varkappa_J \frac{1}{m_1}\frac{\Delta\psi_+^{(1)}}{2\pi}\det(\mathbf{J})\,. \tag{4.2.39}$$

By means of (4.2.8) this relation becomes (see also (4.2.15))

$$\varkappa_D \frac{\Delta\psi_+^{(1)}}{2\pi}\frac{\Delta\psi_-^{(1)}}{2\pi}\big|\det(\mathbf{J})\big| = \varkappa_J \frac{1}{m_1}\frac{\Delta\psi_+^{(1)}}{2\pi}\det(\mathbf{J}) \implies \frac{\Delta\psi_-^{(1)}}{2\pi} = \frac{1}{m_1}\,, \tag{4.2.40}$$

Similarly, the inspection of the components D_{22}, D_{33} and D_{44} gives, respectively,

$$\frac{\Delta\chi_-^{(2)}}{2\pi} = \frac{1}{m_2}\,, \qquad \frac{\Delta\psi_+^{(3)}}{2\pi} = \frac{1}{m_3}\,, \qquad \frac{\Delta\chi_+^{(4)}}{2\pi} = \frac{1}{m_4}\,. \tag{4.2.41}$$

Before computing the off-diagonal terms of the intersection matrix, we combine equations (4.2.29) and (4.2.39). Comparing the results with (4.2.11) we find the useful relations $d_{1,2} = m_1\mathtt{m}_-^{(1)}$ and $d_{4,1} = m_1\mathtt{m}_+^{(1)}$. Equivalently, after using (2.1.55), we have

$$\mathtt{m}_-^{(1)} = m_2\,\hat{d}_{1,2}\,, \quad \mathtt{m}_+^{(1)} = m_4\,\hat{d}_{4,1}\,. \tag{4.2.42}$$

Similar formulas can be proven, relating the order of the orbifold singularities of the remaining divisors to the labels m_a and the positive integers $\hat{d}_{a,a+1}$, namely

$$\begin{aligned}
\mathtt{m}_-^{(2)} &= m_1\,\hat{d}_{1,2}\,, & \mathtt{m}_+^{(2)} &= m_3\,\hat{d}_{2,3}\,,\\
\mathtt{m}_-^{(3)} &= m_2\,\hat{d}_{2,3}\,, & \mathtt{m}_+^{(3)} &= m_4\,\hat{d}_{3,4}\,,\\
\mathtt{m}_-^{(4)} &= m_1\,\hat{d}_{4,1}\,, & \mathtt{m}_+^{(4)} &= m_3\,\hat{d}_{3,4}\,.
\end{aligned} \tag{4.2.43}$$

With these relations it is possible to prove the consistency of all the components of (2.1.59) for the other divisors.

Let us now return to the issue of finding a basis for an effective torus action. With the information that we have obtained so far, it is possible to show that the following $SL(2,\mathbb{R})$ matrix

$$S = \begin{pmatrix} \frac{2\pi}{\Delta\phi}(a_+K_+^{(\phi)} - a_-K_-^{(\phi)}) & -\frac{2\pi}{\Delta\psi}(a_+K_+^{(\psi)} - a_-K_-^{(\psi)}) \\ -\frac{2\pi}{\Delta\phi}\frac{J_-^{(\phi)}}{m_1} & \frac{2\pi}{\Delta\psi}\frac{J_-^{(\psi)}}{m_1} \end{pmatrix}\,, \tag{4.2.44}$$

acting on the "fake" vectors (4.2.10) transforms them in the $\mathbb{Z}^2$-valued set (recall that $d_{1,a} = m_1 m_a\,\hat{d}_{1,a}$ from (2.1.55))

$$\begin{aligned}
\vec{v}_1 &= (m_1, 0)\,, & \vec{v}_2 &= (a_+d_{2,4}, d_{1,2}/m_1)\,,\\
\vec{v}_3 &= (a_-d_{2,3} + a_+d_{3,4}, d_{1,3}/m_1)\,, & \vec{v}_4 &= (a_-d_{2,4}, -d_{4,1}/m_1)\,,
\end{aligned} \tag{4.2.45}$$

where $a_\pm \in \mathbb{Z}$ are integers such that[5]

$$a_-\mathtt{m}_-^{(1)} + a_+\mathtt{m}_+^{(1)} = -1\,, \tag{4.2.46}$$

[5]Here, for simplicity we assumed that $\gcd(\mathtt{m}_+^{(1)}, \mathtt{m}_-^{(1)}) = 1$ and that $\varkappa_D = 1$.

which always exist by Bézout's lemma. The resulting basis reads

$$\begin{aligned} E_1 &= \frac{1}{m_1}\big(J_-^{(\psi)}\partial_\psi + J_-^{(\phi)}\partial_\phi\big)\,, \\ E_2 &= \big(a_+K_+^{(\psi)} - a_-K_-^{(\psi)}\big)\,\partial_\psi + \big(a_+K_+^{(\phi)} - a_-K_-^{(\phi)}\big)\,\partial_\phi\,, \end{aligned} \tag{4.2.47}$$

and the fact that the $\vec{v}_a$ are $\mathbb{Z}^2$-valued indicates that this is a basis for an effectively acting torus. Notice that, remarkably, although the basis $\{e_1, e_2\}$ and the vectors in (4.2.10) depended on the periodicities $\Delta\psi$ and $\Delta\phi$ separately (that we do not know), the new basis $\{E_1, E_2\}$ and the new vectors $\vec{v}_a$ will be explicitly determined in terms of the gauge invariant toric data, once we complete the analysis of the quantization conditions, to which we now turn. Lastly, since $d_{a,b} = m_a m_b\, \hat{d}_{a,b}$, it is straightforward to observe that a factor m_a can be collected from each vector $\vec{v}_a$, confirming that the m_a serve as labels for the divisors $\hat{D}_a$.

4.3 Quantization conditions

The analysis of the toric data of $\mathbb{M}_4$ performed in the previous subsection shows clearly that this space is a well-defined orbifold only if the $d_{a,b}$ computed explicitly in (4.2.11) and (4.2.16) are integers, as observed around (4.2.13). The aim of this section is twofold: first, we have to ensure that

$$\hat{\mathfrak{q}}_i^a \equiv \frac{g_c}{2\pi}\int_{\hat{D}_a} F_i\,, \quad \hat{\mathfrak{q}}_R^a = \frac{n}{\mathtt{m}_-^{(a)}\mathtt{m}_+^{(a)} m_a}\,, \qquad n \in \mathbb{Z}\,, \tag{4.3.1}$$

which means that the R-symmetry gauge field $F_R = F_1 + F_2$ is correctly quantized; second, we will find a Diophantine equation for the $d_{a,b}$, ensuring that $d_{a,b} \in \mathbb{Z}$. This second step will be achieved by expressing all the parameters of the (supersymmetric) solutions, namely $(a, N_x, N_y, \Delta\psi\Delta\phi)$, in terms of the $d_{a,b}$.

Let us start assuming for the moment $d_{a,b} \in \mathbb{Z}$. The field strengths F_i can be conveniently written as

$$F_i = \frac{c\tilde{c}}{\Xi\, sH^2}\big[\partial_x H\,\mathrm{d}x \wedge \big(V_y\,\mathrm{d}\psi - \widetilde{V}_y\,\mathrm{d}\phi\big) + \partial_y H\,\mathrm{d}y \wedge \big(V_x\,\mathrm{d}\psi - \widetilde{V}_x\,\mathrm{d}\phi\big)\big]\,. \tag{4.3.2}$$

We start considering, as an example, the first divisor $\hat{D}_1$, whose associated magnetic fluxes are

$$\hat{\mathfrak{q}}_1^1 = \hat{\mathfrak{q}}_2^1 = -\frac{\Delta\psi_+^{(1)}}{2\pi}\,\frac{g_c c\tilde{c}}{s}\,\frac{|\Delta_x'(x_-)\det(\mathbf{J})|}{2|\Xi|a^2}\left[\frac{1}{H(x_-,y_+)} - \frac{1}{H(x_-,y_-)}\right]. \tag{4.3.3}$$

We can now eliminate $\Delta\psi_+^{(1)}$ in favour of m_1 and $(\Delta\psi\Delta\phi)$, use all the relations around (4.1.26) to simplify the functions computed in the poles $(x_\pm, y_\pm)$ and finally use (4.2.11) to recognize that $\hat{\mathfrak{q}}_R^1$ can be rewritten as

$$\hat{\mathfrak{q}}_R^1 = -\frac{\tau^{(y_+)}}{\mathtt{m}_+^{(1)}} - \frac{\tau^{(y_-)}}{\mathtt{m}_-^{(1)}} - \tau^{(x_-)}\frac{t_K}{\mathtt{m}_+^{(1)}\mathtt{m}_-^{(1)} m_1} = -\tau^{(y_+)}\left[\frac{\mathtt{m}_-^{(1)} + \tau^{(y_+)}\tau^{(y_-)}\mathtt{m}_+^{(1)}}{\mathtt{m}_-^{(1)}\mathtt{m}_+^{(1)}} + \tau^{(y_+)}\tau^{(x_+)}\frac{t_K/m_1}{\mathtt{m}_-^{(1)}\mathtt{m}_+^{(1)}}\right]. \tag{4.3.4}$$

This expression is consistent with the expected quantization condition for a connection on $\mathcal{O}(\mathtt{m}_-^{(1)} + \tau^{(y_+)}\tau^{(y_-)}\mathtt{m}_+^{(1)})$ with also a fibration with Chern number t_K and a transverse singularity $\mathbb{C}/\mathbb{Z}_{m_1}$. Notice in particular that, according to the values of the product $(\tau^{(y_+)}\tau^{(y_-)})$, we can have the twist or the anti-twist for the spindle "located" on $\hat{D}_1$ [42]. Similar computations for the other fluxes $\hat{\mathfrak{q}}_R^a$ show that these can be correctly expressed as

$$\begin{aligned}
\hat{\mathfrak{q}}_R^2 &= -\frac{\tau^{(x_+)}}{\mathtt{m}_+^{(2)}} - \frac{\tau^{(x_-)}}{\mathtt{m}_-^{(2)}} + \tau^{(y_-)}\frac{t_J}{\mathtt{m}_-^{(2)}\mathtt{m}_+^{(2)}m_2}\,,\\
\hat{\mathfrak{q}}_R^3 &= -\frac{\tau^{(y_+)}}{\mathtt{m}_+^{(3)}} - \frac{\tau^{(y_-)}}{\mathtt{m}_-^{(3)}} + \tau^{(x_+)}\frac{t_K}{\mathtt{m}_-^{(3)}\mathtt{m}_+^{(3)}m_3}\,,\\
\hat{\mathfrak{q}}_R^4 &= -\frac{\tau^{(x_+)}}{\mathtt{m}_+^{(4)}} - \frac{\tau^{(x_-)}}{\mathtt{m}_-^{(4)}} - \tau^{(y_+)}\frac{t_J}{\mathtt{m}_-^{(4)}\mathtt{m}_+^{(4)}m_4}\,.
\end{aligned} \tag{4.3.5}$$

Clearly, this does not conclude the analysis, since we have not showed that $\mathtt{m}_\pm^{(a)} \in \mathbb{N}$, or similarly that $d_{a,b} \in \mathbb{Z}$. Before proceeding in this direction, let us notice that the fluxes along the ramification divisors D_a

$$\mathfrak{q}_i^a \equiv \frac{\hat{\mathfrak{q}}_i^a}{m_a} = \frac{g_c}{2\pi}\int_{D_a} F_i\,, \tag{4.3.6}$$

can be similarly written as

$$\begin{aligned}
\mathfrak{q}_R^1 &= -\frac{\tau^{(y_+)}}{d_{4,1}} - \frac{\tau^{(y_-)}}{d_{1,2}} - \tau^{(x_-)}\frac{t_K}{d_{4,1}d_{1,2}}\,, \qquad &\mathfrak{q}_R^2 &= -\frac{\tau^{(x_+)}}{d_{2,3}} - \frac{\tau^{(x_-)}}{d_{1,2}} + \tau^{(y_-)}\frac{t_J}{d_{1,2}d_{2,3}}\,,\\
\mathfrak{q}_R^3 &= -\frac{\tau^{(y_+)}}{d_{3,4}} - \frac{\tau^{(y_-)}}{d_{2,3}} + \tau^{(x_+)}\frac{t_K}{d_{2,3}d_{3,4}}\,, \qquad &\mathfrak{q}_R^4 &= -\frac{\tau^{(x_+)}}{d_{3,4}} - \frac{\tau^{(x_-)}}{d_{4,1}} - \tau^{(y_+)}\frac{t_J}{d_{3,4}d_{4,1}}\,.
\end{aligned} \tag{4.3.7}$$

and satisfy

$$\mathfrak{q}_R^a = \sum_b D_{ab}\,\sigma^b\,, \quad \sigma^a = -\left(\tau^{(x_+)}, \tau^{(y_+)}, \tau^{(x_-)}, \tau^{(y_-)}\right)\,. \tag{4.3.8}$$

As conjectured around (2.2.6), the vector of twists consists of signs, which in this case are related to supersymmetry through (4.1.25), specifically $\Delta_z^{\tau^{(z_\pm)}}(z_\pm) = 0$ for $z = x, y$.

Coming back to the quantization conditions $d_{a,b} \in \mathbb{Z}$, from (4.1.24) we obtain[6]

$$\begin{aligned}
a &= -\kappa\,\frac{x_+^{D-4}(1-gx_+) - x_-^{D-4}(1-gx_-)}{x_+^{D-5}(1-gx_+) - x_-^{D-5}(1-gx_-)}\,,\\
N_x &= (-\kappa a)\,\frac{x_+^{D-5}x_-^{D-5}(x_+-x_-)(1-gx_+)(1-gx_-)}{2[x_+^{D-5}(1-gx_+) - x_-^{D-5}(1-gx_-)]}\,,
\end{aligned} \tag{4.3.9}$$

whereas applying the same method to $\Delta_y^\pm$ we are able to isolate a and N_y in terms of $y_\pm$

$$\begin{aligned}
a &= -\kappa\,\frac{y_+^{D-4}(1-gy_+) + y_-^{D-4}(1+gy_-)}{y_+^{D-5}(1-gy_+) - y_-^{D-5}(1+gy_-)}\,,\\
N_y &= (-\kappa a)\,\frac{y_+^{D-5}y_-^{D-5}(y_++y_-)(1-gy_+)(1+gy_-)}{2[y_+^{D-5}(1-gy_+) - y_-^{D-5}(1+gy_-)]}\,.
\end{aligned} \tag{4.3.10}$$

[6]From now on, to avoid clumsy notation we take $0 < y < x$, which results in $\tau^{(x_+)} = \tau^{(x_-)} = \tau^{(y_+)} = -1$ and $\tau^{(y_-)} = +1$.

Now we have two different expressions for the parameter a, namely $a(x_\pm)$ and $a(y_\pm)$, which should be equal. However, we need first a relation between $x_\pm$ and $y_\pm$. From (4.2.11), we obtain

$$\frac{d_{1,2}\,\mathfrak{q}_R^2}{d_{4,1}\,\mathfrak{q}_R^4} = \frac{x_- + y_-}{x_- - y_+}\,, \qquad \frac{d_{2,3}\,\mathfrak{q}_R^2}{d_{3,4}\,\mathfrak{q}_R^4} = \frac{x_+ + y_-}{x_+ - y_+}\,, \tag{4.3.11}$$

which can be inverted to

$$x_- = \frac{d_{1,2}\mathfrak{q}_R^2\,y_+ + d_{4,1}\mathfrak{q}_R^4\,y_-}{d_{1,2}\mathfrak{q}_R^2 - d_{4,1}\mathfrak{q}_R^4}\,, \qquad x_+ = \frac{d_{2,3}\mathfrak{q}_R^2\,y_+ + d_{3,4}\mathfrak{q}_R^4\,y_-}{d_{2,3}\mathfrak{q}_R^2 - d_{3,4}\mathfrak{q}_R^4}\,. \tag{4.3.12}$$

We now have, schematically, to solve

$$a(y_\pm) = a(y_\pm, d_{a,a+1}, \mathfrak{q}_R^a) = a(y_\pm, d_{a,a+1}, \tau^{(x_\pm)}, \tau^{(y_\pm)}, t_J, t_K) = a(y_\pm, d_{a,b}, \tau^{(x_\pm)}, \tau^{(y_\pm)})\,, \tag{4.3.13}$$

where recall $t_J = d_{1,3}$ and $t_K = d_{2,4}$ from (4.2.16). If we now manage to express $y_\pm$ in terms of $d_{a,b}$, we have a constraint between integers. To do so, we first write down the expressions of t_J and t_K in the supersymmetric case, which simplify to

$$t_J = \frac{\varkappa_D (2\pi)^2}{\Delta\psi\Delta\phi}\frac{4\Xi a^2(1-\kappa a g)(x_+ - x_-)}{g\Delta_x'(x_+)\Delta_x'(x_-)}\,, \qquad t_K = \frac{\varkappa_D (2\pi)^2}{\Delta\psi\Delta\phi}\frac{4\Xi a^2(1-\kappa a g)(y_+ + y_-)}{g\Delta_y'(y_+)\Delta_y'(y_-)}\,, \tag{4.3.14}$$

and give also

$$\frac{\Delta\psi}{2\pi}\frac{\Delta\phi}{2\pi} = -\varkappa_D\,\frac{a^2(1+\kappa\,ag)}{4g\,g_c^2(y_+ + y_-)^3}\,\frac{(t_J\mathfrak{q}_R^1\mathfrak{q}_R^3)^2}{t_K\mathfrak{q}_R^2\mathfrak{q}_R^4}\,. \tag{4.3.15}$$

Having the expressions of t_J and t_K, we immediately obtain

$$\Delta_y'(y_-) = -\frac{4\mathfrak{q}_R^2}{t_J\mathfrak{q}_R^1\mathfrak{q}_R^3}\,g_c(1-\kappa a g)(y_+ + y_-)^2\,, \quad \Delta_y'(y_+) = \frac{4\mathfrak{q}_R^4}{t_J\mathfrak{q}_R^1\mathfrak{q}_R^3}\,g_c(1-\kappa a g)(y_+ + y_-)^2\,. \tag{4.3.16}$$

If we compare these equations with the ones in (4.1.27), and define also $y_\pm = w(1 + \mathtt{x})$,we obtain an expression for w and $\mathtt{x}$, which depends on D. Plugging back to (4.3.13), we finally achieved our goal.

D=6

Comparing (4.1.27) and (4.3.16), we obtain

$$w = -\frac{3(\mathfrak{Q}_+ + \mathfrak{Q}_-)\mathtt{x} - \mathtt{x}^3 \pm \sqrt{3 - 6(\mathfrak{Q}_+ + \mathfrak{Q}_-) - 3(1 - 3(\mathfrak{Q}_+ + \mathfrak{Q}_-)^2)\mathtt{x}^2 + \mathtt{x}^4}}{g[3 + \mathtt{x}^4 - 6(\mathfrak{Q}_+ + \mathfrak{Q}_-)(1 + \mathtt{x}^2)]}\,, \tag{4.3.17}$$

where we defined $\mathfrak{Q}_+ = \frac{\mathfrak{q}_R^4}{t_J\mathfrak{q}_R^1\mathfrak{q}_R^3}$ and $\mathfrak{Q}_- = \frac{\mathfrak{q}_R^2}{t_J\mathfrak{q}_R^1\mathfrak{q}_R^3}$ and $\mathtt{x}$ is solution of the cubic equation

$$\mathtt{x}^3 - 3(\mathfrak{Q}_+ - \mathfrak{Q}_-)\mathtt{x}^2 - 3[1 - 2(\mathfrak{Q}_+ + \mathfrak{Q}_-)]\mathtt{x} + 3(\mathfrak{Q}_+ - \mathfrak{Q}_-) = 0\,. \tag{4.3.18}$$

The two signs in (4.3.17) generate the two sets of physical parameters connected by the inversion symmetry (4.1.9). Equation (4.3.13) gives now the following Diophantine constraint for the integers $d_{a,b}$

$$t_K\mathfrak{q}_R^2\mathfrak{q}_R^4 - 3(\mathfrak{q}_R^1 - \mathfrak{q}_R^3) = 0\,. \tag{4.3.19}$$

The origin of this constraint can be traced back in the specific structure of our solution. Indeed, we have four parameters $(a, N_x, N_y, \Delta\psi\Delta\phi)$ and six minus one integers $d_{a,b}$ (recall (2.1.62)). This means that there must be a redundancy among them and equation (4.3.19) precisely resolves it[7]. It is possible that the constraint (4.3.19) would be eliminated if a more general solution could be constructed, but we will not pursue this here. Similar unnatural Diophantine constraints have been observed in previous solutions associated with orbifolds [61, 65, 1].

D=7

We now focus on $D = 7$ and solve system (4.3.16) following the same method adopted in the six-dimensional case. Parameterizing again the two roots as $y_\pm = w(1 \pm \mathrm{x})$, we obtain

$$w = -\frac{4(\mathfrak{Q}_+ + \mathfrak{Q}_-)\mathrm{x} \pm \sqrt{2}\sqrt{[1-2(\mathfrak{Q}_+ + \mathfrak{Q}_-)][1+(1-4(\mathfrak{Q}_+ + \mathfrak{Q}_-))\mathrm{x}^2]}}{2g[1+\mathrm{x}^2 - 2(\mathfrak{Q}_+ + \mathfrak{Q}_-)(1+3\mathrm{x}^2)]}\,, \tag{4.3.20}$$

$$\mathrm{x} = \frac{\mathfrak{Q}_+ - \mathfrak{Q}_-}{1 - 3(\mathfrak{Q}_+ + \mathfrak{Q}_-)}\,, \tag{4.3.21}$$

where $\mathfrak{Q}_\pm$ are defined as in the previous case. As before, the two signs in (4.3.20) correspond to the two configurations related by (4.1.9). Imposing the consistency condition that comes equating (4.3.13) we obtain the following Diophantine equation

$$\begin{aligned}
&2\mathfrak{q}_R^1\mathfrak{q}_R^3 t_J\big[(\mathfrak{q}_R^1)^3 d_{1,2}d_{4,1}(d_{1,2}-d_{4,1}) - (\mathfrak{q}_R^3)^3 d_{2,3}d_{3,4}(d_{2,3}-d_{3,4})\big]\\
&\quad + \mathfrak{q}_R^1\mathfrak{q}_R^3 t_J\big[8\big((\mathfrak{q}_R^1)^2 d_{1,2}d_{4,1} - (\mathfrak{q}_R^3)^2 d_{2,3}d_{3,4}\big) - \big((\mathfrak{q}_R^1)^4 d_{1,2}^2 d_{4,1}^2 - (\mathfrak{q}_R^3)^4 d_{2,3}^2 d_{3,4}^2\big)\big]\\
&\quad + 4(\mathfrak{q}_R^1)^3 d_{1,2}d_{4,1}\big(\mathfrak{q}_R^2 d_{4,1} - \mathfrak{q}_R^4 d_{1,2}\big) - 4(\mathfrak{q}_R^3)^3 d_{2,3}d_{3,4}\big(\mathfrak{q}_R^2 d_{3,4} - \mathfrak{q}_R^4 d_{2,3}\big)\\
&\quad + 3(\mathfrak{q}_R^2 + \mathfrak{q}_R^4)\big[(\mathfrak{q}_R^1)^4 d_{1,2}^2 d_{4,1}^2 - (\mathfrak{q}_R^3)^4 d_{2,3}^2 d_{3,4}^2\big] = 0\,.
\end{aligned} \tag{4.3.22}$$

Although this is surprisingly much more complicated than the analogous constraint (4.3.19) in $D = 6$, similar comments apply. The analysis of these Diophantine equations (in $(D = 6, 7)$) are reported in appendix D of [2].

4.4 Entropy and central charge

We close this section by considering the regularity of the solutions in $d = 10, 11$ and compute the entropy and central charge of the solutions in $D = 6, 7$ employing the methods developed in section 3.2.

The reduced functions (3.2.9) for the solutions at hand read

$$\Delta_H(x,y) = H_1H_2\mu_0^2 + H_2\mu_1^2 + H_1\mu_2^2\,, \quad U_H(x,y) = 2\big[(1-H_1)(1-H_2)\mu_0^2 - 1\big] - \frac{10-D}{3}\Delta_H\,. \tag{4.4.1}$$

[7]Considering a system with different charges δ_i would increase by one the number of parameters, but would also bring an additional "quantum number" into play, thus not changing the balance.

In term of these, we can write the uplifted metric (in the string frame for the massive type IIA case)

$$\begin{aligned}
\mathrm{d}s^2_{D+4} =&\lambda^2\mu_0^{(D-7)/3}\Delta_H^{1/(D-4)}\Big\{(H_1H_2)^{-1/(D-2)}\mathrm{d}s^2_{(D)} \\
&+g_c^{-2}\Delta_H^{-1}\big[\mathrm{d}\mu_0^2+H_1\big(\mathrm{d}\mu_1^2+\mu_1^2\sigma_1^2\big)+H_2\big(\mathrm{d}\mu_2^2+\mu_2^2\sigma_2^2\big)\big]\Big\}\,, \\
\sigma_i \equiv&\mathrm{d}\phi_i-g_cA_i^{(D)}\,,\quad \Delta\phi_i=2\pi\,,
\end{aligned} \tag{4.4.2}$$

and the relevant terms of the four-form $F_{(4)}$

$$\begin{aligned}
F_{(4)} =& -\frac{\lambda^{(2D-11)}}{g_c^3}\frac{H_1H_2\,U_H}{\Delta_H^2}\frac{\mu_1\mu_2}{\mu_0^{(D-4)/3}}\,\mathrm{d}\mu_1\wedge\mathrm{d}\mu_2\wedge\sigma_1\wedge\sigma_2 \\
&-2(D-5)\lambda^{(2D-11)}\mu_0^{(10-D)/3}\frac{gs_1s_2}{\Xi a^2}\frac{N_yx^{D-3}-N_xy^{D-3}}{(xy)^{D-4}}\,\mathrm{d}y\wedge\mathrm{d}\phi\wedge\mathrm{d}x\wedge\mathrm{d}\psi+\dots\,,
\end{aligned} \tag{4.4.3}$$

The other relevant fields are, as usual, the ten-dimensional dilaton and Romans mass $F_{(0)}$

$$\mathrm{e}^{\Phi}=\lambda^2\mu_0^{-5/6}\Delta_H^{1/4}\,,\quad F_{(0)}=\frac{m}{\lambda^3}=\frac{2g_c}{3\lambda^3}\,. \tag{4.4.4}$$

Writing the metrics in the form (3.2.10), we can extract the factor $\exp(2A)$ and obtain easily

$$S=\frac{1}{(2\pi\ell_s)^8}\frac{48\pi^6\lambda^4}{5|\Xi|a^2g_c^4}\frac{\Delta\psi}{2\pi}\frac{\Delta\phi}{2\pi}\big|(x_+^3-x_-^3)(y_+-y_-)-(x_+-x_-)(y_+^3-y_-^3)\big|\,, \tag{4.4.5}$$

$$c=\frac{1}{(2\pi\ell_s)^9}\frac{64\pi^6\lambda^9}{|\Xi a^3|g_c^4}\frac{\Delta\psi}{2\pi}\frac{\Delta\phi}{2\pi}\big|(x_+^4-x_-^4)(y_+^2-y_-^2)-(x_+^2-x_-^2)(y_+^4-y_-^4)\big|\,. \tag{4.4.6}$$

The number of D4- or M5-branes is, as usual, given by (3.2.17). In terms of it, the entropy (4.4.5) and the central charge (4.4.6) take the form

$$S=\frac{9\sqrt{2}\pi N^{5/2}}{5\sqrt{8-N_f}}\frac{g^4}{|\Xi|a^2}\frac{\Delta\psi}{2\pi}\frac{\Delta\phi}{2\pi}\big|(x_+^3-x_-^3)(y_+-y_-)-(x_+-x_-)(y_+^3-y_-^3)\big|\,, \tag{4.4.7}$$

$$c=4N^3\frac{g^5}{|\Xi a^3|}\frac{\Delta\psi}{2\pi}\frac{\Delta\phi}{2\pi}\big|(x_+^4-x_-^4)(y_+^2-y_-^2)-(x_+^2-x_-^2)(y_+^4-y_-^4)\big|\,. \tag{4.4.8}$$

As last step in the quantization of the fluxes, we consider the second constraint coming from (3.2.12). Fixing $\xi=\pi/2$, namely the pole of the (hemi)sphere $\mathbb{S}^4$, the total flux across $\mathbb{M}_4$ reads

$$K=N\frac{4g_c^3gs_1s_2}{(D-3)(D-5)\Xi a^2}\frac{\Delta\psi}{2\pi}\frac{\Delta\phi}{2\pi}\big[N_y(y_+^{5-D}-y_-^{5-D})(x_+^2-x_-^2)-N_x(x_+^{5-D}-x_-^{5-D})(y_+^2-y_-^2)\big]. \tag{4.4.9}$$

Although the expression seems rather involved, making use of the various results of the section 4.3 and crucially employing the Diophantine constraints in both $D=6,7$, it is possible to write K in the following simple form

$$K=\varkappa_D\,N\,\frac{d_{1,2}d_{2,3}(\mathfrak{q}_R^2)^2-d_{3,4}d_{4,1}(\mathfrak{q}_R^4)^2}{4(D-5)d_{1,3}}=\varkappa_D\,N\,\frac{d_{2,3}d_{3,4}(\mathfrak{q}_R^3)^2-d_{4,1}d_{1,2}(\mathfrak{q}_R^1)^2}{4(D-5)d_{2,4}}\,. \tag{4.4.10}$$

When the physical parameters are properly quantized, all the elements appearing in (4.4.10) are rational, thus N can be tuned appropriately in order to make K integer.

4.5 Entropy functions

We now apply the gravitational blocks prescription of section 2.2 to the $\mathrm{AdS}_{D-4} \times \mathbb{M}_4$ backgrounds with equal charges presented in section 4.1. We will use the toric data computed in section (4.2.1) and the labels of section 4.2.3. Moreover, the vector of twists has been computed in (4.3.8) and, for $x, y > 0$, reduces to

$$\sigma^a = \big(\mathrm{sign}(x^2 - y^2), -\,\mathrm{sign}(x^2 - y^2), +, +\big) \ . \tag{4.5.1}$$

Finally, recall that when the charges are equal we can always pick a gauge with $\vec{W} = 0$ and $\varphi_1^* = \varphi_2^* = 1$ – see the discussion at the end of 2.2.2.

Extremizing the off-shell free energy with respect to the remaining variables $\epsilon_{1,2}$ we obtain the following critical values for the fugacities[8]

$$d = 5: \quad \epsilon_1^* = \frac{2}{g}\frac{2\pi}{\Delta\psi}\,, \quad \epsilon_2^* = -\kappa 2a\,\frac{2\pi}{\Delta\phi}\,, \tag{4.5.2}$$

$$d = 6: \quad \epsilon_1^* = \frac{1}{g}\frac{2\pi}{\Delta\psi}\,, \quad \epsilon_2^* = -\kappa a\,\frac{2\pi}{\Delta\phi}\,. \tag{4.5.3}$$

Plugging these critical values in (2.2.10), we get the off-shell free energy at its extremum[9]

$$F^*_{d=5} = \frac{9\sqrt{2}\pi N^{5/2}}{5\sqrt{8-N_f}}\,\frac{g^3(1-\kappa a g)}{|\Xi| a^2}\,\frac{\Delta\psi}{2\pi}\,\frac{\Delta\phi}{2\pi}\big|(x_+^2 + \tau^{(x_-)}x_-^2)\Delta y - \Delta x(y_+^2 - \tau^{(x_-)}y_-^2)\big|\,, \tag{4.5.4}$$

$$F^*_{d=6} = \kappa\, 4N^3\,\frac{g^4(1-\kappa a g)}{|\Xi| a^3}\,\frac{\Delta\psi}{2\pi}\,\frac{\Delta\phi}{2\pi}\big|(x_+^3 + \tau^{(x_-)}x_-^3)(y_+^2 - y_-^2) - (x_+^2 - x_-^2)(y_+^3 - \tau^{(x_-)}y_-^3)\big|\,. \tag{4.5.5}$$

Even if these formulas seem to be different with respect to their gravitational counterparts of the previous section, the results are indeed equal. This can be seen by explicitly using the following identity

$$\begin{aligned}&\frac{(\kappa a g - 1)\tau^{(x_-)}}{g}\Big[\big(x_+^{D-4} + \tau^{(x_-)}x_-^{D-4}\big)\big(y_+^{D-5} - y_-^{D-5}\big) - \big(x_+^{D-5} - x_-^{D-5}\big)\big(y_+^{D-4} - \tau^{(x_-)}y_-^{D-4}\big)\Big]\\ &\quad = \big[x_+^{D-5} - x_-^{D-5}\big]\big[y_+^{D-5} - y_-^{D-5}\big]\big|x_+^2 + (7-D)x_+x_- + x_-^2 - (y_+^2 + (7-D)y_+y_- + y_-^2)\big|\,,\end{aligned} \tag{4.5.6}$$

which comes from the constraint $a = a$ in (4.3.13). There is only a caveat here: there is a different sign between $F^*_{d=6}$ and the gravitational central charge (4.4.8). We believe that the origin of the opposite sign can be traced back to the chirality of the dual two-dimensional SCFTs, but since we have not solved the Killing spinor equations of $D = 7$ supergravity, we have no control over this information.

4.6 Summary of the chapter

In this chapter, as in chapter 3, we have studied a solution to minimal gauged supergravity in $D = 6$, which can also be extended to be a solution of a similar theory in $D = 7$.

[8] These results hold when $\varkappa_D > 0$. In the opposite case, we need to exchange the two components of the vectors in (4.2.10) and, therefore, to exchange ϵ_1^* and ϵ_2^* accordingly.

[9] In order to take into account a possible exchange in the components of $\vec{v}_a$, which would affect $d_{a,a+1}$ in the denominator of (2.2.10), we multiplied the off-shell free energy by $\varkappa_D$.

Collectively, the solutions take the form $\mathrm{AdS}_{D-4} \times \mathbb{M}_4$, where $\mathbb{M}_4$ is a four-dimensional toric orbifold of similar nature, regardless of whether the spacetime dimension is $D = 6$ or $D = 7$. After presenting the local form of the solution, we restricted our analysis, for technical reasons, to a simplified setup with $\delta_1 = \delta_2 \equiv \delta$, corresponding to a system with equal charges and gauge fields. Despite being simpler, this setup captures all the essential features of the four-dimensional toric orbifold $\mathbb{M}_4$, allowing for a detailed examination of its structure and a reliable test of the gravitational blocks conjecture outlined in section 2.2. In the equal charges case, and only for $D = 6$, we explicitly constructed the Killing spinor solving the Killing spinor equations in section 4.1.1. For the purposes of this chapter, it is sufficient to know that the supersymmetry condition for both cases is $ags^2 = \pm 1$. To extract the toric data, we utilized the tools developed in chapter 2, particularly the adjunction formula in the form of (2.1.59). By comparing the information obtained from analyzing the degeneration of the Killing vectors ξ_a with that derived from the metric on each branch divisor $\hat{D}_a$, we determined the various labels m_a of the system. Section 4.2 reveals that each divisor of $\mathbb{M}_4$ is a spindle, characterized by distinct integers $\mathtt{m}_\pm^{(a)}$. This feature distinguishes the quadrilateral solution from those in chapter 3, where two opposite divisors are equal (in homology), namely $\hat{D}_1 = \hat{D}_3 = \Sigma_1$ (see figure 3.3a). After studying the global regularity of the solution in section 4.3, we computed the entropy and central charge of the black objects with $\mathrm{AdS}_{D-4} \times \mathbb{M}_4$ in their near-horizon geometry in section 4.4. Finally, once again, our conjectural off-shell free energy passed the test in section 4.5, achieving a successful match with the gravitational results.

Chapter 5

Supergravity solutions with non-compact quadrilaterals

In this chapter, we construct a solution to four-dimensional minimal gauged supergravity, which, from a toric perspective, is described by a non-compact quadrilateral. Specifically, the underlying toric orbifold is defined by a two-dimensional (labeled) polytope obtained by removing (or sending to infinity) one of its four facets, as illustrated in figures 2.2c and 2.3b. Consequently, these *spindle bolt* solutions have a boundary and are asymptotically locally the hyperbolic space $\mathbb{H}^4$, allowing for a holographic interpretation [107]. Following standard nomenclature [104], we will say that our solutions feature a spindle bolt, a two-dimensional surface $\Sigma_{(2)}$ fixed by a Killing vector. Indeed, the bulk metric describes the topology of a line bundle over the spindle, *i.e.* $\mathcal{M}_4 = \mathbb{C}/\mathbb{Z}_v \hookrightarrow \mathcal{O}(-t) \to \mathbb{\Sigma}_{[m_-,m_+]}$, where the bolt is $\Sigma_{(2)} = \mathbb{\Sigma}_{[m_-,m_+]}$. Supersymmetric solutions of this type with a spherical bolt $\Sigma_{(2)} = S^2$ can be found in [111], later generalized in [137] to include cases where $\Sigma_{(2)} = \Sigma_{\mathrm{g}}$. Solutions with a bolt generalize those containing a *nut*, a zero-dimensional fixed submanifold $\Sigma_{(0)}$, *i.e.*, a fixed point. Supersymmetric solutions with $SU(2) \times U(1)$-invariant nuts are presented in [111], while $U(1)\times U(1)$-symmetric ones are studied in [96], based on a Carter-Plebański metric [138, 139]. These and other relevant solutions are reviewed in appendix B. Since we are interested in a $U(1)^2$-invariant bulk solution with $\mathcal{M}_4$ as before, we could begin with the same local Carter-Plebański class of solutions. The Carter-Plebański solution was later generalized by Plebański-Demianski [139] through the inclusion of a non-trivial parameter, which, in Lorentzian signature, can be interpreted as an acceleration parameter $\mathbf{A}$. The resulting solution retains the $U(1) \times U(1)$ symmetry, making the Plebański-Demianski class a natural starting point for the searching of the spindle bolt solutions.

We will start presenting the Plebański-Demianski class of solutions and investigating its local properties. Since the presence of $\mathbf{A}$ will make the analysis technically cumbersome, soon we will set $\mathbf{A} = 0$. Although some small progress may also be made in the global analysis of the Plebański-Demianski solutions (see [3] for details), in this chapter we are mainly interested in showing how the topology of a spindle bolt can be obtained and which implications this has on the boundary geometry $\mathcal{M}_3 = L(t, 1)$. For these reasons, our approach will focus on providing as much local information as possible regarding the Plebański-Demianski solution first, conducting then a detailed global analysis in the Carter-Plebański context. Here we will demonstrate how the boundary data, namely a metric $\mathrm{d}s^2_{(3)}$ on a lens space $L(t,1)$, a gauge field $A_{(3)}$ and a spinor $\chi_{(3)}$ are "informed" (through some flat connections for $A_{(3)}$) about (or by) the bulk data. These are the spindle data $m_\pm$, the integer v and a sign $\sigma = \pm 1$ determining if the twist or the anti-twist are realized on the spindle $\Sigma_{[m_-,m_+]}$. Interestingly, the solutions we will present can realize both the twist or the anti-twist, according to the values of the parameters in it. This dichotomy is reflected in the on-shell action, which is completely fixed in terms of $(t, m_\pm, v)$ for the twist ($\sigma = 1$) to

$$\text{twist}: \quad S_{\text{ren}} = \frac{\pi}{8G_4 v}\left[2\,\frac{m_- + m_+}{m_- m_+} - \kappa\frac{t/v}{m_- m_+} - \kappa\frac{v(m_- - m_+)^2}{t\, m_- m_+}\right].$$

Differently, for the anti-twist, the action assumes a complicated form, with the property that it depends on a (single) continuous free parameter. This is the same behaviour observed before in the other solutions containing nuts and bolts. Indeed, in appendix B we show that our solution contains all the previous ones in [109, 110, 111, 96]. Moreover, by an analytic continuation to Lorentzian signature, it is also related to the supersymmetric accelerating four-dimensional black hole [46], whose thermodynamics and action have been studied in [47]. Using the results of equivariant localization [90, 84], we will recover our results, showing the validity of their methods and making also some interesting comments about the extremization of the action.

5.1 Maxwell-Einstein-Λ supergravity and uplift to M-theory

In this section we start presenting the supergravity model of interest, which is the Maxwell-Einstein-Λ theory, or, alternatively, the (bosonic sector of the) $d = 4$, $\mathcal{N} = 2$ minimal gauged supergravity [140], whose Euclidean action reads

$$S_E = -\frac{1}{16\pi G_4}\int \mathrm{d}^4x\sqrt{g}\,\left(R - F_{\mu\nu}F^{\mu\nu} - 2\Lambda\right), \quad \Lambda = -\frac{3}{\ell^2}. \tag{5.1.1}$$

Here R is the Ricci scalar of the four dimensional metric g and $F = \mathrm{d}A$ is the field strength of the abelian graviphoton A. The equations of motion stemming from (5.1.1) are

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = 2\left(F_{\mu\rho}F_\nu{}^\rho - \frac{F^2}{4}g_{\mu\nu}\right), \quad \mathrm{d}\star F = 0\,. \tag{5.1.2}$$

A solution to the equation of motions (5.1.2) is supersymmetric if and only if it admits at least one non-identically zero Dirac spinor ε satisfying the Killing spinor equation

$$\hat{D}_\mu\varepsilon \equiv \left(D_\mu - \frac{\mathrm{i}}{\ell}A_\mu + \frac{1}{2\ell}\Gamma_\mu + \frac{\mathrm{i}}{4}F_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu\right)\varepsilon = 0\,, \quad D_\mu\varepsilon = \partial_\mu\varepsilon + \frac{1}{4}\omega_{\mu ab}\Gamma^{ab}\varepsilon\,. \tag{5.1.3}$$

Here Γ_a, $a = 1,\ldots,4$, generate the Clifford algebra Cliff$(4,0)$, so that $\{\Gamma_a,\Gamma_b\} = 2\delta_{ab}\mathrm{Id}_{4\times 4}$. Alternatively, one can formulate the problem in terms of bilinears in the Killing spinor ε. This was done in Lorentzian signature in [141] and in Euclidean signature in [90]. However, for most of the analysis we are going to perform, we will be only interested in knowing the conditions under which the solution is supersymmetric. In particular, the integrability condition $M_{\mu\nu}\varepsilon \equiv [\hat{D}_\mu, \hat{D}_\nu]\varepsilon = 0$ of (5.1.3) reduces, after some work, to

$$\begin{aligned} M_{\mu\nu}\varepsilon \equiv \Big(\frac{1}{4}R_{\mu\nu}{}^{ab}\Gamma_{ab} + \frac{1}{2}\Gamma_{\mu\nu} - \mathrm{i}F_{\mu\nu} + \frac{\mathrm{i}}{2}\Gamma_{ab}\nabla_{[\mu}F^{ab}\Gamma_{\nu]} + \frac{\mathrm{i}}{4}F_{ab}\Gamma_{[\mu}\Gamma^{ab}\Gamma_{\nu]} \\ - \frac{1}{16}F_{ab}F_{cd}[\Gamma^{ab}\Gamma_\mu, \Gamma^{cd}\Gamma_\nu] + \frac{\mathrm{i}}{4}F_{ab}\Gamma^{ab}\Gamma_{\mu\nu}\Big)\varepsilon = 0\,, \end{aligned} \tag{5.1.4}$$

and we will impose $\det(M_{\mu\nu}) = 0$.

Using the formulas of [142, 143] one can uplift locally any supersymmetric solution of the four-dimensional theory to a supersymmetric solution of M-theory. In particular, it is possible to show that the Ansatz

$$\mathrm{d}s^2_{11\mathrm{d}} = L^2\left[\frac{1}{4}\mathrm{d}s^2_{4\mathrm{d}} + \left(\eta + \frac{1}{2}A_{4\mathrm{d}}\right)^2 + \mathrm{d}s^2_T\right], \quad F_{(4)} = L^3\left[\frac{3}{8}\star_{4\mathrm{d}} 1 - \frac{1}{4}\star_{4\mathrm{d}} F_{4\mathrm{d}} \wedge \mathrm{d}\eta\right], \tag{5.1.5}$$

with $R^{(7)}_{mn} = 6g^{(7)}_{mn}$, indeed leads to a solution of the equations of motion stemming from (3.2.2) for any solution of (5.1.2). Here, as in [111], the effective radius L of AdS_4 is determined by the quantization of $\star_{(11\mathrm{d})}F_{(4)}$ through SE_7, which is a Sasaki-Einstein seven-manifold with contact one-form η and transverse six-dimensional Kähler-Einstein metric $\mathrm{d}s^2_T$. Although for irregular SE_7 the uplift is not possible [111], when SE_7 is a $U(1)$ fibration over $\mathrm{d}s^2_T$ we can always write $\eta = \mathrm{d}\psi + \sigma$ with ψ being the Killing direction such that the Reeb vector is $\xi = \partial_\psi$ and the Ricci form of the Kähler-Einstein metric is $\rho_T = 4\mathrm{d}\sigma$. Then, as usual, we should require that the fibration in (5.1.5) is well-defined. Since, as anticipated, $\mathcal{M}_4 = \mathbb{C}/\mathbb{Z}_v \hookrightarrow \mathcal{O}(-t) \to \Sigma_{[m_-,m_+]}$, we will require (see also (4.3.1))

$$\frac{1}{2\pi}\int_{\mathcal{M}_4}\frac{2\pi}{\Delta\psi}\mathrm{d}\left(\mathrm{d}\psi + \frac{1}{2}A_{4\mathrm{d}}\right) = \frac{1}{2\Delta\psi}\int_{\mathcal{M}_4}F_{4\mathrm{d}} = \frac{k}{\pi I}\int_{\mathcal{M}_4}F_{4\mathrm{d}} \equiv \frac{m}{vm_-m_+}\,, \quad m \in \mathbb{Z}\,. \tag{5.1.6}$$

Here I is the Fano index of the transverse metric $I \equiv I(KE_6)$, which is the largest integer for which it exists a line bundle $\tilde{L}$ such that $L = \tilde{L}^I$, with L the canonical bundle over KE_6. The integer k divides I and is the fundamental group of SE_7, *i.e.* $\pi_1(SE_7) \simeq \mathbb{Z}_k$. Thus, for $k > 1$, the SE_7 is not simply-connected. Some examples are listed in Table 1 of [46]. Finally, the periodicity of the Reeb vector is $\Delta\psi = 2\pi I/4k$, from which the last equation descends. A caveat here is that when $m = 0$, we can always uplift (globally) the four dimensional solution, even if the Sasaki-Einstein is irregular.

5.2 Equivariant localization

One of the aim of this chapter relies in the computation of the on-shell action for the Plebański-Demianski or Carter-Plebański solutions with topology $\mathcal{M}_4 = \mathbb{C}/\mathbb{Z}_v \hookrightarrow \mathcal{O}(-t) \to \Sigma_{[m_-,m_+]}$. As anticipated, we will find that the renormalized action is fixed when the supersymmetry on the spindle is realized through the twist, whilst it depends on a single free parameter for the anti-twist solution. This pattern can be glimpsed *a priori* (*i.e.* for each supersymmetric solution comprising a metric on $\mathcal{M}_4$) by using a G-structure approach [90] or equivariant localization[1] [84].

[1]There are several others papers concerning equivariant localization of the action in various supergravity theories and in various spacetime dimensions, see section 1.2.

In [90] the on-shell action is computed by using the fact that for supersymmetric solutions with a supersymmetric Killing vector $\vec{\epsilon} = (\epsilon_1, \epsilon_2)$ the bulk metric can be written as a $U(1)$ circle fibration generated by $\vec{\epsilon}$ over a non trivial three-dimensional base. The action reduced on this base generalizes [97] by the presence of a gauge field and is then evaluated explicitly employing supersymmetry and the bilinears constructed on top of it. The final formula turns out to be dependent only on quantities at the fixed points of $\vec{\epsilon}$, which resembles the results of equivariant localization, *i.e.* the Duistermaat-Heckman [144], Berline-Vergne [145] or Atiyah-Bott theorems [146]. In fact, the procedure presented later in [84, 91] is based on the equivariant integration of a certain equivariantly closed poly-form Φ with respect to the supersymmetric Killing vector[2] $\epsilon = \varepsilon^\dagger \Gamma^\mu \Gamma_\star \varepsilon\, \partial_\mu$

$$S_{\text{on-shell}} = \frac{\pi}{2G_4}\left[\frac{1}{(2\pi)^2}\int_{\mathcal{M}_4}\Phi\right], \quad (\mathrm{d} - i_\epsilon)\Phi = 0\,, \tag{5.2.1}$$

and depends on the chirality of $\vec{\epsilon}$ at its fixed points p_A or surfaces Σ_i as well. In particular, the "off-shell" action (*i.e.* the action for a generic "Reeb" vector $\vec{\epsilon}$) is computed as

$$I_{\text{off-shell}}(c_A, \vec{\epsilon}) = \frac{\pi}{2G_4}\left[\sum_{A=1}^{f} -c_A \frac{\left[b_1^{(A)} - c_A b_2^{(A)}\right]^2}{4\, d_A\, b_1^{(A)} b_2^{(A)}} + \sum_i \int_{\Sigma_i}\left(\frac{1}{2}c_1(T\Sigma_i) - \frac{c_i}{4}c_1(N\Sigma_i)\right)\right], \tag{5.2.2}$$

where $c_1(X\Sigma_i)$ is the first Chern class of the X bundle of Σ_i, $c_{A,i}$ are signs linked to the chirality of the spinor on p_A or Σ_i and $b_I^{(A)}$ are the weights of $\vec{\epsilon}$ on each copy of $\mathbb{C}_I \subset T_{p_A}\mathcal{M}_4$. The presence of d_A, namely the order of the singularity at p_A, reflects the (possible) orbifold nature of $T_{p_A}\mathcal{M}_4$. For the toric case, the weights of $\vec{\epsilon}$ can be simply expressed in terms of the toric weights as [90]

$$b_{1,2}^{(A)} \equiv \vec{\epsilon}\cdot\vec{\mu}_{1,2}^{(A)}\,. \tag{5.2.3}$$

In particular, (5.2.2) assumes this form because in four dimensions, for a fixed Killing vector, there can be only fixed isolated points $\Sigma_{(0)} = p_A$ or fixed two-dimensional suborbifolds $\Sigma_{(2)} = \Sigma_i \subset \mathcal{M}_4$. Moreover, the (p_A, Σ_i) appearing in (5.2.2) are referred in particular to the supersymmetric Killing vector ϵ, and not to a generic Killing vector of the metric. As we will see, for our solutions ϵ degenerates at two fixed-points p_A, and henceforth we will use the "two-nuts" part of the formula, which simply reads ($f = 2$)

$$I_{\text{off-shell}}(c_A, \vec{\epsilon}) = \frac{\pi}{2G_4}\left[\sum_{A=1}^{2} -c_A \frac{\left[b_1^{(A)} - c_A b_2^{(A)}\right]^2}{4\, d_A\, b_1^{(A)} b_2^{(A)}}\right]. \tag{5.2.4}$$

In [90] it is showed that (5.2.4), when computed on the supersymmetric Killing vector $\vec{\epsilon}_*$ of a solution, reproduces the supergravity computation for the renormalized action, or

$$I_{\text{on-shell}} \equiv I_{\text{off-shell}}(c_A, \vec{\epsilon}_*) = S_{\text{ren}}\,, \tag{5.2.5}$$

in our language. Moreover (5.2.4) is exactly the result found previously in [147] in a different way for (anti-)self-dual solutions with the topology of the four-ball, *i.e.* with a

[2] We use the the notation $\vec{\epsilon}$ when the supersymmetric Killing vector is written in an effective toric basis $\vec{E}_i$, so that in this basis $\epsilon = \vec{\epsilon}\cdot\vec{E}_i$.

single nut. There it is shown that the bulk is regular only if $\epsilon_2/\epsilon_1 > 0$ or $\epsilon_2 = -\epsilon_1$, and these two distinct behaviours will be related to the twist or the anti-twist. We note that at the moment it is not known if, for a generic choice of $\vec{\epsilon}$, there exists a supergravity solution with such a specific vector. However, if it exists, its renormalized on-shell action should be given by (5.2.4). We will come back to this observation in 5.4.6.

Since for a generic set-up of two normal vectors $\vec{v}_{1,2}$, the weights can be written as

$$\vec{\mu}_1 = \pm\frac{(\vec{v}_1\cdot\vec{v}_2)\,\vec{v}_1 - |\vec{v}_1|^2\,\vec{v}_2}{d_{12}}\,, \qquad \vec{\mu}_2 = \mp\frac{|\vec{v}_2|^2\,\vec{v}_1 - (\vec{v}_1\cdot\vec{v}_2)\,\vec{v}_2}{d_{12}}\,, \tag{5.2.6}$$

it turns out that (5.2.4) assumes the following simplified form

$$I_{\text{off-shell}}(c_A, \mathrm{Q}) = \frac{\pi}{8G_4}\left[2\left(\frac{1}{d_{1,2}} + \frac{1}{d_{2,3}}\right) - \frac{c_1}{d_{1,2}}\left(\mathrm{Q}^2 + \frac{1}{\mathrm{Q}^2}\right) - \frac{c_2}{d_{2,3}}\left(\tilde{\mathrm{Q}}^2 + \frac{1}{\tilde{\mathrm{Q}}^2}\right)\right], \tag{5.2.7}$$

where $d_{a,b} = \det(\vec{v}_a, \vec{v}_b)$ as usual and we defined the "off-shell" quantities

$$\mathrm{Q}^2 = -\frac{\det(\vec{\epsilon}, \vec{v}_1)}{\det(\vec{\epsilon}, \vec{v}_2)}\,, \qquad \tilde{\mathrm{Q}}^2 = -\frac{\det(\vec{\epsilon}, \vec{v}_3)}{\det(\vec{\epsilon}, \vec{v}_2)}\,. \tag{5.2.8}$$

From (5.2.8) we deduce at a glance that (5.2.4) is invariant under rescalings of $\vec{\epsilon}$ and under $SL(2,\mathbb{Z})$ transformations (see (2.1.56)). Moreover $(Q, \tilde{Q})$ are not independent, since (2.1.62) implies

$$d_{2,3}\mathrm{Q}^2 + d_{1,2}\tilde{\mathrm{Q}}^2 = -d_{1,3}\,. \tag{5.2.9}$$

A last comment for the time being is that in the twist case, which precisely means that the Killing spinor has the same chirality at its two poles or equivalently $c_1 = c_2 \equiv \kappa$, equation (5.2.7) simplifies since it contains directly (5.2.9) and we get

$$\text{twist}: \quad I_{\text{off-shell}}(c_A, \vec{\epsilon}) = \frac{\pi}{8G_4}\left[\frac{2(d_{1,2} + d_{2,3}) + \kappa d_{1,3}}{d_{1,2}d_{2,3}} + \kappa\frac{d_{1,2}d_{\vec{\epsilon},1} + d_{2,3}d_{\vec{\epsilon},3}}{d_{1,2}d_{2,3}d_{\vec{\epsilon},1}(d_{\vec{\epsilon},2})^{-1}d_{\vec{\epsilon},3}}\right]. \tag{5.2.10}$$

5.3 The Plebański-Demianski solutions

We are interested in the Plebański-Demianski [139] family of solutions to (5.1.2), where the metric takes the form

$$\begin{aligned}\mathrm{d}s_4^2 = \frac{1}{(1-\mathbf{A}pq)^2}\Bigg[&(q^2-\omega^2p^2)\left(\frac{\mathrm{d}q^2}{\mathbf{Q}(q)} + \frac{\mathrm{d}p^2}{-\mathbf{P}(p)}\right)\\ &+ \frac{1}{q^2-\omega^2p^2}\Big(\mathbf{Q}(q)(\mathrm{d}\tau + \omega p^2\mathrm{d}\sigma)^2 - \mathbf{P}(p)(\omega\mathrm{d}\tau + q^2\mathrm{d}\sigma)^2\Big)\Bigg]\,,\end{aligned} \tag{5.3.1}$$

while the graviphoton is

$$A = \frac{\omega pP - qQ}{q^2-\omega^2p^2}\mathrm{d}\tau + pq\frac{qP - \omega pQ}{q^2-\omega^2p^2}\mathrm{d}\sigma\,, \tag{5.3.2}$$

where ω is a scaling parameter usually called "twist" [148], which can be set to one. This parameter has been proved fundamental to perform the analytic continuation from Lorentzian to Euclidean signature [149]. The metric functions are given by

$$\begin{aligned}
\mathbf{P}(p) &= \mathcal{P}(p) - 2\mathbf{A}Mp^3 + \mathbf{A}^2\big[-Q^2 + \alpha\omega^2 - P^2(\omega^2-1)\big]p^4\,,\\
\mathbf{Q}(q) &= \mathcal{Q}(q) - 2\frac{\mathbf{A}N}{\omega}q^3 + \mathbf{A}^2(-P^2+\alpha)q^4\,,
\end{aligned} \tag{5.3.3}$$

where $\mathcal{P}(p)$ and $\mathcal{Q}(q)$ are the "non-accelerating" functions

$$\begin{aligned}
\mathcal{P}(p) &= \omega^2p^4 + Ep^2 - 2\frac{N}{\omega}p - P^2 + \alpha\,,\\
\mathcal{Q}(q) &= q^4 + Eq^2 - 2Mq - Q^2 + \alpha\omega^2 - P^2(\omega^2-1)\,.
\end{aligned} \tag{5.3.4}$$

Explicitly, we have

$$\begin{aligned}
\mathbf{P}(p) &= -P^2 + \alpha - 2\frac{N}{\omega}p + Ep^2 - 2\mathbf{A}Mp^3 + \Big[\mathbf{A}^2\big[\omega^2(-P^2+\alpha) - Q^2 + P^2\big] + \omega^2\Big]p^4\,,\\
\mathbf{Q}(q) &= \big[\omega^2(-P^2+\alpha) - Q^2 + P^2\big] - 2Mq + Eq^2 - 2\frac{\mathbf{A}N}{\omega}q^3 + \big[\mathbf{A}^2(-P^2+\alpha)+1\big]q^4\,.
\end{aligned} \tag{5.3.5}$$

This solution generalizes another one found before (independently) by Carter [138] and Plebański [150] (that we will refer to as Carter-Plebański (CP) solution) by the addition of a parameter that, in Lorentzian signature, may be interpreted as acceleration. The CP solution, as considered for example in [151, 96], is obtained by turning off this parameter, *i.e.* $\mathbf{A} = 0$, and tuning $\omega = 1$: in this case the solution becomes higher symmetric in $p \leftrightarrow q$ and the system results to be more tractable. Even if in Euclidean signature, we will continue to refer to the parameter $\mathbf{A}$ as the "acceleration parameter": the parameters $(\mathbf{A}, N, M, Q, P, \omega)$ can then be loosely identified as the Euclidean counterpart of acceleration, NUT parameter, mass, electric and magnetic charge and rotation, respectively. However, the correct interpretation is more subtle and in the Lorentzian set-up we refer for example to [148, 152] for a more complete classification, where some interesting limits are also discussed. In Euclidean signature their interpretation is even more complicated, and will be a matter of discussion throughout the rest of the chapter.

We recall that the solution has a self-dual Weyl tensor (*i.e.* it is a gravitational instanton) if and only if $P = Q$ and $N = M$, and possesses a scaling symmetry

$$\begin{aligned}
(q,p) \to{}& \lambda\,(q,p)\,, \quad (Q,P,E) \to \lambda^2\,(Q,P,E)\,, \quad (N,M) \to \lambda^3\,(N,M)\,,\\
&\alpha \to \lambda^4\,\alpha\,, \quad \tau \to \lambda^{-1}\,\tau\,, \quad \mathbf{A} \to \lambda^{-2}\,\mathbf{A}\,, \quad \sigma \to \lambda^{-3}\,\sigma\,,
\end{aligned} \tag{5.3.6}$$

with $\lambda \in \mathbb{R} \setminus \{0\}$.

As anticipated, we will require a solution with a lens space $\mathcal{M}_3 = L(t,1)$ in the boundary $\mathcal{M}_3$ and with one (or more) supersymmetric fillings in the bulk $\mathcal{M}_4$ of the form $\mathcal{M}_4 = \mathbb{C}/\mathbb{Z}_v \hookrightarrow \mathcal{O}(-t) \to \Sigma_{[m_-,m_+]}$. For these reasons, we will need a "radial" coordinate (that we take to be q) and an angular one (that will therefore be p). For this set-up we can take the following conditions

$$q^2 - \omega^2p^2 > 0\,, \quad \mathbf{P}(p) < 0\,, \quad \mathbf{Q}(q) > 0\,, \quad \mathbf{q}_+ \leq q \leq \frac{1}{\mathbf{A}p}\,, \quad \mathbf{p}_- \leq p \leq \mathbf{p}_+\,. \tag{5.3.7}$$

The first three conditions ensure the correct Euclidean signature and imply $(\mathbf{Q}'(\mathbf{q}_+) > 0, \mathbf{P}'(\mathbf{p}_+) > 0, \mathbf{P}'(\mathbf{p}_-) < 0)$, whilst the other two define the range of the coordinates. In particular, $\mathbf{q}_+$ is the largest real root of the metric function $\mathbf{Q}(q)$, $\mathbf{p}_\pm$ where $q = (\mathbf{A}p)^{-1}$ is the location of the conformal boundary, $\mathbf{q}_+$ is the largest real root of $\mathbf{Q}(q)$ and $\mathbf{p}_\pm$, similarly, are the largest real zeros of $\mathbf{P}(p)$. Instead, the upper bound of the variable q is given by the location of the conformal boundary, which is pushed to infinity in the CP subcase (when $\mathbf{A} = 0$). For simplicity, and for future comparison with [96], we (partially) use the scaling symmetry (5.3.6) to require $\mathbf{p}_- + \mathbf{p}_+ \geq 0$. We thus introduce the ordering

$$\mathbf{p}_- \leq p \leq \mathbf{p}_+ \leq \mathbf{q}_+ \leq q\,, \quad \mathbf{q}_+^2 - \mathbf{p}_+^2 \geq 0\,, \quad \mathbf{p}_+^2 - \mathbf{p}_-^2 \geq 0\,. \tag{5.3.8}$$

As we will see, and as it happens in the context of black holes [73, 112, 47], we will find many cases in which the metric, as well as the graviphoton, can assume complex values. One may then wonder what the meaning is of imposing "correct Euclidean signature conditions", such as $\mathbf{Q}(q) > 0$, if in general $\mathbf{Q}(q) \in \mathbb{C}$. As in [73], we will assume that for non-supersymmetric solutions $(g_{\mu\nu}, A) \in \mathbb{R}$, so that (5.3.7) is reasonable. It is the imposition of supersymmetry itself that makes $(g_{\mu\nu}, A) \in \mathbb{C}$. However, we will require that in this more general set-up the Killing spinors have correct global properties and the bosonic fields are regular in the neighbourhood of fixed points[3]. Since holography is blind to the reality or complexity of $(g_{\mu\nu}, A)$, from the AdS/CFT point of view it is not natural to require a real metric, as done for example in [111].

5.3.1 Supersymmetry conditions

For $\mathbf{A} \equiv 0$ we will be able to construct the Killing spinor solving (5.1.3) in full generality, showing that the solution is 1/4-BPS as in [96]. However, for many applications, it is sufficient to compute the supersymmetry conditions, which are obtained as $\det(M_{\mu\nu}) = 0$, with $M_{\mu\nu}$ given in (5.1.4). To this end, we choose the coordinates to be ordered as (p, τ, σ, q), the vierbein to be

$$\begin{aligned} e^1 &= \frac{1}{(1-\mathbf{A}pq)}\sqrt{\frac{q^2-\omega^2p^2}{-\mathbf{P}(p)}}\mathrm{d}p\,, & e^2 &= \frac{1}{(1-\mathbf{A}pq)}\sqrt{\frac{-\mathbf{P}(p)}{q^2-\omega^2p^2}}(\omega\mathrm{d}\tau + q^2\mathrm{d}\sigma)\,, \\ e^3 &= \frac{1}{(1-\mathbf{A}pq)}\sqrt{\frac{\mathbf{Q}(q)}{q^2-\omega^2p^2}}(\mathrm{d}\tau + \omega p^2\mathrm{d}\sigma)\,, & e^4 &= \frac{1}{(1-\mathbf{A}pq)}\sqrt{\frac{q^2-\omega^2p^2}{\mathbf{Q}(q)}}\mathrm{d}q\,, \end{aligned} \tag{5.3.9}$$

and adopt the following explicit representation of the Clifford algebra

$$\Gamma_a = \sigma_1 \otimes \sigma_a\,, a = 1, \ldots, 3\,, \quad \Gamma_4 = -\sigma_2 \otimes \mathrm{Id}_2\,, \quad \Gamma_\star = \Gamma_1\Gamma_2\Gamma_3\Gamma_4\,, \tag{5.3.10}$$

with σ_a the standard Pauli matrices. It is then straightforward to show that $\det(M_{1,4}) = 0$ is equivalent to

$$\begin{aligned} 2(q^2 + \omega^2p^2)(N\Sigma_1 - \omega M\Sigma_2) + 4\omega(P^2 - Q^2)(\mathbf{A}qp + 1)(p\Sigma_1 + q\Sigma_2) \\ - 2pq\omega(2M\Sigma_1 - 2\omega N\Sigma_2 + \mathbf{A}\omega\Sigma_3) - \omega^2(\mathbf{A}^2p^2q^2 + 1)\Sigma_3 = 0\,, \end{aligned} \tag{5.3.11}$$

[3] This is the Euclidean counterpart of requiring regularity near the tip of the cigar in the complexified black hole geometries [73].

where

$$\begin{aligned}
-\Sigma_1 &= 2Q\Omega\omega^2 + \mathbf{A}M\Pi\omega + 2\mathbf{A}^2N(P^2-Q^2)\Phi\,,\\
-\Sigma_2 &= 2P\Omega\omega + \mathbf{A}N\Pi + 2\mathbf{A}^2M(P^2-Q^2)(-P^2+\alpha)\omega\,,\\
\Sigma_3 &= \Pi^2 - 4(P^2-Q^2)^2\big[(-P^2+\alpha)\omega^2 + P^2 + \mathbf{A}^2(-P^2+\alpha)\Phi\big]\,,
\end{aligned} \tag{5.3.12}$$

and we defined

$$\Omega = MP - NQ\,,\quad \Pi = M^2 - N^2 - E(P^2-Q^2)\,,\quad \Phi = \omega^2(-P^2+\alpha) - Q^2 + P^2\,. \tag{5.3.13}$$

Moreover it can be verified that $\Sigma_3 = \sum_{i,j=1}^{2} a_{ij}\Sigma_i\Sigma_j$ for some coefficients a_{ij}, thus we only require $\Sigma_{1,2}\equiv 0$. This system is entirely solved by the self-duality condition

$$M = N\,,\quad P = Q\,, \tag{5.3.14}$$

as in [96] or, in general, by fixing

$$\begin{aligned}
\alpha &= \frac{-\omega^2(MP-NQ)^2 + \mathbf{A}^2(P^2-Q^2)\big[N^2(P^2-Q^2) + (M^2-N^2)P^2\omega^2\big]}{\mathbf{A}^2\omega^2(P^2-Q^2)(M^2-N^2)}\,,\\
E &= \frac{-2(MP-NQ)(NP-MQ)\omega^2 + \mathbf{A}\omega(M^2-N^2)^2 + 2\mathbf{A}^2MN(P^2-Q^2)^2}{\mathbf{A}\omega(P^2-Q^2)(M^2-N^2)}\,.
\end{aligned} \tag{5.3.15}$$

Notice that these expressions are ill-defined for $\mathbf{A}=0$. We can then work out another couple of constraints

$$\begin{aligned}
M &= \frac{\omega NPQ + \mathbf{A}N(P^2-Q^2)\sqrt{P^2 - (P^2-\alpha)\Big[\omega^2 - \mathbf{A}^2\big[-P^2+Q^2+\omega^2(P^2-\alpha)\big]\Big]}}{\omega\big[P^2 - \mathbf{A}^2(P^2-Q^2)(P^2-\alpha)\big]}\,,\\
E &= \frac{M^2-N^2}{P^2-Q^2} - \frac{2\mathbf{A}\big[NP(-P^2+Q^2) + \omega^2(NP-MQ)(P^2-\alpha)\big]}{\omega(MP-NQ)}\,,
\end{aligned} \tag{5.3.16}$$

which in the CP case reduces to [151, 109]

$$\mathbf{A}=0:\quad M = \frac{NQ}{P}\,,\quad E = -\frac{N^2}{P^2} + 2\sqrt{\alpha\omega^2 + P^2(1-\omega^2)}\,, \tag{5.3.17}$$

and it further simplify for $\omega = 1$. Equations (5.3.15) and (5.3.16) are the supersymmetry constraints for the PD family of solutions and are compatible with the results in [149] (with $\mathbf{A}=1$).

5.3.2 Ambitoric structure

As introduced at the beginning of chapter 2, the (Euclidean) Plebański-Demianski metric is conformally ambiKähler [153]. Specifically, we introduce two almost symplectic structures

$$\omega_\pm = e^1\wedge e^2 \pm e^3\wedge e^4\,,\quad d\omega_\pm \neq 0\,,\quad \frac{1}{2}\omega_\pm\wedge\omega_\pm = \pm\mathrm{vol}_4\,. \tag{5.3.18}$$

It can be verified that

$$(J_\pm)_\mu{}^\nu = (\omega_\pm)_{\mu\rho}g^{\rho\nu} \implies (J_\pm)_\mu{}^\rho(J_\pm)_\rho{}^\nu = -\delta_\mu^\nu\,, \tag{5.3.19}$$

namely $J_\pm$ are two (commuting) almost complex structures, with also vanishing Nijenhuis tensors ($N_\pm$). Equivalently, it can be seen that the holomorphic (2,0)-forms

$$\Omega_\pm = (e^1 + \mathrm{i}e^2) \wedge (e^3 \pm \mathrm{i}e^4)\,, \tag{5.3.20}$$

satisfy $\mathrm{d}\Omega_\pm = \mathrm{i}P_\pm \wedge \Omega_\pm$, where $P_\pm$ are connections one-forms on the anti-canonical bundles, from which the associated Ricci curvature two-forms are given by $\rho_\pm = \mathrm{d}P_\pm$. Explicitly we find

$$P_\pm = \frac{4\mathbf{A}q\mathbf{P}(p) + (1-\mathbf{A}pq)\mathbf{P}'(p)}{2\sqrt{q^2-\omega^2p^2}\sqrt{-\mathbf{P}(p)}}e^2 \pm \frac{4\mathbf{A}p\mathbf{Q}(q) + (1-\mathbf{A}pq)\mathbf{Q}'(p)}{2\sqrt{q^2-\omega^2p^2}\sqrt{\mathbf{Q}(q)}}e^3\,. \tag{5.3.21}$$

Although the triples $(J_\pm, \omega_\pm, g)$ do not define Kähler structures, since $\omega_\pm$ are not closed, we see that they are conformally closed defining

$$\omega'_\pm = \left(\frac{1-\mathbf{A}pq}{q \mp p\omega}\right)^2 \omega_\pm \equiv \Xi_\pm^2 \omega_\pm\,, \quad \mathrm{d}\omega'_\pm = 0\,. \tag{5.3.22}$$

The rescaled triples $(g'_\pm, J'_\pm, \omega'_\pm) \equiv (\Xi^2_\pm g, J_\pm, \Xi^2_\pm \omega_\pm)$ define an *ambiKähler structure*. Indeed, one can verify the standard relation between the Ricci tensor and the Ricci form $(\mathrm{d}P'_\pm)_{\mu\nu} = (\rho'_\pm)_{\mu\nu} = (J'_\pm)^\sigma{}_\mu R'_{\sigma\nu}$, where $R'_{\mu\nu}$ is computed from the rescaled metrics $g'_\pm$ (see section 2.1.2). Since both Kähler metrics are toric, with common torus action (as will be discussed in detail in Sect. 5.4.3), these define an *ambitoric structure*. This property of the local (Euclidean) Plebański-Demianski solutions was proven in [153]. As follows from the discussion around (5.3.7), the conformal factors $\Xi^2_\pm$ are positive semi-definite, vanishing precisely at the conformal boundary of the orbifolds. This implies that the rescaled Kähler metrics $(g'_\pm, J'_\pm, \omega'_\pm)$ are defined on the "conformal compactifications", which are four-dimensional compact orbifolds with boundaries.

5.3.3 Holographic renormalization

Even if the PD system is much more complicated than the CP solution, we are able to compute the on-shell action in full generality (*i.e.* without imposing supersymmetry). Although the computation is a standard procedure in literature [154, 155], such a general result is not present in the literature, and it has been computed in numerous sub-cases (see *e.g.* [156, 96, 157, 47]). The procedure consists in adding the standard boundary term and the counter-terms to remove the singularities that appears as q approaches the conformal boundary $q = (\mathbf{A}p)^{-1}$. In particular we consider the renormalized action

$$S_{\text{ren}} = S_{E\Lambda} + S_F + S_{GH} + S_{\text{ct}}\,, \tag{5.3.23}$$

evaluated on the solution (5.3.1) and (5.3.2), where the bulk contributions are

$$S_{E\Lambda} = \frac{-1}{16\pi G_4}\int \mathrm{d}^4x\sqrt{g}\,\left(R^{(g)}+6\right)\,, \quad S_F = \frac{-1}{16\pi G_4}\int \mathrm{d}^4x\sqrt{g}\,\left(F^2\right)\,, \tag{5.3.24}$$

whilst the boundary ones are

$$S_{GH} = \frac{-1}{8\pi G_4}\int_{q=(\mathbf{A}p)^{-1}} \mathrm{d}^3x\sqrt{\gamma}\,\left(\gamma^{\mu\nu}K^{(\gamma)}_{\mu\nu}\right)\,, \quad S_{ct} = \frac{1}{8\pi G_4}\int_{q=(\mathbf{A}p)^{-1}} \mathrm{d}^3x\sqrt{\gamma}\,\left(2+\frac{1}{2}R^{(\gamma)}\right)\,. \tag{5.3.25}$$

Here $K^{(\gamma)}_{\mu\nu}$ is the second fundamental form and $R^{(\gamma)}$ is the Ricci of the boundary metric. As usual [96], the contribution of the gauge field from S_F is already finite, and we can anticipate the result

$$\begin{aligned}\frac{(16\pi G_4)S_F}{(\mathbf{p}_+ - \mathbf{p}_-)\Delta\tau\Delta\sigma} =&\mathbf{q}_+\left[\frac{(P-Q)^2}{(\mathbf{q}_+ - \omega\mathbf{p}_+)(\mathbf{q}_+ - \omega\mathbf{p}_-)} + \frac{(P+Q)^2}{(\mathbf{q}_+ + \omega\mathbf{p}_+)(\mathbf{q}_+ + \omega\mathbf{p}_-)}\right], \\ &+ \mathbf{A}(\mathbf{p}_+ - \mathbf{p}_-)\frac{2\mathbf{A}\omega PQ(\mathbf{p}_+^2 + \mathbf{p}_-^2) - (P^2+Q^2)(1+\mathbf{A}^2\omega^2\mathbf{p}_+^2\mathbf{p}_-^2)}{(1-\mathbf{A}^2\omega^2\mathbf{p}_+^4)(1-\mathbf{A}^2\omega^2\mathbf{p}_-^4)},\end{aligned} \tag{5.3.26}$$

where $(\Delta\tau, \Delta\sigma)$ are the periodicities of the angular coordinates (τ, σ) and we have integrated (p, q) simply in their range of definition (5.3.7).

Fefferman-Graham coordinates

To compute correctly the remaining terms, we need to bring the metric in the Fefferman-Graham form [158], following standard literature (see for example [156, 157]), which is

$$\mathrm{d}s^2 \underset{r\to 0^+}{\simeq} \frac{\mathrm{d}r^2}{r^2} + \frac{\mathrm{d}s_3^2}{r^2}, \tag{5.3.27}$$

where

$$\mathrm{d}s_3^2 = \mathrm{d}s^2_{(0)} + \mathrm{d}s^2_{(1)}\, r + \mathrm{d}s^2_{(2)}\, r^2 + \mathrm{d}s^2_{(3)}\, r^3 + O(r^4), \tag{5.3.28}$$

and the leading contribution is (a conformal representative of) the boundary metric, $\mathrm{d}s^2_{(0)} \equiv \mathrm{d}s^2_\gamma$. To this end we change coordinates as[4]

$$q = \frac{1}{\mathbf{A}x} - \sum_{i=1}^{4} f_i(x) r^i, \quad p = x + \sum_{i=1}^{4} g_i(x) r^i, \tag{5.3.29}$$

where all the functions except for $f_1(x)$ are determined by requiring the absence of mixed $(\mathrm{d}r\,\mathrm{d}x)$ terms in the metric[5] and that the coefficient of $\mathrm{d}r^2$ is 1. As an example, we get

$$g_1(x) = \frac{\mathbf{P}(x)}{\mathbf{A}x^2\mathbf{Q}(\frac{1}{\mathbf{A}x})} f_1(x), \tag{5.3.30}$$

where the following identity holds

$$\mathbf{Q}(\mathbf{A}^{-1}p^{-1}) = \frac{\mathbf{P}(p)}{\mathbf{A}^2 p^4} + \frac{p^{-4} - \mathbf{A}^2\omega^2}{\mathbf{A}^4}. \tag{5.3.31}$$

In passing, let us notice that $f_1(x)$ defines a conformal class for the boundary metric, and we expect that the finite terms are independent of it. For future reference, let us define a specific boundary metric $\mathrm{d}s_b^2$ given by

$$\mathrm{d}s^2_{(0)} = \frac{\mathbf{Q}(\frac{1}{\mathbf{A}x})}{f_1(x)^2(1-\omega^2\mathbf{A}^2x^4)^2}\mathrm{d}s_b^2, \tag{5.3.32}$$

[4] By inspection, we noticed that $f_5(x)$ and $g_5(x)$ do not enter in $\mathrm{d}s^2_{(3)}$.

[5] In particular, for the procedure to be correct, we have to expand the metric until $\mathrm{d}s^2_{(5)}$ and require that the Ricci scalar of the metric in (r, x) coordinates is $R^{(g)} = -12 + \mathcal{O}(r^4)$.

where

$$\mathrm{d}s_b^2 = \frac{(1-\omega^2\mathbf{A}^2x^4)^2\mathrm{d}x^2}{-\mathbf{P}(x)} - \frac{\mathbf{A}^8x^8\mathbf{Q}\big(\frac{1}{\mathbf{A}x}\big)^2\mathbf{P}(x)}{\big[1+\mathbf{A}^2\mathbf{P}(x)\big]}\mathrm{d}\sigma^2 \\ + \mathbf{A}^4x^4\big[1+\mathbf{A}^2\mathbf{P}(x)\big]\mathbf{Q}\big(\frac{1}{\mathbf{A}x}\big)\left(\mathrm{d}\tau + \frac{\omega x^2}{1+\mathbf{A}^2\mathbf{P}(x)}\mathrm{d}\sigma\right)^2 . \tag{5.3.33}$$

The strategy now is to perform a mixed analysis, where the bulk integrals will be computed in the original coordinate system (p,q), whilst the boundary ones will be in (x,r) coordinates, taking due care to the Jacobian for (5.3.29). The bulk contribution is

$$S_{E\Lambda} = -\frac{\Delta\tau\Delta\sigma}{16\pi G_4}\int_{\mathbf{p}_-}^{\mathbf{p}_+}\mathrm{d}p\int_{\mathbf{q}_+}^{q_\epsilon(p)}\mathrm{d}q\sqrt{g}\,(-12+6)\,, \tag{5.3.34}$$

where $q_\epsilon(p)$ is determined as follows. The expansion for p in (5.3.29) can be inverted order by order to give

$$x(p,r) = p+\sum_{i=1}^{4}\tilde{g}_i(p)r^i+\mathcal{O}(r^5) = p-g_1(p)r+\big[g_1'(p)g_1(p)-g_2(p)\big]r^2+\ldots+O(r^5)\,, \tag{5.3.35}$$

where the derivatives are taken with respect to p. From this mixed expression for $x(p,r)$ one easily gets $q=q(p,r)$ by inserting (5.3.35) in the q expansion (5.3.29). Finally, we define the limit of integration to be $q_\epsilon(p)\equiv q(p,r)|_{r=\epsilon}$. From this computation we get divergent terms, proportional to $\epsilon^{-3},\epsilon^{-2},\epsilon^{-1}$, and a finite contribution. The infinite parts should be cancelled from the boundary, which gives

$$S_{ct}+S_{GH} = \frac{\Delta\tau\Delta\sigma}{8\pi G_4}\int_{\mathbf{p}_-}^{\mathbf{p}_+}\mathrm{d}p|\partial_p x(p,\epsilon)|\sqrt{\gamma}\left(2+\frac{1}{2}R^{(\gamma)}-K^{(\gamma)}\right)\bigg|_{x=x(p,\epsilon)}\,, \tag{5.3.36}$$

where $\partial_p x(p,\epsilon)$ can be computed from (5.3.35). The divergent terms disappear in the procedure and after some work we can integrate the finite part of the computation, finding

$$S_{\text{ren}} = \frac{\Delta\tau\Delta\sigma}{\mathbf{A}16\pi G_4}\bigg[\frac{-\mathbf{q}_+^2}{(1-\mathbf{A}p\mathbf{q}_+)^2} - \frac{\omega^2(1-2\mathbf{A}p\mathbf{q}_+)}{\mathbf{A}^2\mathbf{q}_+^2(1-\mathbf{A}p\mathbf{q}_+)^2} + \frac{\omega^2p^2(1+16\omega^2\mathbf{A}^2p^4)}{1-\omega^2\mathbf{A}^2p^4} \\ - \frac{\mathbf{A}^2\big[64\omega^4p^6+4\omega^2p^3\mathbf{P}'(p)-[\mathbf{P}'(p)]^2\big]}{4(1-\omega^2\mathbf{A}^2p^4)} - \frac{\mathbf{P}''(p)}{6}\bigg]_{\mathbf{p}_-}^{\mathbf{p}_+} + S_F\,, \tag{5.3.37}$$

with S_F as in (5.3.26). We can now compute some interesting limits

- $\mathbf{A}=0$: despite appearances, after the result is made explicit by substituting $\mathbf{P}(p)$, the expression is finite for $\mathbf{A}\to 0$. Moreover, it coincides with the on-shell action computed in [96] in the self-duality case ($N=M, P=Q$). The CP renormalized action reads

$$\mathbf{A}=0: \quad S_{\text{ren}} = \frac{\Delta\tau\Delta\sigma(\mathbf{p}_+-\mathbf{p}_-)\mathbf{q}_+}{8\pi G_4}\bigg[\omega^2\big[\mathbf{p}_+^2+\mathbf{p}_+\mathbf{p}_-+\mathbf{p}_-^2\big]+\frac{M}{\mathbf{q}_+}-\mathbf{q}_+^2 \\ +\frac{(P-Q)^2}{2(\mathbf{q}_+-\omega\mathbf{p}_+)(\mathbf{q}_+-\omega\mathbf{p}_-)}+\frac{(P+Q)^2}{2(\mathbf{q}_++\omega\mathbf{q}_+)(\mathbf{q}_++\omega\mathbf{p}_-)}\bigg]\,. \tag{5.3.38}$$

- [47]: the PD accelerating black hole studied in [46, 47] can be obtained as a sub-case of (and analytic continuation of) the PD solution (5.3.1) and (5.3.2). In particular, a possible NUT parameter should be set to zero, and we have to impose (see *e.g.* [148])

$$N = -\omega\mathbf{A}M\,, \quad E = \frac{N\omega}{\mathbf{A}M} + \frac{\mathbf{A}}{\omega MN}\left[(M^2+N^2)(-P^2+\alpha)\omega^2 + N^2(P^2-Q^2)\right]. \tag{5.3.39}$$

With these conditions imposed we have

$$\mathbf{P}(p) = (p^2-1)\Big[P^2 - \alpha - 2\mathbf{A}Mp + \Big[\mathbf{A}^2\big[P^2-Q^2-(P^2-\alpha)\omega^2\big] + \omega^2\Big]p^2\Big]\,, \tag{5.3.40}$$

so that the roots are taken to be $\mathbf{p}_\pm = \pm 1$ [159, 46]. Moreover, to compare with [47] we identify

$$\mathbf{q}_+ = r_+\,, \quad \mathbf{A} = \alpha\,, \quad Q = -\mathrm{i}e\,, \quad P = g\,, \quad M = m\,, \quad \omega = -\mathrm{i}a\,, \quad \alpha = P^2 - 1\,. \tag{5.3.41}$$

Summarizing, (5.3.37) is the most general on-shell action for the PD solution, it is very general and is valid independently on the way in which the roots of $\mathbf{P}(p)$ are chosen. Moreover, it is valid also when supersymmetry is not imposed, and thus is more general then the result we will find using the equivariant localization formulas of section 5.2.

5.3.4 Supersymmetric Killing vector

In the next sections we will extensively use the supersymmetric Killing vector ϵ for both the PD and the CP solutions to compare the results for the renormalized on shell-actions to the ones coming from equivariant localization. As a definition, ϵ should be computed from the Killing spinor ε as

$$\epsilon = \varepsilon^\dagger\Gamma^\mu\Gamma_\star\varepsilon\,\partial_\mu\,, \tag{5.3.42}$$

with $\Gamma_\star$ defined in (5.3.10). Since for the PD solution we will not solve (5.1.3) for ε, we should use an alternative way to extract ϵ. In particular we can obtain the supersymmetric Killing vector using the boundary metric (5.3.33) and the boundary gauge field

$$A_b = \frac{\mathbf{A}p(P\omega\mathbf{A}p^2 - Q)\mathrm{d}\tau + (pP - Q\omega\mathbf{A}p^3)\mathrm{d}\sigma}{1-\omega^2\mathbf{A}^2p^4}\,, \tag{5.3.43}$$

obtained from (5.3.2) with $q = (\mathbf{A}p)^{-1}$. Following [48, 49], we write the boundary fields as

$$\begin{aligned} \mathrm{d}s_b^2 &= f^2\mathrm{d}p^2 + h_{11}\mathrm{d}\sigma^2 + 2h_{12}\mathrm{d}\sigma\mathrm{d}\tau + h_{22}\mathrm{d}\tau^2\,, \quad v^2 \equiv h_{11} + 2\mathrm{w}h_{12} + \mathrm{w}^2h_{22}^2\,, \quad h \equiv \det(h_{ij})\,, \\ A_b &= \frac{-v^3}{4f\sqrt{h}}\left[\frac{1}{\mathrm{w}}\partial_p\left(\frac{h_{11}}{v^2}\right)\mathrm{d}\sigma - \partial_p\left(\frac{h_{22}}{v^2}\right)\mathrm{d}\tau\right] + \frac{MP-NQ}{2\mathbf{A}(P^2-Q^2)}\mathrm{d}\sigma + \frac{NP-MQ}{2(P^2-Q^2)}\mathrm{d}\tau\,, \end{aligned} \tag{5.3.44}$$

with f, v and h_{ij} being function of p only. In this formalism, the supersymmetric killing vector reads[6]

$$\epsilon_* \propto \partial_\sigma + \mathrm{w}\partial_\tau\,, \tag{5.3.45}$$

[6] The overall normalization does not play any role in the following discussions.

and by comparison we obtain the relative weight w as

$$\mathrm{w} = \frac{\mathbf{A}\rho}{\omega(MQ-NP)} \frac{\alpha\omega(2\mathbf{A}MN\rho^2+\omega\mu^2) - E\rho(N^2\rho+\omega^2P^2\mu)}{2\mathbf{A}NP\rho^2+\omega(MP-NQ)\mu+2\omega^2\mathbf{A}P^2(MQ-NP)\rho}\,, \tag{5.3.46}$$

where for convenience we defined $\rho = P^2 - Q^2$ and $\mu = M^2 - N^2$. This formula still needs to be evaluated on the supersymmetric relations (5.3.15) or (5.3.16). In the simpler case of (5.3.15), we get

$$\mathbf{A} \neq 0: \quad \mathrm{w} = \frac{1}{\mathbf{A}} \frac{MP-NQ}{NP-MQ}\,, \tag{5.3.47}$$

which is ill-defined in the limit $\mathbf{A} \to 0$. Using instead (5.3.16), we obtain a complicated formula which reduces to

$$\mathbf{A} = 0: \quad \mathrm{w} = \sqrt{\alpha}\,. \tag{5.3.48}$$

We will confirm this last result in section 5.4.1, when we construct the Killing spinor ε for the non-accelerating solution.

5.4 The Carter-Plebański solutions

In this section we focus on the CP case, which is obtained by setting $A = 0$ and $\omega = 1$ on (5.3.1)-(5.3.4). Due to the symmetry of the CP solution, we will be able to complete a general analysis, including the construction of the Killing spinor. The CP solution is given by

$$\mathrm{d}s_4^2 = (q^2-p^2)\left[\frac{\mathrm{d}q^2}{\mathcal{Q}(q)}+\frac{\mathrm{d}p^2}{-\mathcal{P}(p)}\right]+\frac{1}{(q^2-p^2)}\left[\mathcal{Q}(q)(\mathrm{d}\tau+p^2\mathrm{d}\sigma)^2-\mathcal{P}(p)(\mathrm{d}\tau+q^2\mathrm{d}\sigma)^2\right], \tag{5.4.1}$$

and the graviphoton

$$A = \frac{pP-qQ}{q^2-p^2}\mathrm{d}\tau + pq\frac{qP-pQ}{q^2-p^2}\mathrm{d}\sigma\,. \tag{5.4.2}$$

The metric functions coming from (5.3.4) read

$$\mathcal{P}(p) = p^4 + Ep^2 - 2Np - P^2 + \alpha\,, \quad \mathcal{Q}(q) = q^4 + Eq^2 - 2Mq - Q^2 + \alpha\,, \tag{5.4.3}$$

where the high symmetry in (p, q) is now evident and allows for a general analysis. The main difference with respect to the accelerating case is that the conformal boundary is now located at infinity, so that

$$q_+ \leq q \leq +\infty\,, \quad p_- \leq p \leq p_+\,. \tag{5.4.4}$$

When $\mathbf{A} = 0$, the parameter N assumes the interpretation of a NUT parameter [148] and $N = 0$ becomes the condition for having a (possibly) regular Lorentzian signature (see also (5.3.39)). In [3] a complete study of the case $N = 0$ is performed, but here we are interested in the general situation. Henceforth, from now on we assume $N \neq 0$, referring to [3] for all the details.

Firstly, we continue considering local aspects and we construct the Killing spinor ε solving (5.1.3). Global issues will be addressed later on.

5.4.1 Local Killing spinors and reality of the solution

Recall that the supersymmetry conditions (5.3.17) for the Euclidean CP solution (with $\omega = 1$) read

$$MP = NQ\,, \quad E = -\frac{N^2}{P^2} + 2\sqrt{\alpha}\,. \tag{5.4.5}$$

The solution presented in [96] corresponds to the sub-case in which $P = Q$ and $N = M$ by regularity of the metric, so that $\Omega = 0$ identically (see (5.3.13)). As we will discuss, we are in the general situation for which $\Omega = 0$ is a constraint for having a supersymmetric solution. Using these expressions to eliminate M and E in favour of the other parameters, the metric functions (5.4.3) factorize as

$$P^2\mathcal{P}(p) = \mathcal{P}_-(p)\mathcal{P}_+(p)\,, \quad P^2\mathcal{Q}(q) = \mathcal{Q}_-(q)\mathcal{Q}_+(q)\,, \tag{5.4.6}$$

where

$$\mathcal{P}_\pm(p) = Pp^2 \mp Np \mp P^2 + P\sqrt{\alpha}\,, \quad \mathcal{Q}_\pm(q) = Pq^2 \mp Nq \mp PQ + P\sqrt{\alpha}\,. \tag{5.4.7}$$

Writing the four-dimensional Dirac spinor in terms of its chiral components $\varepsilon_\pm$ as

$$\varepsilon = \begin{pmatrix} \varepsilon_+ \\ \varepsilon_- \end{pmatrix} \tag{5.4.8}$$

and employing (5.3.10), the integrability condition (5.1.4) can be used to relate ε_- to ε_+. In particular we find the relation

$$(X^+ + Y^+)\varepsilon_+ + (X^- + Y^-)\varepsilon_- = 0\,, \tag{5.4.9}$$

where $X^\pm$ and $Y^\pm$ are 2×2 matrices with non-vanishing elements

$$\begin{aligned} X^\pm_{1,2} &= \pm\,\mathrm{i}\frac{2\big[\mathcal{P}(p) - \mathcal{Q}(q)\big] + (p \pm q)\Big[2(p \mp q)\big[(q \pm p)^2 - P \mp Q\big] - \mathcal{P}'(p) \pm \mathcal{Q}'(q)\Big]}{4(p \pm q)^2\sqrt{-\mathcal{P}(p)}\sqrt{\mathcal{Q}(q)}}\,, \\ X^\pm_{2,1} &= \pm\,\mathrm{i}\frac{2\big[\mathcal{P}(p) - \mathcal{Q}(q)\big] + (p \pm q)\Big[2(p \mp q)\big[(q \pm p)^2 + P \mp Q\big] - \mathcal{P}'(p) \pm \mathcal{Q}'(q)\Big]}{4(p \pm q)^2\sqrt{-\mathcal{P}(p)}\sqrt{\mathcal{Q}(q)}}\,, \end{aligned} \tag{5.4.10}$$

and

$$Y^\pm = \frac{(p \mp q)(P \pm Q)}{2(q \pm p)^2\sqrt{q^2 - p^2}} \begin{pmatrix} \mp\mathrm{i}/\sqrt{\mathcal{Q}(q)} & \pm 1/\sqrt{-\mathcal{P}(p)} \\ \mp 1/\sqrt{-\mathcal{P}(p)} & \pm\mathrm{i}/\sqrt{\mathcal{Q}(q)} \end{pmatrix}. \tag{5.4.11}$$

Using these relations it is easy to construct the Killing spinor, that is

$$\varepsilon = c \begin{pmatrix} -\mathrm{i}\sqrt{\mathcal{P}_-(p)}\sqrt{\mathcal{Q}_-(q)}/\sqrt{q+p} \\ -\mathrm{i}\sqrt{\mathcal{P}_+(p)}\sqrt{\mathcal{Q}_+(q)}/\sqrt{q+p} \\ \langle P\rangle\sqrt{\mathcal{P}_-(p)}\sqrt{\mathcal{Q}_+(q)}/\sqrt{q-p} \\ \langle P\rangle\sqrt{\mathcal{P}_+(p)}\sqrt{\mathcal{Q}_-(q)}/\sqrt{q-p} \end{pmatrix} \mathrm{e}^{\mathrm{i}\frac{N}{2P}(\tau + \sqrt{\alpha}\sigma)}\,, \tag{5.4.12}$$

where $\langle P\rangle \equiv P/\sqrt{P^2} \in \mathbb{C}\setminus\{0\}$ and $c \in \mathbb{C}$ a normalization constant. Notice that it takes remarkably the same (implicit) form as in [96], and is then the most general form of the Killing spinor for the Euclidean CP solution even when $M \neq N$ and $P \neq Q$. This construction explicitly shows that the CP solution is 1/4-BPS and, as anticipated, we have

$$\epsilon = \varepsilon^\dagger \Gamma^\mu \Gamma_\star \varepsilon \, \partial_\mu = \frac{P}{N} \left(\sqrt{\alpha} \partial_\tau + \partial_\sigma \right) \,, \tag{5.4.13}$$

accordingly to (5.3.48).

Before tackling the regularity of the metric and the spinor, let us express the parameters of the solution (5.4.3) in terms of the real roots $p_\pm$ and q_+, as done in the previous chapters. Specifically we introduce three signs such that

$$(\eta, \delta, \lambda) = \pm 1 \,, \quad \mathcal{P}_\eta(p_+) = 0 \,, \quad \mathcal{P}_\delta(p_-) = 0 \,, \quad \mathcal{Q}_\lambda(q_+) = 0 \,. \tag{5.4.14}$$

Using these expressions we get[7]

$$\begin{aligned} N &= \eta P \frac{(p_+ + p_-)(p_+ + \sigma p_-) + \eta P(\sigma - 1)}{p_+ + p_-} \,, \\ \sqrt{\alpha} &= \sigma \frac{\eta P (p_+ - p_-)(p_+ + \sigma p_-) + p_+ p_- (p_+^2 - p_-^2)}{p_+^2 - p_-^2} \,, \\ Q &= \lambda \frac{(p_+ + p_-)\left[q_+^2 + \sigma p_+ p_- - q_+ (p_+ + \sigma p_-)\kappa\right] + \eta P \left[\sigma p_+ + p_- + \kappa \sigma q_+ (\sigma - 1)\right]}{p_+ + p_-} \,, \end{aligned} \tag{5.4.15}$$

where we have introduced also

$$\sigma = \eta \delta = \pm 1 \,, \qquad \kappa = \eta \lambda = \pm 1 \,. \tag{5.4.16}$$

The sign σ will acquire the interpretation of *twist*, meaning that $\sigma = \pm 1$ corresponds to the twist or the anti-twist realization of the supersymmetry through the spindle bolt, respectively [42]. Diversely, the sign $\kappa = \pm 1$ will lead to different *branches* of the solution, generalizing [111, 96].

For each value of the signs η, δ and λ, it holds

$$\mathcal{P}'(p_+) = \frac{2\,(p_+ - p_-)\left[N + p_+ (p_+ + p_-)^2\right]}{p_+ + p_-} \,, \tag{5.4.17}$$

where recall that $p_+ + p_- > 0$ from the scaling symmetry. A necessary and sufficient condition for this to be positive (as required by the signature of the metric) is $N \in \mathbb{R}_+$. When $\sigma = 1$, $0 < N = (p_+ + p_-)\eta P$ from (5.4.15), and we conclude that

$$\sigma = 1 : \quad P \in \mathbb{R} \,, \quad \eta = \operatorname{sign} P \,. \tag{5.4.18}$$

In the case $\sigma = -1$, from

$$\sigma = -1 : \quad N = \frac{\eta P}{p_+ + p_-} \left[(p_+^2 - p_-^2) - 2\eta P \right] , \tag{5.4.19}$$

[7] In writing the following formulas we assume $p_+ + p_- > 0$. The $p_+ = -p_-$ case is treated throughout section 5.4.2.

we see that in order for N to be real and positive it is necessary that

$$\sigma = -1: \quad \begin{array}{ll} P \in \mathbb{C} \implies \mathrm{Re}(P) = \dfrac{\eta(p_+^2 - p_-^2)}{4}\,, & \eta = \mathrm{sign}\,\mathrm{Re}(P)\,, \\ P \in \mathbb{R} \implies 0 < \eta P < \dfrac{(p_+^2 - p_-^2)}{2}\,, & \eta = \mathrm{sign}\,P\,. \end{array} \tag{5.4.20}$$

When $P \in \mathbb{R}$, which can happen for both $\sigma = \pm 1$, all the parameters of the solution are real from (5.4.15). Then also the metric and the gauge field are real. However, only for $\sigma = -1$, P can be complex with its real part fixed by (5.4.20). However, in this case one can verify that the parameter E and the combination $-P^2 + \alpha$ appearing in the structure function $\mathcal{P}(p)$ in (5.4.3) are nevertheless real. Instead M and $-Q^2+\alpha$ are complex, unless the real root q_+ satisfies the further condition $q_+ = p_+$ with $\kappa = 1$. Finally, $Q \in \mathbb{C}$ iff $P \in \mathbb{C}$.

Summarizing, for $\sigma = 1$ the solution is real, whilst for $\sigma = -1$ the function $\mathcal{P}(p)$ is always real but the metric remains complex except if $q_+ = p_+$. This sub-case corresponds to the solution presented in [96], where the metric was always real but the gauge field can be complex if P (and therefore Q) is complex.

5.4.2 Global analysis: metric

Having established the local form of the solutions of interest, we now examine their global properties. As explained in section 3.5.1 and 4.2.1, when studying the regularity of a metric it is useful to choose coordinates adapted to the degenerating Killing vectors near the fixed points $p_A = (p_\pm, q_+)$ and, as usual, the introduction of patches $U_\pm$ covering $\mathcal{M}_4$ is needed. Indeed, a generic Killing vector $K = a_1\partial_{\tau^\pm} + a_2\partial_{\sigma^\pm}$ of the metric (5.4.1) has norm

$$K_\mu K^\mu = \frac{-(a_1 + a_2 q^2)^2\mathcal{P}(p) + (a_1 + a_2 p^2)^2\mathcal{Q}(q)}{q^2 - p^2}\,, \tag{5.4.21}$$

and since it is the sum of positive quantities, they should be nil separately. We then take $K_{q_+} = (q_+^2\partial_{\tau^\pm} - \partial_{\sigma^\pm})$ and $K_\pm = K_{p_\pm} = (-p_\pm^2\partial_{\tau^\pm} + \partial_{\sigma^\pm})$, which degenerate at q_+ and at $p_\pm$, respectively[8]. These Killing vectors can be conveniently written as $K_{q_+} = \partial_{\theta_1^\pm}$ and $K_\pm = \partial_{\theta_2^\pm}$, with again $(\theta_1^\pm, \theta_2^\pm)$ on $U_\pm$, if we change coordinates as

$$U_\pm: \quad \tau^\pm = q_+^2\theta_1^\pm - p_\pm^2\theta_2^\pm\,, \quad \sigma^\pm = -\theta_1^\pm + \theta_2^\pm\,. \tag{5.4.22}$$

Since the Jacobian of (5.4.22) is $q_+^2 - p_\pm^2$, the periodicities are related simply as

$$\Delta\tau^+\Delta\sigma^+ = \Delta\tau^-\Delta\sigma^- = (q_+^2 - p_+^2)\Delta\theta_1^+\Delta\theta_2^+ = (q_+^2 - p_-^2)\Delta\theta_1^-\Delta\theta_2^-\,. \tag{5.4.23}$$

[8] Notice that if we take $a_1 = 0$, then or $a_2 = 0$ or $q_+ = p_\pm$, which are both degenerate cases. Similarly it happens if we start with $a_2 = 0$, so we continue assuming $a_i \neq 0$.

In these new coordinates $(q,p,\theta_1^\pm,\theta_2^\pm)$, the four-dimensional metric reads

$$
\begin{aligned}
U_\pm: \quad \mathrm{d}s_4^2 =& \frac{q^2-p^2}{-\mathcal{P}(p)}\mathrm{d}p^2 - \frac{(q^2-p_\pm^2)^2\mathcal{P}(p)}{q^2-p^2}\mathrm{d}(\theta_2^\pm)^2 + \frac{q^2-p^2}{\mathcal{Q}(q)}\mathrm{d}q^2 \\
&+ (q^2-q_+^2)\frac{\mathrm{d}(\theta_1^\pm)^2(q^2-q_+^2) - 2(q^2-p_\pm^2)\mathrm{d}\theta_1^\pm\mathrm{d}\theta_2^\pm}{p^2-q^2}\mathcal{P}(p) \\
&- \frac{\left[(p^2-p_\pm^2)\mathrm{d}\theta_2^\pm + (q_+^2-p^2)\mathrm{d}\theta_1^\pm\right]^2}{p^2-q^2}\mathcal{Q}(q)\,.
\end{aligned}
\tag{5.4.24}
$$

This form for the metric (5.4.1) is the most convenient for our purposes. Indeed, as we shall impose later, there can now be a single angular coordinate which degenerates at $p_\pm$, namely $\varphi_2^+ = \varphi_2^-$ with $\varphi_i^\pm = (2\pi/\Delta\theta_i^\pm)\theta_i^\pm$. Then, the first two terms in the first line of (5.4.24) can have the topology of a spindle. We will start analysing the boundary metric, as it is necessarily simpler than (5.4.24). After imposing $\mathcal{M}_3 = L(t,1)$, that is an (eventually branched [49]) lens space, we will come back to (5.4.24). We will have practically for free that the boundary topology results in a (complex) line bundle over the spindle, namely $\mathcal{M}_4 = \mathbb{C}/\mathbb{Z}_v \hookrightarrow \mathcal{O}(-t) \to \Sigma_{[m_-,m_+]}$. Indeed, loosely speaking, we can view the bulk topology as $\mathcal{M}_4 = (\mathbb{R}^+_{(q)}/\mathbb{Z}_v \times S^1_{(\varphi_1)}) \hookrightarrow_t \Sigma_{[m_-,m_+]} = \mathbb{R}^+_{(q)}/\mathbb{Z}_v \times \mathcal{M}_3$. Finally, we will show the regularity of the Killing spinor (5.4.12), both in the bulk and in the boundary.

Metric at the boundary

The boundary metric can be obtained as the $q \to +\infty$ limit of (5.4.24), or alternatively by setting $\mathbf{A}=0$ in (5.3.33). We get

$$
\begin{aligned}
U_\pm: \quad \mathrm{d}s_b^2 =& \frac{\mathrm{d}p^2}{-\mathcal{P}(p)} - \frac{(q_+^2-p_\pm^2)^2\mathcal{P}(p)}{(q_+^2-p^2)^2-\mathcal{P}(p)}\mathrm{d}(\theta_2^\pm)^2 \\
&+ \left[(q_+^2-p^2)^2-\mathcal{P}(p)\right]\left(\mathrm{d}\theta_1^\pm + \frac{(q_+^2-p^2)(p^2-p_\pm^2)+\mathcal{P}(p)}{[(q_+^2-p^2)^2-\mathcal{P}(p)]}\mathrm{d}\theta_2^\pm\right)^2 .
\end{aligned}
\tag{5.4.25}
$$

Notice that the boundary metric (5.4.25) contains only the function $\mathcal{P}(p)$ and is therefore real as a consequence of the discussion at the end of the previous section. We now zoom in near the zeros of $\mathcal{P}(p)$ introducing a new coordinate $R_\pm$

$$
U_\pm: \quad p = p_\pm - \frac{\mathcal{P}'(p_\pm)}{4}R_\pm^2 \implies \mathrm{d}s_b^2 \underset{p\to p_\pm}{\simeq} \mathrm{d}R_\pm^2 + \frac{\mathcal{P}'(p_\pm)^2}{4}R_\pm^2\mathrm{d}(\theta_2^\pm)^2 + (q_+^2-p_\pm^2)^2\mathrm{d}(\theta_1^\pm)^2\,.
\tag{5.4.26}
$$

If we manage to impose

$$
U_\pm: \quad \frac{|\mathcal{P}'(p_\pm)|}{2}\Delta\theta_2^\pm = \frac{2\pi}{m_\pm}\,,
\tag{5.4.27}
$$

the base of this fibration in (5.4.25) can assume the topology of a spindle $\Sigma_{[m_-,m_+]}$, for some co-prime positive integers $m_\pm$, with

$$
m_- > m_+ \implies (q_+^2-p_+^2)\mathcal{P}'(p_-) + (q_+^2-p_-^2)\mathcal{P}'(p_+) > 0\,.
\tag{5.4.28}
$$

We will refer to this spindle as $\Sigma^\infty_{[m_-,m_+]}$, since its shape is different for different values of q, but the topology will remain the same, *i.e.* the volume is a function of q, but the conical deficits remain the same along the flow. We stress that in the boundary the spindle is a topologically trivial cycle, but nevertheless we will see that it plays an important role. Indeed, with reference to the two dimensional base

$$U_\pm: \quad \mathrm{d}s^2_{\Sigma_\infty} = \frac{\mathrm{d}p^2}{-\mathcal{P}(p)} - \frac{(q_+^2 - p_\pm^2)^2\mathcal{P}(p)}{(q_+^2-p^2)^2 - \mathcal{P}(p)}\mathrm{d}(\theta_2^\pm)^2\,, \tag{5.4.29}$$

it can be seen that

$$\sqrt{g_{\Sigma_\infty}}R_{\Sigma_\infty} = (q_+^2-p_\pm^2)\partial_p\Big[(q_+^2-p^2)\Big[4\mathcal{P}(p)p + (q_+^2-p^2)\mathcal{P}'(p)\Big]\Big[(q_+^2-p^2)^2 - \mathcal{P}(p)\Big]^{-3/2}\Big]\,, \tag{5.4.30}$$

and as a consequence

$$U_\pm \;:\quad \frac{1}{4\pi}\int_{\Sigma_\infty}\mathrm{d}p\,\mathrm{d}\theta_2^\pm\,\sqrt{g_{\Sigma_\infty}}R_{\Sigma_\infty} = \frac{q_+^2-p_\pm^2}{4\pi}\left|\frac{\mathcal{P}'(p)}{q_+^2-p^2}\right|_{p_-}^{p_+}\Delta\theta_2^\pm\,. \tag{5.4.31}$$

This computation must agree in the two patches $U_\pm$, and using (5.4.23) we see that necessarily $\Delta\theta_1^+ = \Delta\theta_1^-$. In turn, this implies

$$\frac{1}{4\pi}\int_{\Sigma_\infty}\mathrm{d}p\,\mathrm{d}\theta_2^\pm\,\sqrt{g_{\Sigma_\infty}}R_{\Sigma_\infty} = \frac{m_-+m_+}{m_-m_+} \equiv \chi_\Sigma\,, \tag{5.4.32}$$

where χ_Σ is the (orbifold) Euler characteristic of the spindle. Then, with (5.4.27) and

$$\Delta\theta_1 \equiv \Delta\theta_1^+ = \Delta\theta_1^-\,, \tag{5.4.33}$$

the metric (5.4.29) represents indeed $\Sigma^\infty_{[m_-,m_+]}$. With these conditions imposed, it is also useful to introduce

$$\Delta \equiv (q_+^2-p_+^2)\Delta\theta_2^+ = (q_+^2-p_-^2)\Delta\theta_2^-\,, \tag{5.4.34}$$

in terms of which (5.4.27) is rewritten as

$$\pm\frac{\mathcal{P}'(p_\pm)}{2(q_+^2-p_\pm^2)}\Delta = \frac{2\pi}{m_\pm}\,. \tag{5.4.35}$$

Finally we have to require that the fibration is well-defined in the orbifold sense [42], that is

$$\frac{1}{2\pi}\int_{\Sigma_\infty}\frac{2\pi}{\Delta\theta_1}\mathrm{d}\left(\frac{(q_+^2-p^2)(p^2-p_\pm^2)+\mathcal{P}(p)}{[(q_+^2-p^2)^2-\mathcal{P}(p)]}\mathrm{d}\theta_2^\pm\right) \equiv \frac{t}{m_-m_+}\,, \quad t\in\mathbb{N}\,, \tag{5.4.36}$$

from which

$$U_\pm: \quad t = m_-m_+\frac{p_+^2-p_-^2}{q_+^2-p_\mp^2}\frac{\Delta\theta_2^\pm}{\Delta\theta_1} \implies t = m_-m_+\frac{p_+^2-p_-^2}{(q_+^2-p_-^2)(q_+^2-p_+^2)}\frac{\Delta}{\Delta\theta_1}\,. \tag{5.4.37}$$

Notice that combining (5.4.23), (5.4.33), (5.4.34), (5.4.35) and (5.4.37) we can write

$$\Delta\tau^\pm\Delta\sigma^\pm = \Delta\theta_1\Delta = \frac{m_-m_+}{t}\Delta^2\frac{p_+^2-p_-^2}{(q_+^2-p_-^2)(p_+^2-p_-^2)} = \frac{(4\pi)^2}{t}\frac{p_+^2-p_-^2}{-\mathcal{P}'(p_-)\mathcal{P}'(p_+)}\,, \tag{5.4.38}$$

where the last two expressions are valid only for $t\neq 0$.

When $t = 0$, the boundary is generically a direct product $\Sigma \times S^1$. Since $t \propto (p_+ + p_-)$, we can study separately the case where $p_+ = -p_- \equiv r$. Indeed, the discussion around (5.4.15) assumed $p_+ + p_- > 0$. We can repeat now that analysis, requiring *a priori* $t = 0$. We find

$$\begin{aligned} \sigma = +1 : \quad & N = 0\,, \quad \sqrt{\alpha} = -r^2 + \eta P\,, \quad Q = \lambda(q_+^2 - r^2 + \eta P)\,, \\ \sigma = -1 : \quad & N = 0\,, \quad P = 0\,. \end{aligned} \tag{5.4.39}$$

We see that $t = 0 \implies N = 0$, but the vice-versa is not true. Details of the case $N = 0$ can be found in [3].

Let us conclude this section by working out explicitly the coordinate change between U_+ and U_- from (5.4.22). To this end it is useful to write the Killing vectors expressed in the two patches

$$\begin{aligned} K_{p_+} &= -\frac{p_+^2 - p_-^2}{q_+^2 - p_-^2}\partial_{\theta_1^-} + \frac{q_+^2 - p_+^2}{q_+^2 - p_-^2}\partial_{\theta_2^-} = \partial_{\theta_2^+}\,, \\ K_{q_+} &= \partial_{\theta_1^-} = \partial_{\theta_1^+}\,, \\ K_{p_-} &= \partial_{\theta_2^-} = \frac{p_+^2 - p_-^2}{q_+^2 - p_+^2}\partial_{\theta_1^+} + \frac{q_+^2 - p_-^2}{q_+^2 - p_+^2}\partial_{\theta_2^+}\,, \end{aligned} \tag{5.4.40}$$

which follow from

$$\theta_1^+ = \theta_1^- + \frac{p_+^2 - p_-^2}{q_+^2 - p_+^2}\theta_2^-\,, \quad \theta_2^+ = \frac{q_+^2 - p_-^2}{q_+^2 - p_+^2}\theta_2^-\,. \tag{5.4.41}$$

These can be rewritten introducing 2π-periodic coordinates

$$\varphi_1^\pm = \frac{2\pi}{\Delta\theta_1}\theta_1^\pm\,, \quad \varphi_2^\pm = \frac{2\pi}{\Delta\theta_2^\pm}\theta_2^\pm\,, \tag{5.4.42}$$

and using (5.4.33), (5.4.34) and (5.4.37) to obtain

$$\varphi_1^+ = \varphi_1^- + \frac{t}{m_- m_+}\varphi_2^-\,, \quad \varphi_2^+ = \varphi_2^-\,. \tag{5.4.43}$$

Thus we can define

$$\varphi_2 \equiv \varphi_2^+ = \varphi_2^-\,, \tag{5.4.44}$$

which is identified in the gluing and use it independently of the patch $U_\pm$. Notice that when $p_- = -p_+$, *i.e.* when $t = 0$, we can always choose $\theta_1^+ = \theta_1^- \equiv \theta_1$ and $\theta_2^+ = \theta_2^- \equiv \theta_2$ from (5.4.41). This is the reason for which, for example in [96, 47], it is not needed the description in patches.

Metric in the bulk

We now come back to the bulk metric (5.4.24). The situation is summarized as

$$\begin{aligned} &\mathcal{Q}(q) \geq 0\,, \quad \mathcal{P}(p) \leq 0\,, \quad q^2 > p^2\,, \\ &p_- \leq p \leq p_+ < q_+ \leq q\,, \quad q_+^2 > p_\pm^2\,, \end{aligned} \tag{5.4.45}$$

where the range of p and q is in general disjoint. Indeed, the point $p = q = p_+ = q_+$ contains a curvature singularity unless $M = N$ and $P = Q$. However in this case we would fall into the analysis of [96], which exhibits a four dimensional metric with the topology of the ball, *i.e.* with a single nut in the bulk. Thus we continue with $Q \neq P$, $N \neq M$ and $q > p$.

Noticing that for $q = q_+$ the second line of (5.4.24) is zero, we argue that $(p, \theta_2^\pm)$, when q is fixed at q_+, are again the coordinates on (a different shaped) spindle $\Sigma^{q_+}_{[m_-,m_+]}$. Since q_+ is the position of the bolt, we will refer to this $\Sigma^{q_+}_{[m_-,m_+]}$ as a "spindle bolt". Near the points $(p_\pm, q_+)$ we use

$$U_\pm: \quad q = q_+ + \frac{\mathcal{Q}'(q_+)}{4(q_+^2 - p_\pm^2)} R^2 , \quad p = p_\pm - \frac{\mathcal{P}'(p_\pm)}{4(q_+^2 - p_\pm^2)} R_\pm^2 , \tag{5.4.46}$$

on (5.4.24), obtaining at the leading order

$$U_\pm: \quad \mathrm{d}s_4^2 \underset{q_+, p_\pm}{\simeq} \mathrm{d}R^2 + \frac{\mathcal{Q}'(q_+)^2}{4} R^2 \mathrm{d}(\theta_1^\pm)^2 + \mathrm{d}R_\pm^2 + \frac{R_\pm^2}{m_\pm^2} \mathrm{d}\varphi_2^2 , \tag{5.4.47}$$

where we already used (5.4.42), (5.4.44) and (5.4.27). Locally this metric describes $\mathbb{C}/\mathbb{Z}_v \times \mathbb{C}/\mathbb{Z}_{m_\pm}$, once we impose

$$\frac{\mathcal{Q}'(q_+)}{2} \Delta\theta_1 = \frac{2\pi}{v} , \tag{5.4.48}$$

which is the analogous of (5.4.27) and recall that $\Delta\theta_1 = \Delta\theta_1^+ = \Delta\theta_1^-$. Summarizing, equations (5.4.35), (5.4.37) and (5.4.48) are "quantization conditions", which we will study in detail in section 5.4.5. Thus we have that the (real-valued) spindle bolt metric is the first line of (5.4.24) in $q = q_+$

$$U_\pm \quad \mathrm{d}s^2_{\Sigma_{q_+}} = \frac{q_+^2 - p^2}{-\mathcal{P}(p)} \mathrm{d}p^2 - \frac{(q_+^2 - p_\pm^2)^2 \mathcal{P}(p)}{q_+^2 - p^2} \mathrm{d}(\theta_2^\pm)^2 , \tag{5.4.49}$$

and indeed we have again that $\chi = \chi_\Sigma$.

We conclude this section by considering the collapse of the metric at $q = q_0 > q_+$ fixed. To zoom in near $p_\pm$ we moreover use $p = p_\pm - \frac{\mathcal{P}'(p_\pm)}{4(q_0^2 - p_\pm^2)} R_\pm^2$ for which (5.4.24) becomes

$$U_\pm: \quad \mathrm{d}s^2\big|_{q=q_0} \underset{p\to p_\pm}{\simeq} \mathrm{d}R_\pm^2 + \frac{R_\pm^2}{m_\pm^2}\left(\mathrm{d}\varphi_2 + c_\pm(q_0)\mathrm{d}\theta_1^\pm\right)^2 + \frac{(q_+^2 - p_\pm^2)^2}{q_0^2 - p_\pm^2} \mathcal{Q}(q_0) \mathrm{d}(\theta_1^\pm)^2 , \tag{5.4.50}$$

where $c_\pm(q_0)$ is an irrelevant constant that goes to zero as q_0 approaches q_+. Then, the regularity follows from the previous conditions imposed to the topology of the boundary.

5.4.3 Toric data and equivariant localization formulas

The global analysis performed in the previous section allows us to provide a description of the orbifold $\mathcal{M}_4 = \mathbb{C}/\mathbb{Z}_v \hookrightarrow \mathcal{O}(-t) \to \Sigma_{[m_-,m_+]}$ in the framework of toric geometry, as explained in chapter 2. As for the solutions $\mathrm{AdS}_{D-4} \times \mathbb{M}_4$ with $D = 6, 7$ of the previous chapters, we do not find a closed symplectic two-form. Nevertheless, following section 2.3, we will be able to extract all the relevant toric data. We start defining the loci/hatted divisors (see section 3.5.1)

$$\mathcal{L}_1 = \hat{D}_1 = \{p = p_+\} , \quad \mathcal{L}_2 = \hat{D}_2 = \{q = q_+\} , \quad \mathcal{L}_3 = \hat{D}_3 = \{p = p_-\} , \tag{5.4.51}$$

where $\mathcal{L}_{1,3}$ are non compact whilst $\mathcal{L}_2$ defines the bolt, and it is indeed compact. These meet at the two fixed points

$$p_1 = \{p_+, q_+\}\,, \quad p_2 = \{p_-, q_+\}\,, \tag{5.4.52}$$

where by convention we take $\mathcal{L}_a$ to be the one that joins p_{a-1} and p_a. As explained in section 2.3, we can use (2.3.2) and (3.5.12), namely,

$$\xi_a = \vec{v}_a \cdot (E_1, E_2)\,, \qquad \frac{\partial_\mu |\xi_a|^2 \partial^\mu |\xi_a|^2}{4|\xi_a|^2}\Big|_{\mathcal{L}_a} = 1\,, \tag{5.4.53}$$

to extract the (non-primitive) vectors of the fan $\vec{v}_a \in \mathbb{Z}^2$. Choosing as an effective toric basis[9]

$$U_\pm: \quad E_1 \equiv \partial_{\phi_1} = \partial_{\varphi_1^\pm}\,, \quad E_2 \equiv \partial_{\phi_2} = \partial_{\varphi_2^\pm} \pm \frac{r_\mp}{m_\pm}\partial_{\varphi_1^\pm}\,, \quad r_- m_- + r_+ m_+ = t\,, \tag{5.4.55}$$

the normalized Killing vectors read

$$U_-: \quad \begin{aligned} \xi_1 &= \frac{2}{-\mathcal{P}'(p_+)}(p_+^2 \partial_{\tau^-} - \partial_{\sigma^-}) = m_+\left(\partial_{\varphi_2^-} - \frac{t}{m_+ m_-}\partial_{\varphi_1^-}\right) \\ &= m_+ E_2 + E_1(m_+ r_+ - t)/m_- = m_+ E_2 - r_- E_1\,, \\ \xi_2 &= \frac{2}{\mathcal{Q}'(q_+)}(q_+^2 \partial_{\tau^-} - \partial_{\sigma^-}) = v\partial_{\varphi_1^-} = vE_1\,, \\ \xi_3 &= \frac{2}{\mathcal{P}'(p_-)}(p_-^2 \partial_{\tau^-} - \partial_{\sigma^-}) = m_- \partial_{\varphi_2^-} = m_- E_2 + r_+ E_1\,. \end{aligned} \tag{5.4.56}$$

From the relation (5.4.53) we extract the following non-primitive vectors

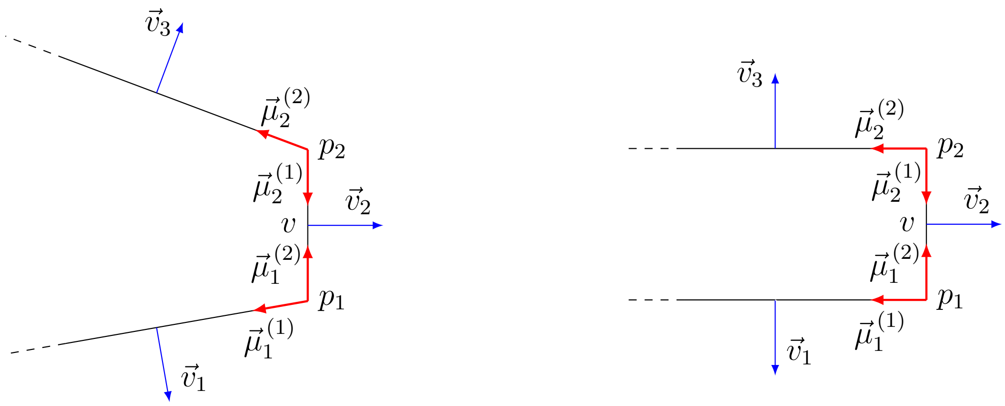
$$\vec{v}_1 = (r_-, -m_+)\,, \quad \vec{v}_2 = (v, 0)\,, \quad \vec{v}_3 = (r_+, m_-)\,, \tag{5.4.57}$$

which generate the non-compact polytope in figure 5.1a. From (5.4.56) we see immediately that when $t = 0$ the polytope is an open rectangle as in figure 5.1b. Notice that when $t = 0$ the integers $r_\pm$ in (5.4.54) loose their meaning. In particular, all the various formulas are applicable by simply setting $r_\pm = 0$. The polytope 5.1a is the same described in [83] as the "blow-up" of $\mathbb{C}/\mathbb{Z}_t$ (see figure 2.2c), and from a supergravity point of view it is the non compact version of the solution presented in chapter 4, with the first facet therein is sent to infinity. As anticipated, v is a label for $\mathcal{L}_2$, whilst $\mathcal{L}_{1,3}$ are (in general) standard toric divisors. As pointed out in 3, one can create singularities along $\mathcal{L}_{1,3}$ as well by tuning the value of the parameter $t = r_- m_- + r_+ m_+$. Indeed it is easy to see from the vectors (5.4.57) that, for the "symmetric" case $t = \bar{t}\, m_+ m_-$, the labels are $m_a = (m_-, v, m_+)$. We will demonstrate this statement explicitly in the next section.

[9] In principle one could be tempted to define $E_i^\pm = \partial_{\phi_i^\pm}$. However, working out the explicit change of coordinates

$$U_\pm: \quad \varphi_1^\pm = \phi_1^\pm \pm \frac{r_\mp}{m_\pm}\phi_2^\pm\,, \quad \varphi_2 = \phi_2^\pm\,, \tag{5.4.54}$$

and using the transition functions (5.4.43), one deduces that $\phi_i^+ = \phi_i^- \equiv \phi_i$.



(a) $t \neq 0$ (b) $t = 0$

Figure 5.1: labeled polytope associated with normal vectors (5.4.57). The normal singularity $\mathbb{C}/\mathbb{Z}_v$ shows up as a label in the polytope. Fig. 5.1a is valid for $t \neq 0$, whilst Fig. 5.1b is for $t = 0$.

Having constructed the polytope in figure 5.1, we can present an explicit formula for the equivariantly localized action (5.2.7). Indeed, using (5.4.57) we have that the action takes the suggestive form

$$I_{\text{off-shell}}(\sigma, z) = -\frac{c_2 \pi}{8G_4 v}\left[\frac{1}{m_-}\left(\beta_- - \frac{c_2}{\beta_-}\right)^2 - \frac{\sigma}{m_+}\left(\beta_+ + \frac{\sigma c_2}{\beta_+}\right)^2\right], \tag{5.4.58}$$

where $c_1 c_2 = \sigma$ and we have defined the (a priori complex) variables

$$\beta_\pm^2 \equiv \frac{1}{v}\left(\frac{m_\pm}{z} \pm r_\mp\right), \quad m_-\beta_+^2 - m_+\beta_-^2 = \frac{t}{v}. \tag{5.4.59}$$

In (5.4.58) the dependence on z is particularly simple and we expect that this form should be suitable for comparisons with large N calculations in the dual field theories.

Entropy functions for supersymmetric black holes

The $t = 0$ case is particularly relevant for supersymmetric black holes. This corresponds to the rectangular polytope in figure 5.1b, and may be obtained from (5.4.58) simply setting $r_\pm = 0 = t$. The resulting expressions for the twist and anti-twist read

$$\sigma = -1: \quad I^{\text{anti-twist}}_{\text{off-shell}}(z) = -\frac{c_2 \pi}{8G_4}\left[\left(\frac{\sqrt{z}}{m_-} - \frac{c_2}{v\sqrt{z}}\right)^2 + \left(\frac{\sqrt{z}}{m_+} - \frac{c_2}{v\sqrt{z}}\right)^2\right], \quad t = 0, \tag{5.4.60}$$

$$\sigma = 1: \quad I^{\text{twist}}_{\text{off-shell}}(z) = \frac{\pi \chi_\Sigma}{8G_4}\left[\frac{2}{v} + c_2\left(\frac{1}{m_+} - \frac{1}{m_-}\right)z\right], \quad t = 0. \tag{5.4.61}$$

It is then straightforward to see that the two expressions above reproduce the gravitational block form of the black hole entropy functions for minimal $d = 4$ gauged supergravity, proposed[10] in [52]. Indeed, defining

$$\mathfrak{n} = \frac{m_+ + \sigma m_-}{4m_- m_+}\,, \qquad 2\varphi - \frac{m_+ - \sigma m_-}{2m_- m_+} z = \frac{1}{v}\,, \tag{5.4.62}$$

we have[11]

$$I^{\text{anti-twist}}_{\text{off-shell}}(z) = F^+(\varphi, z; \mathfrak{n}) = \frac{\pi}{G_4}\left[\frac{\varphi^2}{z} + \mathfrak{n}^2 z\right], \qquad I^{\text{twist}}_{\text{off-shell}}(z) = F^-(\varphi, z; \mathfrak{n}) = \frac{\pi \mathfrak{n}\varphi}{G_4} 2\,, \tag{5.4.63}$$

with

$$F^{-\sigma}(\varphi, z; \mathfrak{n}) = \frac{1}{z}\Big[\mathcal{F}_3(\varphi + \mathfrak{n}z) - \sigma \mathcal{F}_3(\varphi - \mathfrak{n}z)\Big]\,, \tag{5.4.64}$$

where $\mathcal{F}_3$ is proportional to the S^3 off-shell free energy of the ABJM theory (*cf.* Table 2 in [52]) and $\sigma = \pm 1$ is as usual for the twist or the anti-twist, respectively. Notice that for the twist on the sphere, with $c_2 = c_1$ and $m_- = m_+ = 1$, the on-shell action does not depend any more on the supersymmetric Killing vector, and the result is simply $I = \pi/(2vG_4)$.

5.4.4 Global analysis: gauge field, Killing spinor and Seifert orbifolds

Let us come back to the study of the regularity of the spinor (5.4.12), both in the bulk and in the boundary. Throughout this section, we will emphasize how the boundary geometry is affected by the bulk. In particular, even if the spindle Σ_∞ is a trivial two-cycle from the view point of the boundary, its informations (the integers $m_\pm$ and the type of twist $\sigma = \pm 1$) are already contained in the supersymmetric geometry (namely the boundary metric $\mathrm{d}s^2_{(3)}$, the boundary gauge field $A_{(3)}$ and the boundary Killing spinor $\chi_{(3)}$).

Gauge field and Killing spinors in the bulk

We start with the bulk analysis, in which all possible free choices will be fixed by the procedure. The analysis is divided into two parts: firstly, we pick a gauge in which the graviphoton (5.4.2) is regular on the spindle bolt, then we choose an adapted frame in which the spinor is smooth and well-defined:

1. In the patch $U_\pm$ which contains $p_\pm$ we perform the gauge transformation

$$U_\pm \ : \quad A'_\pm = A + q_+ Q \mathrm{d}\theta_1^\pm - p_\pm P \mathrm{d}\theta_2^\pm\,, \tag{5.4.65}$$

so that $A'_\pm(q_+, p_\pm) = 0$. Notice that we do not have any freedom in choosing this gauge transformation. Indeed, near the poles of the spindle bolt, both $\varphi_1^\pm$ and φ_2 are ill-defined and must be subtracted from the gauge field. Correspondingly, the

[10] These expressions have been proved in supergravity using different methods, considering near-horizon geometries with a spindle factor [81, 82, 83, 92]. See also [160] for a discussion in the context of higher derivative supergravity. In the dual field theories these are reproduced by the large N limit of the spindle index [50].

[11] We have adapted the overall normalization and inserted a v factor, to compare to our results. Moreover in both cases $c_2 = -1$.

spinor (5.4.12) acquires a phase. Summing up the contributions, the overall phase $\Phi_\pm$ in the 2π-periodic coordinates $\varphi_i^\pm$ can be written as

$$
\begin{aligned}
U_\pm \ : \quad \Phi_\pm =& \frac{2q_+PQ + N(q_+^2-\sqrt{\alpha})}{2P}\frac{\Delta\theta_1}{2\pi}\varphi_1^\pm - \frac{2p_\pm PQ + N(p_\pm^2-\sqrt{\alpha})}{2P}\frac{\Delta\theta_2^\pm}{2\pi}\varphi_2 \\
=& \frac{\lambda\mathcal{Q}'(q_+)}{4}\frac{\Delta\theta_1}{2\pi}\varphi_1^\pm - \frac{\rho^\pm\mathcal{P}'(p_\pm)}{4}\frac{\Delta\theta_2^\pm}{2\pi}\varphi_2 = \frac{\lambda}{2v}\varphi_1^\pm \mp \frac{\rho_\pm}{2m_\pm}\varphi_2 \,,
\end{aligned}
\tag{5.4.66}
$$

where we have used (5.4.42), (5.4.44), (5.4.6), (5.4.7) and (5.4.27), (5.4.48) in the last step. Moreover we have introduced the sign $\rho_\pm$, defined as $\rho_+ = \eta$ and $\rho_- = \delta$, so that the product $\rho_+\rho_- = \eta\delta = \sigma$. The fact that (5.4.65) is the correct gauge can be seen from (5.4.66): whilst the original phase of (5.4.12) is real, the coefficients of $\Phi_\pm$ are rational numbers;

2. Next, we notice that the frame (5.3.9) is singular at the poles of the bolt. It is therefore convenient to write the frame in $\varphi_i^\pm$ coordinates and focus on $(q_+, p_\pm)$ by means of (5.4.46). We obviously obtain (see (5.4.47))

$$
U_\pm \ : \quad e^a \simeq \left\{\mp \mathrm{d}R_\pm \,, \frac{R_\pm}{m_\pm}\mathrm{d}\varphi_2 \,, \frac{R}{v}\mathrm{d}\varphi_1^\pm \,, \mathrm{d}R\right\} , \tag{5.4.67}
$$

where the sign in e^1 is due to the fact that approaching p_+ (p_-) the coordinate p is increasing (decreasing), while $R_\pm$ is decreasing (increasing). A well-defined frame on the spindle, in the patch $U_\pm$, is given by

$$
U_\pm : \quad (\hat{e}^1) + \mathrm{i}(\hat{e}^2) = \mathrm{d}\left[\mp R_\pm \exp\left(\mp\mathrm{i}\frac{\varphi_2}{m_\pm}\right)\right] = \exp\left(\mp\mathrm{i}\frac{\varphi_2}{m_\pm}\right)(e^1 + \mathrm{i}e^2)\,, \tag{5.4.68}
$$

which is obtained by means of a $U(1) \simeq SO(2)$ rotation. Similarly it happens for the transverse part described by (e^3, e^4), so that the whole $SO(2)\times SO(2)$ transformation is

$$
\begin{pmatrix} \hat{e}^1 \\ \hat{e}^2 \\ \hat{e}^3 \\ \hat{e}^4 \end{pmatrix} = \begin{pmatrix} \cos(\mp\varphi_2/m_\pm) & -\sin(\mp\varphi_2/m_\pm) & 0 & 0 \\ \sin(\mp\varphi_2/m_\pm) & \cos(\mp\varphi_2/m_\pm) & 0 & 0 \\ 0 & 0 & \cos\left(\varphi_1^\pm/v\right) & \sin\left(\varphi_1^\pm/v\right) \\ 0 & 0 & -\sin\left(\varphi_1^\pm/v\right) & \cos\left(\varphi_1^\pm/v\right) \end{pmatrix} \begin{pmatrix} e^1 \\ e^2 \\ e^3 \\ e^4 \end{pmatrix} . \tag{5.4.69}
$$

Since under a transformation of the vielbein $\hat{e}^a = \exp(\lambda^a{}_b)\, e^b$ a four-dimensional Dirac spinor ψ transforms in the spin representation as $\psi' = \exp(1/2\lambda^{ab}\Gamma_{ab})\psi$, with the Γ_a given in (5.3.10), we have that the four dimensional spinor (5.4.12) becomes

$$
U_\pm : \quad \varepsilon' = \exp\left[\frac{\mathrm{i}}{2}\mathrm{diag}\left\{\pm\frac{\varphi_2}{m_\pm} - \frac{\varphi_1^\pm}{v}\,, \mp\frac{\varphi_2}{m_\pm} + \frac{\varphi_1^\pm}{v}\,, \pm\frac{\varphi_2}{m_\pm} + \frac{\varphi_1^\pm}{v}\,, \mp\frac{\varphi_2}{m_\pm} - \frac{\varphi_1^\pm}{v}\right\}\right]\varepsilon\,. \tag{5.4.70}
$$

Explicitly, summing the contributions from (5.4.66) and (5.4.70), the spinor (5.4.12) in the regular gauge and frame reads

$$U_\pm : \quad \varepsilon'(q,p,\varphi_1^\pm,\varphi_2) = c \begin{pmatrix} -\mathrm{i}\frac{\sqrt{\mathcal{P}_-(p)}\sqrt{\mathcal{Q}_-(q)}}{\sqrt{q+p}}\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda-1}{v}\varphi_1^\pm \mp \frac{\rho^\pm-1}{m_\pm}\varphi_2]} \\ -\mathrm{i}\frac{\sqrt{\mathcal{P}_+(p)}\sqrt{\mathcal{Q}_+(q)}}{\sqrt{q+p}}\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda+1}{v}\varphi_1^\pm \mp \frac{\rho^\pm+1}{m_\pm}\varphi_2]} \\ \langle P\rangle \frac{\sqrt{\mathcal{P}_-(p)}\sqrt{\mathcal{Q}_+(q)}}{\sqrt{q-p}}\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda+1}{v}\varphi_1^\pm \mp \frac{\rho^\pm-1}{m_\pm}\varphi_2]} \\ \langle P\rangle \frac{\sqrt{\mathcal{P}_+(p)}\sqrt{\mathcal{Q}_-(q)}}{\sqrt{q-p}}\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda-1}{v}\varphi_1^\pm \mp \frac{\rho^\pm+1}{m_\pm}\varphi_2]} \end{pmatrix} . \tag{5.4.71}$$

We can now see that the spinor in (5.4.71) is smooth everywhere by noticing that, at the fixed points, it can be written as

$$\sigma = 1 : \; \varepsilon'(q_+,p_\pm) = \frac{c}{4} \begin{pmatrix} -\mathrm{i}\frac{\sqrt{\mathcal{P}_-(p_\pm)}\sqrt{\mathcal{Q}_-(q_+)}}{\sqrt{q_++p_\pm}}(1+\eta)(1+\lambda)\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda-1}{v}\varphi_1^\pm \mp \frac{\eta-1}{m_\pm}\varphi_2]} \\ -\mathrm{i}\frac{\sqrt{\mathcal{P}_+(p_\pm)}\sqrt{\mathcal{Q}_+(q_+)}}{\sqrt{q_++p_\pm}}(1-\eta)(1-\lambda)\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda+1}{v}\varphi_1^\pm \mp \frac{\eta+1}{m_\pm}\varphi_2]} \\ \langle P\rangle \frac{\sqrt{\mathcal{P}_-(p_\pm)}\sqrt{\mathcal{Q}_+(q_+)}}{\sqrt{q_+-p_\pm}}(1+\eta)(1-\lambda)\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda+1}{v}\varphi_1^\pm \mp \frac{\eta-1}{m_\pm}\varphi_2]} \\ \langle P\rangle \frac{\sqrt{\mathcal{P}_+(p_\pm)}\sqrt{\mathcal{Q}_-(q_+)}}{\sqrt{q_+-p_\pm}}(1-\eta)(1+\lambda)\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda-1}{v}\varphi_1^\pm \mp \frac{\eta+1}{m_\pm}\varphi_2]} \end{pmatrix} , \tag{5.4.72}$$

or

$$\sigma = -1 : \; \varepsilon'(q_+,p_\pm) = \frac{c}{4} \begin{pmatrix} -\mathrm{i}\frac{\sqrt{\mathcal{P}_-(p_\pm)}\sqrt{\mathcal{Q}_-(q_+)}}{\sqrt{q_++p_\pm}}(1\pm\eta)(1+\lambda)\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda-1}{v}\varphi_1^\pm \mp \frac{\pm\eta-1}{m_\pm}\varphi_2]} \\ -\mathrm{i}\frac{\sqrt{\mathcal{P}_+(p_\pm)}\sqrt{\mathcal{Q}_+(q_+)}}{\sqrt{q_++p_\pm}}(1\mp\eta)(1-\lambda)\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda+1}{v}\varphi_1^\pm \mp \frac{\pm\eta+1}{m_\pm}\varphi_2]} \\ \langle P\rangle \frac{\sqrt{\mathcal{P}_-(p_\pm)}\sqrt{\mathcal{Q}_+(q_+)}}{\sqrt{q_+-p_\pm}}(1\pm\eta)(1-\lambda)\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda+1}{v}\varphi_1^\pm \mp \frac{\pm\eta-1}{m_\pm}\varphi_2]} \\ \langle P\rangle \frac{\sqrt{\mathcal{P}_+(p_\pm)}\sqrt{\mathcal{Q}_-(q_+)}}{\sqrt{q_+-p_\pm}}(1\mp\eta)(1+\lambda)\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda-1}{v}\varphi_1^\pm \mp \frac{\pm\eta+1}{m_\pm}\varphi_2]} \end{pmatrix} . \tag{5.4.73}$$

It is evident from the above expressions that, for a fixed choice of the signs (η,λ), the single component of the spinor which is non-zero does not contain any phase. Choosing for example the roots q_+ and p_+ such that $\mathcal{P}_+(p_+) = \mathcal{Q}_+(q_+) = 0$, which is realized by fixing $\eta = \lambda = 1$, we see from (5.4.72) that all the components are zero but the first, which however does not depend any more on (φ_1^+,φ_2). In the twist case this example is equivalent to

$$\sigma = 1 : \quad \varepsilon'(q_+,p_+) = c \begin{pmatrix} -\mathrm{i}\frac{\sqrt{\mathcal{P}_-(p_+)}\sqrt{\mathcal{Q}_-(q_+)}}{\sqrt{q_++p}} \\ 0 \\ 0 \\ 0 \end{pmatrix} . \tag{5.4.74}$$

Notice also that these spinors are regular at the location of the bolt $q = q_+$, without moving to its poles. Indeed the $\varphi_1^\pm$ are cancelled in the non-vanishing components by requiring $\mathcal{Q}_\pm(q_+) = 0$. Moreover equations (5.4.72) and (5.4.73) show that the spinor has the same chirality at the poles of the spindle in the $\sigma = 1$ case and opposite chiralities when $\sigma = -1$. In turn, this explicitly shows that $(\sigma = -1)$ $\sigma = 1$ corresponds to the (anti-) twist realization of the supersymmetry on the spindle bolt [42].

We end this section by writing the spinor transition function coming from (5.4.70), obtained by the composition of the transformation on U_+ and the (inverse) transformation on U_-

$$\exp\left[\frac{\mathrm{i}\,\varphi_2}{2}\mathrm{diag}\left\{\chi_\Sigma-\frac{t/v}{m_-m_+}\,,-\chi_\Sigma+\frac{t/v}{m_-m_+}\,,\chi_\Sigma+\frac{t/v}{m_-m_+}\,,-\chi_\Sigma-\frac{t/v}{m_-m_+}\right\}\right], \tag{5.4.75}$$

where we have used (5.4.43) and (5.4.44). Notice that this transition function is valid for both the twist and the anti-twist set-up, as for example in [46, 52], even if only χ_Σ appears.

Gauge field and Killing spinors at the boundary

The gauge field $A_{(3)}$ and a Killing spinor $\chi_{(3)}$ induced at the boundary are obtained from the $q\to+\infty$ expansion of (5.4.2) and (5.4.12), respectively. In particular, in the gauge (5.4.65) the boundary gauge field $A_{(3)}=pP\mathrm{d}\sigma$ reads

$$U_\pm\ :\quad \big(A_{(3)}\big)'_\pm=\frac{\Delta\theta_1}{2\pi}(q_+Q-pP)\mathrm{d}\varphi_1^\pm+\frac{\Delta}{2\pi}\frac{P(p-p_\pm)}{(q_+^2-p_\pm^2)}\mathrm{d}\varphi_2\,. \tag{5.4.76}$$

Using the three-dimensional vierbein induced from the bulk (5.3.9), namely

$$e^a_{(3)}=\left\{\frac{\mathrm{d}p}{\sqrt{-\mathcal{P}(p)}}\,,\sqrt{-\mathcal{P}(p)}\mathrm{d}\sigma\,,(\mathrm{d}\tau+p^2\mathrm{d}\sigma)\right\}\underset{p\to p_\pm}{\simeq}\left\{\mp\mathrm{d}R_\pm\,,\frac{R_\pm}{m_\pm}\mathrm{d}\varphi_2\,,(q_+^2-p_\pm^2)\frac{\Delta\theta_1}{2\pi}\mathrm{d}\varphi_1^\pm\right\}, \tag{5.4.77}$$

the boundary spinor is [96]

$$U_\pm\ :\quad \chi_{(3)}=\begin{pmatrix}\sqrt{\mathcal{P}_-(p)}\\ \sqrt{\mathcal{P}_+(p)}\end{pmatrix}\mathrm{e}^{\frac{\mathrm{i}}{2}[\frac{\lambda}{v}\varphi_1^\pm\mp\frac{\rho_\pm}{m_\pm}\varphi_2]}=\begin{pmatrix}\sqrt{\mathcal{P}_-(p)}\\ \sqrt{\mathcal{P}_+(p)}\end{pmatrix}\mathrm{e}^{\frac{\mathrm{i}\eta}{2}[\frac{\kappa}{v}\varphi_1^\pm\mp\frac{\sigma_\pm}{m_\pm}\varphi_2]}\,. \tag{5.4.78}$$

Here we have used the definitions $\rho_+\rho_-=\sigma$, $\kappa=\eta\lambda$ and introduced the symbol $\sigma_\pm=\eta\rho_\pm$ which is $\sigma_+=1$ and $\sigma_-=\sigma$. We emphasise that since we used the *same regular gauge* for the bulk and boundary gauge field, the phase of the spinor is the same as in (5.4.66).

We now show that this boundary spinor is well defined, in each patch. Since, as in (5.4.67), the boundary frame (5.4.77) suffers from singularities at the poles $p=p_\pm$, we perform on $e^{1,2}_{(3)}$ a rotation analogous to (5.4.69)

$$U_\pm:\quad \begin{pmatrix}\hat e^1_{(3)}\\ \hat e^2_{(3)}\\ \hat e^3_{(3)}\end{pmatrix}=\begin{pmatrix}\cos(\mp\varphi_2/m_\pm) & -\sin(\mp\varphi_2/m_\pm) & 0\\ \sin(\mp\varphi_2/m_\pm) & \cos(\mp\varphi_2/m_\pm) & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}e^1_{(3)}\\ e^2_{(3)}\\ e^3_{(3)}\end{pmatrix}. \tag{5.4.79}$$

The corresponding action on the spinor[12] is given by

$$U_\pm:\quad \chi'_{(3)}=\begin{pmatrix}\mathrm{e}^{\pm\frac{\mathrm{i}}{2}\frac{\varphi_2}{m_\pm}} & 0\\ 0 & \mathrm{e}^{\mp\frac{\mathrm{i}}{2}\frac{\varphi_2}{m_\pm}}\end{pmatrix}\chi_{(3)}\,, \tag{5.4.80}$$

[12] We are using the standard Pauli matrices σ_a, $a=1,2,3$ as a basis of the three-dimensional Clifford algebra since they are inherited from (5.3.10).

so that at the two poles we have

$$
\begin{aligned}
\sigma = 1: \quad & \chi'_{(3)}(p_\pm) = \frac{1}{2}\begin{pmatrix} \sqrt{\mathcal{P}_-(p)}(1+\eta)\mathrm{e}^{\mp\frac{\mathrm{i}}{2}\frac{\eta-1}{m_\pm}\varphi_2} \\ \sqrt{\mathcal{P}_+(p)}(1-\eta)\mathrm{e}^{\mp\frac{\mathrm{i}}{2}\frac{\eta+1}{m_\pm}\varphi_2} \end{pmatrix} e^{\frac{\mathrm{i}}{2}\frac{\lambda}{v}\varphi_1^\pm}\,, \\
\sigma = -1: \quad & \chi'_{(3)}(p_\pm) = \frac{1}{2}\begin{pmatrix} \sqrt{\mathcal{P}_-(p)}(1\pm\eta)e^{\mp\frac{\mathrm{i}}{2}\frac{\pm\eta-1}{m_\pm}\varphi_2} \\ \sqrt{\mathcal{P}_+(p)}(1\mp\eta)e^{\mp\frac{\mathrm{i}}{2}\frac{\pm\eta+1}{m_\pm}\varphi_2} \end{pmatrix} e^{\frac{\mathrm{i}}{2}\frac{\lambda}{v}\varphi_1^\pm}\,,
\end{aligned}
\tag{5.4.81}
$$

which are always well defined, by an identical argument as for the bulk. We then conclude that the transition function for the boundary spinor coming from (5.4.80) is given by

$$
\begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}}{2}\chi_\Sigma\varphi_2} & 0 \\ 0 & \mathrm{e}^{-\frac{\mathrm{i}}{2}\chi_\Sigma\varphi_2} \end{pmatrix}\,,
\tag{5.4.82}
$$

which is the same as that for the spinors defined on a spindle [42]! This might be surprising, but it is a direct consequence of the regularity analysis at the bolt.

Seifert orbifolds

In this subsection we collect all the properties of the boundary geometry $(g_{(3)}, A_{(3)}, \chi_{(3)})$ to compare them to the Seifert orbifolds studied for example in [48, 49] from the point of view of rigid supersymmetry. The main difference between [49] and the analysis we will report here is that if one thinks to the boundary data $(g_{(3)}, A_{(3)}, \chi_{(3)})$ as "disconnected" from the bulk ones, more degree of freedoms are present in the boundary. *Vice versa*, if the boundary data comes as a limit of the bulk metric, gauge field and spinor, there exists a mechanism which informs the boundary about the properties of the bulk. Loosely speaking, in our context, $\mathcal{M}_3$ does not depend any more on the coordinate q and as a consequence the angular variable φ_1 is well-defined in the boundary, whilst it is ill-defined at the location of the bolt $q = q_+$. To show this, we will not use the the local coordinates $(\varphi_1^\pm, \varphi_2)$ any more: indeed the toric coordinates (ϕ_1, ϕ_2) introduced in (5.4.54) are more appropriate as they describe correctly the effective action of the torus.

Metric First of all we look at the metric in (ϕ_1, ϕ_2) coordinates. These are the appropriate coordinates to study the singularities of the three-dimensional Seifert orbifold [161, 42]. More precisely, the metric near to the poles (5.4.26) takes the form

$$
U_\pm: \quad \mathrm{d}s_b^2 \underset{p\to p_\pm}{\simeq} \mathrm{d}R_\pm^2 + \frac{R_\pm^2}{m_\pm^2}\mathrm{d}\phi_2^2 + \left(\frac{(q_+^2 - p_\pm^2)\Delta\theta_1}{2\pi}\right)^2\left(\mathrm{d}\phi_1 \pm \frac{r_\mp}{m_\pm}\mathrm{d}\phi_2\right)^2 . \tag{5.4.83}
$$

It is now straightforward to introduce complex coordinates adapted to the poles, namely

$$
U_\pm: \quad z_\pm = R_\pm \mathrm{e}^{\mathrm{i}\frac{\phi_2}{m_\pm}}\,, \quad f_\pm = \frac{(q_+^2 - p_\pm^2)\Delta\theta_1}{2\pi}\mathrm{e}^{\mathrm{i}(\phi_1 \pm \frac{r_\mp}{m_\pm}\phi_2)}\,, \tag{5.4.84}
$$

where $z_\pm$ are local coordinates on the base and $f_\pm$ on the fibre. Performing a 2π-rotation along both ϕ_1 and ϕ_2 we have the following identifications

$$
U_\pm: \quad m\cdot(z_\pm, f_\pm) = \left(\mathrm{e}^{\frac{2\pi\mathrm{i}\, m}{m_\pm}} z_\pm, \mathrm{e}^{\pm\frac{2\pi\mathrm{i}\, m\, r_\mp}{m_\pm}} f_\pm\right)\,, \tag{5.4.85}
$$

with $m \in \mathbb{Z}_{m_\pm}$. The analysis now splits into two cases.

- $\gcd(r_\mp, m_\pm) = 1$: the action is free, the three-dimensional space is smooth, and in particular it is a lens space $L(m_- r_- + m_+ r_+ = t, 1)$. Notice that we can rephrase this statement by saying that the boundary space is smooth if and only if $\gcd(t, m_\pm) = 1$. Indeed given that the Bézout's lemma holds and $\gcd(m_-, m_+) = 1$, we have that $\gcd(t, m_\pm) = \gcd(r_\mp, m_\pm)$;

- $\gcd(r_\mp, m_\pm) \neq 1$: we can parameterize $r_\mp = k_\pm \overline{r}_\mp$ and $m_\pm = k_\pm \overline{m}_\pm$ (with also $\gcd(k_+, k_-) = 1$) so that $\gcd(t, m_\pm) = \gcd(r_\mp, m_\pm) = k_\pm > 0$. Then we can act $m = \overline{m}_\pm$ times on (5.4.85), and the result is

$$U_\pm : \quad \overline{m}_\pm \cdot (z_\pm, f_\pm) = \left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{k_\pm}} z_\pm, f_\pm \right) . \tag{5.4.86}$$

  This explicitly shows that there is a $\mathbb{Z}_{k_\pm}$ sub-action which leaves the fiber fixed and, in turn, that the three-dimensional space is an orbifold, with conical singularities near the poles $p_\pm$ of order $\mathbb{C}/\mathbb{Z}_{k_\pm}$, respectively. In terms of the lens space parameter t, we can state that the space has a singularity $\gcd(t, m_\pm)$ near $p_\pm$. Of course one can also have a singularity at a single pole if one among $k_\pm$ is one, but in the symmetric case with both $k_\pm \neq 1$ the validity of the Bézout's lemma implies $t = k_- k_+ \overline{t}$. This configuration corresponds to the branched lens space $\mathbb{L}_{[k_-, k_+]}(k_- k_+ \overline{t}, 1)$.

Gauge field and spinor In these coordinates, the boundary gauge field (5.4.76) and the three-dimensional Killing spinor (5.4.78) read

$$\begin{aligned} U_\pm : \quad \left(A_{(3)}\right)'_\pm = & \frac{\Delta\theta_1}{2\pi} \Big[(q_+ Q - pP)\mathrm{d}\phi_1 \\ & + \frac{1}{m_- m_+} \left(\frac{t(q_+^2 - p_\mp^2)(p - p_\pm)P}{p_+^2 - p_-^2} \pm (q_+ Q - pP)(t - r_\pm m_\pm) \right) \mathrm{d}\phi_2 \Big] , \end{aligned} \tag{5.4.87}$$

where we have used (5.4.37), and

$$U_\pm : \quad \chi_{(3)} = \begin{pmatrix} \sqrt{\mathcal{P}_-(p)} \\ \sqrt{\mathcal{P}_+(p)} \end{pmatrix} \mathrm{e}^{\frac{\mathrm{i}\eta}{2} [\frac{\kappa}{v}\phi_1 \pm \frac{\kappa r_\mp - v\sigma_\pm}{v m_\pm} \phi_2]} . \tag{5.4.88}$$

Following the steps outlined in (5.4.66), we can easily show that the transition functions for the gauge field (5.4.87) (or also (5.4.76)) across the patches U_+ and U_- are

$$\left(A_{(3)}\right)_- - \left(A_{(3)}\right)_+ = \frac{1}{2} \left[\frac{\rho^+ m_- + \rho^- m_+}{m_- m_+} - \lambda \frac{t/v}{m_- m_+} \right] \mathrm{d}\phi_2 = \frac{\eta}{2} \left[\frac{m_- + \sigma m_+}{m_- m_+} - \kappa \frac{t/v}{m_- m_+} \right] \mathrm{d}\phi_2 . \tag{5.4.89}$$

Notice that this formula is perfectly valid also when $t = 0$, for which the boundary is $\Sigma \times S^1$ and contains the spindle as a non-trivial two-cycle. The transition function for the spinor is essentially unchanged with respect to (5.4.82) since $\varphi_2 = \phi_2$, namely

$$\begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}}{2}\chi_\Sigma \phi_2} & 0 \\ 0 & \mathrm{e}^{-\frac{\mathrm{i}}{2}\chi_\Sigma \phi_2} \end{pmatrix} . \tag{5.4.90}$$

Formula (5.4.89) shows that the choice of the sign σ, that will determine the type of twist for the graviphoton field in the bulk (see (5.4.112) below), is already captured by the boundary geometry. Similarly, from (5.4.88) (or even more explicitly from (5.4.81)) one can see that for $\sigma = 1$ the spinor has the same "transverse chirality"[13] at the poles $p = p_\pm$ of the Seifert orbifold, whilst for $\sigma = -1$ it has opposite chiralities. We therefore refer to the choices $\sigma = \pm 1$ as to *twist and anti-twist for the Seifert orbifold*. Again, even if the transition function for the spinor seems not to be affected by σ, the boundary spinor itself has a dependence on it. In particular, we emphasize that the mechanism by which the bulk "informs" the boundary is through the regularity of the gauge field at the bolt, which then determines a preferred gauge at the boundary[14]. This is the same idea that has proved crucial in the context of supersymmetric black holes, put forward in [73]. In a context analogous to the present one, similar observations were made in [111] and [161].

We can now make a comparison with with the rigid supersymmetry results for a large class of three-dimensional Seifert orbifolds analyzed in [49]. In particular, let us consider the gauge field A^C (*cf.* (2.103) in [49])[15]

$$A^C = \frac{1}{2}\frac{b_1 m_- + \sigma b_2 m_+}{t f(\hat\theta)}\mathrm{d}\phi_1 + \frac{1}{2}\frac{\sigma b_2 r_- - b_1 r_+}{t f(\hat\theta)}\mathrm{d}\phi_2\,, \tag{5.4.91}$$

and its transition functions. Here b_1 and b_2 represent the squashing of the three-sphere from which the lens space is obtained and $f(\hat\theta = 0) = -b_2$ and $f(\hat\theta = \pi) = -b_1$. To ensure that this gauge field is well-defined, we perform a gauge transformation in each patch

$$\begin{aligned}
U_- :\quad A^C_{(0)} &= \frac{1}{2}\left(\frac{b_1 m_- + \sigma b_2 m_+}{t f(\hat\theta)} + \alpha_3^{(0)}\right)\mathrm{d}\phi_1 + \frac{1}{2}\left(\frac{\sigma b_2 r_- - b_1 r_+}{t f(\hat\theta)} + \alpha_2^{(0)}\right)\mathrm{d}\phi_2\,,\\
U_+ :\quad A^C_{(\pi)} &= \frac{1}{2}\left(\frac{b_1 m_- + \sigma b_2 m_+}{t f(\hat\theta)} + \alpha_3^{(\pi)}\right)\mathrm{d}\phi_1 + \frac{1}{2}\left(\frac{\sigma b_2 r_- - b_1 r_+}{t f(\hat\theta)} + \alpha_2^{(\pi)}\right)\mathrm{d}\phi_2\,,
\end{aligned} \tag{5.4.92}$$

and require

$$\alpha_2^{(0)} = \frac{\sigma - r_+\alpha_3^{(0)}}{m_-}\,, \qquad \alpha_2^{(\pi)} = \frac{r_-\alpha_3^{(\pi)} - 1}{m_+}\,. \tag{5.4.93}$$

Notice that A^C is well-defined in the patches $U_\pm$ near $\hat\theta = 0, \pi$ if it depends on the angular coordinates only through the well-defined (in each patch) combinations $\mathrm{d}\phi_1 \pm (r_\mp/m_\pm)\mathrm{d}\phi_2$, which appear in the metric near the poles (5.4.83). Identifying $A^C = \eta A_{(3)}$ and additionally requiring that

$$\alpha_3^{(0)} = \alpha_3^{(\pi)} = \frac{\kappa}{v} \implies \alpha_2^{(0)} = \frac{\sigma v - \kappa r_+}{v m_-}\,, \qquad \alpha_2^{(\pi)} = \frac{\kappa r_- - v}{v m_+}\,, \tag{5.4.94}$$

[13] The boundary spinor (5.4.78) is already decomposed in a product of a two dimensional spinor on the base (with coordinates p, φ_2) and a phase on the Seifert fiber $\varphi_1^\pm$. Accordingly, the two dimensional gamma matrices can be taken to be just σ_1 and σ_2, with chirality matrix $\sigma_\star = -\mathrm{i}\sigma_1\sigma_2 = \sigma_3$. Under $\sigma_\star$ the two-dimensional spinor (5.4.78) is chiral or anti-chiral at the poles of the Seifert orbifold.

[14] In particular, the phase $\mathrm{e}^{\frac{\mathrm{i}}{2}\frac{\Delta}{v}\varphi_1^\pm}$ in the Killing spinor (5.4.78) is precisely the phase $\mathrm{e}^{\frac{\mathrm{i}}{2}n\psi}$ that appears in the spindle index [48], with $n = \pm 1$ for the accelerating black holes. Indeed as remarked the $\Sigma \times S^1$ geometry is a special case $t = 0$ of the general set-up discussed above. See also footnote (11).

[15] The relation between our variables and the ones in section 2.6 of [49] is $n_\pm|_{\text{there}} = m_\pm|_{\text{here}}$, $n|_{\text{there}} = t|_{\text{here}}$, $n|_{\text{there}} = t|_{\text{here}}$, $t_\pm|_{\text{there}} = \mp\frac{r_\mp}{t}|_{\text{here}}$, $\varphi|_{\text{there}} = -\phi_2|_{\text{here}}$, $\psi|_{\text{there}} = \phi_1|_{\text{here}}$ and finally we have mapped $b_1 \to -b_1$ and $b_2 \to \sigma b_2$.

we conclude that indeed $A^C_{(0)} - A^C_{(\pi)}$ coincides with (5.4.89). We emphasize that from the boundary point of view there is no reason to fix $\alpha_3^{(0)}$ and $\alpha_3^{(\pi)}$. Indeed, in principle, they can take any value without affecting the regularity of the gauge field. However, we have a preferred gauge (5.4.65) inherited from the bulk, ensuring that the spinor and the gauge field are regular at the bolt location $q = q_+$. For comparison, the flat connections $(\alpha_3^{(0)}, \alpha_3^{(\pi)})$ chosen in [49] are given by

$$\alpha_3^{(0)} = \alpha_3^{(\pi)} = \frac{m_- + \sigma m_+}{t} \qquad \Longrightarrow \qquad \alpha_2^{(0)} = \alpha_2^{(\pi)} = \frac{\sigma r_- - r_+}{t}\,, \tag{5.4.95}$$

which correspond to trivial gauge transformations across the two patches.

In the case $t = 1$, where $\mathcal{M}_3 = L(1,1) = S^3$ is a smooth (squashed) three-sphere, there exists a different set of effective coordinates for the torus action. The details can be found in [3], but the result is again that the main significant difference with the analysis in [49], both for the squashed three-sphere and for the (possibly branched) lens space, is a distinct choice of gauge field, which is imposed on us from the global analysis in the bulk.

5.4.5 Quantization conditions

Having described in details the bulk and boundary geometry, together with their local and global properties, we now should control if the flux of the graviphoton A is correctly quantized, namely as in (5.1.6). To this end, as in the previous chapters, it proves useful to express all the parameters of the solution in terms of the integers $(t, m_\pm, v)$. Recall that we started with six "local" parameters (E, N, M, P, Q, α) and two "global" ones $(\Delta\theta_i^\pm)$. Among these, two are fixed by supersymmetry, namely (E, M). We are thus left with (N, P, Q, α), which are also exchanged for $(p_\pm, q_+, \alpha)$ and three signs (η, σ, κ) through (5.4.15). These relevant signs are related to the ones introduced around (5.4.15) via

$$\eta = \operatorname{sign}\operatorname{Re}(P)\,, \quad \sigma = \eta\delta\,, \quad \kappa = \eta\lambda\,, \tag{5.4.96}$$

where the value of σ will reflect the twist or anti-twist on the spindle bolt.

Twist

Using the expressions (5.4.15) with $\sigma = 1$, it is easy to take the ratio of (5.4.35). Defining the rescaled parameter

$$\tilde{P} = \frac{P}{p_+ + p_-}\,, \tag{5.4.97}$$

the ratio of (5.4.35) takes the simple form

$$\frac{m_+}{m_-} = \frac{q_+^2 - p_+^2}{q_+^2 - p_-^2}\,\frac{p_- + \eta\tilde{P}}{p_+ + \eta\tilde{P}}\,. \tag{5.4.98}$$

We can parametrize the roots as in the previous chapters

$$p_\pm = w(1 \pm x)\,, \quad q_+ = \frac{p_+ + p_-}{2}\tilde{q}_+ = w\tilde{q}_+\,, \quad 0 < w < q_+\,, \quad x > 0\,, \quad \tilde{q}_+ > 1 + x\,, \tag{5.4.99}$$

where these constraints on w and x come from $p_+ \pm p_- > 0$ and $q_+ > p_+$. From (5.4.98) we get

$$w = -\eta \tilde{P} \frac{\big[\tilde{q}_+^2 - (1-x)^2\big] m_+ - \big[\tilde{q}_+^2 - (1+x)^2\big] m_-}{(1+x)\big[\tilde{q}_+^2 - (1-x)^2\big] m_+ - \big[\tilde{q}_+^2 - (1+x)^2\big](1-x) m_-} \,. \tag{5.4.100}$$

Then, from both (5.4.35) and (5.4.48) we obtain

$$\begin{aligned} \frac{\Delta}{2\pi} &= \frac{1}{8\eta \tilde{P}} \frac{(1+x)\big[\tilde{q}_+^2 - (1-x)^2\big] m_+ - \big[\tilde{q}_+^2 - (1+x)^2\big](1-x) m_-}{x^2 m_+ m_-} \,, \\ \frac{\Delta\theta_1}{2\pi} &= \frac{1}{2v(\tilde{q}_+ - \kappa)} \frac{1}{w^2 \big[2\eta \tilde{P} + w(\tilde{q}_+^2 + 1 - x^2)\big]} \,. \end{aligned} \tag{5.4.101}$$

Finally (5.4.37) is now simply

$$t = \frac{v(\tilde{q}_+ - \kappa)}{x}(m_- - m_+) \implies x = (\tilde{q}_+ - \kappa)\frac{(m_- - m_+)v}{t} \,. \tag{5.4.102}$$

Notice that not all the values of t are admissible from (5.4.102), due to the constraint $\tilde{q}_+ > 1 + x > 1$ and $N \neq 0$. In appendix C of [3] it is showed that necessarily $t \geq 2$. Using the previous expressions it is easy to show that

$$\frac{1}{2\pi}\int_{\mathcal{L}_2} F = \frac{\eta}{2}\left[\chi_\Sigma - \kappa \frac{t/v}{m_- m_+}\right] , \tag{5.4.103}$$

meaning that $2A$ is a connection one-form on the line bundle $\mathcal{O}(m_- + m_+)$ over $\Sigma^{q_+}_{[m_-,m_+]}$, and, in turn, that we have a twist. Observe that (5.4.103) takes the form of (5.1.6).

The additional term $t/(vm_- m_+)$ in (5.4.103) comes from the non trivial fibration on the spindle bolt in (5.4.24), more precisely from the normal bundle to $\mathcal{L}_2$. To see this, we focus on $q = q_+$ in (5.4.24) making the change $q = q_+ + \frac{\mathcal{Q}'(q_+)}{4} r^2$. The result is

$$U_\pm : \quad \mathrm{d}s_4^2 \underset{q_+}{\simeq} \mathrm{d}s^2_{\Sigma_{q_+}} + (q_+^2 - p^2)\left[\mathrm{d}r^2 + \frac{r^2}{v^2}\big(\mathrm{d}\varphi_1^\pm + \mu_f^\pm\big)^2\right] , \tag{5.4.104}$$

where $\mu_f^\pm$ is the fibration one-form, given by

$$U_\pm : \quad \mu_f^\pm = v \frac{2q_+(q_+^2 - p_\pm^2)\mathcal{P}(p) + (p^2 - p_\pm^2)(q_+^2 - p^2)\mathcal{Q}'(q_+)}{2(q_+^2 - p^2)^2} \,\mathrm{d}\theta_2^\pm \,. \tag{5.4.105}$$

Integrating it we get immediately

$$\frac{1}{2\pi}\int_{\mathcal{L}_2} \mathrm{d}\mu_f^\pm = \frac{t}{m_- m_+} \,, \tag{5.4.106}$$

which gives correctly the same value as in (5.4.36). Notice that (5.4.106) and (5.4.103) are analogous to (3.4.13) and (3.4.19), respectively. The role of n_+ in (3.4.19) is here played by v, whilst there is no analogue on n_- since the corresponding divisor has been sent to infinity.

In summary, we started from six quantities $(N, P, Q, \alpha, \Delta, \Delta\theta_1)$ subject to four topological constraint for $(t, m_\pm, v)$. We are left with a solution with *two free continuous parameters*, that we choose to be $(\tilde{P}, \tilde{q}_+) \in \mathbb{R}$.

Anti-twist

We now focus on the other case, when $\sigma = -1$. The ratio of (5.4.35) is much more complicated

$$\frac{m_+}{m_-} = \frac{q_+^2 - p_+^2}{q_+^2 - p_-^2} \frac{\left[2P^2 - (p_+^2 - p_-^2)\eta P - p_-(p_+ + p_-)^3\right]}{\left[\eta P - p_+(p_+ + p_-)\right]\left[2\eta P + (p_+ + p_-)^2\right]}\,, \tag{5.4.107}$$

but it can be simplified redefining

$$\eta P = (p_+ + p_-)\eta\tilde{P} = (p_+ + p_-)\frac{p_+ - p_- + \sqrt{(p_+ - p_-)^2 - 8(p_+ + p_-)\eta\overline{P}}}{4}\,. \tag{5.4.108}$$

With the parametrization (5.4.99), we obtain formally the same expression for w as in (5.4.100) but with $\overline{P}$ instead of $\tilde{P}$, specifically

$$w = -\eta\overline{P}\frac{\left[\tilde{q}_+^2 - (1-x)^2\right]m_+ - \left[\tilde{q}_+^2 - (1+x)^2\right]m_-}{(1+x)\left[\tilde{q}_+^2 - (1-x)^2\right]m_+ - \left[\tilde{q}_+^2 - (1+x)^2\right](1-x)m_-}\,. \tag{5.4.109}$$

The other parameters $(\Delta\theta_1, \Delta, t)$ can be obtained similarly, but the expressions do not have a compact form. In particular, $t = t(x)$ is not invertible. For future reference, we report it here

$$\begin{aligned} t =& v\frac{\left[\tilde{q}_+^2 - (x+1)^2\right]m_- - \left[\tilde{q}_+^2 - (x-1)^2\right]m_+}{\left[\tilde{q}_+^2 - (x+1)^2\right]\left[\tilde{q}_+^2 - (x-1)^2\right]}\times \\ & \frac{\tilde{q}_+ w\left[w(\tilde{q}_+^2 + x^2 - 1) - 2x\eta\tilde{P}\right] + \kappa\left[-2\tilde{P}^2 x - w^2 x(\tilde{q}_+^2 + x^2 - 1) + \eta\tilde{P}w(\tilde{q}_+^2 + 3x^2 - 1)\right]}{w^2 x}\,, \end{aligned} \tag{5.4.110}$$

where w and $\tilde{P}$ are given by (5.4.109) and (5.4.108), respectively. Notice that (5.4.110) is not written in a completely explicit form. However, when expanded, it does not depend on $\overline{P}$, a fact that will play a role later on. Moreover, here as before, the parameter t can not take any value, and in particular it can not happen $t = 1$. From now on the analysis continues as before, but it is more involved. Nevertheless, it can be proved that

$$\frac{1}{2\pi}\int_{\mathcal{L}_2} F = \frac{\eta}{2}\left[\frac{m_- - m_+}{m_- m_+} - \kappa\frac{t/v}{m_- m_+}\right]. \tag{5.4.111}$$

Even though we can not obtain x from (5.4.110), we regard to this implicit expression as a constraint, which leaves us again a solution with *two free continuous parameters* $(\overline{P}, \tilde{q}_+) \in \mathbb{R}$.

The main result of this section is the following formula which encapsulates simultaneously (5.4.103) and (5.4.111)

$$\frac{1}{2\pi}\int_{\mathcal{L}_2} F = \frac{\eta}{2}\left[\frac{m_- + \sigma m_+}{m_- m_+} - \kappa\frac{t/v}{m_- m_+}\right], \tag{5.4.112}$$

and state that $2A$ is a connection on the bundle $\mathcal{O}(m_- + \sigma m_+)$ over the spindle bolt $\Sigma^{q_+}_{[m_-, m_+]}$. It is interesting to notice that both type of twist can be realized in minimal gauged supergravity, as opposite to the black hole solution [46] with a spindly horizon where only the anti-twist is realized. According to the values of the *two* remaining *free continuous parameters* we can have $\sigma = \pm 1$. Recall that the sign σ decides the location of the roots $p_\pm$ describing the poles of the spindle. The situation is, thus, completely analogous to (4.3.4). Notice also that (5.4.112) is identical to the expression of the gauge field transition functions (5.4.89).

We can now compare (5.4.112) with (5.1.6), which encapsulates the cases for which it is possible to uplift *globally* our spindle bolt solution to M-theory. The integer m appearing in (5.1.6) can be read from (5.4.112) and we obtain the following constraint

$$k\eta\,(m_- v + \sigma m_+ v - \kappa t) = mI\,, \tag{5.4.113}$$

where recall that I is the Fano index of the KE_6 base on which the SE_7 is constructed and k is an integer which describes the $U(1)$ fibration over KE_6 and divides I. When $k = I$, which amounts to requiring canonical period $2\pi/4$ for ξ, it is easy to see that the condition (5.4.113) is always satisfied, and then any spindle Bolt solution can be globally uplifted to a M-theory solution on a regular SE manifold. However, in general, the condition (5.4.113) leads to restrictions. For example, consider the case of $M_7 = S^7/\mathbb{Z}_2$, which implies $I = 4$, $k = 2$. By (5.4.113) this means that $m_- v + \sigma m_+ v - \kappa t$ is divisible by 2, which is not satisfied for all values of (m_-, m_+, v, t). Notice that when

$$v(m_- + \sigma m_+) - \kappa t = 0\,, \tag{5.4.114}$$

which is possible only for $\kappa = 1$, it is possible to uplift the solution for any choice of SE_7.

5.4.6 On-shell action

We have now two families of solutions, within the same CP class, which are distinguished by the value of the sign $\sigma = \pm 1$. For both of them, we have expressed all the parameters in terms of the integers $(t, m_\pm, v)$ and computed subsequently the flux (5.4.112). With these informations at disposal we can express the "local" CP (non-supersymmetric) on-shell action (5.3.38), namely

$$\begin{aligned} S_{\rm ren} = \frac{\Delta\tau\Delta\sigma(p_+ - p_-)q_+}{8\pi G_4}\bigg[& p_+^2 + p_+ p_- + p_-^2 + \frac{M}{q_+} - q_+^2 \\ &+ \frac{(P-Q)^2}{2(q_+ - p_+)(q_+ - p_-)} + \frac{(P+Q)^2}{2(q_+ + p_+)(q_+ + p_-)}\bigg]\,, \end{aligned} \tag{5.4.115}$$

in a "global" form, making explicit the dependence on the global parameters $(t, m_\pm, v)$. As anticipated, our spindle bolt solutions contain (through some limits) all the previous (local and global) known solutions with nuts and bolts [109, 110, 111, 96]. A nice check of this can be obtained by performing the limits explicitly on the on-shell actions for $\sigma = \pm 1$, which should give again the ones in [111, 96]. Details can be found in B, but here we can show at a glance, at least locally, that (5.4.115) reduces to the one in [83] for $P \to Q$ and $q_+ \to p_+$, namely

$$S_{\rm ren} = \frac{\Delta\tau\Delta\sigma}{8\pi G_4}\left[p_- p_+(p_+^2 - p_-^2) + M(p_+ - p_-) + Q^2\frac{p_+ - p_-}{p_+ + p_-}\right]. \tag{5.4.116}$$

Recalling that the product $(\Delta\tau\Delta\sigma)$ is equal in the patches $U_\pm$ and given by (5.4.38), employing all the results of the previous section for the twist case ($\sigma = 1$) we can express the renormalized action simply as

$$\sigma = 1: \quad S_{\text{ren}} = \frac{\pi}{8G_4 v}\left[2\chi_\Sigma - \kappa\frac{t/v}{m_- m_+} - \kappa\frac{v(m_- - m_+)^2}{t\, m_- m_+}\right], \quad t \geq 2\,. \tag{5.4.117}$$

In particular, notice that the on-shell action is fixed in terms of the integers $(t, v, m_\pm)$ and $(\tilde{P}, \tilde{q}_+)$ do not appear. The $\sigma = -1$ is, as usual, more involved. Looking explicitly at (5.4.115), it proves useful to define

$$\begin{aligned}\sigma = -1: \quad \mathfrak{q} &= \frac{q_+^2 - p_+^2}{p_+^2 - p_-^2}\,\frac{(p_+ - p_-)\big[\eta P + p_-(p_+ + p_-)\big]}{\big[(p_+ - p_-)\eta P - (p_+ + p_-)(q_+^2 - p_+ p_-)\big]} \\ &= \frac{\big[\tilde{q}_+^2 - (1+x)^2\big]\big[w(x-1) - \eta\tilde{P}\big]}{2w(\tilde{q}_+^2 + x^2 - 1) - 4x\eta\tilde{P}}\,.\end{aligned} \tag{5.4.118}$$

Notice that (5.4.118) seems to depend also on $\overline{P}$, but if one solves for $\tilde{P}$ and w through (5.4.108) and (5.4.109), $\overline{P}$ disappears. Recall also that $x = x(\tilde{q}_+)$ is fixed (non-explicitly) via (5.4.110) which in this context reads

$$\begin{aligned}\sigma = -1: \quad t =& v\big[(\tilde{q}_+^2 - (x+1)^2)m_- - (\tilde{q}_+^2 - (x-1)^2)m_+\big]\times \\ & \frac{-\tilde{q}_+^3 + \tilde{q}_+\big[1 + 2(1+2\mathfrak{q})x + x^2\big] + \kappa\big[\tilde{q}_+^2 - (1+x)^2 + 2\mathfrak{q}(-1 + \tilde{q}_+^2 + x^2)\big]}{x\big[1 - \tilde{q}_+^2 + 2(1+2\mathfrak{q})x + x^2\big]^2}\,.\end{aligned} \tag{5.4.119}$$

Therefore, as $x(\tilde{q}_+)$, also $\mathfrak{q} = \mathfrak{q}(\tilde{q}_+)$. The renormalized action for $\sigma = -1$ can then be written as a function of $\tilde{q}_+$ only

$$\begin{aligned}\sigma = -1: \quad S_{\text{ren}}(\tilde{q}_+) =& \frac{\pi\big[(\tilde{q}_+^2 - (x+1)^2)m_- - (\tilde{q}_+^2 - (x-1)^2)m_+\big]^2}{8G_4 t\, m_- m_+}\times \\ & \frac{2\tilde{q}_+(1 + x + 2\mathfrak{q})x - \kappa\big[\tilde{q}_+^2 + (1+x)^2 + 2\mathfrak{q}(1 + \tilde{q}_+^2 + x^2)\big]}{x^2\big[1 - \tilde{q}_+^2 + 2(1+2\mathfrak{q})x + x^2\big]^2}\,.\end{aligned} \tag{5.4.120}$$

We see that in the anti-twist case S_{ren} not only depends on the choice of the integers $(t, v, m_\pm)$, but contains a real degree of freedom $\tilde{q}_+$ as well.

Comparison with equivariant localization

The on-shell actions (5.4.117) and (5.4.120) computed through holographic renormalization are already valid for supersymmetric solutions, since we have used (5.4.5). For this reason, we can compare these results to the general expectations due to equivariant localization (see section 5.2 and in particular (5.4.58) for our solutions described by the polytope in figure 5.1). From the explicit Killing spinors (5.4.12) we obtain the supersymmetric Killing vector[16]

$$\epsilon_* = \sqrt{\alpha}\partial_\tau + \partial_\sigma\,. \tag{5.4.121}$$

[16]We have normalized differently the spinor ε than in (5.4.13), but the overall normalization will not be important since the action (5.4.58) depends only on z.

Notice that this degenerates only at the points $(q_+, p_\pm)$ (as visible from (5.4.21)), so that the fixed points p_A are "nuts" for ϵ_*. This justifies the use of (5.2.4) instead of (5.2.2). From (5.4.72) and (5.4.73) we recall that

$$\sigma = 1: \quad \begin{array}{l} \Gamma_\star \varepsilon = \kappa\varepsilon \text{ at } p_1\,, \\ \Gamma_\star \varepsilon = \kappa\varepsilon \text{ at } p_2\,, \end{array} \qquad \sigma = -1: \quad \begin{array}{ll} \Gamma_\star \varepsilon = -\kappa\varepsilon & \text{at } p_1\,, \\ \Gamma_\star \varepsilon = \kappa\varepsilon & \text{at } p_2\,, \end{array} \tag{5.4.122}$$

where the chirality matrix has been defined in (5.3.10). The chirality itself is fixed by the value of $\kappa = \pm 1$, so that the correct assignment for the signs c_A is

$$\sigma = 1: \quad c_2 = c_1 = \kappa\,, \qquad \sigma = -1: \quad c_2 = -c_1 = \kappa\,. \tag{5.4.123}$$

That the spinor must be chiral at the fixed points comes from general arguments, in particular as a consequence of the bilinear $\varepsilon^\dagger\varepsilon$ being nowhere vanishing [42, 84]. In the coordinates (5.4.55), the supersymmetric Killing vector (5.4.121) reads

$$\vec{\epsilon}_* = \frac{1}{q_+^2 - p_-^2}\left[\left((p_-^2 + \sqrt{\alpha})\frac{2\pi}{\Delta\theta_1} + \frac{r_+}{m_-}(q_+^2 + \sqrt{\alpha})\frac{2\pi}{\Delta\theta_2^-}\right)E_1 + (q_+^2 + \sqrt{\alpha})\frac{2\pi}{\Delta\theta_2^-}E_2\right]. \tag{5.4.124}$$

It is now a matter of algebra to plug explicitly the vector $\vec{\epsilon}_*$ (5.4.124) in (5.2.7) by using (5.2.8). After some work, using the Bézout's lemma $r_- m_- + r_+ m_+ = t$ and eliminating everywhere $\Delta\theta_1$ from (5.4.124) in favour of the fibration parameter t using (5.4.37), the result takes the form

$$I_{\text{on-shell}}(c_A, \mathsf{q}_*) = \frac{\pi}{8G_4 v}\left[2\chi_\Sigma - \frac{c_1}{m_+}\left(\frac{\mathsf{q}_*^2\, t}{v m_-} + \frac{v m_-}{\mathsf{q}_*^2\, t}\right) - \frac{c_2}{m_-}\left(\frac{\tilde{\mathsf{q}}_*^2\, t}{v m_+} + \frac{v m_+}{\tilde{\mathsf{q}}_*^2\, t}\right)\right], \tag{5.4.125}$$

where we have defined the on-shell reduced quantities

$$\tilde{\mathsf{q}}_*^2 = \frac{q_+^2 - p_+^2}{p_+^2 - p_-^2}\frac{p_-^2 + \sqrt{\alpha}}{q_+^2 + \sqrt{\alpha}}\,, \qquad \mathsf{q}_*^2 = -\frac{q_+^2 - p_-^2}{p_+^2 - p_-^2}\frac{p_+^2 + \sqrt{\alpha}}{q_+^2 + \sqrt{\alpha}}\,, \qquad \tilde{\mathsf{q}}_*^2 + \mathsf{q}_*^2 = -1\,, \tag{5.4.126}$$

such that $(\mathrm{Q}_*^2, \tilde{\mathrm{Q}}_*^2) = (t/v m_- m_+)(m_+ \mathsf{q}_*^2, m_- \tilde{\mathsf{q}}_*^2)$. As a consequence of the reality discussion at the end of Sect. 5.4.1, the parameter $\sqrt{\alpha}$ here can be complex and in turn the action itself can be complex.

It should be no surprise at this point that the equal-signs case ($c_2 = c_1 = \kappa$) is particularly easy to handle. From the point of view of the equivariant localization, this simplification comes from $\tilde{\mathsf{q}}_*^2 + \mathsf{q}_*^2 = -1$ which enters directly in (5.4.125). Indeed we have

$$\text{twist}: \quad (\mathsf{q}_*^2, \tilde{\mathsf{q}}_*^2) = \frac{1}{m_- - m_+}(m_-, -m_+)\,, \tag{5.4.127}$$

and as a consequence

$$\text{twist}: \quad I_{\text{on-shell}}(c_2, \mathsf{q}_*) = \frac{\pi}{8G_4 v}\left[2\chi_\Sigma - c_2\frac{t/v}{m_- m_+} - c_2\frac{v(m_- - m_+)^2}{t\, m_- m_+}\right], \quad t \geq 2\,, \tag{5.4.128}$$

which reproduces precisely (5.4.117) for $c_2 = \kappa$ as evident from (5.4.122) and (5.4.123). Correspondingly, the Reeb vector (5.4.124) is fixed

$$\text{twist}: \quad \vec{\epsilon}_* = f(\tilde{P}, \tilde{q}_+)\big[(r_+ + r_-)E_1 + (m_- - m_+)E_2\big]\,, \tag{5.4.129}$$

where f is a complicated but unimportant function of the free real parameters of the solution $(\tilde{P}, \tilde{q}_+)$.

When $c_2 = -c_1 = \kappa$, using the parametrization (5.4.15) with $\sigma = -1$, it is very easy to recognize that $\mathsf{q} = \mathsf{q}_*$ in (5.4.118). Then, gathering this fact, the expression (5.4.119) and the non-trivial identity

$$\sigma = -1: \quad \frac{m_+}{m_-} = -\frac{\tilde{\mathsf{q}}_\star^2\big[2\tilde{\mathsf{q}}_\star^2(-1+\tilde{q}_+^2-2x+x^2)-(2-x)(\tilde{q}_+^2-(x+1)^2)\big]}{(1-\tilde{\mathsf{q}}_\star^2)\big[2\tilde{\mathsf{q}}_\star^2(-1+\tilde{q}_+^2+2x+x^2)+x(\tilde{q}_+^2-(x+1)^2)\big]}\,, \tag{5.4.130}$$

it is possible to show that indeed (5.4.125) matches (5.4.120). In this case the ratio of the entries of the Reeb vector (5.4.124) is not fixed, and depends on $\tilde{q}_+$ but not on $\overline{P}$, as the action (5.4.120). This is then a highly non-trivial check of the validity of the equivariant procedure of [84].

Comments on extremization

Recall that we have dubbed equation (5.2.4) as "off-shell localized action". The reason is that (5.2.4) holds for each supersymmetric solution of the Maxwell-Einstein-Λ theory. Then, to each solution corresponds a particular $z_* = (\epsilon_2/\epsilon_1)_*$, computed "on-shell". It is then natural to ask if the extremization of the off-shell action reproduces the values we have found, namely (5.4.117) and (5.4.120). Indeed, there are many instances in which the existence of a solution to an extremization problem ensures the existence of a supergravity solutions. This phenomenon can be seen for example in the context of toric Sasaki-Einstein manifolds, where the existence of SE metrics is guaranteed by the extremization of the Sasakian volume [35, 36, 119]. Similarly it happens for [29, 30] in the context of Gauntlett-Kim (GK) geometry, where the key quantity is called "supersymmetric action". In fact, we will show that the extremization of (5.4.58) with respect to z reproduces the gravitational results for the renormalized on-shell action computed previously *only in the twist case* $\sigma = 1$. Choosing again $c_2 = c_1 = \kappa$ for the twist, there are two values of z that extremize I, namely

$$c_2 = c_1 = \kappa \implies z_\pm^* = \frac{\pm m_- - m_+}{\pm r_+ + r_-}\,. \tag{5.4.131}$$

It is then straightforward to compute the action at the extremizing values, using Bézout's lemma $r_- m_- + r_+ m_+ = t$. At the saddle z_+^* we have

$$I_{\text{off-shell}}(\kappa, z_+^*) = S_{\text{ren}}^{(\sigma=1)}\,, \tag{5.4.132}$$

where the latter is given for example in (5.4.117). Indeed, the value of z_+^* is constant and compatible with (5.4.129). Notice that in the limit $m_- \to m_+$, $z_+^* \to v$ from (5.4.102) with $\tilde{q}_+ = 1 + x$ and $\kappa = 1$ as discussed in appendix B. For $v = 1$, this is the expected result for the 1/4-BPS spherical bolt (see (B.0.3)). For the other extremum, with the same choice of signs, the action reads

$$I_-^*(m_\pm, v) \equiv I(m_\pm, v, z_-^*) = \frac{\pi}{8G_4 v}\left[2\chi_\Sigma - \kappa\frac{t/v}{m_- m_+} - \kappa\frac{v(m_- + m_+)^2}{t\, m_- m_+}\right]. \tag{5.4.133}$$

This equation does not reproduce any known result for $m_- = m_+ = 1$. Moreover, whilst the vector $\vec{\epsilon}_+ = -(1, z_+^*)$ always lies inside the polytope, that is $\vec{\epsilon}_+ \cdot \vec{\mu}_{1,2}^{(A)} > 0$, $\vec{\epsilon}_- = -(1, z_-^*)$

is always outside it. This resembles the behaviour of [35, 36]. On the other hand, the extremization of the $\sigma = -1$ off-shell action does not reproduce (5.4.120) and therefore we conclude that there is *no extremization* taking place in this case. At present we do not have an understanding of the reason underlying this difference, but we already observed that this markedly distinct behaviour is consistent with the observations put forward in [147]. Therein, it was showed that (anti-)self dual solutions can be regular only if $z = -1$ or $z > 0$. From this discussion it follows that there can not be any supersymmetric black hole in minimal $d = 4$ gauged supergravity. Indeed (5.4.63), with $(t = 0, \sigma = 1)$ is constant and can not be extremized.

It is interesting to note that the equivariant volume introduced in [83] reproduces the action (5.2.4) *only* in the twist case. The equivariant volume (here in the notation of chapter 2)

$$\mathbb{V}(\lambda_a, \epsilon_I) = (-1)^m \int_{\mathbb{M}_{2m}} \mathrm{e}^{-\vec{\epsilon}\cdot\vec{y}-\frac{\omega}{2\pi}}\,, \quad \mathcal{F}_a = \{\vec{y} \in \mathbb{R}^m : l_a(\vec{y}) = y_I v_a^I - \lambda_a = 0\}\,, \tag{5.4.134}$$

later generalized including higher-times [88], is the (regularized) volume of a symplectic manifold (or orbifold). It has been proved useful in studying the extremal problems linked to supersymmetric solutions, in that it contains (and generalizes) the Sasakian volume and the supersymmetric action (or master volume) of GK geometries [24, 25, 26]. To see the link with our actions, we write the fixed-point formula for the equivariant volume presented in [83] in our conventions (already specialized to the planar case, see (3.35) therein) as[17]

$$\mathbb{V}(\lambda_a, \vec{\epsilon}) = \frac{\pi}{4G_4} \sum_{A=1}^{2} \frac{1}{d_A} \prod_{I=1}^{2} \frac{\mathrm{e}^{\lambda^A_{a_I} b_I^{(A)}}}{b_I^{(A)}}\,, \tag{5.4.135}$$

where the $b_i^{(A)}$ are defined in (5.2.3), the Kähler parameters for each divisor are identified as $\lambda_1^1 \equiv \lambda_1$, $\lambda_2^1 = \lambda_1^2 \equiv \lambda_2$, $\lambda_2^2 \equiv \lambda_3$ and the coefficient $(\pi/4G_4)$ has here been tuned *a posteriori* to have agreement with the supergravity computations. We now Taylor expand this formula around $\lambda_a \sim 0$ and pick the second order term

$$\begin{aligned}\frac{4G_4}{\pi}\mathbb{V}(\lambda_A, \vec{\epsilon})\Big|_{\lambda_a^2} &= \sum_{A=1}^{2} \frac{1}{d_A}\left[\lambda_1^A\lambda_2^A + \frac{1}{2}\frac{b_1^A\lambda_1^A}{b_2^A} + \frac{1}{2}\frac{b_2^A\lambda_2^A}{b_1^A}\right] \\ &= \frac{1}{d_1}\left[\lambda_1\lambda_2 + \frac{1}{2}\frac{b_1^{(1)}\lambda_1}{b_1^{(1)}} + \frac{1}{2}\frac{b_2^{(1)}\lambda_2}{b_1^{(1)}}\right] + \frac{1}{d_2}\left[\lambda_2\lambda_3 + \frac{1}{2}\frac{b_1^{(2)}\lambda_2}{b_2^{(2)}} + \frac{1}{2}\frac{b_2^{(2)}\lambda_3}{b_1^{(2)}}\right]. \end{aligned} \tag{5.4.136}$$

To write each term in square brackets in the form of (5.2.4), we need $\lambda_1 = \lambda_2 = \pm 1$ but also $\lambda_2 = \lambda_3 = \pm 1$. But then $\lambda_1 = \lambda_2 = \lambda_3 \equiv -\kappa = \pm 1$ and the (second order of the) equivariant volume becomes

$$\mathbb{V}(\kappa, \vec{\epsilon})\Big|_{\lambda^2} = \frac{\pi}{2G_4} \sum_{A=2}^{3} \frac{1}{d_A}\left[-\kappa \frac{\big[b_1^{(A)} - \kappa\, b_2^{(A)}\big]^2}{4\, b_1^{(A)} b_2^{(A)}}\right]. \tag{5.4.137}$$

This is indeed (5.2.4) for $c_A = \kappa$, which is the twist case (5.4.123).

[17] No sum over A in the exponent is intended.

5.5 Summary of the chapter

In this chapter, we have examined a simpler theory compared to those discussed in chapters 3 and 4, while still encountering solutions with a rich structure. After introducing the theory under consideration—namely, (Euclidean) minimal gauged supergravity in four dimensions and its uplift to M-theory in section 5.1—we presented general results derived from equivariant localization in section 5.2.

The remaining content of this chapter is primarily divided into two parts. In the first, we analyzed the general Plebański-Demianski solutions, characterized by several parameters, including electric and magnetic charges (Q, P), NUT and mass parameters (N, M), as well as rotation and acceleration $(\omega, \mathbf{A})$. In the second part, we focused on the less general Carter-Plebański solution, obtained by taking the limit $\mathbf{A} \to 0$ in the PD family. The complexity of the PD system compelled us to focus on the simpler CP solution. Nonetheless, we extracted as many insights as possible from the PD family in section 5.3, before setting $\mathbf{A} = 0$ in section 5.4. After establishing that our solutions are complex, not Kähler but conformally ambiKähler, we calculated the main result of section 5.3: a local renormalized on-shell action obtained via holographic renormalization—see (5.3.37). This action is the most general form for the PD solution, with no parameters fixed and no supersymmetry imposed. In section 5.4, we began by determining the most general Killing spinor for the CP solution. We then conducted a thorough global analysis of the solution, focusing on the regularity of the metric, gauge field, and Killing spinor in both the bulk and the boundary. This analysis confirmed that the bulk and boundary topologies are $\mathcal{M}_4 = \mathbb{C}/\mathbb{Z}_v \hookrightarrow \mathcal{O}(-t) \to \Sigma_{[m_-,m_+]}$ and $\mathcal{M}_3 = L(t, 1)$, respectively, with the possibility of an orbifold branching in the underlying three-sphere. An important observation from section 5.4.4 reveals the importance of boundary regularity conditions for the gauge field $A_{(3)}$. From our perspective, where roughly speaking $\mathcal{M}_3 = \lim_{q\to+\infty} \mathcal{M}_4$, the same regularity conditions imposed on $A_{(4)}$ must also apply to $A_{(3)}$. This leads to different flat connections for $A_{(3)}$ compared to those typically chosen in rigid supersymmetry analyses. Another significant finding from the global analysis is that our spindle bolt solutions can exhibit both the twist ($\sigma = +1$) and the anti-twist ($\sigma = -1$), depending on parameter values. Correspondingly, the on-shell actions, reproduced in all cases using equivariant localization, fall into two distinct classes. For the twist case, the action is entirely determined by the topological data $(t, v, m_\pm)$. For the anti-twist case, instead, it depends on a free continuous parameter. The very same behaviour, already observed in earlier solutions involving (spherical) nuts and bolts, is summarized in appendix B.

Chapter 6

Conclusions

In this section we present our conclusions and possible future directions for research, whilst for a summary of each chapter we refer the reader to the specific "Summary of the chapter" sections 2.3, 3.7, 4.6 and 5.5.

In this thesis, we have analyzed various solutions associated with four-dimensional orbifolds $\mathbb{M}_4$, both compact (chapters 3 and 4) and non-compact (chapter 5). Typically, the metrics one finds in explicit constructions solving the supersymmetry equations are incompatible with a Kähler (or even symplectic) structure. Indeed, the solutions presented in [1, 2, 3] are only compatible with an integrable complex structure—meaning that the orbifolds are complex or Hermitian—with the exception of the (non-supersymmetric) $\mathbb{M}_4 = S^2 \ltimes \Sigma$, which admits a symplectic structure and can be thus understood in the context of standard symplectic toric geometry. Even in this more conventional situation, the presence of orbifolds singularities along the divisors implies that the orbifold is described by a labeled polytope [86], obtained as the image of the associated moment map. Moreover, since orbifold singularities can happen only in (complex) co-dimension two for an ordinary four-dimensional toric variety, the fan dual to a labeled polytope leads to a generalization, known as stack [162, 163, 115, 117, 116], which should be the appropriate framework for describing our $\mathbb{M}_4$. In fact, even where we were unable to construct a symplectic two-form, we managed to extract a labeled polytope, which correctly describe the properties of the orbifolds. Therefore, in any case, the $\mathbb{M}_4$ presented in this thesis are not toric in the standard sense, and the simplest description we can give them is in terms of geometries with $U(1)^2$ symmetries, describable by labeled polytopes.

From our perspective, the previous discussion does not pose an obstacle to using the standard symplectic description of toric orbifolds [122, 86, 121, 118]. More broadly, and based on experience, many physical quantities of interest—such as entropies, central charges, and on-shell actions—are topological in nature. Since our primary focus has been on topological quantities that do not rely on the existence of symplectic structures, we have been able to safely employ all the relevant tools of toric geometry introduced in chapter 2. In this sense, it is no coincidence that the *orbifold off-shell free energies* proposed in [1] and detailed in section 2.2, as well as the on-shell actions for the spindle-bolt, can be derived from an equivariant integration that selects contributions solely from the fixed loci of the torus action [144, 145, 146]. For instance, the gravitational block decomposition of the former has been reproduced in [83, 88, 89], using the equivariant integration of the anomaly polynomial, the equivariant (symplectic) volume, or the equivariant localization of the action. Although the derivation for D4-branes wrapped over $\mathbb{M}_4$ from first principles is not available, it is remarkable that (most of) the relevant problems related to extremal functions can be reproduced from (the integration of) simple objects. It would therefore be interesting to generalize the five-dimensional indices [164, 165, 166] to the case of a five-dimensional SCFT defined on $S^1 \times \mathbb{M}_4$ and to apply the technique outlined in [48, 49].

There are many open problems in this direction. First, it would be of utmost interest to reformulate the entire framework in terms of equivariant integration, as was started in [83, 88, 90, 83, 84, 89, 91, 88, 92, 93, 94, 95]. Furthermore, for the moment, it is coincidental that many BPS observables, such as the on-shell action, are equal to the integration of some (equivariantly) closed poly-forms constructed on supersymmetric bilinears[1]. It might be that is possible to reformulate the problem in terms of superfields and supercharges, defining an equivariant differential in the superspace formalism. It would also be interesting to consider the non-toric setting, where the number of expected global $U(1)$ symmetries is strictly less than half of the orbifold's (real) dimension[2]. In this case, the difficulty arises because the localization formulas now receive contributions from extended fixed loci, not just fixed points. Another interesting point concerns supersymmetry. Indeed, since the extremal function is topological in nature, one could wonder whether it is applicable also without supersymmetry. Although this problem seems generally complicated, as supersymmetry brings many simplifications, a simpler question might be to understand what types of twists are allowed in a theory. For example, the spindle in the four-dimensional accelerating black hole [46] realizes only the anti-twist, but on a fundamental basis, it is possible to write down an extremal problem also for the twist [52] (see also (5.4.64)). More generally, for four-dimensional toric orbifolds, one must choose a vector of twists σ^a, as in (2.2.9), but for the moment, only some configurations are found to be "dual" to existing solutions.

This question falls into the category of "extremization and existence" problems. In general, one hopes that the existence of an acceptable critical point with a positive extremal function is sufficient for the (supergravity) solution to exist. In practice, given a polytope with a set of toric data $(\vec{v}_a, m_a)$ in the language of chapter 2, one can run the extremization problem from section 2.2 and obtain a result, but this does not guarantee the existence of a black string solution with that entropy (or central charge). For the time being, statements regarding existence are only available for the Sasaki-Einstein case [35, 36, 119], where the extremization of the Sasakian volume with respect to the trial Reeb vector is sufficient for a Sasaki-Einstein metric to exist. The first step in this direction should be to better understand the existence of Gauntlett-Kim geometries [24, 25, 26], studying the conditions under which the fourth-order PDE characterizing the geometry has solutions, or at least obstructions. Regarding this problem, it should be relevant to acquire a clearer insight into the results of section 5.4.6, where it is shown that the on-shell action for the spindle bolt is reproduced by the extremization of an extremal function $S_{\text{off-shell}}$ only for the twist through the spindle, with a possible connection to the equivariant volume being proportional to $S_{\text{off-shell}}$. A final comment is that it should

[1]Currently, these bilinears are constructed explicitly by trial and error, theory by theory. It would be interesting to explore whether there is a systematic method to find them using a G-structures approach.

[2]One might also consider "non-abelian" localization, or explore why it is sufficient to localize only with respect to some $U(1)$.

also be possible to extend equivariant localization techniques to odd dimensions, with the first work in this direction being [167], where a five-dimensional analogue of nuts and bolts is considered, in the context of five-dimensional ungauged supergravity. For a later development in gauged supergravity see [168].

From the gravitational point of view, solutions of the form $\mathrm{AdS}_a \times \mathbb{M}_b$ naturally lend themselves to an interpretation in terms of branes wrapped (either completely or partially) on $\mathbb{M}_b$, and should be viewed as near-horizon geometries of certain black objects. For instance, our solution $\mathrm{AdS}_2 \times \Sigma_1 \ltimes \Sigma_2$ [1], as well as the more general $\mathrm{AdS}_{2,3} \times \mathbb{M}_4$ [2], should be regarded as simplified models (regardless of how sophisticated they may be!). The prototypical example of this is the $\mathrm{AdS}_2 \times \Sigma$ solution [46], which arises as the near-horizon geometry of the Plebański-Demianski black hole. Identifying the black objects whose near-horizon geometry matches these solutions would be a significant advancement, as it would expand the landscape of black holes to include those with *exotic horizons*. Moreover, the way in which we have written the off-shell free energy in section 2.2 might suggest the existence of orbifold solutions with an arbitrary number of fixed points. Currently, only (holographically relevant) geometries with two or four fixed points have been studied. It is therefore crucial to test our conjecture on different types of orbifolds. For example, finding a solution with three fixed points, such as $\mathrm{AdS}_{2,3} \times \mathbb{WCP}^2_{[n_1,n_2,n_3]}$, within a suitable (super)gravitational theory in $D = 6, 7$ would be an interesting progress. Such a result would, moreover, hold significant value in its own right. In the same spirit, one could try to find spindly bolts also in five-dimensional gauged supergravity, generalizing the results of [167], or search for a multi-bolt solution which generalize the one presented in chapter 5. For example, it is perhaps possible to extend the (anti) self-dual m-pole metrics on the ball [169, 147] and the scalar-flat bolts [170] to more general non self-dual m-bolt solutions, possibly with also orbifold singularities.

Appendix A

Conventions and equations of motion

In this appendix we collect some conventions used throughout the thesis and present the equation of motions for the $D = 6, 7$ gauged supergravities considered in section 3.1.

In our notation the Hodge star action on a p-form $\omega_{(p)}$ in a d-dimensional space(time) with signature s is defined as

$$\begin{aligned} \star_{(d)}\omega_{(p)} &= \frac{\sqrt{|g|}}{p!(d-p)!}\mathrm{d}x^{\nu_1}\wedge\ldots\wedge\mathrm{d}x^{\nu_{d-p}}\tilde{\varepsilon}_{\nu_1\ldots\nu_{d-p}\mu'_1\ldots\mu'_p}g^{\mu'_1\mu_1}\ldots g^{\mu'_p\mu_p}\omega_{\mu_1\ldots\mu_p}\,, \\ \star^2_{(d)}\omega_{(p)} &= (-1)^{s+p(d-p)}\omega_{(p)}\,, \quad \star_{(d)}v_{(p)}\wedge\omega_{(p)} = \frac{1}{p!}v_{\mu_1\ldots\mu_p}\omega^{\mu_1\ldots\mu_p}\sqrt{|g|}\mathrm{d}^dx\,, \end{aligned} \tag{A.0.1}$$

where $\tilde{\varepsilon}$ is the Levi-Civita symbol, with $\tilde{\varepsilon}_{0\ldots(d-1)} = +1$. The Levi-Civita tensor is obtained as $\varepsilon_{\mu_1\ldots\mu_d} = \sqrt{|g|}\tilde{\varepsilon}_{\mu_1\ldots\mu_d}$. Moreover, it holds

$$(-1)^s\tilde{\varepsilon}^{\mu_1\ldots\mu_d}\,\mathrm{d}^dx = \mathrm{d}x^{\mu_1}\wedge\ldots\wedge\mathrm{d}x^{\mu_d}\,. \tag{A.0.2}$$

$D = 6$ gauged supergravity

From the six-dimensional Lagrangian

$$\begin{aligned} \mathcal{L}_{6\mathrm{D}} =& \sqrt{-g}\Big[R - V_6 - \frac{1}{2}\partial_\mu\varphi^i\partial^\mu\varphi_i - \frac{X_i^{-2}}{4}F^i_{\mu\nu}F_i^{\mu\nu} - \frac{(X_1X_2)^2}{48}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{m^2}{8(X_1X_2)}B_{\mu\nu}B^{\mu\nu}\Big] \\ &- \frac{1}{16}\tilde{\varepsilon}^{\mu\nu\rho\sigma\tau\lambda}B_{\mu\nu}\Big(F_{1\,\rho\sigma}F_{2\,\tau\lambda} + \frac{m^2}{12}B_{\rho\sigma}B_{\tau\lambda}\Big) \\ =& (R - V_6)\star 1 - \frac{1}{2}\star\mathrm{d}\varphi_i\wedge\mathrm{d}\varphi^i - \frac{X_i^{-2}}{2}\star F_i\wedge F^i - \frac{(X_1X_2)^2}{8}\star H\wedge H - \frac{m^2}{4(X_1X_2)}\star B\wedge B \\ &+ \frac{1}{2}B\wedge\left(F_1\wedge F_2 + \frac{m^2}{12}B\wedge B\right)\,, \quad X_0 = (X_1X_2)^{-3/2} \end{aligned} \tag{A.0.3}$$

with scalar fields $\vec{\varphi}$ and potential V_6 given by

$$\begin{aligned} X_i = \mathrm{e}^{-\vec{a}_i\cdot\vec{\varphi}}\,, \quad \vec{a}_{1,2} = \left(\pm 2^{-1/2}, 2^{-3/2}\right)\,, \quad \varphi_1 = \frac{-1}{\sqrt{2}}\log\frac{X_1}{X_2}\,, \quad \varphi_2 = -\sqrt{2}\log(X_1X_2)\,, \\ V_6 = m^2X_0^2 - 4g_c^2X_1X_2 - 4mg_c\,, \quad X_0(X_1+X_2)\,, \end{aligned} \tag{A.0.4}$$

we get the following equations of motion

Einstein's equations

$$R_{\mu\nu} - \frac{V_6}{4} g_{\mu\nu} = \frac{1}{2}\partial_\mu\varphi_i\partial_\nu\varphi^i + \frac{X_i^{-2}}{2} T_{\mu\nu}(F^i_{(2)}) + \frac{(X_1X_2)^2}{8} T_{\mu\nu}(H_3) + \frac{m^2}{4(X_1X_2)} T_{\mu\nu}(B_{(2)})\,, \tag{A.0.5}$$

where, for any p-form we have defined the energy-momentum tensor

$$T_{\mu\nu}(\omega_{(p)} \equiv \left(\frac{1}{(p-1)!}\omega_{\mu\rho_2\dots\rho_p}\omega_\nu{}^{\rho_2\dots\rho_p} - \frac{\omega_{\sigma_1\dots\sigma_p}\omega^{\sigma_1\dots\sigma_p}}{(p-2)!p(d-2)} g_{\mu\nu}\right) = |\omega|^2_{\mu\nu} - \frac{p-1}{d-2} g_{\mu\nu}|\omega|^2\,, \tag{A.0.6}$$

Equations for the scalar fields

$$\begin{aligned}
\Box\varphi_1 &= \frac{\delta V}{\delta\varphi_1} + \frac{1}{\sqrt{2}} X_1^{-2}|F_1|^2 - \frac{1}{\sqrt{2}} X_2^{-2}|F_2|^2\,,\\
\Box\varphi_2 &= \frac{\delta V}{\delta\varphi_2} + \frac{1}{2\sqrt{2}} X_1^{-2}|F_1|^2 + \frac{1}{2\sqrt{2}} X_2^{-2}|F_2|^2 - \frac{1}{4\sqrt{2}}(X_1X_2)^2|H|^2 + \frac{m^2}{4\sqrt{2}}(X_1X_2)^{-1}|B|^2\,,
\end{aligned} \tag{A.0.7}$$

Maxwell's equations

$$\begin{aligned}
&\mathrm{d}\big(X_1^{-2}\star F_1\big) - \frac{1}{2} H\wedge F_2 = 0\,, \quad \mathrm{d}\big(X_2^{-2}\star F_2\big) - \frac{1}{2} H\wedge F_1 = 0\,,\\
&\mathrm{d}\big[(X_1X_2)^2\star H\big] + 2m^2(X_1X_2)^{-1}\star B - 2\Big(F_1\wedge F_2 + \frac{m^2}{4} B\wedge B\Big) = 0\,,
\end{aligned} \tag{A.0.8}$$

$D = 7$ gauged supergravity

From the seven-dimensional Lagrangian

$$\begin{aligned}
\mathcal{L}_{7\mathrm{D}} =& \sqrt{-g}\Big[R - V_7 - \frac{1}{2}\partial_\mu\varphi_i\partial_\nu\varphi^i - \frac{X_i^{-2}}{4} F^i_{\mu\nu}F_i^{\mu\nu} - \frac{(X_1X_2)^2}{48} H_{\mu\nu\rho\sigma}H^{\mu\nu\rho\sigma}\Big]\\
&- \frac{1}{24}\varepsilon^{\mu\nu\rho\sigma\tau\lambda\eta} B_{\mu\nu\rho}\Big(F_{1\,\sigma\tau}F_{2\,\lambda\eta} - \frac{g_c}{12} H_{\sigma\tau\lambda\eta}\Big)\,,
\end{aligned} \tag{A.0.9}$$

with scalars and potential given by

$$\begin{gathered}
X_i = \mathrm{e}^{-\vec{a}_i\cdot\vec{\varphi}}\,, \quad \vec{a}_{1,2} = \big(\pm 2^{-1/2}, 10^{-1/2}\big)\,, \quad \varphi_1 = \frac{-\sqrt{2}}{2}\log\frac{X_1}{X_2}\,, \quad \varphi_2 = \frac{-\sqrt{10}}{2}\log(X_1X_2)\,,\\
V_7 = \frac{g_c^2}{2}\big[X_0^2 - 8X_1X_2 - 4X_0(X_1+X_2)\big]\,, \quad X_0 = (X_1X_2)^{-2}\,,
\end{gathered} \tag{A.0.10}$$

we obtain the following equations of motion

Einstein's equations

$$R_{\mu\nu} - \frac{V_7}{5} g_{\mu\nu} = \frac{1}{2}\partial_\mu\varphi_i\,\partial_\nu\varphi^i + \frac{(X_1X_2)^2}{2} T_{\mu\nu}(H_{(4)}) + \frac{X_i^{-2}}{2} T_{\mu\nu}(F^i_{(2)})\,, \tag{A.0.11}$$

Equations for the scalar fields

$$
\begin{aligned}
\Box\varphi_1 &= \frac{\delta V}{\delta\varphi_1} + \frac{1}{\sqrt{2}} X_1^{-2}|F_1|^2 - \frac{1}{\sqrt{2}} X_2^{-2}|F_2|^2\,, \\
\Box\varphi_2 &= \frac{\delta V}{\delta\varphi_2} + \frac{1}{\sqrt{10}} X_1^{-2}|F_1|^2 + \frac{1}{\sqrt{10}} X_2^{-2}|F_2|^2 - \frac{2}{\sqrt{10}}(X_1X_2)^2|H|^2\,,
\end{aligned} \tag{A.0.12}
$$

Maxwell's equations

$$
\begin{aligned}
&\mathrm{d}\big(X_1^{-2}\star F_1\big) - H\wedge F_2 = 0\,, \quad \mathrm{d}\big(X_2^{-2}\star F_2\big) - H\wedge F_1 = 0\,, \\
&\mathrm{d}\big[(X_1X_2)^2\star H\big] - F_1\wedge F_2 - 2gH = 0\,.
\end{aligned} \tag{A.0.13}
$$

Appendix B

Old notable NUTs and Bolts solutions

The aim of this section is to explain the table below in figure B.1, namely how our CP spindle-bolt solution generalizes all the previous known NUTs and Bolts solutions available in literature [109, 110, 111, 96]. Moreover, the PD spindle bolts contain also the accelerating black hole [46, 47]. To this end, we will collect firstly some relevant fact about the old solutions, constructing then a limit which will allow us to recover them both locally and globally.

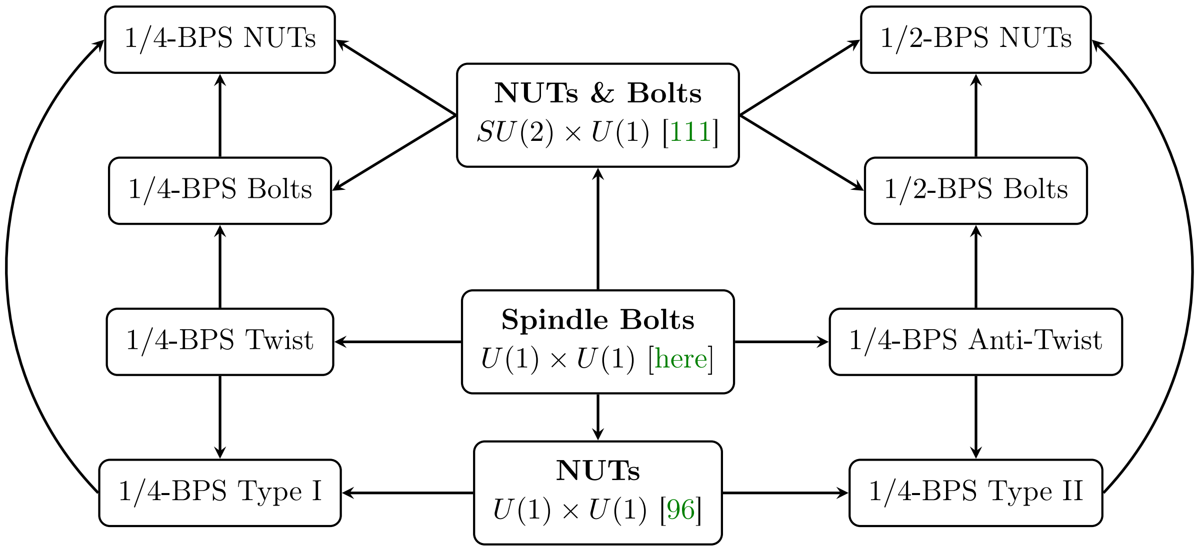


Figure B.1: Summary of supersymmetric euclidean AdS$_4$ solutions to $d = 4$, $\mathcal{N} = 2$ minimal gauged supergravity containing nuts and bolts. In the central column the symmetry and the bulk content are sketched, while the lateral columns show when a certain supersymmetric solution is contained in another one by some limiting procedure which will be explained in detail below.

[96] The first solution we consider is the $U(1)\times U(1)$-invariant two-parameter Euclidean solutions of [96]. These are 1/4-BPS solutions in the CP family ($\mathbf{A} = 0$) where the conformal boundary is a squashed S^3 and have the topology of the ball, namely they contain a single nut in the bulk. They belong to the class of self-dual solutions [147]. According to the values of the parameters, there is a splitting into two distinct families, referred to as Type I and Type II, which are distinguished by the form of their on-shell action and supersymmetric Killing vectors[1]

$$\text{Type I}: \quad \begin{array}{c} S = \dfrac{\pi}{2G_4}\,, \\ \vec{\epsilon} \propto (1,-1)\,, \end{array} \qquad \text{Type II}: \quad \begin{array}{c} S = \dfrac{\pi}{8G_4}\left[\beta + \dfrac{1}{\beta}\right]^2, \\ \vec{\epsilon} \propto (\beta^2, 1)\,. \end{array} \tag{B.0.1}$$

These are consistent with the general results of [147, 171], and also with the results of chapter 5, with fixed action and Killing vector for a family, and with the dependence on a single (out of two) parameter for the other.

[111] A similar behaviour is present in the one-parameter $SU(2)\times U(1)$-invariant solutions of [111], where the boundary, this time, is a squashed lens space $\mathcal{M}_3 = L(\hat{t},1) = S^3/\mathbb{Z}_{\hat{t}}$, while the bulk may be either a nut $\mathcal{M}_4 = \mathbb{R}^4$ or a bolt $\mathcal{M}_4 = \mathbb{C} \hookrightarrow \mathcal{O}(-t) \to S^2$, which is always a round S^2. This has been generalized in [137] including a bolt with topology of $\Sigma_{\rm g}$. For each type of bulk topology, there are again two distinct families, referred to as 1/4-BPS and 1/2-BPS. The Bolt solutions are characterised by their on-shell actions as well as the flux of the graviphoton field through the S^2 bolt, that read

$$\begin{aligned} &\frac{1}{4}\text{-BPS}: \quad \frac{1}{2\pi}\int_{S^2} F = -1 + \kappa\frac{\hat{t}}{2}\,, \qquad S_{\rm bolt} = \frac{\pi}{2G_4}\left[1 - \kappa\frac{\hat{t}}{4}\right], \quad \hat{t} \geq 2\,, \\ &\frac{1}{2}\text{-BPS}: \quad \frac{1}{2\pi}\int_{S^2} F = -\kappa\frac{\hat{t}}{2}\,, \qquad S_{\rm bolt} = \frac{\pi}{2G_4}\left[1 - \kappa\frac{2\sqrt{4s^2-1}}{s\hat{t}}\left[s^2 - \frac{\hat{t}^2}{16}\right]\right], \end{aligned} \tag{B.0.2}$$

where $\hat{t} \in \mathbb{N}$, $\kappa = \pm 1$ is a sign further distinguishing two branches of the solutions (as in chapter 5) and s is the squashing of the lens space boundary. Here $\hat{t} \geq 2$ for $\kappa = -1$ and $\hat{t} \geq 3$ for $\kappa = +1$ [111]. We record here also the supersymmetric Killing vector $\vec{\epsilon}$ for these solutions, which is given (up to an irrelevant normalization constant) by [90]

$$\frac{1}{4}\text{-BPS}: \quad \vec{\epsilon} \propto (1,0)\,, \quad \frac{1}{2}\text{-BPS} \quad : \quad \vec{\epsilon} \propto \left(\frac{\hat{t}}{4s} - 2s - \sqrt{4s^2-1}, 2s + \sqrt{4s^2-1}\right). \tag{B.0.3}$$

For the NUT solutions the on-shell actions and Killing vectors read [111, 147]

$$\frac{1}{4}\text{-BPS}: \quad \begin{array}{c} S_{\rm nut} = \dfrac{\pi}{2G_4}\,, \\ \vec{\epsilon} \propto (1,-1)\,, \end{array} \quad \frac{1}{2}\text{-BPS} \quad : \quad \begin{array}{c} S_{\rm nut} = \dfrac{\pi}{2G_4}4s^2\,, \\ \vec{\epsilon} \propto \left(\dfrac{1}{4s}, -\dfrac{1}{4s} + 2s + \sqrt{4s^2-1}\right). \end{array} \tag{B.0.4}$$

[46, 47] Finally we recall the accelerating black hole solution, which has boundary topology $\mathbb{\Sigma}\times S^1$. In this case the underlying metric is PD ($\mathbf{A} \neq 0$) and it is the acceleration itself responsible for the conical singularities of the spindle. Allowing the parameters of the solution to take complex values one obtains a "non-extremal" supersymmetric solution [47] that in the bulk has a bolt with spindle topology[2]. We introduce some standard notation

$$\mu = \frac{m_- + m_+}{m_- - m_+}\,, \quad G_4 Q_m = \frac{m_- - m_+}{4m_- m_+}\,, \quad \chi_{\mathbb{\Sigma}} = \frac{m_- + m_+}{m_- m_+}\,, \tag{B.0.5}$$

[1] The parameter β is related to the parameter Q that appears in (5.3.4) as $2Q = (\beta^2-1)/(\beta^2+1)$.
[2] In the extremal limit the solution in the near horizon becomes $\mathrm{AdS}_2 \times \mathbb{\Sigma}$.

in terms of which

$$\frac{1}{2\pi}\int_{\Sigma} F = 2G_4 Q_m\,,\quad S_{\pm} = \pm\frac{1}{2\mathrm{i}G_4}\left[\frac{\varphi^2}{z} + (G_4 Q_m)^2\, z\right],\quad \varphi - \frac{\chi_{\Sigma}}{4}z = \pm\mathrm{i}\pi\,,\quad \vec{\epsilon} \propto (2\pi, \mathrm{i}z)\,. \tag{B.0.6}$$

where z is a complicated function of the two parameters of the solution. Notice that the supersymmetric accelerating black hole reduces to the Kerr-Newman-AdS one [149] when the acceleration is turned off. This results in a spherical horizon with $m_- = m_+ = 1$, for which the on-shell action has been computed in [47] and coincides with (B.0.6) for $Q_m = 0$ and $\chi_{S^2} = 2$.

Bolt solutions of [111]

After this preview, we now review briefly in more detail some aspects of the spherical Bolt solutions of [111]. Locally, the solution is given by

$$\mathrm{d}s_4^2 = \frac{r^2 - s^2}{\Omega(r)}\mathrm{d}r^2 + (r^2 - s^2)(\sigma_1^2 + \sigma_2^2) + \frac{4s^2\Omega(r)}{r^2 - s^2}\sigma_3\,,\quad A = \left(\hat{P}\frac{r^2 + s^2}{r^2 - s^2} - \hat{Q}\frac{2rs}{r^2 - s^2}\right)\sigma_3\,. \tag{B.0.7}$$

with metric function

$$\Omega(r) = (r^2 - s^2)^2 + (1 - 4s^2)(r^2 + s^2) - 2\hat{M}r + \hat{P}^2 - \hat{Q}^2\,, \tag{B.0.8}$$

and σ_i given by

$$\sigma_1 + \mathrm{i}\sigma_2 = e^{-\mathrm{i}\hat{\psi}}(\mathrm{d}\hat{\theta} + \mathrm{i}\sin\hat{\theta}\mathrm{d}\hat{\varphi})\,,\quad \sigma_3 = \mathrm{d}\hat{\psi} + \cos\hat{\theta}\mathrm{d}\hat{\varphi}\,. \tag{B.0.9}$$

As demonstrated in [111], this is the most general local form of the supersymmetric solutions with $SU(2)\times U(1)$ symmetry, which is enhanced with respect to the $U(1)\times U(1)$ symmetry of the spindle bolt. The BPS conditions then split this solution into two classes[3]

$$\begin{aligned} &\frac{1}{2}\text{-BPS}: \quad \hat{M} = \eta\hat{Q}\sqrt{4s^2 - 1}\,, && \hat{P} = -\eta s\sqrt{4s^2 - 1}\,, \\ &\frac{1}{4}\text{-BPS}: \quad \hat{M} = 2\eta s\hat{Q}\,, && \hat{P} = -\eta\frac{4s^2 - 1}{2}\,. \end{aligned} \tag{B.0.10}$$

The range of the radial coordinate is taken to be $s \le r_0 \le r < +\infty$, where r_0 is the largest root of $\Omega(r)$, with $r_0 > s$ for the bolt case. Then, the regularity analysis of the metric in the two classes leads to the following conditions

$$\frac{1}{2}\text{-BPS}: \quad \hat{Q} = \eta\frac{(128s^4 - 16s^2 - \hat{t}^2)\kappa}{64s^2}\,,\quad r_0 = \frac{1}{8}\left(\frac{\hat{t}}{s} - 4\kappa\sqrt{4s^2 - 1}\right), \tag{B.0.11}$$

and

$$\frac{1}{4}\text{-BPS}: \quad \begin{aligned} &\hat{Q} = \eta\frac{\hat{t}^2 \pm (\hat{t} - 16s^2)\sqrt{\mathcal{G}_+(\hat{t}, s)}}{128s^2}\,, && r_0 = \frac{\hat{t} \pm \sqrt{\mathcal{G}_+(\hat{t}, s)}}{16s}\,, && \kappa = 1\,, \\ &\hat{Q} = -\eta\frac{\hat{t}^2 \mp (\hat{t} + 16s^2)\sqrt{\mathcal{G}_-(\hat{t}, s)}}{128s^2}\,, && r_0 = \frac{\hat{t} \mp \sqrt{\mathcal{G}_-(\hat{t}, s)}}{16s}\,, && \kappa = -1\,, \end{aligned} \tag{B.0.12}$$

[3]Notice the insertion of the sign $\eta = \pm 1$ here, which accounts in the two possible choices explained in [111].

where

$$\mathcal{G}_\pm(\hat{t}, s) = (\hat{t} \pm 16s^2)^2 - 128s^2 \,. \tag{B.0.13}$$

Notice that here, contrarily to [111], we considered two branches for both the 1/4-BPS conditions, as pointed out in [137].

Limits to the old solutions

As anticipated, our solution (5.4.1)-(5.4.3) generalizes in various ways the ones presented in [111, 96]. In this section we study the limits which reproduce those results and show how to recover their on-shell actions.

$U(1) \times U(1)$-invariant solution of [96]

Let us start from the case [96], which is simpler to be recovered since the local form of the solution is exactly the same. The main difference from this reference is that therein the regularity led to $P = Q$, $N = M$ and $q_+ = p_+$. The conditions to come back to [96] are

$$\tilde{q}_+ \to 1 + x \,, \quad \eta = \lambda \implies \kappa = 1 \,, \quad m_- - m_+ \to 0 \,, \quad \begin{array}{ll} \sigma = 1 : & t \to 0 \,, \\ \sigma = -1 : & t \to \dfrac{-2}{x} v \,. \end{array} \tag{B.0.14}$$

The first comes from the condition $q_+ = p_+$ in terms of the parametrization (5.4.99); as a consequence, q_+ and p_+ must be zeros of the same type of function (5.4.6), hence the second condition. Since the spindle should disappear, we require the third condition which together with previous two implies the last condition. Clearly in this limit the interpretation of t breaks down (and the negative sign is an artefact due to $\mathrm{Re}(\eta P)$ not being positive anymore). In particular, the Type I and Type II solutions of [96] have boundary topology of S^3, for which more properly $t = 1$ in both cases.

We can now recover the actions for Type I and Type II given in (B.0.1). For the Type I case, $m_\pm \to 1$ and $t \to 0$ from (B.0.14), for which only the first contribution in (5.4.117) survives. With $t = -2v/x$ from (B.0.14), instead, and $m_\pm \to 1$, we get

$$\text{Type II} : \quad S_{\text{ren}} = \frac{\pi}{2G_4} \frac{1}{1 - 4[Q/(p_+ + p_-)^2]^2} \,. \tag{B.0.15}$$

This is equivalent to the one presented in (B.0.1) when $p_+ + p_- = 1$, a condition fixed by the authors of [96] taking advantage of the scaling symmetry (5.3.6). Here we have restored that factor, since $p_+ + p_- \neq 1$ for us.

$SU(2) \times U(1)$-invariant solution of [111]

Now we move to the $SU(2) \times U(1)$ invariant solution presented in [111] and summarized in our conventions in Appendix B. In the spirit of appendix C of [147], we firstly construct a scaling limit under which (5.4.1) and (5.4.2) become (B.0.7) and, at least locally. Notice that in our case the solution is not (anti) self-dual, so that [147] must be extended. The limit is obtained for $\epsilon \to 0^+$ on the coordinates

$$p = s(1 - \epsilon x \cos\hat{\theta}) \,, \quad \tau = s\left(2\hat{\psi} + \frac{\hat{\varphi}}{\epsilon x}\right) \,, \qquad \sigma = \frac{\hat{\varphi}}{s} \frac{1}{\epsilon x} \,, \tag{B.0.16}$$

as well as on the parameters

$$N = -s(4s^2-1)\,, \quad E = 1-6s^2\,, \qquad \alpha = \hat{P}^2 - s^2\left(3s^2-1+\epsilon^2 x^2\right)\,, \tag{B.0.17}$$

where we employed the parametrization (5.4.99) for the roots and $\hat{\theta} \in [0,\pi]$. Moreover, we identify

$$q = r\,, \quad (M,P,Q) = (\hat{M},\hat{P},\hat{Q})\,, \quad w = s\,, \tag{B.0.18}$$

where s is the squashing parameter as before. In the limit, the BPS conditions in (5.4.5) boil down to (B.0.10), as expected. Notice that under this scaling limit we get $p_+ \to p_-$, which is in a certain sense the complementary situation to $q_+ \to p_+$ of the previous subsection. As $p_+ \to p_-$, the spindle degenerates and the parametrization variable w is unconstrained, since $m_+/m_- = 1 + O(\epsilon)$. However, the limit procedure instructs us that $w = s + O(\epsilon)$, and applying this on (5.4.100) one obtains

$$\sigma = 1: \quad w = s + O(\epsilon) \iff m_\pm = \frac{1}{2(2s^2+\eta\tilde{P})} \mp \frac{s^2(r_0^2+s^2+\eta\tilde{P})}{(r_0^2-s^2)(2s^2+\eta\tilde{P})^2}\,\epsilon x + O(\epsilon^2)\,. \tag{B.0.19}$$

Even if it looks odd, using the 1/4-BPS conditions (B.0.10) the first term is exactly 1, as expected for a spindle which becomes a sphere. This expression makes (5.4.101) infinite

$$\Delta = \frac{2\pi(r_0^2-s^2)}{s}\frac{1}{\epsilon x} + O(\epsilon)\,, \tag{B.0.20}$$

in the precise way such that $\mathrm{vol}(\Sigma_{q+}) < \infty$ and (5.4.104) coincides with (4.7) in [111]. Plugging the values (B.0.19) in (5.4.102) and inverting the parametrization (5.4.99) to

$$\sigma = 1: \quad \tilde{P} = \frac{P}{2s}\,, \quad \tilde{q}_+ = \frac{r_0}{s}\,, \tag{B.0.21}$$

we obtain correctly that (5.4.102) boils down to $t = v\hat{t} + O(\epsilon)$, respectively, for the appropriate class of regularity conditions (B.0.12) with $\kappa = \pm 1$. The procedure can be now repeated also for the anti-twist case, for which

$$\sigma = -1: \quad m_\pm = 1 \mp \frac{2s\left[s + (r_0^2-s^2)(2s \mp \sqrt{4s^2-1})\right]}{r_0^2-s^2}\,\epsilon x + O(\epsilon^2)\,, \tag{B.0.22}$$

where we already plugged (B.0.10), and with

$$\sigma = -1: \quad \overline{P} = -\frac{P(-2s^2\epsilon x + \eta P)}{4s^3}\,, \quad \tilde{q}_+ = \frac{r_0}{s}\,, \tag{B.0.23}$$

one gets again $w = s + O(\epsilon)$ and $t = v\hat{t} + O(\epsilon)$.

After this local scaling limit, we can recover the action (B.0.2). For the 1/4-BPS case, $m_\pm \to 1 + O(\epsilon)$ from (B.0.19) but $t = v\hat{t} + O(\epsilon)$ remains non-zero. Thus only the last term in (5.4.117) vanishes, giving the correct result. The limit to the 1/2-BPS action in more complicated, since in this limit $\mathfrak{q}$ is infinite. However, $\mathfrak{q}/m_\pm$ is finite and using (B.0.22) and $t = v\hat{t} + O(\epsilon)$ we obtain indeed the second line of (B.0.2).

Finally, let us comment on the relation with the supersymmetric nuts of [111]. These must correspond to the even more degenerate case in which p_-, p_+ and q_+ coincide, thus finishing all the possibilities. We shall refrain to construct explicitly this limit, since it was already observed in [96] that the supersymmetric 1/2 and 1/4-BPS nuts of [111] are subcases of the nuts in [96]. However, the action of the 1/4-BPS nut in (B.0.4) is already equal to the one of Type I in (B.0.1). For 1/2-BPS, we can rewrite the Type II action (B.0.15) using the parametrization (5.4.99) as

$$\text{Type II}: \quad S_{\text{ren}} = \frac{2\pi}{G_4} \frac{w^4}{4w^4 - P^2}\,, \tag{B.0.24}$$

where recall that $Q = P$ for Type I and Type II. Then, since $w = s$ and $P = P(s)$ from (B.0.18) and (B.0.10), we reproduce exactly the second line of (B.0.4).

Bibliography

[1] F. Faedo, A. Fontanarossa and D. Martelli, *Branes wrapped on orbifolds and their gravitational blocks*, Lett. Math. Phys. **113** (2023) [2210.16128].

[2] F. Faedo, A. Fontanarossa and D. Martelli, *Branes wrapped on quadrilaterals*, Lett. Math. Phys. **115** (2025) 26 [2402.08724].

[3] M.K. Crisafio, A. Fontanarossa and D. Martelli, *Nuts, bolts and spindles*, Lett. Math. Phys. **115** (2025) 27 [2412.00428].

[4] J.M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2** (1998) 231 [hep-th/9711200].

[5] I.R. Klebanov and E. Witten, *Superconformal field theory on three-branes at a Calabi-Yau singularity*, Nucl. Phys. B **536** (1998) 199 [hep-th/9807080].

[6] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, Commun. Math. Phys. **313** (2012) 71 [0712.2824].

[7] G. Festuccia and N. Seiberg, *Rigid Supersymmetric Theories in Curved Superspace*, JHEP **06** (2011) 114 [1105.0689].

[8] E. Witten, *Topological Quantum Field Theory*, Commun. Math. Phys. **117** (1988) 353.

[9] F. Benini and A. Zaffaroni, *A topologically twisted index for three-dimensional supersymmetric theories*, JHEP **07** (2015) 127 [1504.03698].

[10] Y. Imamura and S. Yokoyama, *Index for three dimensional superconformal field theories with general R-charge assignments*, JHEP **04** (2011) 007 [1101.0557].

[11] J.M. Maldacena and C. Nunez, *Supergravity description of field theories on curved manifolds and a no go theorem*, Int. J. Mod. Phys. A **16** (2001) 822 [hep-th/0007018].

[12] J.P. Gauntlett, N. Kim and D. Waldram, *M Five-branes wrapped on supersymmetric cycles*, Phys. Rev. D **63** (2001) 126001 [hep-th/0012195].

[13] I. Bah, C. Beem, N. Bobev and B. Wecht, *AdS/CFT Dual Pairs from M5-Branes on Riemann Surfaces*, Phys. Rev. D **85** (2012) 121901 [1112.5487].

[14] I. Bah, C. Beem, N. Bobev and B. Wecht, *Four-Dimensional SCFTs from M5-Branes*, JHEP **06** (2012) 005 [1203.0303].

[15] F. Benini and N. Bobev, *Two-dimensional SCFTs from wrapped branes and c-extremization*, *JHEP* **06** (2013) 005 [1302.4451].

[16] F. Benini, N. Bobev and P.M. Crichigno, *Two-dimensional SCFTs from D3-branes*, *JHEP* **07** (2016) 020 [1511.09462].

[17] S.L. Cacciatori and D. Klemm, *Supersymmetric AdS(4) black holes and attractors*, *JHEP* **01** (2010) 085 [0911.4926].

[18] F. Benini, K. Hristov and A. Zaffaroni, *Black hole microstates in* AdS_4 *from supersymmetric localization*, *JHEP* **05** (2016) 054 [1511.04085].

[19] F. Benini, K. Hristov and A. Zaffaroni, *Exact microstate counting for dyonic black holes in AdS4*, *Phys. Lett. B* **771** (2017) 462 [1608.07294].

[20] F. Benini and N. Bobev, *Exact two-dimensional superconformal R-symmetry and c-extremization*, *Phys. Rev. Lett.* **110** (2013) 061601 [1211.4030].

[21] D.L. Jafferis, *The Exact Superconformal R-Symmetry Extremizes Z*, *JHEP* **05** (2012) 159 [1012.3210].

[22] D.L. Jafferis, I.R. Klebanov, S.S. Pufu and B.R. Safdi, *Towards the F-Theorem: N=2 Field Theories on the Three-Sphere*, *JHEP* **06** (2011) 102 [1103.1181].

[23] K.A. Intriligator and B. Wecht, *The Exact superconformal R symmetry maximizes a*, *Nucl. Phys. B* **667** (2003) 183 [hep-th/0304128].

[24] N. Kim, *AdS(3) solutions of IIB supergravity from D3-branes*, *JHEP* **01** (2006) 094 [hep-th/0511029].

[25] N. Kim and J.-D. Park, *Comments on AdS(2) solutions of D=11 supergravity*, *JHEP* **09** (2006) 041 [hep-th/0607093].

[26] J.P. Gauntlett and N. Kim, *Geometries with Killing Spinors and Supersymmetric AdS Solutions*, *Commun. Math. Phys.* **284** (2008) 897 [0710.2590].

[27] J.P. Gauntlett, D. Martelli and J. Sparks, *Toric geometry and the dual of* $\mathcal{I}$*-extremization*, *JHEP* **06** (2019) 140 [1904.04282].

[28] S.M. Hosseini and A. Zaffaroni, *Geometry of* $\mathcal{I}$*-extremization and black holes microstates*, *JHEP* **07** (2019) 174 [1904.04269].

[29] C. Couzens, J.P. Gauntlett, D. Martelli and J. Sparks, *A geometric dual of c-extremization*, *JHEP* **01** (2019) 212 [1810.11026].

[30] J.P. Gauntlett, D. Martelli and J. Sparks, *Toric geometry and the dual of c-extremization*, *JHEP* **01** (2019) 204 [1812.05597].

[31] S.M. Hosseini and A. Zaffaroni, *Proving the equivalence of c-extremization and its gravitational dual for all toric quivers*, *JHEP* **03** (2019) 108 [1901.05977].

[32] D.Z. Freedman and S.S. Pufu, *The holography of F-maximization*, *JHEP* **03** (2014) 135 [1302.7310].

[33] M. Fluder and J. Sparks, *D2-brane Chern-Simons theories: F-maximization = a-maximization*, *JHEP* **01** (2016) 048 [1507.05817].

[34] C. Couzens and A. Lüscher, *A geometric dual of F-maximization in massive type IIA*, *JHEP* **08** (2024) 218 [2406.15547].

[35] D. Martelli, J. Sparks and S.-T. Yau, *The Geometric dual of a-maximisation for Toric Sasaki-Einstein manifolds*, *Commun. Math. Phys.* **268** (2006) 39 [hep-th/0503183].

[36] D. Martelli, J. Sparks and S.-T. Yau, *Sasaki-Einstein manifolds and volume minimisation*, *Commun. Math. Phys.* **280** (2008) 611 [hep-th/0603021].

[37] C. Beem, T. Dimofte and S. Pasquetti, *Holomorphic Blocks in Three Dimensions*, *JHEP* **12** (2014) 177 [1211.1986].

[38] S.M. Hosseini, K. Hristov and A. Zaffaroni, *Gluing gravitational blocks for AdS black holes*, *JHEP* **12** (2019) 168 [1909.10550].

[39] P. Ferrero, J.P. Gauntlett, J.M. Pérez Ipiña, D. Martelli and J. Sparks, *D3-Branes Wrapped on a Spindle*, *Phys. Rev. Lett.* **126** (2021) 111601 [2011.10579].

[40] P. Scott, *The geometries of 3-manifolds*, *Bulletin of the London Mathematical Society* **15** (1983) 401.

[41] I. SATAKE, *The Gauss-Bonnet Theorem for V-manifolds.*, *Journal of the Mathematical Society of Japan* **9** (1957) 464 .

[42] P. Ferrero, J.P. Gauntlett and J. Sparks, *Supersymmetric spindles*, *JHEP* **01** (2022) 102 [2112.01543].

[43] I. Bah, F. Bonetti, R. Minasian and E. Nardoni, *Anomalies of QFTs from M-theory and Holography*, *JHEP* **01** (2020) 125 [1910.04166].

[44] S.M. Hosseini, K. Hristov, Y. Tachikawa and A. Zaffaroni, *Anomalies, Black strings and the charged Cardy formula*, *JHEP* **09** (2020) 167 [2006.08629].

[45] J.P. Gauntlett, N. Kim and D. Waldram, *Supersymmetric AdS(3), AdS(2) and Bubble Solutions*, *JHEP* **04** (2007) 005 [hep-th/0612253].

[46] P. Ferrero, J.P. Gauntlett, J.M.P. Ipiña, D. Martelli and J. Sparks, *Accelerating black holes and spinning spindles*, *Phys. Rev. D* **104** (2021) 046007 [2012.08530].

[47] D. Cassani, J.P. Gauntlett, D. Martelli and J. Sparks, *Thermodynamics of accelerating and supersymmetric AdS4 black holes*, *Phys. Rev. D* **104** (2021) 086005 [2106.05571].

[48] M. Inglese, D. Martelli and A. Pittelli, *The spindle index from localization*, *J. Phys. A* **57** (2024) 085401 [2303.14199].

[49] M. Inglese, D. Martelli and A. Pittelli, *Supersymmetry and Localization on Three-Dimensional Orbifolds*, 2312.17086.

[50] E. Colombo, S.M. Hosseini, D. Martelli, A. Pittelli and A. Zaffaroni, *Microstates of Accelerating and Supersymmetric AdS4 Black Holes from the Spindle Index*, *Phys. Rev. Lett.* **133** (2024) 031603 [2404.07173].

[51] P. Ferrero, J.P. Gauntlett, D. Martelli and J. Sparks, *M5-branes wrapped on a spindle*, *JHEP* **11** (2021) 002 [2105.13344].

[52] F. Faedo and D. Martelli, *D4-branes wrapped on a spindle*, *JHEP* **02** (2022) 101 [2111.13660].

[53] S.M. Hosseini, K. Hristov and A. Zaffaroni, *Rotating multi-charge spindles and their microstates*, *JHEP* **07** (2021) 182 [2104.11249].

[54] A. Boido, J.M.P. Ipiña and J. Sparks, *Twisted D3-brane and M5-brane compactifications from multi-charge spindles*, *JHEP* **07** (2021) 222 [2104.13287].

[55] I. Bah, F. Bonetti, R. Minasian and E. Nardoni, *M5-brane sources, holography, and Argyres-Douglas theories*, *JHEP* **11** (2021) 140 [2106.01322].

[56] P. Ferrero, M. Inglese, D. Martelli and J. Sparks, *Multicharge accelerating black holes and spinning spindles*, *Phys. Rev. D* **105** (2022) 126001 [2109.14625].

[57] C. Couzens, K. Stemerdink and D. van de Heisteeg, *M2-branes on discs and multi-charged spindles*, *JHEP* **04** (2022) 107 [2110.00571].

[58] S. Giri, *Black holes with spindles at the horizon*, *JHEP* **06** (2022) 145 [2112.04431].

[59] C. Couzens, *A tale of (M)2 twists*, *JHEP* **03** (2022) 078 [2112.04462].

[60] P. Karndumri and P. Nuchino, *Five-branes wrapped on topological disks from 7D N=2 gauged supergravity*, *Phys. Rev. D* **105** (2022) 066010 [2201.05037].

[61] K.C.M. Cheung, J.H.T. Fry, J.P. Gauntlett and J. Sparks, *M5-branes wrapped on four-dimensional orbifolds*, *JHEP* **08** (2022) 082 [2204.02990].

[62] I. Arav, J.P. Gauntlett, M.M. Roberts and C. Rosen, *Leigh-Strassler compactified on a spindle*, *JHEP* **10** (2022) 067 [2207.06427].

[63] C. Couzens and K. Stemerdink, *Universal spindles: D2's on Σ and M5's on $\Sigma \times \mathbb{H}^3$*, 2207.06449.

[64] M. Suh, *M5-branes and D4-branes wrapped on a direct product of spindle and Riemann surface*, *JHEP* **02** (2024) 205 [2207.00034].

[65] C. Couzens, H. Kim, N. Kim, Y. Lee and M. Suh, *D4-branes wrapped on four-dimensional orbifolds through consistent truncation*, *JHEP* **02** (2023) 025 [2210.15695].

[66] M. Suh, *Spindle black holes and mass-deformed ABJM*, *JHEP* **05** (2024) 267 [2211.11782].

[67] M. Suh, *Baryonic spindles from conifolds*, 2304.03308.

[68] A. Amariti, N. Petri and A. Segati, *$T^{1,1}$ truncation on the spindle*, *JHEP* **07** (2023) 087 [2304.03663].

[69] K. Hristov and M. Suh, *Spindle black holes in $AdS_4\times SE_7$*, *JHEP* **10** (2023) 141 [2307.10378].

[70] P. Ferrero, *D6 branes wrapped on a spindle and $Y^{p,q}$ manifolds*, *JHEP* **05** (2024) 182 [2403.03988].

[71] K. Hristov and M. Suh, *Spindle black holes and theories of class $\mathcal{F}$*, *JHEP* **08** (2024) 006 [2405.17432].

[72] M. Suh, *M5-branes and D4-branes wrapped on disk $\times$ disk and spindle $\ltimes$ disk*, 2411.09737.

[73] A. Cabo-Bizet, D. Cassani, D. Martelli and S. Murthy, *Microscopic origin of the Bekenstein-Hawking entropy of supersymmetric AdS_5 black holes*, *JHEP* **10** (2019) 062 [1810.11442].

[74] N. Drukker, M. Marino and P. Putrov, *From weak to strong coupling in ABJM theory*, *Commun. Math. Phys.* **306** (2011) 511 [1007.3837].

[75] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, *N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091 [0806.1218].

[76] D.L. Jafferis and S.S. Pufu, *Exact results for five-dimensional superconformal field theories with gravity duals*, *JHEP* **05** (2014) 032 [1207.4359].

[77] N. Seiberg, *Five-dimensional SUSY field theories, nontrivial fixed points and string dynamics*, *Phys. Lett. B* **388** (1996) 753 [hep-th/9608111].

[78] E. Witten, *Some comments on string dynamics*, in *STRINGS 95: Future Perspectives in String Theory*, pp. 501–523, 7, 1995 [hep-th/9507121].

[79] S.M. Hosseini and A. Zaffaroni, *Universal AdS Black Holes in Theories with 16 Supercharges and Their Microstates*, *Phys. Rev. Lett.* **126** (2021) 171604 [2011.01249].

[80] S.M. Hosseini and K. Hristov, *4d F(4) gauged supergravity and black holes of class $\mathcal{F}$*, *JHEP* **02** (2021) 177 [2011.01943].

[81] A. Boido, J.P. Gauntlett, D. Martelli and J. Sparks, *Entropy Functions For Accelerating Black Holes*, *Phys. Rev. Lett.* **130** (2023) 091603 [2210.16069].

[82] A. Boido, J.P. Gauntlett, D. Martelli and J. Sparks, *Gravitational Blocks, Spindles and GK Geometry*, *Commun. Math. Phys.* **403** (2023) 917 [2211.02662].

[83] D. Martelli and A. Zaffaroni, *Equivariant localization and holography*, *Lett. Math. Phys.* **114** (2024) 15 [2306.03891].

[84] P. Benetti Genolini, J.P. Gauntlett and J. Sparks, *Equivariant Localization in Supergravity*, *Phys. Rev. Lett.* **131** (2023) 121602 [2306.03868].

[85] D. Klemm and W.A. Sabra, *Supersymmetry of black strings in $D = 5$ gauged supergravities*, *Phys. Rev. D* **62** (2000) 024003 [hep-th/0001131].

[86] E. Lerman and S. Tolman, *Hamiltonian torus actions on symplectic orbifolds and toric varieties*, *Trans. Amer. Math. Soc.* **349** (1997) 4201 [dg-ga/9511008].

[87] E. Legendre, *Toric geometry of convex quadrilaterals*, *J. Symp. Geom.* **9** (2011) 343 [0909.4512].

[88] E. Colombo, F. Faedo, D. Martelli and A. Zaffaroni, *Equivariant volume extremization and holography*, *JHEP* **01** (2024) 95 [2309.04425].

[89] P. Benetti Genolini, J.P. Gauntlett and J. Sparks, *Localizing wrapped M5-branes and gravitational blocks*, *Phys. Rev. D* **108** (2023) L101903 [2308.10933].

[90] P. Benetti Genolini, J.M. Perez Ipiña and J. Sparks, *Localization of the action in AdS/CFT*, *JHEP* **10** (2019) 252 [1906.11249].

[91] P. Benetti Genolini, J.P. Gauntlett and J. Sparks, *Equivariant localization for AdS/CFT*, *JHEP* **02** (2024) 015 [2308.11701].

[92] P. Benetti Genolini, J.P. Gauntlett, Y. Jiao, A. Lüscher and J. Sparks, *Localization and attraction*, *JHEP* **05** (2024) 152 [2401.10977].

[93] P. Benetti Genolini, J.P. Gauntlett, Y. Jiao, A. Lüscher and J. Sparks, *Localization of the Free Energy in Supergravity*, *Phys. Rev. Lett.* **133** (2024) 141601 [2407.02554].

[94] P. Benetti Genolini, J.P. Gauntlett, Y. Jiao, A. Lüscher and J. Sparks, *Toric gravitational instantons in gauged supergravity*, 2410.19036.

[95] P. Benetti Genolini, J.P. Gauntlett, Y. Jiao, A. Lüscher and J. Sparks, *Equivariant localization for $D = 4$ gauged supergravity*, 2412.07828.

[96] D. Martelli and A. Passias, *The gravity dual of supersymmetric gauge theories on a two-parameter deformed three-sphere*, *Nucl. Phys. B* **877** (2013) 51 [1306.3893].

[97] G.W. Gibbons and S.W. Hawking, *Classification of Gravitational Instanton Symmetries*, *Commun. Math. Phys.* **66** (1979) 291.

[98] A.H. Taub, *Empty space-times admitting a three parameter group of motions*, *Annals Math.* **53** (1951) 472.

[99] E. Newman, L. Tamburino and T. Unti, *Empty-Space Generalization of the Schwarzschild Metric*, *Journal of Mathematical Physics* **4** (1963) 915.

[100] C.W. Misner, *The Flatter regions of Newman, Unti and Tamburino's generalized Schwarzschild space*, *J. Math. Phys.* **4** (1963) 924.

[101] S.W. Hawking, *Gravitational Instantons*, *Phys. Lett. A* **60** (1977) 81.

[102] T. Eguchi and A.J. Hanson, *Asymptotically Flat Selfdual Solutions to Euclidean Gravity*, *Phys. Lett. B* **74** (1978) 249.

[103] T. Eguchi and A.J. Hanson, *GRAVITATIONAL INSTANTONS*, *Gen. Rel. Grav.* **11** (1979) 315.

[104] G.W. Gibbons and S.W. Hawking, *Gravitational Multi - Instantons*, *Phys. Lett. B* **78** (1978) 430.

[105] T. Eguchi and A.J. Hanson, *Selfdual Solutions to Euclidean Gravity*, *Annals Phys.* **120** (1979) 82.

[106] H. Pedersen, *Eguchi-hanson metrics with cosmological constant*, *Classical and Quantum Gravity* **2** (1985) 579.

[107] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers, *Large N phases, gravitational instantons and the nuts and bolts of AdS holography*, *Phys. Rev. D* **59** (1999) 064010 [`hep-th/9808177`].

[108] M. Dunajski and S.A. Hartnoll, *Einstein-Maxwell gravitational instantons and five dimensional solitonic strings*, *Class. Quant. Grav.* **24** (2007) 1841 [`hep-th/0610261`].

[109] D. Martelli, A. Passias and J. Sparks, *The gravity dual of supersymmetric gauge theories on a squashed three-sphere*, *Nucl. Phys. B* **864** (2012) 840 [`1110.6400`].

[110] D. Martelli and J. Sparks, *The gravity dual of supersymmetric gauge theories on a biaxially squashed three-sphere*, *Nucl. Phys. B* **866** (2013) 72 [`1111.6930`].

[111] D. Martelli, A. Passias and J. Sparks, *The supersymmetric NUTs and bolts of holography*, *Nucl. Phys. B* **876** (2013) 810 [`1212.4618`].

[112] D. Cassani and L. Papini, *The BPS limit of rotating AdS black hole thermodynamics*, *JHEP* **09** (2019) 079 [`1906.10148`].

[113] T. Delzant, *Hamiltoniens périodiques et images convexes de l'application moment*, *Bull. Soc. Math. Fr.* **116** (1988) 315.

[114] L.A. Borisov, L. Chen and G.G. Smith, *The orbifold Chow ring of toric Deligne-Mumford stacks*, *J. Amer. Math. Soc.* **18** (2005) 193 [`math/0309229`].

[115] A. Hochenegger and F. Witt, *On complex and symplectic toric stacks*, *Contributions to Algebraic Geometry* (2012) 305 [`1105.6280`].

[116] A. Geraschenko and M. Satriano, *Toric stacks I: The theory of stacky fans*, *Trans. Amer. Math. Soc.* **367** (2015) 1033 [`1107.1906`].

[117] H. Sakai, *The Symplectic Deligne–Mumford Stack Associated to a Stacky Polytope*, *Results. Math.* **63** (2013) 903 [1009.3547].

[118] M. Abreu, *Kähler metrics on toric orbifolds*, *J. Diff. Geom.* **58** (2001) 151 [math/0105112].

[119] A. Futaki, H. Ono and G. Wang, *Transverse Kahler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds*, *J. Diff. Geom.* **83** (2009) 585 [math/0607586].

[120] W. Wang, *Toric Sheaves on Hirzebruch Orbifolds*, *Doc. Math.* **25** (2020) 655 [1803.11215].

[121] M. Abreu, *Kähler geometry of toric manifolds in symplectic coordinates*, math/0004122.

[122] V.W. Guillemin, *Kaehler structures on toric varieties*, *J. Diff. Geom.* **40** (1994) 285.

[123] C.P. Boyer, K. Galicki and J. Kollar, *Einstein metrics on spheres*, *Ann. Math.* **162** (2005) 557 [math/0309408].

[124] A. Brandhuber and Y. Oz, *The D-4 - D-8 brane system and five-dimensional fixed points*, *Phys. Lett. B* **460** (1999) 307 [hep-th/9905148].

[125] D. Martelli and J. Sparks, *Resolutions of non-regular Ricci-flat Kahler cones*, *J. Geom. Phys.* **59** (2009) 1175 [0707.1674].

[126] D. Martelli and J. Sparks, *Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals*, *Commun. Math. Phys.* **262** (2006) 51 [hep-th/0411238].

[127] L.J. Romans, *The F(4) Gauged Supergravity in Six-dimensions*, *Nucl. Phys. B* **269** (1986) 691.

[128] R. D'Auria, S. Ferrara and S. Vaula, *Matter coupled F(4) supergravity and the AdS(6) / CFT(5) correspondence*, *JHEP* **10** (2000) 013 [hep-th/0006107].

[129] L. Andrianopoli, R. D'Auria and S. Vaula, *Matter coupled F(4) gauged supergravity Lagrangian*, *JHEP* **05** (2001) 065 [hep-th/0104155].

[130] J.T. Liu and R. Minasian, *Black holes and membranes in AdS(7)*, *Phys. Lett. B* **457** (1999) 39 [hep-th/9903269].

[131] M. Pernici, K. Pilch and P. Van Nieuwenhuizen, *Gauged maximally exteded supergravity in seven dimensions*, *Phys. Lett. B* **143** (1984) 103.

[132] Z.W. Chong, M. Cvetic, H. Lu and C.N. Pope, *Non-extremal charged rotating black holes in seven-dimensional gauged supergravity*, *Phys. Lett. B* **626** (2005) 215 [hep-th/0412094].

[133] V. Apostolov, D.M.J. Calderbank and P. Gauduchon, *Ambitoric geometry II: Extremal toric surfaces and Einstein 4-orbifolds*, Ann. Sci. Ec. Norm. Super. **48** (2015) 1075 [1302.6979].

[134] D.D.K. Chow, *Single-rotation two-charge black holes in gauged supergravity*, 1108.5139.

[135] S.-Q. Wu, *Two-charged non-extremal rotating black holes in seven-dimensional gauged supergravity: The Single-rotation case*, Phys. Lett. B **705** (2011) 383 [1108.4158].

[136] W. Chen, H. Lu and C.N. Pope, *General Kerr-NUT-AdS metrics in all dimensions*, Class. Quant. Grav. **23** (2006) 5323 [hep-th/0604125].

[137] C. Toldo and B. Willett, *Partition functions on 3d circle bundles and their gravity duals*, JHEP **05** (2018) 116 [1712.08861].

[138] B. Carter, *Hamilton-Jacobi and Schrodinger separable solutions of Einstein's equations*, Commun. Math. Phys. **10** (1968) 280.

[139] J. Plebanski and M. Demianski, *Rotating, charged, and uniformly accelerating mass in general relativity*, Annals of Physics **98** (1976) 98.

[140] D.Z. Freedman and A.K. Das, *Gauge Internal Symmetry in Extended Supergravity*, Nucl. Phys. B **120** (1977) 221.

[141] M.M. Caldarelli and D. Klemm, *Supersymmetry of Anti-de Sitter black holes*, Nucl. Phys. B **545** (1999) 434 [hep-th/9808097].

[142] J.P. Gauntlett and O. Varela, *Consistent Kaluza-Klein reductions for general supersymmetric AdS solutions*, Phys. Rev. D **76** (2007) 126007 [0707.2315].

[143] J.P. Gauntlett, S. Kim, O. Varela and D. Waldram, *Consistent supersymmetric Kaluza-Klein truncations with massive modes*, JHEP **04** (2009) 102 [0901.0676].

[144] J.J. Duistermaat and G.J. Heckman, *On the Variation in the cohomology of the symplectic form of the reduced phase space*, Invent. Math. **69** (1982) 259.

[145] N. Berline and M. Vergne, *Classes caractéristiques équivariantes. formule de localisation en cohomologie équivariante*, *CR Acad. Sci. Paris* **295** (1982) 539.

[146] M.F. Atiyah and R. Bott, *The Moment map and equivariant cohomology*, Topology **23** (1984) 1.

[147] D. Farquet, J. Lorenzen, D. Martelli and J. Sparks, *Gravity duals of supersymmetric gauge theories on three-manifolds*, JHEP **08** (2016) 080 [1404.0268].

[148] J. Podolsky and A. Vratny, *New form of all black holes of type D with a cosmological constant*, Phys. Rev. D **107** (2023) 084034 [2212.08865].

[149] D. Klemm and M. Nozawa, *Supersymmetry of the C-metric and the general Plebanski-Demianski solution*, *JHEP* **05** (2013) 123 [1303.3119].

[150] J.F. Plebański, *A class of solutions of einstein-maxwell equations*, *Annals of Physics* **90** (1975) 196.

[151] N. Alonso-Alberca, P. Meessen and T. Ortin, *Supersymmetry of topological Kerr-Newman-Taub-NUT-AdS space-times*, *Class. Quant. Grav.* **17** (2000) 2783 [hep-th/0003071].

[152] M. Astorino, *Accelerating and charged type I black holes*, *Phys. Rev. D* **108** (2023) 124025 [2307.10534].

[153] V. Apostolov, D.M.J. Calderbank and P. Gauduchon, *Ambitoric geometry I: Einstein metrics and extremal ambikähler structures*, *J. Reine Angew. Math.* **2016** (2016) 109 [1302.6975].

[154] R. Emparan, C.V. Johnson and R.C. Myers, *Surface terms as counterterms in the AdS / CFT correspondence*, *Phys. Rev. D* **60** (1999) 104001 [hep-th/9903238].

[155] K. Skenderis, *Lecture notes on holographic renormalization*, *Class. Quant. Grav.* **19** (2002) 5849 [hep-th/0209067].

[156] I. Papadimitriou and K. Skenderis, *Thermodynamics of asymptotically locally AdS spacetimes*, *JHEP* **08** (2005) 004 [hep-th/0505190].

[157] A. Anabalón, F. Gray, R. Gregory, D. Kubizňák and R.B. Mann, *Thermodynamics of Charged, Rotating, and Accelerating Black Holes*, *JHEP* **04** (2019) 096 [1811.04936].

[158] C. Fefferman and C.R. Graham, *Conformal invariants*, in *Élie Cartan et les mathématiques d'aujourd'hui - Lyon, 25-29 juin 1984*, no. S131 in Astérisque, pp. 95–116, Société mathématique de France (1985).

[159] J.B. Griffiths and J. Podolsky, *A New look at the Plebanski-Demianski family of solutions*, *Int. J. Mod. Phys. D* **15** (2006) 335 [gr-qc/0511091].

[160] K. Hristov, *Equivariant localization and gluing rules in 4d $\mathcal{N} = 2$ higher derivative supergravity*, 6, 2024 [2406.18648].

[161] C. Closset, H. Kim and B. Willett, *Seifert fibering operators in 3d $\mathcal{N} = 2$ theories*, *JHEP* **11** (2018) 004 [1807.02328].

[162] L.A. Borisov, L. Chen and G.G. Smith, *The orbifold Chow ring of toric Deligne-Mumford stacks*, *J. Amer. Math. Soc.* **18** (2005) 193 [math/0309229].

[163] B. Fantechi, E. Mann and F. Nironi, *Smooth toric Deligne-Mumford stacks*, *J. Reine Angew. Math.* **648** (2010) 201.

[164] S.M. Hosseini, I. Yaakov and A. Zaffaroni, *Topologically twisted indices in five dimensions and holography*, *JHEP* **11** (2018) 119 [1808.06626].

[165] P.M. Crichigno, D. Jain and B. Willett, *5d Partition Functions with A Twist*, *JHEP* **11** (2018) 058 [1808.06744].

[166] S.M. Hosseini, I. Yaakov and A. Zaffaroni, *The joy of factorization at large N: five-dimensional indices and AdS black holes*, *JHEP* **02** (2022) 097 [2111.03069].

[167] D. Cassani, A. Ruipérez and E. Turetta, *Localization of the 5D supergravity action and Euclidean saddles for the black hole index*, *JHEP* **12** (2024) 086 [2409.01332].

[168] E. Colombo, V. Dimitrov, D. Martelli and A. Zaffaroni, *Equivariant localization in supergravity in odd dimensions*, 2502.15624.

[169] D.M.J. Calderbank and H. Pedersen, *Selfdual Einstein metrics with torus symmetry*, *J. Diff. Geom.* **60** (2002) 485 [math/0105263].

[170] D.M.J. Calderbank and M.A. Singer, *Einstein metrics and complex singularities*, *Invent. Math.* **156** (2004) 405 [math/0206229].

[171] P. Benetti Genolini, D. Cassani, D. Martelli and J. Sparks, *Holographic renormalization and supersymmetry*, *JHEP* **02** (2017) 132 [1612.06761].