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# Boundedness, Hypoellipticity and Schatten-von Neumann properties for Global Pseudo-differential Operators on $\mathbb{R}^d$

By

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*There is something weird and magical about Mathematics.*
*One sits, starts to think and gazes into the void.*
*And universes unfold before their eyes.*

# Abstract

This thesis is devoted to study various global properties of pseudo-differential and Fourier integral operators on $\mathbb{R}^d$. We focus on the investigation of continuity results for pseudo-differential and Fourier integral operators in the context of Orlicz spaces, global hypoellipticity and solvability properties of time-periodic evolution partial differential equations and Schatten-$p$ properties of pseudo-differential operators in the Weyl-Hörmander calculus.

We first extend continuity results for pseudo-differential operators from $L^p$ spaces to Orlicz spaces employing a Marcinkiewicz-type interpolation theorem. We also establish continuity properties for Fourier integral operators and define a version of global wave-front set based on Orlicz-Sobolev spaces.

We then investigate the global hypoellipticity and solvability of a certain class of time-periodic evolution operators. Our approach employs Fourier expansions in eigenfunction bases associated with elliptic operators, extending some results to the class of Gevrey-type Sobolev-Kato spaces.

Finally, we explore Schatten-$p$ properties of pseudo-differential operators in the Weyl-Hörmander calculus. By means of the analysis of certain convenient spaces, known as Wiener-Lebesgue spaces, we derive sufficient conditions on symbols to ensure that the corresponding operators belong to the Schatten-$p$ class, in the quasi-Banach case $0 < p < 1$, extending known results for $p \geqslant 1$.

# Sommario

Questa tesi è dedicata allo studio di alcune proprietà globali di operatori pseudo-differenziali e operatori integrali di Fourier in $\mathbb{R}^d$. Il focus riguarda l'analisi di risultati di continuità degli operatori pseudo-differenziali e integrali di Fourier nel contesto degli spazi di Orlicz, di proprietà di ipoellitticità e risolubilità globale riguardanti alcune PDEs periodiche nel tempo e di proprietà di Schatten-$p$ di operatori pseudo-differenziali nel calcolo di Weyl-Hörmander.

In primo luogo, estendiamo i risultati di continuità per gli operatori pseudo-differenziali negli spazi $L^p$ agli spazi di Orlicz, utilizzando un teorema di interpolazione di tipo Marcinkiewicz. Illustriamo inoltre proprietà di continuità per gli operatori integrali di Fourier e definiamo una versione globale del fronte d'onda basato sugli spazi di Orlicz-Sobolev.

Studiamo poi l'ipoellitticità e la risolubilità globale di una certa classe di operatori di evoluzione periodici nel tempo. Il nostro approccio si basa sull'espansione in serie di Fourier associata a un operatore ellittico, estendendo alcuni risultati precedenti alla classe degli spazi di Sobolev-Kato di tipo Gevrey.

Infine, esploriamo le proprietà Schatten-$p$ degli operatori pseudo-differenziali nel calcolo di Weyl-Hörmander. Attraverso l'analisi di alcuni spazi che risultano utili, noti come spazi di Wiener-Lebesgue, deduciamo condizioni sufficienti sui simboli per garantire che gli operatori corrispondenti appartengano alla classe Schatten-$p$, nel caso quasi-Banach $0 < p < 1$, estendendo risultati noti per $p \geqslant 1$.

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# Contents

**Abstract** **3**

**Sommario** **5**

**Acknowledgments** **7**

**Introduction** **13**

**1 Weyl-Hörmander calculus of pseudo-differential operators** **21**

1.1 Symplectic analysis . . . . . . . . . . . . . . . . . . . . . . . 21

1.2 Weyl-Hörmander calculus . . . . . . . . . . . . . . . . . . . . 24

1.3 Hörmander's symbols classes $S^r_{\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$ . . . . . . . . 32

1.4 The SG calculus . . . . . . . . . . . . . . . . . . . . . . . . 33

1.5 Fourier integral operators of SG type . . . . . . . . . . . . 43

1.6 Global wave-front set . . . . . . . . . . . . . . . . . . . . . 44

**2 Continuity of pseudo-differential operators and Fourier integral operators on Orlicz spaces** **51**

2.1 Orlicz Spaces . . . . . . . . . . . . . . . . . . . . . . . . . . 51

2.2 The role of upper and lower Lebesgue exponents for Young functions . . . . . . . . . . . . . . . . . . . . . . . . . 60

2.3 Continuity results and applications to PDEs . . . . . . . 71

**3 Gevrey time-periodic Sobolev - Kato spaces and global hypoellipticity and solvabilty of evolution operators 81**

3.1 Eigenfunction expansions in weighted Sobolev spaces on $\mathbb{R}^d$ . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 82

3.2 Gevrey-periodic weighted Sobolev spaces on $\mathbb{T}^n \times \mathbb{R}^d$ . . 90

3.3 Periodic evolution equations and hypoellipticity on $\mathbb{T} \times \mathbb{R}^d$ 107

3.4 Periodic evolution equations and solvability on $\mathbb{T} \times \mathbb{R}^d$ . . 113

**4 Quasi-Banach Schatten-von Neumann properties in Weyl-Hörmander calculus 119**

4.1 Schatten-von Neumann classes . . . . . . . . . . . . . . . . . 119

4.2 Wiener-Lebesgue spaces . . . . . . . . . . . . . . . . . . . . . 124

4.3 Quasi-Banach Schatten-von Neumann properties in pseudo-differential calculus . . . . . . . . . . . . . . . . . . . . . . . . 131

4.4 Applications to special families of pseudo - differential operators . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 141

**Bibliography 145**

# Introduction

In this thesis we study various properties of classes of pseudo - differential and Fourier integral operators globally defined on $\mathbb{R}^d$. Namely, we prove continuity results for certain pseudo-differential operators and Fourier integral operators in the context of Orlicz spaces (see Chapter 2), study the global hypoellipticity and solvability properties of a class of time-periodic evolution PDEs on suitably defined functional spaces, whose properties we also examine in detail (see Chapter 3), and, finally, investigate Schatten-$p$ properties for pseudo-differential operators in Weyl-Hörmander calculus (see Chapter 4). We remark that all the results discussed in this thesis relate to the global behaviour on $\mathbb{R}^d$ of solutions of PDEs and/or of symbols of operators. All the involved pseudo-differential operators belong to suitable Weyl-Hörmander classes. They are characterised by some control/decay at spatial infinity of their symbol and/or of the associated weight function. The same is true for the elements of the involved spaces of functions or distributions. We summarise below the results we achieve in each one of the dedicated chapters. Definitions and basic properties that we refer to in the sequel of this introduction can be found in Chapter 1.

Orlicz spaces $L^\Phi$ were first introduced by W. Orlicz in 1932 in [88] in order to generalize the usual $L^p$ spaces. The advantage of employing Orlicz spaces lies in the fact that they are more convenient when solving certain problems where $L^p$ spaces are insufficient. As an example, consider the entropy of a probability density function $f$ given by

$$E(f) = -\int f(\xi) \log f(\xi)\, d\xi.$$

In this case, it may be more convenient to work with an Orlicz norm estimate, for instance with $\Phi(t) = t\log(1+t)$, as opposed to $L^1$ norm estimates.

Orlicz spaces have been widely studied, see e.g. [1, 50, 73, 74, 85, 89] and the references therein. In some recent investigations, pseudo-differential operators have been also extended to the framework of Orlicz modulation spaces (cf. [122], and [96, 111] for further properties on Orlicz modulation spaces).

In Chapter 2, we employ a Marcinkiewicz interpolation-type theorem, proved by Liu and Wang in [72], to extend continuity properties of pseudo-differential operators acting an $L^p$ to the context of Orlicz spaces. Namely, we obtain Hörmander's improvement of Mihlin's Fourier multiplier theorem (cf. [51]) in the context of Orlicz spaces (see Theorem 2.37). In a similar manner, in Theorem 2.36, we obtain continuity results for pseudo-differential operators of order 0 in Orlicz spaces as well.

We then further extend this set of results in Theorem 2.38 showing the continuity on Orlicz spaces of a broad class of Fourier integral operators, under a condition on the order of the amplitude (that is, a loss of derivatives and decay).

The analysis focuses also on investigating the role of the so-called Orlicz-Lebesgue exponents $p_\Phi$ and $q_\Phi$, introduced in [72], and constructed from the Young function $\Phi$, which play a crucial role in the interpolation theorem.

Finally, we study a version of global wave-front set involving classes of Sobolev spaces modelled on Orlicz spaces, and investigate its propagation from data to solutions of some Cauchy problems associated with first order hyperbolic equations.

In Chapter 3 we investigate the properties of global hypoellipticity and solvability for operators of the type

$$L = D_t + \omega \mathrm{Op}(p), \quad D_t = -i\partial_t, t \in \mathbb{T} = \mathbb{T}^1, \omega \in \mathbb{C}, \tag{0.1}$$

where the symbol $p$ belongs to a class $S^{m,\mu}$ of (weighted) symbols, satisfying global estimates on $\mathbb{R}^d \times \mathbb{R}^d$.

Our approach is based on the study of the Fourier expansions of a class of time-dependent weighted Sobolev spaces on $\mathbb{T}^n \times \mathbb{R}^d$ by means of a basis of eigenfunctions associated with a positive, normal, elliptic operator having positive order components.

Many authors have explored characterisations of functional spaces through Fourier expansions generated by elliptic operators. Notably, Seeley investigated this topic in the context of smooth and analytic functions on vector bundles in [103, 104]. For Hilbert spaces and closed manifolds, we refer to the work of Delgado and Ruzhansky [37]. Similar ideas have been developed for ultradifferentiable classes on compact manifolds and Lie groups by Dasgupta and Ruzhansky [36], as well as by Kirilov, Moraes, and Ruzhansky in [64, 67]. Additionally, Greenfield and Wallach explored related aspects in [46]. In the context of Gelfand-Shilov classes in Euclidean spaces, a similar approach was introduced by Cappiello, Gramchev, and Pilipović [24, 44].

Furthermore, the Fourier expansions on compact product manifolds, and their applications to periodic evolution equations, have been explored in recent studies. For instance, Ávila, Gramchev, and Kirilov analyzed in [7] smooth functions and distributions on $\mathbb{T} \times M$, where $M$ is a smooth closed manifold, using this kind of expansions to investigate the global hypoellipticity of such equations. In [64, 67], the authors extended this framework to compact Lie groups and their products. The non-compact case $\mathbb{T} \times \mathbb{R}^d$ has been examined by Ávila and Cappiello in [4, 5] within the context of Gelfand-Shilov classes, with applications to hypoellipticity and solvability problems. Further extensions of this approach to the setting $\mathbb{T}^n \times X$, where $X$ is a $d$-dimensional manifold with ends or, more generally, an asymptotically Euclidean manifold, will be the subject of future extensions of the theory we develop in this chapter.

A related relevant contribution is due to Hounie in [58], where he studied the global properties of the evolution operator $\mathcal{L} = D_t + A$. There, $A$ represents an unbounded, densely defined, positive operator on a separable complex Hilbert space $\mathcal{H}$, with eigenvalues tending to $+\infty$. Hounie proved that the spectral structure of $A$ does not influence the regularity of solutions when $t$ lies within an interval in $\mathbb{R}$. However, in the periodic case where $t \in \mathbb{T}$, the spectrum of $A$ plays a crucial role.

This dependence on spectral properties is strongly linked to the so-called *Diophantine approximations*, which relate to Liouville numbers and their generalisations. These ideas have been widely studied in the context of global properties of operators on the torus (see, for example, [11, 38, 44, 45, 92] and the references therein).

In this chapter, we first establish a characterisation of the Sobolev-Kato spaces $H^{r,\rho}$ and of the Schwartz space $\mathscr{S}$ by means of Fourier expansions with respect to the sequence $\{\phi_j\}_{j\in\mathbb{N}}$ of the eigenfunctions of an elliptic operator $\mathrm{Op}(p)$, $p \in S^{m,\mu}$, $m, \mu > 0$, under suitable assumptions. We then introduce and study Gevrey time-periodic Sobolev-Kato spaces on $\mathbb{T}^n \times \mathbb{R}^d$ and their duals, and characterise their elements again by means of Fourier expansions associated with the sequence $\{\phi_j\}_{j\in\mathbb{N}}$. At the end of the chapter, this theory is applied to the study of the hypoellipticity and solvability of operators of the form (0.1).

Finally, in Chapter 4, we explore the Schatten-von Neumann properties of pseudo-differential operators within the Weyl-Hörmander calculus. Such operator classes naturally arise in the study of continuity and compactness properties. In fact, a key objective is to establish suitable conditions on the symbols that guarantee $L^2$-continuity and compactness properties of the corresponding operators.

A thorough analysis of compact operators is possible within the framework of Schatten-von Neumann classes, denoted by $\{\mathscr{I}_p\}_{p\in(0,\infty]}$. For a linear and continuous operator $T$ on $L^2(V)$, $T \in \mathscr{I}_p$, if and only if its singular values

$$\sigma = \{\sigma_j\}_{j=1}^{\infty} = \{\sigma_j(T)\}_{j=1}^{\infty}$$

belong to $\ell^p$. (Recall that if $T$ is compact on $L^2(V)$, then the singular values of $T$ are the eigenvalues of $|T|$ in decreasing order.) Consequently,

$$T \in \mathscr{I}_p \quad \Longrightarrow \quad \sigma_j \leqslant Cj^{-p}. \tag{0.2}$$

We have that $\mathscr{I}_\infty$, $\mathscr{I}_2$, and $\mathscr{I}_1$ are the sets of continuous, Hilbert-Schmidt, and trace-class operators on $L^2(V)$, respectively. As $p$ approaches zero, the class $\mathscr{I}_p$ more closely resembles the set of finite rank operators.

Our goal here is to state sufficient conditions on symbols in the Hörmander class $S(m,g)$ in order for corresponding pseudo-differential operators to be Schatten-von Neumann operators of degree $0 < p \leqslant 1$ on $L^2$. More generally, we assume that our symbols belong to $S_N(m,g)$, where similar regularity conditions are imposed, but only for derivatives up to order $N$.

In the case that $1 \leqslant p \leqslant \infty$, investigations related to ours can be found in [115, 22, 23]. It is then assumed that the weight function $m$ fulfills different types of $L^p$-type conditions. More precisely, suppose that $g$ is strongly feasible on $W$, $p \in [1,\infty]$, and $m$ is $g$-continuous and $(\sigma,g)$-temperate. In [115] it is then proved that

$$m \in L^p \quad \Longleftrightarrow \quad \operatorname{Op}^w(a) \in \mathscr{I}_p, \quad \text{when} \quad a \in S(m,g), \tag{0.3}$$

and in [23], (0.3) it is proved that

$$a \in L^p \quad \Longleftrightarrow \quad \operatorname{Op}^w(a) \in \mathscr{I}_p, \quad \text{when} \quad h_g^{k/2} m \in L^p,\ a \in S(m,g). \tag{0.4}$$

We observe that (0.3) deals with Schatten-von Neumann properties for the whole symbol class $S(m,g)$, while (0.4) is focused on more individual symbols. In the case $p \in (0,1]$, the right implication

$$m \in L^p \quad \Longrightarrow \quad \operatorname{Op}^w(a) \in \mathscr{I}_p, \quad \text{when} \quad a \in S(m,g), \tag{0.5}$$

in (0.3) was proved in [119]. We also remark that the right implication

$$a \in L^p \quad \Longrightarrow \quad \operatorname{Op}^w(a) \in \mathscr{I}_p, \quad \text{when} \quad h_g^{k/2} m \in L^p,\ a \in S(m,g), \tag{0.6}$$

in (0.4) was deduced already in [53] in the case $p = 1$, and in [115] for general $p \in [1,\infty]$. For $p \leqslant 2$, it suffices to assume that $g$ should be feasible instead of strongly feasible, in order for (0.5) and (0.6) to hold.

We recall that the $\mathscr{I}_p$ spaces are Banach spaces when $1 \leqslant p \leqslant \infty$, but only quasi-Banach spaces which fail to be locally convex when $0 < p < 1$. This presents some obstacles when reaching (0.5) and (0.6) in the case $0 < p < 1$, which are absent in the case $1 \leqslant p \leqslant \infty$.

In this chapter, we improve (0.5) and obtain a version of (0.6) in the case $p \in (0,1]$, and with $S_N(m,g)$ in place of $S(m,g)$ for suitable

$N \geqslant 0$. We introduce Wiener-Lebesgue spaces $WL_g^{q,p}$ with respect to a slowly varying metric $g$. By replacing $L^p$ with $WL_g^{1,p}$ in (0.5) and (0.6), we obtain stronger results than in previous investigations, because we neither need to assume that $m$ is $g$-continuous nor $(\sigma, g)$-temperate. At first glance, it might seem that we are more restrictive, since $WL_g^{1,p}$ is contained in $L^p$ when $p \in (0,1]$. However, this is not the case. Indeed, if in addition $m$ is $g$-continuous, which is the case in [119], then $m \in L^p$, if and only if $m \in WL_g^{1,p}$, as we show in Lemma 4.32.

# Chapter 1

# Weyl-Hörmander calculus of pseudo-differential operators

In this introductory chapter we recall the basic definitions and the standard concepts concerning the so-called Weyl-Hörmander pseudo-differential calculus. We also include some preliminary results which concern certain properties of the operators investigated in the subsequent chapters which are defined by means of specific classes of Weyl-Hörmander symbols.

## 1.1 Symplectic analysis

### 1.1.1 Integrations on real vector spaces

Let $V$ be a real vector space of dimension $d$ and let $V'$ be its dual. Denote by $(e_1, \dots, e_d)$ a basis of $V$ and by $(\varepsilon_1, \dots, \varepsilon_d)$ the corresponding dual basis of $V'$, so that

$$\langle e_j, \varepsilon_k \rangle = \delta_{jk},$$

where $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{V,V'}$ denotes the duality map between $V$ and $V'$. For any $f \in L^1(V)$, we denote

$$\int_V f\, dx \equiv \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x_1 e_1 + \cdots + x_d e_d)\, dx_1 \dots dx_d.$$

**Definition 1.1.** Let $f \in L^1(V)$. Then, the Fourier transform on the vector space $V$ is defined by

$$(\mathfrak{F}f)(\xi) = \widehat{f}(\xi) \equiv (2\pi)^{-\frac{d}{2}} \int_V e^{-i\langle x,\xi\rangle} f(x)\, dx, \qquad \xi \in V'.$$

**Remark 1.2.** It follows that $\mathfrak{F}$ restricts to a homeomorphism from $\mathscr{S}(V)$ to $\mathscr{S}(V')$, which in turn is uniquely extendable to a homeomorphism from $\mathscr{S}'(V)$ to $\mathscr{S}'(V')$, and to a unitary map from $L^2(V)$ to $L^2(V')$.

### 1.1.2 Symplectic vector spaces

**Definition 1.3.** Let $W$ be a real vector space of dimension $2d < \infty$. A bilinear form $\sigma$ on $W$ is said to be *symplectic* if it is skew-symmetric, that is

$$\sigma(X,Y) = -\sigma(Y,X) \quad \text{for every} \quad X,Y \in W$$

and non-degenerate, that is

$$\sigma(X,Y) = 0 \quad \text{for every} \quad Y \in W \quad \text{implies} \quad X = 0.$$

**Definition 1.4.** The real vector space $W$ of dimension $2d < \infty$ is called symplectic if it is endowed with with a symplectic form $\sigma$. Moreover, the coordinates $X = (x,\xi)$ are called symplectic if the corresponding basis $(e_1,\dots,e_d,\varepsilon_1,\dots,\varepsilon_d)$ is symplectic, that is, for any $j,k = 1,\dots,d$, it satisfies

(i) $\sigma(e_j,e_k) = \sigma(\varepsilon_j,\varepsilon_k) = 0$;

(ii) $\sigma(e_j,\varepsilon_k) = -\delta_{jk}$.

Notice that $W$ may be identified in a canonical way with $\mathbb{R}^d \oplus \mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d$ with $\sigma$ given by

$$\sigma(X,Y) = \langle y,\xi\rangle - \langle x,\eta\rangle \tag{1.1}$$

where $X = (x,\xi) \in W$, $Y = (y,\eta) \in W$ and $\langle\cdot,\cdot\rangle$ denotes the usual scalar product on $\mathbb{R}^d$.

**Remark 1.5.** Consider the projections on the first and second variable $\pi_1$ and $\pi_2$, defined as

$$\pi_1(x,\xi) = x \quad \text{and} \quad \pi_2(x,\xi) = \xi,$$

and set $V = \pi_1 W$ and $V' = \pi_2 W$, which are identified with

$$V_1 = \{\,(x,0) \in W \,:\, x \in V\,\},$$
$$V_2 = \{\,(0,\xi) \in W \,:\, \xi \in V'\,\}.$$

Then the dual space of $V$ may be identified with $V'$ through the symplectic form $\sigma$, and $W$ agrees with the cotangent bundle (or phase space) $T^*V = V \oplus V'$.

**Remark 1.6.** If instead $V$ is a vector space of dimension $d < \infty$ with dual space $V'$ and duality $\langle\cdot,\cdot\rangle$, then $W = V \oplus V'$ is a symplectic vector space with symplectic form given by (1.1).

**Definition 1.7.** Let $T$ be a linear map on $W$. Then $T$ is called symplectic if $\sigma(TX,TY) = \sigma(X,Y)$ for every $X, Y \in W$.

**Remark 1.8.** Let $(e_1,\dots,e_d,\varepsilon_1,\dots,\varepsilon_d)$ and $(\tilde{e}_1,\dots,\tilde{e}_d,\tilde{\varepsilon}_1,\dots,\tilde{\varepsilon}_d)$ be any pair of symplectic bases. Then, there exists a unique linear symplectic map $T$ such that $Te_j = \tilde{e}_j$ and $T\varepsilon_j = \tilde{\varepsilon}_j$ for every $j = 1,\dots,d$. On the other hand, if $T$ is a linear and symplectic map on $W$ and $(e_1,\dots,e_d,\varepsilon_1,\dots,\varepsilon_d)$ is a symplectic basis, then $(Te_1,\dots,T\varepsilon_d)$ is also a symplectic basis. Consequently, a one-to-one relation has been established between linear symplectic mappings and representations of $W$ as cotangent boundles $T^*V$.

**Definition 1.9.** We define the symplectic volume form by $dX = \sigma^d/d!$. Moreover, if $U \subseteq W$ is measurable, then we shall denote by $|U|$ the measure of $U$ with respect to $dX$.

Definition 1.9 implies that the quantity

$$\int_W f(X)\,dX = \int_{\mathbb{R}\oplus\mathbb{R}} \cdots \int_{\mathbb{R}\oplus\mathbb{R}} f(x_1e_1 + \cdots + x_de_d + \xi_1\varepsilon_1 + \cdots + \xi_d\varepsilon_d)\,dx_1\dots dx_d\,d\xi_1\dots d\xi_d$$

is independent of the choice of symplectic coordinates $X = (x, \xi)$ when $f \in L^1(W)$. As a consequence, the space $\mathscr{D}'(W)$ and its usual subspaces only depend on $\sigma$ and are independent of the choice of symplectic coordinates.

**Definition 1.10.** We define the symplectic Fourier transform on $\mathscr{S}(W)$, denoted by $\mathfrak{F}_\sigma$, via the formula

$$\mathfrak{F}_\sigma a(X) = \widehat{a}(X) \equiv \pi^{-d} \int_W e^{2i\sigma(X,Y)} a(Y)\, dY,$$

when $a \in \mathscr{S}(W)$.

**Remark 1.11.** The symplectic Fourier transform $\mathfrak{F}_\sigma$ is a homeomorphism on $\mathscr{S}(W)$ which extends to a homeomorphism on $\mathscr{S}'(W)$, and to a unitary operator on $L^2(W)$. Moreover, $\mathfrak{F}_\sigma^2$ is the identity operator. Notice also that $\mathfrak{F}_\sigma$ is defined without any reference to symplectic coordinates, hence it is independent of them. Finally, by straightforward computations, it follows that the usual relations between Fourier transform and the convolution, denoted by $*$, hold true, namely:

$$\begin{aligned} \mathfrak{F}_\sigma(a * b)(X) &= \pi^d \widehat{a}(X)\widehat{b}(X), \\ \mathfrak{F}_\sigma(ab)(X) &= \pi^{-d}(\widehat{a} * \widehat{b})(X), \end{aligned}$$

when $a \in \mathscr{S}'(W)$ and $b \in \mathscr{S}(W)$.

## 1.2 Weyl-Hörmander calculus

In this section we introduce the main elements of the Weyl-Hörmander calculus. We first give an overview of some useful properties concerning the metric $g$, then we present the symbol classes with respect to the metric and the related symbolic calculus. Finally, we describe the calculi of pseudo-differential operators that we employ in the subsequent chapters.

### 1.2.1 Weight functions and feasible metrics

Let $g$ be an arbitrary Riemannian metric on the symplectic vector space $W$, and denote by $0 < m \in L^{\infty}_{loc}(W)$ a so-called *weight function*. In this section we state some important properties with respect to the weight function $m$ and the metric $g$ on $W$. We specifically refer to [57]. In particular, the next property follows from Section 18.6 therein.

**Proposition 1.12.** For each fixed $X \in W$, there exist symplectic coordinates $Z = (z, \zeta)$ which diagonalize $g_X$, that is $g_X$ takes the form

$$g_X(Z) = \sum_{j=1}^{d} \lambda_j(X)(z_j^2 + \zeta_j^2), \quad Z = (z, \zeta) \in W, \tag{1.2}$$

where

$$\lambda_1(X) \geqslant \lambda_2(X) \geqslant \cdots \geqslant \lambda_d(X) > 0 \tag{1.3}$$

only depend on $g_X$ and are independent of the choice of symplectic coordinates which diagonalize $g_X$.

Next, we introduce two important concepts which will be essential in our subsequent analysis in Chapter 4.

**Definition 1.13.** Let $g$ be a Riemaniann metric and $\sigma$ a symplectic form. Define the *dual metric* $g^{\sigma}$ with respect to the metric $g$ and the symplectic form $\sigma$ by

$$g_X^{\sigma}(Z) \equiv \sup_{Y \neq 0} \Big( \frac{\sigma(Y, Z)^2}{g_X(Y)} \Big).$$

**Definition 1.14.** Define the *Planck's function* $h_g$ with respect to the metric $g$ by

$$h_g(X) = \sup_{Z \neq 0} \Big( \frac{g_X(Z)}{g_X^{\sigma}(Z)} \Big)^{1/2}.$$

**Proposition 1.15.** Suppose that the conditions (1.2) and (1.3) are fulfilled. Then it follows that $h_g(X) = \lambda_1(X)$ and

$$g_X^{\sigma}(Z) = \sum_{j=1}^{d} \lambda_j(X)^{-1}(z_j^2 + \zeta_j^2), \quad Z = (z, \zeta) \in W. \tag{1.2$'$}$$

We remark that the assumption

$$h_g(X) \leqslant 1 \quad \Longleftrightarrow \quad g_X \leqslant g_X^\sigma, \qquad X \in W, \tag{1.4}$$

is called the *uncertainty principle*.

**Definition 1.16.** The metric $g$ is called *symplectic* if $g_X = g_X^\sigma$ for every $X \in W$.

**Proposition 1.17.** The metric $g$ is symplectic if and only if

$$\lambda_1(X) = \cdots = \lambda_d(X) = 1$$

in equation (1.2). Moreover, if $g_X$ is given by (1.2), then the corresponding *symplectic metric* is given by

$$g_X^0(Z) = \sum_{j=1}^d (z_j^2 + \zeta_j^2),$$

which is defined in a symplectically invariant way (see [115]).

**Definition 1.18.** The Riemannian metric $g$ on $W$ is called *slowly varying* if there exist positive constants $c$ and $C$ such that for every $X, Y \in W$ satisfying

$$g_X(Y - X) \leqslant c$$

it follows

$$C^{-1} g_Y(Z) \leqslant g_X(Z) \leqslant C g_Y(Z) \quad \text{for every} \quad Z \in W. \tag{1.5}$$

We can introduce also the definition of slowly varying symbol with respect to a metric $g$.

**Definition 1.19.** Let $g$ be a Riemannian metric and $m$ a weight function. Then $m$ is called *slowly varying* with respect to the metric $g$ if there exist positive constants $c$ and $C$ such that for every $X, Y \in W$ satisfying

$$g_X(Y - X) \leqslant c$$

it follows

$$C^{-1} \leqslant \frac{m(X)}{m(Y)} \leqslant C. \tag{1.6}$$

**Remark 1.20.** In the literature, a slowly varying weight function $m$ is also called *admissible.*

**Definition 1.21.** Consider two Riemaniann metrics $g$ and $G$. The metric $G$ is called *$g$-continuous* if there exist positive constants $c$ and $C$ such that for every $X, Y \in W$ satisfying

$$g_X(Y - X) \leqslant c$$

it follows

$$C^{-1} G_Y(Z) \leqslant G_X(Z) \leqslant C G_Y(Z) \quad \text{for every} \quad Z \in W. \tag{1.7}$$

**Definition 1.22.** A positive function $m$ is called *$g$-continuous* if there are positive constants $c$ and $C$ such that for every $X, Y \in W$ satisfying

$$g_X(Y - X) \leqslant c$$

it follows

$$C^{-1} m(Y) \leqslant m(X) \leqslant C m(Y). \tag{1.8}$$

**Definition 1.23.** The metric $g$ is called *$\sigma$-temperate* if there are positive constants $C$ and $N$ such that

$$g_Y(Z) \leqslant C g_X(Z)(1 + g_Y^\sigma(X - Y))^N, \qquad X, Y, Z \in W.$$

Moreover, we have the following terminology.

(i) The metric $g$ is called *feasible* if it is slowly varying and satisfies (1.4).

(ii) The metric $g$ is called *strongly feasible* if it is feasible and $\sigma$-temperate.

**Definition 1.24.** The weight function $m$ is called *$(\sigma, g)$-temperate* if there are positive constants $C$ and $N$ such that

$$m(Y) \leqslant C m(X)(1 + g_Y^\sigma(X - Y))^N, \qquad X, Y \in W.$$

### 1.2.2 Symbol classes, symbolic calculus and pseudo - differential operators

We recall here the definition of the symbol classes and their properties in the Weyl-Hörmander calculus, following [53, 54, 57].

We will consider an integer $N \geqslant 0$, a finite-dimensional symplectic vector space $W$ and shall denote by $\mathscr{C}^N(W)$ the set of continuously differentiable functions up to order $N$ on $W$. Also, for $a \in \mathscr{C}^N(W)$, $g$ an arbitrary Riemannian metric on $W$, and $0 < m \in L^\infty_{loc}(W)$, we will employ the following notation: for each $k = 0, \dots, N$, let

$$\begin{aligned} |a|^g_k(X) \equiv \sup\{|a^{(k)}(X; Y_1, \dots, Y_k)| : \\ Y_1, \dots, Y_k \in W,\ g_X(Y_j) \leqslant 1, \forall j = 1, \dots, k\}, \end{aligned} \tag{1.9}$$

where the notation $a^{(k)}$ stands for $D^\alpha a$, for any multi-index $\alpha$ such that $|\alpha| = k$ and $a^{(k)}(X; Y_1, \dots, Y_k)$ denotes the $k$-th differential of $a$ at the point $X$ evaluated on the vectors $Y_1, \dots, Y_k$. Also set

$$\|a\|^g_{N,m} \equiv \sum_{k=0}^{N} \sup_{X \in W} \big(|a|^g_k(X)/m(X)\big). \tag{1.10}$$

**Definition 1.25.** The set of symbols differentiable up to order $N$ with respect to the weight $m$ and the metric $g$, denoted by $S_N(m, g)$, is defined as

$$S_N(m, g) = \{a \in \mathscr{C}^N(W) : \|a\|^g_{N,m} < \infty\}. \tag{1.11}$$

Moreover, we let

$$S(m, g) = S_\infty(m, g) \equiv \bigcap_{N \geqslant 0} S_N(m, g).$$

**Remark 1.26.** The set $S_N(m, g)$ turns out to be a Banach space endowed with the norm $\|\cdot\|^g_{N,m}$, while the set $S(m, g)$ turns out to be a Fréchet space endowed with the family of seminorms $\{\|\cdot\|^g_{N,m} : N \in \mathbb{N}\}$.

**Proposition 1.27.** Let $g$ be a feasible metric on $W$. Let $m_1, m_2$ be two admissible weights (see Remark 1.20) with respect to the metric $g$ and let $a_j \in S(m_j, g)$. Then,

$$a_1 a_2 \in S(m_1 m_2, g).$$

We also introduce the *sharp product* (or *Leibniz product*) of two symbols, which relates with the symbol of a composition of operators.

**Definition 1.28.** Let $g$ be a Riemaniann metric on $W$, $m_1, m_2$ two weight functions, $a_1 \in S(m_1, g)$ and $a_2 \in S(m_2, g)$ two symbols. Then, we define the *sharp product* $a_1 \# a_2$ (or *Leibniz product*) of $a_1$ and $a_2$ by

$$(a\#b)(X) = (\pi)^{-2d} \iint_{W \times W} e^{-2i[X-Y_1, X-Y_2]} a(Y_1)\, b(Y_2)\, dY_1 dY_2. \quad (1.12)$$

**Proposition 1.29.** Suppose that $g$ is a strongly feasible metric on $W$, $m_k$ are $g$-continuous and $(\sigma, g)$-temperate weight functions and $a_k \in S(m_k, g)$, $k = 1, 2$ are symbols. Then,

$$(a\#b) \in S(m_1 m_2, g),$$

that is,

$$S(m_1, g)\#S(m_2, g) \subseteq S(m_1 m_2, g). \quad (1.13)$$

Consider a symplectic vector space $W$ and recall that, by Remark 1.5, the space $W$ can be identified as $T^*V$, where $V = \{ (x, 0) \in W : x \in \pi_1 W \}$. Then, given a symbol $a \in S(m, g)$ we can define the pseudo-differential operator $\mathrm{Op}^w(a)$ associated with the symbol $a$ by

$$\mathrm{Op}^w(a)(x, D)u(x) = (2\pi)^{-d} \iint_{T^*V} e^{i\langle x-y, \xi \rangle} a\left(\frac{x+y}{2}, \xi\right) u(y)\, dy\, d\xi, \quad (1.14)$$

$u \in \mathscr{S}(V)$. We will give a definition of pseudo-differential operators with symbol in $\mathscr{S}'$ in the next Section 1.2.3.

We conclude this section with the definition of an operator of special interest, which will be useful in the subsequent analysis in Chapter 4 (in particular, in Section 4.3.1).

**Definition 1.30.** Let $X \in W$ be fixed, and let $g = g_X$. Then we can define the operator $\Delta_g$ by means of

$$\mathfrak{F}_\sigma(\Delta_g f) = -4 g^\sigma \cdot \widehat{f}$$

when $f \in \mathscr{S}'(W)$.

Notice that the operator $\Delta_g$ is related to the Laplace-Beltrami operator for $g$, and is obviously defined in a symplectically invariant way, since similar facts hold for $\mathfrak{F}_\sigma$ and $g^\sigma$. If $Z = (z, \zeta)$ are symplectic coordinates such that (1.2) holds, then it follows by straightforward computation that

$$\Delta_{g_X} = \sum_{j=1}^{d} \lambda_j(X)^{-1}(\partial_{z_j}^2 + \partial_{\zeta_j}^2).$$

### 1.2.3 The calculus of pseudo-differential operators

In this section we present some of the main features of the pseudo-differential calculus. Consider a real vector space $V$ of dimension $d$ and a function $a \in \mathscr{S}(V \times V')$. Suppose also that $A$ belongs to $\mathscr{L}(V)$, the set of all linear and continuous mappings on $V$. In the special case $V \times V' \equiv \mathbb{R}^d \times \mathbb{R}^d$, $\mathscr{L}(V)$ shall be denoted by $\mathbf{M}(d, \mathbb{R})$, the space of invertible $d \times d$ matrices with entries in $\mathbb{R}$.

**Definition 1.31.** The pseudo-differential operator $\mathrm{Op}_A(a)$ is defined as the linear and continuous operator on $\mathscr{S}(V)$, given by

$$(\mathrm{Op}_A(a)f)(x) = (2\pi)^{-d} \iint_{V \times V'} e^{i\langle x-y, \xi\rangle} a(x - A(x-y), \xi) f(y)\, dy d\xi, \tag{1.15}$$

when $f \in \mathscr{S}(V)$.

This definition can be extended to general $a \in \mathscr{S}'(V \times V')$ in the following way.

**Definition 1.32.** Let $a \in \mathscr{S}'(V \times V')$. Then the pseudo-differential operator $\mathrm{Op}_A(a)$ is defined as the linear and continuous operator from $\mathscr{S}(V)$ to $\mathscr{S}'(V)$ with distribution kernel given by

$$K_{a,A}(x, y) = (2\pi)^{-\frac{d}{2}} (\mathfrak{F}_2^{-1} a)(x - A(x-y), x-y). \tag{1.16}$$

Here $\mathfrak{F}_2 F$ is the partial Fourier transform of $F(x, y) \in \mathscr{S}'(V \times V)$ with respect to the $y$ variable.

This definition is valid because the mappings

$$\mathfrak{F}_2 \quad \text{and} \quad F(x,y) \mapsto F(x - A(x-y), x-y) \tag{1.17}$$

are homeomorphisms on $\mathscr{S}'(V \times V')$ and on $\mathscr{S}'(V \times V)$, respectively. In particular, the map $a \mapsto K_{a,A}$ is a homeomorphism from $\mathscr{S}'(V \times V')$ to $\mathscr{S}'(V \times V)$.

An important special case appears when $A = t \cdot I$, with $t \in \mathbb{R}$. Here $I = I_V$ denotes the identity map on $V$. In this case we write

$$\operatorname{Op}_t(a) = \operatorname{Op}_{t \cdot I}(a).$$

The *normal* or *Kohn-Nirenberg representation*, $a(x,D)$, is obtained when $t = 0$, and the *Weyl quantization*, $\operatorname{Op}^w(a)$, is obtained when $t = \frac{1}{2}$. That is,

$$a(x,D) = \operatorname{Op}_0(a) \quad \text{and} \quad \operatorname{Op}^w(a) = \operatorname{Op}_{1/2}(a).$$

**Remark 1.33.** We recall that if $A \in \mathscr{L}(V)$, then it follows from the kernel theorem of Schwartz and Fourier's inversion formula that the map $a \mapsto \operatorname{Op}_A(a)$ is bijective from $\mathscr{S}'(V \times V')$ to the set of linear and continuous mappings from $\mathscr{S}(V)$ to $\mathscr{S}'(V')$ (cf. e.g. [54, 120]).

We refer to [57, 120] for the proof of the following result, concerning transitions between different pseudo-differential calculi.

**Proposition 1.34.** Let $a_1, a_2 \in \mathscr{S}'(V \times V')$ and $A_1, A_2 \in \mathscr{L}(V)$. Then

$$\operatorname{Op}_{A_1}(a_1) = \operatorname{Op}_{A_2}(a_2) \quad \Longleftrightarrow \quad e^{i\langle A_2 D_\xi, D_x\rangle} a_2(x,\xi) = e^{i\langle A_1 D_\xi, D_x\rangle} a_1(x,\xi). \tag{1.18}$$

Note here that the latter equality in (1.18) makes sense since it is equivalent to

$$e^{i\langle A_2 x,\xi\rangle} \widehat{a}_2(\xi,x) = e^{i\langle A_1 x,\xi\rangle} \widehat{a}_1(\xi,x),$$

and that the map $a \mapsto e^{i\langle Ax,\xi\rangle} a$ is continuous on $\mathscr{S}'(V \times V')$ (cf. e.g. [120]).

**Definition 1.35.** For any $A \in \mathscr{L}(V)$, the $A$-product $a\#_A b$ between $a \in \mathscr{S}'(V \times V')$ and $b \in \mathscr{S}'(V \times V')$ is defined by the formula

$$\mathrm{Op}_A(a\#_A b) = \mathrm{Op}_A(a) \circ \mathrm{Op}_A(b), \tag{1.19}$$

provided that the right-hand side makes sense as a continuous operator from $\mathscr{S}(V)$ to $\mathscr{S}'(V)$.

Notice that, when $a_1 \in S(m_1, g), a_2 \in S(m_2, g)$ and $m_1, m_2, g$ are as in Definition 1.28, Definition 1.35 agrees with Definition 1.28. Since the Weyl case is especially important, we denote by # the particular case when $A = \frac{1}{2} I_V$.

As a consequence of Proposition 1.29, we have the following result.

**Proposition 1.36.** Suppose that $g$ is strongly feasible, $m_k$ are $g$-continuous and $(\sigma, g)$-temperate, and that $a_k \in S(m_k, g)$, $k = 1, 2$. Then there is a unique $a \in S(m_1 m_2, g)$ such that

$$\mathrm{Op}^w(a_1) \circ \mathrm{Op}^w(a_2) = \mathrm{Op}^w(a).$$

## 1.3 Hörmander's symbols classes $S^r_{\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$

Let $\rho, \delta \in \mathbb{R}$ be such that $0 \leqslant \delta \leqslant \rho \leqslant 1$. Then the symbol class $S^r_{\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$ is the set of all $a \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that

$$|(\partial_x^\alpha \partial_\xi^\beta a)(x, \xi)| \lesssim \langle \xi \rangle^{r + \delta|\alpha| - \rho|\beta|}. \tag{1.20}$$

Namely,

$$S^r_{\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d) = S(m, g)$$

when

$$m(x, \xi) = \langle \xi \rangle^r \quad \text{and} \quad g_{(x,\xi)}(y, \eta) = \langle \xi \rangle^{2\delta} |y|^2 + \langle \xi \rangle^{-2\rho} |\eta|^2,$$

where $\langle y \rangle = \sqrt{1 + |y|^2}, y \in \mathbb{R}^d$. Notice that, in this case, the metric $g$ is slowly-varying and the weight $m$ is $g$-continuous and $(\sigma, g)$-temperate.

Also, if we denote by $(e_1, \dots, e_d, \varepsilon_1, \dots, \varepsilon_d)$ a symplectic basis on $W$ and by $k = |\alpha| + |\beta|$, then we have that

$$\begin{aligned}&\left(\partial_x^\alpha \partial_\xi^\beta a\right)(x,\xi) = \\ &= \sum_{\alpha_1+\cdots+\alpha_n+\beta_1+\cdots+\beta_n=k} a_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n}(x,\xi)(e_1^{\alpha_1}\dots e_n^{\alpha_n}\epsilon_1^{\beta_1}\dots\epsilon_n^{\beta_n})\end{aligned}$$

where $a_{\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n}$ is a $k$-multilinear form. Hence the condition (1.10) reads as

$$\begin{aligned}&\left|a^{(k)}(X : e_1^{\alpha_1}, \dots e_n^{\alpha_n}, \epsilon_1^{\beta_1}, \dots \epsilon_n^{\beta_n})\right| \\ &\leqslant C_{\alpha\beta}\langle\xi\rangle^m \prod_{1\leqslant j\leqslant n} g_{x,\xi}(e_j)^{\alpha_j/2} \prod_{1\leqslant j\leqslant n} g_{x,\xi}(\epsilon_j)^{\beta_j/2} \\ &= C_{\alpha\beta}\langle\xi\rangle^m\langle\xi\rangle^{\delta|\alpha|}\langle\xi\rangle^{-\rho|\beta|} \\ &= C_{\alpha\beta}\langle\xi\rangle^{r+\delta|\alpha|-\rho|\beta|},\end{aligned}$$

returning (1.20) as expected.

## 1.4 The SG calculus

We recall here some properties concerning the so-called SG calculus as well as some relevant results.

The SG symbol classes, and the related SG pseudo-differential operators, are a special case of the Weyl-Hörmander calculus $S(m,g)$ with

$$g_{x,\xi}(y,\eta) = \frac{|y|^2}{\langle x\rangle^2} + \frac{|\eta|^2}{\langle\xi\rangle^2} \quad \text{and} \quad m(x,\xi) = \langle x\rangle^r\langle\xi\rangle^\rho. \tag{1.21}$$

Notice that, in this case, the metric $g$ is strongly feasible and the weight $m$ is $g$-continuous and $(\sigma, g)$-temperate.

Since these operators are of major interest in this work, especially in the analysis in Chapter 2 and Chapter 3, we present here the peculiarities of SG calculus independently (see [26] and [90] for details).

**Definition 1.37.** The class $S^{m,\mu} = S^{m,\mu}(\mathbb{R}^d) = S^{m,\mu}(\mathbb{R}^d \times \mathbb{R}^d)$ of SG symbols of order $m, \mu \in \mathbb{R}$ is given by those functions $a(x,\xi) \in \mathscr{C}^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ satisfying the property that, for any multi-indices $\alpha, \beta \in \mathbb{N}^d$, there exist constants $C_{\alpha\beta} > 0$ such that

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \leqslant C_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|} \langle x \rangle^{\mu-|\beta|}. \tag{1.22}$$

We have that

(i) if $a \in S^{m,\mu}$ and $b \in S^{m',\mu'}$, then $ab \in S^{m+m',\mu+\mu'}$;

(ii) if $a \in S^{m,\mu}$ and $b \in S^{m',\mu'}$, then $a+b \in S^{\tilde{m},\tilde{\mu}}$, where $\tilde{m} = \max\{m, m'\}$ and $\tilde{\mu} = \max\{\mu, \mu'\}$.

**Proposition 1.38.** For $m, \mu \in \mathbb{R}$, $\ell \in \mathbb{N}$, the seminorms given by

$$|||a|||_\ell^{m,\mu} = \max_{|\alpha+\beta| \leqslant \ell} \sup_{x,\xi \in \mathbb{R}^d} \langle x \rangle^{-m+|\alpha|} \langle \xi \rangle^{-\mu+|\beta|} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)|, \quad a \in \ S^{m,\mu},$$

define a Fréchet topology on $S^{m,\mu}$.

**Definition 1.39.** The corresponding classes of SG pseudo-differential operators $\mathrm{Op}(S^{m,\mu}) = \mathrm{Op}(S^{m,\mu}(\mathbb{R}^d))$ are given by

$$(\mathrm{Op}(a)u)(x) = (a(\cdot, D)u)(x) = (2\pi)^{-d} \int e^{i\langle x,\xi \rangle} a(x,\xi) \hat{u}(\xi)\, d\xi, \tag{1.23}$$

where $a \in S^{m,\mu}(\mathbb{R}^d), u \in \mathscr{S}(\mathbb{R}^d)$ and extended by duality to $\mathscr{S}'(\mathbb{R}^d)$.

Notice that the operators in (1.23) form a graded algebra with respect to composition, that is,

$$\mathrm{Op}(S^{m_1,\mu_1}) \circ \mathrm{Op}(S^{m_2,\mu_2}) \subseteq \mathrm{Op}(S^{m_1+m_2,\mu_1+\mu_2}).$$

**Definition 1.40.** For any given sequence of symbols

$$(a_j)_j \in S^{m-j,\mu-j}(\mathbb{R}^d), \quad j \in \mathbb{N},$$

we write

$$a(x,\xi) \sim \sum_{j \in \mathbb{N}} a_j(x,\xi) \tag{1.24}$$

if, for every $N \geqslant 1$,

$$a - \sum_{j=0}^{N-1} a_j \in S^{m-N,\mu-N}(\mathbb{R}^d). \tag{1.25}$$

The right-hand side of (1.24) is called *asymptotic expansion* of $a$, while the left-hand side is called *asymptotic sum* of the sequence $(a_j)_j$.

**Proposition 1.41.** The symbol $c \in S^{m_1+m_2,\mu_1+\mu_2}$ of the composed operator $\mathrm{Op}(a) \circ \mathrm{Op}(b)$, $a \in S^{m_1,\mu_1}$, $b \in S^{m_2,\mu_2}$, admits the asymptotic expansion

$$c(x,\xi) \sim \sum_{\alpha} \frac{i^{|\alpha|}}{\alpha!} D_\xi^\alpha a(x,\xi)\, D_x^\alpha b(x,\xi), \tag{1.26}$$

which implies that the symbol $c$ equals $a \cdot b$ modulo $S^{m_1+m_2-1,\mu_1+\mu_2-1}$.

We have that

$$S^{-\infty,-\infty}(\mathbb{R}^d \times \mathbb{R}^d) = \bigcap_{m,\mu\in\mathbb{R}} S^{m,\mu}(\mathbb{R}^d \times \mathbb{R}^d) = \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d),$$

and

$$S^{-\infty,-\infty} \subset S^{m,\mu} \subset S^{s,\sigma}, \quad m, s, \mu, \sigma \in \mathbb{R},$$

whenever $m \leqslant s, \mu \leqslant \sigma$.

**Definition 1.42.** We define the Sobolev-Kato (or *weighted Sobolev*) spaces by

$$H^{r,\rho} = H^{r,\rho}(\mathbb{R}^d) = \{v \in \mathscr{S}'(\mathbb{R}^d) \colon \|v\|_{r,\rho} = \|\langle\cdot\rangle^r \langle D\rangle^\rho v\|_{L^2} < \infty\}, \tag{1.27}$$

where $\langle D\rangle^\rho = \mathrm{Op}(\lambda_\rho)$ is the pseudo-differential operator with symbol $\lambda_\rho(\xi) = \langle\xi\rangle^\rho$.

We have that the spaces $H^{r,\rho}$ are Hilbert if equipped with the norm $\|\cdot\|_{r,\rho}$ associated with the inner product

$$(u,v)_{H^{r,\rho}} = (\langle\cdot\rangle^r \langle D\rangle^\rho u, \langle\cdot\rangle^r \langle D\rangle^\rho v)_{L^2}.$$

**Proposition 1.43.** Let $r, r', \rho, \rho' \in \mathbb{R}$ and suppose that $r \geqslant r'$ and $\rho \geqslant \rho'$. Then $H^{r,\rho} \hookrightarrow H^{r',\rho'}$ with continuous embedding. Moreover, if $r > r'$ and $\rho > \rho'$ the embedding is compact.

Notice that $H^{r,\rho} = \langle\cdot\rangle^r H^{0,\rho} = \langle\cdot\rangle^r H^\rho$, where $H^\rho = H^\rho(\mathbb{R}^d)$ denotes the usual Sobolev space of order $\rho \in \mathbb{R}$. Then, by the Sobolev embedding theorem, it follows that $H^{r,\rho} \hookrightarrow \mathscr{C}^k(\mathbb{R}^d)$, $k \in \mathbb{N}$, provided that $\rho > k + \frac{d}{2}$.

Moreover,

$$\begin{aligned} \bigcap_{r,\rho\in\mathbb{R}} H^{r,\rho}(\mathbb{R}^d) &= H^{\infty,\infty}(\mathbb{R}^d) = \mathscr{S}(\mathbb{R}^d), \\ \bigcup_{r,\rho\in\mathbb{R}} H^{r,\rho}(\mathbb{R}^d) &= H^{-\infty,-\infty}(\mathbb{R}^d) = \mathscr{S}'(\mathbb{R}^d), \end{aligned} \tag{1.28}$$

as well as, for the space of *rapidly decreasing distributions*, see [101, Chap. VII, §5],

$$\mathscr{S}'(\mathbb{R}^d)_\infty = \bigcap_{r\in\mathbb{R}} \bigcup_{\rho\in\mathbb{R}} H^{r,\rho}(\mathbb{R}^d). \tag{1.29}$$

**Proposition 1.44.** Let $a \in S^{m,\mu}$, $m, \mu \in \mathbb{R}$. Then $\mathrm{Op}(a)$ is a linear continuous operator from $H^{r,\rho}(\mathbb{R}^d)$ to $H^{r-m,\rho-\mu}(\mathbb{R}^d)$. Moreover, the map $\mathrm{Op}(a) : H^{r,\rho}(\mathbb{R}^d) \to H^{r',\rho'}(\mathbb{R}^d)$ is compact whenever $r - r' > m$ and $\rho - \rho' > \mu$.

More precisely, the following result holds true.

**Theorem 1.45** ([26, Chap. 3, Theorem 1.1])**.** Let $a \in S^{m,\mu}(\mathbb{R}^d)$, $m, \mu \in \mathbb{R}$. Then, for any $r, \rho \in \mathbb{R}$, $\mathrm{Op}(a) \in \mathscr{L}(H^{r,\rho}(\mathbb{R}^d), H^{r-m,\rho-\mu}(\mathbb{R}^d))$, and there exists a constant $C > 0$, depending only on $d, m, \mu, r, \rho$, such that

$$\|\mathrm{Op}(a)\|_{\mathscr{L}(H^{r,\rho}(\mathbb{R}^d), H^{r-m,\rho-\mu}(\mathbb{R}^d))} \leqslant C |||a|||^{m,\mu}_{\lfloor \frac{d}{2} \rfloor + 1}, \tag{1.30}$$

where $\lfloor t \rfloor$ denotes the integer part of $t \in \mathbb{R}$.

The class $\mathcal{O}(m, \mu)$ of the *operators of order* $m, \mu$ is introduced as follows, see, e.g., [26, Chap. 3, §3].

**Definition 1.46.** A linear continuous operator $A: \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$ belongs to the class $\mathcal{O}(m, \mu)$, $m, \mu \in \mathbb{R}$, of the operators of order $m, \mu$ if, for any $r, \rho \in \mathbb{R}$, it extends to a linear continuous operator $A_{r,\rho}: H^{r,\rho}(\mathbb{R}^d) \to H^{r-m,\rho-\mu}(\mathbb{R}^d)$.

We also define

$$\mathcal{O}(\infty, \infty) = \bigcup_{m,\mu\in\mathbb{R}} \mathcal{O}(m, \mu), \quad \mathcal{O}(-\infty, -\infty) = \bigcap_{m,\mu\in\mathbb{R}} \mathcal{O}(m, \mu).$$

**Remark 1.47.** The following holds true:

(i) any $A \in \mathcal{O}(m,\mu)$ admits a linear continuous extension

$$A_{\infty,\infty}: \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d).$$

In fact, in view of (1.28), it is enough to set

$$A_{\infty,\infty}|_{H^{r,\rho}(\mathbb{R}^d)} = A_{r,\rho};$$

(ii) Theorem 1.45 implies $\mathrm{Op}(S^{m,\mu}(\mathbb{R}^d)) \subset \mathcal{O}(m,\mu)$, $m,\mu \in \mathbb{R}$;

(iii) the spaces $\mathcal{O}(\infty,\infty)$ and $\mathcal{O}(0,0)$ are algebras under operator multiplication, $\mathcal{O}(-\infty,-\infty)$ is an ideal of both $\mathcal{O}(\infty,\infty)$ and $\mathcal{O}(0,0)$, and $\mathcal{O}(m_1,\mu_1) \circ \mathcal{O}(m_2,\mu_2) \subset \mathcal{O}(m_1+m_2,\mu_1+\mu_2)$.

We have also this useful characterization of the class $\mathcal{O}(-\infty,-\infty)$.

**Proposition 1.48** ([26, Ch. 3, Prop. 3.4])**.** The class $\mathcal{O}(-\infty,-\infty)$ coincides with $\mathrm{Op}(S^{-\infty,-\infty}(\mathbb{R}^d))$ and with the class of smoothing operators, that is, the set of all the linear continuous operators $A: \mathscr{S}'(\mathbb{R}^d) \to \mathscr{S}(\mathbb{R}^d)$. All of them coincide with the class of linear continuous operators $A$ admitting a Schwartz kernel $k_A$ belonging to $\mathscr{S}(\mathbb{R}^{2d})$.

The next Theorem 1.49 is a consequence of Theorem 5.1 in [26].

**Theorem 1.49.** Let $m,\mu < 0$ and $a \in S^{m,\mu}(\mathbb{R}^d)$. Then $\mathrm{Op}(a) : H^{s,\sigma} \to H^{s-m,\sigma-\mu}$ is a compact operator.

We now introduce an important concept in the subsequent analysis.

**Definition 1.50.** An operator $A = \mathrm{Op}(a)$ and its symbol $a \in S^{m,\mu}$ are called *elliptic* (or $S^{m,\mu}$*-elliptic* or $m\mu-$*elliptic*) if there exists a constant $R \geqslant 0$ such that

$$C\langle x\rangle^m \langle \xi\rangle^\mu \leqslant |a(x,\xi)|, \qquad |x|+|\xi| \geqslant R, \quad (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d, \tag{1.31}$$

for some constant $C > 0$.

**Proposition 1.51.** Let $A$ be an elliptic SG operator of order $m, \mu$. Then $A$ admits a parametrix $B \in \mathrm{Op}(S^{-m,-\mu})$ such that

$$BA = I + R_1, \quad AB = I + R_2,$$

for suitable $R_1, R_2 \in \mathrm{Op}(S^{-\infty,-\infty})$.

Moreover, if $A$ is elliptic, then it turns out to be a Fredholm operator on the scale of functional spaces $H^{r,\rho}$, $r, \rho \in \mathbb{R}$.

**Proposition 1.52.** If $A \in \mathrm{Op}(S^{m,\mu})$ is elliptic and $B \in \mathrm{Op}(S^{m',\mu'})$ with $m' < m, \mu' < \mu$, then $A + B$ is also elliptic.

Basic examples of SG-elliptic operators are those given by $\Lambda_{r,\rho} = \mathrm{Op}(\lambda_{r,\rho})$, also known as the (standard) *SG order reductions*. Additionally, it is also possible to consider the SG order reductions defined by

$$\Pi_{m,\mu} = \langle D\rangle^{\mu/2}\langle\cdot\rangle^{m}\langle D\rangle^{\mu/2}, \quad m, \mu \in \mathbb{R}.$$

It follows that $\Pi_{m,\mu}$ is self-adjoint and invertible, hence elliptic with constant $R = 0$. Indeed, the operators $\langle D\rangle^{\mu}$ and $\langle\cdot\rangle^{m}$ are self-adjoint, which yields that $\Pi_{m,\mu}$ is self-adjoint as well. Ellipticity is straightforward. Moreover, for $m, \mu > 0$, $u \in H^{m,\mu}$,

$$(\Pi_{m,\mu}u, u) = (\langle\cdot\rangle^{m/2}\langle D\rangle^{\mu/2}u, \langle\cdot\rangle^{m/2}\langle D\rangle^{\mu/2}u) = \|u\|^2_{H^{m/2,\mu/2}} \geqslant 0,$$

which shows that $\Pi_{m,\mu}$ is also positive.

### 1.4.1 The calculus of SG-classical symbols

We present here a subclass of the SG operators, namely the SG-*classical* operators. We begin by giving some definitions of classes of homogeneous functions.

**Definition 1.53.** Let $b \in \mathscr{C}^\infty(\mathbb{R}^d \smallsetminus \{0\})$ be a function, which takes values in a Fréchet space $E$. $b$ is said *positively homogeneous* of degree $z \in \mathbb{C}$ if

$$b(ty) = t^z b(y), \quad \forall t > 0. \tag{1.32}$$

**Definition 1.54.** The space of all positively homogeneous functions on $\mathbb{R}^d \setminus \{0\}$ is denoted by $\mathscr{H}^z(\mathbb{R}^d \setminus \{0\})$.

**Definition 1.55.** (i) The space

$$\mathscr{H}_\xi^\mu(\mathbb{R}^d) = \{b \in \mathscr{C}^\infty(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})) : \\ b(x, t\xi) = t^\mu b(x,\xi) \text{ for all } t > 0\}$$

denotes the space of the smooth functions positively homogeneous of degree $\mu$ with respect to $\xi$.

(ii) The space

$$\mathscr{H}_x^m(\mathbb{R}^d) = \{b \in \mathscr{C}^\infty\left((\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^d\right) : \\ b(tx, \xi) = t^m b(x,\xi) \text{ for all } t > 0\}$$

denotes the space of the smooth functions positively homogeneous of degree $m$ with respect to $x$.

(iii) The space

$$\mathscr{H}_{x,\xi}^{m,\mu}(\mathbb{R}^d \times \mathbb{R}^d) = \{b \in \mathscr{C}^\infty\left((\mathbb{R}^d \setminus \{0\}) \times (\mathbb{R}^d \setminus \{0\})\right) : \\ b(sx, t\xi) = s^m t^\mu b(x,\xi) \text{ for all } s > 0, t > 0\}$$

denotes the space of all smooth functions positively homogeneous of degree $m \in \mathbb{R}$ and $\mu \in \mathbb{R}$ with respect to $x$ and $\xi$, respectively.

We now give the definition of SG symbols which are classical with respect to a variable.

First, recall that a 0-excision function is a real-valued function $\chi \in \mathscr{C}^\infty(\mathbb{R}^d)$ such that $\chi(x) \equiv 0$ in $B_r(0)$, and $\chi(x) \equiv 1$ outside $B_R(0), 0 < r < R$, where $B_r(x_0) = \{x \in \mathbb{R}^d : |x - x_0| < r\}$ denotes the open ball of radius $r$ centered in $x_0$.

**Definition 1.56.** Let $a \in S^{m,\mu}(\mathbb{R}^d \times \mathbb{R}^d)$. Then:

(i) $a$ belongs to the class $S^{m,\mu}_{\mathrm{cl}(x)}(\mathbb{R}^d \times \mathbb{R}^d)$ if there exist $a_{m-i,\cdot}(x,\xi) \in \mathscr{H}_x^{m-i}(\mathbb{R}^d), i \in \mathbb{N}$, homogeneous functions of order $m - i$ with respect to the variable $x$, smooth with respect to the variable $\xi$, such that, for a 0-excision function $\chi$,

$$a(x,\xi) - \sum_{i=0}^{N-1} \chi(x) a_{m-i,\cdot}(x,\xi) \in S^{m-N,\mu}(\mathbb{R}^d \times \mathbb{R}^d), \quad N \in \mathbb{N};$$

(ii) $a$ belongs to the class $S^{m,\mu}_{\mathrm{cl}(\xi)}(\mathbb{R}^d \times \mathbb{R}^d)$ if there exist $a_{\cdot,\mu-k}(x,\xi) \in \mathscr{H}^{\mu-k}_{\xi}(\mathbb{R}^d), k \in \mathbb{N}$, homogeneous functions of order $\mu - k$ with respect to the variable $\xi$, smooth with respect to the variable $x$, such that, for a 0-excision function $\chi$,

$$a(x,\xi) - \sum_{k=0}^{N-1} \chi(\xi) a_{\cdot,\mu-k}(x,\xi) \in S^{m,\mu-N}(\mathbb{R}^d \times \mathbb{R}^d), \quad N \in \mathbb{N}$$

We now give the definition of classical SG symbols.

**Definition 1.57.** A symbol $a$ is SG-classical, and we write $a \in S^{m,\mu}_{\mathrm{cl}(x,\xi)}(\mathbb{R}^d \times \mathbb{R}^d) = S^{m,\mu}_{\mathrm{cl}}(\mathbb{R}^d \times \mathbb{R}^d) = S^{m,\mu}_{\mathrm{cl}}$, if:

(i) there exist $a_{m-j,\cdot}(x,\xi) \in \mathscr{H}^{m-j}_{x}(\mathbb{R}^d)$ such that, for a 0-excision function $\chi$, $\chi(x) a_{m-j,\cdot}(x,\xi) \in S^{m-j,\mu}_{\mathrm{cl}(\xi)}(\mathbb{R}^d \times \mathbb{R}^d)$ and

$$a(x,\xi) - \sum_{j=0}^{N-1} \chi(x) a_{m-j,\cdot}(x,\xi) \in S^{m-N,\mu}(\mathbb{R}^d \times \mathbb{R}^d), \quad N \in \mathbb{N},$$

(ii) there exist $a_{\cdot,\mu-k}(x,\xi) \in \mathscr{H}^{\mu-k}_{\xi}(\mathbb{R}^d)$ such that, for a 0-excision function $\chi$, $\chi(\xi) a_{\cdot,\mu-k}(x,\xi) \in S^{m,\mu-k}_{\mathrm{cl}(x)}(\mathbb{R}^d \times \mathbb{R}^d)$ and

$$a(x,\xi) - \sum_{k=0}^{N-1} \chi(\xi) a_{\cdot,\mu-k}(x,\xi) \in S^{m,\mu-N}(\mathbb{R}^d \times \mathbb{R}^d), \quad N \in \mathbb{N}.$$

Writing

$$(\sigma^{\mu-k}_{\psi} a)(x,\xi) \equiv \chi(\xi) a_{\cdot,\mu-k}(x,\xi)$$

and

$$(\sigma^{m-i}_{e} a)(x,\xi) \equiv \chi(x) a_{m-i,\cdot}(x,\xi),$$

it holds

$$\sigma^{\mu-k}_{\psi} \sigma^{m-i}_{e} a = \sigma^{m-i}_{e} \sigma^{\mu-k}_{\psi} a \equiv \sigma^{m-i,\mu-k}_{\psi,e} a.$$

**Definition 1.58.** The class of all SG-classical symbols is defined as

$$S_{\mathrm{cl(x,\xi)}}(\mathbb{R}^d \times \mathbb{R}^d) = \bigcup_{m,\mu \in \mathbb{R}} S^{m,\mu}_{\mathrm{cl(x,\xi)}}(\mathbb{R}^d \times \mathbb{R}^d).$$

**Proposition 1.59.** Let $a \in S^{m,\mu}_{\mathrm{cl}(\xi,x)}(\mathbb{R}^d \times \mathbb{R}^d)$. Then, the homogeneous components with respect to $\xi$ can be obtained as:

$$a^{\xi}_{\cdot,\mu}(x,\xi) = (\sigma^{\mu}_{\psi} a)(x,\xi) = \lim_{\lambda\to\infty} \lambda^{-\mu} a(x,\lambda\xi);$$
$$a^{\xi}_{\cdot,\mu-1}(x,\xi) = (\sigma^{\mu-1}_{\psi} a)(x,\xi) = \lim_{\lambda\to\infty} \lambda^{-\mu+1}\left[a(x,\lambda\xi) - \chi(x,\lambda\xi) a^{\xi}_{\cdot,\mu}(x,\lambda)\right];$$
$$\vdots$$
$$a^{\xi}_{\cdot,\mu-n}(x,\xi) = (\sigma^{\mu-n} a)(x,\xi)$$
$$= \lim_{\lambda\to\infty} \lambda^{-\mu+n}\left[a(x,\lambda\xi) - \chi(x,\lambda\xi)\sum_{i=0}^{N-1} a^{\xi}_{\cdot,\mu-i}(x,\lambda\xi)\right];$$
$$\vdots$$

Analogous formulae are valid for finding the homogeneous components with respect to $x$ and those homogeneous with respect to both variables.

**Definition 1.60.** The triple $(\sigma_{\psi}(a), \sigma_e(a), \sigma_{\psi,e}(a))$ is called the *principal symbol* of $a$.

### 1.4.2 Eigenvalues asymptotics for elliptic SG - operators

In this section we recall some eigenvalue properties which will be useful in the analysis carried out in Chapter 3.

Consider a classical, positive, self-adjoint, elliptic SG-operator $P \in \mathrm{Op}(S^{m,\mu}_{\mathrm{cl}})$ with order components $m, \mu > 0$. In [78] it has been proved that $N(\lambda) = N_P(\lambda) = |\{\lambda_j \leqslant \lambda : \lambda_j \text{ eigenvalue of } P\}|$, the spectral counting function of the operator $P$, is well-defined and, for $\lambda \to +\infty$,

$$N(\lambda) \sim \begin{cases} C_1\ \lambda^{d/\min\{m,\mu\}}, & m \neq \mu, \\ C_2\ \lambda^{d/m}\log\lambda, & m = \mu. \end{cases} \tag{1.33}$$

The constants $C_1, C_2$ depend on the principal symbol of $P$. Further improvements to the associated Weyl formula for such operators, on manifolds with cylindrical ends and, more generally, on asymptotically Euclidean manifolds, have been subsequently proved in [8, 27, 32].

By the asymptotic behaviour (1.33) of the counting function $N$ it is possibile to obtain, as usual, the asymptotic behaviour of the eigenvalues $\lambda_j$, as showed in the next Theorem 1.61.

**Theorem 1.61.** Let $P \in \mathrm{Op}(S^{m,\mu}_{\mathrm{cl}})$ be a classical, positive, self-adjoint, elliptic SG-operator with order components $m, \mu > 0$. Then, for $j \to \infty$, it follows that there exist constants $\widetilde{C}_1$ and $\widetilde{C}_2$ such that

$$\lambda_j \sim \begin{cases} \widetilde{C}_1\, j^{\min\{m,\mu\}/d}, & m \neq \mu, \\ \widetilde{C}_2 \left( \dfrac{j}{\log j} \right)^{m/d}, & m = \mu. \end{cases} \tag{1.34}$$

*Proof.* The case $m \neq \mu$ follows from analogous argument to Proposition 13.1 in [105]. We then focus here on the case $m = \mu$. Let us first observe that the function $f(\lambda) = (\lambda^{\frac{d}{m}} \log \lambda)^{-1}$ is strictly decreasing on the interval $I = (1, +\infty)$, since $f'(\lambda) < 0$ on $I$. Then, its inverse function $f^{-1}$ is strictly decreasing and continuous as well. By definition of $N(\lambda)$, it follows that, for sufficiently large $j$,

$$(1-\varepsilon)\, C_2\, j^{-1} \leqslant f(\lambda_j) \leqslant (1+\varepsilon)\, C_2\, j^{-1},$$

which, by continuity, implies

$$\begin{aligned} f^{-1}(C_2 j^{-1}) - \widetilde{\varepsilon} \leqslant f^{-1}((1+\varepsilon)\, C_2\, j^{-1}) \leqslant \lambda_j \leqslant \\ \leqslant f^{-1}((1-\varepsilon)\, C_2\, j^{-1}) \leqslant f^{-1}(C_2\, j^{-1}) + \widetilde{\varepsilon}, \end{aligned}$$

that is, $\lambda_j \sim f^{-1}(C_2 j^{-1})$, $j \to \infty$. To conclude, it is enough to prove that, for $t \to 0^+$,

$$f^{-1}(t) \sim h(t) = \left( -\frac{m}{d} t \log t \right)^{-\frac{m}{d}}. \tag{1.35}$$

In fact, (1.35) implies, for $j \to \infty$,

$$\lambda_j \sim f^{-1}(C_2 j^{-1}) \sim \left( \frac{m}{d} C_2 \right)^{-\frac{m}{d}} j^{\frac{m}{d}} (\log j - \log C_2)^{-\frac{m}{d}} \sim \widetilde{C}_2 \left( \frac{j}{\log j} \right)^{\frac{m}{d}},$$

as claimed. To prove (1.35), we compute

$$\begin{aligned}
\lim_{t\to 0^+} \frac{f^{-1}(t)}{h(t)} &= \lim_{\lambda\to+\infty} \frac{f^{-1}(f(\lambda))}{h(f(\lambda))} \\
&= \lim_{\lambda\to+\infty} \frac{\lambda}{\left[-\frac{m}{d}\left(\lambda^{\frac{d}{m}}\log\lambda\right)^{-1}\log\left(\lambda^{\frac{d}{m}}\log\lambda\right)^{-1}\right]^{-\frac{m}{d}}} \\
&= \lim_{\lambda\to+\infty} \frac{\lambda}{\lambda\left[\frac{m}{d}(\log\lambda)^{-1}\left(\frac{d}{m}\log\lambda+\log\log\lambda\right)\right]^{-\frac{m}{d}}} \\
&= \lim_{\lambda\to+\infty}\left(1+\frac{m}{d}\frac{\log\log\lambda}{\log\lambda}\right)^{\frac{m}{d}} = 1,
\end{aligned}$$

which concludes the proof. □

## 1.5 Fourier integral operators of SG type

We present here some properties of a class of Fourier integral operators of SG type on $\mathbb{R}^d$, following [34].

**Definition 1.62.** Let $\varphi \in \mathscr{C}^\infty(\mathbb{R}^d \times (\mathbb{R}^d \smallsetminus 0))$. Then $\varphi$ is called *a phase-function* if:

(i) it is real-valued,

(ii) it is positively 1-homogeneous with respect to $\xi$,

(iii) it satisfies, for all $x, \xi \in \mathbb{R}^d$, $\xi \neq 0$, and $\alpha \in \mathbb{N}^d$

$$\begin{aligned}
|\det \varphi''_{x,\xi}(x,\xi)| &\geqslant C > 0, \\
\partial_x^\alpha \varphi(x,\xi) &\lesssim \langle x\rangle^{1-|\alpha|}|\xi|, \\
\langle \varphi'_\xi(x,\xi)\rangle &\asymp \langle x\rangle, \\
\langle \varphi'_x(x,\xi)\rangle &\asymp \langle \xi\rangle.
\end{aligned} \tag{1.36}$$

The set of all such phase functions will be denoted by $\mathfrak{P}_r^{\mathrm{hom}}$.

**Definition 1.63.** For any $a \in S^{m,\mu}(\mathbb{R}^d \times \mathbb{R}^d)$ and $\varphi \in \mathfrak{P}_r^{\hom}$, the Fourier integral operator $\operatorname{Op}_\varphi(a)$ of type I is defined as the linear and continuous operator from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$ given by

$$(\operatorname{Op}_\varphi(a)f)(x) = \int_{\mathbb{R}^d} e^{i\varphi(x,\xi)} a(x,\xi)\widehat{f}(\xi)\, d\xi, \quad f \in \mathscr{S}(\mathbb{R}^d). \tag{1.37}$$

The next Theorem 1.64 provides the composition properties between Fourier integral operators of SG type and SG pseudo-differential operators.

**Theorem 1.64.** Let $\varphi \in \mathfrak{P}_r^{\hom}$ and assume $a \in S^{s,\sigma}(\mathbb{R}^d \times \mathbb{R}^d)$ and $b \in S^{m,\mu}(\mathbb{R}^d \times \mathbb{R}^d)$. Then,

$$\begin{aligned}
&\operatorname{Op}(a) \circ \operatorname{Op}_\varphi(b) = \operatorname{Op}_\varphi(c_1 + r_1) = \operatorname{Op}_\varphi(c_1) \bmod \operatorname{Op}(S^{-\infty,-\infty}),\\
&\operatorname{Op}_\varphi(a) \circ \operatorname{Op}(p) = \operatorname{Op}_\varphi(c_2 + r_2) = \operatorname{Op}_\varphi(c_2) \bmod \operatorname{Op}(S^{-\infty,-\infty}),
\end{aligned}$$

for some $c_j \in S^{m+s,\mu+\sigma}(\mathbb{R}^d \times \mathbb{R}^d)$, $r_j \in S^{-\infty,-\infty}(\mathbb{R}^d \times \mathbb{R}^d)$, $j = 1, 2$.

## 1.6 Global wave-front set

In this section we recall some aspects of the theory of global wave-front sets developed in [29]. First, we introduce the following definition of type-$k$ invertibility (see Definition 1.10 in [29]).

**Definition 1.65.** Let $a \in S^{m,\mu}(\mathbb{R}^d)$. Then,

(i) $a$ is called locally or type-1 invertible with respect to $(m,\mu)$ at the point $(x_0,\xi_0) \in \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\})$, if there exist a neighborhood $X$ of $x_0$, an open conical neighborhood $\Gamma$ of $\xi_0$ and a positive constant $R$ such that

$$|a(x,\xi)| \gtrsim \langle x\rangle^m \langle\xi\rangle^\mu$$

for $x \in X, \xi \in \Gamma$ and $|\xi| \geqslant R$.

(ii) $a$ is called Fourier-locally or type-2 invertible with respect to $(m,\mu)$ at the point $(x_0,\xi_0) \in (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^d$, if there exist an open conical neighborhood $\Gamma$ of $x_0$, a neighborhood $X$ of $\xi_0$ and a positive constant $R$ such that

$$|a(x,\xi)| \gtrsim \langle x\rangle^m \langle \xi\rangle^\mu$$

for $x \in \Gamma, |x| \geqslant R$ and $\xi \in X$.

(iii) $a$ is called oscillating or type-3 invertible with respect to $(m,\mu)$ at the point $(x_0,\xi_0) \in (\mathbb{R}^d \smallsetminus \{0\}) \times (\mathbb{R}^d \smallsetminus \{0\})$, if there exist open conical neighborhoods $\Gamma_1$ of $x_0$ and $\Gamma_2$ of $\xi_0$, and a positive constant $R$ such that

$$|a(x,\xi)| \gtrsim \langle x\rangle^m \langle \xi\rangle^\mu$$

for $x \in \Gamma_1, |x| \geqslant R, \xi \in \Gamma_2$ and $|\xi| \geqslant R$.

**Definition 1.66.** If $a$ is not type-$k$ invertible at $(x_0,\xi_0)$, then the point $(x_0,\xi_0)$ is called type-$k$ characteristic for $a$ with respect to $(m,\mu)$. The set of type-$k$ characteristic points for $a$ is denoted by $\mathrm{Char}^k(a)$.

The global set of characteristic points, namely the characteristic set, for a symbol $a \in S^{m,\mu}(\mathbb{R}^d)$ is

$$\mathrm{Char}(a) = \mathrm{Char}^1(a) \cup \mathrm{Char}^2(a) \cup \mathrm{Char}^3(a)$$

We now introduce two classes of cut-off functions.

**Definition 1.67.** Let $X \subseteq \mathbb{R}^d$ be an open set, $\Gamma \subseteq \mathbb{R}^d \smallsetminus \{0\}$ be an open cone and consider $x_0 \in X$ and $\xi_0 \in \Gamma$.

(i) A function $\varphi \in \mathscr{C}^\infty(\mathbb{R}^d)$ is called a cut-off function with respect to $x_0$ and $X$, if $0 \leqslant \varphi \leqslant 1$, $\varphi \in \mathscr{C}_0^\infty(X)$ and $\varphi = 1$ in an open neighborhood of $x_0$.

(ii) A function $\psi \in \mathscr{C}^\infty(\mathbb{R}^d)$ is called a directional cut-off function with respect to $\xi_0$ and $\Gamma$, if there is a constant $R > 0$ and open conical neighborhood $\Gamma_1 \subseteq \Gamma$ of $\xi_0$ such that the following is true:

(a) $0 \leqslant \psi \leqslant 1$ and $\mathrm{supp}(\psi) \subseteq \Gamma$;

(b) $\psi(t\xi) = \psi(\xi)$ when $t \geqslant 1$ and $|\xi| \geqslant R$;

(c) $\psi(\xi) = 1$ when $\xi \in \Gamma_1$ and $|\xi| \geqslant R$.

We have the following property for the cut-off functions.

**Proposition 1.68.** Let $X \subseteq \mathbb{R}^d$ be an open set and let $\Gamma, \Gamma_1, \Gamma_2 \subseteq \mathbb{R}^d \setminus \{0\}$ be open cones. Then we have the following.

(i) Pick $x_0 \in X, \xi_0 \in \Gamma$ and let $\varphi$ be a cut-off function with respect to $x_0$ and $X$ and let $\psi$ be a directional cut-off function with respect to $\xi_0$ and $\Gamma$. Then, the function $c_1 = \varphi \otimes \psi$ belongs to $S^{0,0}(\mathbb{R}^d)$, and it is type-1 invertible at $(x_0, \xi_0)$.

(ii) Pick $x_0 \in \Gamma, \xi_0 \in X$ and let $\psi$ be a directional cut-off function with respect to $x_0$ and $\Gamma$ and let $\varphi$ be a cut-off function with respect to $\xi_0$ and $X$. Then, the function $c_2 = \psi \otimes \varphi$ belongs to $S^{0,0}(\mathbb{R}^d)$, and it is type-2 invertible at $(x_0, \xi_0)$.

(iii) Pick $x_0 \in \Gamma_1, \xi_0 \in \Gamma_2$ and let $\psi_1$ be a directional cut-off function with respect to $x_0$ and $\Gamma_1$ and let $\psi_2$ be a directional cut-off function with respect to $\xi_0$ and $\Gamma_2$. Then, the function $c_3 = \psi_1 \otimes \psi_2$ belongs to $S^{0,0}(\mathbb{R}^d)$, and it is type-3 invertible at $(x_0, \xi_0)$.

We also introduce the following convenient notation:

$$\begin{gathered}\Omega_1 = \mathbb{R}^d \times \left(\mathbb{R}^d \setminus \{0\}\right), \quad \Omega_2 = \left(\mathbb{R}^d \setminus \{0\}\right) \times \mathbb{R}^d, \\ \Omega_3 = \left(\mathbb{R}^d \setminus \{0\}\right) \times \left(\mathbb{R}^d \setminus \{0\}\right).\end{gathered} \tag{1.38}$$

We can now give the definition of global wave-front set (see Definition 2.1 in [29]).

**Definition 1.69.** Let $f \in \mathscr{S}'(\mathbb{R}^d)$, let $\mathcal{B}$ a Banach or Fréchet space such that $\mathscr{S}(\mathbb{R}^d) \subseteq \mathcal{B} \subseteq \mathscr{S}'(\mathbb{R}^d)$ and denote by $\Theta^k_{\mathcal{B}}(f)$ the set of all type-$k$ regular points for $f$, that is the set of points $(x_0, \xi_0) \in \Omega_k$ for which $\mathrm{Op}(c_k) f \in \mathcal{B}$, for some $c_k$ as in Proposition 1.68. We then define:

(i) the type-$k$ wave-front set of $f \in \mathscr{S}'(\mathbb{R}^d)$ with respect to $\mathcal{B}$ as the complement of $\Theta^k_{\mathcal{B}}(f)$ in $\Omega_k$, which is denoted by $\mathrm{WF}^k_{\mathcal{B}}(f)$;

(ii) the global wave-front set $\mathrm{WF}_{\mathcal{B}}(f) \subseteq (\mathbb{R}^d \times \mathbb{R}^d) \setminus 0$ as the set

$$\mathrm{WF}_{\mathcal{B}}(f) \equiv \mathrm{WF}^1_{\mathcal{B}}(f) \cup \mathrm{WF}^2_{\mathcal{B}}(f) \cup \mathrm{WF}^3_{\mathcal{B})}(f) \tag{1.39}$$

The sets $\mathrm{WF}^1_{\mathcal{B}}(f), \mathrm{WF}^2_{\mathcal{B}}(f), \mathrm{WF}^3_{\mathcal{B}}(f)$ are called *local*, *Fourier-local* and *oscillating* wave-front set of $f$ with respect to $\mathcal{B}$, respectively.

### 1.6.1 Propagation results for global wave-front sets

We recall here propagation results about global wave-front sets which will be further investigated in the context of Orlicz spaces in Chapter 2. We follow [29] (see also [30, 31] for the general theory).

**Definition 1.70.** Let $t \in \mathbb{R}, \mathcal{B}$ be a topological vector space of distributions on $\mathbb{R}^d$ such that

$$\mathscr{S}\left(\mathbb{R}^d\right) \subseteq \mathcal{B} \subseteq \mathscr{S}'\left(\mathbb{R}^d\right)$$

with continuous embeddings. Then $\mathcal{B}$ is called *SG-admissible* if $\mathrm{Op}_t(a)$ maps $\mathcal{B}$ continuously into itself, for every $a \in S^{0,0}(\mathbb{R}^d)$.If $\mathcal{B}$ and $\mathcal{C}$ are SG-admissible, then the pair $(\mathcal{B}, \mathcal{C})$ is called *SG-ordered* if the maps

$$\mathrm{Op}_t(a) : \mathcal{B} \to \mathcal{C} \quad \text{and} \quad \mathrm{Op}_t(b) : \mathcal{C} \to \mathcal{B}$$

are continuous for every $a \in S^{m,\mu}(\mathbb{R}^d)$ and $b \in S^{-m,-\mu}(\mathbb{R}^d)$.

**Proposition 1.71.** Let $\mathcal{B}$ be SG-admissible, and let $f \in \mathscr{S}'(\mathbb{R}^d)$. Then

$$f \in \mathcal{B} \iff \mathrm{WF}_{\mathcal{B}}(f) = \varnothing.$$

**Proposition 1.72.** Let $k \in \{1,2,3\}, t \in \mathbb{R}$, $a \in S^{m,\mu}(\mathbb{R}^d)$ be SG-elliptic and let $f \in \mathscr{S}'(\mathbb{R}^d)$. Moreover, let $(\mathcal{B}, \mathcal{C})$ be SG-ordered. Then

$$\mathrm{WF}^k_{\mathcal{C}}(\mathrm{Op}_t(a) f) = \mathrm{WF}^k_{\mathcal{B}}(f)$$

**Definition 1.73.** Let $\varphi \in \mathfrak{P}^{\mathrm{hom}}_r$ be a phase function. The canonical transformation, denoted by $\phi$, of the phase space $T^*\mathbb{R}^d$ into itself generated by $\varphi$ is defined by the relations

$$(x, \xi) = \phi(y, \eta) \iff \begin{cases} y = \varphi'_\xi(x, \eta) = \varphi'_\eta(x, \eta), \\ \xi = \varphi'_x(x, \eta), \end{cases} \tag{1.40}$$

**Definition 1.74.** Let $\varphi \in \mathfrak{P}_r^{\mathrm{hom}}$ and let $\phi$ be the canonical transformation generated by $\varphi$. Let $k \in \{1,2,3\}$ and $\Omega_k$ defined by

$$\begin{aligned} \Omega_1 &\equiv \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}), \\ \Omega_2 &\equiv (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^d, \\ \Omega_3 &\equiv (\mathbb{R}^d \setminus \{0\}) \times (\mathbb{R}^d \setminus \{0\}). \end{aligned} \tag{1.41}$$

For any $R > 0$ and $k \in \{1,2,3\}$, we also set

$$\begin{aligned} \Omega_{1,R} &\equiv \{(x,\xi) \in \Omega_1 : |\xi| \geqslant R\}, \\ \Omega_{2,R} &\equiv \{(x,\xi) \in \Omega_2 : |x| \geqslant R\}, \\ \Omega_{3,R} &\equiv \{(x,\xi) \in \Omega_3 : |x|, |\xi| \geqslant R\}. \end{aligned} \tag{1.42}$$

Then,

(i) $\varphi$ is called 1-admissible at $(y_0,\eta_0) \in \Omega_1$ if, for every 1-cone $X \times \Gamma$ containing $\phi(y_0,\eta_0)$ and $r > 0$, there is a 1-cone $Y \times \Gamma_0$ containing $(y_0,\eta_0)$ and $R > 0$ such that

$$\phi(y,\eta) \in (X \times \Gamma) \cap \Omega_{1,r} \quad \text{when} \quad (y,\eta) \in (Y \times \Gamma_0) \cap \Omega_{1,R}.$$

(ii) $\varphi$ is called 2-admissible at $(y_0,\eta_0) \in \Omega_2$ if, for every 2-cone $\Gamma \times X$ containing $\phi(y_0,\eta_0)$ and $r > 0$, there is a 2-cone $\Gamma_0 \times Y$ containing $(y_0,\eta_0)$ and $R > 0$ such that

$$\phi(y,\eta) \in (\Gamma \times X) \cap \Omega_{2,r} \quad \text{when} \quad (y,\eta) \in (\Gamma_0 \times Y) \cap \Omega_{2,R}.$$

(iii) $\varphi$ is called 3-admissible at $(y_0,\eta_0) \in \Omega_3$ if, for every 3-cone $\Gamma_1 \times \Gamma_2$ containing $\phi(y_0,\eta_0)$ and $r > 0$, there is a 3-cone $\Gamma_{0,1} \times \Gamma_{0,2}$ containing $(y_0,\eta_0)$ and $R > 0$ such that

$$\phi(y,\eta) \in (\Gamma_1 \times \Gamma_2) \cap \Omega_{3,r} \quad \text{when} \quad (y,\eta) \in (\Gamma_{0,1} \times \Gamma_{0,2}) \cap \Omega_{3,R}.$$

Furthermore, $\varphi$ is called *k-admissible* if it is $k$-admissible at all points $(y,\eta) \in \Omega_k$, and $\varphi$ is called *admissible* if it is $k$-admissible for all $k = 1,2,3$.

**Definition 1.75.** Let $\varphi \in S^{1,1}(\mathbb{R}^d)$ be a regular phase function, and $\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{C}, \mathcal{C}_1, \mathcal{C}_2$, be SG-admissible. Let also $\Omega \subseteq \mathbb{R}^d$ be open. Then the pair $(\mathcal{B}, \mathcal{C})$ is called *weakly-I SG-ordered* if the mapping

$$\mathrm{Op}_\varphi(a) : \mathcal{B} \to \mathcal{C}$$

is continuous for every $a \in S^{m,\mu}(\mathbb{R}^d)$ which is supported outside $\mathbb{R}^d \times \Omega$. Similarly, the pair $(\mathcal{B}, \mathcal{C})$ is called *weakly-II SG-ordered* if the mapping

$$\mathrm{Op}^*_\varphi(b) : \mathcal{C} \to \mathcal{B}$$

is continuous for every $b \in S^{m,\mu}(\mathbb{R}^d)$ which is supported outside $\Omega \times \mathbb{R}^d$. Furthermore, $(\mathcal{B}_1, \mathcal{C}_1, \mathcal{B}_2, \mathcal{C}_2)$ are called *SG-ordered* if $(\mathcal{B}_1, \mathcal{C}_1)$ is a weakly-I SG-ordered pair and $(\mathcal{B}_2, \mathcal{C}_2)$ is a weakly-II SG-ordered pair.

**Theorem 1.76.** Let $\varphi \in \mathfrak{P}_r^{\mathrm{hom}}$ be $k$-admissible, $k \in \{1,2,3\}$, $a \in S^{m,\mu}(\mathbb{R}^d)$, supported outside $(\mathbb{R}^d \times \Omega), \Omega \subseteq \mathbb{R}^d$ open. Denote by $\phi_1 : x \mapsto (\varphi'_\xi)^{-1}(x,\xi)$ and $\phi_2 : \xi \mapsto (\varphi'_x)^{-1}(x,\xi)$ and assume that

$$\langle \phi_1(x, \eta_1 + \eta_2) \rangle^m \langle \xi \rangle^\mu \lesssim \langle \phi_1(x, \eta_1) \rangle^m \langle \xi \rangle^\mu \tag{1.43}$$

for any $x, \xi, \eta_1, \eta_2 \in \mathbb{R}^d$, uniformly with respect to $\eta_2 \in \mathbb{R}^d$, and

$$\langle x \rangle^m \langle \phi_2(\xi, \eta_1 + \eta_2) \rangle^\mu \lesssim \langle x \rangle^m \langle \phi_2(\xi, \eta_1) \rangle^\mu \tag{1.44}$$

for any $x, \xi, \eta_1, \eta_2 \in \mathbb{R}^d$, uniformly with respect to $\eta_2 \in \mathbb{R}^d$. Assume also that $a$ is SG-elliptic and $(\mathcal{B}, \mathcal{C})$ is a weakly-I SG-ordered pair with respect to $(r, \rho, \omega_0, \varphi, \Omega)$. Then,

$$\mathrm{WF}^k_{\mathcal{C}}\left(\mathrm{Op}_\varphi(a) f\right) = \phi\left(\mathrm{WF}^k_{\mathcal{B}}(f)\right), \quad f \in \mathscr{S}'\left(\mathbb{R}^d\right)$$

where $\phi$ is the canonical transformation generated by $\varphi$.

# Chapter 2

# Continuity of pseudo-differential operators and Fourier integral operators on Orlicz spaces

The results presented in this chapter have been published in [19].

## 2.1 Orlicz Spaces

This first section of this chapter is devoted to recalling the notion of Orlicz space and provide some preliminary results which are needed in order to state our continuity theorems with respect to classes of pseudo-differential operators and Fourier integral operators acting on Orlicz spaces.

### 2.1.1 Young functions

The first fundamental concept we need here is the notion of Young function. (See [50, 93]).

**Definition 2.1.** A function $\Phi : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is called *convex* if it satisfies

$$\Phi\left(\theta t_1 + (1-\theta)\, t_2\right) \leqslant \theta\, \Phi(t_1) + (1-\theta)\, \Phi(t_2)$$

whenever $t_j \in \mathbb{R}$ and $\theta \in [0,1]$.

It is worth remarking that $\Phi$ might not be continuous, because the previous Definition 2.1 allows $\infty$ as function value. For example, for any $a, c \in \mathbb{R}$, the function

$$\Phi(t) = \begin{cases} c, & \text{when } t \leqslant a \\ \infty, & \text{when } t > a \end{cases}$$

is convex but discontinuous at the point $t = a$.

**Definition 2.2.** Let $\Phi$ be a function from $[0,\infty)$ to $[0,\infty]$. Then $\Phi$ is called a *Young function* if it satisfies the following properties:

(i) $\Phi$ is convex,

(ii) $\Phi(0) = 0$,

(iii) $\Phi(t) < \infty$ for some $t > 0$,

(iv) $\lim_{t\to\infty} \Phi(t) = +\infty$.

We highlight the fact that point (iii) has been introduced in order to avoid the function

$$\Phi_1(t) = \begin{cases} 0, & \text{when } t = 0 \\ \infty, & \text{when } t > 0 \end{cases}$$

to be considered a Young function.

**Remark 2.3.** Notice that, according to the previous Definition 2.2, $\Phi$ is non-decreasing. In fact, if we take $0 \leqslant t_1 \leqslant t_2$ and choose $\theta \in [0,1]$ such that $t_1 = \theta\, t_2$, then

$$\Phi(t_1) = \Phi(\theta\, t_2 + (1-\theta)\, 0) \leqslant \theta\, \Phi(t_2) + (1-\theta)\, \Phi(0) \leqslant \Phi(t_2),$$

since $\Phi(0) = 0$ and $\theta \in [0,1]$.

We now introduce the definition of *conjugate Young function* as well as some conditions which will be important in the sequel.

**Definition 2.4.** For any Young function, the *conjugate Young function* is the function from $[0,\infty)$ to $[0,\infty]$, given by

$$\Phi^*(t) = \sup_{s>0} \left( st - \Phi(s) \right). \tag{2.1}$$

**Remark 2.5.** Notice that $\Phi^*$ is also a Young function. Indeed,

$$\begin{aligned} \Phi^*\left(\theta\, t_1 + (1-\theta)\, t_2\right) &= \sup_{s>0} \left( s\left(\theta\, t_1 + (1-\theta)\, t_2\right) - \Phi(s) \right) \\ &= \sup_{s>0} \left( \theta\left(st_1 - \Phi(s)\right) + (1-\theta)\left(st_2 - \Phi(s)\right) \right) \\ &\leqslant \theta\, \sup_{s>0} \left( st_1 - \Phi(s) \right) + (1-\theta)\, \sup_{s>0} \left( st_2 - \Phi(s) \right) \\ &= \theta\, \Phi^*(t_1) + (1-\theta)\, \Phi^*(t_2) \end{aligned}$$

which shows that $\Phi^*$ is convex. The proof of the remaining properties is immediate.

**Definition 2.6.** The Young functions $\Phi_1$ and $\Phi_2$ are called *equivalent*, if there is a constant $C \geqslant 1$ such that

$$C^{-1}\Phi_2(t) \leqslant \Phi_1(t) \leqslant C\Phi_2(t), \quad t \in [0,\infty). \tag{2.2}$$

**Definition 2.7.** A Young function is said to fulfill the $\Delta_2$-*condition* if there exists a constant $C \geqslant 1$ such that

$$\Phi(2t) \leqslant C\Phi(t), \qquad t \in [0,\infty). \tag{2.3}$$

It is clear that if $\Phi$ satisfies the $\Delta_2$-condition, then $\Phi(t) < \infty$ when $t \geqslant 0$.

**Definition 2.8.** A Young function is said to fulfill the $\Lambda$-*condition* if there exists a $p > 1$ such that

$$\Phi(ct) \leqslant c^p \Phi(t), \qquad t \in [0, \infty),\ c \in (0, 1]. \tag{2.4}$$

The $\Lambda$-condition is also called the *lower Matuszewska-Orlicz index* in the literature (cf. p. 117 in [61]).

The following characterization of Young functions fulfilling the $\Delta_2$-condition follows from the fact that any Young function is increasing, that is, $\Phi(t_1) \leqslant \Phi(t_2)$ when $t_1 \leqslant t_2$.

**Proposition 2.9.** Let $\Phi$ be a Young function. Then the following conditions are equivalent:

(i) $\Phi$ satisfies the $\Delta_2$-condition;

(ii) for every constant $c > 0$, the Young function $t \mapsto \Phi(ct)$ is equivalent to $\Phi$;

(iii) for some constant $c > 0$ with $c \neq 1$, the Young function $t \mapsto \Phi(ct)$ is equivalent to $\Phi$.

*Proof.* Since any Young function is increasing, that is, $\Phi(t_1) \leqslant \Phi(t_2)$ when $t_1 \leqslant t_2$, it immediately follows that (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii). It remains to prove that (iii) $\Rightarrow$ (ii).

First, notice that if the Young function $t \mapsto \Phi(ct)$ is equivalent to $\Phi$, then also $t \mapsto \Phi(c^{-1}t)$ is equivalent to $\Phi$. Hence it is not restrictive to assume $c > 1$ and it is sufficient to prove the result only for $t \mapsto \Phi(c_1 t)$ when $c_1 > 1$.

If $c_1 < c$, it follows that there exists $C > 1$ such that

$$C^{-1}\Phi(t) \leqslant \Phi(t) \leqslant \Phi(c_1 t) \leqslant \Phi(ct) \leqslant C\Phi(t)$$

showing that the functions $t \mapsto \Phi(c_1 t)$ and $\Phi$ are equivalent. For $c_1 > c$, choose $n \in \mathbb{N}$ such that $c^n > c_1$. Then,

$$C^{-n}\Phi(t) \leqslant \Phi(t) \leqslant \Phi(ct) \leqslant \Phi(c_1 t) \leqslant \Phi(c^n t) \leqslant C^n\Phi(t)$$

showing again that the functions $t \mapsto \Phi(c_1 t)$ and $\Phi$ are equivalent. □

We introduce now the Lebesgue exponents related to a Young function which will give a useful characterization of the corresponding Orlicz spaces. First, denote by $\Phi'_+$ and $\Phi'_-$ the right and left derivatives of $\Phi$ and set

$$\Phi'_+(t_1) = \Phi'_-(t_2) = \infty \quad \text{when} \quad t_2 > t_1 \geqslant t_0 \equiv \sup\{\, t \geqslant 0 \,:\, \Phi(t) < \infty \,\}.$$

Evidently, if the left limit $\Phi(t_0^-)$ of $\Phi$ at $t_0$ is finite, then $\Phi'_-(t_0)$ is well-defined and finite. If instead $\Phi(t_0^-) = \infty$, then we let

$$\Phi'_-(t_0) = \infty \qquad \left( = \lim_{h \to 0^-} \frac{\Phi(t_0 + h) - \Phi(t_0^-)}{h} \right).$$

respectively.

**Definition 2.10.** For any Young function $\Phi$, let $\Omega$ be the set

$$\Omega = \Omega_\Phi \equiv \{\, t > 0 \,:\, 0 < \Phi(t) < \infty \,\}. \tag{2.5}$$

Then the upper and lower Lebesgue exponents $p_\Phi$ and $q_\Phi$ are defined as

$$p_\Phi \equiv \begin{cases} \sup\limits_{t \in \Omega} \left( \dfrac{t\Phi'_\pm(t)}{\Phi(t)} \right), & \Omega = \mathbb{R}_+, \\ \infty, & \Omega \neq \mathbb{R}_+, \end{cases} \tag{2.6}$$

where $\mathbb{R}_+ = (0, \infty)$, and

$$q_\Phi \equiv \begin{cases} \inf\limits_{t \in \Omega} \left( \dfrac{t\Phi'_\pm(t)}{\Phi(t)} \right), & \Omega \neq \varnothing, \\ \infty, & \Omega = \varnothing. \end{cases} \tag{2.7}$$

These exponents are essential in several investigations of Orlicz spaces and are often called the Simonenko indices (see e. g. [75, 76, 72]).

Notice that, since $\Phi$ is convex, the right and left derivatives $\Phi'_+$ and $\Phi'_-$ are well-defined on $\Omega$.

**Remark 2.11.** It is possible to equivalently define the exponent $p_\Phi$ so that

$$p_\Phi = \begin{cases} \sup\limits_{t\in\Omega} \left( \dfrac{t\Phi'_\pm(t)}{\Phi(t)} \right), & |\Omega| = \infty, \\ \infty, & |\Omega| < \infty. \end{cases} \tag{2.6$'$}$$

**Proposition 2.12.** When $p_\Phi < \infty$, we observe that for any $r_1, r_2 > 0$,

$$t^{p_\Phi} \lesssim \Phi(t) \lesssim t^{q_\Phi} \quad \text{when} \quad t \leqslant r_1 \tag{2.8}$$

and

$$t^{q_\Phi} \lesssim \Phi(t) \lesssim t^{p_\Phi} \quad \text{when} \quad t \geqslant r_2. \tag{2.9}$$

*Proof.* The desired inequalities follow by an argument which will be employed also in the sequel.

By (2.6) we obtain that

$$\frac{t\Phi'_+(t)}{\Phi(t)} - p_\Phi \leqslant 0 \qquad \Longleftrightarrow \qquad \left( \frac{\Phi(t)}{t^{p_\Phi}} \right)'_+ \leqslant 0.$$

Hence $\Phi(t) = t^{p_\Phi} h(t)$ for some decreasing function $h(t) > 0$. This gives

$$\Phi(t) = t^{p_\Phi} h(t) \geqslant t^{p_\Phi} h(r_1) \gtrsim t^{p_\Phi}$$

for $t \leqslant r_1$ and

$$\Phi(t) = t^{p_\Phi} h(t) \leqslant t^{p_\Phi} h(r_2) \lesssim t^{p_\Phi}$$

for $t \geqslant r_2$. This shows the relations between $t^{p_\Phi}$ and $\Phi(t)$ in (2.8) and (2.9). The remaining relations concerning $t^{q_\Phi}$ and $\Phi(t)$ follow by the same argument in analogous way. □

In our investigations below we need to assume that our Young functions are *strict* in the sense of the following Definition 2.13.

**Definition 2.13.** The Young function $\Phi$ from $[0, \infty)$ to $[0, \infty]$ is called *strict* or a *strict Young function*, if

(i) $\Phi(t) < \infty$ for every $t \in [0, \infty)$,

(ii) $\Phi$ satisfies the $\Delta_2$-condition,

(iii) $\Phi$ satisfies the $\Lambda$-condition.

In what follows, we characterize conditions (ii) and (iii) in Definition 2.13 in various ways. In particular we show that (ii) and (iii) in Definition 2.13 are equivalent to $p_\Phi < \infty$ and $q_\Phi > 1$, respectively (see Proposition 2.24).

It will also be useful to rely on regular Young functions, which is possible due to the following Proposition 2.14.

**Proposition 2.14.** Let $\Phi$ be a Young function which satisfies the $\Delta_2$-condition. For every $c \in (0,1)$, there is a Young function $\Psi$ such that the following is true:

(i) $\Psi$ is equivalent to $\Phi$ and $\Psi \leqslant \Phi$;

(ii) $\Psi$ is smooth on $\mathbb{R}_+$;

(iii) $c\,\Phi'_+(0) \leqslant \Psi'_+(0) \leqslant \Phi'_+(0)$;

(iv) $q_\Phi \leqslant q_\Psi$.

*Proof.* Let $\phi \in \mathscr{C}_0^\infty([0,1])$ be such that $\phi \geqslant 0$ and $\int_0^1 \phi(s)\,ds = 1$. Set

$$\Psi(t) = \int_0^1 \Phi(t - \tfrac{1}{2}st)\phi(s)\,ds. \tag{2.10}$$

By straightforward computations it follows that $\Psi$ is a Young function. Let $C \geqslant 1$ be as in (2.3). Since $\Phi$ is increasing we obtain that

$$\frac{1}{C}\Phi(t) \leqslant \Phi(\tfrac{1}{2}t) \leqslant \int_0^1 \Phi(t - \tfrac{1}{2}st)\phi(s)\,ds = \Psi(t)$$

and

$$\Psi(t) = \int_0^1 \Phi(t - \tfrac{1}{2}st)\phi(s)\,ds \leqslant \int_0^1 \Phi(t)\phi(s)\,ds = \Phi(t),$$

which shows (i).

By (2.10) we obtain that

$$\Psi(t) = \int_{t/2}^{t} \Phi(s)\phi(2-2s/t)\frac{2}{t}\,ds,$$

which is evidently smooth on $\mathbb{R}_+$. This gives (ii).

By differentiating (2.10) we obtain

$$\Psi'_+(t) = \int_0^1 \Phi'_+(t-\tfrac{1}{2}st)(1-\tfrac{1}{2}s)\phi(s)\,ds \leqslant \int_0^1 \Phi'_+(t)\phi(s)\,ds = \Phi'_+(t).$$

On the other hand, by choosing the support of $\phi$ to be sufficiently close to the origin, we obtain

$$\Psi'_+(0) = \int_0^1 \Phi'_+(0)(1-\tfrac{1}{2}s)\phi(s)\,ds \geqslant c\Phi'_+(0),$$

and (iii) follows.

By (2.10) we also get

$$\begin{aligned} t\Psi'_+(t) &= \int_0^1 (t-\tfrac{1}{2}st)\Phi'_+(t-\tfrac{1}{2}st)\phi(s)\,ds \geqslant \\ &\geqslant q_\Phi \int_0^1 \Phi(t-\tfrac{1}{2}st)\phi(s)\,ds = q_\Phi\Psi(t), \end{aligned}$$

which implies

$$q_\Phi \leqslant \frac{t\Psi'_+(t)}{\Psi(t)}. \tag{2.11}$$

The assertion (iv) now follows by taking the infimum of the right-hand side of (2.11). □

**Remark 2.15.** It follows that $\Psi$ in Proposition 2.14 fulfills the $\Delta_2$-condition, since $\Phi$ does and $\Psi$ is equivalent to $\Phi$.

### 2.1.2 Orlicz spaces

We can now give the definition of Orlicz spaces with respect to Young functions. We will also highlight the analogies and relations with the usual Lebesgue spaces.

First, we recall the definition of weak $L^p$ spaces.

**Definition 2.16.** Let $p \in (0, \infty]$. The *weak $L^p$ space* $wL^p(\mathbb{R}^d)$ consists of all Lebesgue measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ for which

$$\|f\|_{wL^p} \equiv \sup_{t>0} t\,(\mu_f(t))^{\frac{1}{p}} \tag{2.12}$$

is finite. Here $\mu_f(t)$ is the Lebesgue measure of the level set $\{\, x \in \mathbb{R}^d : |f(x)| > t \,\}$.

**Remark 2.17.** Notice that the $wL^p$-norm is not a true norm, since the triangular inequality fails. Nevertheless, one has that $\|f\|_{wL^p} \leqslant \|f\|_{L^p}$. In particular, $L^p(\mathbb{R}^d)$ is continuously embedded in $wL^p(\mathbb{R}^d)$. We will adopt this slight abuse of notation in the sequel.

**Definition 2.18.** Let $\Phi$ be a Young function. Then, the *Orlicz space* $L^\Phi(\mathbb{R}^d)$ consists of all Lebesgue measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that

$$\|f\|_{L^\Phi} \equiv \inf \left\{ \lambda > 0 \, : \, \int_{\mathbb{R}^d} \Phi\left( \frac{|f(x)|}{\lambda} \right) dx \leqslant 1 \right\}$$

is finite.

**Definition 2.19.** Let $\Phi$ be a Young function. The *weak Orlicz space* $wL^\Phi(\mathbb{R}^d)$ consists of all Lebesgue measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that

$$\|f\|_{wL^\Phi} \equiv \inf \left\{ \lambda > 0 \, : \, \sup_{t>0} \left( \Phi\left( \frac{t}{\lambda} \right) \mu_f(t) \right) \leqslant 1 \right\}$$

is finite. Here $\mu_f(t)$ denotes the Lebesgue measure of the set $\{\, x \in \mathbb{R}^d : |f(x)| > t \,\}$.

As with the usual Lebesgue spaces, $f, g \in wL^\Phi(\mathbb{R}^d)$ are equivalent whenever $f = g$ a. e.

**Remark 2.20.** Suppose that $p \in [1, \infty]$. Then, $L^{\Phi_{[p]}}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$, when

$$\Phi_{[p]}(t) = \frac{t^p}{p}, \; p < \infty \quad \text{and} \quad \Phi_{[\infty]}(t) = \begin{cases} 0, & 0 \leqslant t \leqslant 1, \\ \infty, & t > 1. \end{cases}$$

Moreover, let $p_1, p_2 \in [1, \infty]$ and

$$\Phi(t) = \begin{cases} \frac{t^{p_2}}{p_2}, & 0 \leqslant t \leqslant 1, \\ \frac{t^{p_1}}{p_1} + \frac{1}{p_2} - \frac{1}{p_1}, & t > 1, \end{cases}$$

where $\frac{t^\infty}{\infty}$ is interpreted as

$$\frac{t^\infty}{\infty} \equiv \lim_{p \to \infty} \frac{t^p}{p} = \begin{cases} 0, & 0 \leqslant t \leqslant 1, \\ \infty, & t > 1. \end{cases}$$

Then $\Phi$ is a Young function and it turns out that

$$L^\Phi(\mathbb{R}^d) = L^{p_1}(\mathbb{R}^d) + L^{p_2}(\mathbb{R}^d), \quad p_1 \leqslant p_2,$$

and

$$L^\Phi(\mathbb{R}^d) = L^{p_1}(\mathbb{R}^d) \cap L^{p_2}(\mathbb{R}^d), \quad p_2 \leqslant p_1.$$

In particular, for a general Young function $\Phi$, it follows from (2.8) and (2.9) that

$$L^{p_\Phi}(\mathbb{R}^d) \cap L^{q_\Phi}(\mathbb{R}^d) \subseteq L^\Phi(\mathbb{R}^d) \subseteq L^{p_\Phi}(\mathbb{R}^d) + L^{q_\Phi}(\mathbb{R}^d). \tag{2.13}$$

**Remark 2.21.** Notice that the assignment $p_\Phi = \infty$ when $|\Omega| < \infty$ in (2.6)′ is justified by (2.9). The assignment $q_\Phi = \infty$ when $\Omega = \varnothing$ in (2.7) is justified by the observations in Remark 2.20 and the fact that for $\Phi(t) = \Phi_{[p]}(t) = \frac{t^p}{p}$, we have

$$q_\Phi = \frac{t\Phi'(t)}{\Phi(t)} = p \to \infty, \quad \text{as } p \to \infty.$$

## 2.2 The role of upper and lower Lebesgue exponents for Young functions

In this section we investigate the Orlicz Lebesgue exponents $p_\Phi$ and $q_\Phi$ and link conditions on these exponents to various properties of their

Young functions $\Phi$. In particular, we show that neither of the two implications in

$$q_\Phi > 1 \iff \Phi \text{ is strictly convex} \tag{2.14}$$

can be true, in contrast with what it is stated in [72] (see Proposition 2.26). Instead we deduce other conditions on $\Phi$ which characterize $q_\Phi > 1$ (see Propositions 2.22 and 2.24). We also remark that our investigations are related to the achievements in [69].

In the following Proposition 2.22 we list some basic properties of relations between Young functions and their upper and lower Lebesgue exponents.

**Proposition 2.22.** Let $\Phi$ be a Young function with $\Omega$ as in (2.5), and let $p_\Phi$ and $q_\Phi$ be as in (2.6) and (2.7). Then the following is true:

(i) $1 \leqslant q_\Phi \leqslant p_\Phi$;

(ii) $p_\Phi = 1$, if and only if $\Phi$ is a linear map;

(iii) $p_\Phi < \infty$, if and only if $\Phi$ fulfills the $\Delta_2$-condition;

(iv) $q_\Phi > 1$, if and only if there is a $p > 1$ such that $\frac{\Phi(t)}{t^p}$ increases.

**Remark 2.23.** Taking into account that $\Phi$ in Proposition 2.22 is a Young function, we find that (iv) is equivalent to

(iv)′ $q_\Phi > 1$, *if and only if there is a* $p > 1$ *such that* $\frac{\Phi(t)}{t^p}$ *increases,*

$$\lim_{t\to a^+} \frac{\Phi(t)}{t^p} = 0 \quad \textit{and} \quad \lim_{t\to\infty} \frac{\Phi(t)}{t^p} = \infty$$

*for some* $a \geqslant 0$.

Most assertions in Proposition 2.22 and Remark 2.23 are well-known (see e. g. [17, 75, 76, 77]). For completeness, we include a proof here.

*Proof of Proposition 2.22.* If $\Omega = \varnothing$, (i) is trivially true. Assume therefore that $\Omega \neq \varnothing$, and let $t \in \Omega$. By the fact that $\Phi$ is convex on $\Omega$, we obtain

$$\frac{\Phi(t)}{t} = \frac{\Phi(t) - \Phi(0)}{t} \leqslant \Phi'_-(t) \leqslant \Phi'_+(t).$$

This gives (i).

If $\Phi$ is linear, then $\frac{t\Phi'(t)}{\Phi(t)} = 1$ for all $t \in \mathbb{R}_+$, giving that $q_\Phi = p_\Phi = 1$. Suppose instead that $p_\Phi = 1$. Then $\Omega = \mathbb{R}_+$ and

$$\frac{t\Phi'_\pm(t)}{\Phi(t)} = 1,$$

for all $t \in \mathbb{R}_+$ in view of (i) and its proof. This implies that $\Phi(t) = Ct$ for some constant $C$, and (ii) follows.

In order to prove (iii), we first suppose that $p_\Phi < \infty$. Then

$$\frac{t\Phi'_\pm(t)}{\Phi(t)} \leqslant R \qquad \Longleftrightarrow \qquad t\Phi'_\pm(t) - R\Phi(t) \leqslant 0,$$

for some $R > 0$. Since $\Phi(0) = 0$, we obtain

$$\Phi(t) = t^R h(t), \quad t > 0,$$

for some positive decreasing function $h(t)$. This gives

$$\Phi(2t) = (2t)^R h(2t) \leqslant 2^R t^R h(t) = 2^R \Phi(t),$$

and it follows that $\Phi$ satisfies the $\Delta_2$-condition when $p_\Phi < \infty$.

Suppose instead that $\Phi$ satisfies the $\Delta_2$-condition. Then $\Omega = \mathbb{R}_+$. By the mean-value theorem and the fact that $\Phi'_\pm(t)$ is increasing we obtain

$$\Phi'_\pm(t)t \leqslant \Phi(2t) - \Phi(t) \leqslant \Phi(2t) \leqslant C\Phi(t),$$

for some constant $C > 0$. Here the last inequality follows from the fact that $\Phi$ satisfies the $\Delta_2$-condition. This gives

$$\frac{t\Phi'_\pm(t)}{\Phi(t)} \leqslant C,$$

which implies $p_\Phi \leqslant C < \infty$, and we have proved (iii).

Next we prove (iv). Suppose that $q_\Phi > 1$. Then there is a $p > 1$ such that

$$\frac{t\Phi'_\pm(t)}{\Phi(t)} > p$$

for all $t \in \Omega$, which gives

$$t\Phi'_\pm(t) - p\Phi(t) > 0.$$

Hence

$$\frac{t^p\Phi'_\pm(t) - pt^{p-1}\Phi(t)}{t^{2p}} > 0,$$

or equivalently

$$\left(\frac{\Phi(t)}{t^p}\right)'_\pm > 0.$$

Hence, the desired result holds. If we instead suppose that $\frac{\Phi(t)}{t^p}$ is increasing for some $p > 1$, then applying the arguments above in reverse order yields $q_\Phi \geqslant p > 1$. □

We observe that, besides point (iii) in Proposition 2.22, there are several contributions on characterizations of the $\Delta_2$-condition (see e. g. [17, 69, 75, 76, 77]).

For the equivalence in (iv) of Proposition 2.22 we can prove further results.

**Proposition 2.24.** Let $\Phi$ be a Young function with $\Omega$ as in (2.5), and let $q_\Phi$ be as in (2.7). Then the following conditions are equivalent:

(i) $q_\Phi > 1$;

(ii) there is a $p > 1$ such that $\frac{\Phi(t)}{t^p}$ increases;

(iii) there are $p, q > 1$ such that $\frac{\Phi(t)}{t^p}$ increases near the origin and $\frac{\Phi(t)}{t^q}$ increases at infinity;

(iv) $\Phi$ fulfills the $\Lambda$-condition.

*Proof.* The result is obviously true when $\Omega = \varnothing$. Therefore, suppose that $\Omega \neq \varnothing$. The equivalence of (i) and (ii) was established in Proposition 2.22. Trivially, (ii) implies (iii). Moreover, $\frac{\Phi(t)}{t^p}$ increases if and only if for any $t > 0$ and any $c \in (0,1]$,

$$\frac{\Phi(ct)}{(ct)^p} \leqslant \frac{\Phi(t)}{t^p}$$

which is equivalent to (iv), hence (ii) is equivalent to (iv). We now show that (iii) implies (i), yielding the result.

Suppose that (iii) holds. First we also suppose that $|\Omega| = \infty$. Then, there are $R_1, R_2 > 0$ such that $\frac{\Phi(t)}{t^p}$ is increasing in $\Omega_1 \equiv (0, R_1) \cap \Omega$, and that $\frac{\Phi(t)}{t^p}$ is increasing in $\Omega_2 \equiv (R_2, \infty) \cap \Omega$. By test of differentiation, we obtain

$$q_1 = \inf_{t \in \Omega_1} \left( \frac{t\Phi'_\pm(t)}{\Phi(t)} \right) \geqslant p > 1 \quad \text{and} \quad q_2 = \inf_{t \in \Omega_2} \left( \frac{t\Phi'_\pm(t)}{\Phi(t)} \right) \geqslant q > 1. \tag{2.15}$$

Let

$$q_{1,2} = \inf_{t \in \Omega_{1,2}} \left( \frac{t\Phi'_\pm(t)}{\Phi(t)} \right), \quad \text{where} \quad \Omega_{1,2} = [R_1, R_2] \cap \Omega. \tag{2.16}$$

We intend to show that $q_2 > 1$, which in turn yields

$$q_\Phi = \min\{q_1, q_{1,2}, q_2\} > 1$$

, completing the proof in this case. Here we set $\inf \varnothing = \infty$.

Let $\varphi_1(t) = k_1 t - m_1$ and $\varphi_2(t) = k_2 t - m_2$, with $k_j = \Phi'_+(R_j)$ and $m_j$ chosen so that $\varphi_j(R_j) = \Phi(R_j)$, $j = 1,2$. Given that $\Phi$ is a Young function, is convex, and fulfills (3), it is clear that $k_1 \leqslant k_2$, $m_1 \leqslant m_2$, and $m_j > 0$ for $j = 1,2$.

We now approximate $\Phi(t)$ by linear segments forming polygonal chains for $R_1 \leqslant t \leqslant R_2$. Pick points $R_1 = t_0 < t_1 < \cdots < t_n = R_2$ and define functions $f_j(t) = a_j t - b_j$ such that $f_j(t_j) = \Phi(t_j)$ and $f_j(t_{j+1}) = \Phi(t_{j+1})$. Let $\Phi_n(t)$ be the polygonal chain on $[R_1, R_2]$ formed by connecting the functions $f_j$, meaning $\Phi_n(t) = f_j(t)$ whenever $t \in [t_j, t_{j+1}]$.

Since $\Phi$ is convex and increasing, we have $k_1 \leqslant a_j \leqslant k_2$ and $m_1 \leqslant b_j \leqslant m_2$ for all $j = 1, \dots, n$. Hence, for any $j = 1, \dots, n$,

$$\inf_{t \in [t_j, t_{j+1}]} \left( \frac{t(f_j)'_\pm(t)}{f_j(t)} \right) = \inf_{t \in [t_j, t_{j+1}]} \left( 1 + \frac{b_j}{a_j t - b_j} \right) \geqslant 1 + \frac{m_1}{\Phi(R_2)},$$

where the last inequality follows from the fact that $b_j \geqslant m_1$ and $a_j t_j - b_j = f_j(t_j) \leqslant f_n(t_n) = \Phi(R_2)$. From this, it is clear that

$$q_{\Phi_n} = \inf_{t \in [R_1, R_2]} \left( \frac{t(\Phi_n)'_\pm(t)}{\Phi_n(t)} \right) \geqslant 1 + \frac{m_1}{\Phi(R_2)}$$

independent of the choice of $n$ and the points $t_j$, $j = 1, \dots n-1$, and therefore

$$q_2 = \lim_{n \to \infty} q_{\Phi_n} \geqslant 1 + \frac{m_1}{\Phi(R_2)} > 1.$$

This gives (i), completing the proof when $|\Omega| = \infty$.

Next we consider the case when $|\Omega| < \infty$. Then $a = \sup \Omega$ is finite because $\Phi$ is increasing. First suppose that

$$\lim_{t \to a^-} \Phi(t) < \infty.$$

Letting $R_2 = a$, $q_1$, and $q_{1,2}$ be as in (2.15) and (2.16), the same arguments as in the previous case show that $q_{1,2} > 1$. This gives $q_\Phi = \min\{q_1, q_{1,2}\} > 1$, and (i) follows, giving the result in this case as well.

It remains to consider the case where $|\Omega| < \infty$ and

$$\lim_{t \to a^-} \Phi(t) = \infty.$$

We may assume that $R_1 < a$, and we let $a_\varepsilon = a - \varepsilon$ when $\varepsilon \in (0, a - R_1)$. By convexity we have

$$\frac{\Phi(a_\varepsilon) - \Phi(R_1)}{a_\varepsilon - R_1} \leqslant \Phi'_-(a_\varepsilon),$$

which gives

$$\frac{a_\varepsilon \Phi'_-(a_\varepsilon)}{\Phi(a_\varepsilon)} \geqslant \frac{a_\varepsilon \left(1 - \frac{\Phi(R_1)}{\Phi(a_\varepsilon)}\right)}{a_\varepsilon - R_1} \longrightarrow \frac{a}{a - R_1} > 1,$$

as $\varepsilon \to 0+$. Hence, if $c \in (1, \frac{a}{a-R_1})$, then there is a $\delta \in (0, a - R_1)$ such that

$$\frac{t\Phi'_-(t)}{\Phi(t)} > c > 1, \quad \text{when} \quad t \in (a-\delta, a). \tag{2.17}$$

Let $R_2 = a - \delta$, $\Omega_2$ be redefined as $\Omega_2 = (a-\delta, a)$, and let $q_1$, $q_2$, and $q_{1,2}$ be as in (2.15) and (2.16). Then by the same arguments as in the first case in combination with (2.17), we obtain

$$q_1 \geqslant p > 1, \quad q_{1,2} > 1, \quad \text{and} \quad q_2 > 1.$$

This gives $q_\Phi = \min\{q_1, q_{1,2}, q_2\} > 1$, leading to (i) in this case as well. This completes the proof. □

**Remark 2.25.** Due to Proposition 2.24 and its proof it is evident that $p$ in point (iv) of Proposition 2.24 is strongly linked to the lower Matuszewska-Orlicz index, given in e. g. p. 117 in [61]. It follows that parts of Propositions 2.22 and 2.24 follow from some of the established properties on p. 118–121 in [61], after suitable computations.

The following proposition shows that the condition $q_\Phi > 1$ cannot be linked to strict convexity for the Young function $\Phi$.

**Proposition 2.26.** Let $\Phi$ be a Young function, which is non-zero outside the origin, and let $q_\Phi$ be as in (2.7). Then the following is true:

(i) if $q_\Phi > 1$, then there is an equivalent Young function to $\Phi$ which is strictly convex;

(ii) $\Phi$ can be chosen such that $q_\Phi > 1$ but $\Phi$ is not strictly convex;

(iii) $\Phi$ can be chosen such that $q_\Phi = 1$ but $\Phi$ is strictly convex.

**Remark 2.27.** In [72] it is stated that (i) in Proposition 2.26 can be replaced by

(i)′ $q_\Phi > 1$, if and only if $\Phi$ is strictly convex.

This is equivalent to the following conditions (see the remark after (1.1) in [72]):

(ii)′ if $q_\Phi > 1$, then $\Phi$ is strictly convex;

(iii)′ if $\Phi$ is strictly convex, then $q_\Phi > 1$.

Evidently, the assertion in [72] is (strictly) stronger than Proposition 2.26 (i). On the other hand, Proposition 2.26 (ii) shows that (ii)′ cannot be true and Proposition 2.26 (iii) shows that (iii)′ cannot be true. Consequently, both implications in (i)′ are false.

*Proof of Proposition 2.26.* We begin by proving (i). Therefore assume that $q_\Phi > 1$. By Proposition 2.14, we may assume that $\Phi$ is smooth on $\mathbb{R}_+$. Suppose that $\Phi$ fails to be strictly convex in the whole interval $(0,\varepsilon)$, for some $\varepsilon > 0$. This implies that $\Phi(t) = ct$ when $t \in (0,\varepsilon)$, for some $c \geqslant 0$, which in turn gives $q_\Phi = 1$, violating the condition $q_\Phi > 1$. Hence $\Phi$ must be strictly convex in $(0,\varepsilon)$, for some choice of $\varepsilon > 0$.

Let

$$\Psi(t) = \int_0^t \Phi(t-s)e^{-s}\, ds.$$

Then

$$\Psi''(t) = \Phi'_+(0)e^{-t} + \int_0^t \Phi''(t-s)e^{-s}\, ds \geqslant \int_{t-\varepsilon}^t \Phi''(t-s)e^{-s}\, ds > 0,$$

since $\Phi''(t-s) > 0$ and is continuous when $s \in (t-\varepsilon, t)$. This shows that $\Psi$ is a strictly convex Young function.

Since $\Phi$ is increasing we also have

$$\Psi(t) \leqslant \Phi(t),$$

because

$$\Psi(t) = \int_0^t \Phi(t-s)e^{-s}\, ds \leqslant \Phi(t)\int_0^t e^{-s}\, ds \leqslant \Phi(t)\int_0^\infty e^{-s}\, ds = \Phi(t).$$

It follows that

$$\Phi_1(t) \equiv \Phi(t) + \Psi(t)$$

is a Young function equivalent to $\Phi(t)$. Since $\Psi$ is strictly convex, $\Phi_1$ is strictly convex as well. Consequently, $\Phi_1$ fulfills the required conditions for the searched Young function, and we have shown that (i) holds true.

In order to prove (ii), we choose

$$\Phi(t) = \begin{cases} 2t^2, & \text{when } t \leqslant 1 \\ 4t-2, & \text{when } 1 \leqslant t \leqslant 2 \\ t^2+2, & \text{when } t \geqslant 2 \end{cases}$$

which is not strictly convex. Then

$$\begin{aligned} q_\Phi &= \inf_{t>0} \left( \frac{t\Phi'(t)}{\Phi(t)} \right) \\ &= \min \left\{ \inf_{t\leqslant 1} \left( \frac{4t^2}{2t^2} \right), \inf_{1\leqslant t\leqslant 2} \left( \frac{4t}{4t-2} \right), \inf_{t\geqslant 2} \left( \frac{2t^2}{t^2+2} \right) \right\} = \frac{4}{3} > 1, \end{aligned}$$

which shows that $\Phi$ satisfies all the desired properties. This gives (ii).

Next we prove (iii). Let

$$\Phi(t) = t\ln(1+t), \quad t \geqslant 0.$$

Then $\Phi$ is a Young function, and it follows by straightforward computations that $q_\Phi = 1$. We also have $\Phi''(t) > 0$, giving that $\Phi$ is strictly convex. Consequently, $\Phi$ satisfies all the desired properties and (iii) follows. This completes the proof. □

The exponents $q_\Phi$ and $p_\Phi$ can also be related to corresponding exponents for the conjugate Young function as in the following proposition, which slightly generalizes (2.11) in [69].

**Theorem 2.28.** Suppose $\Phi$ is a Young function and $\Psi$ its corresponding conjugate Young function. Then

$$\frac{1}{p_\Psi} + \frac{1}{q_\Phi} = 1.$$

*Proof.* Suppose that for every $t = t_0 \in (0,\infty)$ there is some $s_0 = s(t)$ such that

$$\Psi(t) = st - \Phi(s).$$

Observe that, necessarily, $\Psi(t) < \infty$. Since for any $h \in \mathbb{R}$,

$$\Psi(t_0 + h) = \sup_{s>0} \left( s(t_0 + h) - \Phi(s) \right) \geqslant s_0(t_0 + h) - \Phi(s_0),$$

we obtain

$$\frac{\Psi(t_0 + h) - \Psi(t_0)}{h} \leqslant s_0$$

when $h < 0$ and

$$\frac{\Psi(t_0 + h) - \Psi(t_0)}{h} \geqslant s_0$$

when $h > 0$. By taking limits, we therefore have $\Psi'_-(t_0) \leqslant s_0 \leqslant \Psi'_+(t_0)$. Hence

$$\frac{t_0 \Psi'_-(t_0)}{\Psi(t_0)} \leqslant \frac{t_0 s_0}{t_0 s_0 - \Phi(s_0)} \leqslant \frac{t_0 \Psi'_+(t_0)}{\Psi(t_0)}$$

and therefore

$$\frac{1}{\left(\frac{t_0 \Psi'_+(t_0)}{\Psi(t_0)}\right)} \leqslant 1 - \frac{\Phi(s_0)}{t_0 s_0} \leqslant \frac{1}{\left(\frac{t_0 \Psi'_-(t_0)}{\Psi(t_0)}\right)}.$$

By rewriting $\Psi(t_0) = t_0 s_0 - \Phi(s_0)$ as $\Phi(s_0) = s_0 t_0 - \Psi(t_0)$ and repeating the arguments above, we obtain (for the same $s_0, t_0$) that $\Phi'_-(s_0) \leqslant t_0 \leqslant \Phi'_+(s_0)$. Hence, for every $t_0$ and corresponding $s_0$, we arrive at the inequalities

$$\frac{1}{\left(\frac{t_0 \Psi'_+(t_0)}{\Psi(t_0)}\right)} \leqslant 1 - \frac{1}{\left(\frac{s_0 \Phi'_+(s_0)}{\Phi(s_0)}\right)} \tag{2.18}$$

and

$$1 - \frac{1}{\left(\frac{s_0 \Phi'_-(s_0)}{\Phi(s_0)}\right)} \leqslant \frac{1}{\left(\frac{t_0 \Psi'_-(t_0)}{\Psi(t_0)}\right)}. \tag{2.19}$$

By minimizing the right-hand side of (2.18) and then maximizing the left-hand side, it is clear that

$$\frac{1}{q_\Psi} \leqslant 1 - \frac{1}{p_\Phi}.$$

The same procedure applied to (2.19) yields

$$1 - \frac{1}{p_\Phi} \leqslant \frac{1}{q_\Psi}.$$

This completes the proof when the supremum of $(st - \Phi(s))$ is attained for some $s \in \mathbb{R}_+$ for every $t \in \mathbb{R}_+$.

Now suppose instead that there is some $t = t_0 \in \mathbb{R}_+$ such that

$$\Psi(t_0) = \sup_{s>0} (st_0 - \Phi(s)) = \lim_{s\to\infty} (st_0 - \Phi(s)).$$

Then

$$\Psi(t_0 + \varepsilon) \geqslant \lim_{s\to\infty} (st_0 - \Phi(s) + \varepsilon s) = \infty,$$

meaning $p_\Psi = \infty$. Moreover,

$$st_0 - \Phi(s) \to \Psi(t_0), \quad s \to \infty,$$

implies that

$$\frac{\Phi(s)}{s} \to t_0, \quad s \to \infty,$$

meaning

$$\frac{\Phi(s)}{s^p} \to 0, \quad s \to \infty$$

for all $p > 1$. Hence $q_\Phi = 1$ by Proposition 2.22(iv). This completes the proof. □

**Corollary 2.29.** Let $\Phi_1$ and $\Phi_2$ be equivalent Young functions. Then the following is true:

(i) $p_{\Phi_1} < \infty$ if and only if $p_{\Phi_2} < \infty$;

(ii) $q_{\Phi_1} > 1$ if and only if $q_{\Phi_2} > 1$.

*Proof.* If $p_{\Phi_1} < \infty$, then by Proposition 2.22 (3), $\Phi_1$ fulfills the $\Delta_2$-condition. Since $\Phi_2$ is equivalent to $\Phi_1$, $\Phi_2$ also fulfills the $\Delta_2$-condition. This proves (i).

Suppose $q_{\Phi_1} > 1$. Then, by Theorem 2.28, $p_{\Psi_1} < \infty$, where $\Psi_1$ is the conjugate Young function to $\Phi_1$. Since $\Psi_1$ is equivalent to $\Psi_2$, where $\Psi_2$ is the conjugate Young function to $\Phi_2$, we get $p_{\Psi_2} < \infty$ by (i). Applying Theorem 2.28 again gives $q_{\Psi_2} > 1$. This completes the proof. □

## 2.3 Continuity results and applications to PDEs

In this section we extend properties on $L^p$ continuity for various types of Fourier type operators into continuity on Orlicz spaces. Especially we perform such extensions for Hörmander's improvement of Mihlin's Fourier multiplier theorem (see Theorem 2.37). We also deduce Orlicz space continuity for suitable classes of pseudo-differential and Fourier integral operators (see Theorems 2.36 and 2.38). Our investigations are based on a special case of Marcinkiewicz type interpolation theorem for Orlicz spaces, deduced in [72]. Subsequently, we introduce a scale of Sobolev spaces modelled on Orlicz spaces, and show boundedness results involving them, relying on our $L^\Phi$ continuity results and on lift properties. Finally, we illustrate a notion of (global) wave-front set associated with Sobolev-Orlicz spaces, along the lines in [29] (see also [30]), and show its propagation for suitable classes of PDEs.

### 2.3.1 Continuity results on $L^p$

We recall continuity results already known on $L^p$ for pseudo-differential opeartors, Fourier multipliers and Fourier integral operators.

The first result concerns pseudo-differential operators acting on $L^p$-spaces (see, e. g., [132]).

**Proposition 2.30.** Let $p \in (1,\infty)$, $A \in \mathbf{M}(d,\mathbb{R})$, and $a \in S^0_{1,0}(\mathbb{R}^{2d})$. Then $\operatorname{Op}_A(a)$ is continuous on $L^p(\mathbb{R}^d)$.

In the next proposition we essentially recall Hörmander's improvement of Mihlin's Fourier multiplier theorem.

**Proposition 2.31.** Let $p \in (1,\infty)$ and $a \in L^\infty(\mathbb{R}^d \setminus \{0\})$ be such that

$$\sup_{R>0} \left( R^{-d+2|\alpha|} \int_{A_R} |\partial^\alpha a(\xi)|^2 \, d\xi \right) \tag{2.20}$$

is finite for every $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant [\frac{d}{2}] + 1$, where $A_R$ is the annulus $\{\, \xi \in \mathbb{R}^d : R < |\xi| < 2R \,\}$. Then $a(D)$ is continuous on $L^p(\mathbb{R}^d)$.

The next Theorem 2.32 concerns $L^p$ boundedness of SG Fourier integral operators (see [34] and recall Section 1.5 for definitions and notations).

**Theorem 2.32.** Let $p \in (1,\infty)$ and $m_0, \mu_0 \in \mathbb{R}$ be such that

$$m_0, \mu_0 \leqslant -(d-1)\left|\frac{1}{p} - \frac{1}{2}\right|. \tag{2.21}$$

Suppose that $\varphi \in \mathfrak{P}_r^{\mathrm{hom}}$ and $a \in S^{m_0,\mu_0}(\mathbb{R}^{2d})$ is such that $|\xi| \geqslant \varepsilon$, for some $\varepsilon > 0$, on the support of $a$. Then $\mathrm{Op}_\varphi(a)$ from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$ extends uniquely to a continuous operator from $L^p(\mathbb{R}^d)$ to itself.

**Remark 2.33.** If $\varphi(x,\xi) = \langle x,\xi\rangle$ in Theorem 2.32, then $\mathrm{Op}_\varphi(a)$ becomes a pseudo-differential operator with symbol $a$, which we still denote by $\mathrm{Op}(a)$. In this case, Proposition 2.30 is a strict improvement of Theorem 2.32. For $p \neq 2$ and general $\varphi \in \mathfrak{P}_r^{\mathrm{hom}}$, uniform boundedness of the amplitude $a$ is not enough to guarantee that $\mathrm{Op}_\varphi(a)$ maps $L^p$ continuously into itself, even if the support of $f$ is compact (see [102]). In (2.21), the condition on the $x$-order $m_0$ can be viewed as a *loss of decay*. Similarly, the condition on the $\xi$-order $\mu_0$ can be considered a *loss of smoothness*. Notice also that no condition of compactness of the support of $f$ is needed in Theorem 2.32 (see [34] and the references therein for more details). We also mention [39], where further local and global $L^p$-boundedness results for Fourier integral operators have been proved, under hypotheses different from those assumed in Theorem 2.32.

### 2.3.2 Continuity results in Orlicz spaces

We now state and prove our continuity results on Orlicz spaces. We first recall the following Marcinkiewicz type interpolation theorem on Orlicz spaces. Notice that the result is a special case of [72, Theorem 5.1].

**Proposition 2.34.** Let $\Phi$ be a strict Young function and $p_0, p_1 \in (0,\infty]$ be such that $p_0 < q_\Phi \leqslant p_\Phi < p_1$, where $p_\Phi$ and $q_\Phi$ are defined in (2.6) and (2.7). Also let

$$T : L^{p_0}(\mathbb{R}^d) + L^{p_1}(\mathbb{R}^d) \to wL^{p_0}(\mathbb{R}^d) + wL^{p_1}(\mathbb{R}^d) \tag{2.22}$$

be a linear and continuous map which restricts to linear and continuous mappings

$$T : L^{p_0}(\mathbb{R}^d) \to wL^{p_0}(\mathbb{R}^d) \quad \text{and} \quad T : \ L^{p_1}(\mathbb{R}^d) \to wL^{p_1}(\mathbb{R}^d).$$

Then (2.22) restricts to linear and continuous mappings

$$T : L^{\Phi}(\mathbb{R}^d) \to L^{\Phi}(\mathbb{R}^d) \qquad \text{and} \quad T : wL^{\Phi}(\mathbb{R}^d) \to wL^{\Phi}(\mathbb{R}^d). \tag{2.23}$$

**Remark 2.35.** Let $\Phi$ and $T$ be the same as in Proposition 2.34. Then the continuity of the mappings in (2.23) means

$$\|Tf\|_{L^{\Phi}} \lesssim \|f\|_{L^{\Phi}}, \qquad f \in L^{\Phi}(\mathbb{R}^d)$$

and

$$\|Tf\|_{wL^{\Phi}} \lesssim \|f\|_{wL^{\Phi}}, \quad f \in wL^{\Phi}(\mathbb{R}^d).$$

A combination of Propositions 2.30 and 2.34 gives the following result on continuity properties for pseudo-differential operators on $L^{\Phi}$-spaces.

**Theorem 2.36.** Let $\Phi$ be a strict Young function, $A \in \mathbf{M}(d, \mathbb{R})$, and $a \in S^0_{1,0}(\mathbb{R}^{2d})$. Then

$$\operatorname{Op}_A(a) : L^{\Phi}(\mathbb{R}^d) \to L^{\Phi}(\mathbb{R}^d) \quad \text{and} \quad \operatorname{Op}_A(a) : wL^{\Phi}(\mathbb{R}^d) \to wL^{\Phi}(\mathbb{R}^d)$$

are continuous.

*Proof.* By Propositions 2.22 and 2.24 it follows that $q_{\Phi} > 1$ and $p_{\Phi} < \infty$. Choose $p_0, p_1 \in (1, \infty)$ such that $p_0 < q_{\Phi}$ and $p_1 > p_{\Phi}$. In view of Remark 2.17 and Proposition 2.30,

$$\begin{aligned} \|\operatorname{Op}(a)f\|_{wL^{p_j}} \leqslant \|\operatorname{Op}(a)f\|_{L^{p_j}} &\leqslant C\|f\|_{L^{p_j}}, \\ f \in L^{p_j}(\mathbb{R}^d),\ &j = 0, 1. \end{aligned} \tag{2.24}$$

Then it follows that $\operatorname{Op}_A(a)$ extends uniquely to a continuous map from $L^{p_0}(\mathbb{R}^d) + L^{p_1}(\mathbb{R}^d)$ to $wL^{p_0}(\mathbb{R}^d) + wL^{p_1}(\mathbb{R}^d)$ (see e.g. [15]). Hence the conditions of Proposition 2.34 are fulfilled and the result follows. □

By using Proposition 2.31 instead of Proposition 2.30 in the previous proof we obtain the following extension of Hörmander's improvement of Mihlin's Fourier multiplier theorem.

**Theorem 2.37.** Let $\Phi$ be a strict Young function and $a \in L^\infty(\mathbb{R}^d \setminus 0)$ be such that

$$\sup_{R>0} \left( R^{-d+2|\alpha|} \int_{A_R} |\partial^\alpha a(\xi)|^2 \, d\xi \right) \tag{2.25}$$

is finite for every $\alpha \in \mathbb{N}^d$ with $|\alpha| \leqslant [\frac{d}{2}] + 1$, where $A_R$ is the annulus $\{\, \xi \in \mathbb{R}^d \, : \, R < |\xi| < 2R \,\}$. Then $a(D)$ is continuous on $L^\Phi(\mathbb{R}^d)$ and on $wL^\Phi(\mathbb{R}^d)$.

Finally, employing Theorem 2.32, we prove the following continuity result for Fourier integral operators on $L^\Phi$-spaces. Here we let

$$\mathfrak{L}_{\Phi,d} \equiv (d-1) \max \left( \left| \frac{1}{p_\Phi} - \frac{1}{2} \right|, \left| \frac{1}{q_\Phi} - \frac{1}{2} \right| \right). \tag{2.26}$$

**Theorem 2.38.** Let $\Phi$ be a strict Young function, $\mathfrak{L}_{\Phi,d}$ be as in (2.26) and $m, \mu \in \mathbb{R}$ be such that

$$m \leqslant -\mathfrak{L}_{\Phi,d} \quad \text{and} \quad \mu \leqslant -\mathfrak{L}_{\Phi,d}, \tag{2.27}$$

with strict inequalities when $q_\Phi < p_\Phi$. Suppose that $\varphi \in \mathfrak{P}_r^{\text{hom}}$ and $a \in S^{m,\mu}(\mathbb{R}^{2d})$ is such that $|\xi| \geqslant \varepsilon$, for some $\varepsilon > 0$, on the support of $a$. Then $\operatorname{Op}_\varphi(a)$ from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$ extends uniquely to a continuous operator from $L^\Phi(\mathbb{R}^d)$ to itself, and from $wL^\Phi(\mathbb{R}^d)$ to itself.

**Remark 2.39.** In contrast to condition (2.21) in Theorem 2.32, strict inequality is required in (2.27) when $q_\Phi < p_\Phi$.

*Proof.* First suppose that $q_\Phi = p_\Phi = p$. Then $1 < p < \infty$, $L^\Phi = L^p$, $wL^\Phi = wL^p$ and the result follows from Remark 2.20, Theorem 2.32 and Proposition 2.34.

Next suppose $q_\Phi < p_\Phi$. As above, by Proposition 2.22 it follows that $q_\Phi > 1$ and $p_\Phi < \infty$. Choose $p_0, p_1 \in (1, \infty)$ such that $p_0 < q_\Phi$, $p_1 > p_\Phi$ and

$$m, \mu < -(d-1) \left| \frac{1}{p_j} - \frac{1}{2} \right|, \qquad j = 0, 1.$$

Notice that this is possible since, in view of (2.26) and (2.27), for any $\epsilon > 0$ we can choose $\delta, \delta' > 0$ such that

$$\min\left(\left|\frac{1}{p_\Phi + \delta} - \frac{1}{2}\right|, \left|\frac{1}{q_\Phi - \delta} - \frac{1}{2}\right|\right) - \max\left(\left|\frac{1}{p_\Phi} - \frac{1}{2}\right|, \left|\frac{1}{q_\Phi} - \frac{1}{2}\right|\right) < \epsilon.$$

In view of Remark 2.17 and Theorem 2.32,

$$\|\mathrm{Op}_\varphi(a)f\|_{wL^{p_j}} \leqslant \|\mathrm{Op}_\varphi(a)f\|_{L^{p_j}} \leqslant C\|f\|_{L^{p_j}}, \quad f \in L^{p_j}(\mathbb{R}^d),\ j = 0, 1. \tag{2.28}$$

By Proposition 2.34, the claim follows, arguing as in the final step of the proof of Theorem 2.36. □

### 2.3.3 Sobolev-Kato-Orlicz spaces, adapted global wave-front sets, and applications to PDEs

We introduce here the notion of Sobolev-Kato-Orlicz spaces. The next Definition 2.40 is a natural one, in view of the results in the previous section.

**Definition 2.40.** Let $s, \sigma \in \mathbb{R}$, $\Phi$ be a Young function and $\vartheta_{s,\sigma}(x,\xi) = \langle x\rangle^s \langle \xi\rangle^\sigma$. Then the *Sobolev-Kato-Orlicz* space $H^\Phi_{s,\sigma}(\mathbb{R}^d)$ is given by

$$H^\Phi_{s,\sigma}(\mathbb{R}^d) \equiv \{\, f \in \mathscr{S}'(\mathbb{R}^d) : \mathrm{Op}(\vartheta_{s,\sigma})f \in L^\Phi(\mathbb{R}^d)\,\},$$

with topology induced by the norm

$$\|f\|_{H^\Phi_{s,\sigma}} = \|\mathrm{Op}(\vartheta_{s,\sigma})f\|_{L^\Phi}.$$

By straightforward computations it follows that $H^\Phi_{s,\sigma}(\mathbb{R}^d)$ is a Banach space. Obviously, $H^\Phi_{0,0}(\mathbb{R}^d) = L^\Phi(\mathbb{R}^d)$.

The next result is a consequence of Theorems 2.36 and 2.38 (see, e.g., [29, 31]), in view of the calculus of SG FIOs (see Section 1.5).

**Theorem 2.41.** Let $\Phi$ be a strict Young function, $\mathfrak{L}_{\Phi,d}$ be as in (2.26), $l, \lambda, m, \mu,\ s, \sigma \in \mathbb{R}$ be such that

$$m \leqslant l - \mathfrak{L}_{\Phi,d} \quad \text{and} \quad \mu \leqslant \lambda - \mathfrak{L}_{\Phi,d}, \tag{2.29}$$

with strict inequalities when $q_\Phi < p_\Phi$. Suppose that $A \in \mathbf{M}(d,\mathbb{R})$, $\varphi \in \mathfrak{P}^{\mathrm{hom}}_r$ and $a \in S^{m,\mu}(\mathbb{R}^{2d})$. Then the following is true:

(i) the map $\operatorname{Op}_A(a)$ on $\mathscr{S}'(\mathbb{R}^d)$ restricts to a linear and continuous map from $H^\Phi_{s+m,\sigma+\mu}(\mathbb{R}^d)$ to $H^\Phi_{s,\sigma}(\mathbb{R}^d)$;

(ii) if, additionally, $|\xi| \geqslant \varepsilon$, for some $\varepsilon > 0$, on the support of $a$, then, $\operatorname{Op}_\varphi(a)$ from $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}'(\mathbb{R}^d)$ extends uniquely to a continuous operator from $H^\Phi_{s+l,\sigma+\lambda}(\mathbb{R}^d)$ to $H^\Phi_{s,\sigma}(\mathbb{R}^d)$.

*Proof.* We prove only (ii), since the assertion (i) follows by similar arguments. Let $f \in H^\Phi_{s+l,\sigma+\lambda}$ and $u = \operatorname{Op}(\vartheta_{s+l,\sigma+\lambda})f$. Then, $u \in L^\Phi$, $\|f\|_{H^\Phi_{s+l,\sigma+\lambda}} = \|u\|_{L^\Phi}$, and

$$\begin{aligned}
\|\operatorname{Op}_\varphi(a)f\|_{H^\Phi_{s,\sigma}} &= \|\operatorname{Op}(\vartheta_{s,\sigma})\operatorname{Op}_\varphi(a)(\operatorname{Op}(\vartheta_{s+l,\sigma+\lambda}))^{-1}u\|_{L^\Phi} \\
&\leqslant \|\operatorname{Op}(\vartheta_{s,\sigma})\operatorname{Op}_\varphi(a)(\operatorname{Op}(\vartheta_{s+l,\sigma+\lambda}))^{-1}\|_{\mathcal{L}(L^\Phi)} \cdot \|u\|_{L^\Phi} \\
&= \|\operatorname{Op}_\varphi(b) + \mathcal{K}\|_{\mathcal{L}(L^\Phi)} \cdot \|f\|_{H^\Phi_{s+l,\sigma+\lambda}},
\end{aligned}$$

where $\mathcal{K}$ is an operator with kernel in $\mathscr{S}(\mathbb{R}^{2d})$ (see Theorem 1.64). In particular, $\mathcal{K}$ is continuous from $\mathscr{S}'(\mathbb{R}^d)$ to $\mathscr{S}(\mathbb{R}^d)$. Hence (2.13) gives $\|\mathcal{K}\|_{\mathcal{L}(L^\Phi)} < \infty$. By Theorem 1.64, we have $b \in S^{m-l,\mu-\lambda}$. Since $b$ is the asymptotic sum of an expansion involving $a$, $\vartheta_{s,\sigma}$, $\vartheta_{s+l,\sigma+\lambda}$, their derivatives, and suitable compositions with $\varphi'_x$ and $\varphi'_\xi$ (see [31] and the references therein), on its support it holds $|\xi| \geqslant \varepsilon' > 0$, and the hypotheses of Theorem 2.38 are satisfied. We conclude that it also holds $\|\operatorname{Op}_\varphi(b)\|_{\mathcal{L}(L^\Phi)} < \infty$, and the claim follows. □

**Remark 2.42.** We observe that $H^\Phi_{s,\sigma}(\mathbb{R}^d)$ in Theorem 2.41 is SG-admissible and that the pair $(H^\Phi_{s+m,\sigma+\mu}(\mathbb{R}^d), H^\Phi_{s,\sigma}(\mathbb{R}^d))$ is SG-ordered with respect to $(m,\mu)$.

We recall that here this means

$$\operatorname{Op}(a) : H^\Phi_{s+m,\sigma+\mu}(\mathbb{R}^d) \to H^\Phi_{s,\sigma}(\mathbb{R}^d)$$

continuously for any $a \in S^{m,\mu}(\mathbb{R}^d)$,

$$\operatorname{Op}(b) : H^\Phi_{s,\sigma}(\mathbb{R}^d) \to H^\Phi_{s+m,\sigma+\mu}(\mathbb{R}^d)$$

continuously for any $b \in S^{-m,-\mu}(\mathbb{R}^d)$ and

$$\operatorname{Op}(c) : H^\Phi_{s+m,\sigma+\mu}(\mathbb{R}^d) \to H^\Phi_{s+m,\sigma+\mu}(\mathbb{R}^d),$$

$$\mathrm{Op}(c) : H^{\Phi}_{s,\sigma}(\mathbb{R}^d) \to H^{\Phi}_{s,\sigma}(\mathbb{R}^d),$$

continuously for any $c \in S^{0,0}(\mathbb{R}^d)$ (the detailed theory has been developed in [29, 30, 31]).

It is then possible to apply the theory of global wave-front sets developed in [29] and [31], choosing the Sobolev-Kato-Orlicz spaces from Definition 2.40 as reference spaces.

We adapt Definition 1.69 to define the global wave-front set as follows.

**Definition 2.43.** Let $f \in \mathscr{S}'(\mathbb{R}^d)$ and denote by $\Theta^m_{H^{\Phi}_{s,\sigma}(\mathbb{R}^d)}(f)$ the set of all type-$m$ regular points for $f$, that is, the set of points $(x_0, \xi_0) \in \Omega_m$ for which $\mathrm{Op}(c_m) f \in H^{\Phi}_{s,\sigma}(\mathbb{R}^d)$, for some $c_m$ as in Proposition 1.68. We then define:

(i) the type-$m$ wave-front set of $f \in \mathscr{S}'(\mathbb{R}^d)$ with respect to $H^{\Phi}_{s,\sigma}(\mathbb{R}^d)$ as the complement of $\Theta^m_{H^{\Phi}_{s,\sigma}(\mathbb{R}^d)}(f)$ in $\Omega_m$, which is denoted by $\mathrm{WF}^m_{H^{\Phi}_{s,\sigma}(\mathbb{R}^d)}(f)$;

(ii) the global wave-front set $\mathrm{WF}_{H^{\Phi}_{s,\sigma}(\mathbb{R}^d)}(f) \subseteq (\mathbb{R}^d \times \mathbb{R}^d) \setminus 0$ as the set

$$\begin{aligned} \mathrm{WF}_{H^{\Phi}_{s,\sigma}(\mathbb{R}^d)}(f) &\equiv \mathrm{WF}^{\Phi}_{s,\sigma}(f) \\ &\equiv \mathrm{WF}^1_{H^{\Phi}_{s,\sigma}(\mathbb{R}^d)}(f) \cup \mathrm{WF}^2_{H^{\Phi}_{s,\sigma}(\mathbb{R}^d)}(f) \cup \mathrm{WF}^3_{H^{\Phi}_{s,\sigma}(\mathbb{R}^d)}(f) \end{aligned} \tag{2.30}$$

We have the following characterization of Sobolev-Kato-Orlicz spaces in terms of the global wave-front sets as a consequence of the general theory (see Theorem 2.6 in [29]).

**Theorem 2.44.** Let $f \in \mathscr{S}'(\mathbb{R}^d)$. Then,

$$f \in H^{\Phi}_{s,\sigma}(\mathbb{R}^d) \iff \mathrm{WF}^{\Phi}_{s,\sigma}(f) = \varnothing.$$

We show now two results about propagation of singularities with respect to the $H^{\Phi}_{s,\sigma}$ spaces. These follow by Theorem 2.41 and by the theory developed in [29, 30, 31].

The first result concerns mapping properties of wave-front sets of pseudo-differential operators with symbols in SG classes and is a direct application of Theorem 3.1 in [29] (see Section 1.6.1).

**Theorem 2.45.** Let $a \in S^{m,\mu}(\mathbb{R}^d)$, $f \in H^{\Phi}_{s,\sigma}(\mathbb{R}^d)$, $u \in \mathscr{S}'(\mathbb{R}^d)$, $m, s, \mu, \sigma \in \mathbb{R}$, and consider the equation $\mathrm{Op}(a)u = f$. Then,

$$\mathrm{WF}^{\Phi}_{s,\sigma}(f) \subseteq \mathrm{WF}^{\Phi}_{s+m,\sigma+\mu}(u) \subseteq \mathrm{WF}^{\Phi}_{s,\sigma}(f) \cup \mathrm{Char}(a). \tag{2.31}$$

In addition, if $a$ in Theorem 2.45 is elliptic, then $\mathrm{Char}(a) = \varnothing$ and (2.31) gives

$$\mathrm{WF}^{\Phi}_{s+m,\sigma+\mu}(u) = \mathrm{WF}^{\Phi}_{s,\sigma}(f).$$

The next two results concerns properties of solutions to hyperbolic Cauchy problems. We employ the notation introduced in Section 1.5.

**Proposition 2.46.** Let $h \in S^{1,1}_{\mathrm{cl}}(\mathbb{R}^d)$ be such that $h = h_p + h_0$, $h_p \in S^{1,1}_{\mathrm{cl}}(\mathbb{R}^d)$ real-valued, $h_0 \in S^{0,0}_{\mathrm{cl}}(\mathbb{R}^d)$, $u_0 \in H^{\Phi}_{s+\mathfrak{L}_{\Phi,d},\sigma+\mathfrak{L}_{\Phi,d}}(\mathbb{R}^d)$, with $s, \sigma \in \mathbb{R}$ and $\mathfrak{L}_{\Phi,d}$ as in (2.26). Then, the solution of

$$\begin{cases} D_t u(t) = \mathrm{Op}(h)u(t), \ t \in [-T,T], T > 0, \\ \quad u(0) = u_0, \end{cases} \tag{2.32}$$

is given by

$$u(t) \equiv \mathrm{Op}_{\varphi(t)}(a(t))u_0 \bmod (\mathscr{C}^\infty([-T',T'];\mathscr{S}(\mathbb{R}^d))), \quad t \in [-T',T'],$$

for suitable families of regular phase functions $\varphi(t)$ and symbols $a(t)$ of order $(0,0)$.

Under the same hypotheses as Proposition 2.46, but neglecting low frequencies via a suitable cut-off function

$$\chi(\xi) = \begin{cases} 0, & |\xi| \geqslant R, \\ 1, & |\xi| \leqslant r, \end{cases} \tag{2.33}$$

for the solution $u$ of the Cauchy problem (2.32), Theorem 2.41 implies that $(1 - \chi(D))u \in \mathscr{C}^\infty([-T',T'];H^{\Phi}_{s,\sigma}(\mathbb{R}^d))$. Moreover, Theorem 5.2 in [34], combined with Theorem 2.41 and the theory developed in [29, 30, 31] (see Section 1.6.1), implies the next Theorem 2.47, with which we conclude this chapter.

**Theorem 2.47.** Let $u$ be the solution of (2.32) from Proposition 2.46 and $\chi \in \mathscr{C}_0^\infty(\mathbb{R}^d)$ satisfy (2.33) for some suitable $R > 0$ and $r \in (0, R)$. Then

$$\mathrm{WF}^{\Phi}_{s,\sigma}((1-\chi(D))u(t)) = \Psi(t)(\mathrm{WF}^{\Phi}_{s+\mathfrak{L}_{\Phi,d},\sigma+\mathfrak{L}_{\Phi,d}}(u_0)),\ t \in [-T', T'],$$

where $\Psi(t)$ is a smooth family of transformations, generated by the smooth family of phase functions $\varphi(t)$.

**Remark 2.48.** Notice that local counterparts of Theorem 2.45, Proposition 2.46, and Theorem 2.47, in terms of Hörmander's classical wave-front sets, also hold true, when $x$ belongs to a bounded open subset of $\mathbb{R}^d$.

# Chapter 3

# Gevrey time-periodic Sobolev-Kato spaces and global hypoellipticity and solvabilty of evolution operators

In this chapter we introduce a class of Gevrey time-periodic weighted Sobolev spaces and study hypoellipticity and solvability properties of a naturally associated class of evolution operators, with coefficients growing polynomially with respect to the space variable.

By employing a strategy based on Fourier decomposition, we consider an operator $P = \mathrm{Op}(p)$ on $\mathbb{R}^d$, associated with an elliptic symbol $p \in S^{m,\mu}(\mathbb{R}^d)$ and satisfying suitable assumptions (see Section 3.1 below). Using the corresponding orthonormal basis of eigenfunctions $\{\phi_j\}_{j\in\mathbb{N}^*}$, we derive expansions

$$u(t) = \sum_{j\in\mathbb{N}^*} u_j(t)\phi_j, \tag{3.1}$$

where $u_j(t)$, $j \in \mathbb{N}^*, \mathbb{N}* = \mathbb{N} \setminus \{0\}$, is a suitable sequence of Gevrey periodic functions.

We subsequently employ representations of the form (3.1) in the study of the equation

$$(D_t + \omega \mathrm{Op}(p))\, u(t) = f(t), \quad \omega \in \mathbb{C},$$

in the above-mentioned spaces.

## 3.1 Eigenfunction expansions in weighted Sobolev spaces on $\mathbb{R}^d$

In this section we aim at characterising the Sobolev-Kato spaces by means of the eigenfunction expansions related to a suitable elliptic, normal SG operator $P$. This approach was originally considered by Seeley [103, 104] in the context of smooth and analytic functions on vector bundles, and by Cappiello, Gramchev, Pilipović and Rodino in [24, 44] in Gelfand-Shilov classes on the Euclidean spaces. Here, we precisely characterise smoothness and polynomial decay of temperate distributions on $\mathbb{R}^d$, in terms of the behaviour at infinity of the coefficients of eigenfunction expansions, related to the corresponding behaviour of the eigenvalues of $P$. As a byproduct, we also similarly characterise when a temperate distribution is actually a rapidly decreasing Schwartz function.

Let $P \in \mathrm{Op}(S^{m,\mu})$ be an elliptic, normal SG operator with order components $m, \mu > 0$. By Theorem 4.1, Section 3.4 in [26], we have that

$$\ker P = \ker P^* \subset \mathscr{S}.$$

Moreover, the ellipticity of $P$ implies that it is Fredholm, and then it follows that

$$\dim \ker P = N < \infty$$

and

$$\mathrm{ind}\, P = \dim \ker P - \dim \ker P^* = 0.$$

Let now $\{\phi_j\}_{j=1}^N \subset \mathscr{S}$ be an orthonormal basis of $\ker P$ and consider the operator $P_0$ with kernel

$$K(x,y) = \sum_{j=1}^N \phi_j(x)\overline{\phi_j(y)} \in \mathscr{S}(\mathbb{R}^d \times \mathbb{R}^d),$$

that is, the orthogonal projection on $\ker P$. We then get

$$P_0 u = \int K(x,y)u(y)dy = \sum_{j=1}^{N} (u, \phi_j)\, \phi_j,$$

so that $P_0 \in \mathrm{Op}(S^{-\infty,-\infty})$ and it is a compact operator $P_0\colon L^2 \to L^2$.

Notice that the operator $\widetilde{P} = P + P_0 \in \mathrm{Op}(S^{m,\mu})$ is elliptic, normal and injective with $\mathrm{ind}\,\widetilde{P} = \mathrm{ind}\,P$. Therefore, $\widetilde{P}\colon H^{m,\mu} \to L^2$ is bijective with inverse $Q \in \mathscr{L}(L^2, H^{m,\mu})$. In particular, $Q \in \mathrm{Op}(S^{-m,-\mu})$ is normal and compact (see Theorem 1.49). Therefore, we have proved that the next Proposition 3.1 holds true (see, e.g., [87, Chapter 3], (3.1.8), (3.1.9), for the last statement).

**Proposition 3.1.** Let $P \in \mathrm{Op}(S^{m,\mu}(\mathbb{R}^d))$ be an elliptic, normal SG operator with order components $m, \mu > 0$. Denote by $P_0$ the orthogonal projection on $\ker P$ and let $\widetilde{P} = P + P_0$. Then,

$$u \in H^{m,\mu}(\mathbb{R}^d) \Longleftrightarrow Pu \in L^2(\mathbb{R}^d)$$

and

$$\|u\|_{H^{m,\mu}(\mathbb{R}^d)} \equiv \|\mathrm{Op}(\lambda_{m,\mu})u\|_{L^2(\mathbb{R}^d)} \asymp \|\widetilde{P}u\|_{L^2(\mathbb{R}^d)}.$$

Since $Q\colon L^2 \to L^2$ is a compact normal operator, there exists a basis $\{\phi_j\}_{j\in\mathbb{N}^*}$ of orthonormal eigenfunctions, associated with eigenvalues $\{\mu_j\}_{j\in\mathbb{N}^*}$ such that

$$\mu_j \to 0, \quad j \to \infty.$$

The injectivity of $Q$ ensures that $\mu_j \neq 0$, for all $j \in \mathbb{N}^*$. We claim that $\phi_j$ is still an eigenfunction of $\widetilde{P}$, with eigenvalue $\widetilde{\lambda}_j = \mu_j^{-1}$. Indeed,

$$Q\phi_j = \mu_j \phi_j,$$

and, since $\phi_j \in \mathrm{Im}\, Q = H^{m,\mu} = \mathrm{dom}\,\widetilde{P}$, and $\widetilde{P}Q = \mathrm{Id}$ on $H^{m,\mu}$, we get

$$\widetilde{P}\phi_j = \mu_j^{-1}\phi_j = \lambda_j \phi_j.$$

We can of course choose an orthornormal basis $\{\phi_j\}_{j\in\mathbb{N}^*}$ which completes the basis $\{\phi_j\}_{j=1}^N$ of $\ker P$, so that, clearly,

$$Q\phi_j = \mu_j\phi_j \iff \mu_j^{-1}\phi_j = \widetilde{P}\phi_j = \phi_j \Rightarrow \widetilde{\lambda}_j = \mu_j^{-1} = 1, \quad j = 1, \dots, N,$$

and

$$Q\phi_j = \mu_j\phi_j \iff \mu_j^{-1}\phi_j = \widetilde{P}\phi_j = P\phi_j \Rightarrow \widetilde{\lambda}_j = \mu_j^{-1} = \lambda_j, \quad j \geqslant N+1.$$

Then, the eigenvalues $\widetilde{\lambda}_j$ of $\widetilde{P}$ are given by

$$\widetilde{\lambda}_j = \begin{cases} 1, & j = 1, \dots, N, \\ \lambda_j & j \geqslant N+1, \end{cases} \tag{3.2}$$

where $\{\lambda_j\}_{j\geqslant N+1}$ are the non-vanishing eigenvalues of $P$ and, of course, $\lambda_1 = \lambda_2 = \dots = \lambda_N = 0$. Finally, $\lim\limits_{j\to\infty} \mu_j = 0 \Rightarrow \lim\limits_{j\to\infty} |\lambda_j| = \infty$.

**Definition 3.2.** Let $P \in \mathrm{Op}(S^{m,\mu}(\mathbb{R}^d))$ be an elliptic, normal SG operator with order components $m, \mu > 0$, and denote by $\{\phi_j\}_{j\in\mathbb{N}^*}$ a basis of orthonormal eigenfunctions of $P$. Given $f \in \mathscr{S}(\mathbb{R}^d)$ we set

$$f_j = (f, \phi_j)_{L^2(\mathbb{R}^d)}, \quad j \in \mathbb{N}^*,$$

which implies $f = \sum_{j\in\mathbb{N}^*} f_j\phi_j$. By duality, for $u \in \mathscr{S}'(\mathbb{R}^d)$ we set

$$u_j = u\big(\overline{\phi_j}\big) = \langle u, \overline{\phi_j}\rangle, \tag{3.3}$$

which implies $u = \sum_{j\in\mathbb{N}^*} u_j\phi_j$.

**Remark 3.3.** Indeed, since $\{\overline{\phi_j}\}_{j\in\mathbb{N}^*}$ is also a basis for $L^2$, for any $\psi \in \mathscr{S}(\mathbb{R}^d)$

$$\langle u, \psi\rangle = \left\langle u, \sum_{j\in\mathbb{N}^*} (\psi, \overline{\phi_j})\,\overline{\phi_j}\right\rangle = \sum_{j\in\mathbb{N}^*} \langle u, \overline{\phi_j}\rangle \left(\int_{\mathbb{R}^d} \psi\phi_j\right) = \left\langle \sum_{j\in\mathbb{N}^*} u_j\phi_j, \psi\right\rangle.$$

In the next Theorem 3.4 we show that a temperate distribution $u$ belongs to a certain Sobolev-Kato space if and only if its associated Fourier coefficients $\{u_j\}_{j\in\mathbb{N}^*}$ satisfy a certain behaviour for $j \to \infty$, in relation with the eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}^*}$ of $P$.

**Theorem 3.4.** Let $P \in \mathrm{Op}(S^{m,\mu}(\mathbb{R}^d))$ be an elliptic, normal SG operator with order components $m, \mu > 0$, and denote by $\{\phi_j\}_{j\in\mathbb{N}^*}$ a basis of orthonormal eigenfunctions of $P$ with corresponding eigenvalues

$\{\lambda_j\}_{j\in\mathbb{N}^*}$. Let $r \in \mathbb{N}$. Then, $u \in \mathscr{S}'(\mathbb{R}^d)$ belongs to $H^{rm,r\mu}(\mathbb{R}^d)$ if and only if

$$\sum_{j\in\mathbb{N}} |u_j|^2 |\lambda_j|^{2r} < \infty,$$

with $u_j$ defined in (3.3). Moreover,

$$\|u\|^2_{H^{rm,r\mu}(\mathbb{R}^d)} \asymp \sum_{j\in\mathbb{N}^*} |u_j|^2 |\widetilde{\lambda}_j|^{2r}, \tag{3.4}$$

with the eigenvalues $\{\widetilde{\lambda}_j\}_{j\in\mathbb{N}^*}$ of $\widetilde{P} = P + P_0$, $P_0$ the projection on $\ker P$, given in (3.2).

*Proof.* Let $u = \sum_{j\in\mathbb{N}^*} u_j \phi_j \in \mathscr{S}'$. Then,

$$\begin{aligned} \|P^r u\|^2_{L^2} &= \sum_{j\in\mathbb{N}^*} |(P^r u, \phi_j)|^2 = \sum_{j\in\mathbb{N}^*} \left| \sum_{k\in\mathbb{N}} u_k \left(P^r \phi_k, \phi_j\right) \right|^2 \\ &= \sum_{j\in\mathbb{N}^*} \left| \sum_{k\in\mathbb{N}} u_k \lambda_k^r \left(\phi_k, \phi_j\right) \right|^2 = \sum_{j\in\mathbb{N}^*} |u_j|^2 |\lambda_j|^{2r}. \end{aligned}$$

By normality of $P$, and the fact that $P_0$ is the projection on $\ker P$,

$$PP_0 = P_0 P = 0 \Rightarrow \widetilde{P}^r = (P + P_0)^r = P^r + P_0^r = P^r + P_0,$$

and the claims follow, in view of Proposition 3.1. □

**Remark 3.5.** Notice that (3.4) actually holds true for $u \in \mathscr{S}'$, $r \in \mathbb{Z}$. Indeed, for $r \in \mathbb{Z}$, $r < 0$, we have $\|u\|_{H^{rm,r\mu}} \asymp \|Q^r u\|_{L^2}$. Incidentally, we also observe that,

$$N_2 |\widetilde{\lambda}_j|^r \leqslant \|\phi_j\|_{H^{rm,r\mu}} \leqslant N_1 |\widetilde{\lambda}_j|^r, \quad j \in \mathbb{N}^*, r \in \mathbb{Z},$$

with constants $N_1, N_2 > 0$ depending only on $P, r, d, m, \mu$.

**Corollary 3.6.** Under the same hypotheses of Theorem 3.4, we have, for $u \in \mathscr{S}'(\mathbb{R}^d)$,

$$u \in \mathscr{S}(\mathbb{R}^d) \iff \sum_{j\in\mathbb{N}^*} |u_j|^2 |\lambda_j|^{2M} < \infty \text{ for any } M \in \mathbb{N}.$$

*Proof.* It follows immediately by Theorem 3.4, in view of the equality $\mathscr{S} = \bigcap_{r,\rho\in\mathbb{R}} H^{r,\rho}$ (see Chapter 1 and [26, Chapter 3]). □

The next Corollary 3.7 follows by combining Theorem 3.4 with results concerning the spectral asymptotics of self-adjoint SG classical operators and the properties of their fractional powers (see, e.g., [79], [87, Chapter 4] and the references quoted therein).

**Corollary 3.7.** Let $P \in \mathrm{Op}(S^{m,\mu}_{\mathrm{cl}}(\mathbb{R}^d))$ be an elliptic, invertible, self-adjoint, positive, classical SG operator with order components $m, \mu > 0$, and denote by $\{\phi_j\}_{j\in\mathbb{N}^*}$ a basis of orthonormal eigenfunctions of $P$ with corresponding eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}^*}$. Let $r \in \mathbb{R}$. Then, $u \in \mathscr{S}'(\mathbb{R}^d)$ belongs to $H^{rm,r\mu}(\mathbb{R}^d)$ if and only if

$$\begin{cases} \sum\limits_{j\in\mathbb{N}} |u_j|^2\, j^{\frac{2r\min\{m,\mu\}}{d}} < \infty, & \text{if } m \neq \mu, \\ \sum\limits_{j\in\mathbb{N}} |u_j|^2 \left(\dfrac{j}{\log j}\right)^{\frac{2rm}{d}} < \infty, & \text{if } m = \mu, \end{cases}$$

with $u_j$ defined in (3.3). Moreover,

$$\|u\|^2_{H^{rm,r\mu}(\mathbb{R}^d)} \asymp \sum_{j\in\mathbb{N}^*} |u_j|^2 |\lambda_j|^{2r} \asymp \begin{cases} \sum\limits_{j\in\mathbb{N}} |u_j|^2\, j^{\frac{2r\min\{m,\mu\}}{d}} < \infty, & \text{if } m \neq \mu, \\ \sum\limits_{j\in\mathbb{N}} |u_j|^2 \left(\dfrac{j}{\log j}\right)^{\frac{2rm}{d}} < \infty, & \text{if } m = \mu. \end{cases} \tag{3.5}$$

*Proof.* The operator $P^r$ is self-adjoint and it holds $P^r \in \mathrm{Op}(S^{rm,r\mu}_{\mathrm{cl}})$ (cf. [79]). It has eigenfunctions $\{\phi_j\}_{j\in\mathbb{N}^*}$ with eigenvalues $\{\lambda_j^r\}_{j\in\mathbb{N}^*}$, and satisfies $P^r u = \sum_{j\in\mathbb{N}^*} \lambda_j^r u_j \phi_j$. It is then bijective as an operator $P^r\colon H^{rm,r\mu} \to L^2$, so that, for any $u \in H^{rm,r\mu}$, $\|u\|_{H^{rm,r\mu}} \asymp \|P^r u\|_{L^2}$. The claims follow by $\|P^r u\|^2_{L^2} = \sum\limits_{j\in\mathbb{N}^*} |u_j|^2 |\lambda_j|^{2r}$, as in Theorem 3.4, taking into account (1.34) in Chapter 1. □

**Remark 3.8.** Under the hypotheses of Corollary 3.7, we observe that, for $r \in \mathbb{Z}$, $j \to \infty$,

$$\|\phi_j\|_{H^{rm,r\mu}} \asymp |\lambda_j|^r \asymp \begin{cases} j^{\frac{r\min\{m,\mu\}}{d}}, & \text{if } m \neq \mu, \\ \left(\dfrac{j}{\log j}\right)^{\frac{rm}{d}}, & \text{if } m = \mu, \end{cases} \tag{3.6}$$

where the constants tacitly involved in (3.6) depend only on $P, r, d, m, \mu$.

The next Lemma 3.9 is a property of the kernel of the powers $P^M$ of a SG operator of negative orders, for sufficiently large $M \in \mathbb{N}$.

**Lemma 3.9.** Let $P \in \mathrm{Op}(S^{m,\mu}(\mathbb{R}^d))$ be an SG operator with order components $m, \mu < 0$. Then $P^M$ is Hilbert-Schmidt for any $M \in \mathbb{N}$ such that $\max\{Mm, M\mu\} < -\frac{d}{2}$.

*Proof.* Denote by $K_M(x,y)$ the Schwartz kernel of the operator $P^M$ and by $p_M(x,\xi)$ its symbol. In order to prove that $P^M$ is Hilbert-Schmidt we have to show that the kernel

$$K_M(x,y) = (2\pi)^{-d} \int e^{i(x-y)\,\xi} p_M(x,\xi)\, d\xi$$

satisfies

$$\iint |K_M(x,y)|^2 \; dx\, dy < \infty.$$

By the calculus, $P^{Mm,M\mu}$ is an operator of order $Mm, M\mu$, hence it follows that $p_M(x,\xi) \in L^2(\mathbb{R}^{2d})$, since

$$\iint |p_M(x,\xi)|^2 \, dx\, d\xi \lesssim \iint \langle x\rangle^{2Mm} \langle \xi \rangle^{2M\mu} \, dx\, d\xi < \infty.$$

Moreover, notice that $K_M(x,y) = \left(\mathfrak{F}^{-1}_{\xi\to x-y}\, p_M\right)(x,\cdot) = b(x,z) \in L^2(\mathbb{R}^{2d})$, which yields

$$\iint |K_M(x,y)|^2 \; dx\, dy = \iint |b(x,z)|^2 \; dx\, dz < \infty,$$

showing that $P^M$ is Hilbert-Schmidt. □

We now prove a characterization of the elements of the space $\mathscr{S}(\mathbb{R}^d)$ by means of uniform convergence of series generated by SG operators. This is an analogue of Theorem 10.2 in [103].

**Theorem 3.10.** Let $P \in \mathrm{Op}(S^{m,\mu}_{\mathrm{cl}}(\mathbb{R}^d))$ be an elliptic, normal operator with order components $m, \mu > 0$ or $m, \mu < 0$, and denote by $\{\phi_j\}_{j\in\mathbb{N}^*}$ a basis of orthonormal eigenfunctions of $P$. Then, $f \in \mathscr{S}(\mathbb{R}^d)$ if and only if

$$\sum_{j\in\mathbb{N}^*} \left|(f, \phi_j)_{L^2(\mathbb{R}^d)}\right| |(A\phi_j)(x)|, \quad x \in \mathbb{R}^d, \tag{3.7}$$

converges uniformly on $\mathbb{R}^d$ for every SG operator $A$.

*Proof.* Notice that, since (3.7) involves only the eigenfunctions of $P$, it is possible to consider the operator $\widetilde{P} = P + P_0 \in \mathrm{Op}(S^{m,\mu})$ in place of $P$ as the eigenfunctions are the same, with eigenvalues $\{\widetilde{\lambda}_j\}_{j\in\mathbb{N}^*}$. Then, it is not restrictive to assume $P$ invertible, and even with order components $m, \mu < 0$. Indeed, if that is not the case, since $\widetilde{P}$ has no vanishing eigenvalues, we can instead employ $Q = (P + P_0)^{-1} \in \mathrm{Op}(S^{-m,-\mu})$, which has again the same eigenfunctions, with eigenvalues $\{\widetilde{\lambda}_j^{-1}\}_{j\in\mathbb{N}^*}$. Having negative order components, as an operator from $L^2$ to itself, $P$ is then compact. Finally, $P^M$, $M \in \mathbb{N}$, has again the same eigenfunctions of $P$, $P+P_0$, and $(P+P_0)^{-1}$, with eigenvalues $\{\lambda_j^M\}_{j\in\mathbb{N}^*}$. Summing up, for this proof we may assume that $P$ is also Hilbert-Schmidt (cf. Lemma 3.9), that is,

$$m, \mu < -\frac{d}{2} \Rightarrow \sum_{j\in\mathbb{N}^*} |\lambda_j|^2 < \infty, \quad \lambda_j \neq 0, j \in \mathbb{N}^*,$$

and the topology of $H^{Mm,M\mu}$ is given by the equivalent norm $\|P^M u\|_{L^2}$, $M \in \mathbb{Z}$, $M < 0$.

Recall that, by the embeddings of the Sobolev-Kato spaces (cf. Chapter 1 and [26, Chapter 3]) and Sobolev's Lemma, for $t \geqslant 0$, $\tau > \frac{d}{2}$, one has $\mathscr{S} \hookrightarrow H^{t,\tau} \hookrightarrow H^{0,\tau} \equiv H^\tau \hookrightarrow L^\infty$.

Assume $A \in \mathrm{Op}(S^{pm,p\mu})$, $p \in \mathbb{Z}$, $p < 0$. Again, this is no restriction, given the inclusions among the SG symbols and operators spaces. It follows that, for any $j \in \mathbb{Z}$, $u \in H^{(j+p)m,(j+p)\mu}$,

$$\|Au\|_{H^{jm,j\mu}} \lesssim \|u\|_{H^{(j+p)m,(j+p)\mu}},$$

and, by the observations above,

$$
\begin{aligned}
|(A\phi_j)(x)| \leqslant \|A\phi_j\|_{L^\infty} \lesssim \|A\phi_j\|_{H^{-m,-\mu}} &\lesssim \|\phi_j\|_{H^{m(p-1),\mu(p-1)}} \\
&\asymp \|P^{p-1}\phi_j\|_{L^2} = |\lambda_j|^{p-1}.
\end{aligned}
$$

Now, in view of Corollary 3.6, the sequence $\{|(f,\phi_j)|\,|\lambda_j|^{p-3}\}_{j\in\mathbb{N}^*}$ is bounded for any $f \in \mathscr{S}$: indeed, its elements are the terms of a convergent series. We then conclude

$$
\begin{aligned}
\sum_{j\in\mathbb{N}^*} |(f,\phi_j)|\,|(A\phi_j)(x)| &\lesssim \sum_{j\in\mathbb{N}^*} |(f,\phi_j)|\,|\lambda_j|^{p-3}\,|\lambda_j|^2 \\
&\lesssim \sum_{j\in\mathbb{N}^*} |\lambda_j|^2 < \infty, \quad x \in \mathbb{R}^d.
\end{aligned}
$$

We have showed that, for any $f \in \mathscr{S}$ and any SG operator $A$, (3.7) is totally convergent, which implies the first part of the claim.

Conversely, assume that (3.7) converges uniformly for every operator $A \in \mathrm{Op}(S^{s,\sigma})$, $s, \sigma \in \mathbb{R}$. Then we can take $A = M^\alpha D^\beta \colon \mathscr{S} \to \mathscr{S}$, where $D^\beta = (-i)^{|\beta|}\partial^\beta$ and, for $\phi \in \mathscr{S}$, $(M^\alpha\phi)(x) = x^\alpha\phi(x)$, $x \in \mathbb{R}^d$, $\alpha, \beta \in \mathbb{Z}^d_+$, so that $A \in \mathrm{Op}(S^{|\alpha|,|\beta|}_{\mathrm{cl}})$. In principle, we assume $f \in \mathscr{S}'$ and the coefficients $(f,\phi_j)$ understood in the sense of (3.3). However, defining

$$
f_{\alpha\beta}(x) = \sum_{j\in\mathbb{N}^*} (f,\phi_j)\,(M^\alpha D^\beta\phi_j)(x) = \sum_{j\in\mathbb{N}^*} (f,\phi_j)\,x^\alpha D^\beta\phi_j(x), x \in \mathbb{R}^d, \tag{3.8}
$$

the hypotheses clearly imply that $f_{\alpha\beta} \in \mathscr{C}$, and actually, setting $g = f_{00}$, it also follows $g \in \mathscr{C}^\infty$ and $M^\alpha D^\beta g = f_{\alpha\beta}$, $\alpha, \beta \in \mathbb{Z}^d_+$. Moreover, by the hypothesis of uniform convergence on $\mathbb{R}^d$ of (3.7), we also see that for any $\alpha, \beta \in \mathbb{Z}^d_+$ there exists $\nu_{\alpha\beta} \in \mathbb{N}$ such that, for any $N > \nu_{\alpha\beta}$,

$$
\begin{aligned}
\sup_{x\in\mathbb{R}^d} &\left| x^\alpha D^\beta g(x) - \sum_{j=1}^{N} (f,\phi_j)\,x^\alpha D^\beta\phi_j(x) \right| \\
&= \sup_{x\in\mathbb{R}^d} \left| \sum_{j\geqslant N+1} (f,\phi_j)\,x^\alpha D^\beta\phi_j(x) \right| \leqslant 1.
\end{aligned}
$$

Since $\phi_j \in \mathscr{S}$, $j \in \mathbb{N}^*$, it follows that, for any $\alpha, \beta \in \mathbb{Z}^d_+$, choosing

$N > \nu_{\alpha\beta}$, there exists $C_{\alpha\beta} > 0$ such that

$$\sup_{x\in\mathbb{R}^d} |x^\alpha D^\beta g(x)| \leqslant \sup_{x\in\mathbb{R}^d} \left| \sum_{j=1}^{N} (f, \phi_j)\, x^\alpha D^\beta \phi_j(x) \right| + \sup_{x\in\mathbb{R}^d} \left| \sum_{j\geqslant N+1} (f, \phi_j)\, x^\alpha D^\beta \phi_j(x) \right| \leqslant C_{\alpha\beta},$$

that is, $g \in \mathscr{S} \subset L^2$. Since $g$ and $f$ have the same Fourier coefficients $(f, \phi_j)$, $j \in \mathbb{N}^*$, we conclude $f = g \in \mathscr{S}$, as claimed.

The proof is complete. □

## 3.2 Gevrey-periodic weighted Sobolev spaces on $\mathbb{T}^n \times \mathbb{R}^d$

In this section we introduce a classes of Gevrey time-periodic Sobolev-Kato spaces and investigate their properties. Our goal consists in characterizing these spaces by means of eigenfunction expansions generated by an elliptic, normal SG operator, see below Theorem 3.18 and its corollaries, Theorem 3.22, and Theorem 3.26.

### 3.2.1 Gevrey classes on the torus

We begin by recalling the standard characterisation of the Gevrey classes on the $n$-dimensional torus $\mathbb{T}^n$. Given $\eta > 0$ and $\sigma \geqslant 1$, $\mathcal{G}^{\sigma,\eta} = \mathcal{G}^{\sigma,\eta}(\mathbb{T}^n)$ denotes the space of all smooth functions $u \in \mathscr{C}^\infty(\mathbb{T}^n)$ such that there exists $C = C_{\sigma\eta} > 0$ for which

$$\sup_{t\in\mathbb{T}^n} |\partial_t^\gamma u(t)| \leqslant C\eta^{|\gamma|}(\gamma!)^\sigma, \quad \gamma \in \mathbb{Z}_+^n.$$

The space $\mathcal{G}^{\sigma,\eta}$ is a Banach space endowed with the norm

$$\|u\|_{\mathcal{G}^{\sigma,\eta}} = \sup_{\gamma\in\mathbb{Z}_+^n} \left\{ \sup_{t\in\mathbb{T}^n} \eta^{-|\gamma|}(\gamma!)^{-\sigma} |\partial_t^\gamma u(t)| \right\}.$$

The space of periodic Gevrey functions of order $\sigma$ is defined by

$$\mathcal{G}^\sigma(\mathbb{T}^n) = \operatorname*{ind\,lim}_{\eta\to+\infty} \mathcal{G}^{\sigma,\eta}(\mathbb{T}^n).$$

The dual space $(\mathcal{G}^\sigma(\mathbb{T}^n))'$ is defined as the set of all linear maps $\theta: \mathcal{G}^\sigma(\mathbb{T}^n) \to \mathbb{C}$ with the property that (cf. [94]) for every $\eta > 0$ there exists $C = C_\eta > 0$ such that, for any $u \in \mathcal{G}^{\sigma,\eta}$,

$$|\langle\theta, u\rangle| \leqslant C \sup_{\gamma\in\mathbb{Z}^n_+} \left\{ \sup_{t\in\mathbb{T}^n} \eta^{-|\gamma|}(\gamma!)^{-\sigma}|\partial_t^\gamma u(t)| \right\} = C\|u\|_{\mathcal{G}^{\sigma,\eta}}.$$

### 3.2.2 Inductive limits of Gevrey time-periodic Sobolev-Kato spaces and their duals

We now define families of Banach spaces of smooth maps on the torus, taking values in weighted Sobolev spaces, and satisfying Gevrey-type estimates with respect to the variable $t \in \mathbb{T}^n$.

**Definition 3.11.** Let $\sigma > 1$ and $r, \rho \in \mathbb{R}$ be fixed. Given a constant $C > 0$ we denote by

$$\mathcal{G}^\sigma\mathcal{H}^C_{r,\rho} \equiv \mathcal{G}^\sigma\mathcal{H}^C_{r,\rho}(\mathbb{T}^n \times \mathbb{R}^d) \tag{3.9}$$

the space of all functions $u \in \mathscr{C}^\infty(\mathbb{T}^n; \mathscr{S}'(\mathbb{R}^d))$ such that

$$\begin{aligned}\|u\|_{\mathcal{G}^\sigma\mathcal{H}^C_{r,\rho}(\mathbb{T}^n\times\mathbb{R}^d)} &= \|u\|_{\sigma,C,r,\rho} \\ &\equiv \sup_{\gamma\in\mathbb{Z}^n_+} \left\{ C^{-|\gamma|}(\gamma!)^{-\sigma} \sup_{t\in\mathbb{T}^n} \|\partial_t^\gamma u(t)\|_{H^{r,\rho}(\mathbb{R}^d)} \right\} < +\infty.\end{aligned} \tag{3.10}$$

If $u \in \mathcal{G}^\sigma\mathcal{H}^C_{r,\rho}$, then:

(i) for any $\gamma \in \mathbb{Z}^n_+$ it is well defined the map

$$\mathbb{T}^n \ni t \mapsto \partial_t^\gamma u(t) \in H^{r,\rho}; \tag{3.11}$$

in particular, $u(t) \in H^{r,\rho}$ for all $t \in \mathbb{T}^n$;

(ii) for any $\gamma \in \mathbb{Z}_+^n$ and $r, \rho \in \mathbb{R}$ there exists $C_{r,\rho} > 0$ such that

$$\sup_{t \in \mathbb{T}^n} \|\partial_t^\gamma u(t)\|_{H^{r,\rho}} \leqslant C_{r,\rho}^{|\gamma|+1} (\gamma!)^\sigma,$$

implying a Gevrey estimate to (3.11).

We now investigate the topological properties of these spaces. First, notice that $\mathcal{G}^\sigma \mathcal{H}_{r,\rho}^C$ is a Banach space endowed with the norm $\|\cdot\|_{\sigma,C,r,\rho}$ given by (3.10). Moreover, we have the immediate inclusions

$$\mathcal{G}^\sigma \mathcal{H}_{r,\rho}^C \subset \mathcal{G}^\mu \mathcal{H}_{r,\rho}^C, \; 1 < \sigma \leqslant \mu, \tag{3.12}$$

$$\mathcal{G}^\sigma \mathcal{H}_{r,\rho}^C \subset \mathcal{G}^\sigma \mathcal{H}_{r,\rho}^{\widetilde{C}}, \; 0 < C \leqslant \widetilde{C}, \tag{3.13}$$

$$\mathcal{G}^\sigma \mathcal{H}_{r,\rho}^C \subset \mathcal{G}^\sigma \mathcal{H}_{t,\tau}^C, \; t \leqslant r \text{ and } \tau \leqslant \rho. \tag{3.14}$$

**Definition 3.12.** For $\sigma > 1$ and $r, \rho \in \mathbb{R}$ we define the spaces

$$\mathcal{G}^\sigma \mathcal{H}_{r,\rho} = \mathcal{G}^\sigma \mathcal{H}_{r,\rho}(\mathbb{T}^n \times \mathbb{R}^d) \equiv \bigcup_{C>0} \mathcal{G}^\sigma \mathcal{H}_{r,\rho}^C,$$

and

$$\mathcal{G}\mathcal{H}_{r,\rho} = \mathcal{G}\mathcal{H}_{r,\rho}(\mathbb{T}^n \times \mathbb{R}^d) \equiv \bigcup_{\sigma>1} \mathcal{G}^\sigma \mathcal{H}_{r,\rho} = \bigcup_{\sigma>1} \left( \bigcup_{C>0} \mathcal{G}^\sigma \mathcal{H}_{r,\rho}^C \right), \tag{3.15}$$

equipped with the induced inductive limit topologies.

In this way, $\mathcal{G}\mathcal{H}_{r,\rho}$ turns out to be an inductive limit of an inductive limit, which is (mildly) inconvenient. However, in view of the next Theorem 3.13, we may see it as an inductive limit of Banach spaces, that is, as an (LB)-space.

**Theorem 3.13.** For each $r, \rho \in \mathbb{R}$ the following equality holds true, both among sets as well as topological spaces:

$$\mathcal{G}\mathcal{H}_{r,\rho}(\mathbb{T}^n \times \mathbb{R}^d) = \bigcup_{\sigma>1} \mathcal{G}^\sigma \mathcal{H}_{r,\rho}^{\sigma-1}(\mathbb{T}^n \times \mathbb{R}^d), \tag{3.16}$$

where the set in the right-hand side is endowed with the induced inductive limit topology.

*Proof.* Since $\mathcal{G}^\sigma\mathcal{H}^{\sigma-1}_{r,\rho} = \mathcal{G}^\sigma\mathcal{H}^{C_\sigma}_{r,\rho}$, with $C_\sigma = \sigma - 1 > 0$, we see that the right-hand side of (3.16) is a subset of the left-hand side of (3.15). Conversely, let $u \in \mathcal{G}\mathcal{H}_{r,\rho}$. Then, $u \in \mathcal{G}^\sigma\mathcal{H}^C_{r,\rho}$, for some $\sigma > 1$ and $C > 0$. If $C < \sigma - 1$, then

$$u \in \mathcal{G}^\sigma\mathcal{H}^C_{r,\rho} \subset \mathcal{G}^\sigma\mathcal{H}^{\sigma-1}_{r,\rho},$$

in view (3.13). On the other hand, if $\sigma \leqslant C + 1$, we obtain

$$u \in \mathcal{G}^\sigma\mathcal{H}^C_{r,\rho} \subset \mathcal{G}^{C+1}\mathcal{H}^C_{r,\rho},$$

in view of (3.12). Hence, $u \in \mathcal{G}^\delta\mathcal{H}^{\delta-1}_{r,\rho}$, where $\delta = C + 1 > 1$. Therefore, we have proved the claimed equality as sets.

Now, let $\{u_j\}$ be a convergent sequence in $\bigcup\limits_{\sigma>1}\mathcal{G}^\sigma\mathcal{H}^{\sigma-1}_{r,\rho}$. Then, $u_j \to 0$ in $\mathcal{G}^\sigma\mathcal{H}^{C_\sigma}_{r,\rho}$, with $C_\sigma = \sigma - 1$, implying the convergence in $\mathcal{G}\mathcal{H}_{r,\rho}$. On the other hand, suppose now that $\{u_j\}$ is a convergent sequence in $\mathcal{G}\mathcal{H}_{r,\rho}$ defined as in (3.15), namely, there are $\sigma > 1$ and $C > 0$ such that

$$\lim_{j\to\infty}\left[\sup_{\gamma\in\mathbb{Z}^n_+} C^{-|\gamma|}(\gamma!)^{-\sigma}\sup_{t\in\mathbb{T}^n}\|\partial_t^\gamma u_j(t)\|_{H^{r,\rho}}\right] = 0.$$

If $C \leqslant \sigma - 1$, then

$$u_j \in \mathcal{G}^\sigma\mathcal{H}^C_{r,\rho} \subset \mathcal{G}^\sigma\mathcal{H}^{\sigma-1}_{r,\rho},\ j \in \mathbb{N}^*,$$

and

$$(\sigma - 1)^{-|\gamma|}(\gamma!)^{-\sigma}\sup_{t\in\mathbb{T}^n}\|\partial_t^\gamma u_j(t)\|_{H^{r,\rho}} \leqslant C^{-|\gamma|}(\gamma!)^{-\sigma}\sup_{t\in\mathbb{T}^n}\|\partial_t^\gamma u_j(t)\|_{H^{r,\rho}},$$

which yields

$$\lim_{j\to\infty}\left[\sup_{\gamma\in\mathbb{Z}^n_+}(\sigma - 1)^{-|\gamma|}(\gamma!)^{-\sigma}\sup_{t\in\mathbb{T}^n}\|\partial_t^\gamma u_j(t)\|_{H^{r,\rho}(\mathbb{R}^d)}\right] = 0.$$

If $\sigma < C + 1$, then

$$u_j \in \mathcal{G}^\sigma\mathcal{H}^C_{r,\rho} \subset \mathcal{G}^{C+1}\mathcal{H}^C_{r,\rho},\ j \in \mathbb{N}^*,$$

and

$$C^{-|\gamma|}(\gamma!)^{-(C+1)}\sup_{t\in\mathbb{T}^n}\|\partial_t^\gamma u_j(t)\|_{H^{r,\rho}} \leqslant C^{-|\gamma|}(\gamma!)^{-\sigma}\sup_{t\in\mathbb{T}^n}\|\partial_t^\gamma u_j(t)\|_{H^{r,\rho}},$$

which implies

$$\lim_{j\to\infty}\left[\sup_{\gamma\in\mathbb{Z}_+^n}C^{-|\gamma|}(\gamma!)^{-(C+1)}\sup_{t\in\mathbb{T}^n}\|\partial_t^\gamma u_j(t)\|_{H^{r,\rho}(\mathbb{R}^d)}\right]=0.$$

Therefore, $u_j\to 0$ in $\bigcup_{\sigma>1}\mathcal{G}^\sigma\mathcal{H}_{r,\rho}^{\sigma-1}$, as desired. This completes the proof.

□

In view of Theorem 3.13, we are allowed to consider the space $\mathcal{G}\mathcal{H}_{r,\rho}$ equipped with the equivalent inductive limit topology defined by (3.16). In particular, we denote by

$$\mathcal{G}\mathcal{H}'_{r,\rho}=\mathcal{G}\mathcal{H}'_{r,\rho}(\mathbb{T}^n\times\mathbb{R}^d)=(\mathcal{G}\mathcal{H}_{r,\rho}(\mathbb{T}^n\times\mathbb{R}^d))'$$

the space of all linear and continuous maps $\theta:\mathcal{G}\mathcal{H}_{r,\rho}\to\mathbb{C}$. Then, for every $\sigma>1$, there exists $A=A_\sigma>0$ such that

$$|\langle\theta,u\rangle|\leqslant A\sup_{\gamma\in\mathbb{Z}_+^n}\left\{(\sigma-1)^{-|\gamma|}(\gamma!)^{-\sigma}\sup_{t\in\mathbb{T}^n}\|\partial_t^\gamma u(t)\|_{H^{r,\rho}}\right\},$$

for every $u\in\mathcal{G}^\sigma\mathcal{H}_{r,\rho}^{\sigma-1}$, which is equivalent to the assertion that, for every $\sigma>1, C>0$, there exists $B=B_{\sigma C}>0$ such that

$$|\langle\theta,u\rangle|\leqslant B\sup_{\gamma\in\mathbb{Z}_+^n}\left\{C^{-|\gamma|}(\gamma!)^{-\sigma}\sup_{t\in\mathbb{T}^n}\|\partial_t^\gamma u(t)\|_{H^{r,\rho}}\right\},$$

for every $u\in\mathcal{G}^\sigma\mathcal{H}_{r,\rho}^{C}$. Therefore, the next Proposition 3.14 holds true.

**Proposition 3.14.** A linear functional $\theta:\mathcal{G}\mathcal{H}_{r,\rho}(\mathbb{T}^n\times\mathbb{R}^d)\to\mathbb{C}$ is an element of $\mathcal{G}\mathcal{H}'_{r,\rho}(\mathbb{T}^n\times\mathbb{R}^d)$ if and only if, for every $\sigma>1, C>0$, there exists $B=B_{\sigma C}>0$ such that, for any $u\in\mathcal{G}^\sigma\mathcal{H}_{r,\rho}^{C}(\mathbb{T}^n\times\mathbb{R}^d)$,

$$|\langle\theta,u\rangle|\leqslant B\sup_{\gamma\in\mathbb{Z}_+^n}\left\{C^{-|\gamma|}(\gamma!)^{-\sigma}\sup_{t\in\mathbb{T}^n}\|\partial_t^\gamma u(t)\|_{H^{r,\rho}(\mathbb{R}^d)}\right\}.$$

We can actually describe $\mathcal{GH}'_{r,\rho}$ as a projective limit, dual to the inductive limit definition (3.16) of $\mathcal{GH}_{r,\rho}$.

**Theorem 3.15.** For each $r, \rho \in \mathbb{R}$ we have

$$\mathcal{GH}'_{r,\rho}(\mathbb{T}^n \times \mathbb{R}^d) = \bigcap_{\sigma>1} (\mathcal{G}^\sigma \mathcal{H}^{\sigma-1}_{r,\rho}(\mathbb{T}^n \times \mathbb{R}^d))',$$

equipped with the projective limit topology.

*Proof.* If $\theta \in \mathcal{GH}'_{r,\rho}$, then $\theta$ is well defined as linear map on $\mathcal{G}^\sigma \mathcal{H}_{r,\rho}$, for each $\sigma > 1$. Hence, for $\sigma > 1$ and $C > 0$ there is $B = B_{\sigma C} > 0$ such that

$$|\langle \theta, u\rangle| \leqslant B \sup_{\gamma \in \mathbb{Z}^n_+} \left\{ C^{-|\gamma|} (\gamma!)^{-\sigma} \sup_{t \in \mathbb{T}^n} \|\partial_t^\gamma u(t)\|_{H^{r,\rho}} \right\},$$

for all $u \in \mathcal{G}^\sigma \mathcal{H}^C_{r,\rho}$. This shows, in particular, that the restriction of $\theta$ to $\mathcal{G}^\sigma \mathcal{H}^C_{r,\rho}$ belongs to $(\mathcal{G}^\sigma \mathcal{H}^{\sigma-1}_{r,\rho})'$.

Conversely, let $\theta \in \bigcap_{\sigma>1} (\mathcal{G}^\sigma \mathcal{H}^{\sigma-1}_{r,\rho})'$. Then, for each $\sigma > 1$ we have that $\theta \in (\mathcal{G}^\sigma \mathcal{H}^{\sigma-1}_{r,\rho})'$ and that $\theta$ is well defined on $\mathcal{GH}_{r,\rho}$. Moreover, if $u, v \in \mathcal{GH}_{r,\rho}$, there exists a $\delta$ such that $u, v \in \mathcal{G}^\delta \mathcal{H}_{r,\rho}$ and for every $\alpha \in \mathbb{C}$ we have

$$\langle \theta\, ,\, \alpha u + v\rangle = \alpha \langle \theta\, ,\, u\rangle + \langle \theta\, ,\, v\rangle.$$

Then, $\theta$ is linear on $\mathcal{GH}_{r,\rho}$. Also, given $\sigma > 1$ and $C > 0$, there is $B = B_{\sigma C} > 0$ satisfying

$$|\langle \theta\, ,\, u\rangle| \leqslant B \sup_{\gamma \in \mathbb{Z}^n_+} \left\{ C^{-|\gamma|} (\gamma!)^{-\sigma} \sup_{t \in \mathbb{T}^n} \|\partial_t^\gamma u(t)\|_{H^{r,\rho}} \right\}, \quad u \in \mathcal{G}^\sigma \mathcal{H}^C_{r,\rho}.$$

Therefore, $\theta \in \mathcal{GH}'_{r,\rho}$. □

### 3.2.3 The spaces $\mathscr{F}$ and $\mathscr{F}'$

We introduce now the target spaces with respect to our analysis, namely those spaces which will be considered in studying the hypoellipticity of suitable operators in Section 3.3 below.

We observe that, in view of (3.14), we have

$$\mathcal{GH}_{r,\rho} \subset \mathcal{GH}_{t,\tau},$$

whenever $t \leqslant r$ and $\tau \leqslant \rho$.

**Definition 3.16.** We define the space $\mathscr{F}$ as

$$\mathscr{F} = \mathscr{F}(\mathbb{T}^n \times \mathbb{R}^d) \equiv \bigcap_{r,\rho\in\mathbb{R}} \mathcal{G}\mathcal{H}_{r,\rho}(\mathbb{T}^n \times \mathbb{R}^d), \tag{3.17}$$

endowed with the projective limit topology.

In particular, $f_j \to 0$, $j \to \infty$, in $\mathscr{F}$ if, and only if, for each $r, \rho \in \mathbb{R}$ there is $\sigma = \sigma_{r\rho} > 1$ such that

$$\lim_{j\to\infty} f_j = 0 \quad \text{in} \quad \mathcal{G}^\sigma\mathcal{H}_{r,\rho}.$$

By $\mathscr{F}'$ we denote the space of all linear continuous functionals $\theta: \mathscr{F} \to \mathbb{C}$. It follows, by [68, (6), page 290], that

$$\mathscr{F}' = \mathscr{F}'(\mathbb{T}^n \times \mathbb{R}^d) = \bigcup_{r,\rho\in\mathbb{R}} \mathcal{G}\mathcal{H}'_{r,\rho}(\mathbb{T}^n \times \mathbb{R}^d), \tag{3.18}$$

endowed with the inductive limit topology.

### 3.2.4 Eigenfunctions expansions in $\mathscr{F}$

In this section we characterise the elements of the space $\mathscr{F}$ in terms of the behaviour of their time-dependent coefficients of the eigenfunctions expansion generated by a SG operator of the type considered in Section 3.1. Indeed, as above, here we will often denote by $P \in \mathrm{Op}(S^{m,\mu})$ a normal, elliptic SG operator with order components $m, \mu > 0$, and by $\{\phi_j\}_{j\in\mathbb{N}^*} \subset \mathscr{S}$ its associated orthonormal basis of eigenfunctions, with eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}^*}$.

**Remark 3.17.** Notice that if we also assume $P \in \mathrm{Op}(S^{m,\mu}_{\mathrm{cl}})$, self-adjoint and positive, both in the case $m \neq \mu$ and in the case $m = \mu$, we have that, for a suitable choice of $\varrho, \varrho' > 0$ and constants $K, K' > 0$

$$K' j^{\varrho'} \leqslant |\lambda_j| = \lambda_j \leqslant K j^{\varrho}, \ j \to \infty.$$

Moreover, $\varrho, \varrho'$ can be chosen such that $\varrho - \varrho' \leqslant \epsilon$, for any $\epsilon > 0$ (see Section 1.4.2).

If $f \in \mathscr{F}$, then for all $r, \rho \in \mathbb{R}$ there are $\sigma = \sigma_{r\rho} > 1$ and $C = C_{r\rho} > 0$ such that

$$\|f\|_{r,\rho,\sigma,C} \equiv \sup_{\gamma \in \mathbb{Z}_+^n} \left\{ C^{-|\gamma|} (\gamma!)^{-\sigma} \sup_{t \in \mathbb{T}^n} \|\partial_t^\gamma f(t)\|_{H^{r,\rho}} \right\} < +\infty.$$

Since, for any $\gamma \in \mathbb{Z}_+^n$,

$$\mathbb{T}^n \ni t \mapsto \partial_t^\gamma f(t) \in \bigcap_{r,\rho \in \mathbb{R}} H^{r,\rho} = \mathscr{S},$$

we see that $f \in \mathscr{C}^\infty(\mathbb{T}^n; \mathscr{S})$ and for all $r, \rho \in \mathbb{R}$ there are $\sigma > 1$ and $C > 0$ as above such that, for all $\gamma \in \mathbb{Z}_+^n$,

$$\sup_{t \in \mathbb{T}^n} \|\partial_t^\gamma f(t)\|_{H^{r,\rho}} \leqslant C^{|\gamma|+1} (\gamma!)^\sigma.$$

Given $f \in \mathscr{F}$, we then have, for any $t \in \mathbb{T}^n$, the Schwartz function $f(t) : \mathbb{R}^d \to \mathbb{C}$. Its Fourier coefficients $f_j : \mathbb{T}^n \to \mathbb{C}$, given by

$$f_j(t) \equiv (f(t), \phi_j), \quad j \in \mathbb{N}^*, \tag{3.19}$$

are well-defined, we actually have $f_j \in \mathscr{C}^\infty(\mathbb{T}^n)$ and, for any $\gamma \in \mathbb{Z}_+^n$,

$$f_j^\gamma(t) \equiv (\partial_t^\gamma f(t), \phi_j) = \partial_t^\gamma f_j(t). \tag{3.20}$$

Indeed, denoting by $\|\!|\cdot|\!\|_N$, $N = 0, 1, \dots$, the $N$-th seminorm of $\mathscr{S}$, namely,

$$|\!|\!|g|\!|\!|_N = \sum_{|\alpha+\beta| \leqslant N} \sup_{x \in \mathbb{R}^d} |x^\alpha \partial^\beta g(x)|, \quad g \in \mathscr{S},$$

for any $f \in \mathscr{F}$, $\gamma \in \mathbb{Z}_+^n$, $N = 0, 1, \dots$, we of course have

$$|\!|\!|\partial_t^\gamma f(t)|\!|\!|_N \in \mathscr{C}(\mathbb{T}^n) \Rightarrow \sup_{t \in \mathbb{T}^n} |\!|\!|\partial_t^\gamma f(t)|\!|\!|_N = \max_{t \in \mathbb{T}^n} |\!|\!|\partial_t^\gamma f(t)|\!|\!|_N = S_N^\gamma < \infty.$$

Then,

$$|\partial_t^\gamma f(t,x)| \leqslant S_{2N}^\gamma (1 + |x|^2)^{-N}, \quad f \in \mathscr{F}, \gamma \in \mathbb{Z}_+^n, N \in \mathbb{N}, t \in \mathbb{T}^n, x \in \mathbb{R}^d,$$

and (3.20) follows by dominated convergence. We have shown, for any $f \in \mathscr{F}$, $\gamma \in \mathbb{Z}_+^n$, $t \in \mathbb{T}^n$,

$$\partial_t^\gamma f(t) = \partial_t^\gamma \sum_{j \in \mathbb{N}^*} f_j(t) \phi_j = \sum_{j \in \mathbb{N}^*} f_j^\gamma(t) \phi_j = \sum_{j \in \mathbb{N}^*} \partial_t^\gamma f_j(t) \phi_j, \tag{3.21}$$

The next Theorem 3.18, together with its Corollaries 3.19, 3.20, 3.21, is our third main result of this chapter.

**Theorem 3.18.** Let $P \in \mathrm{Op}(S^{m,\mu}(\mathbb{R}^d))$ be a normal, elliptic SG operator with order components $m, \mu > 0$. Then $f \in \mathscr{F}(\mathbb{T}^n \times \mathbb{R}^d)$ if and only if it can be represented as

$$f(t) = \sum_{j \in \mathbb{N}^*} f_j(t)\phi_j$$

with $f_j(t)$, $j \in \mathbb{N}^*$, defined by (3.19) satisfying the condition

for every $M \in \mathbb{N}$ there exist $\sigma = \sigma_M > 1$ and $C = C_M > 0$ such that
$$\sup_{t \in \mathbb{T}^n} \sum_{j \in \mathbb{N}^*} \widetilde{\lambda}_j^{2M} |\partial_t^\gamma f_j(t)|^2 = \sup_{t \in \mathbb{T}^n} \sum_{j \in \mathbb{N}^*} \widetilde{\lambda}_j^{2M} |f_j^\gamma(t)|^2 \leqslant C^{2(|\gamma|+1)}(\gamma!)^{2\sigma}, \tag{*}$$
for every $\gamma \in \mathbb{Z}_+^n$.

*Proof.* Let $f \in \mathscr{F}$. By (3.21), for any $\gamma \in \mathbb{Z}_+^n$ we can write

$$\partial_t^\gamma f(t) = \sum_{j \in \mathbb{N}^*} f_j^\gamma(t)\phi_j,$$

with $f_j^\gamma(t)$ in (3.20). Moreover, denoting by $P_0 \in \mathrm{Op}(S^{-\infty,-\infty})$ the projection on $\ker P$ by the definition of $\mathscr{F}$ and Corollary 3.6,

$$\begin{aligned}
f \in \mathscr{F} &\iff \forall M \in \mathbb{N}\ f \in \mathcal{GH}_{Mm,M\mu} \\
&\iff \forall M \in \mathbb{N}\ \exists \sigma = \sigma_M > 1, C = C_M > 0 \text{ such that } \forall \gamma \in \mathbb{Z}_+^n \\
&\qquad \sup_{t \in \mathbb{T}^n} \|\partial_t^\gamma f(t)\|^2_{H^{Mm,M\mu}} \leqslant C^{2(|\gamma|+1)}(\gamma!)^{2\sigma} \\
&\iff \forall M \in \mathbb{N}\ \exists \sigma > 1, C > 0 \text{ such that } \forall \gamma \in \mathbb{Z}_+^n \\
&\qquad \sup_{t \in \mathbb{T}^n} \|(P + P_0)^M \partial_t^\gamma f(t)\|^2_{L^2} \leqslant C^{2(|\gamma|+1)}(\gamma!)^{2\sigma} \\
&\iff \forall M \in \mathbb{N}\ \exists \sigma > 1, C > 0 \text{ such that } \forall \gamma \in \mathbb{Z}_+^n \\
&\qquad \sup_{t \in \mathbb{T}^n} \left[\|P^M \partial_t^\gamma f(t) + P_0^M \partial_t^\gamma f(t)\|^2_{L^2}\right] \leqslant C^{2(|\gamma|+1)}(\gamma!)^{2\sigma} \\
&\iff \forall M \in \mathbb{N}\ \exists \sigma > 1, C > 0 \text{ such that } \forall \gamma \in \mathbb{Z}_+^n \\
&\qquad \sup_{t \in \mathbb{T}^n} \left[\left\|\sum_{j \geqslant N+1} f_j^\gamma(t) P^M \phi_j\right\|^2_{L^2} + \left\|\sum_{j=1}^{N} f_j^\gamma(t) P_0^M \phi_j\right\|^2_{L^2}\right] \\
&\qquad \leqslant C^{2(|\gamma|+1)}(\gamma!)^{2\sigma},
\end{aligned}$$

which yields

$$\begin{aligned} f \in \mathscr{F} \iff & \forall M \in \mathbb{N}\, \exists \sigma > 1, C > 0 \text{ such that } \forall \gamma \in \mathbb{Z}^n_+ \\ & \sup_{t\in\mathbb{T}^n} \left[ \sum_{j \geqslant N+1} |\lambda_j|^{2M} |f_j^\gamma(t)|^2 + \sum_{j=1}^N |f_j^\gamma(t)|^2 \right] \leqslant C^{2(|\gamma|+1)} (\gamma!)^{2\sigma} \\ \iff & \forall M \in \mathbb{N}\, \exists \sigma > 1, C > 0 \text{ such that } \forall \gamma \in \mathbb{Z}^n_+ \\ & \sup_{t\in\mathbb{T}^n} \sum_{j\in\mathbb{N}^*} |\widetilde{\lambda}_j|^{2M} |f_j^\gamma(t)|^2 \leqslant C^{2(|\gamma|+1)} (\gamma!)^{2\sigma}, \end{aligned}$$

as claimed. □

**Corollary 3.19.** Let $P \in \mathrm{Op}(S^{m,\mu}_{\mathrm{cl}}(\mathbb{R}^d))$ be a self-adjoint, positive, elliptic SG operator with order components $m, \mu > 0$. In this case, condition (*) in Theorem 3.18 is equivalent to the condition

for every $M \in \mathbb{N}$ there exist $\sigma = \sigma_M > 1$ and $C = C_M > 0$ such that

$$\sup_{t\in\mathbb{T}^n} |\partial_t^\gamma f_j(t)| = \sup_{t\in\mathbb{T}^n} |f_j^\gamma(t)| \leqslant C^{|\gamma|+1} (\gamma!)^\sigma |\widetilde{\lambda}_j|^{-M}, \tag{**}$$

for every $j \in \mathbb{N}^*, \gamma \in \mathbb{Z}^n_+$.

*Proof.* The implication (*) ⇒ (**) is immediate, and it holds true even under only the less restrictive hypotheses of Theorem 3.18. To show that the opposite implication (**) ⇒ (*) holds true as well, recalling Remark 3.17, we first observe that for any $M \in \mathbb{N}$ there exists $M' = M'_M \in \mathbb{N}$ such that

$$0 < \sum_{j\in\mathbb{N}^*} \widetilde{\lambda}_j^{\,2(M-M')} = S = S_{MM'} < \infty. \tag{3.22}$$

Then, for any $M \in \mathbb{N}$ we choose $M' \in \mathbb{N}$ satisfying (3.22) and, by hypothesis, there are $\sigma = \sigma_{M'} > 1$ and $C = C_{M'} > 0$ satisfying condition (**), which implies

$$\begin{aligned} \sup_{t\in\mathbb{T}^n} \sum_{j\in\mathbb{N}^*} \widetilde{\lambda}_j^{2M} |f_j^\gamma(t)|^2 &\leqslant \sum_{j\in\mathbb{N}^*} \widetilde{\lambda}_j^{2M} \sup_{t\in\mathbb{T}^n} |f_j^\gamma(t)|^2 \leqslant \sum_{j\in\mathbb{N}^*} \widetilde{\lambda}_j^{2M} C^{2(|\gamma|+1)} (\gamma!)^{2\sigma} \widetilde{\lambda}_j^{-2M'} \\ &= C^{2(|\gamma|+1)} (\gamma!)^{2\sigma} \sum_{j\in\mathbb{N}^*} \lambda_j^{2(M-M')} = S C^{2(|\gamma|+1)} (\gamma!)^{2\sigma}. \end{aligned}$$

It follows that (*) holds true for any $\gamma \in \mathbb{Z}^n_+$ with $\widetilde{C}$ in place of $C$, setting $\widetilde{C} = C$ if $S \leqslant 1$ or $\widetilde{C} = SC$ if $S > 1$. □

**Corollary 3.20.** Assume that $f \in \mathscr{F}(\mathbb{T}^n \times \mathbb{R}^d)$.

i) Under the hypotheses of Theorem 3.18, for every $M \in \mathbb{N}$ there exist $\sigma = \sigma_M > 1$ and $C = C_M > 0$ such that

$$\|f_j\|_{\mathcal{G}^{\sigma,C}(\mathbb{T}^n)} \leqslant C|\widetilde{\lambda}_j|^{-M}, \quad j \in \mathbb{N}^*. \tag{3.23}$$

ii) Under the hypotheses of Corollary 3.19, for every $M \in \mathbb{N}$ there exist $\sigma = \sigma_M > 1$ and $C = C_M > 0$ such that

$$\|f_j\|_{\mathcal{G}^{\sigma,C}(\mathbb{T}^n)} \leqslant Cj^{-M\varrho}, \ j \to \infty,$$

for some $\varrho > 0$. In particular, $\{f_j\}_{j\in\mathbb{N}^*} \subset \mathcal{G}^{\sigma_1}(\mathbb{T}^n)$.

*Proof.* i) By the proof of Theorem 3.18 and the definition of $\mathcal{G}^{\sigma,C}$ (cf. Section 3.2.1), it immediately follows that for every $M \in \mathbb{N}$ there exist $\sigma = \sigma_M > 1$ and $C = C_M > 0$ such that

$$\sup_{t\in\mathbb{T}^n} |f_j^\gamma(t)| \leqslant C^{|\gamma|+1}(\gamma!)^\sigma |\widetilde{\lambda}_j|^{-M}, \quad j \in \mathbb{N}^*, \gamma \in \mathbb{Z}^n_+,$$

that is, $\{f_j\}_{j\in\mathbb{N}^*} \subset \mathcal{G}^{\sigma,C}$ and

$$\|f_j\|_{\mathcal{G}^{\sigma,C}} \leqslant C|\widetilde{\lambda}_j|^{-M}, \quad j \in \mathbb{N}^*.$$

ii) Recalling Remark 3.17, the estimates for the Gevrey norms of the $f_j$ with respect to $j \to \infty$ follow from the previous point. In particular, for $M = 1$, from $\lambda_j \to \infty$, we obtain $j_0 \in \mathbb{N}$ such that

$$\sup_{t\in\mathbb{T}^n} |f_j^\gamma(t)| \leqslant C_1 C^{|\gamma|+1}(\gamma!)^{\sigma_1}, \quad j \geqslant j_0, \gamma \in \mathbb{Z}^n_+.$$

On the other hand,

$$\sup_{t\in\mathbb{T}^n} |f_j^\gamma(t)| \leqslant C_1 C' C^{|\gamma|+1}(\gamma!)^{\sigma_1}, \quad j = 1, \dots, j_0 - 1, \gamma \in \mathbb{Z}^n_+,$$

where $C' = \max\{\widetilde{\lambda}_j^{-1}, \ j = 1, \dots, j_0 - 1\}$. Therefore, $\{f_j\}_{j\in\mathbb{N}^*} \subset \mathcal{G}^{\sigma_1}$. ☐

**Corollary 3.21.** Under the hypotheses of Corollary 3.19, let $\{f_j\}_{j\in\mathbb{N}^*} \subset \mathcal{G}^{\sigma,C}(\mathbb{T}^n)$ be a sequence with the property that for every $M' \in \mathbb{N}$ there exists $B = B_{M'} > 0$ such that

$$\|f_j\|_{\mathcal{G}^{\sigma,C}(\mathbb{T}^n)} \leqslant Bj^{-M'}, \quad j \to \infty.$$

Then, setting $f(t) = \sum_{j\in\mathbb{N}^*} f_j(t)\phi_j$, $t \in \mathbb{T}^n$, it follows $f \in \mathscr{F}(\mathbb{T}^n \times \mathbb{R}^d)$.

*Proof.* For any $M \in \mathbb{N}$, choose $M' = M'_M \in \mathbb{N}$ so that $\sum_{j\in\mathbb{N}^*} j^{2(M\varrho - M')} < \infty$, recalling that, in view of Remark 3.17, it holds $\widetilde{\lambda}_j \leqslant K j^{\varrho}$, $j \to \infty$. Then, for a suitable $j_0 \in \mathbb{N}$,

$$\begin{aligned}\sup_{t\in\mathbb{T}^n} \sum_{j\geqslant j_0} \widetilde{\lambda}_j^{2M} |\partial_t^\gamma f_j(t)|^2 &\leqslant \sum_{j\geqslant j_0} \lambda_j^{2M} C^{2(|\gamma|+1)} (\gamma!)^{2\sigma} B^2 j^{-2M'} \\ &\leqslant K^{2M} C^{2(|\gamma|+1)} (\gamma!)^{2\sigma} B^2 \sum_{j\geqslant j_0} j^{2M\varrho - 2M'} \\ &\leqslant \widetilde{C}^{2(|\gamma|+1)} (\gamma!)^{2\sigma}, \quad \gamma \in \mathbb{Z}_+^n,\end{aligned}$$

which implies the claim. □

The next Theorem 3.22 is the analogue for the space $\mathscr{F}$ of Theorem 3.10 for the Schwartz space $\mathscr{S}$: it is a consequence of Theorem 3.18 and its corollaries.

**Theorem 3.22.** Let $P \in \mathrm{Op}(S^{m,\mu}(\mathbb{R}^d))$ be a normal, elliptic SG operator with order components $m, \mu > 0$ or $m, \mu < 0$, and let $\{\phi_j\}$ be a corresponding orthonormal basis of eigenfunctions.

(i) For any $f \in \mathscr{F}$ the series

$$\sum_{j\in\mathbb{N}} \left|\partial_t^\gamma (f(t), \phi_j)_{L^2(\mathbb{R}^d)}\right| |A\phi_j(x)|, \quad \gamma \in \mathbb{Z}_+^n, \tag{3.24}$$

converges uniformly on $\mathbb{T}^n \times \mathbb{R}^d$ for every SG operator $A$ and there exist $B = B_{APnd} > 0$, $\sigma = \sigma_{APnd} > 1$, $C = C_{APnd} > 0$, depending only on $A, P, n$, and $d$, such that the sums $S_{\gamma A}$ of (3.24) satisfy the condition

$$S_{\gamma A}(t, x) \leqslant B C^{|\gamma|+1} (\gamma!)^\sigma, \quad t \in \mathbb{T}^n, x \in \mathbb{R}^d. \tag{3.25}$$

(ii) Additionally, assume $P \in \mathrm{Op}(S^{m,\mu}(\mathbb{R}^d))$, $m, \mu > 0$, to be self-adjoint, positive and SG classical. Then, for $f \in \mathscr{C}^\infty(\mathbb{T}^n, L^2(\mathbb{R}^d))$ the converse of the implication in point (i) above holds true as well.

*Proof.* Following the proof of Theorem 3.10, we may assume $P$ invertible, with order components $m, \mu < -\frac{d}{2}$, and so Hilbert-Schmidt, with nonvanishing eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}^*}$ such that $\sum_{j\in\mathbb{N}^*}|\lambda_j|^2<\infty$, and $A\in\mathrm{Op}(S^{pm,p\mu})$, $p\in\mathbb{Z}$, $p<0$ so that $|A\phi_j(x)|\leqslant\widetilde{B}|\lambda_j|^{p-1}$, $x\in\mathbb{R}^d$, $\widetilde{B}>0$, and $\widetilde{B}$ depends only on $A,P,n$, and $d$.

(i) Assume $f\in\mathscr{F}$. Then, by Theorem 3.18, for any $M\in\mathbb{Z}$, for suitable $\sigma>1$, $C>0$,

$$\sup_{t\in\mathbb{T}^n}|\partial_t^\gamma(f(t),\phi_j)|\leqslant C^{(|\gamma|+1)}(\gamma!)^\sigma|\lambda_j|^{-M},\quad \gamma\in\mathbb{Z}^n_+.$$

Choosing $M=p-3$, we then find, for any $t\in\mathbb{T}^n$, $x\in\mathbb{R}^d$, $\gamma\in\mathbb{Z}^n_+$,

$$\begin{aligned}\sum_{j\in\mathbb{N}}|\partial_t^\gamma(f(t),\phi_j)|\,|A\phi_j(x)| &\leqslant \sum_{j\in\mathbb{N}^*}C^{(|\gamma|+1)}(\gamma!)^\sigma|\lambda_j|^{-p+3}\widetilde{B}|\lambda_j|^{p-1}\\ &\leqslant \widetilde{B}C^{(|\gamma|+1)}(\gamma!)^\sigma\sum_{j\in\mathbb{N}^*}|\lambda_j|^2=BC^{(|\gamma|+1)}(\gamma!)^\sigma,\end{aligned}$$

which proves the claim.

(ii) We proceed as in the second part of the proof of Theorem 3.10, setting $A=M^\alpha D^\beta$, $\alpha,\beta\in\mathbb{Z}^d_+$,

$$f^\gamma_{\alpha\beta}(t,x)=\sum_{j\in\mathbb{N}^*}\partial_t^\gamma(f(t),\phi_j)\,(M^\alpha D^\beta\phi_j)(x),\quad\gamma\in\mathbb{Z}^n_+,$$

and $g=f^0_{00}$. By the hypotheses, it follows $g\in\mathscr{C}^\infty(\mathbb{T}^n\times\mathbb{R}^d)$, with $f^\gamma_{\alpha\beta}(t,x)=\partial_t^\gamma[x^\alpha D_x^\beta g(t,x)]$. As in the proof of Theorem 3.10, we actually find $g\in\mathscr{C}^\infty(\mathbb{T}^n,\mathscr{S})$, and, of course, $g=f$. It then also follows

$$\partial_t^\gamma f_j(t)=\partial_t^\gamma(f(t),\phi_j)=(\partial_t^\gamma f(t),\phi_j)=f_j^\gamma(t),\ t\in\mathbb{T}^n, j\in\mathbb{N}^*,\gamma\in\mathbb{Z}^n_+.$$

For any $M\in\mathbb{N}$ there exist $\sigma>1$, $C>0$, such that

$$\begin{aligned}&\|P^M\partial_t^\gamma f(t)\|_{L^2}\leqslant C_d\left|\left|\left|\sum_{j\in\mathbb{N}^*}f_j^\gamma(t)P^M\phi_j\right|\right|\right|_{2[\frac{d}{2}]+2}\\ &\leqslant\widetilde{C}_d\sum_{|\alpha+\beta|\leqslant2[\frac{d}{2}]+2}\sup_{\mathbb{R}^d}\left|\sum_{j\in\mathbb{N}^*}f_j^\gamma(t)\,M^\alpha D^\beta P^M\phi_j\right|\\ &\leqslant\widetilde{C}_d\sum_{|\alpha+\beta|\leqslant2[\frac{d}{2}]+2}\sup_{\mathbb{R}^d}\sum_{j\in\mathbb{N}^*}|f_j^\gamma(t)|\,|M^\alpha D^\beta P^M\phi_j|\\ &\leqslant\widetilde{C}_d\sum_{|\alpha+\beta|\leqslant2[\frac{d}{2}]+2}B_{\alpha\beta Pnd}\,C^{|\gamma|+1}_{\alpha\beta Pnd}\,(\gamma!)^{\sigma_{\alpha\beta Pnd}}\leqslant C^{|\gamma|+1}\,(\gamma!)^\sigma, t\in\mathbb{T}^n,\gamma\in\mathbb{Z}^n_+.\end{aligned}$$

It follows that for any $M \in \mathbb{N}$ there exist $\sigma > 1$, $C > 0$ such that

$$\begin{aligned} |\partial_t^\gamma f_j(t)|\,|\widetilde{\lambda}_j|^M &= |\partial_t^\gamma f_j(t)|\,|\widetilde{\lambda}_j|^M \,\|\phi_j\|_{L^2} = \|P^M \partial_t^\gamma [(f(t),\phi_j)\phi_j]\|_{L^2} \\ &\leqslant \|P^M \partial_t^\gamma f(t)\|_{L^2} \leqslant C^{|\gamma|+1}\,(\gamma!)^\sigma,\ t \in \mathbb{T}^n, j \in \mathbb{N}^*, \gamma \in \mathbb{Z}_+^n \\ &\iff \sup_{t\in\mathbb{T}^n} |\partial_t^\gamma f_j(t)| \leqslant C^{|\gamma|+1}\,(\gamma!)^\sigma |\widetilde{\lambda}_j|^{-M}, \quad j \in \mathbb{N}^*, \gamma \in \mathbb{Z}_+^n. \end{aligned}$$

By Corollary 3.19, this proves $f \in \mathscr{F}$.

□

**Remark 3.23.** Since, for any $g \in \mathscr{S}'$, $\mathrm{Op}(\lambda_\rho)\overline{g} = \langle D\rangle^\rho \overline{g} = \overline{\langle D\rangle^\rho g}$, it also follows

$$\langle\cdot\rangle^r \langle D\rangle^\rho \overline{g} = \overline{\langle\cdot\rangle^r \langle D\rangle^\rho g} \Rightarrow \|\overline{g}\|_{H^{r,\rho}} = \|g\|_{H^{r,\rho}}, \quad g \in \mathscr{S}', r, \rho \in \mathbb{R}.$$

Then, by Definition 3.11, $u \in \mathcal{G}^\sigma\mathcal{H}^C_{r,\rho} \Leftrightarrow \overline{u} \in \mathcal{G}^\sigma\mathcal{H}^C_{r,\rho}$, with $\|u\|_{\sigma,C,r,\rho} = \|\overline{u}\|_{\sigma,C,r,\rho}$, for arbitrary $\sigma > 1$, $C > 0$, $r, \rho \in \mathbb{R}$, and, of course, $f \in \mathscr{F} \Leftrightarrow \overline{f} \in \mathscr{F}$. Recalling that also $\{\overline{\phi_j}\}_{j\in\mathbb{N}^*}$ is an orthonormal basis, and observing that

$$\widetilde{f}_j(t) \equiv (f(t), \overline{\phi_j}) = \int_{\mathbb{R}^d} f(t)\,\phi_j = \overline{(\overline{f}(t), \phi_j)} = \overline{\overline{f}_j(t)}, \quad t \in \mathbb{T}^n, j \in \mathbb{N}^*, f \in \mathscr{F},$$

we notice that the statements of Theorem 3.18 and Corollaries 3.19, 3.20, 3.21 hold true also with the coefficients $\widetilde{f}_j$ in place of the coefficients $f_j$, $j \in \mathbb{N}^*$. This will be useful in the subsequent Section 3.2.5.

### 3.2.5 Eigenfunctions expansions in $\mathscr{F}'$

Let $\theta \in \mathscr{F}'$ and $M \in \mathbb{Z}$ such that $\theta \in \mathcal{GH}'_{Mm,M\mu}$. For any $\psi \in \mathcal{G}^\sigma(\mathbb{T}^n)$ we consider $\psi \otimes \overline{\phi_j} \in \mathcal{G}^\sigma\mathcal{H}_{Mm,M\mu}$ by setting

$$\mathbb{T}^n \ni t \mapsto \psi(t)\overline{\phi_j} : \mathbb{R}^d \to \mathbb{C} : x \mapsto \psi(t)\overline{\phi_j(x)}, \quad j \in \mathbb{N}^*.$$

Then, there are well-defined linear maps $\theta_j : \mathcal{G}^\sigma(\mathbb{T}^n) \to \mathbb{C}$, given by

$$\langle \theta_j, \psi\rangle \equiv \langle \theta, \psi \otimes \overline{\phi_j}\rangle, \quad j \in \mathbb{N}^*. \tag{3.26}$$

We claim that $\theta_j \in (\mathcal{G}^\sigma(\mathbb{T}^n))'$. Indeed, given any constant $C > 0$, by Proposition 3.14 there exists $B = B_{\sigma C} > 0$ such that

$$\begin{aligned}|\langle \theta_j, \psi\rangle| &= |\langle \theta, \psi \otimes \overline{\phi_j}\rangle| \\ &\leqslant B \sup_{\gamma\in\mathbb{Z}_+^n} \left\{ C^{-|\gamma|}(\gamma!)^{-\sigma} \sup_{t\in\mathbb{T}^n} \|\partial_t^\gamma \psi(t)\overline{\phi_j}\|_{H^{Mm,M\mu}} \right\} \\ &= B\|\overline{\phi_j}\|_{H^{Mm,M\mu}} \sup_{\gamma\in\mathbb{Z}_+^n} \left\{ C^{-|\gamma|}(\gamma!)^{-\sigma} \sup_{t\in\mathbb{T}^n} |\partial_t^\gamma \psi(t)| \right\} \\ &= B\|\phi_j\|_{H^{Mm,M\mu}} \|\psi\|_{\mathcal{G}^{\sigma,C}} \leqslant \widetilde{B}|\widetilde{\lambda}_j|^M \|\psi\|_{\mathcal{G}^{\sigma,C}}, \quad j\in\mathbb{N}^*,\end{aligned} \tag{3.27}$$

which proves the assertion, recalling Remarks 3.5 and 3.23. This allows us to decompose any $\theta \in \mathscr{F}'$ into a series of tensor products, whose first factors satisfy the estimates (3.27), as shown in the subsequent Lemma 3.24. Next, we study convergence in $\mathscr{F}'$.

**Lemma 3.24.** Let $\theta \in \mathscr{F}'(\mathbb{T}^n \times \mathbb{R}^d)$. Then

$$\theta = \sum_{j\in\mathbb{N}^*} \theta_j \otimes \phi_j, \tag{3.28}$$

with $\{\theta_j\}_{j\in\mathbb{N}^*} \subset (\mathcal{G}^\sigma(\mathbb{T}^n))'$ given by (3.26), and so satisfying (3.27).

*Proof.* Since also $\{\overline{\phi_j}\}_{j\in\mathbb{N}^*}$ is an orthonormal basis, for any $f \in \mathscr{F}$ we can write

$$f(t,x) = \sum_{j\in\mathbb{N}^*} (f(t), \overline{\phi_j})\,\overline{\phi_j(x)} = \left[\int_{\mathbb{R}^d} f(t,y)\phi_j(y)\,dy\right]\overline{\phi_j(x)}.$$

Then, recalling the properties of tensor products of distributions, for any $f \in \mathscr{F}$,

$$\begin{aligned}\sum_{j\in\mathbb{N}^*} \langle \theta_j \otimes \phi_j, f\rangle &= \sum_{j\in\mathbb{N}^*} \left\langle \theta_j, \int_{\mathbb{R}^d} f(\cdot,y)\phi_j(y)\,dy \right\rangle \\ &= \sum_{j\in\mathbb{N}^*} \left\langle \theta, (f(\cdot), \overline{\phi_j}) \otimes \overline{\phi_j}(\cdot\cdot)\right\rangle \\ &= \left\langle \theta, \sum_{j\in\mathbb{N}^*} (f(\cdot), \overline{\phi_j}) \otimes \overline{\phi_j}(\cdot\cdot) \right\rangle = \langle \theta, f\rangle,\end{aligned}$$

as claimed. □

**Theorem 3.25.** Let $\{\tau_j\}_{j\in\mathbb{N}^*} \subset \mathscr{F}'(\mathbb{T}^n \times \mathbb{R}^d)$ be such that $\{\langle\tau_j, f\rangle\}_{j\in\mathbb{N}^*}$ is a Cauchy sequence in $\mathbb{C}$, for all $f \in \mathscr{F}(\mathbb{T}^n \times \mathbb{R}^d)$. Then there exists $\tau \in \mathscr{F}'(\mathbb{T}^n \times \mathbb{R}^d)$ such that $\tau = \lim_{j\to\infty} \tau_j$, that is,

$$\langle\tau, f\rangle = \lim_{j\to\infty} \langle\tau_j, f\rangle, \quad f \in \mathscr{F}(\mathbb{T}^n \times \mathbb{R}^d).$$

*Proof.* By hypothesis, it is well defined the linear map $\tau : \mathscr{F} \to \mathbb{C}$ given by

$$\langle\tau, f\rangle = \lim_{j\to\infty} \langle\tau_j, f\rangle, \quad f \in \mathscr{F}.$$

In order to verify the continuity, we first recall that

$$\mathscr{F}' = \bigcup_{r,\rho\in\mathbb{R}} \mathcal{G}\mathcal{H}'_{r,\rho} = \bigcup_{r,\rho\in\mathbb{R}} \left[\bigcap_{\sigma>1} (\mathcal{G}^\sigma \mathcal{H}^{\sigma-1}_{r,\rho})'\right].$$

Hence, for each $j \in \mathbb{N}^*$ we can find $r_j, \rho_j \in \mathbb{R}$ such that $\tau_j \in (\mathcal{G}^\sigma \mathcal{H}^{\sigma-1}_{r_j,\rho_j})'$ for any $\sigma > 1$. Now, let $\{f_\ell\}_{\ell\in\mathbb{N}}$ be a sequence in $\mathscr{F}$ converging to 0. This means that for every $r, \rho \in \mathbb{R}$ there is $\sigma = \sigma_{r,\rho} > 1$ such that

$$f_\ell \to 0 \ \text{ in } \ \mathcal{G}^{\sigma_{r,\rho}} \mathcal{H}^{\sigma_{r,\rho}-1}_{r,\rho}.$$

In particular, $\{f_\ell\}_{\ell\in\mathbb{N}}$ converges to zero in each $\mathcal{G}^{\sigma_{r_j,\rho_j}} \mathcal{H}^{\sigma_{r_j,\rho_j}-1}_{r,\rho_j}$. Consider the complete metrizable space

$$\mathcal{B} = \bigcap_{j\in\mathbb{N}^*} \mathcal{G}^{\sigma_{r_j,\rho_j}} \mathcal{H}^{\sigma_{r_j,\rho_j}-1}_{r,\rho_j} \subset \mathscr{F},$$

and define by $\omega_j$ the restrictions

$$\omega_j = \tau_j|_{\mathcal{B}} : \mathcal{B} \to \mathbb{C}, \ j \in \mathbb{N}^*,$$

which clearly are linear and continuous. Notice that $\{\langle\omega_j, b\rangle\}$ is a Cauchy sequence in $\mathbb{C}$, for every $b \in \mathcal{B}$. Therefore, by the Banach-Steinhaus theorem, $\{\omega_j\}_{j\in\mathbb{N}^*}$ is equicontinuous. Since $f_\ell$ converges to zero in $\mathcal{B}$, for every $\epsilon > 0$ there exists some $\ell_0 \in \mathbb{N}$ such that

$$|\langle\omega_j, f_\ell\rangle| \leqslant \frac{\epsilon}{2}, \quad \ell \geqslant \ell_0, j \in \mathbb{N}^*. \tag{3.29}$$

Now, by $\langle \tau, f_\ell \rangle = \lim_{j\to\infty} \langle \tau_j, f_\ell \rangle$, we may assume that for $\ell \geqslant \ell_0$ there exists an index $j_\ell$ satisfying

$$|\langle \tau, f_\ell \rangle - \langle \tau_{j_\ell}, f_\ell \rangle| < \frac{\epsilon}{2}. \tag{3.30}$$

Finally, for all $\ell \geqslant \ell_0$, it follows from (3.29) and (3.30) that

$$\begin{aligned} |\langle \tau, f_\ell \rangle| &\leqslant |\langle \tau, f_\ell \rangle - \langle \tau_{j_\ell}, f_\ell \rangle| + |\langle \tau_{j_\ell}, f_\ell \rangle| \\ &= |\langle \tau, f_\ell \rangle - \langle \tau_{j_\ell}, f_\ell \rangle| + |\langle \omega_{j_\ell}, f_\ell \rangle| \\ &< \epsilon, \end{aligned}$$

that is, $\tau \in \mathscr{F}'$. The proof is complete. □

The subsequent Theorem 3.26 provides a sufficient condition on the coefficients of an expansion with respect to the basis $\{\phi_j\}_{j\in\mathbb{N}^*}$ to indeed produce an element of $\mathscr{F}'$. Together with Lemma 3.24 it completes the characterisation of $\mathscr{F}'$ in terms of eigenfunctions expansions associated with a classical, self-adjoint, positive SG operator, and is a further main result of this chapter.

**Theorem 3.26.** Let $P \in \mathrm{Op}(S^{m,\mu}_{\mathrm{cl}}(\mathbb{R}^d))$ be an elliptic, self-adjoint, positive, classical SG operator with order components $m, \mu > 0$, and denote by $\{\phi_j\}_{j\in\mathbb{N}^*}$ a basis of orthonormal eigenfunctions of $P$ with corresponding eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}^*}$. Let $\{\vartheta_j\}_{j\in\mathbb{N}^*} \subset \bigcap_{\sigma>1}(\mathcal{G}^\sigma(\mathbb{T}^n))'$ be a sequence such that there exist $M \in \mathbb{Z}$, $B > 0$, satisfying

$$|\langle \vartheta_j, \psi \rangle| \leqslant B \|\psi\|_{\mathcal{G}^{\sigma,C}(\mathbb{T}^n)} |\widetilde{\lambda}_j|^M, \quad j \in \mathbb{N}^*,$$

for all $\sigma > 1$, $C > 0$, $\psi \in \mathcal{G}^{\sigma,C}(\mathbb{T}^n)$. Then

$$\vartheta = \sum_{j\in\mathbb{N}^*} \vartheta_j \otimes \phi_j \in \mathscr{F}'(\mathbb{T}^n \times \mathbb{R}^d). \tag{3.31}$$

Moreover,

$$\langle \vartheta_j, \psi \rangle = \langle \vartheta, \psi \otimes \overline{\phi_j} \rangle, \quad \psi \in \mathcal{G}^\sigma(\mathbb{T}^n), j \in \mathbb{N}^*.$$

*Proof.* Set, for each $J \in \mathbb{N}$,

$$\varsigma_J = \sum_{j=1}^{J} \vartheta_j \otimes \phi_j \in \mathscr{F}'.$$

Then, choose $M' \in \mathbb{Z}$ such that $\sum_{j \in \mathbb{N}^*} j^{\varrho(M-M')} < +\infty$. Recall that, by Corollary 3.21 and Remark 3.23, for any $f = \sum_{j \in \mathbb{N}^*} \widetilde{f}_j \overline{\phi_j} \in \mathscr{F}$ there are $\sigma = \sigma_{M'} > 1$ and $C = C_{M'} > 0$ such that

$$\|\widetilde{f}_j\|_{\mathcal{G}^{\sigma,C}} \leqslant C |\widetilde{\lambda}_j|^{-M'}, \quad j \in \mathbb{N}^*.$$

Then, for arbitrary $\varepsilon > 0$, suitable $J_0 = J_0(\varepsilon)$, any $J > J_0$ and $\ell \geqslant 1$, recalling also Remark 3.17,

$$\begin{aligned}
|\langle \varsigma_{J+\ell} - \varsigma_J, f \rangle| &\leqslant \sum_{k=J+1}^{J+\ell} \sum_{j \in \mathbb{N}^*} |\langle \vartheta_k, \widetilde{f}_j \rangle| \, |\langle \phi_k, \overline{\phi_j} \rangle| = \sum_{k=J+1}^{J+\ell} \sum_{j \in \mathbb{N}^*} |\langle \vartheta_k, \widetilde{f}_j \rangle| \, |(\phi_k, \phi_j)| \\
&= \sum_{k=J+1}^{J+\ell} |\langle \vartheta_k, \widetilde{f}_k \rangle| \leqslant B \sum_{k=J+1}^{J+\ell} \|\widetilde{f}_k\|_{\mathcal{G}^{\sigma,C}} |\widetilde{\lambda}_k|^M \\
&\leqslant BCK^{M-M'} \sum_{k=J+1}^{J+\ell} j^{\varrho(M-M')} < \varepsilon.
\end{aligned}$$

Hence, $\{\langle \varsigma_J, f \rangle\}_{J \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$. Therefore, by Theorem 3.25, there exists $\vartheta \in \mathscr{F}'$ such that

$$\langle \vartheta, f \rangle = \lim_{J \to \infty} \langle \varsigma_J, f \rangle, \; f \in \mathscr{F} \Leftrightarrow \vartheta = \sum_{j \in \mathbb{N}^*} \vartheta_j \otimes \phi_j.$$

It also follows, for any $\psi \in \mathcal{G}^\sigma$, $j \in \mathbb{N}^*$,

$$\begin{aligned}
\langle \vartheta, \psi \otimes \overline{\phi_j} \rangle &= \sum_{k \in \mathbb{N}} \langle \vartheta_k \otimes \phi_k, \psi \otimes \overline{\phi_j} \rangle = \sum_{k \in \mathbb{N}} \langle \vartheta_k, \psi \rangle \cdot \langle \phi_k, \overline{\phi_j} \rangle \\
&= \sum_{k \in \mathbb{N}} \langle \vartheta_k, \psi \rangle \cdot (\phi_k, \phi_j) = \langle \vartheta_j, \psi \rangle.
\end{aligned}$$

The proof is complete. □

## 3.3 Periodic evolution equations and hypoellipticity on $\mathbb{T} \times \mathbb{R}^d$

In this section we discuss the global hypoellipticity on $\mathbb{T} \times \mathbb{R}^d$ of the operator

$$L = D_t + \omega P, \; t \in \mathbb{T}, x \in \mathbb{R}^d, \tag{3.32}$$

where $\omega = \alpha + i\beta \in \mathbb{C}$, and $P = \mathrm{Op}(p) \in \mathrm{Op}(S^{m,\mu})$ is a positive, self-adjoint, elliptic SG operator with order components $m, \mu > 0$.

More precisely, we study the global regularity on $\mathbb{T} \times \mathbb{R}^d$ of the solutions $u \in \mathscr{F}'$ of the equation $Lu = f$. Our strategy is based on the Fourier expansions of $u \in \mathscr{F}'$ and $f \in \mathscr{F}$ with respect to the eigenfunctions of the operator $P$. Therefore, writing, for $t \in \mathbb{T}$,

$$u(t) = \sum_{j \in \mathbb{N}^*} u_j(t)\phi_j = \sum_{j \in \mathbb{N}^*} u_j(t) \otimes \phi_j$$

and

$$f(t) = \sum_{j \in \mathbb{N}^*} f_j(t)\phi_j = \sum_{j \in \mathbb{N}^*} f_j(t) \otimes \phi_j,$$

we obtain that the equation $Lu = f$ is equivalent to the infinite system of ODEs

$$D_t u_j(t) + \omega\lambda_j u_j(t) = f_j(t), \ t \in \mathbb{T}, \ j \in \mathbb{N}^*. \tag{3.33}$$

**Definition 3.27.** We say that the operator $L$ defined in (3.32) is globally hypoelliptic on $\mathbb{T} \times \mathbb{R}^d$ if

$$u \in \mathscr{F}'(\mathbb{T} \times \mathbb{R}^d) \ \text{ and } \ Lu \in \mathscr{F}(\mathbb{T} \times \mathbb{R}^d) \Rightarrow u \in \mathscr{F}(\mathbb{T} \times \mathbb{R}^d).$$

By standard arguments, it follows that the solutions of (3.33) are given by

$$u_j(t) = u_{j0} \exp\left(-i\lambda_j \omega t\right) + i \int_0^t \exp\left(i\lambda_j \omega (s-t)\right) f_j(s) ds, \tag{3.34}$$

for some $u_{j0} \in \mathbb{C}$, $j \in \mathbb{N}^*$.

We may assume that the coefficients $f_j$ belong to a Gevrey class $\mathcal{G}^\sigma = \mathcal{G}^\sigma(\mathbb{T})$ for all $j \in \mathbb{N}^*$. Then, by the properties of equation (3.33), also $u_j$ is in $\mathscr{G}^\sigma$ for all $j \in \mathbb{N}^*$. Now consider the set

$$\mathcal{Z} = \{j \in \mathbb{N}^*; \ \omega\lambda_j \in \mathbb{Z}\}. \tag{3.35}$$

**Remark 3.28.** Notice that if $\beta \neq 0$ the set $\mathcal{Z}$ is finite.

If $j \notin \mathcal{Z}$, then $u_{j0}$ is uniquely defined and (3.34) can be written in either of the two equivalent forms:

$$u_j(t) = \frac{i}{1 - e^{-2\pi i \lambda_j \omega}} \int_0^{2\pi} \exp\left(-i\lambda_j \omega s\right) f_j(t-s) ds \tag{3.36}$$

or

$$u_j(t) = \frac{i}{e^{2\pi i \lambda_j \omega} - 1} \int_0^{2\pi} \exp\left(i\lambda_j \omega s\right) f_j(t+s) ds. \tag{3.37}$$

We point out that, in view of the Fourier characterisations obtained in Section 3.2, to study the solutions of $Lu = f$ we must perform an analysis of $|\partial_t^k u_j(t)|$ as $j \to \infty$, for all $k \in \mathbb{Z}_+$. In particular, we need to consider the size of set $\mathcal{Z}$, as well the growth of the sequences

$$\Theta_j = |1 - e^{-2\pi i \lambda_j \omega}|^{-1} \quad \text{and} \quad \Gamma_j = |e^{2\pi i \lambda_j \omega} - 1|^{-1},$$

as $j \to \infty$. Loosely speaking, the behaviour of $\{\Theta_j\}_{j \in \mathbb{N}^*}$ or $\{\Gamma_j\}_{j \in \mathbb{N}^*}$ must not destroy the growth of $f_j$ and its derivatives. For instance, when $\beta \neq 0$ it is easy to see that both sequences converge to 1 as $j \to \infty$. On the other hand, the case $\beta = 0$ is more delicate, and is connected with the so-called Diophantine approximations. To deal with this situation, we need the next Definition 3.29.

**Definition 3.29** (Condition ($\mathscr{A}$))**.** We say that a real number $\alpha$ satisfies Condition ($\mathscr{A}$) if there are positive constants $\epsilon$ and $C$ such that

$$|\tau - \alpha \lambda_j| \geqslant C j^{-\epsilon},$$

for all $(j, \tau) \in \mathbb{N} \times \mathbb{Z}$.

In literature Condition ($\mathscr{A}$) is also known as the property for the real number $\alpha$ of *not being Liouville* with respect to the sequence $\{\lambda_j\}$.

If Condition ($\mathscr{A}$) fails we may obtain that for any $C > 0$ and any $\delta > 0$ there exist a subsequence $\{\lambda_{j_k}\}_{k \in \mathbb{N}^*}$ and a sequence $\{\tau_k\}_{k \in \mathbb{N}^*} \subset \mathbb{Z}$, depending on $C$ and $\delta$, such that

$$|\alpha \lambda_{j_k} - \tau_k| < C {j_k}^{-\delta}, \quad k \in \mathbb{N}^*. \tag{3.38}$$

Indeed, it is enough to choose a decreasing sequence $\{C_k\}_{k \in \mathbb{N}^*} \subset (0, C)$. Actually, it is possibile to obtain a better statement, as we now show.

**Lemma 3.30.** If Condition ($\mathscr{A}$) fails, there are a subsequence $\{\lambda_{j_k}\}_{k\in\mathbb{N}^*}$ and a sequence $\{\tau_k\}_{k\in\mathbb{N}^*} \subset \mathbb{Z}$ such that, for any $C > 0$ and any $n \in \mathbb{N}$,

$$|\alpha\lambda_{j_k} - \tau_k| < Cj_k^{-n}, \quad k \to \infty. \tag{3.39}$$

*Proof.* We know, by (3.38), that for any $C > 0$ and $n \in \mathbb{N}$ there exist a subsequence $\{\lambda_{j_\ell}^{(n)}\}_{\ell\in\mathbb{N}^*}$ and a sequence $\{\tau_\ell^{(n)}\} \subset \mathbb{Z}$ such that

$$|\alpha\lambda_{j_\ell}^{(n)} - \tau_\ell^{(n)}| < Cj_\ell^{-n}, \quad \ell \in \mathbb{N}^*. \tag{3.40}$$

Then, employing a diagonalization argument, we define the subsequence $\{\lambda_{j_k}\}_{k\in\mathbb{N}^*}$ and the sequence $\{\tau_k\}_{k\in\mathbb{N}^*} \subset \mathbb{Z}$ as

$$(\lambda_{j_k}, \tau_k) \equiv \left(\lambda_{j_k}^{(k)}, \tau_k^{(k)}\right).$$

Hence, for any $n \in \mathbb{N}$ we get $|\alpha\lambda_{j_k} - \tau_k| < Cj_k^{-n}$, provided that $k \geqslant n$. This completes the proof. □

The next result will be useful in the final step of the proof of Theorem 3.32 below.

**Lemma 3.31.** Let $\{\beta_j\}_{j\in\mathbb{N}^*}$ be a sequence of real numbers. Then, for each $j \in \mathbb{N}^*$ there exist $l(j) \in \mathbb{Z}$ such that

$$|1 - e^{2\pi i\beta_j}| \geqslant 4\ |\beta_j + l(j)|.$$

*Proof.* See Proposition 5.7 in [7]. □

We can now characterise the global hypoellipticity on $\mathbb{T} \times \mathbb{R}^d$ of the operator $L$ in (3.32). The subsequent Theorem 3.32 is further main result of this chapter.

**Theorem 3.32.** The operator $L$ defined in (3.32) is globally hypoelliptic on $\mathbb{T} \times \mathbb{R}^d$ if and only if either $\beta \neq 0$ or $\beta = 0$ and $\alpha$ satisfies Condition ($\mathscr{A}$).

First, we observe that if $\beta \neq 0$, by the previous considerations, (3.36) or (3.37) imply

$$\sup_{t\in\mathbb{T}} |\partial_t^\gamma u_j(t)| \lesssim \sup_{t\in\mathbb{T}} |\partial_t^\gamma f_j(t)|, \quad \gamma \in \mathbb{Z}_+,$$

showing that $u$ satisfies the same Gevrey estimates of $f$. The global hypoellipticity of $L$ on $\mathbb{T}\times\mathbb{R}^d$ then follows, in view of Corollary 3.19. Hence, we need focusing our analysis only on the case $\beta = 0$.

Second, when $\beta = 0$ the next Lemma 3.33 provides a necessary condition for the global hypoellipticity of $L$ on $\mathbb{T}\times\mathbb{R}^d$.

**Lemma 3.33.** If $\beta = 0$ and the set $\mathcal{Z}$ defined in (3.35) is infinite, the operator $L$ defined in (3.32) is not globally hypoelliptic on $\mathbb{T}\times\mathbb{R}^d$.

*Proof.* Since $\beta = 0$ we consider $L = D_t + \alpha P$. By the hypothesis that $\mathcal{Z}$ is infinite, there is a subsequence $\{\lambda_{j_k}\}_{k\in\mathbb{N}^*}$ such that $\alpha\lambda_{j_k} \in \mathbb{Z}$. Then, let $\{u_j\}_{j\in\mathbb{N}^*} \subset \mathscr{C}^\infty(\mathbb{T})$ be defined, for any $j \in \mathbb{N}^*$, by

$$u_j(t) = \begin{cases} \exp\left(-i\alpha\lambda_{j_k} t\right), & j = j_k, k \in \mathbb{N}^*, \\ 0, & j \neq j_k, k \in \mathbb{N}^*. \end{cases}$$

Clearly, $u_j \in \mathcal{G}^\sigma(\mathbb{T})$ for every $\sigma > 1$, and

$$D_t u_j(t) + \lambda_j \alpha u_j(t) = 0, \quad j \in \mathbb{N}^*.$$

Moreover, by Theorem 3.26, setting $u(t) = \sum_{j\in\mathbb{N}^*} u_j(t) \otimes \phi_j$, it also immediately follows $u \in \mathscr{F}'$. Similarly, Corollary 3.19 shows that $u \notin \mathscr{F}$, observing that $|u_{j_k}(t)| \equiv 1$ for all $k \in \mathbb{N}^*$. Then, since $Lu = 0$, we conclude that $L$ is not globally hypoelliptic on $\mathbb{T}\times\mathbb{R}^d$. □

It follows that to conclude the proof of Theorem 3.32 it is enough to prove the next Proposition 3.34.

**Proposition 3.34.** The operator $L = D_t + \alpha P$, $\alpha \in \mathbb{R}$, defined in (3.32) is globally hypoelliptic on $\mathbb{T}\times\mathbb{R}^d$ if and only if $\alpha$ satisfies Condition $(\mathscr{A})$.

*Proof.* We start showing sufficiency. By Lemma 3.33, the set $\mathcal{Z}$ in (3.35) must be finite. Then we may consider the solutions of equations

$$D_t u_j(t) + \alpha\lambda_j u_j(t) = f_j(t),$$

given by (3.36), for any $j$ large enough. In view of Lemma 3.31, there are $C > 0$ and $\epsilon > 0$ such that

$$|1 - e^{-2\pi i\alpha\lambda_j}| \geqslant Cj^{-\epsilon}.$$

For $j$ large enough and any $\gamma \in \mathbb{Z}_+$ we then have

$$\sup_{t\in\mathbb{T}} |\partial_t^\gamma u_j(t)| \leqslant \frac{2\pi}{|1 - e^{-2\pi i\alpha\lambda_j}|} \sup_{t\in\mathbb{T}} |\partial_t^\gamma f_j(t)| \leqslant C' j^\epsilon \sup_{t\in\mathbb{T}} |\partial_t^\gamma f_j(t)|, \quad t \in \mathbb{T},$$

for another suitable constant $C' > 0$. Now, let $M \in \mathbb{N}$ be fixed and let $N = N_M \in \mathbb{N}$ satisfy $-N\varrho' + \epsilon < -M$, where $\varrho'$ is given by Remark 3.17. For this $N$, there are $C_N > 0$ and $\sigma_N > 1$ such that

$$\sup_{t\in\mathbb{T}} |\partial_t^k f_j(t)| \leqslant C_N^{k+1}(k!)^{\sigma_N} j^{-N}, \quad k \in \mathbb{Z}_+,$$

which implies

$$\sup_{t\in\mathbb{T}} |\partial_t^\gamma u_j(t)| \leqslant C_{N_M}^{\gamma+1}(\gamma!)^{\sigma_{N_M}} j^{-M}, \quad \gamma \in \mathbb{Z}_+.$$

By Corollary 3.21, it follows $u \in \mathscr{F}$ and we conclude that $L$ is globally hypoelliptic on $\mathbb{T} \times \mathbb{R}^d$.

To prove necessity, we proceed by contradiction, that is, we assume that Condition $(\mathscr{A})$ fails and exhibit a solution $u \in \mathscr{F}' \setminus \mathscr{F}$ of $Lu = f$ with $f \in \mathscr{F}$. Indeed, by Lemma 3.30, there exist $\{\lambda_{j_k}\}_{k\in\mathbb{N}^*}$ and $\{\tau_k\} \subset \mathbb{Z}$ such that for any $n \in \mathbb{N}$ and for any $C > 0$ it holds

$$0 < |\tau_k - \alpha\lambda_{j_k}| < C j_k^{-n}, \quad k \to \infty.$$

Consider the sequences

$$u_j(t) = \begin{cases} e^{-i\tau_k t}, & \text{if } j = j_k, k \in \mathbb{N}^*, \\ 0, & \text{if } j \neq j_k, k \in \mathbb{N}^*, \end{cases}$$

$$f_j(t) = \begin{cases} (\alpha\lambda_{j_k} - \tau_k)e^{-i\tau_k t}, & \text{if } j = j_k, k \in \mathbb{N}^*, \\ 0, & \text{if } j \neq j_k, k \in \mathbb{N}^*. \end{cases}$$

Since $|u_{j_k}(t)| \equiv 1$, $k \in \mathbb{N}^*$, Theorem 3.26 and Corollary 3.19 imply $u = \sum_{j \in \mathbb{N}^*} u_j \phi_j \in \mathscr{F}' \setminus \mathscr{F}$. On the other hand, it follows from Remark 3.17 that $|\lambda_j| \leqslant C j^{\varrho}$, $j \to \infty$. Since $|\tau_k| \leqslant 1 + C_1 j_k^{\varrho}$, we can estimate, for $\gamma \in \mathbb{Z}_+$,

$$|\tau_k|^{\gamma} = \sum_{s=0}^{\gamma} \binom{\gamma}{s} C_1^s j_k^{s\varrho} \leqslant 2^{\gamma} C_1^{\gamma+1} j_k^{\gamma\varrho} \leqslant C_2^{\gamma+1} j_k^{\gamma\varrho}, \quad k \to \infty,$$

for a new constant $C_2$ not depending on $\gamma \in \mathbb{Z}_+$. Finally, given $N \in \mathbb{N}$ we choose $n$ such that $\gamma\varrho - n < -N$. Then,

$$|\partial_t^{\gamma} f_{j_k}(t)| \leqslant C_2^{\gamma+1} C j_k^{\gamma\varrho - n} \leqslant C_3^{\gamma+1} j_k^{-N},$$

for every $\gamma \in \mathbb{Z}_+$, which shows $f \in \mathscr{F}$. Since $Lu = f$, we conclude that $L$ is not globally hypoelliptic on $\mathbb{T} \times \mathbb{R}^d$. □

## 3.4 Periodic evolution equations and solvability on $\mathbb{T} \times \mathbb{R}^d$

We now discuss the global solvability on $\mathbb{T} \times \mathbb{R}^d$ of the operator

$$L = D_t + \omega P, \ t \in \mathbb{T}, x \in \mathbb{R}^d, \tag{3.41}$$

where $\omega = \alpha + i\beta \in \mathbb{C}$ and $P = \mathrm{Op}(p) \in \mathrm{Op}(S_{\mathrm{cl}}^{m,\mu})$ is a classical, positive, self-adjoint, elliptic SG-operator with order components $m, \mu > 0$.

First, we point out that if $Lu = f \in \mathscr{F}$, then, by periodicity of $u_j$, we must have

$$\int_0^{2\pi} \exp\left(i\lambda_j \omega t\right) f_j(t) dt = 0, \tag{3.42}$$

whenever $j \in \mathcal{Z}$, where $f(t) = \sum_{j \in \mathbb{N}^*} f_j(t)\phi_j$. Hence, we introduce the space of *admissible functions*

$$\mathbb{E} = \{f \in \mathscr{F}; \ (3.42) \ \text{ holds, whenever } j \in \mathcal{Z}\}. \tag{3.43}$$

**Definition 3.35.** We say that the operator $L$ defined in (3.41) is globally solvable on $\mathbb{T} \times \mathbb{R}^d$ if for every $f \in \mathbb{E}$ there exists $u \in \mathscr{F}'$ such that $Lu = f$.

The connection between hypoellipticity and solvability is given by the next Proposition 3.36.

**Proposition 3.36.** If $L$ is globally hypoelliptic, then it is globally solvable.

*Proof.* By Theorem 3.32 we have either $\beta \neq 0$, or $\alpha$ satisfies Condition $(\mathscr{A})$. Employing eventually (3.37) in place of (3.36), we can assume, without loss of generality, that $\beta \leqslant 0$, .

Now, let $f \in \mathbb{E}$ be fixed. If $j \notin \mathcal{Z}$, we define

$$u_j(t) = \frac{i}{1 - e^{-2\pi i \lambda_j \omega}} \int_0^{2\pi} \exp\left(-i\lambda_j \omega s\right) f_j(t-s) ds, \tag{3.44}$$

and for $j \in \mathcal{Z}$ we put

$$u_j(t) = \exp(-i\lambda_j \omega t) \int_0^t \exp\left(i\lambda_j \omega s\right) f_j(s) ds. \tag{3.45}$$

Clearly, each $u_j(t)$ is a Gevrey function on $\mathbb{T}$ and

$$D_t u_j(t) + \lambda_j(\alpha + i\beta) u_j(t) = f_j(t), \ \forall j \in \mathbb{N}^*.$$

Since $\mathcal{Z}$ is a finite set, cf. Remark 3.28 and Lemma 3.33, estimates for $u_j(t)$ in the case $j \in \mathcal{Z}$ have no influence. For $j \notin \mathcal{Z}$ we may use similar arguments as in the proof of Proposition 3.34 to conclude that $u(t) = \sum_{j \in \mathbb{N}^*} u_j(t)\phi_j \in \mathscr{F}$. Then, by $Lu = f$, we see that $L$ is globally solvable. □

To study solvability, we need the following Definition 3.37.

**Definition 3.37** (Condition $(\mathscr{B})$)**.** We say that a real number $\alpha$ satisfies Condition $(\mathscr{B})$ if there are positive constants $\epsilon$ and $C$ such that

$$|\tau - \alpha\lambda_j| \geqslant C j^{-\epsilon},$$

for all $(j, \tau) \in \mathbb{N}^* \times \mathbb{Z}$ such that $\tau - \alpha\lambda_j \neq 0$.

**Proposition 3.38.** If the operator $L$ is globally solvable on $\mathbb{T} \times \mathbb{R}^d$, then either $\beta \neq 0$ or $\beta = 0$ and $\alpha$ satisfies Condition $(\mathscr{B})$.

*Proof.* We employ a contradiction argument, that is suppose that $L = D_t + \alpha P$ and $\alpha$ does not satisfy Condition $(\mathscr{B})$. By Lemma 3.30 there is a sequence $(j_\ell, \tau_\ell) \in \mathbb{N}^* \times \mathbb{Z}$ such that $(j_\ell)_\ell$ and $(|\tau_\ell|)_\ell$ are increasing and

$$0 < |\tau_\ell - \alpha\lambda_{j_\ell}| < j_\ell^{-\ell} \exp(-\ell).$$

Because of Remark 3.17 the sequence

$$f_j(t) = \begin{cases} 0, \ j \neq j_\ell, \\ j_\ell^{\ell/2} \exp(\ell/2)|\tau_\ell - \alpha\lambda_{j_\ell}| \exp(-i\tau_\ell t), \ j = j_\ell. \end{cases}$$

satisfies

$$|\partial_t^\gamma f_{j_\ell}(t)| \leqslant C^{\gamma+1} j_\ell^{-\ell/2+\gamma\varrho} \exp(-\ell/2),$$

where $C$ does not depend on $\gamma \in \mathbb{Z}_+$. Given $N \in \mathbb{N}$ we choose $\ell_0$ so that $-\ell/2 + \gamma\varrho < -N$, for all $\ell \geqslant \ell_0$. Then, $f = \sum_{j\in\mathbb{N}^*} f_j(t)\phi_j \in \mathscr{F}$. Since $f_j \equiv 0, j \in \mathcal{Z}$, we conclude that $f \in \mathbb{E}$.

Now, if $u \in \mathscr{F}'$ is a solution of $Lu = f$ we should have

$$\begin{aligned} u_{j_\ell}(t) &= \frac{i}{1 - e^{-2\pi i\lambda_{j_\ell}\alpha}} \int_0^{2\pi} \exp\left(-i\lambda_{j_\ell}\alpha s\right) f_{j_\ell}(t-s) ds \\ &= \frac{i}{1 - e^{-2\pi i\lambda_{j_\ell}\alpha}} j_\ell^{\ell/2} |\tau_\ell - \alpha\lambda_{j_\ell}| \int_0^{2\pi} \exp\left(-i\lambda_{j_\ell}\alpha s\right) \exp(-i\tau_\ell(t-s)) ds \\ &= -j_\ell^{\ell/2} \exp(\ell/2) \frac{|\tau_\ell - \alpha\lambda_{j_\ell}|}{\tau_\ell - \alpha\lambda_{j_\ell}} \exp(-i\tau_\ell t). \end{aligned}$$

Choosing $\sigma > 1$, $M \in \mathbb{N}$, and $\psi \in \mathcal{G}^\sigma(\mathbb{T})$ defined as

$$\psi(t) = (2\pi)^{-1} \sum_{\ell\in\mathbb{N}^*} \exp(-\ell/4) \exp(i\tau_l t),$$

we see directly that

$$|\langle u_{j_\ell}, \psi\rangle| \lambda_{j_\ell}^{-M} = \lambda_{j_\ell}^{-M} j_\ell^{\ell/2} \exp(\ell/4) \to \infty, \quad \text{as } \ell \to \infty,$$

which contradicts (3.27). So, $u \notin \mathscr{F}'$, which shows that $L$ is not globally solvable on $\mathbb{T} \times \mathbb{R}^d$. □

The next Theorem 3.39 is the last main result of this chapter.

**Theorem 3.39.** The operator $L$ defined in (3.41) is globally solvable on $\mathbb{T} \times \mathbb{R}^d$ if and only if either $\beta \neq 0$ or $\beta = 0$ and $\alpha$ satisfies Condition $(\mathscr{B})$.

In view of Proposition 3.38, we only need to prove that $\beta = 0$ and $\alpha$ satisfying Condition $(\mathscr{B})$ imply global solvability, which we do in the subsequent Proposition 3.40.

**Proposition 3.40.** If $\beta = 0$ and $\alpha$ satisfies Condition $(\mathscr{B})$, then $L$ is globally solvable.

*Proof.* Consider $f \in \mathbb{E}$ be fixed. We set

$$u_j(t) = \exp(-i\lambda_j \alpha t) \int_0^t \exp(i\lambda_j \alpha s) f_j(s) ds, \tag{3.46}$$

in case $j \in \mathcal{Z}$ and

$$u_j(t) = \frac{i}{1 - e^{-2\pi i \lambda_j \alpha}} \int_0^{2\pi} \exp(-i\lambda_j \alpha s) f_j(t - s) ds, \tag{3.47}$$

if $j \notin \mathcal{Z}$. Then,

$$|u_j(t)| \leqslant 2\pi \sup_{t \in \mathbb{T}} |f_j(t)|, \ \ j \in \mathcal{Z},$$

and

$$|u_j(t)| \leqslant C j^{\epsilon} \sup_{t \in \mathbb{T}} |f_j(t)|, \ \ j \notin \mathcal{Z}.$$

It follows from Theorem 3.26 that $u(t) = \sum_{j \in \mathbb{N}^*} u_j(t) \phi_j \in \mathscr{F}'$ and $Lu = f$.

□

# Chapter 4

# Quasi-Banach Schatten-von Neumann properties in Weyl-Hörmander calculus

In this chapter we present the Schatten-von Neumann classes investigating the Schatten-$p$ properties of pseudo-differential operators for Weyl-Hörmander calculus in the quasi-Banach context. We first recall some basic definitions and results, then examine the properties of the Wiener-Lebesgue spaces and, finally, state the main results, giving some examples.

## 4.1 Schatten-von Neumann classes

Schatten-von Neumann classes are related to the study of the asymptotics of an operator. We start by recalling the definition of *singular numbers* (or *singular values*).

**Definition 4.1.** Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and let $T$ be a linear map from $\mathcal{H}_1$ to $\mathcal{H}_2$. For every integer $j \geqslant 1$, the *singular number* of $T$ of order $j$ is given by

$$\sigma_j(T) = \sigma_j(\mathcal{H}_1, \mathcal{H}_2, T) \equiv \inf \|T - T_0\|_{\mathcal{H}_1 \to \mathcal{H}_2},$$

where the infimum is taken over all linear operators $T_0$ from $\mathcal{H}_1$ to $\mathcal{H}_2$ with rank at most $j-1$. Therefore, $\sigma_1(T)$ equals $\|T\|_{\mathcal{H}_1\to\mathcal{H}_2}$, while $\sigma_j(T)$ is non-negative and decreases with $j$.

We can now give the definition of Schatten-$p$ classes.

**Definition 4.2.** Let $p\in(0,\infty]$. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and let $T$ be a linear map from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then the *Schatten-p* norm of the operator T is defined as

$$\|T\|_{\mathscr{I}_p} = \|T\|_{\mathscr{I}_p(\mathcal{H}_1,\mathcal{H}_2)} \equiv \|\{\sigma_j(\mathcal{H}_1,\mathcal{H}_2,T)\}_{j=1}^{\infty}\|_{\ell^p}$$

(notice that it might attain $+\infty$). The operator $T$ is called a *Schatten-von Neumann operator* of order $p$ from $\mathcal{H}_1$ to $\mathcal{H}_2$, if $\|T\|_{\mathscr{I}_p}$ is finite, that is

$$\{\sigma_j(\mathcal{H}_1,\mathcal{H}_2,T)\}_{j=1}^{\infty}\in\ell^p.$$

The set of all Schatten-von Neumann operators of order $p$ from $\mathcal{H}_1$ to $\mathcal{H}_2$ is denoted by $\mathscr{I}_p=\mathscr{I}_p(\mathcal{H}_1,\mathcal{H}_2)$.

The next Proposition 4.3 is an immediate consequence of the properties of $\ell^p$ spaces.

**Proposition 4.3.** The spaces $\mathscr{I}_p(\mathcal{H}_1,\mathcal{H}_2)$ for $p\in(0,\infty]$ and $\mathcal{K}(\mathcal{H}_1,\mathcal{H}_2)$ are quasi-Banach spaces which are Banach spaces when $p\geqslant 1$.

We remark that, if $p<\infty$, then $\mathscr{I}_p(\mathcal{H}_1,\mathcal{H}_2)$ is contained in $\mathcal{K}(\mathcal{H}_1,\mathcal{H}_2)$, the set of linear and compact operators from $\mathcal{H}_1$ to $\mathcal{H}_2$.

In the next remark we highlight some significative Schatten-von Neumann classes.

**Remark 4.4.** We notice that $\mathscr{I}_\infty(\mathcal{H}_1,\mathcal{H}_2)$ agrees with $\mathcal{B}(\mathcal{H}_1,\mathcal{H}_2)$ (also in norms), the set of linear and bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. Furthermore, $\mathscr{I}_2(\mathcal{H}_1,\mathcal{H}_2)$ is a Hilbert space and agrees with the set of Hilbert-Schmidt operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ (also in norms). We set $\mathscr{I}_p(\mathcal{H})=\mathscr{I}_p(\mathcal{H},\mathcal{H})$. The set $\mathscr{I}_1(\mathcal{H}_1,\mathcal{H}_2)$ is the set of trace-class operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, and $\|\cdot\|_{\mathscr{I}_1(\mathcal{H}_1,\mathcal{H}_2)}$ coincides with the trace-norm. If in addition $\mathcal{H}_1=\mathcal{H}_2=\mathcal{H}$, then the trace

$$\mathrm{Tr}_{\mathcal{H}}(T)\equiv\sum_\alpha (Tf_\alpha,f_\alpha)_{\mathcal{H}}$$

is well-defined and independent of the orthonormal basis $\{f_\alpha\}_\alpha$ in $\mathcal{H}$.

For the next Proposition 4.5, see, e. g., [108, 117].

**Proposition 4.5.** Let $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ be Hilbert spaces, and let $T_k$ be linear and continuous operators from $\mathcal{H}_k$ to $\mathcal{H}_{k+1}$, $k = 1,2$. Then, the Hölder inequality

$$\|T_2 \circ T_1\|_{\mathscr{I}_r(\mathcal{H}_1,\mathcal{H}_3)} \leqslant \|T_1\|_{\mathscr{I}_{p_1}(\mathcal{H}_1,\mathcal{H}_2)} \|T_2\|_{\mathscr{I}_{p_2}(\mathcal{H}_2,\mathcal{H}_3)} \tag{4.1}$$

holds when

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r}.$$

In particular, the map $(T_1, T_2) \mapsto T_2^* \circ T_1$ is continuous from $\mathscr{I}_p(\mathcal{H}_1, \mathcal{H}_2) \times \mathscr{I}_{p'}(\mathcal{H}_1,\mathcal{H}_2)$ to $\mathscr{I}_1(\mathcal{H}_1)$, giving that

$$(T_1, T_2)_{\mathscr{I}_2(\mathcal{H}_1,\mathcal{H}_2)} \equiv \mathrm{Tr}_{\mathcal{H}_1}(T_2^* \circ T_1) \tag{4.2}$$

is well-defined and continuous from $\mathscr{I}_p(\mathcal{H}_1,\mathcal{H}_2) \times \mathscr{I}_{p'}(\mathcal{H}_1,\mathcal{H}_2)$ to $\mathbb{C}$. If $p = 2$, then the product, defined by (4.2) agrees with the scalar product in $\mathscr{I}_2(\mathcal{H}_1,\mathcal{H}_2)$.

The next proposition can be found in [16] and [108].

**Proposition 4.6.** Let $p \in [1,\infty]$, $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and let $T$ be a linear and continuous map from $\mathcal{H}_1$ to $\mathcal{H}_2$. Then the following is true:

(i) if $q \in [1, p']$, then

$$\|T\|_{\mathscr{I}_p(\mathcal{H}_1,\mathcal{H}_2)} = \sup |(T, T_0)_{\mathscr{I}_2(\mathcal{H}_1,\mathcal{H}_2)}|,$$

where the supremum is taken over all $T_0 \in \mathscr{I}_q(\mathcal{H}_1,\mathcal{H}_2)$ such that $\|T_0\|_{\mathscr{I}_{p'}(\mathcal{H}_1,\mathcal{H}_2)} \leqslant 1$;

(ii) if in addition $p < \infty$, then the dual of $\mathscr{I}_p(\mathcal{H}_1,\mathcal{H}_2)$ can be identified through the form (4.2).

In the next investigations, we will be interested in finding necessary and sufficient conditions on symbols in order for the corresponding pseudo-differential operators to belong to $\mathscr{I}_p(\mathcal{H}_1, \mathcal{H}_2)$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ satisfy

$$\mathscr{S}(V) \hookrightarrow \mathcal{H}_1, \mathcal{H}_2 \hookrightarrow \mathscr{S}'(V),$$

where $V$ denotes a real vector space. Then, we introduce the following definition.

**Definition 4.7.** Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces satisfying

$$\mathscr{S}(V) \hookrightarrow \mathcal{H}_1, \mathcal{H}_2 \hookrightarrow \mathscr{S}'(V),$$

and let $p \in (0, \infty]$. Then, we define

$$s_{A,p}(\mathcal{H}_1, \mathcal{H}_2) \equiv \{ a \in \mathscr{S}'(V \times V') : \operatorname{Op}_A(a) \in \mathscr{I}_p(\mathcal{H}_1, \mathcal{H}_2) \}$$

and

$$\|a\|_{s_{A,p}(\mathcal{H}_1, \mathcal{H}_2)} \equiv \| \operatorname{Op}_A(a) \|_{\mathscr{I}_p(\mathcal{H}_1, \mathcal{H}_2)}. \tag{4.3}$$

Notice that the map $a \mapsto \operatorname{Op}_A(a)$ is bijective from $\mathscr{S}'(V \times V')$ to the set of all linear and continuous operators from $\mathscr{S}(V)$ to $\mathscr{S}'(V)$. It then follows from the definitions that the map $a \mapsto \operatorname{Op}_A(a)$ restricts to a bijective and isometric map from $s_{A,p}(\mathcal{H}_1, \mathcal{H}_2)$ to $\mathscr{I}_p(\mathcal{H}_1, \mathcal{H}_2)$. Hence, we can set

$$s_{A,p}(W) = s_{A,p}(\mathcal{H}_1, \mathcal{H}_2) \quad \text{when} \quad \mathcal{H}_1 = \mathcal{H}_2 = L^2(V).$$

For convenience we also put $s_p^w = s_{A,p}$ in the Weyl case (i. e. when $A = \frac{1}{2} \cdot I_V$). We recall that if $T$ is an affine symplectic map, then the pullback map $a \mapsto T^* a$ is bijective and norm preserving on $s_p^w(W)$, that is

$$\|a(T\cdot)\|_{s_p^w} = \|a\|_{s_p^w}$$

(see e. g. [115]).

### 4.1.1 Modulation spaces

Some subsequent investigations in this chapter involve certain properties for a small class of modulation spaces, defined on the phase space $W = V \times V'$ (cf. Section 1.1.2, [43, 47] and the references therein).

We first recall the definition of quasi-Banach spaces.

**Definition 4.8.** A *quasi-norm* $\|\cdot\|_{\mathcal{B}}$ of order $p \in (0,1]$, also denoted as *p-norm*, to the vector space $\mathcal{B}$, is a functional on $\mathcal{B}$ such that the following is true:

(i) $\|f\|_{\mathcal{B}} \geqslant 0$, when $f \in \mathcal{B}$, with equality only for $f = 0$;

(ii) $\|\alpha f\|_{\mathcal{B}} = |\alpha|\,\|f\|_{\mathcal{B}}$, when $f \in \mathcal{B}$ and $\alpha \in \mathbb{C}$;

(iii) $\|f+g\|_{\mathcal{B}}^p \leqslant \|f\|_{\mathcal{B}}^p + \|g\|_{\mathcal{B}}^p$, when $f, g \in \mathcal{B}$.

We equip $\mathcal{B}$ with the topology induced by $\|\cdot\|_{\mathcal{B}}$. The space $\mathcal{B}$ is called a *quasi-Banach space of order* $p$, or a $p$*-Banach space*, if $\mathcal{B}$ is complete under this topology.

**Remark 4.9.** It follows that a topological vector space is a Banach space if and only if it is a quasi-Banach space of order 1.

We then recall the definition of modulation spaces.

**Definition 4.10.** Let $\phi \in \mathscr{S}(W) \setminus \{0\}$. Then the symplectic short-time Fourier transform of the function $u \in L^1(W)$, is defined as the continuous function on $W \times W$, given by

$$(\mathcal{V}_\phi u)(X,Y) = \pi^{-d} \int_W e^{2i\sigma(Y,Z)} u(Z) \overline{\phi(Z-X)}\, dZ,$$

where $d$ denotes the dimension of $V$.

**Definition 4.11.** Let $p \in (0,1]$. The modulation space $M^p(W)$ consists of all functions $u \in L^1(W)$ such that

$$\|u\|_{M^p} = \|u\|_{M^p(W)} \equiv \|V_\phi u\|_{L^p(W\times W)} \tag{4.4}$$

is finite.

In the following Proposition 4.12, we recall the essential properties concerning the spaces $M^p(W)$. Their proofs and a more systematic discussion can be found in [43].

**Proposition 4.12.** Let $p \in (0,1]$. Then the following is true:

(i) $M^p(W)$ is a quasi-Banach space under the quasi-norm (4.4), which is independent of the choice of $\phi \in \mathscr{S}(W) \setminus \{0\}$ in (4.4). Furthermore, different choices of $\phi$ give rise to equivalent quasi-norms;

(ii) $\mathscr{S}(W)$ is continuously embedded and dense in $M^p(W)$;

(iii) $\mathfrak{F}_\sigma$ is a homeomorphism from $M^p(W)$ to $M^p(W)$.

A part of our investigations of Schatten-von Neumann properties later on is based on the following lemma. The result can be found in e. g. [119].

**Lemma 4.13.** Let $p \in (0,1]$ and $A \in \mathscr{L}(V)$. Then $M^p(W)$ is continuously embedded in $s_{A,p}(W)$, and

$$\|a\|_{s_{A,p}(W)} \leqslant C\|a\|_{M^p(W)}, \quad a \in M^p(W), \tag{4.5}$$

where the constant $C$ only depends on $p$, $d$ and $\phi$ in (4.4). Additionally, if $A = \frac{1}{2}I$, then the constant $C$ in (4.5) does not depend on the choice of symplectic coordinates.

## 4.2 Wiener-Lebesgue spaces

In this section we introduce the notion of Wiener-Lebesgue spaces with respect to slowly varying metrics and investigate their structural properties.

### 4.2.1 Wiener-Lebesgue spaces with respect to slowly varying metrics

We start by recalling some facts about $g$-balls, which are given by

$$U_{X,R} = U_{g,X,R} \equiv \{ Y \in W \,:\, g_X(Y - X) < R^2 \}, \tag{4.6}$$

when $X \in W$ and $R > 0$. The following lemma is a consequence of Lemma 1.4.9 and the proof of Theorem 1.4.10 in [57].

**Lemma 4.14.** Let $g$ be slowly varying on $W$ and let $c$ and $C$ be as in (1.8). Then there exists a sequence $\{X_j\}_{j=1}^{\infty}$ such that if

$$U_j = U_{X_j,R},$$

for some $R > 0$ such that $\frac{c}{2} < R^2 < c$, then the following is true:

(i) $g_{X_j}(X_j - X_k) \geqslant \frac{c}{2C}$ for every $j, k = 1, 2, \dots$ such that $j \neq k$;

(ii) $W = \bigcup_{j=1}^{\infty} U_j$;

(iii) if $j \in \mathbb{Z}_+$ is fixed, then $U_j \cap U_k \neq \varnothing$ for at most $(4C^3+1)^{2d}$ numbers of $k$.

**Definition 4.15.** Let $g$ be slowly varying on $W$, $c$ and $C$ be as in (1.8). Then the family of $g$-balls $\{U_j\}_{j=1}^{\infty}$ in Lemma 4.14 is called an *admissible $g$-covering* of $W$.

**Remark 4.16.** Let $\{X_j\}_{j=1}^{\infty}$ be as in Lemma 4.14. For future reference, we observe that if $Y \in W$, $r, R_1, R_2 > 0$ satisfy

$$\frac{c}{2} < R_1^2 < R_2^2 < c, \quad r < \frac{R_2 - R_1}{2C},$$

and $U_{Y,r} \cap U_{X_j,R_1} \neq \varnothing$ for some $j \in \mathbb{Z}_+$, then $U_{Y,r} \subseteq U_{X_j,R_2}$.

As a consequence of Lemma 4.14 there are at most $(4C^3 + 1)^{2d}$ numbers of $U_{X_j,R_1}$ or $U_{X_j,R_2}$ which intersect with $U_{Y,r}$.

In fact, suppose $Z \in U_{Y,r} \cap U_{X_j,R_1}$. Then, for every $X \in U_{Y,r}$ we have that

$$\begin{aligned}\big(g_{X_j}(X - X_j)\big)^{\frac{1}{2}} &= \big(g_{X_j}(X - Z + Z - X_j)\big)^{\frac{1}{2}}\\ &\leqslant \big(g_{X_j}(Z - X_j)\big)^{\frac{1}{2}} + \big(g_{X_j}(X - Z)^{\frac{1}{2}}.\end{aligned}$$

By the fact that $g$ is slowly varying, we obtain that $g_{X_j} \leqslant C g_Z \leqslant C^2 g_Y$. Hence, we have

$$\begin{aligned}\big(g_{X_j}(Z - X_j)\big)^{\frac{1}{2}} + \big(g_{X_j}(X - Z)\big)^{\frac{1}{2}} &\leqslant R_1 + C\big(g_Y(Z - X)\big)^{\frac{1}{2}}\\ &\leqslant R_1 + 2Cr < R_2,\end{aligned}$$

which shows that $X \in U_{X_j,R_2}$, and the assertion follows.

**Definition 4.17.** Let $p, q \in (0, \infty]$, $\theta \in \mathbb{R}$, $g$ be a slowly varying metric on $W$, $\{U_j\}_{j=1}^{\infty}$ be an admissible $g$-covering, and let $U \subseteq \mathbb{R}^d$ be an open ball such that $\{j + U\}_{j\in\mathbb{Z}^d}$ covers $\mathbb{R}^d$.

(i) The *Wiener-Lebesgue space* $WL^{q,p}(\mathbb{R}^d)$ (with respect to $p$ and $q$) consists of all measurable functions $f$ such that $\|f\|_{WL^{q,p}}$ is finite, where

$$\|f\|_{WL^{q,p}} \equiv \big\|\{\|f\|_{L^q(j+U)}\}_{j\in\mathbb{Z}^d}\big\|_{\ell^p(\mathbb{Z}^d)}.$$

(ii) The *Wiener-Lebesgue space* $WL^{q,p}_{g,\theta}(W)$ (with respect to $p$, $q$, $\theta$ and $g$) consists of all measurable functions $a$ such that $\|a\|_{WL^{q,p}_{g,\theta}}$ is finite, where

$$\|a\|_{WL^{q,p}_{g,\theta}} \equiv \big\|\{\|a\|_{L^q(U_j)} \cdot |U_j|^{\theta}\}_{j\in\mathbb{Z}_+}\big\|_{\ell^p(I)}.$$

We remark that $WL^{q,p}_{g,\theta}(W)$ is a quasi-Banach space of order given by $\min(1, p, q)$, and independent of the choice of admissible $g$-covering $\{U_j\}_{j\in\mathbb{Z}_+}$ in Definition 4.17 (cf. Proposition 4.18 below). In particular, it follows that $WL^{q,p}(\mathbb{R}^d)$ is independent of the choice of $U$ in Definition 4.17. (This follows from [47] as well.) If $p, q \geqslant 1$, then $WL^{q,p}_{g,\theta}(W)$ is a Banach space.

For $p \in (0,1]$ and $q \in (0,\infty]$, the choice of parameter $\theta = \frac{1}{p} - \frac{1}{q}$ in the $WL^{q,p}_{g,\theta}$ spaces is of special interest. For this reason we let

$$WL^{q,p}_{g} = WL^{q,p}_{g,\theta} \quad \text{when} \quad \theta = \frac{1}{p} - \frac{1}{q}.$$

### 4.2.2 Properties of Wiener-Lebesgue spaces

We show here some basic properties for $WL^{q,p}_{g,\theta}$ spaces. First, we show that such spaces are invariantly defined with respect to the choice of admissible $g$-covering. Then, we show that such spaces increase if we replace the metrics with corresponding symplectic metrics.

**Proposition 4.18.** Let $p, q \in (0,\infty]$, $\theta \in \mathbb{R}$, and $g$ be slowly varying on $W$. Then $WL^{q,p}_{g,\theta}(W)$ is independent of the choice of admissible $g$-covering $\{U_j\}_{j\in\mathbb{Z}_+}$ in Definition 4.17.

**Remark 4.19.** Since $WL^{q,p}_{g,\theta}(W)$ is defined through quasi-norm estimates, it follows from Proposition 4.18 that different admissible coverings give rise to equivalent quasi-norms for $WL^{q,p}_{g,\theta}(W)$.

*Proof of Proposition 4.18.* We only prove the result when $p \leqslant q < \infty$. The other cases follow by analogous arguments. By considering $b(X) = |a(X)|^q$, we reduce ourselves to the case when $q = 1$ and $p \leqslant 1$. We may also replace $\theta$ by $\theta/p$.

Let $\mathcal{U} = \{U_j\}_{j\in\mathbb{Z}_+}$ and $\mathcal{V} = \{V_k\}_{k\in\mathbb{Z}_+}$ be admissible $g$-coverings, let

$$\|a\|^p_{\mathcal{U}} = \sum_{j=0}^{\infty} \left( \int_{U_j} |a(X)|\, dX \right)^p |U_j|^\theta,$$

and let

$$\|a\|^p_{\mathcal{V}} = \sum_{k=0}^{\infty} \left( \int_{V_k} |a(X)|\, dX \right)^p |V_k|^\theta.$$

By [57, Lemma 18.4.4], there is a bounded sequence $\{\varphi_k\}_{k=0}^\infty$ in $S(1,g)$ such that $\varphi_k \geqslant 0$, $\operatorname{supp}\varphi_k \subseteq V_k$ for every $k$, and $\sum_{k=0}^\infty \varphi_k = 1$.

We have

$$
\begin{aligned}
\|a\|_{\mathcal{U}}^p &= \sum_{j=0}^{\infty} \left( \int_{U_j} |a(X)|\, dX \right)^p |U_j|^\theta \\
&\asymp \sum_{j=0}^{\infty} \left( \int_{U_j} \sum_{k=0}^{\infty} |\varphi_k(x) a(X)|\, dX \right)^p |U_j|^\theta \\
&\leqslant \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left( \int_{U_j} |\varphi_k(x) a(X)|\, dX \right)^p |U_j|^\theta \\
&\asymp \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left( \int_{U_j} |\varphi_k(x) a(X)|\, dX \right)^p |V_k|^\theta,
\end{aligned}
$$

where the last relation follows from the fact that $|U_j| \asymp |V_k|$ when $U_j \cap V_k \neq \varnothing$ in combination with the fact that $g$ is slowly varying. Since there is an upper bound of intersections between $U_j$ and $V_k$ in view of Remark 4.16, we obtain

$$
\begin{aligned}
\|a\|_{\mathcal{U}}^p &\lesssim \sum_{k=0}^{\infty} \left( \sum_{j=0}^{\infty} \left( \int_{U_j} |\varphi_k(x) a(X)|\, dX \right)^p |V_k|^\theta \right) \\
&\asymp \sum_{k=0}^{\infty} \left( \int_{W} |\varphi_k(x) a(X)|\, dX \right)^p |V_k|^\theta \\
&\leqslant \sum_{k=0}^{\infty} \left( \int_{V_k} |a(X)|\, dX \right)^p |V_k|^\theta \\
&= \|a\|_{\mathcal{V}}^p. \qquad \square
\end{aligned}
$$

Next we show that $WL_g^{q,p}(W)$ is contained in $WL_{g^0}^{q,p}(W)$, when $g$ is feasible. For that reason we need the following proposition.

**Proposition 4.20.** Let $g$ be a slowly varying metric on $W$, $G$ be a $g$-continuous metric such that $g \leqslant G$, and let $\{U_{X_j,R}\}_{j=1}^{\infty}$ be an admissible $g$-covering of $W$. Then there exists an admissible $G$-covering $\{U_{G,k}\}_{k=1}^{\infty}$ of $W$ given by

$$U_{G,k} = U_{G,Y_k,r} = \{\, X \in W \, : \, G_{Y_k}(X - Y_k) < r^2 \,\}, \quad k \in \mathbb{Z}_+$$

such that

$$C_1 \frac{|U_{X_j,R}|}{|U_{G,X_j,r}|} \leqslant N_j \leqslant C_2 \frac{|U_{X_j,R}|}{|U_{G,X_j,r}|} \tag{4.7}$$

when $N_j$ is the number of $U_{G,k}$ intersecting $U_{X_j,R}$, and the constants $C_1, C_2 > 0$ are independent of $j \in \mathbb{Z}_+$.

*Proof.* Let $U_{X_j,R_1}$ and $U_{X_j,R_2}$ be as in Remark 4.16. If $r > 0$ is chosen small enough, then there is an admissible $G$-covering of $W$, given by

$$U_{G,k} = \{\, X \in W \,:\, G_{Y_k}(X - Y_k) < r^2 \,\}, \quad k \in \mathbb{Z}_+$$

such that $U_{G,k} \subseteq U_{X_j,R_2}$ when $U_{G,k}$ intersects $U_{X_j,R_1}$. The fact that $G$ is $g$-continuous and $g \leqslant G$ guarantees that such $r$ exists. Also, let $\Omega_j$ be the set of all $k \in \mathbb{Z}_+$ such that $U_{G,k}$ intersects $U_j$ and let $N_j = |\Omega_j|$.

We have

$$U_{X_j,R_1} \subseteq \bigcup_{k \in \Omega_j} U_{G,k} \subseteq U_{X_j,R_2}.$$

Since the balls $\{U_{G,k}\}_{k \in I}$ form an admissible covering of $W$, there is an upper bound $M$ of overlapping $U_{G,k}$. This gives

$$\frac{1}{M} \sum_{k \in \Omega_j} |U_{G,k}| \leqslant |U_{X_j,R_2}| \quad \Longleftrightarrow \quad \sum_{k \in \Omega_j} |U_{G,k}| \leqslant M |U_{X_j,R_2}|.$$

Since $G$ is $g$-continuous, we have

$$C_3 |U_{G,X_j,r}| \leqslant |U_{G,k}| \leqslant C_4 |U_{G,X_j,r}|$$

for some constants $C_3, C_4 > 0$ which are independent of $k$. A combination of these estimates gives

$$C_3 N_j \, |U_{G,X_j,r}| = C_3 |\Omega_j| \, |U_{G,X_j,r}| \leqslant \sum_{k \in \Omega_j} |U_{G,k}| \leqslant M |U_{X_j,R_2}|,$$

which leads to the second inequality in (4.7).

We also have

$$|U_{X_j,R_1}| \leqslant \left| \bigcup_{k \in \Omega_j} U_{G,k} \right| \leqslant \sum_{k \in \Omega_j} |U_{G,k}| \leqslant C_4 N_j |U_{G,X_j,r}|,$$

giving the first inequality in (4.7), which in turn gives the result. □

Since all $g$-balls are of the same size when $g$ is symplectic, the previous proposition takes the following form.

**Corollary 4.21.** Let $g$ be a feasible metric on $W$ and let $\{U_{X_j,R}\}_{j=1}^{\infty}$ be an admissible $g$-covering of $W$. Then there exists an admissible $g^0$-covering $\{U_k^0\}_{k=1}^{\infty}$ of $W$ given by

$$U_k^0 = \{\, X \in W \,:\, g_{Y_k}(X - Y_k) < r^2 \,\}, \quad k \in \mathbb{Z}_+$$

such that $N_j \leqslant C|U_{X_j,R}|$, where $N_j$ is the number of $U_k^0$ intersecting $U_{X_j,R}$, and the constant $C > 0$ is independent of $j \in \mathbb{Z}_+$.

**Proposition 4.22.** Let $g$ be a slowly varying metric on $W$ and let $G$ be a $g$-continuous metric such that $g \leqslant G$. Also, suppose that and $0 < p \leqslant q < \infty$. Then

$$WL_g^{q,p}(W) \subseteq WL_G^{q,p}(W).$$

*Proof.* Let $p_0 = \frac{p}{q} \in (0,1]$ and $b(X) = |a(X)|^q$. The inequalities in (4.7) shall be combined with

$$\sum_{k=1}^{N} x_k^{p_0} \leqslant N^{1-p_0} \left( \sum_{k=1}^{N} x_k \right)^{p_0}, \qquad x_1, \ldots, x_N \geqslant 0, \tag{4.8}$$

which follows by concavity of $t \mapsto t^{p_0}$.

We use the same notations as in the proof of Proposition 4.20. Since $\|a\|_{WL_G^{q,p}}^p = \|b\|_{WL_G^{1,p_0}}^{p_0}$, we obtain

$$\begin{aligned}
\|a\|_{WL_G^{q,p}}^p &\asymp \sum_{k=1}^{\infty} \left( \int_{U_{G,k}} |b(X)|\, dX \right)^{p_0} |U_{G,k}|^{1-p_0} \\
&\leqslant \sum_{j=1}^{\infty} \left( \sum_{k \in \Omega_j} \left( \int_{U_{G,k}} |b(X)|\, dX \right)^{p_0} |U_{G,k}|^{1-p_0} \right) \\
&\leqslant \sum_{j=1}^{\infty} \left( |\Omega_j|^{1-p_0} \left( \sum_{k \in \Omega_j} \int_{U_{G,k}} |b(X)|\, dX \right)^{p_0} |U_{G,k}|^{1-p_0} \right),
\end{aligned}$$

where the last inequality follows from (4.8). Since there is a bound $M$ of overlapping $U_{G,k}$,

$$|U_{X_j,R_1}| \asymp |U_{X_j,R_2}|, \quad \text{and} \quad |U_{G,k}| \asymp |U_{G,X_j,r}|,$$

when $U_{G,k}$ intersects with $U_{X_j,R_1}$, Proposition 4.20 gives

$$\begin{aligned}\|a\|^p_{WL^{q,p}_G} &\lesssim \sum_{j=1}^{\infty} \left( \left( \frac{|U_{X_j,R_2}|}{|U_{G,X_j,r}|} \right)^{1-p_0} \left( \sum_{k\in\Omega_j} \int_{U_{G,k}} |b(X)|\,dX \right)^{p_0} |U_{G,X_j,r}|^{1-p_0} \right) \\ &\leqslant \sum_{j=1}^{\infty} \left( \left( M \int_{U_{X_j,R_2}} |b(X)|\,dX \right)^{p_0} |U_{X_j,R_2}|^{1-p_0} \right) \\ &\asymp \|b\|^{p_0}_{WL^{1,p_0}_g} = \|a\|^p_{WL^{q,p}_g} ,\end{aligned}$$

and the result follows from these estimates. □

Since $g^0$ is $g$-continuous and $g \leqslant g^0$ whenever $g$ is feasible, the following corollary is an immediate consequence of Proposition 4.22.

**Corollary 4.23.** Let $g$ be feasible on $W$ and $0 < p \leqslant q < \infty$. Then

$$WL^{q,p}_g(W) \subseteq WL^{q,p}_{g^0}(W).$$

## 4.3 Quasi-Banach Schatten-von Neumann properties in pseudo-differential calculus

In this section we deduce Schatten-von Neumann properties, with respect to $p \in (0,1]$, for pseudo-differential operators with symbols in $S(m,g)$ and with $m$ or $a$ belonging to $WL^{1,p}_g(W)$. At the beginning, in Section 4.3.1, we deal with Weyl operators, where in the first part the assumptions on $m$ and $g$ are minimal, and the operators are acting on $L^2(V)$. The second part of Section 4.3.1 is devoted to operators acting between (different) Bony-Chemin Sobolev-type spaces $H(m,g)$. Here, we restrict ourselves and assume that $m$ and $g$ satisfy the usual conditions in the Weyl-Hörmander calculus. Subsequently, in Section 4.3.2, we consider more general pseudo-differential calculi, but with some additional restrictions on $g$.

### 4.3.1 The case of Weyl-Hörmander calculus

First we have the following result, related to [119, Theorem 4.1]. Here $\lfloor t \rfloor$ is the integer part of $t \geqslant 0$. That is, $\lfloor t \rfloor$ is the largest integer smaller than or equal to $t$.

**Theorem 4.24.** Let $p \in (0,1]$, $N > 2\lfloor \frac{d}{p} \rfloor + 1$ be an integer, $g$ be feasible on $W$, and $m \in WL^{1,p}_g(W)$ be a positive function on $W$. Then $S_N(m,g) \subseteq s^w_p(W)$.

Theorem 4.24 agrees with [119, Theorem 4.1], except that the assumption

$$m \in WL^{1,p}_g(W)$$

in Theorem 4.24 is replaced with

$$m \in L^p(W) \quad \text{and} \quad m \text{ is } g\text{-continuous.}$$

In view of Lemma 4.32 below, it follows that [119, Theorem 4.1] is a special case of Theorem 4.24.

For the proof of Theorem 4.24, we need the following lemma on embeddings between $s^w_p(W)$ and Sobolev-type spaces of distributions with suitable numbers of derivatives belonging to $WL^{1,p}(W)$.

**Lemma 4.25.** Let $p \in (0,1]$, $W$ be a symplectic vector space of dimension $2d$, $g$ be a constant symplectic metric on $W$, and $N > 2\lfloor \frac{d}{p} \rfloor + 1$ be an integer. Also let $a \in \mathscr{C}^N_0(U)$ with

$$U = \{\, X \,:\, g(X - X_0) < r^2 \,\},$$

for some $r > 0$ and $X_0 \in W$. Then there is a constant $C > 0$ which only depends on $p$ and $d$ such that

$$\|a\|_{s^w_p} \leqslant C \sum_{|\alpha| \leqslant N} \|\partial^\alpha_g a\|_{L^1(U)}. \tag{4.9}$$

For the proof we recall that $C$ and $\|a\|_{s_p^w}$ in (4.5) are independent of the choice of symplectic coordinates. Hence, if $\phi \in \mathscr{S}(W) \setminus 0$ is fixed, then (4.5) can be refined as

$$\begin{aligned}&\|T^*a\|_{s_p^w} = \|a\|_{s_p^w}\\ &\leqslant C \min\left(\|\mathcal{V}_\phi a\|_{L^p}, \|\mathcal{V}_\phi(T^*a)\|_{L^p}, \|\mathcal{V}_{T^*\phi}a\|_{L^p}, \|\mathcal{V}_{T^*\phi}(T^*a)\|_{L^p}\right),\end{aligned} \tag{4.5$'$}$$

for some constant $C>0$ which is independent of the choice of affine symplectic map $T$.

*Proof.* Let $N$ be chosen as small as possible, still satisfying $N > 2\lfloor \frac{d}{p} \rfloor + 1$. Then $N = 2n$, for some integer $n > \frac{d}{p}$. Let $T$ be a linear and symplectic map such that

$$g(TY) = \sum_{j=1}^{d} (y_j^2 + \eta_j^2), \qquad Y = (y,\eta) \in W. \tag{4.10}$$

Also let $\phi \in \mathscr{C}_0^\infty(W) \setminus 0$ be fixed, and let $b$ be such that $a = T^*b$. Then

$$\partial_g^\alpha a = T^*(\partial^\alpha b) \quad \text{and} \quad \operatorname{supp} b \subseteq \Omega,$$

where $\Omega$ is the open ball with radius $r$ and center at $TX_0$, with respect to the Euclidean metric on the right-hand side in (4.10).

Let $K = \operatorname{supp}\phi$. By (4.5)$'$, we obtain

$$\begin{aligned}
\|a\|_{s_p^w}^p &= \|b\|_{s_p^w}^p \leqslant C\|b\|_{M^p}^p\\
&= C \iint \left( \int_\Omega b(Z)\overline{\phi(Z-X)} e^{2i\sigma(Y,Z)}\, dZ \right)^p dX\,dY\\
&= C \iint \left( \langle Y\rangle^{-2n} \int_\Omega b(Z)\overline{\phi(Z-X)} (1-\tfrac{1}{2}\Delta_Z)^n e^{2i\sigma(Y,Z)}\, dZ \right)^p dX\,dY\\
&\leqslant C_1 \sum_{|\alpha+\beta|\leqslant N} \int_{K+\Omega} \left( \int_\Omega |\partial^\alpha b(Z)||\partial^\beta \phi(Z-X)|\, dZ \right)^p dX\\
&\leqslant C_2 \left( |K+\Omega| \sup_{|\beta|\leqslant N} (\|\partial^\beta\phi\|_{L^\infty}^p) \right) \sum_{|\alpha|\leqslant N} \left( \int_\Omega |\partial^\alpha b(Z)| dZ \right)^p,\\
&= C_3 \sum_{|\alpha|\leqslant N} \left( \int_U |\partial_g^\alpha a(Z)| dZ \right)^p,
\end{aligned}$$

which gives (4.9), and the result follows. □

*Proof of Theorem 4.24.* By $g \leqslant g^0$, Corollary 4.23, and the fact that $S(m,g)$ increases with $g$, it suffices to prove the result with $g^0$ in place of $g$. Hence we may assume that $g$ is symplectic.

Let $\{U_j\}_{j=0}^\infty$ be an admissible $g$-covering and let $\{\varphi_j\}_{j=0}^\infty$ be a bounded sequence in $S(1,g)$ as in the proof of Proposition 4.18, with $\operatorname{supp}\varphi_j \subseteq U_j$. Also, let $g_j = g_{X_j}$. By Lemma 4.25, we obtain

$$\|a\|_{s_p^w}^p = \Big\|\sum_{j=1}^\infty(\varphi_j a)\Big\|_{s_p^w}^p \leqslant \sum_{j=1}^\infty \|\varphi_j a\|_{s_p^w}^p \lesssim \sum_{j=1}^\infty \sum_{|\alpha|\leqslant 2N}\left(\int_{U_j}|(\partial_{g_j}^\alpha(\varphi_j a))(X)|\,dX\right)^p$$
$$\lesssim \sum_{j=1}^\infty \sum_{|\alpha|\leqslant 2N}\left(\int_{U_j}|(\partial_{g_j}^\alpha a)(X)|\,dX\right)^p \lesssim \|a\|^p \sum_{j=1}^\infty\left(\int_{U_j}|m(X)|\,dX\right)^p.$$

Here $\|a\| = \|a\|_{N,m}^g$, the norm of $a$ in $S_N(m,g)$. Since there is a bound of overlapping $U_j$, it follows from these estimates that

$$\|a\|_{s_p^w}^p \lesssim \|a\|^p \sum_{j=1}^\infty\left(\int_{U_j}|m(X)|\,dX\right)^p \asymp \|a\|^p\|m\|_{WL_g^{1,p}}^p,$$

which gives the result. □

The next result improves Theorem 4.24. It also extends [53, Theorem 3.9].

**Theorem 4.26.** Let $p \in (0,1]$, $N > 2\lfloor \frac{d}{p}\rfloor + 1$ be an integer, $g$ be feasible on $W$, $m$ be a positive function on $W$ such that $h_g^{k/2}m \in WL_g^{1,p}(W)$ for some integer $k \in [0,N]$, and suppose $a \in S_N(m,g) \cap WL_g^{1,p}(W)$. Then $a \in s_p^w(W)$.

For the proof we need the following lemmas.

**Lemma 4.27.** Let $p \in (0,\infty]$, $q \in [1,\infty]$, $N \in \mathbf{N}$ and $f \in WL^{q,p}(\mathbb{R}^d) \cap \mathscr{C}^N(\mathbb{R}^d)$. Then there exists a constant $C > 0$ such that

$$\|\partial^\alpha f\|_{WL^{q,p}}^p \leqslant C\left(\|f\|_{WL^{q,p}}^p + \sum_{|\beta|=N}\|\partial^\beta f\|_{WL^{q,p}}^p\right), \quad |\alpha| \leqslant N. \qquad (4.11)$$

**Lemma 4.28.** Let $g$ be a feasible metric on $W$, $\alpha \in [0,1]$ and set $G = h_g^{-\alpha} g$. Also, assume that $N \geqslant 0$ is an integer which is fixed, $m > 0$ is a weight function on $W$, $a \in \mathscr{C}^N(W)$, and set

$$m_0 = \sum_{n=0}^{N-1} |a|_n^G + h_g^{\alpha N/2} m.$$

Then the following statements are true:

(i) if $p \in (0,1]$, then

$$\|m_0\|_{WL_g^{1,p}} \leqslant C(\|a\|_{WL_g^{1,p}} + \|h_g^{\alpha N/2} m\|_{WL_g^{1,p}}); \tag{4.12}$$

(ii) if $a \in WL_g^{1,p}(W)$ and $h_g^{\alpha N/2} m \in WL_g^{1,p}(W)$, then $m_0 \in WL_g^{1,p}(W)$.

*Proof of Lemma 4.27.* Let $U$ be as in Definition 4.17. Then there exists a constant $C > 0$ such that, for any $|\alpha| \leqslant N$ and $j \in \mathbb{Z}^d$,

$$\|\partial^\alpha f\|_{L^q(j+U)} \leqslant C \left( \|f\|_{L^q(j+U)} + \sum_{|\beta|=N} \|\partial^\beta f\|_{L^q(j+U)} \right).$$

(See e. g. [10].) Hence for a (possibly new) constant $C > 0$, we obtain

$$\|\partial^\alpha f\|_{L^q(j+U)}^p \leqslant C \left( \|f\|_{L^q(j+U)}^p + \sum_{|\beta|=N} \|\partial^\beta f\|_{L^q(j+U)}^p \right).$$

Summing up with respect to $j \in \mathbb{Z}^d$ we have

$$\begin{aligned} \|\partial^\alpha f\|_{WL^{q,p}}^p &= \sum_{j\in\mathbb{Z}^d} \|\partial^\alpha f\|_{L^q(j+U)}^p \\ &\leqslant C \left( \sum_{j\in\mathbb{Z}^d} \|f\|_{L^q(j+U)}^p + \sum_{j\in\mathbb{Z}^d} \sum_{|\beta|=N} \|\partial^\beta f\|_{L^q(j+U)}^p \right) \\ &= C \left( \|f\|_{WL^{q,p}}^p + \sum_{|\beta|=N} \|\partial^\beta f\|_{WL^{q,p}}^p \right). \qquad \square \end{aligned}$$

*Proof of Lemma 4.28.* It suffices to prove (i). By [115, Lemma 6.1], it follows that $|a|_k^G \leqslant Cm_0$ for some constant $C>0$. Let $V_j=U_j$, and let $\varphi_j$ and $U_j$ for $j\in \mathbb{Z}_+$ be as in the proof of Proposition 4.18. Also, let $\{\psi_j\}_{j=1}^\infty$ be a bounded sequence in $S(1,g)$ such that $\psi_j\in \mathscr{C}_0^\infty(U_j)$ and $\psi_j=1$ in the support of $\varphi_j$. Lastly, let $g_j=g_{X_j}$ and $G_j=G_{X_j}$. Then

$$|\varphi_j a|_N^{G_j} = h_{g_j}^{\alpha N/2}|\varphi_j a|_N^{g_j} \leqslant C h_{g_j}^{\alpha N/2}\psi_j m,$$

where the constant $C$ is independent of $j\in\mathbb{Z}_+$. For every $j\in\mathbb{Z}_+$, let $G_j$ define the Euclidean structure on $W$. By Lemma 4.27, and the fact that $C$ in (4.11) is invariant under changes of symplectic structures on $W$, it follows that

$$\|\,|\varphi_j a|_n^{G_j}\|_{L^1} \leqslant C\big(\|\varphi_j a\|_{L^1} + \|h_{g_j}^{\alpha N/2}\psi_j m\|_{L^1}\big),$$

where the constant $C$ neither depends on $j\in\mathbb{Z}_+$ nor on $n\in\{0,\dots,N\}$.

We have

$$\|\,|a|_n^G\|_{WL_g^{1,p}}^p = \Big\|\Big|\sum_{l=1}^\infty \varphi_l a\Big|_n^G\Big\|_{WL_g^{1,p}}^p = \sum_{j=1}^\infty \left(\int_{U_j}\Big|\sum_{l=1}^\infty \varphi_l a\Big|_n^G(X)\,dX\right)^p |U_j|^{1-p}.$$

Since there is a bound of overlapping sets $U_j$ when $j\in\mathbb{Z}_+$, we get

$$\left(\int_{U_j}\Big|\sum_{l=1}^\infty \varphi_l a\Big|_n^G(X)\,dX\right)^p \leqslant C_1\left(\sum_{k=0}^n \int_{U_j}|a|_k^G(X)\,dX\right)^p,$$

where the constant $C_1$ is independent of $j$. By Lemma 4.27, and the fact that there is a bound of overlapping $U_j$, we obtain

$$\begin{aligned}
\|\,|a|_n^G\|_{WL_g^{1,p}}^p &\leqslant C_1\sum_{j=1}^\infty\left(\sum_{k=0}^n\int_{U_j}|a|_k^G(X)\,dX\right)^p|U_j|^{1-p}\\
&\leqslant C_2\sum_{j=1}^\infty\left(\int_{U_j}\big(|a(X)|+|a|_N^G(X)\big)\,dX\right)^p|U_j|^{1-p}\\
&\leqslant C_3\Bigg(\sum_{j=1}^\infty\left(\int_{U_j}|a(X)|\,dX\right)^p|U_j|^{1-p}\\
&\qquad\qquad + \sum_{j=1}^\infty\left(\int_{U_j}h_{g_j}^{\alpha N/2}(X)m(X)\,dX\right)^p|U_j|^{1-p}\Bigg)\\
&\asymp \|a\|_{WL_g^{1,p}}^p + \|h_g m\|_{WL_g^{1,p}}^p,
\end{aligned}$$

for some constants $C_2$ and $C_3$. This gives (4.12), and the proof is complete. □

**Remark 4.29.** By the proof of Lemma 4.27, it follows that the constant $C$ in (4.11) only depends on the dimension of $W$ and on $N$.

In particular, by changing the coordinates in suitable ways, and using that there is a bound of overlapping $U_j$, it follows that

$$\| |a|^g_k \|^p_{WL^{q,p}_{g,\theta}} \leqslant C \left( \|a\|^p_{WL^{q,p}_{g,\theta}} + \| |a|^g_N \|^p_{WL^{q,p}_{g,\theta}} \right), \quad k = 0, 1, \dots, N. \tag{4.13}$$

*Proof of Theorem 4.26.* Let $G$ and $m_0$ be as in Lemma 4.28. We observe that if $a \in S(m, g)$, then $a \in S(m_0, G)$, in view of [115, Lemma 6.1]. The result now follows from Theorem 4.24. □

If the involved weight functions are $g$-continuous, we can replace the conditions on them as in the next two theorems, where the first one agrees with [119, Theorem 4.1] when $p \leqslant 1$.

**Theorem 4.30.** Let $p \in (0, 1]$, $g$ be feasible on $W$, and $m \in L^p(W)$ be a positive $g$-continuous function on $W$. Then $S(m, g) \subseteq s^w_p(W)$.

**Theorem 4.31.** Let $p \in (0, 1]$, $g$ be feasible on $W$, $m$ be a positive $g$-continuous function on $W$ such that $h_g^{k/2} m \in L^p(W)$ for some $k \geqslant 0$, and suppose $a \in S(m, g) \cap WL^{1,p}_g(W)$. Then $a \in s^w_p(W)$.

Theorems 4.30 and 4.31 are straightforward consequences of Theorems 4.24 and 4.26, combined with the following Lemma 4.32. We omit the details.

**Lemma 4.32.** Let $p, q \in (0, \infty]$, $g$ be slowly varying, and $m$ be $g$-continuous on $W$. Then

$$m \in L^p(W) \quad \Longleftrightarrow \quad m \in WL^{q,p}_g(W).$$

*Proof.* Suppose $m \in L^p(W)$, and let $\{U_j\}_{j \in \mathbb{Z}_+}$ be an admissible $g$-covering of $W$ with centers in $X_j \in W$, $j \in \mathbb{Z}_+$. Since $m$ is $g$-continuous and $g$ is slowly varying, it follows that

$$\|m\|^p_{L^p} \asymp \sum_{j=1}^{\infty} m(X_j)^p |U_j|.$$

By using the $g$-continuity again, it also follows that

$$\|m\|^p_{WL^{q,p}_{g,\theta}} \asymp \sum_{j=1}^{\infty} m(X_j)^p |U_j|^{\frac{p}{q}} |U_j|^{\theta p} = \sum_{j=1}^{\infty} m(X_j)^p |U_j|,$$

and the asserted equivalence follows from these relations. □

**Remark 4.33.** Suppose that, in addition to the assumptions of Theorem 4.30, the metric $g$ and the weight $m$ are $\sigma$-temperate and $(\sigma, g)$-temperate, respectively. Then there is a natural extension of Theorem 4.30 to Weyl operators acting on Sobolev-type Hilbert spaces, $H(m, g)$, introduced by Bony and Chemin in [21], which is especially suitable for the Weyl-Hörmander calculus. (See also Section 2.6 in [71].)

We recall here the definition of such spaces along with some necessary preliminary concepts (cf. [21]).

**Definition 4.34.** Let $g$ be a feasible and $\sigma$-temperate metric and $m$ $g$-continuous and $(\sigma, g)$-temperate. Then the spaces of confined symbols within the balls $U_{Y,r}$, denoted by $\mathrm{Conf}(g, Y, r)$, is the space $\mathscr{S}(W)$ endowed with the family of semi-norms given by

$$\|a\|_{k;\mathrm{Conf}(g,Y,r)} = \sup_{\ell \leqslant k; X \in W} |a|^g_k(X) \left(1 + g^\sigma_Y \left(X - U_{Y,r}\right)\right)^{k/2}$$

**Definition 4.35.** A family of symbols $(a_{Y,\lambda})$, indexed by $Y$ and (possibly) by another parameter $\lambda$, is said to be uniformly confined within the balls $U_{Y,r}$ if the norms $\|a_{Y,\lambda}\|_{k;\mathrm{Conf}(g,Y,r)}$ are bounded by constants $C_k$ that are independent of $Y$ and $\lambda$.

For instance, a family of symbols $(a_Y)$ supported in $(U_{Y,r})$, which is bounded in $S(1, g)$, is uniformly confined within the balls $(U_Y)$.

**Theorem 4.36.** Let $(a_Y)$ be a family of symbols uniformly confined within the balls $U_{Y,r}$. Then, there exist a constant $C$ independent of $(a_Y)$ such that, for all $r$ sufficiently small there exist functions $b_{Y,\nu}$ and $c_{Y,\nu}$ belonging to $\mathscr{S}(W)$, for $Y \in W$ and $\nu \in \mathbb{N}$, such that the following decomposition holds true

$$a_Y = \sum_{\nu=0}^{\infty} b_{Y,\nu} \# c_{Y,\nu}$$

and for all $N \in \mathbb{N}$, the families $(1+\nu)^N b_{Y,\nu}$ and $(1+\nu)^N c_{Y,\nu}$ are uniformly confined within the balls $U_{Y,Cr}$.

We can give the following definition.

**Definition 4.37.** Let $g$ be a feasible and $\sigma$-temperate metric and $m$ $g$-continuous and $(\sigma, g)$-temperate. Then the Sobolev space $H(m, g)$ is the space of $u \in \mathscr{S}'(W)$ satisfying:

$$\sum_{\nu=0}^{\infty} \int_W m(Y)^2 \left\| \theta_{Y,\nu}^w u \right\|_{L^2(W)}^2 |g_Y|^{1/2} \, \mathrm{d}Y < \infty$$

The space $H(m, g)$ turns out to be a Hilbert space, equipped with the inner product:

$$(u \mid v)_{H(m)} = \sum_{\nu=0}^{\infty} \int_W m(Y)^2 \left( \theta_{Y,\nu}^w u, \theta_{Y,\nu}^w v \right)_{L^2(W)} |g_Y|^{1/2} \, \mathrm{d}Y < \infty$$

Now, suppose that $m$ and $m_0$ are $g$-continuous and $(\sigma, g)$-temperate, and $a \in S(m, g)$. Then

$$\mathrm{Op}^w(a) : H(m_0, g) \to H(m_0/m, g)$$

is continuous. In [21, 71] it is also shown that there are $a_0 \in S(m, g)$ and $b_0 \in S(1/m, g)$ such that

$$\mathrm{Op}^w(b_0) = \mathrm{Op}^w(a_0)^{-1}, \qquad a_0 \in S(m, g), \; b_0 \in S(1/m, g). \tag{4.14}$$

Especially, it follows that

$$\mathrm{Op}^w(a_0) : H(m_0, g) \to H(m_0/m, g)$$

and

$$\mathrm{Op}^w(b_0) : H(m_0/m, g) \to H(m_0, g)$$

are continuous bijections, which are inverses to each other. In particular, from these mapping properties it follows that equality is attained in (1.13).

Now let $p \in (0,1]$, $g$ be strongly feasible on $W$, and $m$, $m_1$, and $m_2$ be positive $g$-continuous and $(\sigma,g)$-temperate functions on $W$ such that

$$\frac{m_2 m}{m_1} \in L^p(W).$$

A combination of Theorem 4.30 and (4.14) then gives

$$S(m,g) \subseteq s_{A,p}(\mathcal{H}_1,\mathcal{H}_2), \quad \text{when} \quad \mathcal{H}_1 = H(m_1,g),\ \mathcal{H}_2 = H(m_2,g).$$

(See also [119, Theorem 4.4].) Since $H(1,g) = L^2(V)$, in view of [21, 71], we regain Theorem 4.30 in the case when $m$ is $g$-continuous and $(\sigma,g)$-temperate, by choosing $m_1 = m_2 = 1$.

### 4.3.2 Split metrics and more general pseudo-differential calculi

In order to state analogous results for more general pseudo-differential calculi, we impose further restrictions on the metric $g$ and weight function $m$.

We recall that a feasible metric $g$ on $W$ is called *split* if there are global symplectic coordinates $Y = (y,\eta)$ such that

$$g_X(y,-\eta) = g_X(y,\eta),$$

for all $X \in W$.

The next proposition follows from [57, Theorem 18.5.10] and its proof.

**Proposition 4.38.** Let $A, B \in \mathscr{L}(V)$, $g$ be strongly feasible and split on $W = T^*V$, and let $m$ be $g$-continuous and $(\sigma,g)$-temperate weight function. Then

$$\operatorname{Op}_A(S(m,g)) = \operatorname{Op}_B(S(m,g)).$$

A combination of Theorem 4.30, Theorem 4.31, and Proposition 4.38 gives the following.

**Theorem 4.39.** Let $A \in \mathscr{L}(V)$, $p \in (0,1]$, $g$ be strongly feasible and split on $W$, and $m \in L^p(W)$ be a positive $g$-continuous and $(\sigma,g)$-temperate function on $W$. Then $S(m,g) \subseteq s_{A,p}(W)$.

**Theorem 4.40.** Let $A \in \mathscr{L}(V)$, $p \in (0,1]$, $g$ be strongly feasible and split on $W$, $m$ be a positive $g$-continuous and $(\sigma,g)$-temperate function on $W$ such that $h_g^{k/2}m \in L^p(W)$ for some $k \geqslant 0$. Also, suppose $a \in S(m,g) \cap WL_g^{1,p}(W)$. Then $a \in s_{A,p}(W)$.

## 4.4 Applications to special families of pseudo - differential operators

In this section we apply the results from previous sections to obtain Schatten-von Neumann properties for pseudo-differential operators with symbols in the well known Shubin classes and SG classes (see [107]).

**Remark 4.41.** Let $p \in (0,1]$. For the Shubin and SG symbol classes, we observe the following.

(i) If $r \in \mathbb{R}$, then $S(m,g) = \mathrm{Sh}^r(\mathbb{R}^{2d})$ when

$$g_{x,\xi}(y,\eta) = \frac{|y|^2 + |\eta|^2}{\langle (x,\xi)\rangle^2} \quad \text{and} \quad m(x,\xi) = \langle (x,\xi)\rangle^r. \tag{4.15}$$

Furthermore, $h_g(x,\xi) = \langle (x,\xi)\rangle^{-2}$ and

$$h_g^{k/2} m \in L^p(\mathbb{R}^{2d}), \quad \text{when} \quad k > r + \frac{2d}{p};$$

(ii) As in Section 1.4, if $r,\rho \in \mathbb{R}$, then $S(m,g) = S^{r,\rho}(\mathbb{R}^{2d})$ when

$$g_{x,\xi}(y,\eta) = \frac{|y|^2}{\langle x\rangle^2} + \frac{|\eta|^2}{\langle \xi\rangle^2} \quad \text{and} \quad m(x,\xi) = \langle x\rangle^r \langle \xi\rangle^\rho. \tag{4.16}$$

Furthermore, $h_g(x,\xi) = (\langle x\rangle\langle \xi\rangle)^{-1}$ and

$$h_g^{k/2} m \in L^p(\mathbb{R}^{2d}), \quad \text{when} \quad k > 2\max(r,\rho) + \frac{2d}{p}.$$

In both (i) and (ii), $g$ is strongly feasible and $m$ is $g$-continuous and $(\sigma, g)$-temperate.

In the next result we show how Lemma 4.32 and Theorem 4.40 can be combined with Remark 4.41, in order to obtain quasi-Banach Schatten-von Neumann properties for the Shubin classes and the SG classes.

**Proposition 4.42.** Let $p \in (0,1]$, $A$ be a real $d \times d$-matrix, and $r, \rho \in \mathbb{R}$. Then the following is true:

(i) if $m$ and $g$ are given by (4.15), then

$$\mathrm{Sh}^r(\mathbb{R}^{2d}) \cap WL_g^{1,p}(\mathbb{R}^{2d}) \subseteq s_{A,p}(\mathbb{R}^{2d});$$

(ii) if $m$ and $g$ are given by (4.16), then

$$S^{r,\rho}(\mathbb{R}^{2d}) \cap WL_g^{1,p}(\mathbb{R}^{2d}) \subseteq s_{A,p}(\mathbb{R}^{2d}).$$

**Example 4.43.** Let $p \in (0,1]$,

$$a(X) = \langle X \rangle^{-\frac{2d}{p}} (\log(1 + \langle X \rangle))^{-\frac{4d+1}{p}} e^{i(\log(1+\langle X \rangle))^r},$$

$$g_X(Y) = \frac{|Y|^2}{\langle X \rangle^2} \quad \text{and} \quad m(X) = \langle X \rangle^{-\frac{2d-1}{p}}.$$

Then $a \in WL_g^{1,p}(\mathbb{R}^{2d})$, and $h_g(X) = \langle X \rangle^{-2}$ and by observing that

$$|\partial_g^\alpha a| \asymp \langle X \rangle^{-\frac{2d}{p}} (\log(1 + \langle X \rangle))^{|\alpha|(r-1)+\frac{1-4d}{p}}$$

we obtain $a \in \mathrm{Sh}^{-\frac{2d-1}{p}}(\mathbb{R}^{2d}) = S(m,g)$. Since $h_g^{\frac{k}{2}} m \in WL_g^{1,p}(\mathbb{R}^{2d})$ for $k = 2$, we obtain $a \in s_{A,p}(\mathbb{R}^{2d})$ by Proposition 4.42.

**Example 4.44.** Let $W = \mathbb{R}^{2d}$, $p \in (0,1]$, $t < 2d(1 - \frac{1}{p})$, $g(Y) = |Y|^2$,

$$m_j(X) = \begin{cases} |j|^t & \text{whenever } |X - j| < \frac{1}{|j|}, \\ 0 & \text{otherwise} \end{cases}$$

for $j \in \mathbb{Z}^{2d} \setminus 0$, and

$$m(X) = \sum_{j \in \mathbb{Z}^{2d} \setminus 0} m_j(X).$$

Let $N > 2\lfloor \frac{d}{p} \rfloor + 1$ be an integer, let $r \in \mathbb{R}$ fulfill $r + N < t$, and let

$$a(X) = \sum_{j \in \mathbb{Z}^{2d}} |j|^r \phi_j(X),$$

where $\phi_j(X) = \phi(|j|(X - j))$ for some smooth positive function $\phi$ with support in the unit ball. Then $m$ is *not* $g$-continuous, but $m \in WL^{1,p}_g(W)$. Since for $|\alpha| \leqslant N$,

$$|\partial^{(\alpha)} a(X)| \leqslant \|\phi^\alpha\|_{L^\infty} m(X),$$

we have $a \in S_N(m, g)$ so that $a \in s^w_p(W)$ by Theorem 4.24. Hence $\mathrm{Op}^w(a)$ is a Schatten-von Neumann operator on $L^2(\mathbb{R}^d)$ of order $p$.

Observe that this conclusion cannot be reached with [119, Theorem 4.1]. In fact, neither is $m$ $g$-continuous, nor does $a$ belong to $S(m, g)$, both of which are necessary conditions in [119, Theorem 4.1].

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