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Some results in set theory

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Summary

This thesis is divided into five main chapters, preceded by a short introductory chapter that briefly recalls some preliminary concepts. Each chapter is self-consistent, although the last three share some underlying questions and concepts.

The first chapter is, for the most part, taken from [57], and its argument lies in descriptive set theory. A particularly effective tool in descriptive set theory is games, specifically infinite two-player, perfect information games. Unlike game theorists, who typically seek to find equilibrium strategies, set theorists are interested in the existence of winning strategies for either of the two players and in the implications that this has on the mathematical entities (e.g., topological spaces, functions, measure spaces) involved in the game. This chapter introduces a game that characterizes Baire class 1 functions between separable metrizable spaces. We show that the determinacy of our game (i.e. the statement "one of the two players has a winning strategy") is equivalent to a generalization of Baire's characterization theorem for Baire class 1 functions, and that both these statements hold under AD (the axiom of determinacy) and in Solovay's model.

The second chapter is largely derived from [6], co-authored with Alessandro Andretta. This chapter is about the relationship between two weak variants of the axiom of choice, specifically the axiom of countable choice (AC_ω) and the axiom of dependent choice (DC). These two axioms have a local version: given a set X, $\mathsf{AC}_\omega(X)$ asserts that every countable collection of nonempty subsets of X admits a choice function, while $\mathsf{DC}(X)$ asserts that every binary and total relation on X has an infinite chain. It is well-known that DC implies AC_ω. We show that it is consistent with ZF that there exists a set $A \subseteq \mathbb{R}$ such that $\mathsf{DC}(A)$ holds but $\mathsf{AC}_\omega(A)$ fails.

The third chapter lies at the intersection of lattice theory and combinatorial set theory. It addresses an open question posed by S. Z. Ditor in 1984 [22]: Given a positive integer n, is there a lattice of cardinality $\aleph_n$ whose principal ideals are finite and whose elements have at most $n+1$ lower covers? We show that such lattices exist in the constructible universe and, therefore, that their existence is consistent with ZFC.

The fourth chapter is for the most part taken from [7], co-authored with Alessandro Andretta. In this chapter, we generalize a result of A. Törnquist and W. Weiss [70] by studying the connection between the existence of Σ^1_2 Sierpiński's coverings of $\mathbb{R}^n$, and a numeric invariant of the join-semilattice of constructibility real degrees known as breadth. Additionally, we investigate the relationship between the breadth of the constructibility real degrees and the size of the continuum.

In the last chapter, we analyze the structure of the constructibility real degrees in the side-by-side Sacks model. The key feature of Sacks forcing is adding a particularly tame generic real of minimal degree of constructibility. Hence, it is natural to seek an understanding of the structure of the constructibility real degrees in models of ZFC obtained by forcing (over the constructible universe) with products or iterations of Sacks forcing. Much is already known regarding iterations and finite products. In this chapter, we address the case that has been less explored: the structure of the constructibility degrees of the reals in the model of ZFC obtained by forcing over L with a countable-support product of infinitely many Sacks forcings. Among other results, we show that, in such a model, the join-semilattice of the constructibility real degrees is rigid, meaning that it has no non-trivial automorphisms.

Contents

Chapter 0

Preliminaries

We briefly discuss some preliminary notions that are used across different chapters.

Trees

Let us recall the basic operations on finite sequences and the definition of a (descriptive) tree. All the relevant notations and definitions in this section are taken from the monograph [47].

Let X be a nonempty set. We denote by ${}^{<\omega}X$ the set of finite sequences of elements of X, with $\emptyset$ being the empty sequence. Let $u \in {}^{<\omega}X$ and $v \in {}^{\leq\omega}X$. If $u \subseteq v$ we say that u is an *initial segment* of v, and we say that v *extends* u. For each $n \leq \text{length}(u)$, we let $u{\restriction}n$ be the initial segment $\langle u(0), u(1), \ldots, u(n-1)\rangle$ if $n > 0$; otherwise, we let it be the empty sequence.

The *concatenation* of u with v is the sequence $u^\frown v$ obtained by listing all elements of u and then all elements of v. Clearly, $u \subseteq u^\frown v$. Given an $x \in X$, we write $u^\frown x$ instead of $u^\frown\langle x\rangle$. Sometimes we use the notation $\vec{s}$ to denote a sequence.

A *tree*[1] *T on X* is a subset of ${}^{<\omega}X$ closed under initial segments, i.e. for every $u, v \in {}^{<\omega}X$ with $u \subseteq v$, if $v \in T$, then $u \in T$. If u is an initial segment of $v \in T$ and $\mathrm{length}(v) = \mathrm{length}(u) + 1$, then we say that v is an *immediate successor of u in T*. A tree is said to be *pruned* if all its nodes have a proper extension. The *body* of a tree T, denoted by $[T]$, is the set $\{r \in {}^{\omega}X \mid \forall n\ r{\restriction}n \in T\}$, whose elements are called *branches* of T. A tree T is *well-founded* if $[T] = \emptyset$; otherwise, it is *ill-founded*.

The Baire space ${}^{\omega}\omega$ and the Cantor space ${}^{\omega}2$

Our main reference for descriptive set theory is the monograph [47]. The Baire space ${}^{\omega}\omega$ is the space of all infinite sequences of natural numbers endowed with the topology generated by the prebase $\{N_s \mid s \in {}^{<\omega}\omega\}$, where

$$N_s := \{r \in {}^{\omega}\omega \mid s \subset r\}.$$

Each N_s is called a *basic open set* of ${}^{\omega}\omega$. The Cantor space ${}^{\omega}2$ is the space of all infinite binary sequences endowed with a topology generated by a prebase analogous to the one of the Baire space. Since we can associate in a natural way to each infinite binary sequence a subset of ω and vice-versa, we also sometimes identify the Cantor space with $\mathcal{P}(\omega)$, i.e. the space of all subsets of ω endowed with the topology induced by the natural bijection with the Cantor space.

Both the Baire and the Cantor spaces are Polish spaces, i.e. separable completely metrizable spaces. Moreover, the fact that they have a base made of clopen sets makes them a combinatorically convenient setting for descriptive set theory, especially since many questions that arise in this area are invariant under Borel isomorphisms and any two uncountable Polish spaces are Borel isomorphic (see [4, 47]).

The Baire space ${}^{\omega}\omega$, the Cantor space ${}^{\omega}2$ (the power set of ω) and the Euclidean line $\mathbb{R}$ are not only Borel isomorphic, but they are *effectively* Borel-isomorphic (i.e. Δ^1_1-isomorphic) [55]. This is why, following set-theoretic

[1] Note that this definition comes from descriptive set theory [47, Definition 2.1], and is not to be confused with the more general order-theoretic notion of trees as posets whose principal ideals are well-ordered [40, Definition 9.10].

practice, we sometimes tacitly identify the three spaces and refer to infinite sequences of natural number or to subsets of ω as "reals".

Join-semilattices

The monograph [29] is our reference for all classical definitions and results in lattice theory. We treat join-semilattices as posets (i.e. partially ordered sets) or algebraic structures, depending on what representation is more suited for the given context.

Given a join-semilattice $(P, \leq)$ and a set $F \subseteq P$, we denote by $P \downarrow F$ and $P \uparrow F$ the sets $\{q \in P \mid q \leq p \text{ for some } p \in F\}$ and $\{q \in P \mid q \geq p \text{ for some } p \in F\}$, respectively. Sometimes, instead of $P \downarrow F$ we write $\leq \downarrow F$ or simply $\downarrow F$, when no ambiguity arises. If $F = \{p\}$, for some $p \in P$, then we write $P \downarrow p$ instead of $P \downarrow \{p\}$. A nonempty subset $D \subseteq P$ is *downward-closed* if for every $p \in D$, $\downarrow p \subseteq D$. Recall also that a nonempty subset $S \subseteq P$ is called an *sub-join-semilattice* of P if it is closed under joins. Furthermore, a sub-join-semilattice which is also downward-closed is called an *ideal*. Note that $P \downarrow p$ is an ideal of P for every $p \in P$; such ideals are known as *principal ideals*. An ideal I that does not coincide with the whole join-semilattice is called a *proper ideal*.

The greatest element (resp. least element) of a join-semilattice P, if it exists, is denoted by $\mathbf{1}_P$ (resp. $\mathbf{0}_P$), or simply $\mathbf{1}$ (resp. $\mathbf{0}$) if there is no risk of ambiguity. If we write $\bigvee \emptyset$, we are tacitly assuming that P has a least element, and therefore that $\bigvee \emptyset = \mathbf{0}$. Furthermore, an element $a \in P$ is an *atom* if $\mathbf{0} < a$ and there is no $b \in P$ with $\mathbf{0} < b < a$. More generally, given two elements $p, q \in P$, q is a *lower cover of* p (or, equivalently, p is an *upper cover of* q) if $q < p$ and there is no $x \in P$ with $q < x < p$. In particular, the atoms are the upper covers of $\mathbf{0}$.

Given two posets $(P, \leq)$ and $(Q, \leq)$, a map $f : P \to Q$ is *monotone* if $p \leq p'$ implies $f(p) \leq f(p')$ for every $p, p' \in P$. A map $f : P \to Q$ is said to be an *order-embedding* if it is monotone, injective, and its inverse is monotone. An *isomorphism* is a surjective order-embedding. Furthermore, when P and Q are join-semilattices f is an *homomorphism* if $f(p \vee p') = f(p) \vee f(p')$ for every

$p, p' \in P$. An injective homomorphism is called an *embedding*. Note that every homomorphism is monotone and that every embedding is an order-embedding, but the converse is not true in general.

Let us briefly review the notion of quotient join-semilattice. Given a join-semilattice $(P, \vee)$ an equivalence relation $\sim$ on P is a *congruence relation* if for all x_0, x_1, y_0, y_1 in P,

$$x_0 \sim y_0 \text{ and } x_1 \sim y_1 \Rightarrow x_0 \vee x_1 \sim y_0 \vee y_1.$$

Given a congruence relation $\sim$ on P, we can define the join operator $\vee$ on the quotient $P/\sim$ as follows: for every $x, y \in P$,

$$[x]_\sim \vee [y]_\sim := [x \vee y]_\sim$$

It is easy to check that this operator satisfies all the properties of a join. The resulting join-semilattice $P/\sim$ is called the *quotient join-semilattice of P modulo $\sim$*. We denote the quotient map by $\pi_\sim : P \to P/\sim$.

Any ideal I of P induces the following natural congruence relation $\sim_I$ on P: for $x, y \in P$, $x \sim_I y$ if there exists a $z \in I$ such that $x \vee z = y \vee z$. In this case, we simply write P/I and π_I instead of $P/\sim_I$ and $\pi_{\sim_I}$. Not every congruence relation on a join-semilattice is induced by an ideal.

Definition (Breadth)**.** Let P be a join-semilattice and $n \in \omega$. We say that P has *breadth at most n* if, for every nonempty finite subset X of P, there exists $Y \subseteq X$ with at most n elements such that $\bigvee X = \bigvee Y$. The *breadth* of P is the least $n \in \omega$ such that P has breadth at most n, if such n exists.

Note that a join-semilattice P has breadth 0 if and only if $P = \{\mathbf{0}\}$, and has breadth at most 1 if and only if it is a linear order.

Lemma. *Given a join-semilattice P and an $n \in \omega$, the following are equivalent:*

(1) *P has breadth at most n.*

(2) *For every $X \in [P]^{n+1}$, there exists $Y \in [X]^n$ such that $\bigvee X = \bigvee Y$.*

Proof. The only direction to be proven is (2) $\Rightarrow$ (1). Fix a nonempty finite subset X of P, towards finding some $Y \subseteq X$ of size at most n such that

$\bigvee Y = \bigvee X$. If $|X| \leq n$ there is nothing to prove, as we can set $Y = X$. Hence, suppose $X = \{x_0, x_1, \ldots, x_k\}$ for some $k \geq n$. We inductively define a finite sequence of sets $Y_0, \ldots, Y_{k-n}$ in $[X]^n$ as follows: first let $Y_0 \in [\{x_0, \ldots, x_n\}]^n$ be such that $\bigvee Y_0 = x_0 \vee x_1 \vee \ldots \vee x_n$, which exists by hypothesis; for each $i < k-n$, let $Y_{i+1} \in [Y_i \cup \{x_{n+i+1}\}]^n$ be such that $\bigvee Y_{i+1} = (\bigvee Y_i) \vee x_{n+i+1}$, which, again, exists by hypothesis. Note that the set Y_{k-n} is such that $Y_{k-n} \in [X]^n$ and $\bigvee Y_{k-n} = \bigvee X$. Thus, P has breadth at most n. □

Forcing

Forcing was first introduced by Paul Cohen in his breakthrough work [18] to prove the independence of the continuum hypothesis from Zermelo-Fraenkel set theory ZF. Since then, it has become one of the fundamental tools of modern set theory to prove consistency results. The monograph [50] is our reference for all classical results and notions in forcing theory.

If $\mathbb{P}$ is a forcing notion, i.e. a preordered set with a maximum $\mathbf{1}_{\mathbb{P}}$, we convene that $p \leq_{\mathbb{P}} q$ means that p is *stronger* than q, or, in other words, that p *extends* q. When there is no danger of confusion, we drop the subscript $\mathbb{P}$. Dotted letters $\dot{x}, \dot{y}, \ldots$ vary over the class of $\mathbb{P}$-names, $\check{x}$ is the canonical $\mathbb{P}$-name for x, while $\dot{G}$ is the $\mathbb{P}$-name for the generic filter. In practice, we often abuse the notation by dropping the háček and confusing a set x in the ground model with its name $\check{x}$. If F is a set of $\mathbb{P}$-names, then, following Karagila's notation [44], $F^\bullet$ is the $\mathbb{P}$-name $\{(\dot{x}, \mathbf{1}) \mid \dot{x} \in F\}$. This notation extends naturally to ordered pairs and sequences, so $(\dot{x}, \dot{y})^\bullet := \{\{\dot{x}\}^\bullet, \{\dot{x}, \dot{y}\}^\bullet\}^\bullet$ and so on. If G is $\mathbb{P}$-generic over V, then $\dot{x}_G$ is the object in V$[G]$ obtained by evaluating $\dot{x}$ with G.

Let $\mathbb{P}$ be a forcing notion. Every automorphism $\pi \in \mathrm{Aut}(\mathbb{P})$ acts canonically on $\mathbb{P}$-names as follows: given $\dot{x}$ a $\mathbb{P}$-name,

$$\pi\dot{x} = \{(\pi\dot{y}, \pi p) \mid (\dot{y}, p) \in \dot{x}\}.$$

The following classical lemma will play an important role in Chapter 2.

Lemma (Symmetry Lemma, [40, Lemma 14.37])**.** *Let* $\mathbb{P}$ *be a forcing notion,* $\pi \in \mathrm{Aut}(\mathbb{P})$ *and* $\dot{x}_1, \dots, \dot{x}_n$ *be* $\mathbb{P}$*-names. For every formula* $\varphi(x_1, \dots, x_n)$*,*

$$p \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n) \iff \pi p \Vdash \varphi(\pi\dot{x}_1, \dots, \pi\dot{x}_n).$$

Constructibility degrees

The monograph [20] is our reference for all classical definitions and results regarding the constructibility hierarchy and the constructible universe L. The *constructibility preorder* $\leq_c$ on the universe of sets is defined by

$$x \leq_c y \iff x \in \mathrm{L}(y).$$

Let $\equiv_c$ be the induced equivalence relation. We are interested mainly in the restriction of the contructibility preorder $\leq_c$ on $\mathcal{P}(\omega)$. The quotient $\mathcal{P}(\omega)/{\equiv_c}$ with the induced order is the set of *constructibility real degrees*, and it is denoted by $\mathcal{D}_c$. We will often refer to the constructibility real degrees simply as *real degrees*. Given a set $a \subseteq \omega$, we denote in bold $\mathbf{a}$ its equivalence class $[a]_\mathrm{c}$. If $\mathbf{a} \in \mathcal{D}_\mathrm{c}$, then set $\mathrm{L}[\mathbf{a}]$ to be $\mathrm{L}[a]$ for some/any $a \in \mathbf{a}$. Note that $(\mathcal{D}_\mathrm{c}, \leq_c)$ is a poset with minimum $\mathbf{0} = \mathcal{P}(\omega) \cap \mathrm{L}$, every degree $\mathbf{a}$ has size $(\aleph_1)^{\mathrm{L}[\mathbf{a}]} \leq \aleph_1$, and $\{\mathbf{b} \mid \mathbf{b} \leq_\mathrm{c} \mathbf{a}\}$, the set of all predecessors of $\mathbf{a} \in \mathcal{D}_\mathrm{c}$ has size $\leq (\aleph_1)^{\mathrm{L}[\mathbf{a}]} \leq \aleph_1$. Our official definition of $\leq_\mathrm{c}$ takes place on $\mathcal{P}(\omega)$, but since the Cantor space and the Euclidean line $\mathbb{R}$ are Δ^1_1-isomorphic, we could have used the "true" reals without any problem. The poset $\mathcal{D}_\mathrm{c}$ is actually a join-semilattice, with $\mathbf{a} \vee \mathbf{b} = [a \oplus b]_\mathrm{c}$ for some/any $a \in \mathbf{a}$ and $b \in \mathbf{b}$, where

$$a \oplus b = \{2n \mid n \in a\} \cup \{2n+1 \mid n \in b\}.$$

More generally, if $\langle\cdot,\cdot\rangle \colon \omega \times \omega \to \omega$ is a recursive bijection, and $a_n \subseteq \omega$, then letting

$$\textstyle\bigoplus_{n\in\omega} a_n := \{\langle n, k\rangle \mid n \in \omega \text{ and } k \in a_n\}$$

we have that $a_i \leq_\mathrm{c} \bigoplus_{n\in\omega} a_n$ for all $i < \omega$. Thus $\mathcal{D}_\mathrm{c}$ is σ-directed—any countable set of elements $\{\mathbf{a}_n \mid n \in \omega\}$ has an upper bound $[\bigoplus_{n\in\omega} a_n]_\mathrm{c}$. On the other hand $\{\mathbf{a}_n \mid n \in \omega\}$ need not have a least upper bound, i.e. $\mathcal{D}_\mathrm{c}$ need not be a

σ-complete join-semilattice [71]. Again in [71], it is also shown that $\mathcal{D}_c$ need not be a meet-semilattice.

The structure of $\mathcal{D}_c$ is highly non-absolute, being very sensitive to the ambient model. If $V = L$, then $\mathcal{D}_c$ is the singleton $\{\mathbf{0}\}$, while if $V = L[r]$ where r is a Cohen real, then $\mathcal{D}_c$ has the size of the continuum and a rich structure [2].

Adamowicz [67] has shown that for every constructible, constructibly countable and well-founded join-semilattice with a lowest element, there is a generic extension of the constructible universe in which $\mathcal{D}_c$ is isomorphic to the given join-semilattice—see [33, 64] for stronger results and more discussion on this.

Even though there is not yet a general result *à la* Adamowicz for uncountable join-semilattices, there are some scattered, yet interesting, results in this regard: in the iterated perfect set model, also known as iterated Sacks model, the continuum is $\aleph_2$ and $\mathcal{D}_c$ is well-ordered of order-type ω_2 [11, 30]; Groszek [31] has shown that $\mathcal{D}_c$ can, consistently with ZFC, be isomorphic to the reverse copy of $\omega_1 + 1$; see also [42, 43] for similar results.

Chapter 1

A game for Baire's grand theorem

1.1 Introduction

This chapter is, for the most part, taken from [57]. Given two topological spaces X, Y, a function $f : X \to Y$ is said to be *Baire class* 1 if, for every open subset V of Y, the pre-image $f^{-1}(V)$ is an F_σ subset of X, i.e. a countable union of closed subsets. If X is metrizable, then the open subsets of X are also F_σ subsets. All continuous functions with metrizable domain are Baire class 1.

A classical result concerning this class of functions is the following theorem of Baire—known as Baire's grand theorem—which provides a characterization of Baire class 1 functions from a Polish space to a separable metrizable space (e.g. see [47, Theorem 24.15]).

Theorem (Baire)**.** *Let X be a Polish space, Y a separable metrizable space, and $f : X \to Y$. Then the following are equivalent:*

1) *f is Baire class 1*

2) *$f{\restriction}K$ has a point of continuity for every compact $K \subseteq X$*

Actually, the separability hypothesis on X can be avoided (e.g. see [51, Ch. II, §31, X] and [26]), but here we are interested in the separable case.

We note that Baire class 1 functions have been, and still are, sometimes defined as pointwise limits of continuous functions—e.g. [36, 58, 59], Baire himself originally stated his grand theorem for pointwise limits of continuous real functions [8]. This definition and ours are equivalent only under certain hypotheses—e.g. see [47, Theorem 24.10] and [46].

In this chapter, we study the generalization of Baire's grand theorem in which the domain's hypothesis is weakened from Polish to separable metrizable, and its relationship with the determinacy of a two-player game. The use of infinite two-player, perfect information games to characterize certain classes of functions has a long and established history—e.g. [3, 9, 14, 15, 23, 24, 48, 72], see [56] for a detailed introduction on this subject.

In Section 1.2, we define our game $G(f)$, where f is a function between separable metrizable spaces. We prove that Player II has a winning strategy in $G(f)$ if and only if f is Baire class 1 (Theorem 1.2.2). Then we show that Player I has a winning strategy in $G(f)$ if and only if there is a compact $K \subseteq X$ such that $f \restriction K$ has no points of continuity (Theorem 1.2.3).

In Section 1.3, we discuss the determinacy of our game. We start by observing that the determinacy of our game for every function is equivalent to GBT, the generalization of Baire's grand theorem in which the domain's hypothesis is weakened from Polish to separable metrizable (Corollary 1.3.1). We note that AC (i.e. the axiom of choice) and GBT are mutually inconsistent (Proposition 1.3.2). Then we show that GBT is equivalent to a separation property coming from descriptive set theory (Theorem 1.3.4) and that both these statements hold under AD, the axiom of determinacy, and in Solovay's model.

1.2 The game

Given X a topological space and $(U_n)_{n\in\omega}$ a sequence of open subsets of X, we say that $(U_n)_{n\in\omega}$ is *convergent* if it is decreasing with respect to $\subseteq$, and if it is a local basis of some $x \in X$. In that case we write $\lim_{n\to\infty} U_n = x$.

Definition 1.2.1. Let X, Y be separable metrizable spaces and let $f : X \to Y$. In our game $G(f)$, at the nth round, Player I plays a nonempty open subset U_n of X, and then Player II plays $y_n \in \operatorname{ran}(f)$,

$$\begin{array}{llllllll} \text{I} & U_0 & & U_1 & & U_2 & & \dots \\ \text{II} & & y_0 & & y_1 & & y_2 & \dots \end{array}$$

with the rule: $U_{n+1} \subseteq U_n$ for each $n \in \omega$. At the end of a game run, Players I and II have produced a sequence $(U_n)_{n\in\omega}$ of nonempty open subsets of X and a sequence $(y_n)_{n\in\omega}$ in $\operatorname{ran}(f)$, respectively. Player II *wins* the run if either the sequence $(U_n)_{n\in\omega}$ is not convergent or it converges to an $x \in X$ and $(y_n)_{n\in\omega}$ converges to $f(x)$.

This game is an elaboration of Kiss' game [48] and a further generalization of Duparc's eraser game [23]. A *partial play* of Player I (resp. Player II) in $G(f)$ is a nonempty finite sequence of open subsets of X decreasing with respect to $\subseteq$ (resp. a finite sequence of elements of $\operatorname{ran}(f)$). A *strategy* for Player I (resp. Player II) in $G(f)$ is a function that maps each partial play of Player II (resp. Player I) to a nonempty open subset of X (resp. to an element of $\operatorname{ran}(f)$). A strategy σ for Player I is *winning* if for every infinite sequence $\vec{y} = (y_n)_{n\in\omega}$ of elements of $\operatorname{ran}(f)$, Player I wins the run of the game in which at each turn n Player I plays $\sigma(\vec{y} \restriction n)$ and Player II plays y_n. Winning strategies for Player II are analogously defined.

Kiss [48] used his game to characterize Baire class 1 functions between separable complete metric spaces. The following theorem generalizes this result by providing an analogous characterization for Baire class 1 functions between arbitrary separable metrizable spaces.

Theorem 1.2.2. *Let X, Y be separable metrizable spaces and $f : X \to Y$. Then Player II has a winning strategy in $G(f)$ if and only if f is Baire class* 1.

Proof. ($\Longleftarrow$): Assume that the function f is Baire class 1. As every separable metrizable space embeds into ${}^{\omega}\mathbb{R}$ (i.e. the space of infinite sequences of real numbers with the product of the Euclidean topology), we can assume $Y \subseteq {}^{\omega}\mathbb{R}$ without loss of generality.

By a theorem of Lebesgue, Hausdorff and Banach [47, Theorem 24.10] there exists a sequence $(f_n)_{n\in\omega}$ of continuous functions from X to ${}^{\omega}\mathbb{R}$ (with range not necessarily in Y) converging pointwise to f. Fix such a sequence, and also fix a compatible metric d on ${}^{\omega}\mathbb{R}$ and a sequence $(q_n)_{n\in\omega} \subset \operatorname{ran}(f)$ dense in $\operatorname{ran}(f)$. Given two nonempty $A, B \subseteq {}^{\omega}\mathbb{R}$, we let $d(A,B)$ be $\inf\{d(z_0,z_1) \mid z_0 \in A, z_1 \in B\}$.

In the next paragraph, we define by induction a map $\sigma^\star$ that maps partial plays of Player I into $Y \times \omega$. Then $\pi_Y \circ \sigma^\star$ is shown to be a winning strategy for Player II, where π_Y, π_ω are the canonical projections. We denote by $\sigma^\star_Y$ and $\sigma^\star_\omega$ the functions $\pi_Y \circ \sigma^\star$ and $\pi_\omega \circ \sigma^\star$, respectively.

Here is the definition on $\sigma^\star$ by induction on the lengths of Player I's partial plays: set $\sigma^\star(U_0) = (q_0, 0)$ for each nonempty open subset U_0 of X; fix $k \in \omega$, suppose that we have defined $\sigma^\star$ for all Player I's partial plays of length up to k and consider a partial play $\vec{U}^\frown U_k$ of length $k+1$, then

1) if there is an $n > \sigma^\star_\omega(\vec{U})$ such that $\operatorname{diam}(f_n[U_k]) \leq 2^{-n}$: fix an n satisfying the condition and an m such that $d(q_m, f_n[U_k]) \leq d(\operatorname{ran}(f), f_n[U_k]) + 2^{-n}$; set $\sigma^\star(\vec{U}^\frown U_k) = (q_m, n)$.
2) otherwise: we set $\sigma^\star(\vec{U}^\frown U_k) = \sigma^\star(\vec{U})$.

We now show that $\sigma^\star_Y$ is a winning strategy for Player II in $G(f)$. Fix an infinite play $(U_k)_{k\in\omega}$ of Player I in $G(f)$. If $(U_k)_{k\in\omega}$ is not convergent, then Player II wins. Assume that $(U_k)_{k\in\omega}$ converges to an $x \in X$, and set $y_k = \sigma^\star_Y(U_0,\dots,U_k)$, $n_k = \sigma^\star_\omega(U_0,\dots,U_k)$ for each $k \in \omega$. We need to show $\lim_{k\to\infty} y_k = f(x)$.

Claim 1.2.2.1. *The sequence $(n_k)_{k\in\omega}$ is nondecreasing and unbounded in ω.*

Proof. The fact that $(n_k)_{k\in\omega}$ is nondecreasing is a direct consequence of the definition of $\sigma^\star$. Next, note that, for all $n \in \omega$, the diameters of the sets in the sequence $(f_n[U_k])_{k\in\omega}$ converge to 0, as f_n is continuous and $(U_k)_{k\in\omega}$ is a local basis of x, decreasing with respect to $\subseteq$.

Fix a $k \in \omega$ and an $n > n_k$. By the previous observation, there exist a $k' > k$ such that $\operatorname{diam}(f_n[U_{k'}]) \leq 2^{-n}$. Fix one such k', there are two cases: either $n_{k'-1} > n_k$ or $n_{k'-1} = n_k$. In the latter case, the first condition in the inductive definition of $\sigma^\star$ happens at the k'-th round, hence $n_{k'} > n_{k'-1} = n_k$. In either case, $n_{k'} > n_k$. We just proved that for every k there is a $k' > k$ such that $n_{k'} > n_k$, therefore $(n_k)_{k\in\omega}$ is unbounded. $\square$

Let $\bar{k}$ be the least k such that $n_k > 0$.

Claim 1.2.2.2. *For all* $k \geq \bar{k}$, $d(y_k, f_{n_k}(x)) \leq d(f(x), f_{n_k}(x)) + 2^{1-n_k}$.

Proof. Fix a $k \geq \bar{k}$ and pick the smallest $l \leq k$ such that $n_l = n_k$. Note that $y_k = y_l$, as from the l-th round to the k-th $\sigma^\star$ does not change its response. From the minimality of l it follows that the first condition of the inductive definition of $\sigma^\star$ happens at the l-th round, therefore $d(y_l, f_{n_l}[U_l]) \leq d(\mathrm{ran}(f), f_{n_l}[U_l]) + 2^{-n_l}$ and $\mathrm{diam}(f_{n_l}[U_l]) \leq 2^{-n_l}$. Since we assumed $x = \lim_{n\to\infty} U_n$, x belongs to U_l and $f_{n_l}(x)$ belongs to $f_{n_l}[U_l]$, hence $d(\mathrm{ran}(f), f_{n_l}[U_l]) \leq d(f(x), f_{n_l}(x))$, and, overall,

$$d(y_l, f_{n_l}(x)) \leq d(y_l, f_{n_l}[U_l]) + \mathrm{diam}(f_{n_l}[U_l]) \leq d(f(x), f_{n_l}(x)) + 2^{1-n_l}.$$

As $n_l = n_k$ and $y_l = y_k$ we are done. □

Then, for each $k \geq \bar{k}$,

$$d(y_k, f(x)) \leq d(f_{n_k}(x), f(x)) + d(y_k, f_{n_k}(x)) \leq 2d(f_{n_k}(x), f(x)) + 2^{1-n_k}.$$

Since $(n_k)_{k\in\omega}$ is unbounded and the f_n's converge pointwise to f, these inequalities imply that $(y_k)_{k\in\omega}$ converges to $f(x)$ and therefore $\sigma^\star_Y$ wins the run. As $(U_k)_{k\in\omega}$ was an arbitrary play of Player I, we have shown that $\sigma^\star_Y$ is a winning strategy for Player II in $G(f)$.

($\Longrightarrow$): Suppose that Player II has a winning strategy in $G(f)$, we show that the function f is Baire class 1.

Fix a winning strategy σ for Player II in $G(f)$ and fix a compatible metric d on X. As X is separable, there exists a scheme $(U_s)_{s\in{}^{<\omega}\omega}$ of open subsets of X satisfying the following properties:

1) $U_\emptyset = X$.

2) For all $s \in {}^{<\omega}\omega$, $\bigcup_n U_{s^\frown n} = U_s$.

3) For all $s \in {}^{<\omega}\omega$, $\mathrm{diam}(U_s) \leq 2^{-\mathrm{length}(s)}$.

For each $s \in {}^{<\omega}\omega$ let

$$y_s = \sigma(U_{s\restriction 0}, U_{s\restriction 1}, U_{s\restriction 2}, \dots, U_s).$$

In other words, y_s is the response of Player II following σ to the partial play $(U_{s\restriction 0}, \dots, U_s)$ of Player I. For every $x \in X$, let T_x be the tree $\{s \in {}^{<\omega}\omega \mid x \in U_s\}$. It follows directly from properties 1) and 2) of the scheme that all such trees are nonempty and pruned.

Claim 1.2.2.3. *For all $x \in X$ and all open neighborhoods V of $f(x)$, there is an $s \in T_x$ such that for all $t \in T_x$ if $s \subseteq t$ then $y_t \in V$.*

Proof. Assume towards a contradiction that there is an $x \in X$ and a V, open neighborhood of $f(x)$, such that for all $s \in T_x$ there is a $t \in T_x$ that extends s and such that $y_t \notin V$. Then there exists a branch $r \in [T_x]$ such that $\{n \in \omega \mid y_{r\restriction n} \notin V\}$ is infinite. Fix one and note that Player I wins in $G(f)$ by playing the sequence $(U_{r\restriction n})_{n\in\omega}$, as, by property 3) of the scheme, this sequence converges to x, while the corresponding play of Player II according to σ does not converge to $f(x)$. Since we have assumed σ to be a winning strategy for Player II, we have reached a contradiction. □

Fix V open subset of Y and a sequence $(V_n)_{n\in\omega}$ of open subsets of V such that $V = \bigcup_{n\in\omega} V_n = \bigcup_{n\in\omega} \overline{V_n}$.

Claim 1.2.2.4.

$$f^{-1}(V) = \bigcup_{n\in\omega} \bigcup_{s\in {}^{<\omega}\omega} \Big(U_s \setminus \bigcup \{U_t \mid t \supseteq s \text{ and } y_t \notin V_n\}\Big).$$

Proof. Take x in the set on the right-hand side. By definition, there exists an $n \in \omega$ and an $s \in {}^{<\omega}\omega$ such that $s \in T_x$ and for all $t \in T_x$ extending s, $y_t \in V_n$. Fix a branch $r \in [T_x] \cap N_s$, then the sequence $(y_{r\restriction k})_{k\in\omega}$ is eventually in V_n, i.e. $y_{r\restriction k} \in V_n$ for all k greater than some $m \in \omega$. As $(U_{r\restriction k})_{k\in\omega}$ converges to x by property 3) of the scheme, and σ is a winning strategy for Player II, we have $\lim_{k\to\infty} y_{r\restriction k} = f(x)$, and therefore $f(x) \in \overline{V_n} \subseteq V$.

Now pick an x in $f^{-1}(V)$. There must be an n such that $f(x) \in V_n$. By Claim 1.2.2.3, there exists an $s \in {}^{<\omega}\omega$ such that for all $t \in T_x$ extending s, $y_t \in V_n$. But this means that x belongs to the set on the right-hand side. □

Since the set on the right-hand side is an F_σ subset of X and V was an arbitrary open subset of Y, we have shown that pre-images of open subsets of Y by f are F_σ sets. So f is Baire class 1. $\square$

Theorem 1.2.3. *Let X, Y be separable metrizable spaces and $f : X \to Y$. Then Player I has a winning strategy in $G(f)$ if and only if there exists a compact set $K \subseteq X$ such that $f \restriction K$ has no points of continuity.*

To prove this theorem, we need the notion of pointwise oscillation of a function. Given X a topological space, Y a metric space and $f : A \to Y$ for some nonempty $A \subseteq X$, we define $\mathrm{osc}_f(x)$ for each $x \in X$ as

$$\mathrm{osc}_f(x) := \inf \Big\{ \mathrm{diam}(f[U \cap A]) \mid U \subseteq X \text{ open neighborhood of } x \Big\}.$$

The function osc_f is upper semi-continuous, i.e. for every $\epsilon > 0$ the set $\{x \in X \mid \mathrm{osc}_f(x) \geq \epsilon\}$ is closed.

Lemma 1.2.4. *Let X, Y be separable metric spaces, $\epsilon > 0$ and $f : X \to Y$ such that $\mathrm{osc}_f(x) \geq \epsilon$ for all $x \in X$. Then there is a countable $Q \subseteq X$ such that $\mathrm{osc}_{f \restriction Q}(x) \geq \epsilon$ for all $x \in X$.*

Proof. Let d_Y be the metric on Y and fix a sequence $(y_n)_{n \in \omega}$ dense in Y. For each $n, m \in \omega$, let $Q_{n,m}$ be a countable and dense subset of $f^{-1}(B(y_n, 2^{-m}))$. We claim that the countable set $Q = \bigcup_{n,m} Q_{n,m}$ has the desired property.

Fix $x \in X$, $m \in \omega$ and an open neighborhood U of x. By assumption, there are $x_0, x_1 \in U$ such that $d_Y(f(x_0), f(x_1)) \geq \epsilon - 2^{-m}$. Let n_0, n_1 be such that $f(x_i) \in B(y_{n_i}, 2^{-m})$ for $i = 0, 1$. In particular, $U \cap f^{-1}(B(y_{n_i}, 2^{-m})) \neq \emptyset$, and therefore $U \cap Q_{n_i,m} \neq \emptyset$ for $i = 0, 1$. Pick q_0, q_1 in $U \cap Q_{n_0,m}$ and $U \cap Q_{n_1,m}$, respectively. Then,

$$\begin{aligned} d_Y(f(q_0), f(q_1)) &\geq d_Y(f(x_0), f(x_1)) - d_Y(f(x_0), f(q_0)) - d_Y(f(x_1), f(q_1)) \\ &\geq (\epsilon - 2^{-m}) - 2^{1-m} - 2^{1-m} = \epsilon - 5 \cdot 2^{-m}. \end{aligned}$$

Indeed, for $i = 0, 1$, $f(x_i)$ and $f(q_i)$ both belong to $B(y_{n_i}, 2^{-m})$, and therefore their distance is less than 2^{1-m}.

We have shown that for each $x \in X$, for every open neighborhood U of x and for all m there are $q_0, q_1 \in U \cap Q$ such that $d_Y(f(q_0), f(q_1))$ is greater than $\epsilon - 5 \cdot 2^{-m}$. In particular, $\mathrm{diam}(f[U \cap Q]) \geq \epsilon$. Hence, for all $x \in X$, $\mathrm{osc}_{f\restriction Q}(x) \geq \epsilon$. □

Proof of Theorem 1.2.3. ($\Longleftarrow$) : Fix a compact set $K \subseteq X$ such that $f\restriction K$ has no points of continuity. The winning strategy for Player I that we define is essentially the one defined by Kiss[1] in [48], the only difference being that we deal with a bit more care the amount of choice used in the construction (see Remark 1.2.5).

Fix a compatible metric d_X on X and d_Y on Y. Since $f\restriction K$ has no points of continuity, it follows that $\mathrm{osc}_{f\restriction K}(x) > 0$ for every $x \in K$. In particular, $K = \bigcup_n K_n$, where

$$K_n := \left\{ x \in K \mid \mathrm{osc}_{f\restriction K}(x) \geq \frac{1}{n} \right\}.$$

By Baire's category theorem, there is a nonempty open $U \subseteq X$ and an n such that $K_n \cap U = K \cap U$. Let C be the closure of $K_n \cap U$ and $\epsilon = 1/n$, then $\mathrm{osc}_{f\restriction C}(x) \geq \epsilon$ for every $x \in C$.

By Lemma 1.2.4, we know that there is a countable $Q \subseteq C$ such that $\mathrm{osc}_{f\restriction Q}(x) \geq \epsilon$ for every $x \in Q$. Let $(q_n)_{n\in\omega}$ be an enumeration of Q. We now define a winning strategy τ for Player I by induction on the lengths of Player II's partial plays. In particular, the map τ ranges among the open balls of X centered in Q, i.e. open sets of the form $B(x, \rho)$ for some $x \in Q$ and radius $\rho > 0$: first set $\tau(\emptyset) = B(q_0, 1)$—we are setting the first move of Player I; fix $k \in \omega$, suppose that we have defined τ for all partial plays of Player II of lengths up to k and consider the partial play $\vec{y}\,^\frown y_k$ of length $k+1$ with $B(q_{n_k}, \rho_k) = \tau(\vec{y})$, then

1) if $d_Y(y_k, f(q_{n_k})) \leq \epsilon/8$:
   let n_{k+1} be an n such that $q_n \in B(q_{n_k}, \rho_k)$ and $d_Y(f(q_n), f(q_{n_k})) \geq \epsilon/3$; let $\overline{\rho}$ be the greatest $\rho \leq \rho_k$ such that $B(q_{n_{k+1}}, \rho) \subseteq B(q_{n_k}, \rho_k)$ and set $\tau(\vec{y}\,^\frown y_k) = B(q_{n_{k+1}}, \overline{\rho}/2)$.

2) otherwise:
   $\tau(\vec{y}\,^\frown y_k) = B(q_{n_k}, \rho_k/2)$.

[1] Kiss' strategy, in turn, is based on the one defined by Carroy in [15].

We now prove that τ is a winning strategy for Player I. Fix an infinite play $\vec{y} = (y_k)_{k\in\omega}$ of Player II and set $B_k = B(x_k, \rho_k) = \tau(\vec{y} \restriction k)$ for every $k \in \omega$. First we show that the sequence $(B_k)_{k\in\omega}$ converges to an $x \in K$. Indeed, it follows directly from τ's inductive definition that $\bigcap_k B_k = \bigcap_k \overline{B_k}$; the compactness of K guarantees that $K \cap \left(\bigcap_k \overline{B_k}\right) \neq \emptyset$; finally, the radii of $(B_k)_{k\in\omega}$ converge to 0, hence $K \cap \left(\bigcap_k B_k\right)$ is a singleton $\{x\}$ and $(B_k)_{k\in\omega}$ converges to x.

So we are left to prove that the sequence $(y_k)_{k\in\omega}$ does not converge to $f(x)$. Suppose first that condition 1) of τ's inductive definition happens only finitely many times during this game run. This means that there exists an n such that for all $k \geq n$, $x_k = x$, and therefore $d_Y(y_k, f(x)) > \epsilon/8$ for all $k \geq n$. In this case $(y_k)_{k\in\omega}$ certainly does not converge to $f(x)$.

Now suppose otherwise, and let the increasing sequence $(k_n)_{n\in\omega}$ be such that condition 1) happens at the k_n+1-th round for each n. More precisely, $(k_n)_{n\in\omega}$ is the increasing sequence such that $d_Y(y_k, f(x_k)) \leq \epsilon/8$ if and only if $k = k_n$ for some (unique) n. For every n,

$$\begin{aligned} d_Y(y_{k_n}, y_{k_{n+1}}) &\geq d_Y(f(x_{k_n}), f(x_{k_{n+1}})) - d_Y(f(x_{k_n}), y_{k_n}) - d_Y(f(x_{k_{n+1}}), y_{k_{n+1}}) \\ &= d_Y(f(x_{k_n}), f(x_{k_n+1})) - d_Y(f(x_{k_n}), y_{k_n}) - d_Y(f(x_{k_{n+1}}), y_{k_{n+1}}) \\ &\geq \epsilon/3 - \epsilon/8 - \epsilon/8 = \epsilon/12 \end{aligned}$$

where the equality follows from $x_{k_{n+1}} = x_{k_n+1}$, which holds because in the rounds between $k_n + 1$ and k_{n+1} the strategy τ does not change the center of its balls; the last inequality follows directly from the definition of τ. Therefore, as $(k_n)_{n\in\omega}$ is unbounded, the sequence $(y_k)_{k\in\omega}$ does not converge.

In either case $(y_k)_{k\in\omega}$ does not converge to $f(x)$, therefore τ wins the run. As $(y_k)_{k\in\omega}$ was an arbitrary play of Player II, we have shown that τ is a winning strategy for Player I in $G(f)$.

$(\Longrightarrow)$: Suppose that Player I has a winning strategy in $G(f)$, we want to prove that there exists a compact set $K \subseteq X$ such that $f\restriction K$ has no points of continuity. We show instead that there exists a compact $K \subseteq X$ such that Player I has a winning strategy in $G(f\restriction K)$. Indeed, if we do so, it would mean that the function $f\restriction K$ is not Baire class 1, as otherwise Player II would have a winning strategy in $G(f\restriction K)$ by Theorem 1.2.2. Then, by Baire's grand theorem—which can be applied as K, being a compact separable metrizable space, is a Polish space—there would be a compact $K' \subseteq K$ such that $f \upharpoonright K'$ has no points of continuity.

Fix a winning strategy τ for Player I and also fix an enumeration $(q_n)_{n\in\omega}$ of a countable dense subset of $\operatorname{ran}(f)$. Denote by S the tree $\{s \in {}^{<\omega}\omega \mid s(n) \leq n \text{ for all } n < \operatorname{length}(s)\}$. Note that $[S]$ is a compact subset of the Baire space.

Consider the following map:

$$\begin{aligned} \varphi : [S] &\longrightarrow X \\ r &\longmapsto \lim_{n\to\infty} \tau\big(q_{r(0)}, q_{r(1)}, \ldots, q_{r(n)}\big). \end{aligned}$$

Since we are assuming τ winning for Player I, the limits in the definition always exist, and the map φ is well-defined. We now show that φ is continuous. Given an $r \in [S]$ and V open neighborhood of $\varphi(r)$, there exists an $n \in \omega$ such that $\tau\big(q_{r(0)}, q_{r(1)}, \ldots, q_{r(n-1)}\big) \subseteq V$, by definition of limit of sequences of open sets. But then the rules of the game force every $t \in [S] \cap N_{r\upharpoonright n}$ to be mapped by φ into $\tau\big(q_{r(0)}, q_{r(1)}, \ldots, q_{r(n-1)}\big) \subseteq V$. Therefore φ is continuous and $K = \operatorname{ran}(\varphi)$ is a compact subset of X.

Next, we show that Player I has a winning strategy in $G(f\upharpoonright K)$. Fix d_Y compatible metric on Y. For each $y \in Y$ and $k \in \omega$, pick an $n \leq k$ such that $d_Y(q_n, y) = \min_{m\leq k} d_Y(q_m, y)$ and let $q(y, k) := q_n$.

We define the strategy τ' for Player I in $G(f\upharpoonright K)$ as follows: for each $(y_0, \ldots, y_k)$ partial play of Player II in $G(f\upharpoonright K)$, we let

$$\tau'(y_0, y_1, \ldots, y_k) = \tau(q(y_0, 0), q(y_1, 1), \ldots, q(y_k, k)) \cap K.$$

We claim that τ' is a winning strategy for Player I in $G(f\upharpoonright K)$. Take an infinite play $(y_k)_{k\in\omega}$ of Player II. For each k, let n_k be such that $n_k \leq k$ and $q_{n_k} = q(y_k, k)$. Then $(n_k)_{k\in\omega}$ belongs to $[S]$, and the limit of the sequence $(\tau'(y_0, \ldots, y_k))_{k\in\omega}$ is $\varphi((n_k)_{k\in\omega}) \in K$, by definition of φ.

If $(y_k)_{k\in\omega}$ is not convergent, then Player I wins the run. So suppose that $(y_k)_{k\in\omega}$ converges to $y \in \overline{\operatorname{ran}(f)}$. Then,

$$\begin{aligned} d_Y(q(y_k,k),y) \leq d_Y(y_k,y) + d_Y(q(y_k,k),y_k) &= d_Y(y_k,y) + \min_{m\leq k} d_Y(q_m,y_k) \\ &\leq 2d_Y(y_k,y) + \min_{m\leq k} d_Y(q_m,y). \end{aligned}$$

Since $\lim_{k\to\infty} y_k = y$ by assumption, and $\lim_{k\to\infty} \min_{m\leq k} d_Y(q_m,y) = 0$ by the density of $(q_n)_{n\in\omega}$ in $\operatorname{ran}(f)$, it follows from the above inequalities that $(q(y_k,k))_{k\in\omega}$ converges to y. Therefore,

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} q(y_k,k) \neq f(\lim_{k\to\infty} \tau(q(y_0,0),\dots,q(y_k,k))) = f(\lim_{k\to\infty} \tau'(y_0,\dots y_k)).$$

The $\neq$ follows from having assumed τ winning strategy for Player I in $G(f)$, and the last equality instead comes directly from having defined $\tau'(y_0,\dots,y_k)$ as $\tau(q(y_0,0),\dots,q(y_k,k)) \cap K$.

Hence $(y_k)_{k\in\omega}$ does not converge to $f(\lim_{k\to\infty} \tau'(y_0,\dots y_k))$, and τ' wins the run. Since $(y_k)_{k\in\omega}$ was an arbitrary play of Player II, τ' is a winning strategy for Player I in $G(f\restriction K)$. □

Remark 1.2.5. The careful reader may have noticed that in this section we did not use the axiom of choice, or even the axiom of dependent choice, in their full potential. Indeed, all the proofs contained or cited in this section go through assuming only $\mathsf{AC}_\omega(\mathbb{R})$, the axiom of countable choice over the reals: "Every countable family of nonempty subsets of $\mathbb{R}$ has a choice function".

1.3 On the determinacy of $G(f)$

Recall that a two-player game G is *determined* if either Player I or Player II has a winning strategy. Carroy [15] proved that Duparc's eraser game $G_e(f)$, which characterizes Baire class 1 functions from and into ${}^\omega\omega$, is determined for every function f, and used this determinacy result to give a new game-theoretic proof of Baire's grand theorem restricted to functions between 0-dimensional Polish spaces. On the other hand, Kiss [48] used Baire's grand theorem to prove the determinacy of his game. Our game $G(f)$ is a further generalization of both these games, and, again, a strong relationship between its determinacy and

Baire's grand theorem emerges as a direct corollary of the two main theorems of the previous section. Let us introduce the following statement, which is the same as Baire's grand theorem with the hypothesis on the domain weakened from Polish to separable metrizable:

(GBT) For all X, Y separable metrizable spaces and $f : X \to Y$, f is Baire class 1 if and only if $f \restriction K$ has a point of continuity for every compact $K \subseteq X$.

The following is a direct corollary of Theorems 1.2.2 and 1.2.3.

Corollary 1.3.1. *The following are equivalent:*

1) *$G(f)$ is determined for every f.*
2) GBT

But unlike Duparc's and Kiss' games, ours is not determined in general, as the next folklore proposition shows.

Proposition 1.3.2. (AC) GBT *is false.*

Proof. Under the axiom of choice, there exists a set of reals with cardinality of the continuum that does not contain any uncountable closed set. Let X be such a set. Since the family of the F_σ subsets of a second countable space has at most the cardinality of the continuum, it follows from Cantor's theorem that there must be a subset $A \subset X$ which is not an F_σ subset of X.

The function $\mathbb{1}_A : X \to 2$, with $\mathbb{1}_A(x) = 1$ iff $x \in A$, is not Baire class 1, as A is not an F_σ subset of X. Nonetheless, we claim that $\mathbb{1}_A \restriction K$ has a point of continuity for every compact $K \subseteq X$. Fix a compact $K \subseteq X$, then K needs to be countable, as we have assumed X not to contain any uncountable closed set. But then K, being a countable and compact subset of $\mathbb{R}$, has an isolated point, which is, in particular, a continuity point of $\mathbb{1}_A \restriction K$. □

A separable metrizable space is (*absolutely*) *analytic* precisely when it is the continuous image of a Polish space. Gerlits and Laczkovich [26] showed that Baire's grand theorem holds if the domain is assumed only to be an absolutely analytic metrizable space—actually, they stated this generalization for real functions, but their argument goes through assuming only separable metrizable codomains. From the theorems of the previous section, it follows that the game $G(f)$ is determined for every function f with analytic domain.

We cannot hope to extend *tout court* this determinacy result to functions with co-analytic domains, where a separable metrizable space is said to be *co-analytic* if it is homeomorphic to the complement of an analytic set in a Polish space. In fact, the existence of a co-analytic set of cardinality of the continuum that does not contain any uncountable closed set is consistent with ZFC—in particular, it follows from $\mathrm{V} = \mathrm{L}$, see [41, Theorem 13.12]—and the example defined in Proposition 1.3.2 would give us a function f with separable metrizable co-analytic domain witnessing the failure of GBT and the undeterminacy of $G(f)$.

We now focus on GBT, which, by Proposition 1.3.2, is inconsistent with AC. We first introduce a couple of statements coming from descriptive set theory that are strictly related to GBT. We recall that, given three sets A, B, S, we say that S *separates* A *from* B if $A \subseteq S$ and $B \cap S = \emptyset$.

(HSP) For every disjoint $A, B \subseteq {}^{\omega}\omega$ such that there is no F_σ set separating A from B, there is a Cantor set $\mathcal{C} \subseteq A \cup B$ with $\mathcal{C} \cap B$ countable and dense in $\mathcal{C}$.

where HSP stands for Hurewicz' Separation Property. The fact that the trace of B on $\mathcal{C}$ (i.e. $B \cap \mathcal{C}$) is countable and dense not only means that $B \cap \mathcal{C}$ is F_σ in $\mathcal{C}$, but also that it is F_σ-complete [47, Theorem 22.10]. HSP is known to hold under AD (see [47, §21.F] and [16, Theorem 4.2]).

Fact 1.3.3. HSP is a strong statement, in the sense that HSP + DC is equiconsistent with the existence of an inaccessible cardinal. Indeed, HSP + DC implies PSP, the perfect set property for every subset of ${}^{\omega}\omega$, and it is well-known that PSP + DC implies the consistency of an inaccessible cardinal [41, Propositions 11.4 and 11.5]; on the other hand, Todorčević and Di Prisco [21] proved that HSP holds in Solovay's model, hence the equiconsistency.

Consider now this seemingly weaker statement:

(WHSP) For every disjoint $A, B \subseteq {}^{\omega}\omega$ such that there is no F_σ set separating A from B, there is a Cantor set $\mathcal{C} \subseteq A \cup B$ with $\mathcal{C} \cap A$ dense and codense in $\mathcal{C}$.

This statement is clearly a consequence of HSP, but it does not tell us anything about the definability of the trace of A or B on $\mathcal{C}$.

Theorem 1.3.4. *The following are equivalent:*

1) GBT

2) WHSP

Proof. 1) $\implies$ 2): let $A, B \subseteq {}^{\omega}\omega$ be disjoint subsets of the Baire space such that A cannot be separated from B by an F_σ set. Equivalently, A is not F_σ with respect to the relative topology on $A \cup B$. Therefore, the function $\mathbb{1}_A : A \cup B \to 2$, with $\mathbb{1}_A(x) = 1$ iff $x \in A$, is not Baire class 1, and by GBT there exists a compact set $K \subseteq A \cup B$ such that $\mathbb{1}_A \restriction K$ has no points of continuity. This means that $A \cap K$ is both dense and codense in K, as otherwise $\mathbb{1}_A$ would have a point of continuity. Finally, notice that K, being a compact and perfect subset of ${}^{\omega}\omega$, is actually a Cantor set [47, Theorem 7.4]. Hence WHSP holds.

2) $\implies$ 1): let X, Y be separable metrizable spaces and $f : X \to Y$ a function which is not Baire class 1. Every Polish space is the image of ${}^{\omega}\omega$ by a continuous and closed map [25]. As every separable metrizable space embeds into a Polish space, there exists a closed and continuous surjection $g : X' \to X$ for some $X' \subseteq {}^{\omega}\omega$. Since the image of a closed set by a closed function is still closed by definition, images of F_σ sets by a closed function remain F_σ. Therefore, the function $h = f \circ g : X' \to Y$ is still not Baire class 1.

As h is not Baire class 1, there is an open set $V \subseteq Y$ such that $h^{-1}(V)$ is not an F_σ set of X'. Fix such V and also fix a sequence of closed sets $(F_n)_{n\in\omega}$ such that $V = \bigcup_n F_n$. It must be the case that, for some n, $h^{-1}(F_n)$ is not separable from $h^{-1}(Y \setminus V)$ by an F_σ set, as otherwise $h^{-1}(V)$ would be a countable union of F_σ sets, which is still F_σ. Fix such an n, then, by WHSP, there is a Cantor set $\mathcal{C} \subseteq X'$ with both $h^{-1}(F_n) \cap \mathcal{C}$ and $h^{-1}(Y \setminus V) \cap \mathcal{C}$ dense in $\mathcal{C}$.

By continuity of g, the set $g[\mathcal{C}]$ is compact in X and $f^{-1}(F_n) \cap g[\mathcal{C}]$, $f^{-1}(Y \setminus V) \cap g[\mathcal{C}]$ are both dense in $g[\mathcal{C}]$.

We claim that the function $f \restriction g[\mathcal{C}]$ has no points of continuity. Take an $x \in g[\mathcal{C}]$, and fix two sequences $(x_k)_{k\in\omega} \subset f^{-1}(F_n) \cap g[\mathcal{C}]$, $(x'_k)_{k\in\omega} \subset f^{-1}(Y \setminus V) \cap g[\mathcal{C}]$ converging to x. Such sequences exist because $f^{-1}(F_n) \cap g[\mathcal{C}]$ and $f^{-1}(Y \setminus V) \cap g[\mathcal{C}]$ are both dense in $g[\mathcal{C}]$. If the sequences $(f(x_k))_{k\in\omega}$ and $(f(x'_k))_{k\in\omega}$ converged in Y, they would converge in F_n and in $Y \setminus V$, respectively, as both these sets are closed. Thus, even if their limits were to exist, they could not coincide. In particular, x is not a point of continuity of $f \restriction g[\mathcal{C}]$. Since $x \in g[\mathcal{C}]$ was arbitrary, we have that no $x \in g[\mathcal{C}]$ is a continuity point of $f \restriction g[\mathcal{C}]$.

Given a function $f : X \to Y$ between separable metrizable spaces which is not Baire class 1, we have found a compact $K \subseteq X$ such that $f \restriction K$ has no points of continuity. On the other hand, if $f : X \to Y$ is Baire class 1, then the classical argument used in the proof of Baire's grand theorem shows that $f \restriction K$ has a point of continuity for every compact $K \subseteq X$ (e.g. see [47, Theorem 24.15]), with no need to invoke WHSP. Hence GBT holds. □

As HSP, and in particular WHSP, holds under AD (see [47, §21.F] and [16, Theorem 4.2]) and in Solovay's model [21], we can say the same of GBT and the full determinacy of our game, by Corollary 1.3.1 and Theorem 1.3.4. However, the precise consistency strength of these three statements (+DC) is still unknown.

WHSP, compared to HSP, seems to be weak enough to be proved consistent relative to ZF, with no large cardinals needed (see Fact 1.3.3). Hence, the following conjecture.

Conjecture 1.3.5. GBT + DC is consistent relative to ZF.

Lastly, notice that the definition of our game (Definition 1.2.1) does not rely on the separability or the metrizability of the function's domain and codomain, and it would make perfect sense to study it on broader classes of functions. Future research can shed light on how our game behaves on the class of functions with metrizable (not necessarily separable) domains and separable metrizable codomains. Would our results of Section 1.2 still hold? How much choice would be needed to prove them?

Chapter 2

Does DC imply AC_ω uniformly?

2.1 Introduction

This chapter is largely derived from [6], co-authored with Alessandro Andretta. The *axiom of choice* AC is the statement $\forall X\, \mathsf{AC}(X)$, where

$$(\mathsf{AC}(X)) \qquad X \neq \emptyset \Rightarrow \exists f \colon \mathcal{P}(X) \to X\ \forall A \subseteq X\, (A \neq \emptyset \Rightarrow f(A) \in A).$$

The function f is a *choice function* for X. Observe that $\mathsf{AC}(X)$ holds if and only if X is well-orderable. By restricting the choice function we have $\mathsf{AC}(X) \Rightarrow \mathsf{AC}_I(X)$, where

($\mathsf{AC}_I(X)$) For any sequence $(A_i)_{i\in I}$ of nonempty subsets of X there is $(a_i)_{i\in I}$ such that $\forall i \in I\, (a_i \in A_i)$.

Of particular interest is the case when $I = \omega$: the *axiom of countable choice* AC_ω is $\forall X\, \mathsf{AC}_\omega(X)$.

Let R be a binary relation on a set X.

- An *R-chain* is a sequence $(x_n)_{n\in\omega}$ of elements of X such that $x_i\ R\ x_{i+1}$ for all $i \in \omega$.
- An *R-cycle* is a finite sequence $(x_0, \dots, x_n)$ of elements of X such that $x_i\ R\ x_{i+1}$ for all $i < n$ and $x_n\ R\ x_0$.
- R is *total on X* if for every $x \in X$ there is a $y \in Y$ such that $x\ R\ y$.

The *axiom of dependent choice* DC is $\forall X\, \mathsf{DC}(X)$, where

($\mathsf{DC}(X)$) For any nonempty and total $R \subseteq X^2$ there is an R-chain.

The axioms DC and AC_ω are ubiquitous in set theory and figure prominently in many areas of mathematics, including analysis and topology. They are probably the most popular among weak forms of the axiom of choice, since they are powerful enough to enable standard mathematical constructions, yet they are weak enough to avoid the pathologies given by AC. We refer to the monograph [54] for a comprehensive treatment of the history of the axiom of choice and its weakenings.

It is well-known that $\mathsf{DC} \Rightarrow \mathsf{AC}_\omega$, so one may ask if this result holds uniformly: is the statement

$$\forall X\, (\mathsf{DC}(X) \Rightarrow \mathsf{AC}_\omega(X)) \tag{2.1}$$

a theorem of ZF? The main result of this chapter is that the answer to this question is negative.

Theorem 2.1.1. *It is consistent with* ZF *that there is a set* $A \subseteq \mathbb{R}$ *such that* $\mathsf{DC}(A)$ *and* $\neg\mathsf{AC}_\omega(A)$.

Section 2.2 discusses some preliminary facts about the relationship between $\mathsf{AC}_\omega(X)$ and $\mathsf{DC}(X)$, and also discusses a variation of the axiom $\mathsf{DC}(X)$. Section 2.3 is a brief introduction to *symmetric extensions*, the key notion involved in the proof of Theorem 2.1.1. Section 2.4 is devoted to the proof of Theorem 2.1.1 while Section 2.5 contains some complementary results.

2.2 Basic constructions

For the reader's convenience, let us recall a few notions and results that will be used throughout the chapter. Our set-theoretic notation is standard: we write $X \approx Y$ if there exists a bijection between X and Y, and we write $X \precsim Y$ if there exists an injective function from X to Y.

A set X is *finite* if $X \approx n$ for some $n \in \omega$; otherwise it is *infinite*. A set X is *Dedekind-infinite* if $\omega \precsim X$; otherwise it is *Dedekind-finite*. Every finite set is Dedekind-finite, and assuming AC_ω the converse holds. It is consistent with ZF that infinite Dedekind-finite sets exist (see Section 2.3.1). By [45], it is even consistent that every set is the surjective image of a Dedekind-finite set.

Let R be a binary relation. With abuse of notation, we write

$$R(x) := \{y \mid x \, R \, y\}$$

for the set of all ys that are related to x, and

$$R \restriction A := R \cap (A \times A)$$

for the restriction of R to the set A. The *transitive closure* of R

$$R^+ := \{(x, y) \mid \exists \langle y_0, \dots, y_n \rangle \, (x \, R \, y_0 \, R \, y_1 \, R \cdots R \, y_n \, R \, y)\}$$

is the smallest transitive relation containing R. The next few results are folklore.

Proposition 2.2.1. *Let X be a set.*

(a) *If Y is the surjective image of X, then $\mathsf{DC}(X) \Rightarrow \mathsf{DC}(Y)$.*

(b) *$\mathsf{DC}(X)$ is equivalent to the seemingly stronger statement: For any nonempty and total $R \subseteq X \times X$ and for any $a \in X$, there is an R-chain $(x_n)_{n\in\omega}$ with $x_0 = a$.*

(c) *Given $(A_n)_{n\in\omega}$ with $\emptyset \neq A_n \subseteq X$ and $A_n \cap A_m = \emptyset$ for all distinct n, m, then $\mathsf{DC}(X)$ implies that $(A_n)_{n\in\omega}$ has a choice function.*

(d) *$\mathsf{DC}(X \times \omega) \Rightarrow \mathsf{AC}_\omega(X)$.*

Proof. (a) Assume $\mathsf{DC}(X)$ and let R be a total relation on Y and let $F \colon X \to Y$ be a surjection. The relation $S = \{(x, x') \in X^2 \mid (F(x), F(x')) \in R\}$ is total on X, so by assumption there is an S-chain $(x_n)_{n\in\omega}$. Then $(F(x_n))_{n\in\omega}$ is an R-chain.

(b) Suppose $R \subseteq X^2$ is nonempty and total and let $a \in X$. Observe that $S = R \restriction R^+(a)$ is total on $R^+(a)$. By part (a), $\mathsf{DC}(R^+(a))$ holds, hence there is an S-chain $(y_n)_{n\in\omega}$. Let $(x_0,\dots,x_{k+1})$ witness that $y_0 \in R^+(a)$, i.e. $x_0 = a$, $x_{k+1} = y_0$ and $x_i \mathrel{R} x_{i+1}$ for all $i \le k$: then $(x_0,\dots,x_k)^\frown(y_n)_{n\in\omega}$ is an R-chain starting from a.

(c) Let R be the relation on $\bigcup_n A_n \subseteq X$ defined by

$$x \mathrel{R} y \Leftrightarrow \exists n \in \omega\ (x \in A_n \wedge y \in A_{n+1})$$

By part (a), $\mathsf{DC}(\bigcup_n A_n)$ holds, hence by part (b) there is an R-chain $(a_n)_{n\in\omega}$ in $\bigcup_n A_n$ with $a_0 \in A_0$. Observe that any R-chain $(a_n)_{n\in\omega}$ with $a_0 \in A_0$ is such that $a_n \in A_n$ for all $n \in \omega$.

(d) Given $\emptyset \neq A_n \subseteq X$, let $\bar{A}_n = A_n \times \{n\} \subseteq X \times \omega$. By hypothesis and part (c), there is a sequence $(a_n, n)_{n\in\omega}$ such that $(a_n, n) \in \bar{A}_n$, hence $a_n \in A_n$. $\square$

The gist of part (c) of Proposition 2.2.1 is that we can use dependent choice rather than countable choice whenever the sets we choose from are disjoint. Here is an example of such an application.

Lemma 2.2.2. *Suppose that X is a first countable* T_1 *space and that $A \subseteq X$ is not closed in X. If* $\mathsf{DC}(A)$ *holds, then A is Dedekind-infinite.*

Proof. Since A is not closed, we can fix some $a \in \mathrm{cl}(A)\setminus A$. Let $\{U_n \mid n \in \omega\}$ be a neighborhood basis for a, with $U_{n+1} \subseteq U_n$ for every n. Given that X is T_1, we can assume, by passing to a subsequence if needed, that $A_n = (U_n \setminus U_{n+1}) \cap A$ is nonempty for every n. Since the A_ns are pairwise disjoint and nonempty, by Proposition 2.2.1(c) there is a sequence of $(a_n)_{n\in\omega}$ of distinct elements of A such that $a_n \in A_n$ for every n. In particular, A is Dedekind-infinite. $\square$

Lemma 2.2.3. *Let X be a set.*

(a) $X \times 2 \precsim X \Rightarrow X \times \omega \precsim X$.

(b) *If $X \neq \emptyset$, then* ${}^{<\omega}({}^{<\omega}X) \precsim {}^{<\omega}X$, *so* ${}^{<\omega}X \times 2 \precsim {}^{<\omega}X$.

(c) $\forall X\, \exists Y\, (X \subseteq Y \wedge {}^{<\omega}Y \precsim Y)$.

Proof. (a) If $f_0, f_1 \colon X \to X$ are injections with $\mathrm{ran}(f_0) \cap \mathrm{ran}(f_1) = \emptyset$, then define an injection $F \colon X \times \omega \to X$ as follows:

$$F(x,0) = f_0(x), \qquad F(x,n+1) = \underbrace{f_1 \circ \cdots \circ f_1}_{n+1 \text{ times}} \circ f_0(x).$$

(b) If X is a singleton, then ${}^{<\omega}X \approx \omega$, and the result follows at once. If X has at least two elements, the result follows from [5, Proposition 2.1].

(c) Given X take $Y = \mathrm{V}_\lambda$ with sufficiently large limit λ. □

From Lemma 2.2.3 and Proposition 2.2.1(d) we obtain at once:

Proposition 2.2.4. (a) *If $X \times 2 \precsim X$, then $\mathsf{DC}(X) \Rightarrow \mathsf{AC}_\omega(X)$. In particular: $\mathsf{DC}(\mathbb{R}) \Rightarrow \mathsf{AC}_\omega(\mathbb{R})$.*

(b) $\forall X\, \exists Y\, (X \subseteq Y \wedge (\mathsf{DC}(Y) \Rightarrow \mathsf{AC}_\omega(Y))$.

(c) $\mathsf{DC} \Rightarrow \mathsf{AC}_\omega$.

Lemma 2.2.5. (a) *Let $A \subseteq \mathbb{R}$. Then $\mathsf{AC}_\omega(A)$ implies that A is separable.*

(b) $\mathsf{AC}_\omega(\mathbb{R}) \Leftrightarrow \forall A \subseteq \mathbb{R}\, (A$ *is separable*$)$.

(c) *Suppose $A \subseteq \mathbb{R}$ contains a nonempty perfect set, and assume $\mathsf{DC}(A)$. Then $\mathsf{DC}(\mathbb{R})$ holds, and hence $\mathsf{AC}_\omega(A)$ holds.*

Proof. As A is second countable, part (a) of Lemma 2.2.5 follows.

(b) The direction $(\Rightarrow)$ is a direct consequence of part (a). For the other direction, fix a sequence $(A_n)_{n\in\omega}$ of nonempty subsets of ${}^\omega\omega$ and consider the set $A = \{\langle n\rangle^\frown x \mid n \in \omega \text{ and } x \in A_n\}$. From an enumeration of a dense subset of A (which exists by assumption), we can extract a choice function for $(A_n)_{n\in\omega}$.

(c) If $P \subseteq A$ is perfect, then $P \approx \mathbb{R}$, and since A surjects onto P, then $\mathsf{DC}(\mathbb{R})$ holds, and hence $\mathsf{AC}_\omega(\mathbb{R})$ holds. □

Note that the implication in part (a) of Lemma 2.2.5 cannot be reversed: if $A \subseteq \mathbb{R}$ is a witness of the failure of countable choice, then the same is true of the separable set $A \cup \mathbb{Q}$.

2.2.1 Real reflection

In this section we prove that if (2.1) fails, then we can find a witness of the failure already among the sets of reals. This is precisely the content of the following proposition.

Proposition 2.2.6. *Suppose that for all $X \subseteq \mathbb{R}$, $\mathsf{DC}(X) \Rightarrow \mathsf{AC}_\omega(X)$. Then for all X, $\mathsf{DC}(X) \Rightarrow \mathsf{AC}_\omega(X)$.*

Proof. Assume that for all $X \subseteq \mathbb{R}$, $\mathsf{DC}(X) \Rightarrow \mathsf{AC}_\omega(X)$. Fix a nonempty set Y such that $\mathsf{DC}(Y)$ and fix also a sequence $(Y_n)_{n\in\omega}$ of nonempty subsets of Y towards showing that the sequence $(Y_n)_{n\in\omega}$ has a choice function.

Let $F : \bigcup_{n\in\omega} Y_n \to \mathcal{P}(\omega)$, $F(y) = \{n \in \omega \mid y \in Y_n\}$ and let $A_n = \{a \in \operatorname{ran}(F) \mid n \in a\}$. Observe that for all $y \in \bigcup_{n\in\omega} Y_n$ and all $n \in \omega$

$$y \in Y_n \Leftrightarrow F(y) \in A_n.$$

In particular, $\emptyset \neq A_n \subseteq \mathcal{P}(\omega)$ for all $n \in \omega$. By Proposition 2.2.1(a), $\mathsf{DC}(\operatorname{ran}(F))$ holds, and therefore also $\mathsf{AC}_\omega(\operatorname{ran}(F))$ holds, by assumption. Pick a choice sequence $(a_n)_{n\in\omega}$ for $(A_n)_{n\in\omega}$, i.e. $a_n \in \operatorname{ran}(F)$ and $n \in a_n$ for all $n \in \omega$. Let $Z_n = F^{-1}(\{a_n\})$. Then

$$Z_n = \{y \mid F(y) = a_n\} = \{y \mid \{m \mid y \in Y_m\} = a_n\},$$

and since $n \in a_n$, then $Z_n \subseteq Y_n$. The sets Z_n need not be distinct as the a_ns need not be distinct, but if $a_n \neq a_m$, then $Z_n \cap Z_m = \emptyset$.

Let $I \subseteq \omega$ be such that $\{Z_n \mid n \in I\} = \{Z_n \mid n \in \omega\}$ and $Z_n \cap Z_m = \emptyset$ for every distinct $n, m \in I$. If we can find $y_n \in Z_n$ for all $n \in I$, then we can extend this to a choice sequence $y_n \in Z_n \subseteq Y_n$ for all $n \in \omega$ as required. If I is finite, then the y_is can be found without any appeal to choice. If I is infinite, then $I \approx \omega$ so we can find the y_ns by Proposition 2.2.1(c). □

Since the proof of Proposition 2.2.6 does not use the axiom of foundation, an important consequence of Proposition 2.2.6 is that we cannot prove Theorem 2.1.1 using solely Fraenkel-Mostowski permutation models [38]. This is because the kernel of any such model would reflect (2.1). Thus, employing a symmetric extension (see Section 2.3) is, arguably, necessary to prove Theorem 2.1.1. Furthermore, as a direct corollary of Proposition 2.2.6 we obtain the following:

Corollary 2.2.7. *Assume* $\mathsf{AC}_\omega(\mathbb{R})$*, then* $\forall X\,(\mathsf{DC}(X) \Rightarrow \mathsf{AC}_\omega(X))$.

2.2.2 Some considerations on (2.1)

By Proposition 2.2.4 and Corollary 2.2.7, the statement (2.1) follows from either one of the following assumptions:

- $X \times 2 \precsim X$ for all infinite X,
- $\mathsf{AC}_\omega(\mathbb{R})$.

Sageev in [61] proved that "$X \times 2 \precsim X$ for all infinite X" does not imply $\mathsf{AC}_\omega(\mathbb{R})$, while Monro in [53] proved that DC (and hence the weaker $\mathsf{AC}_\omega(\mathbb{R})$) does not imply "$X \times 2 \precsim X$ for all infinite X". So neither assumption implies the other.

Suppose now that (2.1) does not hold. By Proposition 2.2.6, there exists a set $A \subseteq \mathbb{R}$ such that $\mathsf{DC}(A)$ holds but $\mathsf{AC}_\omega(A)$ fails. What can we say about A? By Lemma 2.2.5(c), any such A cannot contain a nonempty perfect set. Moreover, such a set A also needs to be Dedekind-infinite: indeed, A cannot be closed, as otherwise, by the usual Cantor-Bendixson argument, it would either be countable, contradicting $\neg\mathsf{AC}_\omega(A)$, or else it would contain a nonempty perfect set, which we already excluded; therefore A is not closed, and, by Lemma 2.2.2, A is Dedekind-infinite.

It can be shown that (2.1) holds both in Cohen's first model (Proposition 2.3.3) and in the Feferman-Levy model (Proposition 2.5.3). Thus, these two classical models of $\mathsf{ZF} + \neg\mathsf{AC}$ do not directly yield Theorem 2.1.1, even though Cohen's first model has a crucial role in its proof.

2.2.3 An equivalent formulation of DC

For every (nonempty) set X, let

($\mathsf{DC}_\omega(X)$) Any nonempty pruned tree on X is ill-founded

and let DC_ω be $\forall X\, \mathsf{DC}_\omega(X)$. As DC is equivalent to DC_ω (Corollary 2.2.9 below), the axiom of dependent choice is often stated as DC_ω. The advantage of this formulation is that it can be generalized to ordinals larger than ω.

Proposition 2.2.8. $\mathsf{DC}_\omega(X) \Leftrightarrow \mathsf{DC}({}^{<\omega}X)$ *for every nonempty set* X*.*

Proof. ($\Longrightarrow$) Suppose R is a binary relation on ${}^{<\omega}X$ such that $\forall s\, \exists t\, (s\ R\ t)$. If $\emptyset\ R\ \emptyset$, then $\langle \emptyset, \emptyset, \dots \rangle$ is an R-chain as required, so we may assume otherwise. Let $R' \subseteq R$ be the sub-relation on ${}^{<\omega}X$ defined by

$$s\ R'\ t \Leftrightarrow s\ R\ t \wedge \forall t' \left(s\ R\ t' \Rightarrow \operatorname{length}(t') \geq \operatorname{length}(t)\right).$$

The relation R' is total and any R'-chain is an R-chain. Then

$$T = \left\{t \in {}^{<\omega}X \mid \exists s_0, \dots, s_n\, (\emptyset\ R'\ s_0\ R'\ \dots\ R'\ s_n\ \wedge\ t \subseteq s_0{}^\frown s_1{}^\frown \dots {}^\frown s_n)\right\}$$

is a pruned tree on X, so it has a branch by hypothesis. By the minimality assumption of R', given a branch b of T one can construct inductively an R'-chain $(s_n)_{n\in\omega}$ such that $s_0{}^\frown s_1{}^\frown \dots {}^\frown s_n \subseteq b$ for all n.

($\Longleftarrow$) If T is a pruned tree on X, let $R \subseteq T \times T$ be defined by

$$s\ R\ t \Leftrightarrow s \subset t \wedge \operatorname{length}(s) + 1 = \operatorname{length}(t).$$

As $T \subseteq {}^{<\omega}X$ then $\mathsf{DC}(T)$ holds, and since R is total on T, as T is pruned, there is an R-chain. Any such chain yields a branch of T. □

Corollary 2.2.9. $\mathsf{DC} \Leftrightarrow \mathsf{DC}_\omega$.

Proposition 2.2.10. *Let* X *be a set.*

(a) $\mathsf{DC}_\omega(X) \Rightarrow \mathsf{DC}(X)$.

(b) $\mathsf{DC}_\omega(X) \Rightarrow \mathsf{AC}_\omega(X)$.

Proof. X injects into ${}^{<\omega}X$, so part (a) holds by Proposition 2.2.8.

For part (b) argue as follows. If $(A_n)_{n\in\omega}$ is a sequence of nonempty subsets of X, then $\{\langle x_0,\dots,x_{n-1}\rangle \mid \forall i<n\,(x_i\in A_i)\}$ is a pruned tree on X, and any branch of it is a sequence $(a_n)_{n\in\omega}$ such that $a_n\in A_n$ for all $n\in\omega$. □

In light of Proposition 2.2.10, our main result, Theorem 2.1.1, tells us it is consistent with ZF that there is a set $A\subseteq\mathbb{R}$ for which $\mathsf{DC}(A)$ holds but $\mathsf{DC}_\omega(A)$ fails.

2.3 Symmetric extensions

The model we construct in Section 2.4 to prove Theorem 2.1.1 is an iterated symmetric extension. For the reader's convenience, let us recall a few facts about symmetric extensions.

Let $\mathbb{P}$ be forcing notion and let $\mathcal{G}$ be a subgroup of $\mathrm{Aut}(\mathbb{P})$. A nonempty collection $\mathcal{F}$ of subgroups of $\mathcal{G}$ is a *filter* on $\mathcal{G}$ if it is closed under supergroups and finite intersections. A filter $\mathcal{F}$ on $\mathcal{G}$ is said to be *normal* if for every $H\in\mathcal{F}$ and $\pi\in\mathcal{G}$, the conjugated subgroup $\pi H\pi^{-1}$ belongs to $\mathcal{F}$ as well.

We say that the triple $\langle\mathbb{P},\mathcal{G},\mathcal{F}\rangle$ is a *symmetric system* if $\mathbb{P}$ is a forcing notion, $\mathcal{G}$ is a subgroup of $\mathrm{Aut}(\mathbb{P})$ and $\mathcal{F}$ is a normal filter on $\mathcal{G}$. Given a $\mathbb{P}$-name $\dot{x}$, we say that $\dot{x}$ is *$\mathcal{F}$-symmetric* if there exists $H\in\mathcal{F}$ such that for all $\pi\in H$, $\pi\dot{x}=\dot{x}$. This definition extends by recursion: $\dot{x}$ is *hereditarily $\mathcal{F}$-symmetric*, if $\dot{x}$ is $\mathcal{F}$-symmetric and every name $\dot{y}\in\mathrm{dom}(\dot{x})$ is hereditarily $\mathcal{F}$-symmetric. We denote by $\mathsf{HS}_\mathcal{F}$ the class of all hereditarily $\mathcal{F}$-symmetric names. The fundamental theorem of symmetric extensions is the following.

Theorem 2.3.1 ([40, Lemma 15.51])**.** *Suppose that $\langle\mathbb{P},\mathcal{G},\mathcal{F}\rangle$ is a symmetric system and G is a $\mathbb{P}$-generic filter over* V*. Denote by $\mathcal{N}$ the class $\{\dot{x}_G \mid \dot{x}\in\mathsf{HS}_\mathcal{F}\}$, then $\mathcal{N}$ is a transitive model of* ZF*, and* $\mathrm{V}\subseteq\mathcal{N}\subseteq\mathrm{V}[G]$.

The class $\mathcal{N}$ is also known as a *symmetric extension* of V. Symmetric extensions are often used to produce models of ZF in which the axiom of choice fails. We focus on this notion by discussing the construction due to Cohen of a symmetric extension in which there is an infinite, Dedekind-finite set of reals. This model will be the first step in the forcing iteration in the proof of Theorem 2.1.1.

2.3.1 The first Cohen model

Let $\mathbb{P}_0$ be the forcing that adds countably many Cohen reals, i.e.

$$\mathbb{P}_0 = \Big\{p \mid \exists I \subseteq \omega\, (p\colon I \to {}^{<\omega}2, \text{ and } I \text{ is finite})\Big\},$$

with $p \leq q$ if $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$ and $p(n) \supseteq q(n)$ for all $n \in \mathrm{dom}(q)$. Although this is not the standard presentation of such a forcing, this way of defining $\mathbb{P}_0$ will become useful in Section 2.4. Let $\dot{a}_n$ be the canonical name for the n-th Cohen real, that is

$$\dot{a}_n := \{((\check{k, i}), p) \mid p \in \mathbb{P}_0 \text{ and } n \in \mathrm{dom}\, p \text{ and } p(n)(k) = i\}. \tag{2.2}$$

Observe that $\dot{A} := \{\dot{a}_n \mid n \in \omega\}^\bullet$ is forced to be a dense subset of ${}^\omega 2$.

Every permutation π on ω induces an automorphism of $\mathbb{P}_0$ as follows: given $p \in \mathbb{P}_0$, we let $\pi p \in \mathbb{P}_0$ be defined by

$$\forall n \in \mathrm{dom}(p)\, (\pi p(\pi n) = p(n)).$$

We conflate the notation by using the same symbol π to denote both the permutation and the automorphism on $\mathbb{P}_0$ it induces. Let $\mathcal{G}_0$ be the group of all such automorphisms. For every finite $E \subset \omega$, let $\mathrm{Fix}(E)$ be the subgroup of $\mathcal{G}_0$ of all those automorphisms induced by permutations that pointwise fix the set E. Let $\mathcal{F}_0$ be the filter on $\mathcal{G}_0$ generated by $\{\mathrm{Fix}(E) \mid E \subset \omega \text{ finite}\}$. It is easy to check that $\mathcal{F}_0$ is actually a normal filter on $\mathcal{G}_0$, and hence $\langle \mathbb{P}_0, \mathcal{G}_0, \mathcal{F}_0 \rangle$ is a symmetric system. Let G be a $\mathbb{P}_0$-generic filter over V, and let $\mathcal{N}_0$ be the corresponding symmetric extension, which is known as the *first Cohen model*.

Denote by A the realization of the name $\dot{A}$ in $\mathrm{V}[G]$, i.e. the set $\dot{A}_G$. Note that every $\dot{a}_n$ is in $\mathsf{HS}_{\mathcal{F}_0}$ and so is $\dot{A}$.

Proposition 2.3.2 ([40, Example 15.52]). $\mathcal{N}_0 \vDash$ "*A is Dedekind-finite*".

In $\mathcal{N}_0$, the set A, being infinite and Dedekind-finite, is certainly not separable as a subspace of $\mathbb{R}$—indeed, every infinite, separable T_1 space is Dedekind-infinite. Moreover, $\mathsf{DC}(A)$ also fails, as otherwise A would be Dedekind-infinite (see the penultimate paragraph of Section 2.2.2).

The simultaneous local failure of both AC_ω and DC is not accidental—the next proposition shows that, in the first Cohen model, a statement stronger than (2.1) holds.

Proposition 2.3.3. $\mathcal{N}_0 \vDash \forall X\,(\mathsf{DC}(X) \Rightarrow \mathsf{AC}(X))$.

Lemma 2.3.4. *Let X be a linearly ordered set, and let $Y \subseteq [X]^{<\omega}$. If Y is Dedekind-infinite, then so is $\bigcup Y$.*

Proof. Let $\leq$ be a linear ordering of X, and let $(Z_n)_{n\in\omega}$ be a sequence of distinct elements of Y. By passing to a subsequence, we may assume that $Z_{n+1} \nsubseteq Z_0 \cup \cdots \cup Z_n$, and that $Z_0 \neq \emptyset$. Let z_0 be the $\leq$-least element of Z_0, and z_{n+1} be the $\leq$-least element of $Z_{n+1} \setminus (Z_0 \cup \cdots \cup Z_n)$ for every n. The z_ns are distinct, and belong to $\bigcup Y$. In particular, $\bigcup Y$ is Dedekind-infinite. □

Lemma 2.3.5. *If $\mathsf{DC}(Y)$ with $Y \subseteq [\mathbb{R}]^{<\omega}$ infinite, then $\bigcup Y$ is Dedekind-infinite.*

Proof. If $\bigcup Y$ has no limit points, then it is discrete, so $\omega \precsim \bigcup Y$. Suppose otherwise, and let $x \in \mathbb{R}$ be a limit point of $\bigcup Y$. It is enough to show that $\omega \precsim Y$ and then apply Lemma 2.3.4 with $X = \mathbb{R}$. Without loss of generality, we may assume that $\{x\}, \emptyset \notin Y$. For all $Z \in Y$ let $d(x, Z) = \min\{|r - x| \mid r \in Z \setminus \{x\}\}$ be the distance of x from the rest of Z. Let $R \subseteq Y^2$ be the binary relation defined as follows: for every $Z, W \in Y$,

$$R(Z, W) \Leftrightarrow d(x, W) < d(x, Z).$$

The relation R is acyclic, and, by our hypothesis on x, it is total. It follows from $\mathsf{DC}(Y)$ that R has an infinite chain, and hence, since R is acyclic, $\omega \precsim Y$. □

Proof of Proposition 2.3.3. In the first Cohen model, for every set X, there is a map $s_X \colon X \to [A]^{<\omega}$, known as the least support map, such that $s^{-1}(\{B\})$ is well-orderable for every $B \in [A]^{<\omega}$ [38, Theorem 5.21, Lemma 5.25].

Let X be such that $\mathsf{DC}(X)$ holds. Then also $\mathsf{DC}(\operatorname{ran}(s_X))$ holds. If $\operatorname{ran}(s_X)$ were infinite, then letting $Y = \operatorname{ran}(s_X)$ in Lemma 2.3.5 we would have $\omega \precsim$

$\bigcup \operatorname{ran}(s_X) \subseteq A$, against the fact that A is Dedekind-finite. Hence $\operatorname{ran}(s_X)$ is finite, and X, being a finite union of well-orderable sets, is well-orderable. $\square$

2.4 The main result

This section is devoted to the proof of Theorem 2.1.1.

2.4.1 Outline of the proof

We prove the theorem via an iteration of symmetric extensions of length ω. We start the iteration with the first Cohen model $\mathcal{N}_0$, with $A \in \mathcal{N}_0$ being the generic Dedekind-finite set of reals (see Section 2.3.1). As noted right after Proposition 2.3.2, in $\mathcal{N}_0$ the set A is not separable (in particular $\mathsf{AC}_\omega(A)$ fails) and $\mathsf{DC}(A)$ fails. Next, we define a chain of models $\mathcal{N}_0 \subset \mathcal{N}_1 \subset \cdots \subset \mathcal{N}_\omega$ such that, for each n, $\mathcal{N}_{n+1}$ is a symmetric extension of $\mathcal{N}_n$ that contains a generic set of chains for all binary relations in $\mathcal{N}_n$ that are total and acyclic on A. At the final stage, $\mathcal{N}_\omega$, which is our model, is going to be something resembling "the model of sets definable from finitely many elements of $\bigcup_n \mathcal{N}_n$". If we do the construction properly, we can prove that in $\mathcal{N}_\omega$ we have added enough countable subsets of A (or, equivalently, enough sequences over A) to guarantee $\mathsf{DC}(A)$ (Theorem 2.4.9), but A is still not separable, in particular $\mathsf{AC}_\omega(A)$ fails (Corollary 2.4.7).

Actually, we don't only show that A is not separable in our model, but we give a topological characterization of its separable subsets: among the subsets of A, the separable ones are precisely those which are scattered with finite scattered height (Definition 2.4.4, Theorem 2.4.6).

2.4.2 The symmetric system

We define recursively a sequence $\langle \mathbb{P}_n, \mathcal{G}_n, \mathcal{F}_n \rangle_{n\in\omega}$ of symmetric systems. Let $\langle \mathbb{P}_0, \mathcal{G}_0, \mathcal{F}_0 \rangle$ be the symmetric system defined in Section 2.3.1, i.e. the one that induces the first Cohen model. For each n, we denote by $\leq_n, \Vdash_n$ the ordering and the forcing relation of $\mathbb{P}_n$, respectively, and by HS_n the class $\mathsf{HS}_{\mathcal{F}_n}$, i.e. the class of all hereditarily $\mathcal{F}_n$-symmetric $\mathbb{P}_n$-names. We also let

$$\mathcal{R}_n := \Big\{\dot{R} \in \mathsf{HS}_n \mid \forall \dot{x} \in \mathrm{dom}(\dot{R})\, \exists m, k \in \omega \ (\dot{x} = (\dot{a}_m, \dot{a}_k)^\bullet)\Big\}, \tag{2.3}$$

where the $\dot{a}_i$s are as in (2.2). Then, $\mathcal{R}_n$ is the set of all "good" hereditarily $\mathcal{F}_n$-symmetric $\mathbb{P}_n$-names for binary relations on $\dot{A}$.

Recursively on n, we define $\mathbb{P}_{n+1}$ to be the set of all the sequences $p = \langle p_k \mid k \leq n+1\rangle$ such that

1) $p \restriction n+1 \in \mathbb{P}_n$,
2) $p_{n+1} \colon \mathrm{dom}(p_{n+1}) \to \mathcal{R}_n \times {}^{<\omega}\omega$ with $\mathrm{dom}(p_{n+1})$ a finite subset of ω,
3) For each $k \in \mathrm{dom}(p_{n+1})$ with $p_{n+1}(k) = (\dot{R}, \langle m_0, \dots, m_h\rangle)$ we have

$$p \restriction n+1 \Vdash_n \text{“}\dot{R} \text{ is total, acyclic and } \dot{a}_{m_0}\ \dot{R}\ \dot{a}_{m_1}\ \dot{R} \dots \dot{R}\ \dot{a}_{m_h}\text{”},$$

where, at stage $n = 0$, we identify the conditions $p \in \mathbb{P}_0$ with their singleton sequence $\langle p\rangle$.

For each $p \in \mathbb{P}_{n+1}$ and $k \in \mathrm{dom}(p_{n+1})$ with $p_{n+1}(k) = (\dot{R}, s)$, we denote $\dot{R}$ and s by $p^R_{n+1}(k)$ and $p^s_{n+1}(k)$, respectively. Given $p, q \in \mathbb{P}_{n+1}$ we let $p \leq_{n+1} q$ if and only if

- $p \restriction n+1 \leq_n q \restriction n+1$,
- $\mathrm{dom}(p_{n+1}) \supseteq \mathrm{dom}(q_{n+1})$,
- $\forall k \in \mathrm{dom}(q_{n+1}) \left(p^R_{n+1}(k) = q^R_{n+1}(k) \text{ and } p^s_{n+1}(k) \supseteq q^s_{n+1}(k)\right)$.

This defines the forcing $\mathbb{P}_{n+1}$. Now we are left to define the subgroup $\mathcal{G}_{n+1}$ of $\mathrm{Aut}(\mathbb{P}_{n+1})$, and the filter $\mathcal{F}_{n+1}$.

Consider a sequence $\vec{\pi} = \langle \pi_0, \dots, \pi_{n+1}\rangle$ with each π_i being a permutation of ω. By induction hypothesis[1], $\vec{\pi} \restriction n+1$ induces an automorphism $\vec{\pi} \restriction n+1 \in \mathcal{G}_n$. Note that, as in Section 2.3.1, we conflate the notation by using the same symbol to denote both sequences of permutations and the automorphisms they

[1]At $n = 0$ we identify each $\pi \in \mathcal{G}_0$ with the singleton sequence $\langle \pi\rangle$.

induce. Now, the sequence $\vec{\pi}$ induces an automorphism on $\mathbb{P}_{n+1}$ as follows: given $p \in \mathbb{P}_{n+1}$, we let $\vec{\pi}p$ be the condition in $\mathbb{P}_{n+1}$ such that

$$(\vec{\pi}p) \restriction n+1 = (\vec{\pi} \restriction n+1)\,(p \restriction n+1)$$

and, for each $k \in \mathrm{dom}(p_{n+1})$ with $p^s_{n+1}(k) = \langle m_0, \ldots, m_h \rangle$ and $p^R_{n+1}(k) = \dot{R}$,

$$(\vec{\pi}p)^R_{n+1}(\pi_{n+1}(k)) = (\vec{\pi} \restriction n+1)\,(\dot{R}),$$
$$(\vec{\pi}p)^s_{n+1}(\pi_{n+1}(k)) = \langle \pi_0(m_0), \ldots, \pi_0(m_h) \rangle.$$

Let $\mathcal{G}_{n+1}$ be the group of all such automorphisms on $\mathbb{P}_{n+1}$, i.e. the ones induced by sequences (of length $n+2$) of permutations of ω. For each sequence $\vec{H} = \langle H_0, \ldots, H_{n+1} \rangle$ of subsets of ω, we let $\mathrm{Fix}(\vec{H})$ be the subgroup of all those $\vec{\pi} \in \mathcal{G}_{n+1}$ such that π_k pointwise fixes H_k for all $k \leq n+1$. We define $\mathcal{F}_{n+1}$ as the filter on $\mathcal{G}_{n+1}$ generated by $\{\mathrm{Fix}(\vec{H}) \mid H_k \text{ is finite for all } k \leq n+1\}$. From now on, we use the symbol $\vec{H}$ to denote *finite sequences of finite subsets of* ω.

This ends the inductive definition of the sequence $\langle \mathbb{P}_n, \mathcal{G}_n, \mathcal{F}_n \rangle_{n \in \omega}$. Note that, for each $n < m$, there is a natural complete embedding $i_{n,m} \colon \mathbb{P}_n \to \mathbb{P}_m$ and a natural embedding $j_{n,m} \colon \mathcal{G}_n \to \mathcal{G}_m$. Thus we let $\mathbb{P}$ and $\mathcal{G}$ be the direct limits of the forcings $\mathbb{P}_n$ and of the groups $\mathcal{G}_n$, respectively.

We now define the normal filter $\mathcal{F}$ on $\mathcal{G}$ in the expected way: we let $\mathcal{F}$ be the filter generated by

$$\Big\{ \mathrm{Fix}(\vec{H}) \mid H_k \text{ is finite for all } k < \mathrm{length}(\vec{H}) \Big\},$$

where, given any $\vec{H}$, $\mathrm{Fix}(\vec{H})$ is the subgroup of $\mathcal{G}$ made of all those $\vec{\pi}$ such that π_k pointwise fixes H_k for all $k < \mathrm{length}(\vec{H})$.

A condition p of the direct limit $\mathbb{P}$ is a finite sequence $\langle p_0, \ldots, p_n \rangle$, and it is identified with $\langle p_0, \ldots, p_n, \emptyset, \ldots, \emptyset \rangle$ and with $\langle p_0, \ldots, p_n, \emptyset, \emptyset \ldots \rangle$, that is a sequence obtained by concatenating p with a finite sequence of empty sets or with the infinite sequence of empty sets. We treat analogously the $\vec{H}$s.

Henceforth $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is our symmetric system, with HS being the class of all $\mathcal{F}$-symmetric $\mathbb{P}$-names and $\leq, \Vdash$ being the ordering and the forcing relation of $\mathbb{P}$, respectively.

Remark 2.4.1. Our iterative construction fits into the general framework developed by Asaf Karagila [44] to deal with iterations of symmetric extensions.

Given an $\dot{x} \in \mathsf{HS}$, we say that $\vec{H}$ is a *support* of $\dot{x}$ if $\vec{\pi}\dot{x} = \dot{x}$ for all $\vec{\pi} \in \mathrm{Fix}(\vec{H})$. Also, given $p = \langle p_0, \dots, p_n \rangle \in \mathbb{P}$ and $\vec{H} = \langle H_0, \dots, H_n \rangle$, we write $p \restriction \vec{H}$ to denote the sequence $\langle p_0 \restriction H_0, \dots, p_n \restriction H_n \rangle$. Note that the latter sequence does not necessarily belong to $\mathbb{P}$.

Lemma 2.4.2 (Restriction Lemma)**.** *Let $\varphi(x_1, \dots, x_n)$ be a formula, and let $\dot{x}_1, \dots, \dot{x}_n \in \mathsf{HS}$. For any $p \in \mathbb{P}$ and for any $\vec{H}$, if $\vec{H}$ is a support for each $\dot{x}_i$ and, for all $m > 0$, for all $k \in H_m \cap \mathrm{dom}(p_m)$, $\vec{H} \restriction m$ is a support for $p^R_m(k)$ and* $\mathrm{ran}(p^s_m(k)) \subseteq H_0$*, then $p \restriction \vec{H} \in \mathbb{P}$ and*

$$p \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n) \iff p \restriction \vec{H} \Vdash \varphi(\dot{x}_1, \dots, \dot{x}_n).$$

Proof. We prove the lemma by induction on the length of $\vec{H}$. Let's first assume $\vec{H} = \langle H_0 \rangle$ for some finite $H_0 \subset \omega$, then $p \restriction \vec{H} \in \mathbb{P}_0$. Assume towards a contradiction that $p \restriction \vec{H} \nVdash \varphi(\dot{x}_1, \dots, \dot{x}_n)$, then there is a $q \leq p \restriction \vec{H}$ such that $q \Vdash \neg\varphi(\dot{x}_1, \dots, \dot{x}_n)$. Let $\vec{\pi} \in \mathcal{G}$ such that π_0 pointwise fixes H_0 and such that $\pi_0[\mathrm{dom}(q_0)] \cap \mathrm{dom}(p_0) = H_0 \cap \mathrm{dom}(p_0)$ and $\pi_m[\mathrm{dom}(q_m)] \cap \mathrm{dom}(p_m) = \emptyset$ for all $m > 0$. In particular, $\vec{\pi} \in \mathrm{Fix}(\vec{H})$. By hypothesis, $\vec{H}$ is a support for all the $\dot{x}_i$'s. Thus, by the Symmetry Lemma, $\vec{\pi}q \Vdash \neg\varphi(\dot{x}_1, \dots, \dot{x}_n)$. However, p and $\vec{\pi}q$ are compatible, contradiction.

Now let's assume that $\vec{H} = \langle H_0, \dots, H_m \rangle$.

Claim 2.4.2.1. *$p \restriction \vec{H} \in \mathbb{P}_m$.*

Proof. By induction hypothesis, $p \restriction (\vec{H} \restriction m) \in \mathbb{P}_{m-1}$. Fix a $k \in H_m \cap \mathrm{dom}(p_m)$. Let $\dot{R} = p^R_m(k)$ and $\langle n_0, \dots, n_h \rangle = p^s_m(k)$. Then, by assumption, $\vec{H} \restriction m$ is a support of $\dot{R}$ and $n_i \in H_0$ for every $i \leq h$, or, equivalently, H_0 is a support for $\dot{a}_{n_i}$. By definition of $\mathbb{P}_m$,

$$p \restriction m \Vdash \text{“}\dot{R} \text{ is total, acyclic and } \dot{a}_{n_0}\ \dot{R}\ \dot{a}_{n_1}\ \dot{R} \dots \dot{R}\ \dot{a}_{n_h}\text{”}.$$

By induction hypothesis,

$$(p \restriction \vec{H}) \restriction m = p \restriction (\vec{H} \restriction m) \Vdash \text{“}\dot{R} \text{ is total, acyclic and } \dot{a}_{n_0}\ \dot{R}\ \dot{a}_{n_1}\ \dot{R} \dots \dot{R}\ \dot{a}_{n_h}\text{”}.$$

Therefore, $p \restriction \vec{H} \in \mathbb{P}_m$. □

Assume towards a contradiction that $p \restriction \vec{H} \nVdash \varphi(\dot{x}_1, \ldots, \dot{x}_n)$, then there is a $q \leq p \restriction \vec{H}$ such that $q \Vdash \neg\varphi(\dot{x}_1, \ldots, \dot{x}_n)$. Let $\vec{\pi} \in \mathcal{G}$ such that π_l pointwise fixes H_l for each $l \leq m$ and such that $\pi_l[\operatorname{dom}(q_l)] \cap \operatorname{dom}(p_l) = H_l \cap \operatorname{dom}(p_l)$ for all $l \leq m$ and $\pi_l[\operatorname{dom}(q_l)] \cap \operatorname{dom}(p_l) = \emptyset$ for all $l > m$. In particular, $\vec{\pi} \in \operatorname{Fix}(\vec{H})$. Thus, by the Symmetry Lemma, $\vec{\pi}q \Vdash \neg\varphi(\dot{x}_1, \ldots, \dot{x}_n)$.

Claim 2.4.2.2. *p and $\vec{\pi}q$ are compatible.*

Proof. It suffices to show that $p_l^R(k) = (\vec{\pi}q)_l^R(k)$ and that the sequence $p_l^s(k)$ is extended by $(\vec{\pi}q)_l^s(k)$, for every $l \leq m$ and for every $k \in \operatorname{dom}(p_l) \cap \operatorname{dom}((\vec{\pi}q)_l)$. Note that $\operatorname{dom}((\vec{\pi}q)_l) = \pi_l[\operatorname{dom}(q_l)]$. Fix an $l \leq m$ and a $k \in \operatorname{dom}(p_l) \cap \pi_l[\operatorname{dom}(q_l)]$. By the way we chose $\vec{\pi}$, we must have $k \in H_l$. Also, as we assumed $q \leq p$, we have $q_l^R(k) = p_l^R(k)$ and $q_l^s(k) \supseteq p_l^s(k)$. Moreover, we assumed $\vec{H} \restriction l$ to be a support for $p_l^R(k)$, and we have picked $\vec{\pi}$ so that $\vec{\pi} \in \operatorname{Fix}(\vec{H})$. In particular, $(\vec{\pi} \restriction l)\,(p_l^R(k)) = p_l^R(k)$ and $\pi_l(k) = k$. Therefore, by the definition of the induced automorphism $\vec{\pi} \in \mathcal{G}$,

$$(\vec{\pi}q)_l^R(k) = (\vec{\pi} \restriction l)\,(q_l^R(\pi_l^{-1}(k))) = (\vec{\pi} \restriction l)\,(q_l^R(k)) = (\vec{\pi} \restriction l)\,(p_l^R(k)) = p_l^R(k).$$

Moreover, since we assumed $\operatorname{ran}(p_l^s(k)) \subseteq H_0$, and $\pi_0 \in \operatorname{Fix}(H_0)$, we have, for every $i < \operatorname{length}(p_l^s(k))$,

$$(\vec{\pi}q)_l^s(k)(i) = \pi_0(q_l^s(\pi_l^{-1}(k))(i)) = \pi_0(q_l^s(k)(i)) = \pi_0(p_l^s(k)(i)) = p_l^s(k)(i),$$

and therefore $(\vec{\pi}q)_l^s(k) \supseteq p_l^s(k)$. □

As before, the fact that p and $\vec{\pi}q$ are compatible yields the desired contradiction and concludes the proof. □

2.4.3 The model

For each $n, k \in \omega$, we let

$$\dot{f}_{n,k} := \left\{((\check{l}, \dot{a}_m)^\bullet, p) \mid l, m \in \omega \text{ and } p \in \mathbb{P}_{n+1} \text{ and } p_{n+1}^s(k)(l) = m\right\},$$
$$\dot{F}_n := \left\{\dot{f}_{n,k} \mid k \in \omega\right\}^\bullet.$$

Note that, for each $\vec{\pi} \in \mathcal{G}$, $\vec{\pi}(\dot{f}_{n,k}) = \dot{f}_{n,\pi_{n+1}(k)}$. In particular, both $\dot{f}_{n,k}$ and $\dot{F}_n$ belong to HS for every $n, k \in \omega$. These $\mathbb{P}$-names, together with $\dot{A}$ and the $\dot{a}_n$s, are the (hereditarily symmetric) names for all the generic sets we are interested in. Observe that $\dot{f}_{n,k}$ is a $\mathbb{P}_{n+1}$-name for an R-chain belonging to $\mathcal{N}_{n+1}$, where R is the relation with $\mathbb{P}_{n+1}$-name $\{(p^R_{n+1}(k), p) \mid p \in \mathbb{P}_{n+1}\}$.

Fix a $\mathbb{P}$-generic filter G over V and, for all n, let $\mathcal{N}_n$ be the symmetric extension obtained from $\langle \mathbb{P}_n, \mathcal{G}_n, \mathcal{F}_n \rangle$, and $\mathcal{N}$ be the symmetric extension, obtained from $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$. Clearly we have

$$\mathrm{V} \subseteq \mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \cdots \subseteq \mathcal{N} = \mathcal{N}_\omega \subseteq \mathrm{V}[G].$$

For each $\mathbb{P}$-name (e.g. $\dot{A}$), we let its symbol without the dot (i.e. A) be its evaluation according to G (i.e. $\dot{A}_G$).

Lemma 2.4.3. *For every $n \in \omega$, for every total and acyclic binary relation $R \in \mathcal{N}_n$ on A, there is an an R-chain in $\mathcal{N}_{n+1}$.*

Proof. Let $p \in G$ and $\dot{R} \in \mathsf{HS}_n$ such that

$$p \Vdash \dot{R} \subseteq \dot{A} \times \dot{A} \text{ total and acyclic,}$$

and without loss of generality we may assume that $p \in \mathbb{P}_n$. We show that $\dot{R}_G = \dot{S}_G$ for some $\dot{S} \in \mathcal{R}_n$ as in (2.3). Let

$$\dot{S} := \Big\{ ((\dot{a}_m, \dot{a}_k)^\bullet, q) \mid m, k \in \omega, q \in \mathbb{P}_n, \text{ and } q \Vdash \dot{a}_m \; \dot{R} \; \dot{a}_k \Big\}.$$

It readily follows that $\dot{S}$ is in $\mathcal{R}_n$ and $p \Vdash \dot{R} = \dot{S}$. Fix any $q \in \mathbb{P}_{n+1}$ with $q \leq p$. Pick an $m \in \omega \setminus \mathrm{dom}(q_{n+1})$ and consider the finite sequence q' such that $q'_l = q_l$ for every $l \neq n+1$ and $q'_{n+1} = q_{n+1} \cup \{(m, (\dot{S}, \emptyset))\}$. Then $q' \in \mathbb{P}_{n+1}$, $q' \leq q$ and

$$q' \Vdash \dot{f}_{n,m} \text{ is an } \dot{S}\text{-chain, and } \dot{S} = \dot{R}.$$

By density,

$$p \Vdash \exists f \in \dot{F}_n \text{ which is an } \dot{R}\text{-chain.}$$

Since $F_n \in \mathcal{N}_{n+1}$ we are done. □

Since A is not closed as a set of reals, there exists a total and acyclic binary relation over A in $\mathcal{N}_0$ (see Lemma 2.2.2). Therefore, by Lemma 2.4.3, the set A becomes Dedekind-infinite already in $\mathcal{N}_1$. However, the next proposition tells us that the range of any generic chain introduced by the iteration is far from being dense in A. This result is crucial in showing that A is not separable in $\mathcal{N}$. But before stating this key proposition, we need to recall the well-known notion of scattered space (e.g. see [63, §8.5]).

Definition 2.4.4. Given a topological space X, we let by ordinal induction

$$\begin{aligned} X^{(0)} &= X, \\ X^{(\alpha+1)} &= \left\{x \in X^{(\alpha)} \mid x \text{ is a limit point of } X^{(\alpha)}\right\}, \\ X^{(\lambda)} &= \bigcap_{\alpha<\lambda} X^{(\alpha)} \qquad \text{for } \lambda \text{ a limit ordinal.} \end{aligned}$$

For every space X there is necessarily an ordinal α such that $X^{(\alpha)} = X^{(\alpha+1)}$, and we call the least such ordinal the *scattered height* of the space. A topological space X is *scattered* if there is an α such that $X^{(\alpha)} = \emptyset$.

It is easy to check in ZF that every second countable scattered space is countable [63, Proposition 8.5.5]. If $\emptyset \neq X \subseteq \mathbb{R}$ is dense in itself, then $X^{(1)} = X$ and X is not scattered. In particular, A is not scattered.

For each $t \in {}^{<\omega}2$, we denote by $\dot{N}_t$ the canonical name for the basic open set N_t of ${}^{\omega}2$. Furthermore, we denote by cl_A the closure with respect to the subspace topology of A.

Proposition 2.4.5. *For each $n, k \in \omega$, $\mathcal{N} \models \left(\mathrm{cl}_A(\mathrm{ran}(f_{n,k}))\right)^{(n+2)} = \emptyset$.*

Proof. In other words, we want to show that in $\mathcal{N}$ (or, equivalently[2], in $\mathrm{V}[G]$), for every n, k, the closure with respect to A of the range of $f_{n,k}$ is scattered of height at most $n+2$. For any $H \subseteq \omega$, let us introduce the $\mathbb{P}$-name

$$\dot{A}_H := \{\dot{a}_m \mid m \in H\}^\bullet. \tag{2.4}$$

[2] Note that the formula $\varphi(x, y, \alpha) :=$ "α is an ordinal, $x \subseteq y$ are sets of reals and $\mathrm{cl}_y(x)$ is scattered of height $\leq \alpha$" is a Δ_1^{ZF}-formula. In particular, it is absolute between models of ZF.

Let (†) be the statement $\forall n \in \omega\, (\dagger)_n$, where $(\dagger)_n$ is the following statement:

$(\dagger)_n$ Let $k \in \omega$, $p = \langle p_0, \dots p_n\rangle \in \mathbb{P}_n$, $\dot{R} \in \mathcal{R}_n$ with support $\vec{H} = \langle H_0, \dots, H_n\rangle$ such that $p \Vdash \dot{R}$ is total and acyclic". Assume also that, for all $i \le n$, $\mathrm{dom}(p_i) = H_i$, and, for all $0 < i \le n$, for all $j \in H_i$, $\vec{H} \restriction i$ is a support for $p_i^R(j)$. Then

$$p^\frown\{(k,(\dot{R},\emptyset))\} \Vdash \left(\mathrm{cl}_{\dot{A}}(\mathrm{ran}(\dot{f}_{n,k}))\right)^{(1)} \subseteq \dot{A}_{H_0} \cup \bigcup_{\substack{i<n\\ j\in H_{i+1}}} \mathrm{ran}(\dot{f}_{i,j}).$$

Remark. (a) The condition $p^\frown\{(k,(\dot{R},\emptyset))\}$ in the statement of $(\dagger)_n$ belongs to $\mathbb{P}_{n+1}$ and it is obtained by extending p with the function with domain $\{k\} \subseteq \omega$ such that $k \mapsto (\dot{R},\emptyset) \in \mathcal{R}_n \times {}^{<\omega}\omega$.

(b) Note that $p^\frown\{(k,(\dot{R},\emptyset))\}$ is the $\le$-maximum among the conditions $q \le p$ such that $q_{n+1}^R(k) = \dot{R}$. Therefore, for any fixed n, k, the set of conditions $p^\frown\{(k,(\dot{R},\emptyset))\}$ we are considering in $(\dagger)_n$ is pre-dense in $\mathbb{P}_{n+1}$ (and also in $\mathbb{P}$).

Claim 2.4.5.1. *Assume* (†). *For each $n, k \in \omega$, $\Vdash \left(\mathrm{cl}_{\dot{A}}(\mathrm{ran}(\dot{f}_{n,k}))\right)^{(n+2)} = \emptyset$.*

Proof. We prove the claim by induction on n. Let $n = 0$ and fix $k \in \omega$, $p \in \mathbb{P}_0$, $\dot{R} \in \mathcal{R}_0$, $\vec{H} = \langle H_0\rangle$ satisfying the hypotheses of $(\dagger)_0$. By $(\dagger)_0$,

$$p^\frown\{(k,(\dot{R},\emptyset))\} \Vdash (\mathrm{cl}_{\dot{A}}(\mathrm{ran}(\dot{f}_{n,k})))^{(1)} \subseteq \dot{A}_{H_0}.$$

Since H_0 is finite, we have that $\Vdash$ "$\dot{A}_{H_0}$ is finite", so our condition forces that $\mathrm{cl}_A(\mathrm{ran}(f_{n,k}))$ has scattered height ≤ 2, that is

$$p^\frown\{(k,(\dot{R},\emptyset))\} \Vdash \left(\mathrm{cl}_{\dot{A}}(\mathrm{ran}(\dot{f}_{n,k}))\right)^{(2)} = \emptyset.$$

By density (see the remark right after the definition of $(\dagger)_n$), the base case follows.

Now the induction step. Let $n > 0$ and fix $k \in \omega$, $p \in \mathbb{P}_n$, $\dot{R} \in \mathcal{R}_n$, $\vec{H} = \langle H_0, \dots, H_n\rangle$ satisfying the hypotheses of $(\dagger)_n$. By $(\dagger)_n$,

$$p^\frown\{(k,(\dot{R},\emptyset))\} \Vdash \left(\mathrm{cl}_{\dot{A}}(\mathrm{ran}(\dot{f}_{n,k}))\right)^{(1)} \subseteq \dot{A}_{H_0} \cup \bigcup_{\substack{i<n\\ j\in H_{i+1}}} \mathrm{ran}(\dot{f}_{i,j}).$$

As the H_is are all finite,

$$p^\frown\{(k,(\dot R,\emptyset))\} \Vdash \left(\mathrm{cl}_{\dot A}(\mathrm{ran}(\dot f_{n,k}))\right)^{(n+2)} \subseteq \bigcup_{\substack{i<n\\ j\in H_{i+1}}} \left(\mathrm{cl}_{\dot A}(\mathrm{ran}(\dot f_{i,j}))\right)^{(n+1)}.$$

By induction hypothesis, for all $i<n$ and all $j\in H_{i+1}$

$$\Vdash \left(\mathrm{cl}_{\dot A}(\mathrm{ran}(\dot f_{i,j}))\right)^{(n+1)} = \emptyset,$$

and hence,

$$p^\frown\{(k,(\dot R,\emptyset))\} \Vdash \left(\mathrm{cl}_{\dot A}(\mathrm{ran}(\dot f_{n,k}))\right)^{(n+2)} = \emptyset.$$

By density, the induction step follows. □

The statement (†) is proved by induction on $n\in\omega$. Let $n=0$ and fix $k\in\omega$, $p\in\mathbb{P}_0$, $\dot R\in\mathcal{R}_0$, $\vec H=\langle H_0\rangle$ satisfying the hypotheses of $(\dagger)_0$.

Claim 2.4.5.2. $p^\frown\{(k,(\dot R,\emptyset))\} \Vdash \mathrm{ran}(\dot f_{0,k})\setminus \dot A_{H_0}$ *is discrete.*

Proof. Assume towards a contradiction that there are $q\le p^\frown\{(k,(\dot R,\emptyset))\}$ and $l\in\omega$ such that

$$q\Vdash \dot f_{0,k}(l)\notin \dot A_{H_0} \text{ and } \dot f_{0,k}(l) \text{ is a limit point of } \mathrm{ran}(\dot f_{0,k})\setminus \dot A_{H_0}.$$

Without loss of generality suppose that $\mathrm{length}(q_1^s(k))>l+1$ and let $m=q_1^s(k)(l)$, $t=q_0(m)$—in particular, $m\notin H_0$ and $q\Vdash \dot f_{0,k}(l)=\dot a_m\in \dot N_t$. From our assumption and from the fact that H_0 is a finite set, it follows that there must be a $z\le q$ and an $h>l$ such that

$$z\Vdash \dot f_{0,k}(h)\in \dot N_t\setminus \dot A_{H_0}.$$

Assume without loss of generality $\mathrm{length}(z_1^s(k))>h$ and let $m'=z_1^s(k)(h)$, $t'=z_0(m')$—in particular, $m'\notin H_0$, $t'\supseteq t$ and $z_0\Vdash \dot a_m\ \dot R^+\ \dot a_{m'}$. Note that $m'\neq m$, as otherwise z_0 would force $\dot R$ to have a cycle, which is a contradiction, as z_0 extends p and by hypothesis p forces $\dot R$ to be acyclic. Let $p':=z_0\restriction(\omega\setminus\{m\})\cup\{(m,t')\}$.

Subclaim 2.4.5.2.1. $p'\Vdash \dot a_m\ \dot R^+\ \dot a_{m'}$.

A quick observation: since $z_0 \leq q_0$, $z_0(m)$ surely extends $t = q_0(m)$, but a priori $z_0(m)$ could be incompatible with $t' = z_0(m') = p'(m)$, making p' incompatible with z_0. Thus, our subclaim needs some care.

Proof of the Subclaim. Let $m_0, m_1, \ldots, m_{h-l} \in \omega$ be such that $m_i = z_1^s(k)(l+i)$ for all $i \leq h-l$. Note that $m_0 = m = q_1^s(k)(l)$ and $m_1 = q_1^s(k)(l+1)$, since $z \leq q$. Moreover, $m_{h-l} = m'$, by definition of m'. We can assume that the m_is are all distinct, as otherwise z_0 would force $\dot{R}$ to have a cycle.

Clearly $q_0 \Vdash \dot{a}_{m_0}\ \dot{R}\ \dot{a}_{m_1}$. By the Restriction Lemma, $q_0 \restriction H_0 \cup \{m_0, m_1\}$ forces the same.

On the other hand, $z_0 \Vdash \dot{a}_{m_1}\ \dot{R}\ \dot{a}_{m_2}\ \dot{R} \ldots \dot{R}\ \dot{a}_{m_{h-l}}$. Again by the Restriction Lemma, $z_0 \restriction H_0 \cup \{m_1, m_2, \ldots, m_{h-l}\}$ forces the same.

The condition $p' = z_0 \restriction (\omega \setminus \{m\}) \cup \{(m, t')\}$ extends both $q_0 \restriction H_0 \cup \{m_0, m_1\}$ and $z_0 \restriction H_0 \cup \{m_1, m_2, \ldots, m_{h-l}\}$—here use the fact that $m_0 \notin H_0$ and that all the m_is are distinct. Hence p' forces $\dot{a}_{m_0}\ \dot{R}\ \dot{a}_{m_1}\ \dot{R}\ \dot{a}_{m_2}\ \dot{R} \ldots \dot{R}\ \dot{a}_{m_{h-l}}$. □

Let $\pi_0 \colon \omega \to \omega$ be the permutation that swaps m and m' fixing everything else—in particular, $\pi_0 \in \mathrm{Fix}(H_0)$ and $\pi_0 \dot{R} = \dot{R}$. Then, by the Symmetry Lemma,

$$\pi_0 p' = p' \Vdash \dot{a}_{m'}\ \dot{R}^+\ \dot{a}_m,$$

but then p' both extends p and forces $\dot{a}_m\ \dot{R}^+\ \dot{a}_m$, which is a contradiction, since we assumed that p forces $\dot{R}$ to be acyclic. □

By Claim 2.4.5.2, condition $p^\frown\{(k, (\dot{R}, \emptyset))\}$ forces that the limit points of $\mathrm{ran}(f_{0,k})$ (in $\mathrm{ran}(f_{0,k})$) belong to the finite set A_{H_0}. The next claim shows that the same is true for the larger set $\mathrm{cl}_A(\mathrm{ran}(f_{0,k}))$.

Claim 2.4.5.3. $p^\frown\{(k, (\dot{R}, \emptyset))\} \Vdash \left(\mathrm{cl}_{\dot{A}}(\mathrm{ran}(\dot{f}_{0,k}))\right)^{(1)} \subseteq \dot{A}_{H_0}$.

Proof. Suppose towards a contradiction that the claim is false. Then there is a $q \leq p^\frown\{(k, (\dot{R}, \emptyset))\}$ and an $m \notin H_0$ such that

$$q \Vdash \dot{a}_m \text{ is a limit point of } \mathrm{ran}(\dot{f}_{0,k}). \tag{2.5}$$

Note that, since H_0 is finite, q actually forces $\dot{a}_m$ to be a limit point of $\mathrm{ran}(\dot{f}_{0,k}) \setminus \dot{A}_{H_0}$. Hence, it follows from Claim 2.4.5.2 that q forces $\dot{a}_m$ not to be in the range of $\dot{f}_{0,k}$. In particular, $m \notin \mathrm{ran}(q_1^s(k))$.

The condition $q' := \langle q_0, q_1 \restriction \{k\}\rangle$ extends p and, by the Restriction Lemma, still forces (2.5). Let t be $q_0(m)$—in particular, $q' \Vdash \dot{a}_m \in \dot{N}_t$. We now show that $q' \Vdash \dot{N}_t \subseteq \mathrm{cl}(\mathrm{ran}(\dot{f}_{0,k}) \setminus \dot{A}_{H_0})$, which clearly contradicts Claim 2.4.5.2, as every discrete set of reals is nowhere dense. Pick any $z \le q'$ and a $t' \supseteq t$. Fix an $m' \notin H_0 \cup \mathrm{dom}(z_0) \cup \mathrm{ran}(q_1^s(k))$. Define z' to be the condition such that $z'_0 = z_0 \cup \{(m', t')\}$ and $z'_i = z_i$ for every $i > 0$.

Now, z' clearly extends z. Moreover, if we let π_0 be the permutation of ω that swaps m and m', z' also extends $\langle \pi_0\rangle q'$. Indeed, since $t' \supseteq t$, it's clear that z'_0 extends $\pi_0 q_0$. But since both m and m' do not belong to $H_0 \cup \mathrm{ran}(q_1^s(k))$, we also have $(\langle \pi_0\rangle q')_1 = q'_1$, and therefore $\langle z'_0, z'_1\rangle = \langle z'_0, z_1\rangle$ extends $\langle \pi_0\rangle q' = \langle \pi_0 q_0, q'_1\rangle$. Overall, z' extends $\langle \pi_0\rangle q'$. By (2.5) and the Symmetry Lemma,

$$\langle \pi_0\rangle q' \Vdash \dot{a}_{m'} \in \mathrm{cl}(\mathrm{ran}(\dot{f}_{0,k}) \setminus \dot{A}_{H_0}).$$

Since z' extends $\langle \pi_0\rangle q'$ and $z' \Vdash \dot{a}_{m'} \in \dot{N}_{t'}$, we have

$$z' \Vdash \dot{N}_{t'} \cap \mathrm{cl}(\mathrm{ran}(\dot{f}_{0,k}) \setminus \dot{A}_{H_0}) \neq \emptyset.$$

By density,

$$q' \Vdash \dot{N}_t \subseteq \mathrm{cl}(\mathrm{ran}(\dot{f}_{0,k}) \setminus \dot{A}_{H_0}). \qquad \square$$

We have just proven $(\dagger)_0$. Here comes the induction step: fix an $n > 0$ and suppose $(\dagger)_i$ holds for every $i < n$, towards proving $(\dagger)_n$. Fix $k \in \omega$, $p \in \mathbb{P}_n$, $\dot{R} \in \mathcal{R}_n$, $\vec{H} = \langle H_0, \dots, H_n\rangle$ satisfying the hypotheses of $(\dagger)_n$. The next claim is the analogue of Claim 2.4.5.2.

Claim 2.4.5.4.

$$p^\frown\{(k, (\dot{R}, \emptyset))\} \Vdash \mathrm{ran}(\dot{f}_{n,k}) \setminus \Big(\dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j \in H_{i+1}}} \mathrm{ran}(\dot{f}_{i,j})\Big) \textit{ is discrete.}$$

Proof. Suppose towards a contradiction that there are $q \leq \langle p, \{(k, (\dot{R}, \emptyset))\}\rangle$ and $l \in \omega$ such that

$$q \Vdash \dot{f}_{n,k}(l) \notin \dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j \in H_{i+1}}} \operatorname{ran}(\dot{f}_{i,j}) \text{ and } \dot{f}_{n,k}(l) \text{ is a limit point of } \operatorname{ran}(\dot{f}_{n,k}) \setminus \left(\dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j \in H_{i+1}}} \operatorname{ran}(\dot{f}_{i,j})\right).$$

Suppose without loss of generality that $\operatorname{length}(q^s_{n+1}(k)) > l + 1$ and let $m = q^s_{n+1}(k)(l)$, and $t = q_0(m)$—in particular, $q \Vdash \dot{f}_{n,k}(l) = \dot{a}_m \in \dot{N}_t$. By assumption there must be a $z \leq q$ and an $h > l$ such that

$$z \Vdash \dot{f}_{n,k}(h) \in \dot{N}_t \setminus \left(\dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j \in H_{i+1}}} \operatorname{ran}(\dot{f}_{i,j})\right).$$

Assume without loss of generality $\operatorname{length}(z^s_{n+1}(k)) > h$ and let $m' = z^s_{n+1}(k)(h)$, and $t' = z_0(m')$—in particular $t' \supseteq t$ and $z \restriction n+1 \Vdash \dot{a}_m \ \dot{R}^+ \ \dot{a}_{m'}$. Since

$$z \Vdash \dot{a}_m, \dot{a}'_m \notin \left(\dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j \in H_{i+1}}} \operatorname{ran}(\dot{f}_{i,j})\right),$$

then, in particular,

$$m, m' \notin H_0 \cup \bigcup_{\substack{i<n \\ j \in H_{i+1}}} \operatorname{ran}\left(z^s_{i+1}(j)\right). \tag{2.6}$$

Now let

$$p' = \langle z_0 \restriction (\omega \setminus \{m\}) \cup \{(m, t')\}, z_1 \restriction H_1, \ldots, z_n \restriction H_n\rangle.$$

By the Restriction Lemma, $p' \in \mathbb{P}_n$. Moreover, by an argument analogous to the one used in the proof of Subclaim 2.4.5.2.1, we can show that

$$p' \Vdash \dot{a}_m \ \dot{R}^+ \ \dot{a}_{m'}.$$

If we let $\pi_0 \colon \omega \to \omega$ be the permutation that swaps m and m', then $\langle \pi_0 \rangle p' = p'$. Indeed, it directly follows from the definition of p' that $\pi_0 p'_0 = p'_0$. Moreover, by (2.6), both m and m' do not belong to H_0, hence $\langle \pi_0 \rangle \in \operatorname{Fix}(\vec{H})$. As such, $(\langle \pi_0 \rangle p')^R_i = (p')^R_i$ for every $1 \leq i \leq n$. Again by (2.6), m and m' do not belong to the range of $(p')^s_i(j)$ for any $1 \leq i \leq n$ and $j \in \mathrm{dom}(p'_i) = H_i$, and therefore $(\langle \pi_0 \rangle p')^s_i = (p')^s_i$ for every $1 \leq i \leq n$. Overall, $\langle \pi_0 \rangle p' = p'$.

Next note that $\langle \pi_0 \rangle \dot{R} = \dot{R}$, as $\langle \pi_0 \rangle \in \mathrm{Fix}(\vec{H})$. By the Symmetry Lemma,

$$\langle \pi_0 \rangle p' = p' \Vdash \dot{a}_{m'} \ \dot{R}^+ \ \dot{a}_m,$$

but then p' both extends p and forces $\dot{a}_m \ \dot{R}^+ \ \dot{a}_m$, which is a contradiction, since we assumed that p forces $\dot{R}$ to be acyclic. □

Claim 2.4.5.5.

$$p \Vdash \dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j \in H_{i+1}}} \mathrm{ran}(\dot{f}_{i,j}) \textit{ is closed in } \dot{A}.$$

Proof. Fix $q \leq p$ and m such that

$$q \Vdash \dot{a}_m \in \mathrm{cl}_{\dot{A}}\Big(\dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j \in H_{i+1}}} \mathrm{ran}(\dot{f}_{i,j})\Big).$$

We would like to prove that there is a condition $z \leq q$ such that

$$z \Vdash \dot{a}_m \in \dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j \in H_{i+1}}} \mathrm{ran}(\dot{f}_{i,j}),$$

so, to avoid trivialities, we assume

$$q \Vdash \dot{a}_m \in \Big(\mathrm{cl}_{\dot{A}}\Big(\dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j \in H_{i+1}}} \mathrm{ran}(\dot{f}_{i,j})\Big)\Big)^{(1)}.$$

As the H_is are finite, there exists a $z \leq q$, an $i < n$ and some $j \in H_{i+1}$ such that $z \Vdash \dot{a}_m \in (\mathrm{cl}_{\dot{A}}(\mathrm{ran}(\dot{f}_{i,j})))^{(1)}$. But then, by $(\dagger)_i$ (here we use our induction hypothesis),

$$z \Vdash \dot{a}_m \in \Big(\mathrm{cl}_{\dot{A}}(\mathrm{ran}(\dot{f}_{i,j}))\Big)^{(1)} \subseteq \dot{A}_{H_0} \cup \bigcup_{\substack{l<i \\ h \in H_{l+1}}} \mathrm{ran}(\dot{f}_{l,h}).$$

By density, the claim follows. □

The next claim is the analogue of Claim 2.4.5.3.

Claim 2.4.5.6.

$$p^\frown\{(k,(\dot R,\emptyset))\} \Vdash \left(\mathrm{cl}_{\dot A}(\mathrm{ran}(\dot f_{n,k}))\right)^{(1)} \subseteq \dot A_{H_0} \cup \bigcup_{\substack{i<n\\ j\in H_{i+1}}} \mathrm{ran}(\dot f_{i,j}).$$

Proof. Suppose towards a contradiction that this is not the case, then there is a $q \leq p^\frown\{(k,(\dot R,\emptyset))\}$ and an m such that

$$q \Vdash \dot a_m \text{ is a limit point of } \mathrm{ran}(\dot f_{n,k}) \text{ and } \dot a_m \notin \dot A_{H_0} \cup \bigcup_{\substack{i<n\\ j\in H_{i+1}}} \mathrm{ran}(\dot f_{i,j}).$$

From Claim 2.4.5.5 it follows that

$$q \Vdash \dot a_m \text{ is a limit point of } \mathrm{ran}(\dot f_{n,k}) \setminus \left(\dot A_{H_0} \cup \bigcup_{\substack{i<n\\ j\in H_{i+1}}} \mathrm{ran}(\dot f_{i,j})\right). \tag{2.7}$$

But then, by Claim 2.4.5.4, q also forces $\dot a_m$ not to be in the range of $\dot f_{n,k}$. In particular,

$$m \notin H_0 \cup \mathrm{ran}(q^s_{n+1}(k)) \cup \bigcup_{\substack{i<n\\ j\in H_{i+1}}} \mathrm{ran}(q^s_{i+1}(j)). \tag{2.8}$$

Let

$$q' = \langle q_0, q_1 \restriction H_1, \ldots, q_n \restriction H_n, q_{n+1} \restriction \{k\}\rangle.$$

Then q' extends p and, by the Restriction Lemma, still forces (2.7). Let t be $q_0(m)$—in particular $q' \Vdash \dot a_m \in \dot N_t$.

We now show that

$$q' \Vdash \dot N_t \subseteq \mathrm{cl}\Big(\mathrm{ran}(\dot f_{n,k}) \setminus \Big(\dot A_{H_0} \cup \bigcup_{\substack{i<n\\ j\in H_{i+1}}} \mathrm{ran}(\dot f_{i,j})\Big)\Big),$$

which contradicts Claim 2.4.5.4. Pick any $z \leq q'$ and $t' \supseteq t$. Fix an $m' \in \omega$ such that

$$m' \notin H_0 \cup \mathrm{dom}(z_0) \cup \mathrm{ran}(q^s_{n+1}(k)) \cup \bigcup_{\substack{i<n\\ j\in H_{i+1}}} \mathrm{ran}(q^s_{i+1}(j)). \tag{2.9}$$

Define z' to be the condition such that $z'_0 = z_0 \cup \{(m', t')\}$ and $z'_i = z_i$ for all $i > 0$. Now, z' clearly extends z. Moreover, if we let π_0 be the permutation of ω that swaps m and m', z' also extends $\langle \pi_0 \rangle q'$. Indeed, since $t' \supseteq t$, it's clear that z'_0 extends $\pi_0 q_0$. But from (2.8) and (2.9), it follows that $(\langle \pi_0 \rangle q')_i = q'_i$ for every $1 \leq i \leq n+1$, and therefore z' extends $\langle \pi_0 \rangle q'$.

By (2.7) and the Symmetry Lemma,

$$\langle \pi_0 \rangle q' \Vdash \dot{a}_{m'} \in \mathrm{cl}\Big(\mathrm{ran}(\dot{f}_{n,k}) \setminus \Big(\dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j \in H_{i+1}}} \mathrm{ran}(\dot{f}_{i,j})\Big)\Big).$$

Since z' extends $\langle \pi_0 \rangle q'$ and $z' \Vdash \dot{a}_{m'} \in \dot{N}_{t'}$, we have

$$z' \Vdash \dot{N}_{t'} \cap \mathrm{cl}\Big(\mathrm{ran}(\dot{f}_{n,k}) \setminus \Big(\dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j \in H_{i+1}}} \mathrm{ran}(\dot{f}_{i,j})\Big)\Big) \neq \emptyset.$$

By density,

$$q' \Vdash \dot{N}_t \subseteq \mathrm{cl}\Big(\mathrm{ran}(\dot{f}_{n,k}) \setminus \Big(\dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j \in H_{i+1}}} \mathrm{ran}(\dot{f}_{i,j})\Big)\Big). \qquad \square$$

This completes the proof of $(\dagger)_n$, so by induction $(\dagger)$ holds. By Claim 2.4.5.1, we are done. $\square$

In light of Proposition 2.4.5, we can prove that in $\mathcal{N}$ every separable subset of A is scattered with finite scattered height.

Theorem 2.4.6. *In the model $\mathcal{N}$ the following holds: for every separable $S \subseteq A$ there is an $n \in \omega$ such that $S^{(n)} = \emptyset$.*

Proof. Let $S \in \mathcal{N}$ be a separable subset of A and fix in $\mathcal{N}$ a function $f\colon \omega \to A$ such that $S \subseteq \mathrm{cl}(\mathrm{ran}(f))$. Then there must be a $p \in G$ such that

$$p \Vdash \dot{f}\colon \omega \to \dot{A},$$

where $\dot{f} \in \mathsf{HS}$ is a symmetric name for f, with support $\vec{H} = \langle H_0, \ldots, H_n \rangle$. We can assume without loss of generality that $\mathrm{dom}(p_i) = H_i$ for each i, and that for all $i > 0$, for all $j \in H_i$, $\vec{H} \restriction i$ is a support for $p_i^R(j)$. We claim that

$$p \Vdash \operatorname{ran}(\dot{f}) \subseteq \dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j \in H_{i+1}}} \operatorname{ran}(\dot{f}_{i,j}),$$

where $\dot{A}_{H_0}$ is the $\mathbb{P}$-name as in (2.4). If we manage to do so, then Proposition 2.4.5 ensures that $\mathrm{cl}_A(\operatorname{ran}(f))$ is scattered of height $\leq n+2$, and, a fortiori, that $S^{(n+2)} = \emptyset$, as required.

Suppose that the claim is false, then there exist $q \leq p$ and $l, m \in \omega$ such that

$$q \Vdash \dot{f}(l) = \dot{a}_m \notin \dot{A}_{H_0} \cup \bigcup_{\substack{i<n \\ j \in H_{i+1}}} \operatorname{ran}(\dot{f}_{i,j}). \tag{2.10}$$

In particular,

$$m \notin H_0 \cup \bigcup_{\substack{i<n \\ j \in H_{i+1}}} \operatorname{ran}(q_{i+1}^s(j)). \tag{2.11}$$

Let $q' = \langle q_0, q_1 \restriction H_1, \ldots, q_n \restriction H_n \rangle$. Then, by the Restriction Lemma, q' still forces (2.10).

Fix an $m' \in \omega$ such that

$$m' \notin H_0 \cup \operatorname{dom}(q_0) \cup \bigcup_{\substack{i<n \\ j \in H_{i+1}}} \operatorname{ran}(q_{i+1}^s(j)). \tag{2.12}$$

Let π_0 be the permutation of ω that swaps m and m', then $\langle \pi_0 \rangle q'$ and q' are compatible. Indeed, since $m' \notin \operatorname{dom}(q_0')$, then q_0' and $\pi_0 q_0'$ are clearly compatible. Moreover, it follows from (2.11) and (2.12) that $(\langle \pi_0 \rangle q')_i = q_i'$ for every $1 \leq i \leq n$, and therefore $\langle \pi_0 \rangle q'$ and q' are compatible. By the Symmetry Lemma,

$$\langle \pi_0 \rangle q' \Vdash \dot{f}(l) = \dot{a}_{m'}.$$

So q' and $\langle \pi_0 \rangle q'$, while being compatible, force $\dot{f}$ to take different values at l, but they both extend p, which forces $\dot{f}$ to be a function. Contradiction. □

Corollary 2.4.7. *$\mathcal{N} \vDash \neg\mathsf{AC}_\omega(A)$.*

Proof. Assume towards a contradiction that $\mathsf{AC}_\omega(A)$ holds, then A is certainly separable. By Theorem 2.4.6, A would be scattered. But A has no isolated points. Contradiction. □

Now we are left to prove that $\mathsf{DC}(A)$ holds in $\mathcal{N}$. Let $\dot{\mathcal{N}}_n$ be the canonical name[3] for the intermediate model $\mathcal{N}_n$.

Lemma 2.4.8. *Let $n \in \omega$ and $\dot{x} \in \mathsf{HS}$ with support $\vec{H} = \langle H_0, \dots, H_n \rangle$, then*

$$\Vdash \dot{x} \subseteq \dot{\mathcal{N}}_n \Rightarrow \dot{x} \in \dot{\mathcal{N}}_n\,.$$

Proof. Fix $(\dot{y}, p) \in \dot{x}$. As the set of qs such that either $q \Vdash \dot{y} \in \dot{\mathcal{N}}_n$ or else $q \Vdash \dot{y} \notin \dot{\mathcal{N}}_n$ is dense below p, there is a maximal antichain $D_{(\dot{y},p)}$ below p and a map $h_{(\dot{y},p)} \colon D_{(\dot{y},p)} \to \mathsf{HS}_n$ such that, for each $q \in D_{(\dot{y},p)}$, either $q \Vdash \dot{y} = h_{(\dot{y},p)}(q)$ or $q \Vdash \dot{y} \notin \dot{\mathcal{N}}_n$. Let $D'_{(\dot{y},p)} = \{q \in D_{(\dot{y},p)} \mid q \Vdash \dot{y} \in \dot{\mathcal{N}}_n\}$ and let

$$C := \Big\{\vec{\pi}\Big(h_{(\dot{y},p)}(q)\Big) \mid (\dot{y}, p) \in \dot{x},\ q \in D'_{(\dot{y},p)},\ \vec{\pi} \in \mathrm{Fix}(\vec{H})\Big\}.$$

Consider the following $\mathbb{P}_n$-name:

$$\dot{w} := \{(\dot{y}, q) \mid \dot{y} \in C,\ q \in \mathbb{P}_n \text{ and } q \Vdash \dot{y} \in \dot{x}\}.$$

Claim 2.4.8.1. *$\dot{w} \in \mathsf{HS}_n$ with support $\vec{H}$.*

Proof. Let $\vec{\pi} \in \mathrm{Fix}(\vec{H})$ and $(\dot{y}, q) \in \dot{w}$. By definition, $q \Vdash \dot{y} \in \dot{x}$, hence $\vec{\pi}q \Vdash \vec{\pi}\dot{y} \in \dot{x}$. Since $\vec{\pi}\dot{y} \in C$, this means that $(\vec{\pi}\dot{y}, \vec{\pi}q) \in \dot{w}$. Hence $\vec{\pi}\dot{w} = \dot{w}$. □

Fix $p \in \mathbb{P}$ such that $p \Vdash \dot{x} \subseteq \dot{\mathcal{N}}_n$.

Claim 2.4.8.2. *$p \Vdash \dot{w} = \dot{x}$.*

Proof. Let $q \leq p$ and $\dot{z} \in \mathsf{HS}$ such that $q \Vdash \dot{z} \in \dot{x}$. By definition of C and our hypothesis on p, there is an $r \leq q$ and a $\dot{y} \in C$ such that $r \Vdash \dot{z} = \dot{y} \in \dot{x}$. By the Restriction Lemma, $r \restriction n+1 \Vdash \dot{y} \in \dot{x}$, hence $(\dot{y}, r \restriction n+1) \in \dot{w}$ and, in

[3] $\dot{\mathcal{N}}_n$ is not a name in the standard sense, as it would be a class-sized name. Formally, the "name" $\dot{\mathcal{N}}_n$ is a constant of the forcing language, with $p \Vdash \dot{x} \in \dot{\mathcal{N}}_n$ if and only if $\forall q \leq p\ \exists r \leq q\ \exists \dot{y} \in \mathsf{HS}_n\ (r \Vdash \dot{x} = \dot{y})$.

particular, $r \Vdash \dot{z} = \dot{y} \in \dot{w}$. By density, $p \Vdash \dot{x} \subseteq \dot{w}$. The other inclusion is immediate from the definition of $\dot{w}$. $\square$

Therefore $p \Vdash \dot{x} \in \dot{\mathcal{N}}_n$. By density, $\Vdash \dot{x} \subseteq \dot{\mathcal{N}}_n \Rightarrow \dot{x} \in \dot{\mathcal{N}}_n$. $\square$

Theorem 2.4.9. $\mathcal{N} \vDash \mathsf{DC}(A)$.

Proof. Since every binary relation $R \in \mathcal{N}$ on A is a subset of $A \times A \in \mathcal{N}_0$, it follows from Lemma 2.4.8 that $R \in \mathcal{N}_n$ for some n. Now, either R is cyclic, but then it surely has a chain, or it is acyclic, but then Lemma 2.4.3 says that in $\mathcal{N}_{n+1} \subseteq \mathcal{N}$ there is a chain for this relation. $\square$

This finishes the proof of Theorem 2.1.1.

2.5 Some complementary results

We collect some facts related to our main results, and conclude the chapter with an open question.

2.5.1 Dependent Choice is closed under finite unions.

By Proposition 2.2.1, the axiom $\mathsf{DC}(X)$ is closed under surjective images and, hence, under subsets. The next result shows that it is also closed under finite unions.

Theorem 2.5.1. $\mathsf{DC}(X) \wedge \mathsf{DC}(Y) \Rightarrow \mathsf{DC}(X \cup Y)$.

Corollary 2.5.2. $\mathsf{DC}(X) \Rightarrow \mathsf{DC}(X \times n)$, *for all sets* X *and all* $n \in \omega$.

The natural progression from Corollary 2.5.2 would be to prove that $\mathsf{DC}(X) \Rightarrow \mathsf{DC}(X \times \omega)$, but this cannot be established in ZF, since $\mathsf{DC}(X \times \omega)$ implies $\mathsf{AC}_\omega(X)$ (part (d) of Proposition 2.2.1) and we know from Theorem 2.1.1 that $\mathsf{DC}(X)$ does not necessarily imply $\mathsf{AC}_\omega(X)$.

If a binary relation R is such that $\operatorname{ran}(R) \subseteq \operatorname{dom}(R)$, then it is total on its domain. The largest $R' \subseteq R$ such that $\operatorname{ran}(R') \subseteq \operatorname{dom}(R')$ is

$$\mathcal{D}(R) = \bigcup \Big\{ S \subseteq R \mid \operatorname{ran}(S) \subseteq \operatorname{dom}(S) \Big\}.$$

By part (a) of Proposition 2.2.1 it is easy to see that

$$\mathsf{DC}(X) \Leftrightarrow \forall R \subseteq X^2 \left(\mathcal{D}(R) \neq \emptyset \Rightarrow \text{there is a } \mathcal{D}(R)\text{-chain}\right). \tag{2.13}$$

Proof of Theorem 2.5.1. Suppose $\mathsf{DC}(X)$ and $\mathsf{DC}(Y)$, and let $R \subseteq (X \cup Y)^2$ be total, towards proving that there is an R-chain. Without loss of generality, we may assume that X and Y are nonempty and disjoint. If $\mathcal{D}(R \restriction X) \neq \emptyset$, then by $\mathsf{DC}(X)$ and (2.13) there is a $\mathcal{D}(R \restriction X)$-chain, which is, in particular an R-chain. Similarly, if $\mathcal{D}(R \restriction Y) \neq \emptyset$, then there is an R-chain. Therefore, without loss of generality, we may assume that R is acyclic, and that

$$\mathcal{D}(R \restriction X) = \mathcal{D}(R \restriction Y) = \emptyset. \tag{2.14}$$

Recall that R^+ is the smallest transitive relation containing R. If $x \in X \cup Y$ and $R^+(x) \subseteq X$, then $R \restriction R^+(x)$ would witness that $\mathcal{D}(R \restriction X) \neq \emptyset$, against (2.14). Similarly $R^+(x)$ cannot be included in Y. Therefore

$$\forall x \in X \cup Y \left(R^+(x) \nsubseteq X \text{ and } R^+(x) \nsubseteq Y\right). \tag{2.15}$$

Here is the idea of the proof. By (2.14), any R-chain $(z_n)_{n\in\omega}$ must visit both X and Y infinitely often, so $(z_n)_{n\in\omega}$ can be seen as the careful merging of two sequences $(x_n)_{n\in\omega}$ in X and $(y_n)_{n\in\omega}$ in Y. The sequence $(x_n)_{n\in\omega}$ is obtained by applying $\mathsf{DC}(X)$ to a total relation R_X on X such that $R \restriction X \subseteq R_X \subseteq R^+$. Using $(x_n)_{n\in\omega}$, a suitable total relation R_Y on some $Y' \subseteq Y$ is defined, and by $\mathsf{DC}(Y)$ the required sequence $(y_n)_{n\in\omega}$ is obtained.

Let R_X be the relation on X given by $R \restriction X$, together with all pairs (x, x') such that $x \mathrel{R} y_0 \mathrel{R} y_1 \mathrel{R} \cdots \mathrel{R} y_n \mathrel{R} x'$ for some finite sequence of elements of Y:

$$\begin{aligned} R_X := (R \restriction X) \cup \Big\{&(x, x') \in X^2 \mid \exists m \geq 1\, \exists s \in {}^m Y \\ &(x \mathrel{R} s(0) \text{ and } s(m-1) \mathrel{R} x' \text{ and } \forall i < m-1\, (s(i) \mathrel{R} s(i+1)))\Big\}. \end{aligned}$$

It is immediate that $R_X \subseteq R^+$.

Claim 2.5.2.1. *R_X is total on X.*

Proof. We must show that $\mathrm{dom}(R_X) = X$. Let $x \in X$. If $R(x) \cap X \neq \emptyset$, then $x \in \mathrm{dom}(R \restriction X) \subseteq \mathrm{dom}(R_X)$. Now suppose otherwise. By (2.15) $R^+(x) \nsubseteq Y$, so there are $y_0, \ldots, y_n \in Y$ and $x' \in X$ such that $x \mathrel{R} y_0 \mathrel{R} \ldots \mathrel{R} y_n \mathrel{R} x'$. Thus $(x, x') \in R_X$, so $x \in \mathrm{dom}(R_X)$. □

By $\mathsf{DC}(X)$ there is an R_X-chain $(x_n)_{n\in\omega}$.

Claim 2.5.2.2. $\forall n \, \exists m > n \, \neg(x_m \mathrel{R} x_{m+1})$.

Proof. Towards a contradiction, suppose that there is $\bar{n} \in \omega$ such that $x_m \mathrel{R} x_{m+1}$ for every $m \geq \bar{n}$. Then $R \restriction \{x_m \mid m \geq \bar{n}\}$ is total on $\{x_m \mid m \geq \bar{n}\}$ and contained in $R \restriction X$, against (2.14). □

Let $(n_k)_{k\in\omega}$ be the increasing sequence enumerating the set of ms such that $\neg(x_m \mathrel{R} x_{m+1})$. By the definition of R_X, each x_{n_k} is linked to x_{n_k+1} via R through some finite path in Y, and let Y_k be the collection of all places visited by these paths:

$$Y_k := \bigcup \Big\{ \mathrm{ran}(s) \mid \exists m \left(s \in {}^{m+1}Y \text{ and } x_{n_k} \mathrel{R} s(0) \text{ and } s(m) \mathrel{R} x_{n_k+1} \text{ and } \forall i < m \, (s(i) \mathrel{R} s(i+1)) \right) \Big\}.$$

Claim 2.5.2.3. *The Y_ks are nonempty, pairwise disjoint subsets of Y.*

Proof. For each k we have $(x_{n_k}, x_{n_k+1}) \in R_X \setminus R$. This means that there is some $\langle y_0, \ldots, y_m \rangle \in {}^{<\omega}Y$ such that $x_{n_k} \mathrel{R} y_0 \mathrel{R} \ldots \mathrel{R} y_m \mathrel{R} x_{n_k+1}$. In particular, $Y_k \neq \emptyset$.

Towards a contradiction, suppose there are indices $k < j$ such that $Y_k \cap Y_j \neq \emptyset$. Pick $y \in Y_k \cap Y_j$. Then $y \mathrel{R^+} x_{n_k+1} \mathrel{R^+} x_{n_j} \mathrel{R^+} y$, if $x_{n_k+1} \neq x_{n_j}$, or $y \mathrel{R^+} x_{n_k+1} = x_{n_j} \mathrel{R^+} y$ otherwise. Either way, this contradicts our assumption that R is acyclic. □

Now we let R_Y be the following relation on $\bigcup_{k\in\omega} Y_k$:

$$R_Y := \bigcup_{k\in\omega} (R \restriction Y_k) \cup \bigcup_{k\in\omega} \{(y, y') \in Y_k \times Y_{k+1} \mid y \mathrel{R} x_{n_k+1} \text{ and } x_{n_{k+1}} \mathrel{R} y'\}.$$

It readily follows from the definition that $R_Y \subseteq R^+$.

Claim 2.5.2.4. *R_Y is total on $\bigcup_{k\in\omega} Y_k$.*

Proof. Pick $k \in \omega$ and $y \in Y_k$, towards proving that $y \in \mathrm{dom}(R_Y)$. Then there is a finite sequence $\langle y_0, \dots, y_m\rangle$ of elements of Y_k such that $x_{n_k}\ R\ y_0\ R \cdots R\ y_m\ R\ x_{n_k+1}$, and $y = y_i$ for some $0 \leq i \leq m$. If $i < m$, then $y\ R\ y_{i+1}$. If $i = m$ then $y\ R_Y\ y'$ for any $y' \in Y_{k+1}$ such that $x_{n_{k+1}}\ R\ y'$. In either case $y \in \mathrm{dom}(R_Y)$. □

By $\mathsf{DC}(Y)$, there is an R_Y-chain $(y_n)_{n\in\omega}$. By part (b) of Proposition 2.2.1 we can suppose that $y_0 \in Y_0$ and that $x_{n_0}\ R\ y_0$. As the Y_ks are disjoint, for every n there is a unique k such that $y_n \in Y_k$, and let $i(n)$ be this k.

Claim 2.5.2.5. *The set $I_k = \{n \in \omega \mid i(n) = k\}$ is a finite interval of natural numbers.*

Proof. By definition of R_Y it follows that either $i(n+1) = i(n)$ or else $i(n+1) = i(n) + 1$, so it is enough to show that I_k is finite. Towards a contradiction, suppose $I_{\bar{k}}$ is infinite, for some $\bar{k} \in \omega$. This means that there is $\bar{n}$ such that $i(n) = i(\bar{n})$ for all $n \geq \bar{n}$, that is $\{y_n \mid n \geq \bar{n}\} \subseteq Y_{\bar{k}}$. But then $R \restriction \{y_n \mid n \geq \bar{n}\}$ would be a total on $\{y_n \mid n \geq \bar{n}\}$ and contained in $R \restriction Y$, against (2.14). □

Let $m_k = \max(I_k)$ so that $I_0 = [0; m_0]$ and $I_{k+1} = [m_k + 1; m_{k+1}]$. Then

$$\langle x_0, \dots, x_{n_0}\rangle^\frown\langle y_0, \dots, y_{m_0}\rangle^\frown\langle x_{n_0+1}, \dots, x_{n_1}\rangle^\frown\langle y_{m_0+1}, \dots, y_{m_1}\rangle^\frown \cdots$$
$$\cdots{}^\frown\langle x_{n_k+1}, \dots, x_{n_{k+1}}\rangle^\frown\langle y_{m_k+1}, \dots, y_{m_{k+1}}\rangle^\frown \cdots$$

is the required R-chain. □

2.5.2 The Feferman-Levy model

Feferman and Levy showed that the following statement is consistent relative to ZF:

(FL) $\mathbb{R}$ is the countable union of countable sets.

We refer the reader to [38, p. 142] for an exposition of the Feferman-Levy model. In this section, we prove that FL implies (2.1).

Proposition 2.5.3. FL *implies that if* $\mathsf{DC}(A)$ *holds with* $A \subseteq \mathbb{R}$*, then* A *is countable.*

Lemma 2.5.4. *Assume* FL*. Then there is a sequence of nonempty, countable, pairwise disjoint sets* $(X_n)_{n\in\omega}$ *such that* $\mathbb{R} = \bigcup_n X_n$*, and no infinite subsequence of* $(X_n)_{n\in\omega}$ *has a choice function.*

Proof. Fix a bijection $\pi \colon \mathbb{R} \to \mathbb{R}^\omega$, and for each $m \in \omega$ let $\pi_m \colon \mathbb{R} \to \mathbb{R}$ be defined as $\pi_m(x) = \pi(x)_m$. If $Y \subseteq \mathbb{R}$ and $f \colon \omega \to Y$ is surjective, then $\tilde{Y}$, the closure of Y under the π_ms, is also countable, as

$$\tilde{f} \colon {}^{<\omega}\omega \times \omega \to \tilde{Y} \qquad (\langle n_0, \dots, n_k \rangle, m) \mapsto \pi_{n_k} \circ \cdots \circ \pi_{n_0} \circ f(m)$$

is surjective. By FL, let $(Y_n)_{n\in\omega}$ be a sequence of countable sets such that $\mathbb{R} = \bigcup_n Y_n$, and without loss of generality we may assume that each Y_n is closed under every π_m. Then let $X_n = Y_n \setminus \bigcup_{m<n} Y_m$ for each $n \in \omega$. If necessary, we can pass to a subsequence to get them to be nonempty.

We claim that no infinite subsequence of $(X_n)_{n\in\omega}$ has a choice function. Otherwise there would be an infinite sequence $(x_n)_{n\in\omega} \in \mathbb{R}^\omega$ whose range intersects infinitely many X_ns. Let $x \in \mathbb{R}$ be such that $\pi(x) = (x_n)_{n\in\omega}$. Then $x \in X_k \subseteq Y_k$ for some $k \in \omega$, and hence

$$\forall n \in \omega \ (x_n = \pi_n(x) \in Y_k \subseteq X_0 \cup \cdots \cup X_k)$$

as Y_k is closed under the π_ns. But this contradicts the assumption that $\{x_n \mid n \in \omega\}$ intersects infinitely many X_ns. □

Proof of Proposition 2.5.3. Fix $(X_n)_{n\in\omega}$ as in Lemma 2.5.4. Let $A \subseteq \mathbb{R}$ such that $\mathsf{DC}(A)$ holds, and let $I = \{n \in \omega \mid A \cap X_n \neq \emptyset\}$. If I is infinite then, by part (c) of Proposition 2.2.1, $\mathsf{DC}(A)$ would imply the existence of a choice function for the family $\{A \cap X_n \mid n \in I\}$, which is, in particular, a choice function for $\{X_n \mid n \in I\}$, against Lemma 2.5.4. So I must be finite, that is $A \subseteq X_0 \cup \cdots \cup X_k$ for some k. But the finite union of countable sets is countable, so A is countable. □

Corollary 2.5.5. *Assume* FL*. Then* $\forall X\,(\mathsf{DC}(X) \Rightarrow \mathsf{AC}_\omega(X))$

Proof. Immediate by Proposition 2.2.6 and Proposition 2.5.3. □

2.5.3 Definability of the counterexample

Theorem 2.1.1 shows that the negation of (2.1) is consistent with ZF. The set A constructed in the proof of Theorem 2.1.1, witnessing the failure of (2.1), is a set of Cohen reals, so it is not ordinal definable in our model. But what is the possible descriptive complexity of a set A as above?

Recall that a subset of $\mathbb{R}$ is $\boldsymbol{\Pi}^1_n$ if it is the complement of a $\boldsymbol{\Sigma}^1_n$, and it is $\boldsymbol{\Sigma}^1_n$ if it is the projection of a $\boldsymbol{\Pi}^1_{n-1}$ set $C \subseteq \mathbb{R} \times \mathbb{R}$, where $\boldsymbol{\Pi}^1_0$ is the collection of closed sets. The lightface hierarchy Σ^1_n, Π^1_n is obtained by replacing $\boldsymbol{\Pi}^1_0$ with Π^1_0, the collection of recursively-closed sets, see [41, Ch. 3, §12]. Working in ZF, every $\boldsymbol{\Sigma}^1_2$ set is either well-orderable, or else it contains a perfect set by a theorem of Mansfield and Solovay [41, Ch. 3, Corollary 14.9]. But by part (c) of Proposition 2.2.5, the set A cannot contain a perfect set. Hence, we obtain:

Corollary 2.5.6. *If* $A \subseteq \mathbb{R}$ *is* $\boldsymbol{\Sigma}^1_2$ *and* $\mathsf{DC}(A)$ *holds, then* $\mathsf{AC}_\omega(A)$*.*

We conclude with the following question.

Question 2.5.7. Is it consistent with ZF that there is a Π^1_2 set $A \subseteq \mathbb{R}$ such that $\mathsf{DC}(A)$ and $\neg\mathsf{AC}_\omega(A)$?

Chapter 3

Squares and ladders

3.1 Introduction

We use Grätzer's monograph [29] as our reference for all classical definitions and results in lattice theory. For a positive integer n, an *n-ladder*[1] is a lattice whose principal ideals are finite (i.e. a lower-finite lattice) and whose elements have at most n lower covers [29]—see Chapter 0 for the definition of lower cover. In 1984, S. Z. Ditor proved that every $(n+1)$-ladder, for some $n \in \omega$, has cardinality at most $\aleph_n$ [22]. More generally, he proved that every join-semilattice of breadth at most $n+1$ whose principal ideals have cardinality $< \kappa$, for some infinite cardinal κ, has cardinality $\leq \kappa^{+n}$ (see Theorem 3.2.1). He then posed the following questions, which ask whether his cardinal bounds are sharp ([22] and [29, p. 291]):

(A) *For each $n \in \omega$ and infinite cardinal κ, is there a join-semilattice of breadth $n+1$ and cardinality κ^{+n} whose principal ideals have cardinality $< \kappa$?*

(B) *For each $n \in \omega$, is there an $(n+1)$-ladder of cardinality $\aleph_n$?*

Since every n-ladder has breadth at most n [22, Proposition 4.1] (see Lemma 3.3.3 for a more general result), (B) can be regarded as a more demanding version of (A) when $\kappa = \aleph_0$.

[1] This notion was first introduced by S. Z. Ditor in [22] as *n-lattice*. We stick to Grätzer's terminology and refer the reader to [74, p. 387] for a brief history of the name *n-ladder*.

The case $n = 0$ is trivial for both (A) and (B). Ditor answered both questions positively when $n = 1$ and κ is regular. Recently, progress was made in addressing the remaining cases. F. Wehrung [73] provided a positive answer in ZFC to question (A) when κ is uncountable regular, and, under some additional set-theoretic assumptions, to question (B) when $n = 2$. These set-theoretic assumptions consist of either $\mathsf{MA}(\aleph_1; \text{precaliber } \aleph_1)$—i.e. a weakening of Martin's axiom $\mathsf{MA}(\aleph_1)$—or the existence of an $(\omega_1, 1)$-morass. Notably, the non-existence of $(\omega_1, 1)$-morasses implies that ω_2 is inaccessible in L [20], and therefore the same holds for the non-existence of 3-ladders of size $\aleph_2$.

In this chapter, we introduce a generalization of the notion of n-ladder that encompasses join-semilattices that are not lower-finite. In particular, we introduce the concepts of (n, κ)-semiladder and (n, κ)-ladder, where κ is an infinite cardinal and $n \in \omega$ (Definition 3.3.5). By definition, an (n, κ)-semiladder (resp. (n, κ)-ladder) is a join-semilattice (resp. lattice) whose principal ideals have cardinality $< \kappa$, and that satisfies a property similar to "every element has at most n lower covers". The notion of $(n, \aleph_0)$-ladder coincides with the standard notion of n-ladder, and each (n, κ)-semiladder has breadth at most n (Lemma 3.3.3).

Here are the main results of this chapter:

Theorem 3.1.1. *Let κ be an uncountable regular cardinal and $n \in \omega$. Then, there exists an $(n + 1, \kappa)$-semiladder of cardinality κ^{+n}.*

Theorem 3.1.2. *Let κ be a singular cardinal and $n \in \omega$. If $\boxminus_{\kappa^{+m}, \geq \mathrm{cf}(\kappa)}$ holds for every $m < n$, then there exists an $(n + 1, \kappa)$-semiladder of cardinality κ^{+n}.*

Theorem 3.1.3. *Let κ be an infinite cardinal and $n \in \omega$. If $\square_{\kappa^{+m}}$ holds for every $m < n$, then there exists an $(n + 1, \kappa)$-ladder of cardinality κ^{+n}.*

In particular, Theorem 3.1.3 implies that Ditor's questions (A) and (B) have positive answers in the constructible universe:

Corollary 3.1.4. *Assume* $\mathrm{V} = \mathrm{L}$. *Then:*

a) *For every $n \in \omega$ and infinite cardinal κ, there exists a join-semilattice of breadth $n + 1$ and cardinality κ^{+n} whose principal ideals have cardinality $< \kappa$.*

b) *For every* $n \in \omega$*, there exists an* $(n+1)$*-ladder of cardinality* $\aleph_n$*.*

Furthermore, since it is well-known that the failure of $\square_{\omega_1}$ implies that ω_2 is Mahlo in L, we obtain the following strengthening of Wehrung's consistency result as an immediate corollary of Theorem 3.1.3:

Corollary 3.1.5. *If there is no* 3*-ladder of cardinality* $\aleph_2$*, then* ω_2 *is Mahlo in* L*.*

In Section 3.2, we discuss some preliminary results. In Section 3.3 we introduce and examine (n, κ)-ladders. In Section 3.4, we prove Theorem 3.1.1. In Section 3.5 we introduce the notion of *special* (n, κ)-ladder, which is crucial for employing the square's machinery to prove Theorems 3.1.2 and 3.1.3. The proofs of Theorems 3.1.2 and 3.1.3 are contained in Sections 3.6 and 3.7, respectively.

3.2 Ditor's Theorem

We have already mentioned the following theorem at the beginning of the introduction, as it underpins questions (A) and (B). It shows that the breadth, together with cardinality of the principal ideals, provides a neat upper bound on the cardinality of the join-semilattice.

Theorem 3.2.1 (S. Z. Ditor, [22])**.** *Given some* $n \in \omega$ *and an infinite cardinal* κ*, if* P *is a join-semilattice of breadth at most* $n+1$ *whose principal ideals have cardinality* $< \kappa$*, then*

(a) $|P| \leq \kappa^{+n}$*, and*

(b) $|I| < \kappa^{+n}$ *for every proper ideal* I *of* P*.*

As an application of Theorem 3.2.1, let us prove the following proposition, which states that every join-semilattice witnessing the sharpness of Theorem 3.2.1(a) for some n and some cardinal κ, must be $\mathrm{cf}(\kappa)$-directed. Recall that a poset P is said to be μ*-directed*, for some infinite cardinal μ, if every subset of P of cardinality $< \mu$ has an upper bound.

Proposition 3.2.2. *Given some $n \in \omega$ and an infinite cardinal κ, every join-semilattice of cardinality κ^{+n}, breadth $n+1$ and whose principal ideals have cardinality $< \kappa$ is* $\mathrm{cf}(\kappa)$*-directed.*

Proof. We prove the statement by contraposition. Let $(P, \vee)$ be a join-semilattice of breadth at most $n+1$ and whose principal ideals have cardinality $< \kappa$ which is not $\mathrm{cf}(\kappa)$-directed, towards showing that $|P| < \kappa^{+n}$.

If $n = 0$ the claim is easy. Indeed, in this case, P is a linear order, and saying that it is not $\mathrm{cf}(\kappa)$-directed is equivalent to saying that it has a cofinal subset of cardinality $< \mathrm{cf}(\kappa)$. But this, together with the fact that every principal ideal has cardinality $< \kappa$, implies that P itself has cardinality $< \kappa$.

So assume $n > 0$ and fix a subset $A \subseteq P$ of cardinality $< \mathrm{cf}(\kappa)$ which is unbounded above. Let J be the ideal generated by A, i.e.

$$J := \bigcup \left\{ \downarrow \bigvee F \;\middle|\; F \in [A]^{<\omega} \right\}.$$

Let us focus on the quotient P/J. Since $|P| \leq \max(|P/J|, \kappa)$, we are done once we show that $|P/J| < \kappa^{+n}$. Let us first observe that every principal ideal of P/J has cardinality $< \kappa$. Indeed, for every $p \in P$,

$$\downarrow [p]_J = \pi_J \left[\bigcup \left\{ \downarrow \bigvee \left(F \cup \{p\} \right) \;\middle|\; F \in [A]^{<\omega} \right\} \right] \tag{3.1}$$

and the cardinality of the set on the right-hand side of (3.1) is $< \kappa$, being the union of $< \mathrm{cf}(\kappa)$-many sets of cardinality $< \kappa$. Moreover, we claim that P/J has breadth at most n. This, together with Ditor's Theorem 3.2.1(a), implies that $|P/J| \leq \kappa^{+n-1} < \kappa^{+n}$.

Suppose towards a contradiction that there exists a set $X \in [P/J]^{n+1}$ such that for all $Y \in [X]^n$, $\bigvee Y \neq \bigvee X$. Fix a $K \in [P]^{n+1}$ such that $\pi_J[K] = X$. Now fix any $p \in A$. It follows from our assumption on X and from the definition of the quotient join that, for every $L \in [K]^n$,

$$\begin{aligned} \pi_J \left(p \vee \bigvee K \right) = \pi_J(p) \vee \bigvee \pi_J[K] &= \bigvee X \\ &\neq \bigvee \pi_J[L] \\ &= \pi_J \left(\bigvee L \right) \vee \pi_J(p) = \pi_J \left(p \vee \bigvee L \right) \end{aligned}$$

In particular, $p \vee \bigvee K \neq p \vee \bigvee L$ for every $L \in [K]^n$. Consequently, $p \notin K$ or, equivalently, $K \cup \{p\}$ has size $n+2$. Since P has breadth at most $n+1$ by hypothesis, at least one of the following must hold:

a) $\bigvee K = p \vee \bigvee K$.

b) There is an $L \in [K]^n$ such that $p \vee \bigvee L = p \vee \bigvee K$

Only a) is possible by our previous considerations. Thus, we have shown that for every $p \in A$, $\bigvee K = p \vee \bigvee K$ or, equivalently, that $\bigvee K$ is an upper bound of A. However, we assumed A to be unbounded above, hence the contradiction. □

3.3 Generalizing ladders

An *n-ladder* is a lower-finite lattice whose elements have at most n lower covers [29]. For lower-finite lattices, the property of having at most n lower covers for each element is (strictly) stronger than the property of having breadth at most n [22, Proposition 4.1]. However, this implication does not hold in general for join-semilattices that are not lower-finite. For example, consider the set of rational numbers $\mathbb{Q}$ with its usual ordering: it is, in particular, a join-semilattice of breadth 1, yet no rational number has a lower cover.

In this section, we present a notion that generalizes the one of n-ladders by comprising (join-semi)lattices that are not lower-finite, while retaining all the main features of n-ladders. This notion is being *$(n+1)_\top$-free*.

Given a set X, we slightly modify[2] Davey and Priestley's notation [19] and denote by $X_\top$ the join-semilattice whose domain is $X \sqcup \{\mathbf{1}\}$ and such that $x \vee y = \mathbf{1}$ for all distinct $x, y \in X \sqcup \{\mathbf{1}\}$. For example, given an $n \in \omega$, the Hasse diagram of $n_\top$ is depicted in Figure 3.1.

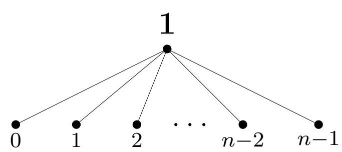

Fig. 3.1 Hasse diagram of $n_\top$

[2]The correct notation would be $\overline{X}_\top$, but it quickly becomes cumbersome when the underlying set (i.e. X) has a long expression.

The join-semilattice $n_\top$ is the unique, up to isomorphism, join-semilattice of cardinality $n+1$ and length[3] at most 1. We call a join-semilattice $n_\top$*-free* if it does not have a sub-join-semilattice isomorphic to $n_\top$.

Let us also introduce the notion of *lower covering*, not to be confused with the one of lower cover. Its connection to $n_\top$-free join-semilattices is the content of Proposition 3.3.2.

Definition 3.3.1. Given a join-semilattice $(P, \leq)$ and an element $x \in P$, a *lower covering of* x is a set $\mathcal{I}$ of ideals of P such that $\{y \in P \mid y < x\} = \bigcup \mathcal{I}$.

Every upper-bounded ideal of a lower-finite join-semilattice is a principal ideal. Therefore, if P is a lower-finite join-semilattice, then every lower covering of some $p \in P$ is made of principal ideals. In particular, the assertion "p has at most n lower covers" is, in this case, equivalent to "p has a lower covering of size at most n".

Proposition 3.3.2. *Given a join-semilattice* $(P, \leq)$ *and an* $n \in \omega$*, the following are equivalent:*

(1) *Every element of* P *has a lower covering of size at most* n*.*

(2) P *is* $(n+1)_\top$*-free.*

Proof. (1)⇒(2): Suppose that every element of P has a lower covering of size at most n and assume towards a contradiction that $\phi : (n+1)_\top \to P$ is an embedding. By assumption, we can pick a lower covering $\mathcal{I}$ of $\phi(\mathbf{1})$ of size at most n. By the pigeonhole principle, there must be some $I \in \mathcal{I}$ and two distinct $i, j \leq n$ such that $\phi(i), \phi(j) \in I$. Since I is an ideal, $\phi(i) \vee \phi(j) \in I$, but $\phi(i) \vee \phi(j) = \phi(i \vee j) = \phi(\mathbf{1})$, and hence $\phi(\mathbf{1}) \in I$, which is a contradiction, as it means that $\phi(\mathbf{1}) < \phi(\mathbf{1})$.

(2)⇒(1): Suppose that P is $(n+1)_\top$-free. Pick any $x \in P$ and let m be the greatest natural number such that there exists an embedding $\phi : m_\top \to P$ with $\phi(\mathbf{1}) = x$. Clearly, $m \leq n$. Fix an embedding $\phi : m_\top \to P$ with $\phi(\mathbf{1}) = x$. Now, for each $k < m$ let

$$I_k := \{y \in P \mid y \vee \phi(k) < x\}.$$

[3] The length of a poset P is defined as $\sup\{|C| - 1 \mid C \text{ is a chain of } P\}$.

We claim that $\{I_k \mid k < m\}$ is a lower covering of x. Let us first prove that the I_ks are ideals of P. They are all downward-closed. We now show that I_{m-1} is closed under joins, as the same argument applies to all I_ks. Towards a contradiction, suppose that there are $y_0, y_1 \in I_{m-1}$ such that $y_0 \vee y_1 \notin I_{m-1}$, or equivalently, such that $y_0 \vee y_1 \vee \phi(m-1) = x$. Let $\varphi : (m+1)_\top \to P$ be the map defined by: $\varphi(i) = \phi(i)$ for every $i < m-1$; $\varphi(m-1) = y_0 \vee \phi(m-1)$ and $\varphi(m) = y_1 \vee \phi(m-1)$. Then, φ is an embedding, against the maximality of m.

We left to show that $\bigcup_{k<m} I_k = \{y \in P \mid y < x\}$. Pick some $y < x$ and assume towards a contradiction that $y \notin I_k$ for every $k < m$. Equivalently, $y \vee \phi(k) = x$ for every $k < m$. We can extend ϕ to $\varphi : (m+1)_\top \to P$ by setting $\varphi(k) = \phi(k)$ for every $k < m$ and $\varphi(m) = y$. Again, φ is an embedding, against the maximality of m. □

Lemma 3.3.3. *An $(n+1)_\top$-free join-semilattice has breadth at most n.*

Proof. By contraposition, let $(P, \leq)$ be a join-semilattice of breadth greater than n, towards showing that it is not $(n+1)_\top$-free. Fix some $X \in [P]^{n+1}$ such that for every $Y \in [X]^n$, $\bigvee X \neq \bigvee Y$. Fix an enumeration $S_0, S_1, \dots, S_n$ of $[X]^n$. Consider the map $\phi : (n+1)_\top \to P$ defined by $\phi(\mathbf{1}) = \bigvee X$ and $\phi(k) = \bigvee S_k$ for every $k \leq n$. It immediately follows that ϕ is an embedding. □

Suppose that an element of a join-semilattice has a finite lower covering. In that case, the following lemma tells us that, among the finite lower coverings of the given element, one exists that is least with respect to the inclusion relation.

Lemma 3.3.4. *Given a join-semilattice $(P, \leq)$ and some $p \in P$, if p has a finite lower covering, then there exists a (unique) lower covering $\mathcal{I}$ of p such that $\mathcal{I} \subseteq \mathcal{J}$ for every finite lower covering $\mathcal{J}$ of p.*

Proof. Fix a finite lower covering $\mathcal{I}$ of p which is $\subseteq$-minimal—i.e. such that there is no finite lower covering $\mathcal{I}'$ of p with $\mathcal{I}' \subsetneq \mathcal{I}$. We want to prove that $\mathcal{I} \subseteq \mathcal{J}$ for every finite lower covering $\mathcal{J}$ of p. To this end, fix an $I \in \mathcal{I}$ and some finite covering $\mathcal{J}$ of p towards showing that $I \in \mathcal{J}$.

We first claim that there exists some $J \in \mathcal{J}$ such that $I \subseteq J$. Suppose otherwise towards a contradiction and for each $J \in \mathcal{J}$ pick a $p_J \in I \setminus J$. Then, $\bigvee\{p_J \mid J \in \mathcal{J}\} \in I$, since $\mathcal{J}$ is finite and I is closed under joins. Moreover, since

$\mathcal{J}$ is a lower covering of p, there exists $W \in \mathcal{J}$ such that $\bigvee\{p_J \mid J \in \mathcal{J}\} \in W$. As W is downward-closed, we must have $p_W \in W$, hence the contradiction.

So fix a $J \in \mathcal{J}$ such that $I \subseteq J$. We now claim that $I = J$. By the same argument of the previous paragraph, an $I' \in \mathcal{I}$ exists such that $J \subseteq I'$. In particular, $I \subseteq J \subseteq I'$. If $I \subsetneq I'$, then we contradict the minimality of $\mathcal{I}$, as the family $\mathcal{I} \setminus \{I\}$ would still be a lower covering of p. Hence, $I = I'$, and therefore $I = J \in \mathcal{J}$. □

We are ready to introduce the main notion of this chapter.

Definition 3.3.5. Given an $n \in \omega$ and an infinite cardinal κ, a nonempty join-semilattice (resp. lattice) is said to be an (n, κ)*-semiladder* (resp. (n, κ)*-ladder*) if it is $(n+1)_{\mathsf{T}}$-free and its principal ideals have cardinality $< \kappa$.

In particular, it directly follows from Proposition 3.3.2 and our remark after Definition 3.3.1 that the notion of $(n, \aleph_0)$-ladder coincides with the one of n-ladder.

The notion of being n_{T}-free behaves as expected with respect to products: for every two join-semilattices P and Q that are n_{T}-free and m_{T}-free, respectively, their product $P \times Q$ (with the product ordering) is $(n+m-1)_{\mathsf{T}}$-free. Therefore, the product of an (n, κ)-semiladder with an (m, λ)-semiladder is an $(n+m, \max(\kappa, \lambda))$-semiladder. However, in the following sections, we will be interested in combining an (n, κ)-semiladder with an (m, λ)-semiladder to produce an $(n+m, \min(\kappa, \lambda))$-semiladder. This is done by using the following weaker notion of product.

Definition 3.3.6. Let $(X, \vee)$ and $(Y, \vee)$ be two join-semilattices. A join-semilattice $(X \times Y, \vee)$ is said to be a *quasi-product* of X and Y if the following conditions hold:

(q1) For every $x \in X$, the map $F_x : Y \to X \times Y$, $y \mapsto (x, y)$ is an order-embedding.

(q2) The canonical projection $\pi_X : X \times Y \to X$ is an homomorphism.

Note that if we were to substitute (q1) with the stronger requirement "The canonical projection π_Y is a homomorphism," then we would end up with the standard definition of product join-semilattice.

Lemma 3.3.7. *For every two join-semilattices X, Y, for every $x \in X$ and every quasi-product of X and Y, the map F_x is an embedding.*

Proof. Fix a quasi-product $(X \times Y, \vee)$ of X and Y. Fix also $x \in X$ and $y, z \in Y$ towards showing that $(x, y) \vee (x, z) = (x, y \vee z)$. Let $(p, q) \in X \times Y$ be such that $(x, y) \vee (x, z) = (p, q)$. From (q2), it follows that $x = p$. But then, it immediately follows from (q1) that $q = y \vee z$. □

The following proposition shows that the notion of being $n_\top$-free behaves also well with respect to quasi-products.

Proposition 3.3.8. *Let X and Y be two join-semilattices and let n, m be positive integers such that X and Y are $n_\top$-free and $m_\top$-free, respectively. Then, every quasi-product of X and Y is $(n+m-1)_\top$-free.*

Proof. Fix a quasi-product $(X \times Y, \vee)$ of X and Y and suppose by contraposition that there exists a $k \in \omega$ and an embedding $\varphi : k_\top \to X \times Y$ towards finding $n, m \in \omega$ and two embeddings $\varphi_X : n_\top \to X$ and $\varphi_Y : m_\top \to Y$ such that $n + m = k$.

Let
$$A := \Big\{ l < k \mid \pi_X \circ \varphi(l) = \pi_X \circ \varphi(\mathbf{1}) \Big\}.$$
By treating $A_\top$ and $(k \setminus A)_\top$ as sub-join-semilattices of $k_\top$, we define the maps $\varphi_X : (k \setminus A)_\top \to X$ and $\varphi_Y : A_\top \to Y$ as $\pi_X \circ \varphi \restriction (k \setminus A)_\top$ and $\pi_Y \circ \varphi \restriction A_\top$, respectively. We claim that both maps are embeddings, which suffices to finish the proof.

By definition of A, we have $\pi_Y \circ \varphi \restriction A_\top = F^{-1}_{\pi_X \circ \varphi(\mathbf{1})} \circ \varphi \restriction A_\top$. Since φ is an embedding by hypothesis, and $F^{-1}_{\pi_X \circ \varphi(\mathbf{1})} \restriction \varphi[A_\top]$ is an embedding by Lemma 3.3.7, we conclude that φ_Y is embedding.

Since, by (q2), π_X is an homomorphism, the map φ_X is an homomorphism. Therefore, in order to show that φ_X is an embedding, it suffices to prove that $\pi_X \restriction \varphi[(k \setminus A)_\top]$ is injective. Suppose otherwise. Then there must exist two distinct $i, j \in k \setminus A$ such that $\pi_X \circ \varphi(i) = \pi_X \circ \varphi(j)$. Then, by (q2), $\pi_X(\varphi(i) \vee \varphi(j)) = \pi_X \circ \varphi(i)$, but $\varphi(i) \vee \varphi(j) = \varphi(\mathbf{1})$, and thus $\pi_X \circ \varphi(i) = \pi_X \circ \varphi(\mathbf{1})$ which contradicts $i \notin A$. □

3.4 Proof of Theorem 3.1.1

This section is devoted to the proof of Theorem 3.1.1. The following proposition does all the work.

Proposition 3.4.1. *Given an uncountable regular cardinal κ, an $n \in \omega$ and a well-founded (n, κ^+)-semiladder $(P, \leq)$, there exists a quasi-product of P and κ whose principal ideals have cardinality $< \kappa$.*

Proof. We define inductively a system $\langle \trianglelefteq_p \mid p \in P\rangle$ of join-semilattice orderings such that:

i) $\mathrm{dom}(\trianglelefteq_p) = (P \downarrow p) \times \kappa$.

ii) $(p, \alpha) \trianglelefteq_p (p, \beta)$ if and only if $\alpha \leq \beta$.

iii) The principal ideals of $\trianglelefteq_p$ have cardinality $< \kappa$.

iv) If $p \leq q$ then $\trianglelefteq_p = \trianglelefteq_q \restriction \mathrm{dom}(\trianglelefteq_p)$ and $\mathrm{dom}(\trianglelefteq_p)$ is an ideal of $\trianglelefteq_q$.

Suppose we have defined such a system and let $\trianglelefteq := \bigcup_{p \in P} \trianglelefteq_p$. We denote the join operator associated to $\trianglelefteq_p$ and $\trianglelefteq$ by $\vee_p$ and $\vee$, respectively. We also denote the join operator of P by $\vee$, but this does not lead to ambiguities in what follows.

Claim 3.4.1.1. *$(P \times \kappa, \trianglelefteq)$ is a quasi-product of P and κ whose principal ideals have cardinality $< \kappa$.*

Proof. By condition iv), $(P \times \kappa, \trianglelefteq)$ is a direct limit of join-semilattices. In particular, $(P \times \kappa, \trianglelefteq)$ is a join-semilattice. Furthermore, as a direct consequence of iii), its principal ideals have cardinality less than κ.

Let us show that $(P \times \kappa, \trianglelefteq)$ is a quasi-product of P and κ. Condition (q1) directly follows from ii). In order to show that (q2) holds, fix (p_0, α_0) and (p_1, α_1) in $P \times \kappa$ towards proving that there exists a $\beta \in \kappa$ with $(p_0 \vee p_1, \beta) = (p_0, \alpha_0) \vee (p_1, \alpha_1)$. Let $q \in P$ and $\beta \in \kappa$ be such that $(p_0, \alpha_0) \vee (p_1, \alpha_1) = (q, \beta)$. By i), (p_0, α_0) and (p_1, α_1) belong to the domain of $\trianglelefteq_{p_0 \vee p_1}$. By iv), also their join (in $(P \times \kappa, \trianglelefteq)$) belongs to the domain of $\trianglelefteq_{p_0 \vee p_1}$. In particular, $q \leq p_0 \vee p_1$. On the other hand, by iv) again, (p_0, α_0) and (p_1, α_1) must belong to the domain of $\trianglelefteq_q$, and thus, by i), $p_0, p_1 \leq q$, or equivalently $p_0 \vee p_1 \leq q$. Therefore $q = p_0 \vee p_1$ as we wanted to show. □

We proceed with the inductive definition. We carry out the induction over $(P, \leq)$, i.e. over the set of indices ordered by $\leq$, which is well-founded by hypothesis. First let $(p, \alpha) \trianglelefteq_p (p, \beta)$ for every minimal $p \in P$ and every $\alpha \leq \beta$.

Now fix a $p \in P$ and suppose that we have defined $\trianglelefteq_q$ for every $q < p$, towards defining $\trianglelefteq_p$. By hypothesis and Proposition 3.3.2, p has a lower covering of size at most n. We assume for simplicity that the lower covering of p has size exactly n. Thus, fix a lower covering $\{I_0, I_1, \ldots, I_{n-1}\}$ of p.

Claim 3.4.1.2. *There exists a sequence $\langle J^i_\alpha \mid i < n, \alpha < \kappa \rangle$ such that, for every $q, z \in P$ and $\alpha, \beta, \gamma < \kappa$ and $i < n$:*

a) $\bigcup_{\alpha \in \kappa} J^i_\alpha = I_i \times \kappa$.

b) *If $\alpha \leq \beta$, then $J^i_\alpha \subseteq J^i_\beta$.*

c) $|J^i_\alpha| < \kappa$.

d) *If $(q, \beta) \trianglelefteq_z (z, \gamma) \in J^i_\alpha$, then $(q, \beta) \in J^i_\alpha$.*

e) *If $q \vee z \in I_i$ and $(q, \beta), (z, \gamma) \in J^0_\alpha \cup \ldots \cup J^{n-1}_\alpha$, then $(q, \beta) \vee_{q \vee z} (z, \gamma) \in J^i_\alpha$.*

Proof. We construct this sequence by induction on $\alpha < \kappa$. First, for each $i < n$ fix an enumeration $\langle (q^i_\alpha, \gamma^i_\alpha) \mid \alpha < \kappa \rangle$ of $I_i \times \kappa$. Set $J^i_0 = \trianglelefteq_{q^i_0} \downarrow (q^i_0, \gamma^i_0)$. Suppose we have defined J^i_β for every $i < n$ and $\beta < \alpha$. Define the sequence $\langle H^i_k \mid i < n, k \in \omega \rangle$ as follows: for each $i < n$, let

$$H^i_0 = \left(\trianglelefteq_{q^i_\alpha} \downarrow (q^i_\alpha, \gamma^i_\alpha) \right) \cup \bigcup_{\beta < \alpha} J^i_\beta.$$

Then, for each $i, k \in \omega$, we let inductively

$$H^i_{k+1} = H^i_k \cup \bigcup \Big\{ \trianglelefteq_{z_0 \vee z_1} \downarrow \left((z_0, \beta_0) \vee_{z_0 \vee z_1} (z_1, \beta_1) \right) \Big| \; (z_0, \beta_0), (z_1, \beta_1) \in \bigcup_{l < n} H^l_k \text{ and } z_0 \vee z_1 \in I_i \Big\}.$$

Finally let $J^i_\alpha = \bigcup_{k \in \omega} H^i_k$. The sequence $\langle J^i_\alpha \mid i < n, \alpha \in \kappa \rangle$ satisfies a)-e): properties a) and b) directly follow from our construction; properties d) and e) are guaranteed by the way we defined the H^i_ks; finally, to see that also property c) holds, note that each H^i_k has cardinality less than κ by the regularity of κ, and that J^i_α, being the union of the H^i_ks, has cardinality less than κ since κ is uncountable and regular. □

We are ready to define $\trianglelefteq_p$:

- Let $(p,\alpha) \trianglelefteq_p (p,\beta)$ for every $\alpha \leq \beta < \kappa$.
- Let $\trianglelefteq_p \restriction \operatorname{dom}(\trianglelefteq_q) = \trianglelefteq_q$ for every $q < p$.
- Let $(q,\beta) \trianglelefteq_p (p,\alpha)$ for every $\alpha < \kappa$, $i < n$ and $(q,\beta) \in J^i_\alpha$.

By induction hypothesis, $\trianglelefteq_r = \trianglelefteq_q \restriction \operatorname{dom}(\trianglelefteq_r)$ for every $r \leq q < p$. Therefore $\trianglelefteq_p$ is well-defined.

It follows from the transitivity of $\trianglelefteq_q$ for $q < p$ and from properties b) and d) of the J^i_αs that $\trianglelefteq_p$ is transitive. Since $\trianglelefteq_p$ is reflexive, we conclude it is a partial order. Moreover, for every $\alpha \in \kappa$, we have, by construction,

$$\trianglelefteq_p \downarrow (p,\alpha) = \Big(\{p\} \times (\alpha+1)\Big) \cup \bigcup_{i<n} J^i_\alpha$$

which has cardinality less than κ by c). In particular, every principal ideal of $\trianglelefteq_p$ has cardinality less than κ.

Towards showing that $\trianglelefteq_p$ is an join-semilattice, fix $(q_0,\beta_0),(q_1,\beta_1)$ with $q_0,q_1 \leq p$. Suppose first that $q_0 \vee q_1 < p$. We claim that that $(q_0,\beta_0) \vee_{q_0\vee q_1} (q_1,\beta_1)$ is the $\trianglelefteq_p$-least upper bound of $\{(q_0,\beta_0),(q_1,\beta_1)\}$—note that the claim also implies that $\operatorname{dom}(\trianglelefteq_q)$ is an ideal of $\trianglelefteq_p$ for every $q < p$. It suffices to show that if $(q_0,\beta_0),(q_1,\beta_1) \trianglelefteq_p (p,\alpha)$ for some α, then $(q_0,\beta_0) \vee_{q_0\vee q_1} (q_1,\beta_1) \trianglelefteq_p (p,\alpha)$. By hypothesis and by definition of $\trianglelefteq_p$, there must be $i,j,k < n$ such that $q_0 \vee q_1 \in I_i$, $(q_0,\beta_0) \in J^j_\alpha$ and $(q_1,\beta_1) \in J^k_\alpha$. But then, it follows from property e) of the J^i_αs that $(q_0,\beta_0) \vee_{q_0\vee q_1} (q_1,\beta_1) \in J^i_\alpha$. Thus, $(q_0,\beta_0) \vee_{q_0\vee q_1} (q_1,\beta_1) \trianglelefteq_p (p,\alpha)$ by definition of $\trianglelefteq_p$.

If, on the other hand, $q_0 \vee q_1 = p$, then

$$\Big(p, \min\Big\{\alpha \in \kappa \mid (q_0,\beta_0),(q_1,\beta_1) \trianglelefteq_p (p,\alpha)\Big\}\Big)$$

is easily seen to be the $\trianglelefteq_p$-least upper bound of $\{(q_0, \beta_0), (q_1, \beta_1)\}$—note that, by property a) of the J^i_αs, there always exist an $\alpha \in \kappa$ such that $(q_0, \beta_0), (q_1, \beta_1) \trianglelefteq_p (p, \alpha)$. □

The proof of Theorem 3.1.1 follows by a repeated application of Proposition 3.4.1.

Proof of Theorem 3.1.1. Let κ be an uncountable regular cardinal and $n \in \omega$ some natural number. We inductively define a finite sequence $(P_i)_{i \leq n}$ of join-semilattices such that P_i is a well-founded $(i+1, \kappa^{+n-i})$-semiladder of cardinality κ^{+n} for every $i \leq n$. The definition goes as follows:

- Let $P_0 = \kappa^{+n}$ with its usual ordering—in particular, P_0 is a well-founded $(1, \kappa^{+n})$-semiladder.
- For every $i < n$, let P_{i+1} be a quasi-product of P_i and κ^{+n-i-1} whose principal ideals have cardinality less than κ^{+n-i-1}, which exists by Proposition 3.4.1. Since a quasi-product of two well-founded join-semilattices is still well-founded, and since P_i is well-founded, P_{i+1} is also well-founded. Moreover, as P_i is $(i+2)_\top$-free, it follows from Proposition 3.3.8 that P_{i+1}, being a quasi-product of P_i and a linear order, is $(i+3)_\top$-free. Thus, P_{i+1} is a well-founded $(i+2, \kappa^{+n-i-1})$-semiladder of cardinality κ^{+n}.

At the end, P_n is an $(n+1, \kappa)$-semiladder of cardinality κ^{+n}. □

3.5 Special ladders

In Section 3.4, we have proven Theorem 3.1.1 by iterating the construction of an $(n+1, \kappa)$-semiladder as a quasi-product of P and κ, where P is an (n, κ^+)-semiladder. To prove Theorems 3.1.2 and 3.1.3 we follow a similar strategy: we iterate the construction of an $(n+1, \kappa)$-semiladder as a quasi-product of $|P|^+$ and P for some (n, κ)-semiladder P. These quasi-products are induced by maps from $[|P|^+]^2$ into P that satisfy certain triangular inequalities that are reminiscent[4] of the ones satisfied by Todorčević ρ-functions [69]. Consequently, we adopt his terminology in the following definition.

Definition 3.5.1. Consider a join-semilattice $(B, \leq)$, an ordinal γ and a map $\varrho : [\gamma]^2 \to B$, then:

- ϱ is a said be *transitive* if $\varrho(\alpha, \delta) \leq \varrho(\alpha, \beta) \vee \varrho(\beta, \delta)$ for every $\alpha < \beta < \delta < \gamma$.
- ϱ is said to be *subadditive* if $\varrho(\alpha, \beta) \leq \varrho(\alpha, \delta) \vee \varrho(\beta, \delta)$ for every $\alpha < \beta < \delta < \gamma$.

Given $\alpha < \beta < \gamma$ and a map $\varrho : [\gamma]^2 \to B$, we simply write $\varrho(\alpha, \beta)$ instead of $\varrho(\{\alpha, \beta\})$, and we write $\varrho \restriction \alpha$ instead of $\varrho \restriction [\alpha]^2$. Moreover, it will be convenient to assume that B has a least element and to convene that $\varrho(\alpha, \alpha) = \mathbf{0}$ for every $\alpha < \gamma$.

For each $\alpha < \gamma$ and $p \in B$ we let

$$\begin{aligned} \mathrm{ht}(\alpha, p) &:= \alpha, \\ D_\varrho(\alpha, p) &:= \Big\{\eta < \alpha \mid \varrho(\eta, \alpha) \leq p\Big\}. \end{aligned}$$

We are interested in the following binary relation $\trianglelefteq_\varrho$ on $\gamma \times B$ induced by ϱ: for every $p, q \in B$ and $\alpha, \beta < \gamma$,

$$(\alpha, p) \trianglelefteq_\varrho (\beta, q) \text{ if and only if } \alpha \leq \beta \text{ and } p \vee \varrho(\alpha, \beta) \leq q. \tag{3.2}$$

The relation $\trianglelefteq_\varrho$ is locally close to the product ordering. In particular, if $\varrho : [\gamma]^2 \to B$ is constant with value $\mathbf{0}$, then $\trianglelefteq_\varrho$ is exactly the product ordering of γ and B. The next proposition studies the relationship between the properties of ϱ and the properties of the induced relation $\trianglelefteq_\varrho$. We drop the subscript ϱ from $\trianglelefteq$ and $D(\alpha, p)$ as the map ϱ is fixed.

[4] It is unclear to us if and to what extent our Definition 3.5.1 and the subsequent results fit into Todorčević's *Walks on ordinals* framework [69].

Proposition 3.5.2. *The structure* $(\gamma \times B, \trianglelefteq)$ *is:*

1) *a poset if and only if* ϱ *is transitive.*
2) *a join-semilattice if and only if* ϱ *is transitive and subadditive. In this case, it is a quasi-product of* γ *and* B.
3) *a lattice if and only if* ϱ *is transitive and subadditive,* B *is a lattice,* $\varrho(0,\alpha) = \mathbf{0}$ *and* $D(\alpha, p)$ *is closed in* α *for every* $\alpha < \gamma$ *and* $p \in B$.

Proof. 1): Let us first prove the "only if" direction. Pick some $\alpha < \beta < \delta < \gamma$. By definition of $\trianglelefteq$,

$$(\alpha, \varrho(\alpha,\beta) \vee \varrho(\beta,\delta)) \trianglelefteq (\beta, \varrho(\alpha,\beta) \vee \varrho(\beta,\delta))$$

and

$$(\beta, \varrho(\alpha,\beta) \vee \varrho(\beta,\delta)) \trianglelefteq (\delta, \varrho(\alpha,\beta) \vee \varrho(\beta,\delta)).$$

Since we are assuming that $\trianglelefteq$ is transitive, it follows

$$(\alpha, \varrho(\alpha,\beta) \vee \varrho(\beta,\delta)) \trianglelefteq (\delta, \varrho(\alpha,\beta) \vee \varrho(\beta,\delta)),$$

and therefore, by definition of $\trianglelefteq$, $\varrho(\alpha,\delta) \leq \varrho(\alpha,\beta) \vee \varrho(\beta,\delta)$. Hence, ϱ is transitive.

Now to the "if" direction. Pick $\alpha \leq \beta \leq \delta < \gamma$ and $p, q, r \in B$ such that $(\alpha, p) \trianglelefteq (\beta, q) \trianglelefteq (\delta, r)$, towards showing that $(\alpha, p) \trianglelefteq (\delta, r)$. Since $\leq$ on B is transitive, the only non-trivial case to check is when $\alpha < \beta < \delta$. From the definition of $\trianglelefteq$ and our assumption it follows that $p \leq q \leq r$ and $\varrho(\alpha,\beta) \leq q$ and $\varrho(\beta,\delta) \leq r$. By transitivity of ϱ, $\varrho(\alpha,\delta) \leq \varrho(\alpha,\beta) \vee \varrho(\beta,\delta) \leq r$, and therefore $(\alpha, p) \trianglelefteq (\delta, r)$. Hence, $\trianglelefteq$ is transitive.

2): Let us first prove the "only if" direction. Suppose that $(\gamma \times B, \trianglelefteq)$ is a join-semilattice towards showing that ϱ is subadditive. First, we need the following claim, which amounts to saying that $(\gamma \times B, \trianglelefteq)$ satisfies condition (q2) of being a quasi-product (see Definition 3.3.6):

Claim 3.5.2.1. *The map* $\mathrm{ht} : (\gamma \times B, \trianglelefteq) \to (\gamma, \leq)$ *is an homomorphism.*

Proof. Fix $\alpha \leq \beta < \gamma$ and $p, q \in B$. Let $\eta < \gamma$ and $z \in B$ be such that $(\alpha, p) \vee (\beta, q) = (\eta, z)$. By definition of $\trianglelefteq$, $(\beta, p \vee q \vee \varrho(\alpha,\beta))$ is an upper bound of $\{(\alpha, p), (\beta, q)\}$. Therefore, $(\eta, z) \trianglelefteq (\beta, p \vee q \vee \varrho(\alpha, \beta))$. In particular, $\eta \leq \beta$. On the other hand, as $(\beta, q) \trianglelefteq (\eta, z)$, it follows that $\beta \leq \eta$. Therefore, $\eta = \beta$. We have shown that ht is an homomorphism. □

Note that $(\gamma \times B, \trianglelefteq)$ satisfies also (q1) by definition of $\trianglelefteq$. Therefore, $(\gamma \times B, \trianglelefteq)$ is a quasi-product of γ and B.

Going back to the "only if" direction, fix $\alpha < \beta < \delta < \gamma$ and let $\bar{p} = \varrho(\alpha, \delta) \vee \varrho(\beta, \delta)$. By Claim 3.5.2.1, there exists some $z \in B$ such that

$$(\alpha, \bar{p}) \vee (\beta, \bar{p}) = (\beta, z). \tag{3.3}$$

Note that $\bar{p} \leq z$. By $\bar{p}$'s definition, $(\alpha, \bar{p}), (\beta, \bar{p}) \trianglelefteq (\delta, \bar{p})$. It follows that $(\beta, z) \trianglelefteq (\delta, \bar{p})$. Then, $z \leq \bar{p}$ and we conclude that $z = \bar{p}$. This, together with (3.3), implies that $(\alpha, \bar{p}) \trianglelefteq (\beta, \bar{p})$, and thus $\varrho(\alpha, \beta) \leq \bar{p} = \varrho(\alpha, \delta) \vee \varrho(\beta, \delta)$. We have shown that ϱ is subadditive.

Let us prove the "if" direction. The transitivity of ϱ and case 1) of this proposition imply that $\trianglelefteq$ is a partial order. Now pick $p, q \in B$ and $\alpha \leq \beta < \gamma$ towards finding a $\trianglelefteq$-least upper bound for $\{(\alpha, p), (\beta, q)\}$. We claim that $(\beta, p \vee q \vee \varrho(\alpha, \beta))$ is the $\trianglelefteq$-least upper bound of $\{(\alpha, p), (\beta, q)\}$.

From $\trianglelefteq$'s definition, $(\alpha, p), (\beta, q) \trianglelefteq (\beta, p \vee q \vee \varrho(\alpha, \beta))$. Now suppose that $(\alpha, p), (\beta, q) \trianglelefteq (\delta, r)$ for some $\delta < \gamma$ and $r \in B$. In particular, $\alpha, \beta \leq \delta$ and $p \vee \varrho(\alpha, \delta) \leq r$ and $q \vee \varrho(\beta, \delta) \leq r$. From the subadditivity of ϱ it follows that $\varrho(\alpha, \beta) \leq r$. Altogether, we have $p \vee q \vee \varrho(\alpha, \beta) \vee \varrho(\beta, \delta) \leq r$, and therefore $(\beta, p \vee q \vee \varrho(\alpha, \beta)) \trianglelefteq (\delta, r)$. Thus, our claim is true.

3): Let us first prove the "only if" direction. Towards showing that B is a meet-semilattice, fix some $q, r \in B$ and let us find a $\leq$-greatest lower bound of $\{q, r\}$. Let $p \in B$ and $\alpha < \gamma$ be such that $(\alpha, p) = (0, q) \wedge (0, r)$. By $\trianglelefteq$'s definition, we must have $p \leq q, r$ and $\alpha = 0$. We claim that p is the $\leq$-greatest lower bound of $\{q, r\}$. Pick any $x \in B$ such that $x \leq q, r$. Then, $(0, x) \trianglelefteq (0, q), (0, r)$. It follows from $(0, p)$ being the greatest lower bound of $(0, q)$ and $(0, r)$ that $(0, x) \trianglelefteq (0, p)$. In particular, $x \leq p$. Thus, p is the $\leq$-greatest lower bound of $\{q, r\}$.

Next, fix an $\alpha < \gamma$ toward showing that $\varrho(0, \alpha) = \mathbf{0}$. This immediately follows once we note the following: by definition of $\trianglelefteq$ (see (3.2)), $(0, \mathbf{0})$ is a minimal element of $(\gamma \times B, \trianglelefteq)$; but since we are assuming $(\gamma \times B, \trianglelefteq)$ to be lower-directed, this implies that $(0, \mathbf{0})$ is actually the minimum of $(\gamma \times B, \trianglelefteq)$. Hence, for every $\alpha < \gamma$, $(0, \mathbf{0}) \trianglelefteq (\alpha, \mathbf{0})$ or, equivalently, $\varrho(0, \alpha) = \mathbf{0}$.

Now fix an $\alpha < \gamma$ and some $p \in B$ towards showing that $D(\alpha, p)$ is a closed subset of α. Fix some $\eta < \alpha$ such that $\eta = \sup(D(\alpha, p) \cap \eta)$. We want to prove that $\eta \in D(\alpha, p)$. Fix a $\nu \in D(\alpha, p) \cap \eta$. By definition of D, $\varrho(\nu, \alpha) \leq p$. Since we are assuming $(\gamma \times B, \trianglelefteq)$ to be lattice, it follows from claim 2) of this proposition that ϱ is subadditive. By the subadditivity of ϱ, $\varrho(\nu, \eta) \leq \varrho(\nu, \alpha) \vee \varrho(\eta, \alpha)$. Combining these last observations, we get that $\varrho(\nu, \eta) \leq p \vee \varrho(\eta, \alpha)$. Therefore,

$$\forall \nu \in D(\alpha, p) \cap \eta, \quad (\nu, p) \trianglelefteq (\alpha, p) \wedge (\eta, p \vee \varrho(\eta, \alpha)). \tag{3.4}$$

Let $\mu < \gamma$ and $q \in B$ be such that $(\mu, q) = (\alpha, p) \wedge (\eta, p \vee \varrho(\eta, \alpha))$. Clearly, $\mu \leq \eta$. Moreover, it follows from (3.4) that $\mu \geq \sup D(\alpha, p) \cap \eta$. But since we are assuming $\sup D(\alpha, p) \cap \eta = \eta$, we conclude $\mu = \eta$. From $(\eta, q) \trianglelefteq (\alpha, p)$ and the definition of $\trianglelefteq$, we have $\varrho(\eta, \alpha) \leq p$, or, equivalently, $\eta \in D(\alpha, p)$, as we wanted to show.

Let us prove the "if" direction. Fix $\beta, \delta < \gamma$ with $\beta \leq \delta$ and $q, r \in B$, towards showing that there exists a $\trianglelefteq$-greatest lower bound for $\{(\beta, q), (\delta, r)\}$. If $\beta = \delta$, it is straightforward that $(\beta, q \wedge r)$ is the greatest lower bound. Hence suppose that $\beta < \delta$.

Note that the set $(D(\beta, q) \cup \{\beta\}) \cap D(\delta, r)$ is a closed nonempty subset of δ: it is easy to see that it is closed, as by assumption both $D(\beta, q) \cup \{\beta\}$ and $D(\delta, r)$ are closed in δ; moreover, since $\varrho(0, \beta) = \varrho(0, \delta) = \mathbf{0}$, it follows that $0 \in (D(\beta, q) \cup \{\beta\}) \cap D(\delta, r) \neq \emptyset$.

Let α be the maximum of $(D(\beta, q) \cup \{\beta\}) \cap D(\delta, r)$, which exists since the set is nonempty, closed and bounded in δ. We claim that $(\alpha, q \wedge r)$ is the $\trianglelefteq$-greatest lower bound of $\{(\beta, q), (\delta, r)\}$. Indeed, pick an $\eta < \gamma$ and some $p \in B$ such that $(\eta, p) \trianglelefteq (\beta, q), (\delta, r)$ towards showing that $(\eta, p) \trianglelefteq (\alpha, q \wedge r)$. By definition of $\trianglelefteq$, $p \leq q \wedge r$. Moreover, as η must belong to $(D(\beta, q) \cup \{\beta\}) \cap D(\delta, r)$, we conclude that $\eta \leq \alpha$. If $\eta = \alpha$, then $(\eta, p) = (\alpha, p) \trianglelefteq (\alpha, q \wedge r)$. So suppose

$\eta < \alpha$. The following holds:

$$\varrho(\eta, \alpha) \leq \varrho(\eta, \beta) \vee \varrho(\alpha, \beta) \leq q,$$

where the first inequality comes from the subadditivity of ϱ and the second one follows from both η and α belonging to $D(\beta, q)$. Analogously,

$$\varrho(\eta, \alpha) \leq \varrho(\eta, \delta) \vee \varrho(\alpha, \delta) \leq r.$$

Thus, $\varrho(\eta, \alpha) \leq q \wedge r$. As we already noted that $p \leq q \wedge r$, we conclude $(\eta, p) \leq (\alpha, q \wedge r)$. □

Corollary 3.5.3. *If B is an $n_\top$-free join-semilattice and $\varrho : [\gamma]^2 \to B$ is transitive and subadditive, then $(\gamma \times B, \trianglelefteq_\varrho)$ is an $(n+1)_\top$-free join-semilattice.*

Proof. Immediate by claim 2) of Proposition 3.5.2 and Proposition 3.3.8. □

Consider now the following recursive definition, where a special $(0, \kappa)$-semiladder is simply the trivial join-semilattice $\{\mathbf{0}\}$.

Definition 3.5.4. Given a positive integer n and an infinite cardinal κ, an (n, κ)-(semi)ladder S is *special* if there exists an ordinal γ, a special $(n-1, \kappa)$-(semi)ladder B and a map $\varrho : [\gamma]^2 \to B$ such that $S = (\gamma \times B, \trianglelefteq_\varrho)$.

Note that a join-semilattice $(P, \trianglelefteq)$ is a special $(1, \kappa)$-semiladder if and only if there exists an ordinal $\alpha \leq \kappa$ such that $P = \alpha \times \{\mathbf{0}\}$ with $(\gamma, \mathbf{0}) \trianglelefteq (\beta, \mathbf{0})$ if and only if $\gamma \leq \beta < \alpha$. This is the reason why we sometimes identify the special $(1, \kappa)$-semiladders with the ordinals less or equal to κ in the following sections.

3.6 Proof of Theorem 3.1.2

In this section, we prove Theorem 3.1.2. Let us first recall the definition of the combinatorial principle $\boxminus_{\kappa,\geq\chi}$, where κ, and χ are infinite cardinals and $\chi \leq \kappa$:

($\boxminus_{\kappa,\geq\chi}$) There exists a sequence $\langle C_\alpha \mid \alpha < \kappa^+ \text{ and } \mathrm{cf}(\alpha) \geq \chi\rangle$ such that:

(i) C_α is a closed and unbounded subset of α;

(ii) $\mathrm{otp}(C_\alpha) \leq \kappa$;

(iii) If $\beta \in \mathrm{Lim}(C_\alpha)$ and $\mathrm{cf}(\beta) \geq \chi$, then $C_\beta = C_\alpha \cap \beta$.

The principle $\boxminus_{\kappa,\geq\chi}$, originally due to Baumgartner (see e.g. [13]), is a weakening of Jensen's $\square_\kappa$ and as such holds in L for every infinite cardinals $\chi \leq \kappa$. Furthermore, note that for every infinite cardinal κ, the principle $\boxminus_{\kappa,\geq\omega}$ is precisely $\square_\kappa$, and that $\boxminus_{\kappa,\geq\kappa}$ trivially holds in ZFC.

Theorem 3.1.2 follows from the following theorem.

Theorem 3.6.1. *Let κ be an infinite cardinal and $n \in \omega$. If $\boxminus_{\kappa^{+m},\geq \mathrm{cf}(\kappa)}$ holds for every $m < n$, then there exists a special $(n+1,\kappa)$-semiladder of cardinality κ^{+n}*

Proof. Let us fix an infinite cardinal κ. We prove the result by induction on $n \in \omega$. The case $n = 0$ is trivially true, as κ is, in particular, a special $(1,\kappa)$-ladder of cardinality κ. So let us fix a special $(n+1,\kappa)$-semiladder $(B,\leq)$ of cardinality κ^{+n} and a $\boxminus_{\kappa^{+n},\geq \mathrm{cf}(\kappa)}$-sequence $\langle C_\alpha \mid \alpha < \kappa^{+n+1} \text{ and } \mathrm{cf}(\alpha) \geq \mathrm{cf}(\kappa)\rangle$ towards constructing a special $(n+2,\kappa)$-semiladder of cardinality κ^{+n+1}. Without loss of generality, we can suppose that B has a least element.

We want to define a map $\varrho : [\kappa^{+n+1}]^2 \to B$ such that, for every $\alpha < \kappa^{+n+1}$ and $p \in B$:

(a) ϱ is transitive and subadditive.

(b) $|D_\varrho(\alpha, p)| \leq |{\downarrow} p| + \aleph_0$.

(c) $|D_\varrho(\alpha, p)| < \kappa$.

Note that (b) implies (c) when κ is uncountable, as we are assuming that the principal ideals of B have cardinality $< \kappa$. Indeed, condition (c) plays a role only when $\kappa = \aleph_0$.

Let us first argue that if we manage to construct a map ϱ satisfying (a)-(c), then the induced $(\kappa^{+n+1} \times B, \trianglelefteq_\varrho)$ is a special $(n+2, \kappa)$-semiladder. By Corollary 3.5.3, $(\kappa^{+n+1} \times B, \trianglelefteq_\varrho)$ is an $(n+3)_\top$-free join-semilattice. We are left to argue that its principal ideals have cardinality $< \kappa$. If $\alpha < \kappa^{+n+1}$ and $p \in B$, then, by definition of $\trianglelefteq_\varrho$,

$$\trianglelefteq_\varrho \downarrow (\alpha, p) = (D_\varrho(\alpha, p) \cup \{\alpha\}) \times (\downarrow p).$$

It follows from condition (c) and our hypotheses on B that the principal ideal of (α, p) in $(\kappa^{+n+1} \times B, \trianglelefteq_\varrho)$ has cardinality $< \kappa$.

We define $\varrho \restriction \gamma$ by induction on $\gamma < \kappa^{+n+1}$, and in doing so, we make sure that conditions (a)-(c) are satisfied by $\varrho \restriction \gamma$. For clarity, when we say "$\varrho \restriction \gamma$ satisfies (a)-(c)" we mean that the map $\varrho \restriction \gamma$ satisfies statements (a)-(c) for all $\alpha < \gamma$ and $p \in B$.

We fix a well-ordering $\lessdot$ on a sufficiently large set, which will be used to guarantee uniformity in our choices during the inductive construction.

If λ is limit and we have defined $\varrho \restriction \gamma$ so to satisfy (a)-(c) for every $\gamma < \lambda$, then clearly $\varrho \restriction \lambda = \bigcup_{\gamma<\lambda} \varrho \restriction \gamma$ still satisfies (a)-(c). Thus, let us take care of the successor case. Suppose that we have defined ϱ on $[\gamma]^2$ towards extending it to $[\gamma + 1]^2$. There are two cases:

Case 1 $\mathrm{cf}(\gamma) < \mathrm{cf}(\kappa)$: Fix an increasing sequence $\langle \gamma_\nu \mid \nu < \mathrm{cf}(\gamma) \rangle$ cofinal in γ. Let p_γ be an upper bound in B of the set

$$\left\{ \varrho(\gamma_\mu, \gamma_\nu) \mid \mu \leq \nu < \mathrm{cf}(\gamma) \right\}$$

such that $|{\downarrow} p_\gamma| \geq \mathrm{cf}(\gamma)$. Note that such an upper bound always exists as B is $\mathrm{cf}(\kappa)$-directed by Proposition 3.2.2. For each $\alpha < \gamma$, let $\nu < \mathrm{cf}(\gamma)$ be the least such that $\alpha \leq \gamma_\nu$ and set $\varrho(\alpha, \gamma) = p_\gamma \vee \varrho(\alpha, \gamma_\nu)$.

Case 2 $\mathrm{cf}(\gamma) \geq \mathrm{cf}(\kappa)$: Let $\vec{\theta}_\gamma = \langle \theta_\gamma(\nu) \mid \nu < \lambda_\gamma \rangle$ be the monotone enumeration of C_γ. We inductively define a sequence $\vec{p}_\gamma = \langle p_\gamma(\nu) \mid \nu < \lambda_\gamma \rangle$ of elements of B as follows:

(i) Let $p_\gamma(0) = \mathbf{0}_B$.

(ii) If $\mathrm{cf}(\nu) < \mathrm{cf}(\kappa)$, then let $\langle \nu_\iota \mid \iota < \mathrm{cf}(\nu)\rangle$ be the $\lessdot$-least increasing sequence cofinal in ν and let $p_\gamma(\nu)$ be the $\lessdot$-least $p \in B$ such that p is an upper bound in B of the set

$$\left\{p_\gamma(\nu_\iota) \vee \varrho(\theta_\gamma(\nu_\iota), \theta_\gamma(\nu)) \mid \iota < \mathrm{cf}(\nu)\right\}, \tag{3.5}$$

and such that $p \neq p_\gamma(\mu)$ for every $\mu < \nu$. Such a p always exists: as B is $\mathrm{cf}(\kappa)$-directed, there exists a $p' \in B$ which is an upper bound of the set (3.5); moreover, since $\uparrow p'$ has cardinality κ^{+n}, and since $|\nu| < \kappa^{+n}$, we can find a $p \in B$ such that $p' \leq p$ and $p \neq p_\gamma(\mu)$ for every $\mu < \nu$.

(iii) If $\mathrm{cf}(\nu) \geq \mathrm{cf}(\kappa)$, then let $p_\gamma(\nu)$ be the $\lessdot$-least $p \in B$ such that $p \neq p_\gamma(\mu)$ for all $\mu < \nu$.

Now that we have defined the sequence $\vec{p}_\gamma$, we can extend ϱ: for each $\alpha < \gamma$, let $\mu_\gamma(\alpha)$ be the least $\nu < \lambda_\gamma$ such that $\alpha \leq \theta_\gamma(\nu)$ and set $\varrho(\alpha, \gamma) = p_\gamma(\mu_\gamma(\alpha)) \vee \varrho(\alpha, \theta_\gamma(\mu_\gamma(\alpha)))$.

We assume that $\varrho \restriction \gamma$ satisfies (a)-(c). The rest of the proof consists of showing that $\varrho \restriction \gamma + 1$ also satisfies (a)-(c).

Claim 3.6.1.1. *Suppose that* $\mathrm{cf}(\gamma) \geq \mathrm{cf}(\kappa)$. *Then* $\varrho(\theta_\gamma(\mu), \theta_\gamma(\nu)) \leq p_\gamma(\mu) \vee p_\gamma(\nu)$ *for every* $\mu < \nu < \lambda_\gamma$.

Proof. We prove our claim by induction on ν. It vacuously holds when $\nu = 0$.

Suppose $\nu > 0$ and $\mathrm{cf}(\nu) < \mathrm{cf}(\kappa)$. Pick any $\iota < \mathrm{cf}(\nu)$ such that $\mu \leq \nu_\iota$. By transitivity of $\varrho \restriction \gamma$,

$$\varrho(\theta_\gamma(\mu), \theta_\gamma(\nu)) \leq \varrho(\theta_\gamma(\mu), \theta_\gamma(\nu_\iota)) \vee \varrho(\theta_\gamma(\nu_\iota), \theta_\gamma(\nu)).$$

By induction hypothesis, $\varrho(\theta_\gamma(\mu), \theta_\gamma(\nu_\iota)) \leq p_\gamma(\mu) \vee p_\gamma(\nu_\iota)$. Moreover, $p_\gamma(\nu_\iota) \vee \varrho(\theta_\gamma(\nu_\iota), \theta_\gamma(\nu)) \leq p_\gamma(\nu)$ by definition of $p_\gamma(\nu)$ (see (3.5)). Combining these observations, we get $\varrho(\theta_\gamma(\mu), \theta_\gamma(\nu)) \leq p_\gamma(\mu) \vee p_\gamma(\nu)$.

Finally, suppose that $\mathrm{cf}(\nu) \geq \mathrm{cf}(\kappa)$. By the properties of our square sequence, we have $\vec{\theta}_{\theta_\gamma(\nu)} = \vec{\theta}_\gamma \restriction \nu$ and $\vec{p}_{\theta_\gamma(\nu)} = \vec{p}_\gamma \restriction \nu$. The following holds:

$$\varrho(\theta_\gamma(\mu), \theta_\gamma(\nu)) = \varrho(\theta_{\theta_\gamma(\nu)}(\mu), \theta_\gamma(\nu)) = p_{\theta_\gamma(\nu)}(\mu) = p_\gamma(\mu), \tag{3.6}$$

where the first and last equalities follow from the abovementioned properties of our square sequence, and the middle one comes directly from the definition of $\varrho \restriction \theta_\gamma(\nu)+1$. It follows from (3.6) that $\varrho(\theta_\gamma(\mu), \theta_\gamma(\nu)) = p_\gamma(\mu) \leq p_\gamma(\mu) \vee p_\gamma(\nu)$. □

Claim 3.6.1.2. *Suppose that* $\mathrm{cf}(\gamma) \geq \mathrm{cf}(\kappa)$. *For every* $\alpha < \gamma$ *and every* ν *with* $\mu_\gamma(\alpha) \leq \nu < \lambda_\gamma$, *we have* $p_\gamma(\mu_\gamma(\alpha)) \leq p_\gamma(\nu) \vee \varrho(\alpha, \theta_\gamma(\nu))$.

Proof. We prove the claim by induction on ν. Clearly, the claim holds when $\nu = \mu_\gamma(\alpha)$.

Suppose that $\mu_\gamma(\alpha) < \nu$ and $\mathrm{cf}(\nu) < \mathrm{cf}(\kappa)$. Fix some $\iota < \mathrm{cf}(\nu)$ such that $\mu_\gamma(\alpha) \leq \nu_\iota$. By induction hypothesis, $p_\gamma(\mu_\gamma(\alpha)) \leq p_\gamma(\nu_\iota) \vee \varrho(\alpha, \theta_\gamma(\nu_\iota))$. By subadditivity of $\varrho \restriction \gamma$, we have $\varrho(\alpha, \theta_\gamma(\nu_\iota)) \leq \varrho(\alpha, \theta_\gamma(\nu)) \vee \varrho(\theta_\gamma(\nu_\iota), \theta_\gamma(\nu))$. But since, by definition, $p_\gamma(\nu_\iota) \vee \varrho(\theta_\gamma(\nu_\iota), \theta_\gamma(\nu)) \leq p_\gamma(\nu)$, we conclude that $p_\gamma(\mu_\gamma(\alpha)) \leq p_\gamma(\nu) \vee \varrho(\alpha, \theta_\gamma(\nu))$.

Finally, suppose that $\mu_\gamma(\alpha) < \nu$ and $\mathrm{cf}(\nu) \geq \mathrm{cf}(\kappa)$. By definition of $\varrho \restriction \theta_\gamma(\nu) + 1$, we have

$$\varrho(\alpha, \theta_\gamma(\nu)) = p_{\theta_\gamma(\nu)}(\mu_{\theta_\gamma(\nu)}(\alpha)) \vee \varrho(\alpha, \theta_{\theta_\gamma(\nu)}(\mu_{\theta_\gamma(\nu)}(\alpha))). \tag{3.7}$$

By the properties of our square sequence, $\vec{\theta}_{\theta_\alpha(\nu)} = \vec{\theta}_\alpha \restriction \nu$ and $\vec{p}_{\theta_\alpha(\nu)} = \vec{p}_\alpha \restriction \nu$. In particular, (3.7) becomes:

$$\varrho(\alpha, \theta_\gamma(\nu)) = p_\gamma(\mu_\gamma(\alpha)) \vee \varrho(\alpha, \theta_\gamma(\mu_\gamma(\alpha))).$$

Thus $p_\gamma(\mu_\gamma(\alpha)) \leq \varrho(\alpha, \theta_\gamma(\nu))$ and, a fortiori, $p_\gamma(\mu_\gamma(\alpha)) \leq p_\gamma(\nu) \vee \varrho(\alpha, \theta_\gamma(\nu))$. □

Claim 3.6.1.3. $\varrho \restriction \gamma + 1$ *is transitive.*

Proof. Since we are assuming that $\varrho \restriction \gamma$ is transitive, it suffices to show that $\varrho(\alpha, \gamma) \leq \varrho(\alpha, \beta) \vee \varrho(\beta, \gamma)$ for every α, β with $\alpha < \beta < \gamma$. Hence, fix α, β with $\alpha < \beta < \gamma$.

First suppose that $\mathrm{cf}(\gamma) < \mathrm{cf}(\kappa)$. Let μ, ν be the least such that $\alpha \leq \gamma_\mu$ and $\beta \leq \gamma_\nu$, respectively. Clearly, $\mu \leq \nu$. If $\mu = \nu$, our claim follows straightforwardly from the transitivity of $\varrho \restriction \gamma$. Indeed we would have $\varrho(\alpha, \gamma_\mu) \leq \varrho(\alpha, \beta) \vee \varrho(\beta, \gamma_\mu) = \varrho(\alpha, \beta) \vee \varrho(\beta, \gamma_\nu)$, and thus

$$\varrho(\alpha, \gamma) = p_\gamma \vee \varrho(\alpha, \gamma_\mu) \leq p_\gamma \vee \varrho(\alpha, \beta) \vee \varrho(\beta, \gamma_\nu) = \varrho(\alpha, \beta) \vee \varrho(\beta, \gamma),$$

where the equalities hold by definition of $\varrho \restriction \gamma + 1$. Hence suppose that $\mu < \nu$. In this case $\gamma_\mu < \beta$ must hold by the minimality of ν. The following holds:

$$\begin{aligned}
\varrho(\alpha, \gamma) &= p_\gamma \vee \varrho(\alpha, \gamma_\mu) \\
&\leq p_\gamma \vee \varrho(\alpha, \beta) \vee \varrho(\gamma_\mu, \beta) \\
&\leq p_\gamma \vee \varrho(\alpha, \beta) \vee \varrho(\gamma_\mu, \gamma_\nu) \vee \varrho(\beta, \gamma_\nu) \\
&= p_\gamma \vee \varrho(\alpha, \beta) \vee \varrho(\beta, \gamma_\nu) \\
&= \varrho(\alpha, \beta) \vee \varrho(\beta, \gamma),
\end{aligned}$$

where the first and last equality hold by definition of $\varrho \restriction \gamma + 1$; the two inequalities follow from the subadditivity of $\varrho \restriction \gamma$ and, lastly, the second equality holds because $\varrho(\gamma_\mu, \gamma_\nu) \leq p_\gamma$ by definition of p_γ.

Now suppose that $\mathrm{cf}(\gamma) \geq \mathrm{cf}(\kappa)$ and $\mu_\gamma(\alpha) = \mu_\gamma(\beta)$. By the transitivity of $\varrho \restriction \gamma$ we have

$$\varrho(\alpha, \theta_\gamma(\mu_\gamma(\alpha))) \leq \varrho(\alpha, \beta) \vee \varrho(\beta, \theta_\gamma(\mu_\gamma(\alpha)))$$

and thus

$$\begin{aligned}
\varrho(\alpha, \gamma) &= p_\gamma(\mu_\gamma(\alpha)) \vee \varrho(\alpha, \theta_\gamma(\mu_\gamma(\alpha))) \\
&\leq p_\gamma(\mu_\gamma(\alpha)) \vee \varrho(\alpha, \beta) \vee \varrho(\beta, \theta_\gamma(\mu_\gamma(\alpha))) \\
&= \varrho(\alpha, \beta) \vee p_\gamma(\mu_\gamma(\beta)) \vee \varrho(\beta, \theta_\gamma(\mu_\gamma(\beta))) \\
&= \varrho(\alpha, \beta) \vee \varrho(\beta, \gamma),
\end{aligned}$$

where the second equality comes from the hypothesis $\mu_\gamma(\alpha) = \mu_\gamma(\beta)$ and the other two equalities hold by definition of $\varrho \restriction \gamma + 1$.

Finally, suppose that $\mathrm{cf}(\gamma) \geq \mathrm{cf}(\kappa)$ and $\mu_\gamma(\alpha) < \mu_\gamma(\beta)$. Note that $\theta_\gamma(\mu_\gamma(\alpha))$ must be strictly less than β by the minimality of $\mu_\gamma(\beta)$. The following holds:

$$\begin{aligned}
\varrho(\alpha,\gamma) &= p_\gamma(\mu_\gamma(\alpha)) \vee \varrho(\alpha, \theta_\gamma(\mu_\gamma(\alpha))) \\
&\leq p_\gamma(\mu_\gamma(\alpha)) \vee \varrho(\alpha,\beta) \vee \varrho(\theta_\gamma(\mu_\gamma(\alpha)),\beta) \\
&\leq p_\gamma(\mu_\gamma(\alpha)) \vee \varrho(\alpha,\beta) \vee \varrho(\theta_\gamma(\mu_\gamma(\alpha)), \theta_\gamma(\mu_\gamma(\beta))) \vee \varrho(\beta, \theta_\gamma(\mu_\gamma(\beta))) \\
&\leq p_\gamma(\mu_\gamma(\alpha)) \vee \varrho(\alpha,\beta) \vee p_\gamma(\mu_\gamma(\beta)) \vee \varrho(\beta, \theta_\gamma(\mu_\gamma(\beta))) \\
&= p_\gamma(\mu_\gamma(\alpha)) \vee \varrho(\alpha,\beta) \vee \varrho(\beta,\gamma),
\end{aligned} \tag{3.8}$$

where the first two inequalities hold because of the subadditivity of $\varrho \restriction \gamma$, while the last inequality holds by Claim 3.6.1.1. To finish the argument, note the following:

$$\begin{aligned}
p_\gamma(\mu_\gamma(\alpha)) &\leq p_\gamma(\mu_\gamma(\beta)) \vee \varrho(\alpha, \theta_\gamma(\mu_\gamma(\beta))) \\
&\leq p_\gamma(\mu_\gamma(\beta)) \vee \varrho(\alpha,\beta) \vee \varrho(\beta, \theta_\gamma(\mu_\gamma(\beta))) \\
&= \varrho(\alpha,\beta) \vee \varrho(\beta,\gamma),
\end{aligned} \tag{3.9}$$

where the first inequality holds by Claim 3.6.1.2 and the second inequality follows from the transitivity of $\varrho \restriction \gamma$. By combining (3.8) and (3.9) we get $\varrho(\alpha,\gamma) \leq \varrho(\alpha,\beta) \vee \varrho(\beta,\gamma)$. □

Claim 3.6.1.4. $\varrho \restriction \gamma + 1$ *is subadditive.*

Proof. Since we are assuming that $\varrho \restriction \gamma$ is subadditive, it suffices to show that $\varrho(\alpha,\beta) \leq \varrho(\alpha,\gamma) \vee \varrho(\beta,\gamma)$ for every α, β with $\alpha < \beta < \gamma$. Hence fix some α, β with $\alpha < \beta < \gamma$.

First suppose that $\mathrm{cf}(\gamma) < \mathrm{cf}(\kappa)$. Let μ, ν be the least such that $\alpha \leq \gamma_\mu$ and $\beta \leq \gamma_\nu$, respectively. If $\mu = \nu$, our claim follows from the subadditivity of $\varrho \restriction \gamma$. Indeed we would have $\varrho(\alpha,\beta) \leq \varrho(\alpha,\gamma_\mu) \vee \varrho(\beta,\gamma_\mu) = \varrho(\alpha,\gamma_\mu) \vee \varrho(\beta,\gamma_\nu)$ and, a fortiori,

$$\varrho(\alpha,\beta) \leq \varrho(\alpha,\gamma_\mu) \vee \varrho(\beta,\gamma_\nu) \vee p_\gamma = \varrho(\alpha,\gamma) \vee \varrho(\beta,\gamma),$$

where the equality directly follows from the definition of $\varrho \restriction \gamma + 1$. Suppose now that $\mu < \nu$. In this case, γ_μ must be strictly less than β by the minimality of ν. The following holds:

$$\begin{aligned}\varrho(\alpha,\beta) &\le \varrho(\alpha,\gamma_\mu)\vee\varrho(\gamma_\mu,\beta)\\ &\le \varrho(\alpha,\gamma_\mu)\vee\varrho(\gamma_\mu,\gamma_\nu)\vee\varrho(\beta,\gamma_\nu)\\ &\le \varrho(\alpha,\gamma_\mu)\vee p_\gamma\vee\varrho(\beta,\gamma_\nu)\\ &= \varrho(\alpha,\gamma)\vee\varrho(\beta,\gamma),\end{aligned}$$

where the first inequality holds by the transitivity of $\varrho\restriction\gamma$, the second inequality instead holds by the subadditivity of $\varrho\restriction\gamma$, and the last one follows from $\varrho(\gamma_\mu,\gamma_\nu)\le p_\gamma$, which holds by definition of p_γ.

Now suppose that $\mathrm{cf}(\gamma)\ge\mathrm{cf}(\kappa)$ and $\mu_\gamma(\alpha)=\mu_\gamma(\beta)$. By assumption and the subadditivity of $\varrho\restriction\gamma$,

$$\begin{aligned}\varrho(\alpha,\beta) &\le \varrho(\alpha,\theta_\gamma(\mu_\gamma(\alpha)))\vee\varrho(\beta,\theta_\gamma(\mu_\gamma(\alpha)))\\ &= \varrho(\alpha,\theta_\gamma(\mu_\gamma(\alpha)))\vee\varrho(\beta,\theta_\gamma(\mu_\gamma(\beta)))\end{aligned}$$

and, a fortiori,

$$\begin{aligned}\varrho(\alpha,\beta) &\le \varrho(\alpha,\theta_\gamma(\mu_\gamma(\alpha)))\vee\varrho(\beta,\theta_\gamma(\mu_\gamma(\beta)))\vee p_\gamma(\mu_\gamma(\alpha))\\ &= \varrho(\alpha,\gamma)\vee\varrho(\beta,\gamma).\end{aligned}$$

Finally, suppose $\mathrm{cf}(\gamma)\ge\mathrm{cf}(\kappa)$ and $\mu_\gamma(\alpha)<\mu_\gamma(\beta)$. Note that $\theta_\gamma(\mu_\gamma(\alpha))$ must be strictly less than β by the minimality of $\mu_\gamma(\beta)$. The following holds:

$$\begin{aligned}\varrho(\alpha,\beta) &\le \varrho(\alpha,\theta_\gamma(\mu_\gamma(\alpha)))\vee\varrho(\theta_\gamma(\mu_\gamma(\alpha)),\beta)\\ &\le \varrho(\alpha,\theta_\gamma(\mu_\gamma(\alpha)))\vee\varrho(\theta_\gamma(\mu_\gamma(\alpha)),\theta_\gamma(\mu_\gamma(\beta)))\vee\varrho(\beta,\theta_\gamma(\mu_\gamma(\beta)))\\ &\le \varrho(\alpha,\theta_\gamma(\mu_\gamma(\alpha)))\vee p_\gamma(\mu_\gamma(\alpha))\vee p_\gamma(\mu_\gamma(\beta))\vee\varrho(\beta,\theta_\gamma(\mu_\gamma(\beta)))\\ &= \varrho(\alpha,\gamma)\vee\varrho(\beta,\gamma),\end{aligned}$$

where the first inequality follows from the transitivity of $\varrho\restriction\gamma$, the second one follows from the subadditivity of $\varrho\restriction\gamma$, while the last holds by Claim 3.6.1.1. □

Claim 3.6.1.5. *$|D_\varrho(\gamma,p)| \leq |{\downarrow}p| + \aleph_0$ for every $p \in B$*

Proof. Let us assume first that $\mathrm{cf}(\gamma) < \mathrm{cf}(\kappa)$. If $p_\gamma \not\leq p$ then, by definition of $\varrho \restriction \gamma+1$, $D_\varrho(\gamma,p) = \emptyset$. Thus, we can suppose $p_\gamma \leq p$. We claim that

$$D_\varrho(\gamma,p) = \bigcup_{\mu<\mathrm{cf}(\gamma)} D_\varrho(\gamma_\mu,p) \cup \{\gamma_\mu\}. \tag{3.10}$$

The fact that $D_\varrho(\gamma,p)$ is included in the set on the right-hand side directly follows from the definition of $\varrho \restriction \gamma+1$. Now we focus on the other inclusion. Since, by definition of $\varrho \restriction \gamma+1$, $\varrho(\gamma_\mu,\gamma) = p_\gamma$, and since $p_\gamma \leq p$, we have directly that $\gamma_\mu \in D_\varrho(\gamma,p)$ for every $\mu < \mathrm{cf}(\gamma)$. Now pick some $\mu < \mathrm{cf}(\gamma)$ and $\alpha \in D_\varrho(\gamma_\mu,p)$ towards showing that $\alpha \in D_\varrho(\gamma,p)$. By transitivity of $\varrho \restriction \gamma+1$ (proved in Claim 3.6.1.3), we have that

$$\varrho(\alpha,\gamma) \leq \varrho(\alpha,\gamma_\mu) \vee \varrho(\gamma_\mu,\gamma) = \varrho(\alpha,\gamma_\mu) \vee p_\gamma,$$

Since we are assuming $p_\gamma \leq p$ and $\varrho(\alpha,\gamma_\mu) \leq p$, we conclude that $\varrho(\alpha,\gamma) \leq p$, or, equivalently, $\alpha \in D_\varrho(\gamma,p)$. Thus, (3.10) holds.

By induction hypothesis, $\varrho \restriction \gamma$ satisfies (b), and therefore $|D_\varrho(\gamma_\mu,p)| \leq |{\downarrow}p| + \aleph_0$ for every $\mu < \mathrm{cf}(\gamma)$. This last observation, together with (3.10), implies that $|D_\varrho(\gamma,p)| \leq \mathrm{cf}(\gamma) \cdot |{\downarrow}p| + \aleph_0$. But since, by the definition of p_γ, $|{\downarrow}p_\gamma| \geq \mathrm{cf}(\gamma)$, we conclude that $|D_\varrho(\gamma,p)| \leq |{\downarrow}p| + \aleph_0$.

Now assume $\mathrm{cf}(\gamma) \geq \mathrm{cf}(\kappa)$. We claim that

$$D_\varrho(\gamma,p) = \bigcup \Big\{ D_\varrho(\theta_\gamma(\mu),p) \cup \{\theta_\gamma(\mu)\} \mid p_\gamma(\mu) \leq p \Big\}. \tag{3.11}$$

The fact that $D_\varrho(\gamma,p)$ is included in the set on the right-hand side directly follows from the definition of $\varrho \restriction \gamma+1$. Now we focus on the other inclusion. Since, by definition, $\varrho(\theta_\gamma(\mu),\gamma) = p_\gamma(\mu)$ for every $\mu < \lambda_\gamma$, we automatically have that $\theta_\gamma(\mu) \in D_\varrho(\gamma,p)$ for every μ such that $p_\gamma(\mu) \leq p$. Now pick some $\mu < \lambda_\gamma$ and an α such that $\alpha \in D_\varrho(\theta_\gamma(\mu),p)$ and $p_\gamma(\mu) \leq p$, towards showing that $\alpha \in D_\varrho(\gamma,p)$. By transitivity of $\varrho \restriction \gamma+1$ (proved in Claim 3.6.1.3), we have that

$$\varrho(\alpha,\gamma) \leq \varrho(\alpha,\theta_\gamma(\mu)) \vee \varrho(\theta_\gamma(\mu),\gamma) = \varrho(\alpha,\theta_\gamma(\mu)) \vee p_\gamma(\mu).$$

Since we picked μ such that $p_\gamma(\mu) \leq p$ and, by assumption, $\varrho(\alpha, \theta_\gamma(\mu)) \leq p$, we conclude that $\varrho(\alpha, \gamma) \leq p$, or, equivalently, $\alpha \in D_\varrho(\gamma, p)$. Thus, (3.11) holds.

By induction hypothesis, $\varrho \restriction \gamma$ satisfies (b), and therefore $|D_\varrho(\theta_\gamma(\mu), p)| \leq |{\downarrow}p| + \aleph_0$ for every $\mu < \lambda_\gamma$. This last observation, together with (3.11), implies that

$$|D_\varrho(\gamma, p)| \leq |\{\mu < \lambda_\gamma \mid p_\gamma(\mu) \leq p\}| \cdot |{\downarrow}p| + \aleph_0.$$

By construction, $\vec{p_\gamma}$ is injective, and therefore $|\{\mu < \lambda_\gamma \mid p_\gamma(\mu) \leq p\}| \leq |{\downarrow}p|$. Thus, we conclude that $|D_\varrho(\gamma, p)| \leq |{\downarrow}p| + \aleph_0$. □

Claim 3.6.1.6. *$|D_\varrho(\gamma, p)| < \kappa$ for every $p \in B$.*

Proof. If κ is uncountable, the result follows directly from Claim 3.6.1.5 and our hypotheses on B. So assume $\kappa = \aleph_0$.

If $\gamma = \beta + 1$ for some β, then, by (3.10), $D_\varrho(\gamma, p) = D_\varrho(\beta, p) \cup \{\beta\}$. Since by induction hypothesis $D_\varrho(\beta, p)$ is finite, we conclude that also $D_\varrho(\gamma, p)$ is finite.

If γ is limit, then it follows from (3.11) and from the injectivity of $\vec{p_\gamma}$ that $D_\varrho(\gamma, p)$, being a finite union of finite sets, is finite. □

□

3.7 Proof of Theorem 3.1.3

This section is devoted to the proof of the following theorem, which implies Theorem 3.1.3

Theorem 3.7.1. *Let κ be an infinite cardinal and $n \in \omega$. If $\square_{\kappa^{+m}}$ holds for every $m < n$, then there exists a special $(n+1, \kappa)$-ladder of cardinality κ^{+n}.*

Proof. The proof is very much analogous to the one of Theorem 3.6.1. Moreover, since the case $\kappa = \aleph_0$ already follows from Theorem 3.6.1, we can suppose that κ is uncountable.

We prove the result by induction on $n \in \omega$. The case $n = 0$ is trivially true. So let us fix a special $(n+1, \kappa)$-ladder $(B, \leq)$ of cardinality κ^{+n} and a

$\square_{\kappa^{+n}}$-sequence $\langle C_\alpha \mid \alpha < \kappa^{+n+1}$ and α limit$\rangle$ towards constructing a special $(n+2,\kappa)$-ladder of cardinality κ^{+n+1}.

Let P be the lattice such B is a quasi-product of κ^{+n} and P. We want to define a map $\varrho : [\kappa^{+n+1}]^2 \to B$ such that, for every $\alpha < \kappa^{+n+1}$ and every $p \in B$:

(a) ϱ is transitive and subadditive.

(b) $|D_\varrho(\alpha, p)| \leq |{\downarrow}p| + \aleph_0$.

(c) $D_\varrho(\alpha, p)$ is closed in α.

(d) $\varrho(0, \alpha) = \mathbf{0}_B$.

Note that conditions (c) and (d) are new with respect to the conditions used in the proof of Theorem 3.6.1. With these new conditions, the same argument used at the beginning of the proof of Theorem 3.6.1 yields that $(B \times \kappa^{+n+1}, \trianglelefteq_\varrho)$ is a special $(n+2,\kappa)$-ladder.

We define $\varrho \restriction \gamma$ by induction on $\gamma < \kappa^{+n+1}$, and in doing so, we make sure that conditions (a)-(d) are satisfies for $\varrho \restriction \gamma$. As in the proof of Theorem 3.6.1, when we say "$\varrho \restriction \gamma$ satisfies (a)-(d)" we mean that the map $\varrho \restriction \gamma$ satisfies statements (a)-(d) for all $\alpha < \gamma$.

Fix a well-ordering $\lessdot$ on a sufficiently large set, which guarantees uniformity in our choices during the inductive construction.

If λ is limit and we have defined $\varrho \restriction \gamma$ so to satisfy (a)-(d) for every $\gamma < \lambda$, then clearly $\varrho \restriction \lambda = \bigcup_{\gamma<\lambda} \varrho \restriction \gamma$ still satisfies (a)-(d). Thus, let us take care of the successor case.

Suppose that we have defined ϱ on $[\gamma]^2$ towards extending it to $[\gamma+1]^2$. There are two cases:

Case 1 γ is a successor ordinal: For each $\alpha < \gamma$, set $\varrho(\alpha,\gamma) = \varrho(\alpha, \gamma - 1)$.

Case 2 γ is a limit ordinal: Let $\vec{\theta}_\gamma = \langle \theta_\gamma(\nu) \mid \nu < \lambda_\gamma \rangle$ be the monotone enumeration of C_γ. We inductively define a sequence $\vec{p}_\gamma = \langle p_\gamma(\nu) \mid \nu < \lambda_\gamma \rangle$ of elements of B as follows:

(i) Let $p_\gamma(0) = \mathbf{0}_B = (0, \mathbf{0}_P)$.

(ii) If ν is a successor ordinal, let $p_\gamma(\nu)$ be the $\lessdot$-least $p \in B$ such that $p_\gamma(\nu-1) \vee \varrho(\theta_\gamma(\nu-1), \theta_\gamma(\nu)) \leq p'$ and $\mathrm{ht}(p) > \mathrm{ht}(p_\gamma(\nu-1))$ and $p \neq p_\gamma(\mu)$ for all $\mu < \nu$. Note that such a p always exists: first pick a p' such that $p_\gamma(\nu-1) \vee \varrho(\theta_\gamma(\nu-1), \theta_\gamma(\nu)) \leq p$; then, as B is a quasi-product of κ^{+n} and P, there exists $p'' > p'$ with $\mathrm{ht}(p'') > \mathrm{ht}(p')$, and, a fortiori, $\mathrm{ht}(p'') > \mathrm{ht}(p_\gamma(\nu-1))$; finally, since $|{\uparrow}p''| = \kappa^{+n}$ and $|\nu| < \kappa^{+n}$, we conclude that there exists a $p \geq p''$ such that $p \neq p_\gamma(\mu)$ for all $\mu < \nu$.

(iii) If ν is a limit ordinal and[5] $\liminf_{\mu<\nu} \mathrm{ht}(p_\gamma(\mu)) < \kappa^{+n}$, then let

$$p_\gamma(\nu) = \Big(\liminf_{\mu<\nu} \mathrm{ht}(p_\gamma(\mu)), \mathbf{0}_P\Big).$$

(iv) If ν is a limit ordinal and $\liminf_{\mu<\nu} \mathrm{ht}(p_\gamma(\mu)) = \kappa^{+n}$ (we will argue that this can happen only when $n = 0$), then let $p_\gamma(\nu)$ be the $\lessdot$-least $p \in B$ such that $p \neq p_\gamma(\mu)$ for all $\mu < \nu$.

Now that we have defined the sequence $\vec{p}_\gamma$, we can extend ϱ: for each $\alpha < \gamma$, let $\mu_\gamma(\alpha)$ be the least $\nu < \lambda_\gamma$ such that $\alpha \leq \theta_\gamma(\nu)$, and set $\varrho(\alpha,\gamma) = p_\gamma(\mu_\gamma(\alpha)) \vee \varrho(\alpha, \theta_\gamma(\mu_\gamma(\alpha)))$.

We assume that $\varrho \restriction \gamma$ satisfies (a)-(d). The rest of the proof consists of showing that $\varrho \restriction \gamma+1$ also satisfies (a)-(d). If γ is limit, then for every $p \in B$ we let

$$\Delta_\gamma(p) := \Big\{\nu < \lambda_\gamma \mid p_\gamma(\nu) \leq p\Big\}.$$

Claim 3.7.1.1. $\varrho(0,\gamma) = \mathbf{0}_B$.

Proof. If γ is a successor ordinal, then, by definition of $\varrho \restriction \gamma+1$, $\varrho(0,\gamma) = \varrho(0,\gamma-1)$. Since we are assuming $\varrho(0,\gamma-1) = \mathbf{0}_B$, the claim follows.

Now suppose that γ is limit. Clearly, $\mu_\gamma(0) = 0$; therefore, by definition of $\varrho \restriction \gamma+1$, $\varrho(0,\gamma) = p_\gamma(0) \vee \varrho(0, \theta_\gamma(0))$. By construction, $p_\gamma(0) = \mathbf{0}_B$ and, by assumption, $\varrho(0, \theta_\gamma(0)) = \mathbf{0}_B$. Hence, the claim follows. □

[5] Given a sequence $(x_\mu)_{\mu<\nu}$, by $\liminf_{\mu<\nu} x_\mu$ we mean $\sup_{\mu<\nu} \inf_{\mu\leq\xi<\nu} x_\xi$.

Claim 3.7.1.2. *Suppose that γ is limit. Then, for each $p \in B$, the set $\Delta_\gamma(p)$ is closed in λ_γ.*

Proof. Pick any limit point ν of $\Delta_\gamma(p)$, towards showing that $\nu \in \Delta_\gamma(p)$. There must be cofinally many $\mu < \nu$ such that $p_\gamma(\mu) \leq p$. In particular,

$$\liminf_{\mu<\nu} \operatorname{ht}(p_\gamma(\mu)) \leq \operatorname{ht}(p) < \kappa^{+n}, \tag{3.12}$$

and hence, by definition of $\vec{p}_\gamma$,

$$p_\gamma(\nu) = \Big(\liminf_{\mu<\nu} \operatorname{ht}(p_\gamma(\mu)), \mathbf{0}_P\Big). \tag{3.13}$$

If $n = 0$, then B can be identified with κ (see the remark after Definition 3.5.4). In particular, the inequality (3.12) becomes

$$\liminf_{\mu<\nu} p_\gamma(\mu) \leq p < \kappa,$$

and (3.13) becomes

$$p_\gamma(\nu) = \liminf_{\mu<\nu} p_\gamma(\mu).$$

Therefore, $p_\gamma(\nu) \leq p$, or, equivalently, $\nu \in \Delta_\gamma(p)$.

Now suppose $n > 0$. Since κ^{+n} is regular, $\vec{p}_\gamma \restriction \nu$ does not satisfy the hypothesis of case (iv) of the definition of $\vec{p}_\gamma$ for any $\nu < \lambda_\gamma$. In other words, $\liminf_{\mu<\nu} \operatorname{ht}(p_\gamma(\mu)) < \kappa^{+n}$ for all $\nu < \lambda_\gamma$. It quickly follows from the definition of $\vec{p}_\gamma$ that the sequence $\operatorname{ht} \circ \vec{p}_\gamma = \langle \operatorname{ht}(p_\gamma(\mu)) \mid \mu < \lambda_\gamma \rangle$ is increasing. In particular,

$$\begin{aligned} \liminf_{\mu<\nu} \operatorname{ht}(p_\gamma(\mu)) &= \sup_{\mu<\nu} \operatorname{ht}(p_\gamma(\mu)) \\ &= \sup\Big\{\operatorname{ht}(p_\gamma(\mu)) \mid \mu < \nu \text{ and } p_\gamma(\mu) \leq p\Big\}, \end{aligned} \tag{3.14}$$

where the first equality follows from the monotonicity of $\operatorname{ht} \circ \vec{p}_\gamma$ and the second one from the already noted fact that for cofinally many $\mu < \nu$, $p_\gamma(\mu) \leq p$. Since B is a lattice, then, by 3) of Proposition 3.5.2, $\{\operatorname{ht}(q) \mid q \leq p\}$ is a closed subset of κ^{+n}. By this observation, by (3.14) and (3.13), we conclude

$$\operatorname{ht}(p_\gamma(\nu)) \in \{\operatorname{ht}(q) \mid q \leq p\}.$$

Equivalently, there exists a $z \in B$ such that $\operatorname{ht}(z) = \operatorname{ht}(p_\gamma(\nu))$ and $z \leq p$. Hence, $p_\gamma(\nu) = (\operatorname{ht}(z), \mathbf{0}_P)$. As $(B, \leq) = (\kappa^{+n} \times P, \leq)$ is a quasi-product of κ^{+n} and P, we conclude that $p_\gamma(\nu) \leq z$, and therefore $p_\gamma(\nu) \leq p$, or, equivalently, $\nu \in \Delta_\gamma(p)$. □

Claim 3.7.1.3. *Suppose that γ is limit. Then, for each $p \in B$, $|\Delta_\gamma(p)| \leq |{\downarrow}p|$.*

Proof. Fix some $p \in B$ towards showing that $|\Delta_\gamma(p)| \leq |{\downarrow}p|$. If $n > 0$, we have already noted in the proof of Claim 3.7.1.2 that $\operatorname{ht} \circ \vec{p}_\gamma$ is strictly monotone. In particular, $\vec{p}_\gamma$ is injective, and it directly follows that $|\Delta_\gamma(p)| \leq |{\downarrow}p|$.

We are left to deal with $n = 0$. Recall that in this case we identify B with κ. Let

$$\Delta_\gamma^-(p) := \Big\{\nu < \lambda_\gamma \mid p_\gamma(\nu) \leq p \text{ and } (\nu \text{ successor or } \liminf_{\mu<\nu} p_\gamma(\mu) = \kappa)\Big\}.$$

We claim that $\Delta_\gamma(p) \subseteq \operatorname{cl}(\Delta_\gamma^-(p))$, where $\operatorname{cl}(\Delta_\gamma^-(p))$ is the closure of the set $\Delta_\gamma^-(p)$. This suffices to prove that $|\Delta_\gamma(p)| \leq |{\downarrow}p|$, as $\vec{p}_\gamma \restriction \Delta_\gamma^-(p)$ is injective by definition of $\vec{p}_\gamma$, and hence, if our claim is true, we have $|{\downarrow}p| \geq |\Delta_\gamma^-(p)| = |\operatorname{cl}(\Delta_\gamma^-(p))| \geq |\Delta_\gamma(p)|$.

We prove that $\Delta_\gamma(p) \cap \nu \subseteq \operatorname{cl}(\Delta_\gamma^-(p)) \cap \nu$ for all $\nu < \lambda_\gamma$ by induction on ν. As the limit case is trivial, we can fix a $\nu \in \Delta_\gamma(p)$ and suppose that $\Delta_\gamma(p) \cap \nu \subseteq \operatorname{cl}(\Delta_\gamma^-(p)) \cap \nu$ towards showing that $\nu \in \operatorname{cl}(\Delta_\gamma^-(p))$. If ν is a successor ordinal or $\liminf_{\mu<\nu} p_\gamma(\mu) = \kappa$, then $\nu \in \Delta_\gamma^-(p)$ by definition; otherwise, $p_\gamma(\nu) = \liminf_{\mu<\nu} p_\gamma(\mu)$, and thus $\liminf_{\mu<\nu} p_\gamma(\mu) \leq p$ since we are assuming $p_\gamma(\nu) \leq p$. But note that

$$\liminf_{\mu<\nu} p_\gamma(\mu) = \sup_{\mu<\nu} \min_{\mu\leq\xi<\nu} p_\gamma(\xi), \tag{3.15}$$

as $B = \kappa$ is well-ordered. Hence, there must be cofinally many $\mu < \nu$ such that $p_\gamma(\mu) \leq p$. In other words, ν is a limit point of $\Delta_\gamma(p)$. Since we assumed $\Delta_\gamma(p) \cap \nu \subseteq \operatorname{cl}(\Delta_\gamma^-(p)) \cap \nu$, we conclude that ν belongs to $\operatorname{cl}(\Delta_\gamma^-(p))$, being a limit point of it. □

Claims 3.7.1.4-3.7.1.8 are just minor variations of Claims 3.6.1.1-3.6.1.5, respectively.

Claim 3.7.1.4. *Suppose that γ is limit. Then $\varrho(\theta_\gamma(\mu), \theta_\gamma(\nu)) \leq p_\gamma(\mu) \vee p_\gamma(\nu)$ for every $\mu < \nu < \lambda_\gamma$.*

Proof. We prove our claim by induction on ν. It vacuously holds when $\nu = 0$.

Suppose that $\nu > 0$ and that ν is a successor ordinal. By transitivity of $\varrho \restriction \gamma$,

$$\varrho(\theta_\gamma(\mu), \theta_\gamma(\nu)) \leq \varrho(\theta_\gamma(\mu), \theta_\gamma(\nu - 1)) \vee \varrho(\theta_\gamma(\nu - 1), \theta_\gamma(\nu)).$$

By induction hypothesis, $\varrho(\theta_\gamma(\mu), \theta_\gamma(\nu - 1)) \leq p_\gamma(\mu) \vee p_\gamma(\nu - 1)$. Moreover, $p_\gamma(\nu - 1) \vee \varrho(\theta_\gamma(\nu - 1), \theta_\gamma(\nu)) \leq p_\gamma(\nu)$ by definition of $p_\gamma(\nu)$. Combining these observations, we get $\varrho(\theta_\gamma(\mu), \theta_\gamma(\nu)) \leq p_\gamma(\mu) \vee p_\gamma(\nu)$.

Now suppose that ν is limit. By the properties of our square sequence, we have $\vec{\theta}_{\theta_\gamma(\nu)} = \vec{\theta}_\gamma \restriction \nu$ and $\vec{p}_{\theta_\gamma(\nu)} = \vec{p}_\gamma \restriction \nu$. The following holds:

$$\varrho(\theta_\gamma(\mu), \theta_\gamma(\nu)) = \varrho(\theta_{\theta_\gamma(\nu)}(\mu), \theta_\gamma(\nu)) = p_{\theta_\gamma(\nu)}(\mu) = p_\gamma(\mu), \tag{3.16}$$

where the first and last equalities follow from the abovementioned properties of our square sequence, and the middle one comes directly from the definition of $\varrho \restriction \theta_\gamma(\nu)+1$. It follows from (3.16) that $\varrho(\theta_\gamma(\mu), \theta_\gamma(\nu)) = p_\gamma(\mu) \leq p_\gamma(\mu) \vee p_\gamma(\nu)$. □

Claim 3.7.1.5. *Suppose that γ is limit. For every $\alpha < \gamma$ and every ν with $\mu_\gamma(\alpha) \leq \nu < \lambda_\gamma$, we have $p_\gamma(\mu_\gamma(\alpha)) \leq p_\gamma(\nu) \vee \varrho(\alpha, \theta_\gamma(\nu))$.*

Proof. We prove the claim by induction on ν. Clearly, the claim holds when $\nu = \mu_\gamma(\alpha)$.

Suppose that $\mu_\gamma(\alpha) < \nu$ and that ν is a successor ordinal. By induction hypothesis, $p_\gamma(\mu_\gamma(\alpha)) \leq p_\gamma(\nu - 1) \vee \varrho(\alpha, \theta_\gamma(\nu - 1))$. By subadditivity of $\varrho \restriction \gamma$, we have $\varrho(\alpha, \theta_\gamma(\nu - 1)) \leq \varrho(\alpha, \theta_\gamma(\nu)) \vee \varrho(\theta_\gamma(\nu - 1), \theta_\gamma(\nu))$. But since, by definition, $p_\gamma(\nu - 1) \vee \varrho(\theta_\gamma(\nu - 1), \theta_\gamma(\nu)) \leq p_\gamma(\nu)$, we conclude that $p_\gamma(\mu_\gamma(\alpha)) \leq p_\gamma(\nu) \vee \varrho(\alpha, \theta_\gamma(\nu))$.

Finally, suppose that $\mu_\gamma(\alpha) < \nu$ and that ν is limit. By definition of $\varrho \restriction \theta_\gamma(\nu) + 1$, we have

$$\varrho(\alpha, \theta_\gamma(\nu)) = p_{\theta_\gamma(\nu)}(\mu_{\theta_\gamma(\nu)}(\alpha)) \vee \varrho(\alpha, \theta_{\theta_\gamma(\nu)}(\mu_{\theta_\gamma(\nu)}(\alpha))). \tag{3.17}$$

By the properties of our square sequence, $\vec{\theta}_{\theta_\alpha(\nu)} = \vec{\theta}_\alpha \restriction \nu$ and $\vec{p}_{\theta_\alpha(\nu)} = \vec{p}_\alpha \restriction \nu$. In particular, (3.17) becomes:

$$\varrho(\alpha, \theta_\gamma(\nu)) = p_\gamma(\mu_\gamma(\alpha)) \vee \varrho(\alpha, \theta_\gamma(\mu_\gamma(\alpha))).$$

Thus $p_\gamma(\mu_\gamma(\alpha)) \leq \varrho(\alpha, \theta_\gamma(\nu))$ and, a fortiori, $p_\gamma(\mu_\gamma(\alpha)) \leq p_\gamma(\nu) \vee \varrho(\alpha, \theta_\gamma(\nu))$. □

Claim 3.7.1.6. *$\varrho \restriction \gamma + 1$ is transitive.*

Proof. Since we are assuming that $\varrho \restriction \gamma$ is transitive, it suffices to show that $\varrho(\alpha, \gamma) \leq \varrho(\alpha, \beta) \vee \varrho(\beta, \gamma)$ for every α, β with $\alpha < \beta < \gamma$. Hence, fix α, β with $\alpha < \beta < \gamma$.

First suppose that γ is a successor ordinal. By the transitivity of $\varrho \restriction \gamma$, $\varrho(\alpha, \gamma - 1) \leq \varrho(\alpha, \beta) \vee \varrho(\beta, \gamma - 1)$, and thus

$$\varrho(\alpha, \gamma) = \varrho(\alpha, \gamma - 1) \leq \varrho(\alpha, \beta) \vee \varrho(\beta, \gamma - 1) = \varrho(\alpha, \beta) \vee \varrho(\beta, \gamma),$$

where the equalities hold by definition of $\varrho \restriction \gamma + 1$.

Now suppose that γ is limit and $\mu_\gamma(\alpha) = \mu_\gamma(\beta)$. By the transitivity of $\varrho \restriction \gamma$ we have

$$\varrho(\alpha, \theta_\gamma(\mu_\gamma(\alpha))) \leq \varrho(\alpha, \beta) \vee \varrho(\beta, \theta_\gamma(\mu_\gamma(\alpha)))$$

and thus

$$\begin{aligned}
\varrho(\alpha, \gamma) &= p_\gamma(\mu_\gamma(\alpha)) \vee \varrho(\alpha, \theta_\gamma(\mu_\gamma(\alpha))) \\
&\leq p_\gamma(\mu_\gamma(\alpha)) \vee \varrho(\alpha, \beta) \vee \varrho(\beta, \theta_\gamma(\mu_\gamma(\alpha))) \\
&= \varrho(\alpha, \beta) \vee p_\gamma(\mu_\gamma(\beta)) \vee \varrho(\beta, \theta_\gamma(\mu_\gamma(\beta))) \\
&= \varrho(\alpha, \beta) \vee \varrho(\beta, \gamma),
\end{aligned}$$

where the second equality comes from the hypothesis $\mu_\gamma(\alpha) = \mu_\gamma(\beta)$ and the other two equalities hold by definition of $\varrho \restriction \gamma + 1$.

Finally, suppose that γ is limit and $\mu_\gamma(\alpha) < \mu_\gamma(\beta)$. Note that $\theta_\gamma(\mu_\gamma(\alpha))$ must be strictly less than β by the minimality of $\mu_\gamma(\beta)$. The following holds:

$$\begin{aligned}\varrho(\alpha,\gamma) &= p_\gamma(\mu_\gamma(\alpha)) \vee \varrho(\alpha,\theta_\gamma(\mu_\gamma(\alpha))) \\ &\leq p_\gamma(\mu_\gamma(\alpha)) \vee \varrho(\alpha,\beta) \vee \varrho(\theta_\gamma(\mu_\gamma(\alpha)),\beta) \\ &\leq p_\gamma(\mu_\gamma(\alpha)) \vee \varrho(\alpha,\beta) \vee \varrho(\theta_\gamma(\mu_\gamma(\alpha)),\theta_\gamma(\mu_\gamma(\beta))) \vee \varrho(\beta,\theta_\gamma(\mu_\gamma(\beta))) \\ &\leq p_\gamma(\mu_\gamma(\alpha)) \vee \varrho(\alpha,\beta) \vee p_\gamma(\mu_\gamma(\beta)) \vee \varrho(\beta,\theta_\gamma(\mu_\gamma(\beta))) \\ &= p_\gamma(\mu_\gamma(\alpha)) \vee \varrho(\alpha,\beta) \vee \varrho(\beta,\gamma),\end{aligned} \tag{3.18}$$

where the first two inequalities hold because of the subadditivity of $\varrho \restriction \gamma$, while the last inequality holds by Claim 3.7.1.4. To finish the argument, note the following:

$$\begin{aligned}p_\gamma(\mu_\gamma(\alpha)) &\leq p_\gamma(\mu_\gamma(\beta)) \vee \varrho(\alpha,\theta_\gamma(\mu_\gamma(\beta))) \\ &\leq p_\gamma(\mu_\gamma(\beta)) \vee \varrho(\alpha,\beta) \vee \varrho(\beta,\theta_\gamma(\mu_\gamma(\beta))) \\ &= \varrho(\alpha,\beta) \vee \varrho(\beta,\gamma),\end{aligned} \tag{3.19}$$

where the first inequality holds by Claim 3.7.1.5 and the second inequality follows from the transitivity of $\varrho \restriction \gamma$. By combining (3.18) and (3.19) we get $\varrho(\alpha,\gamma) \leq \varrho(\alpha,\beta) \vee \varrho(\beta,\gamma)$. □

Claim 3.7.1.7. *$\varrho \restriction \gamma + 1$ is subadditive.*

Proof. Since we are assuming that $\varrho \restriction \gamma$ is subadditive, it suffices to show that $\varrho(\alpha,\beta) \leq \varrho(\alpha,\gamma) \vee \varrho(\beta,\gamma)$ for every α, β with $\alpha < \beta < \gamma$. Hence fix some α, β with $\alpha < \beta < \gamma$.

First suppose that γ is a successor ordinal. By the subadditivity of $\varrho \restriction \gamma$, $\varrho(\alpha,\beta) \leq \varrho(\alpha,\gamma-1) \vee \varrho(\beta,\gamma-1)$. It immediately follows by definition of $\varrho \restriction \gamma + 1$ that $\varrho(\alpha,\beta) \leq \varrho(\alpha,\gamma) \vee \varrho(\beta,\gamma)$.

Now suppose that γ is limit and $\mu_\gamma(\alpha) = \mu_\gamma(\beta)$. By assumption and the subadditivity of $\varrho \restriction \gamma$,

$$\begin{aligned}\varrho(\alpha,\beta) &\leq \varrho(\alpha,\theta_\gamma(\mu_\gamma(\alpha))) \vee \varrho(\beta,\theta_\gamma(\mu_\gamma(\alpha))) \\ &= \varrho(\alpha,\theta_\gamma(\mu_\gamma(\alpha))) \vee \varrho(\beta,\theta_\gamma(\mu_\gamma(\beta)))\end{aligned}$$

and a fortiori

$$\begin{aligned}\varrho(\alpha,\beta) &\leq \varrho(\alpha,\theta_\gamma(\mu_\gamma(\alpha))) \vee \varrho(\beta,\theta_\gamma(\mu_\gamma(\beta))) \vee p_\gamma(\mu_\gamma(\alpha)) \\ &= \varrho(\alpha,\gamma) \vee \varrho(\beta,\gamma).\end{aligned}$$

Finally, suppose γ is limit and $\mu_\gamma(\alpha) < \mu_\gamma(\beta)$. Note that $\theta_\gamma(\mu_\gamma(\alpha))$ must be strictly less than β by the minimality of $\mu_\gamma(\beta)$. The following holds:

$$\begin{aligned}\varrho(\alpha,\beta) &\leq \varrho(\alpha,\theta_\gamma(\mu_\gamma(\alpha))) \vee \varrho(\theta_\gamma(\mu_\gamma(\alpha)),\beta) \\ &\leq \varrho(\alpha,\theta_\gamma(\mu_\gamma(\alpha))) \vee \varrho(\theta_\gamma(\mu_\gamma(\alpha)),\theta_\gamma(\mu_\gamma(\beta))) \vee \varrho(\beta,\theta_\gamma(\mu_\gamma(\beta))) \\ &\leq \varrho(\alpha,\theta_\gamma(\mu_\gamma(\alpha))) \vee p_\gamma(\mu_\gamma(\alpha)) \vee p_\gamma(\mu_\gamma(\beta)) \vee \varrho(\beta,\theta_\gamma(\mu_\gamma(\beta))) \\ &= \varrho(\alpha,\gamma) \vee \varrho(\beta,\gamma),\end{aligned}$$

where the first inequality follows from the transitivity of $\varrho \restriction \gamma$, the second one follows from the subadditivity of $\varrho \restriction \gamma$, while the last holds by Claim 3.7.1.4. $\square$

Claim 3.7.1.8. $|D_\varrho(\gamma,p)| \leq |{\downarrow}p| + \aleph_0$ *for every* $p \in B$

Proof. Let us assume first that γ is a successor ordinal. By definition of $\varrho \restriction \gamma+1$, $D_\varrho(\gamma,p) = D_\varrho(\gamma-1,p) \cup \{\gamma-1\}$, and therefore the claim follows by induction hypothesis.

Now assume γ is limit. We claim that

$$D_\varrho(\gamma,p) = \bigcup_{\mu\in\Delta_\gamma(p)} D_\varrho(\theta_\gamma(\mu),p) \cup \{\theta_\gamma(\mu)\}. \tag{3.20}$$

The fact that $D_\varrho(\gamma,p)$ is included in the set on the right-hand side directly follows from the definition of $\varrho \restriction \gamma+1$. Now we focus on the other inclusion. Since, by definition, $\varrho(\theta_\gamma(\mu),\gamma) = p_\gamma(\mu)$ for every $\mu < \lambda_\gamma$, we automatically have that $\theta_\gamma(\mu) \in D_\varrho(\gamma,p)$ for every $\mu \in \Delta_\gamma(p)$. Now pick some $\mu \in \Delta_\gamma(p)$ and an α such that $\alpha \in D_\varrho(\theta_\gamma(\mu),p)$, towards showing that $\alpha \in D_\varrho(\gamma,p)$. By transitivity of $\varrho \restriction \gamma+1$ (proved in Claim 3.7.1.6), we have that

$$\varrho(\alpha,\gamma) \leq \varrho(\alpha,\theta_\gamma(\mu)) \vee \varrho(\theta_\gamma(\mu),\gamma) = \varrho(\alpha,\theta_\gamma(\mu)) \vee p_\gamma(\mu).$$

Since we picked μ such that $p_\gamma(\mu) \leq p$ and, by assumption, $\varrho(\alpha,\theta_\gamma(\mu)) \leq p$, we conclude that $\varrho(\alpha,\gamma) \leq p$, or, equivalently, $\alpha \in D_\varrho(\gamma,p)$. Thus, (3.20) holds.

By induction hypothesis, $\varrho \restriction \gamma$ satisfies (b), and therefore $|D_\varrho(\theta_\gamma(\mu), p)| \leq |{\downarrow}p| + \aleph_0$ for every $\mu < \lambda_\gamma$. This last observation, together with (3.20), implies that

$$|D_\varrho(\gamma, p)| \leq |\Delta_\gamma(p)| \cdot |{\downarrow}p| + \aleph_0.$$

By Claim 3.7.1.3, $|\Delta_\gamma(p)| \leq |{\downarrow}p|$. Thus, we conclude that $|D_\varrho(\gamma, p)| \leq |{\downarrow}p| + \aleph_0$. □

Claim 3.7.1.9. *$D_\varrho(\gamma, p)$ is closed in γ for every $p \in B$.*

Proof. If γ is a successor ordinal, then, by definition of $\varrho \restriction \gamma + 1$, $D_\varrho(\gamma, p) = D_\varrho(\gamma - 1, p) \cup \{\gamma - 1\}$. By induction hypothesis, $D_\varrho(\gamma - 1, p)$ is closed in $\gamma - 1$, and therefore $D_\varrho(\gamma, p)$ is closed in γ.

Now suppose that γ is limit. We first claim that

$$\forall \mu \in \Delta_\gamma(p),\ D_\varrho(\gamma, p) \cap \theta_\gamma(\mu) = D_\varrho(\theta_\gamma(\mu), p). \tag{3.21}$$

Towards showing that (3.21) holds, fix some $\mu \in \Delta_\gamma(p)$. By (3.20), $D_\varrho(\theta_\gamma(\mu), p) \subseteq D_\varrho(\gamma, p)$. Now, given some $\alpha \in D_\varrho(\gamma, p) \cap \theta_\gamma(\mu)$, let us show that α belongs to $D_\varrho(\theta_\gamma(\mu), p)$. By (3.20) again, there exists $\nu \in \Delta_\gamma(p)$ such that $\alpha \in D_\varrho(\theta_\gamma(\nu), p) \cup \{\theta_\gamma(\nu)\}$. If $\mu \leq \nu$, then, by subadditivity of $\varrho \restriction \gamma$,

$$\varrho(\alpha, \theta_\gamma(\mu)) \leq \varrho(\alpha, \theta_\gamma(\nu)) \vee \varrho(\theta_\gamma(\mu), \theta_\gamma(\nu));$$

if $\nu < \mu$ instead, then, by transitivity of $\varrho \restriction \gamma$,

$$\varrho(\alpha, \theta_\gamma(\mu)) \leq \varrho(\alpha, \theta_\gamma(\nu)) \vee \varrho(\theta_\gamma(\nu), \theta_\gamma(\mu)).$$

In either case, we may conclude, by Claim 3.7.1.4, that

$$\varrho(\alpha, \theta_\gamma(\mu)) \leq \varrho(\alpha, \theta_\gamma(\nu)) \vee p_\gamma(\mu) \vee p_\gamma(\nu).$$

Therefore $\varrho(\alpha, \theta_\gamma(\mu)) \leq p$, since both μ and ν belong to $\Delta_\gamma(p)$ and since we are assuming $\varrho(\alpha, \theta_\gamma(\nu)) \leq p$. Thus, $\alpha \in D_\varrho(\theta_\gamma(\mu), p)$ as we wanted to show.

We are ready to prove that $D_\varrho(\gamma, p)$ is closed in γ. Pick an $\alpha < \gamma$ such that α is a limit point of $D_\varrho(\gamma, p)$, towards showing that $\alpha \in D_\varrho(\gamma, p)$. There are two cases: either α is a limit point of $\{\theta_\gamma(\mu) \mid \mu \in \Delta_\gamma(p)\}$ or it isn't. In the first case, by the closure of $\Delta_\gamma(p)$ (Claim 3.7.1.2) and the continuity of $\vec{\theta}_\gamma$, there exists a $\mu \in \Delta_\gamma(p)$ such that $\alpha = \theta_\gamma(\mu)$; but also in the second case there must be some $\mu \in \Delta_\gamma(p)$ such that $\alpha \leq \theta_\gamma(\mu)$, as otherwise, by (3.20), $D_\varrho(\gamma, p)$ would be bounded below α, contradicting α being a limit point of $D_\varrho(\gamma, p)$. Therefore, in either case, there exists a $\mu \in \Delta_\gamma(p)$ such that $\alpha \leq \theta_\gamma(\mu)$. Fix one such μ. If $\alpha = \theta_\gamma(\mu)$, then $\alpha \in D_\varrho(\gamma, p)$ by (3.20). If $\alpha < \theta_\gamma(\mu)$, then, by (3.21), α is a limit point of $D_\varrho(\theta_\gamma(\mu), p)$. Furthermore, since $D_\varrho(\theta_\gamma(\mu), p)$ is closed in $\theta_\gamma(\mu)$ by induction hypothesis, α belongs to $D_\varrho(\theta_\gamma(\mu), p)$ and, by (3.20) again, we conclude that also in this case $\alpha \in D_\varrho(\gamma, p)$. Overall, $\alpha \in D_\varrho(\gamma, p)$, and thus $D_\varrho(\gamma, p)$ is a closed subset of γ. □

□

3.8 Conclusions

There are still many open questions. They can be divided into two groups: the first one is concerned with whether Theorems 3.1.2 and 3.1.3 can be proved in ZFC; the second one focuses on finding “simply definable” ladders in the constructible universe.

3.8.1 Independence

The main question stemming from Ditor’s work [22] and explicitly posed by Wehrung [74] remains open:

Question 3.8.1 (Ditor’s Problem). Is the existence of a 3-ladder of cardinality $\aleph_2$ a theorem of ZFC?

The next natural questions, in the light of our results, are:

Question 3.8.2. Is the existence of a $(2, \aleph_1)$-ladder of cardinality $\aleph_2$ a theorem of ZFC?

Question 3.8.3. Is the existence of a special $(2, \aleph_1)$-ladder of cardinality $\aleph_2$ a theorem of ZFC?

Question 3.8.4. Does the existence of a $(2, \aleph_1)$-ladder of cardinality $\aleph_2$ imply the existence of a 3-ladder of cardinality $\aleph_2$ and vice-versa?

Question 3.8.5. Does the existence of a $(2, \aleph_1)$-ladder of cardinality $\aleph_2$ imply the existence of a special $(2, \aleph_1)$-ladder of cardinality $\aleph_2$?

Question 3.8.1 may also be of interest from a universal-algebraic perspective, in addition to its intrinsic set-theoretic significance. We refer the reader to [74] for further discussion on this topic.

It is quite surprising to us that the existence of a 3-ladder of cardinality $\aleph_2$ can be derived from either $\square_{\omega_1}$ (Theorem 3.1.3) or from $\mathsf{MA}(\aleph_1)$ [73]. It means that the existence of this structure follows from two axioms that are usually considered, in some sense, "orthogonal", as already noted by Wehrung in [73].

3.8.2 Definability

In the proofs of our Theorems 3.1.2 and 3.1.3, we have constructed our (semi)ladders by employing the square principles $\square_\kappa$. We feel that, in the constructible universe, there may be ladders that both witness the sharpness of Ditor's Theorem 3.2.1, and that have a much simpler definition than the one we gave. For this direction, we propose the following test question:

Question 3.8.6. Suppose $\mathrm{V} = \mathrm{L}$, and consider the set of all the countable Σ_1-elementary submodels of $\langle \mathrm{L}_{\omega_2}, \in \rangle$ ordered by the membership relation (i.e. $\in$). Is it a $(2, \aleph_1)$-ladder?

Chapter 4

Sierpiński coverings and the real degrees' breadth

4.1 Introduction

This chapter is for the most part taken from [7], co-authored with Alessandro Andretta.

In 1919 Sierpiński proved that CH, the Continuum Hypothesis, is equivalent to the existence of two sets A_0, A_1 covering $\mathbb{R}^2$ and such that every line parallel to the x-axis intersects A_0 in a countable set, and every line parallel to the y-axis intersects A_1 in a countable set. Three decades later, in 1951, Sierpiński obtained another geometric statement equivalent to CH: $\mathbb{R}^3$ is the union of three sets A_0, A_1, A_2 such that every line parallel to $\mathbf{e}_i$ has finite intersection with A_i, where $(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2)$ is the canonical basis for $\mathbb{R}^3$. These results were generalized to higher dimensions by Kuratowski and Sierpiński (see [65] for a detailed account of the history of these results): for all $n, k \in \omega$

$$2^{\aleph_0} \leq \aleph_{n+k} \Leftrightarrow \exists A_0, \ldots, A_{n+1} \Big(\mathbb{R}^{n+2} = \bigcup_{i \leq n+1} A_i \text{ and } \forall i \leq n+1 \, \forall \ell \in \mathcal{L}_i(\mathbb{R}^{n+2}) \, (|A_i \cap \ell| < \aleph_k) \Big). \tag{4.1}$$

Here and below $\mathcal{L}_i(\mathbb{R}^n)$ is the subset of all lines in $\mathbb{R}^n$ that are parallel to the vector $\mathbf{e}_i$, the i-th vector of the canonical basis for $\mathbb{R}^n$. Setting $n = 0$ and $k = 1$, or $n = 1$ and $k = 0$ in (4.1) the two theorems of Sierpiński from 1919 and 1951 are obtained.

The A_is in (4.1) are constructed using a transfinite induction of length $|\mathbb{R}|$, so their descriptive complexity is bounded by the complexity of the well-ordering of $\mathbb{R}$. On the other hand, some of the A_is are neither measurable nor have the property of Baire (Lemma 4.2.1), and therefore they can't be $\mathbf{\Delta}^1_2$ (provably in ZFC, see [41, p. 180]). In Gödel's constructible universe L, there is a good Σ^1_2 well-ordering of $\mathbb{R}$, and Törnquist and Weiss in [70] proved that $\mathbb{R} \subseteq \mathrm{L}$ is equivalent to either one of the following statements:

$$\exists A_0, A_1 \in \Sigma^1_2 \big(A_0 \cup A_1 = \mathbb{R}^2 \text{ and } \forall i < 2\ \forall \ell \in \mathcal{L}_i(\mathbb{R}^2)(|\ell \cap A_i| < \aleph_1) \big)$$
$$\exists A_0, A_1, A_2 \in \Sigma^1_2 \big(A_0 \cup A_1 \cup A_2 = \mathbb{R}^3 \text{ and } \forall i < 3\ \forall \ell \in \mathcal{L}_i(\mathbb{R}^3)(|\ell \cap A_i| < \aleph_0) \big).$$

In other words, Törnquist and Weiss showed that by asking the A_is to be definable in the best possible way (i.e. Σ^1_2) in both Sierpiński's 1919 and 1951 results, we get the "strongest" version of CH (i.e. $\mathbb{R} \subseteq \mathrm{L}$).

In this chapter, we generalize their result. In Section 4.3, we study the relationship between the breadth of the join-semilattice of real degrees and the size of the continuum (Theorem 4.3.6). In Section 4.4, we generalize Törnquist and Weiss' result by showing that the existence of a Σ^1_2 covering as in (4.1), with $n + k > 1$ and $k \leq 1$, is equivalent to the breadth of the join-semilattice of real degrees having a certain upper bound. Finally, in Section 4.5, we conclude with some open questions.

Notation

Our notation is standard—see e.g. [40]. When we treat a transitive set M as a model-theoretic structure, we use the language of set theory, identifying M with the structure $\langle M, \in \rangle$. We write $M \prec_1 N$ to mean that $\langle M, \in \rangle$ is a Σ_1-elementary substructure of $\langle N, \in \rangle$.

4.2 Sierpiński coverings

The theorem of Sierpiński-Kuratowski (4.1) deals with sets $A_0, \ldots, A_{n-1}$ such that $\bigcup_{i<n} A_i = \mathbb{R}^n$ and such that $\ell \cap A_i$ is small, for any line ℓ with direction $\mathbf{e}_i$. In the applications below, small means being at most countable, or thin (Lemma 4.2.1). Recall that a set of reals is *thin* if it does not contain a non-empty perfect set.

Sets A_i as in (4.1) form a *Sierpiński covering* of $\mathbb{R}^n$. As mentioned in the introduction, we let

$$\mathcal{L}_i(\mathbb{R}^n) := \{\ell \subseteq \mathbb{R}^n \mid \ell \text{ is a line parallel to } \mathbf{e}_i\}.$$

A line $\ell \in \mathcal{L}_i(\mathbb{R}^n)$ is uniquely determined by a point $\mathbf{p}$ in

$$H_i = H_i^n := \{(x_0, \ldots, x_{n-1}) \in \mathbb{R}^n \mid x_i = 0\}$$

the coordinate hyperplane orthogonal to $\mathbf{e}_i$. We denote by $\ell_{\mathbf{p}}$ the line determined by $\mathbf{p}$.

Lemma 4.2.1. *Let $n \in \omega$ and suppose that $\mathbb{R}^n = \bigcup_{i<n} A_i$ and*

$$\forall i < n \ \Big(\{\mathbf{p} \in H_i \mid \ell_{\mathbf{p}} \cap A_i \text{ is thin}\} \text{ is comeager in } H_i\Big).$$

Then, the A_is cannot all have the property of Baire.

Proof. Suppose otherwise, and fix $i < n$. By Kuratowski-Ulam theorem [47, Theorem 8.41] the set of all the $\mathbf{p} \in H_i$ such that $\ell_{\mathbf{p}} \cap A_i$ has the property of Baire in H_i, is comeager in H_i. Therefore, there is a comeager set (in H_i) of $\mathbf{p}$'s such that $\ell_{\mathbf{p}} \cap A_i$ is thin and has the property of Baire (in H_i). However, a set that is thin and has the property of Baire is necessarily meager. Therefore, for comeager-many $\mathbf{p}$'s, the set $\ell_{\mathbf{p}} \cap A_i$ is meager. It follows again from Kuratowski-Ulam theorem that the set A_i is meager in $\mathbb{R}^n$. However, this leads to a contradiction, as it implies that $\mathbb{R}^n$, being the union of finitely many meager sets, is meager. □

A similar result holds with Lebesgue-measurability in place of the property of Baire.

4.3 The real degrees' breadth

In this section, we explore the relationship between the breadth of the join-semilattice of real degrees and the size of the continuum. Consider the following statement:

$$(\star) \qquad \forall M \prec_1 \mathrm{H}_{\omega_1} \left(|M| = 2^{\aleph_0} \Rightarrow M = \mathrm{H}_{\omega_1} \right),$$

where $\mathrm{H}_{\omega_1} := \{x \mid |\mathrm{TC}(x)| \leq \aleph_0\}$ is the set of all hereditarily countable sets. The property ($\star$) may be interpreted as a kind of minimality principle: as soon as a Σ_1-elementary substructure of $\langle \mathrm{H}_{\omega_1}, \in \rangle$ has size of the continuum, it already coincides with H_{ω_1}. This interpretation is supported by the following result.

Proposition 4.3.1. *Assume* ($\star$). *Then, there are no* L*-generic Cohen reals.*

In order to prove Proposition 4.3.1 we need the following theorem, known as *Lévy-Shoenfield Absoluteness Theorem*.

Theorem 4.3.2 (Lévy-Shoenfield [39, Theorem 36])**.** *For every $a \in \mathbb{R}$, if $\theta = \omega_1^{\mathrm{L}[a]}$, then*

$$\mathrm{L}_\theta[a] \prec_1 \mathrm{H}_{\omega_1}.$$

Proof of Proposition 4.3.1*.* Suppose otherwise towards a contradiction. Then, it is well-known that there exists a perfect set $\mathcal{C} \subseteq \mathbb{R}$ such that any finite $F \subseteq \mathcal{C}$ is a set of mutually Cohen generic reals over L. It follows that $\omega_1^{\mathrm{L}[F]} = \omega_1^{\mathrm{L}}$, for any such F. Now fix an $x \in \mathcal{C}$ and consider the transitive set

$$M = \bigcup_{F \in [\mathcal{C} \setminus \{x\}]^{<\omega}} \mathrm{L}_{\omega_1^{\mathrm{L}}}[F].$$

By Lévy-Shoenfield absoluteness theorem, $\mathrm{L}_{\omega_1^{\mathrm{L}}}[F] = \mathrm{L}_{\omega_1^{\mathrm{L}[F]}}[F] \prec_1 \mathrm{H}_{\omega_1}$ for every $F \in [\mathcal{C}]^{<\omega}$. Therefore, $M \prec_1 \mathrm{H}_{\omega_1}$. Clearly $|M| = |\mathcal{C}| = 2^{\aleph_0}$, but $x \notin M$, which contradicts ($\star$). □

We next explore the relationship between ($\star$) and the breadth of the real degrees. We need the following well-known fact.

Lemma 4.3.3. *For every $M \prec_1 \mathrm{H}_{\omega_1}$ the following hold:*

(a) *M is transitive.*

(b) $|M| = |M \cap \mathbb{R}|$.

(c) *If $\mathbb{R} \subseteq M$, then $M = \mathrm{H}_{\omega_1}$.*

Proof. (a). Since $M \prec_1 \mathrm{H}_{\omega_1}$, we must have $\omega \in M$. Moreover, since $\mathrm{H}_{\omega_1} \vDash$ "Every set is countable", the same sentence holds in M. Fix a nonempty $x \in M$, then there is a function $f \in M$ such that

$$M \vDash f\colon \omega \to x \text{ is a surjection.}$$

Hence the function f is a surjection of ω onto x also according to H_{ω_1} (and V). Therefore $x \subseteq M$. Thus, M is transitive.

(b), (c). It is well known (e.g. see [66, §VII.3] or [40, §25]) that the map G which, given any real that recursively encodes an extensional and well-founded relation on ω, returns the Mostowski collapse of such relation, is Δ_1-definable over H_{ω_1}. Moreover,

$$\mathrm{H}_{\omega_1} \vDash \forall x\, \exists y\, (y \subseteq \omega \text{ and } G(y) = x).$$

Since $M \prec_1 \mathrm{H}_{\omega_1}$, the same sentence is satisfied by M, thus (b) and (c) follow. □

Proposition 4.3.4. *The following are equivalent:*

(a) $\mathbb{R} \subseteq \mathrm{L}$.

(b) $\mathsf{CH} + (\star)$.

Proof. (a)⇒(b). Since $\mathbb{R} \subseteq \mathrm{L}$, then $\mathrm{H}_{\omega_1} = \mathrm{L}_{\omega_1}$ and CH holds. Fix an $M \prec_1 \mathrm{L}_{\omega_1}$. By a direct corollary of Gödel's Condensation Lemma [20, Lemma 5.10], $M = \mathrm{L}_\alpha$ for some $\alpha \leq \omega_1$. When $\alpha < \omega_1$, M is countable, otherwise it coincides with $\mathrm{L}_{\omega_1} = \mathrm{H}_{\omega_1}$. Therefore $(\star)$ holds.

(b)⇒(a). Note that we must have $\omega_1^{\mathrm{L}} = \omega_1$, as otherwise there would exists an L-generic Cohen real, against Proposition 4.3.1. By Lévy-Shoenfield, $\mathrm{L}_{\omega_1} \prec_1 \mathrm{H}_{\omega_1}$. By CH, L_{ω_1} has size the continuum, and therefore, by $(\star)$, $\mathrm{L}_{\omega_1} = \mathrm{H}_{\omega_1}$. Hence $\mathbb{R} \subseteq \mathrm{L}$. □

Note that the statement $\mathbb{R} \subseteq \mathrm{L}$ is equivalent to saying that $\mathcal{D}_\mathrm{c}$ has breadth 0. A direct consequence of Ditor's Theorem 3.2.1(a) applied to the join-semilattice of real degrees is the following:

Proposition 4.3.5. *Suppose that $\mathcal{D}_\mathrm{c}$ has breath at most n for some $n \in \omega$. Then, $2^{\aleph_0} \leq \aleph_{n+1}$.*

However, using the definability of the constructibility preorder, we can say something more (cf. Proposition 4.3.4).

Theorem 4.3.6. *Suppose that $\mathcal{D}_\mathrm{c}$ has breath at most n for some $n \in \omega$. Then, either $2^{\aleph_0} \leq \aleph_n$ or ($\star$).*

Proof. We already proved the case $n = 0$ in Proposition 4.3.4. So fix an $n > 0$ and suppose that $2^{\aleph_0} = \aleph_{n+1}$ and that $\mathcal{D}_\mathrm{c}$ has breadth n, towards proving ($\star$).

Fix some $M \prec_1 \mathrm{H}_{\omega_1}$. By part (a) of Lemma 4.3.3, $\omega_1 \cap M$ is an ordinal $\alpha \leq \omega_1$. Let $(\mathcal{D}_\mathrm{c}^M, \leq_\mathrm{c}^M)$ be the join-semilattice of real degrees relativized to M. For every $x, y \in \mathbb{R} \cap M$, we have

$$x \leq_\mathrm{c}^M y \Leftrightarrow x \in \mathrm{L}_\alpha[y]. \tag{4.2}$$

There are two cases:

Case 1 $\alpha < \omega_1$. As the sentence "$\mathcal{D}_\mathrm{c}$ has breadth at most n" is Π_2 over H_{ω_1} and $M \prec_1 \mathrm{H}_{\omega_1}$, we have that $\mathcal{D}_\mathrm{c}^M$ has breadth at most n. By (4.2) and case assumption, any element of $\mathcal{D}_\mathrm{c}^M$ has at most countably many $\leq_\mathrm{c}^M$-predecessors. By part (a) of Ditor's Theorem 3.2.1, we have $|\mathcal{D}_\mathrm{c}^M| \leq \aleph_n$ and therefore $|\mathbb{R} \cap M| \leq |\mathcal{D}_\mathrm{c}^M| \cdot \aleph_0 \leq \aleph_n$. By part (b) of Lemma 4.3.3, we conclude that $|M| \leq \aleph_n < 2^{\aleph_0}$.

Case 2 $\alpha = \omega_1$. If $x, y \in \mathbb{R}$ and $y \in M$, and $x \leq_\mathrm{c} y$, then $x \in \mathrm{L}_{\omega_1}[y]$ so $x \leq_\mathrm{c}^M y$ by (4.2). In particular, $\mathcal{D}_\mathrm{c}^M$ is an ideal of $\mathcal{D}_\mathrm{c}$. By (b) of Ditor's Theorem 3.2.1, if $\mathcal{D}_\mathrm{c}^M$ is a proper ideal, then $|\mathcal{D}_\mathrm{c}^M| \leq \aleph_n$. In this case $|\mathbb{R} \cap M| \leq |\mathcal{D}_\mathrm{c}^M| \cdot \aleph_1 \leq \aleph_n$ and therefore $|M| \leq \aleph_n < 2^{\aleph_0}$. Otherwise, we would have $\mathbb{R} \subseteq M$ and then $M = \mathrm{H}_{\omega_1}$.

Either way ($\star$) holds. □

Remark 4.3.7. Theorem 4.3.6 is a meaningful extension of Proposition 4.3.5 only if it is consistent that $2^{\aleph_0} = \aleph_{n+1}$ and that $\mathcal{D}_c$ has breadth n. The natural question is whether this assumption is consistent for every $n > 0$. We know that for $n = 1$ this is the case, with the iterated Sacks model witnessing the consistency (see Chapter 0). It is open whether it is the case also for $n > 1$ (see Section 4.5.2).

Remark 4.3.8. In the light of Proposition 4.3.4, one may be tempted to think that $2^{\aleph_0} = \aleph_{n+1} + (\star)$ may imply that $\mathcal{D}_c$ has breadth at most n. This is certainly true when $n = 0$ (which is the content of Proposition 4.3.4), but it fails badly already for $n = 1$. Indeed, let $\mathbb{Q}$ be the countable-support iteration $\langle \mathbb{Q}_\alpha \mid \alpha < \omega_2 \rangle$ such that $\mathbb{Q}_\alpha$ is forced to be $\mathbb{S} \times \mathbb{S}$, the product of two Sacks forcing—see [32, Proposition 2.4] for some general properties of this kind of forcing. If we let G be a $\mathbb{Q}$-generic filter over L, then it can be shown that, in $\mathrm{L}[G]$, the continuum is $\aleph_2$ and $(\star)$ holds, but the breadth of $\mathcal{D}_c$ in $\mathrm{L}[G]$ is 2, cofinally—i.e. $\mathcal{D}_c$ has breadth 2 above $\mathbf{a}$ for every $\mathbf{a} \in \mathcal{D}_c$.

4.4 Definable Sierpiński coverings

The next theorem shows the connection between the existence of Σ^1_2 Sierpiński coverings and the breadth of the join-semilattice of the real degrees. The case $n = 0$ has already been shown by Törnquist and Weiss in [70]—see the introduction of this chapter.

Theorem 4.4.1. *For every $n \in \omega$, the following are equivalent:*

(a) *$\mathcal{D}_c$ has breadth at most n.*

(b) *There are Σ^1_2 sets $A_0, \dots, A_{n+1} \subseteq \mathbb{R}^{n+2}$ such that $\mathbb{R}^{n+2} = \bigcup_{i \leq n+1} A_i$ and for all $i \leq n+1$, for all $\ell \in \mathcal{L}_i(\mathbb{R}^{n+2})$, $\ell \cap A_i$ is countable.*

(c) *There are Σ^1_2 sets $A_0, \dots, A_{n+2} \subseteq \mathbb{R}^{n+3}$ such that $\mathbb{R}^{n+3} = \bigcup_{i \leq n+2} A_i$ and for all $i \leq n+2$, for all $\ell \in \mathcal{L}_i(\mathbb{R}^{n+3})$, $\ell \cap A_i$ is finite.*

Before delving into its proof, we need the following lemma.

Lemma 4.4.2. *Given a Σ_1 formula $\varphi(x, y)$ in the language of set theory, the set*

$$\left\{ (x, y) \in \mathbb{R}^2 \mid x \leq_c y \text{ and } \mathrm{L}[y] \vDash \varphi(x, y) \right\}$$

is Σ^1_2.

Proof. It suffices to prove that our set is Σ_1 over H_{ω_1} [40, Lemma 25.25]. In other words, we need to show that there is a Σ_1 formula $\psi(x, y)$ in the language of set theory such that

$$\big(x \leq_{\mathrm{c}} y \text{ and } \mathrm{L}[y] \vDash \varphi(x, y)\big) \Leftrightarrow \mathrm{H}_{\omega_1} \vDash \psi(x, y).$$

By Gödel's Condensation Lemma, for every $x \in \mathbb{R} \cap \mathrm{L}[y]$,

$$\mathrm{L}[y] \vDash \varphi(x, y) \Leftrightarrow \mathrm{L}_\delta[y] \vDash \varphi(x, y), \text{ for some countable ordinal } \delta.$$

Then, for any $x, y \in \mathbb{R}$,

$$(4.3)\quad (x \leq_{\mathrm{c}} y \text{ and } \mathrm{L}[y] \vDash \varphi(x, y)) \Leftrightarrow \mathrm{H}_{\omega_1} \vDash \exists\delta\big(x \in \mathrm{L}_\delta[y] \text{ and } \mathrm{L}_\delta[y] \vDash \varphi(x, y)\big).$$

The sentence "$x \in \mathrm{L}_\delta[y]$" is $\Delta_1(x, y, \delta)$ over H_{ω_1}. Regarding the sentence "$\mathrm{L}_\delta[y] \vDash \varphi(x, y)$", it does not matter if we interpret it as a genuine satisfaction relation [20, Ch. 1, §9] or as a relativization [40, Definition 12.6], because in both cases, the complexity of the sentence is at most $\Delta_1(x, y, \delta)$ over H_{ω_1}. Hence our set is Σ_1 over H_{ω_1}. □

We also need the following theorem.

Theorem 4.4.3 (Mansfield-Solovay, [41, Corollary 14.9]). *If $X \subseteq \mathbb{R}$ is $\Sigma^1_2(c)$ and $X \not\subseteq \mathrm{L}[c]$, then X contains a nonempty perfect set.*

Now, we are ready to prove our main theorem.

Proof of Theorem 4.4.1. (a)⇒(b). For each $i \leq n + 1$, let A_i be the set

$$\{(x_0, \ldots, x_{n+1}) \in \mathbb{R}^{n+2} \mid \exists j \neq i\, (x_i, x_j \leq_{\mathrm{c}} \textstyle\bigoplus_{k \neq i,j} x_k \text{ and } \mathrm{L}\big[\bigoplus_{k \neq i,j} x_k\big] \vDash x_i \trianglelefteq x_j)\}$$

where $\trianglelefteq$ is the canonical Σ_1 well-ordering of $\mathrm{L}[\bigoplus_{k \neq i,j} x_k]$. This is a Σ^1_2 definition by Lemma 4.4.2.

Next, we show that the A_i's cover $\mathbb{R}^{n+2}$ and that for each $i \leq n + 1$ and for each line $\ell \in \mathcal{L}_i(\mathbb{R}^{n+2})$, $\ell \cap A_i$ is countable. Pick any $(x_0, \ldots, x_{n+1}) \in \mathbb{R}^{n+2}$. As, by hypothesis, $\mathcal{D}_{\mathrm{c}}$ has breadth at most n, it follows that there are distinct

$i, j \leq n+1$ such that $x_i, x_j \leq_{\mathrm{c}} \bigoplus_{k \neq i,j} x_k$. Now, either $\mathrm{L}[\bigoplus_{k \neq i,j} x_k] \models x_i \trianglelefteq x_j$, and then, by definition, $(x_0, \ldots, x_{n+1}) \in A_i$, or $\mathrm{L}[\bigoplus_{k \neq i,j} x_k] \models x_i \triangleright x_j$, and then $(x_0, \ldots, x_{n+1}) \in A_j$. Thus $\mathbb{R}^{n+2} = \bigcup_{i \leq n+1} A_i$.

Fix an $i \leq n+1$ and an $n+1$-tuple $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$. By definition, for each $y \in \mathbb{R}$,

$$(x_0, \ldots, x_{i-1}, y, x_i, \ldots, x_n) \in A_i \Leftrightarrow \\ \exists j \leq n \Big(y, x_j \leq_{\mathrm{c}} \bigoplus_{k \neq j} x_k \text{ and } \mathrm{L}\Big[\bigoplus_{k \neq j} x_k\Big] \models y \trianglelefteq x_j \Big).$$

Since the choices of the js in the formula above are finite, and each initial segment of $\trianglelefteq \restriction \mathbb{R}$ is countable, it follows that the set

$$\{y \in \mathbb{R} \mid (x_0, \ldots, x_{i-1}, y, x_i, \ldots, x_n) \in A_i\}$$

is countable.

(a)$\Rightarrow$(c). For each $i \leq n+2$ let A_i be the set

$$\Big\{(x_0, \ldots, x_{n+2}) \in \mathbb{R}^{n+3} \mid \exists j, l \neq i \, (j \neq l \text{ and } x_i, x_j, x_l \leq_{\mathrm{c}} \bigoplus_{k \neq i,j,l} x_k \text{ and} \\ \mathrm{L}\Big[\bigoplus_{k \neq i,j,l} x_k\Big] \models \text{“}x_i, x_j \trianglelefteq x_l \text{ and } f(x_i) \leq f(x_j), \text{ where } f \text{ is the} \\ \trianglelefteq\text{-least bijection between the } \trianglelefteq\text{-predecessors of } x_l \text{ and } \omega\text{”}\Big\}$$

where, as before, $\trianglelefteq$ is the canonical Σ_1 well-ordering of $\mathrm{L}[\oplus_{k \neq i,j,l} x_k]$. These are Σ^1_2 sets by Lemma 4.4.2.

Arguing as in case (a), it follows that $\mathbb{R}^{n+3} = \bigcup_{i \leq n+2} A_i$ and that for any $i \leq n+2$, for any $(x_0, \ldots, x_{n+1}) \in \mathbb{R}^{n+2}$, the set

$$\{y \in \mathbb{R} \mid (x_0, \ldots, x_{i-1}, y, x_i, \ldots, x_{n+1}) \in A_i\}$$

is finite.

(b)$\Rightarrow$(a). Towards a contradiction, let $B = \{[b_0]_{\mathrm{c}}, \ldots, [b_n]_{\mathrm{c}}\}$ be a set of $n+1$ real degrees such that for every $L \in [B]^n$, $\bigoplus L \neq \bigoplus B$—if $n = 0$ this reads as: let $b_0 \in \mathbb{R}$ be such that $b_0 \notin \mathrm{L}$. Fix an $i \leq n$. For $u_{i+1}, \ldots, u_{n+1} \in \mathrm{L}[b_{i+1}, \ldots, b_n]$, the set

$$X_i(u_{i+1}, \ldots, u_{n+1}) = \{(b_0, \ldots, b_{i-1}, y, u_{i+1}, \ldots, u_{n+1}) \mid y \in \mathbb{R}\} \cap A_i$$

is countable by assumption, and Σ^1_2-definable with parameters in $\mathrm{L}[\bigoplus_{k \neq i} b_k]$. A straightforward consequence of Theorem 4.4.3 is that $X_i(u_{i+1}, \ldots, u_{n+1}) \subseteq \mathrm{L}[\bigoplus_{k \neq i} b_k]$. As $b_i \notin \mathrm{L}[\bigoplus_{k \neq i} b_k]$, it follows that

$$(b_0, \ldots, b_i, u_{i+1}, \ldots, u_{n+1}) \notin A_i.$$

Therefore,

$$\forall i \leq n \, \forall u_{i+1}, \ldots, u_{n+1} \in \mathrm{L}[b_{i+1}, \ldots, b_n] \Big((b_0, \ldots, b_i, u_{i+1}, \ldots, u_{n+1}) \notin A_i \Big).$$

As $\mathrm{L}[b_{i+1}, \ldots, b_n] \subseteq \mathrm{L}[b_i, \ldots, b_n]$ for all $i \leq n$, we have that

$$\forall i \leq n \, \forall u_{i+1}, \ldots, u_{n+1} \in \mathrm{L}[b_{i+1}, \ldots, b_n] \Big((b_0, \ldots, b_i, u_{i+1}, \ldots, u_{n+1}) \notin \textstyle\bigcup_{k \leq i} A_k \Big).$$

In particular, when $i = n$, we have $\forall u_{n+1} \in \mathrm{L} \Big((b_0, \ldots, b_n, u_{n+1}) \notin \bigcup_{k \leq n} A_k \Big)$. Since, by hypothesis the A_is cover $\mathbb{R}^{n+2}$, it follows that

$$\forall u_{n+1} \in \mathrm{L} \Big((b_0, \ldots, b_n, u_{n+1}) \in A_{n+1} \Big).$$

If $\omega_1^{\mathrm{L}} = \omega_1$, then this would imply that the line determined by $(b_0, \ldots, b_n)$ intersects A_{n+1} in an uncountable set, against our assumption. If $\omega_1^{\mathrm{L}} < \omega_1$, then there is r, a Cohen real over L. Note that $\omega_1^{\mathrm{L}} = \omega_1^{\mathrm{L}[r]}$ and that, in $\mathrm{L}[r]$, the breadth of $\mathcal{D}_{\mathrm{c}}$ is infinite. By Shoenfield, $\mathbb{R}^{n+2} \cap \mathrm{L}[r] = \bigcup_{i<n+2} \bar{A}_i$ where $\bar{A}_i = A_i \cap \mathrm{L}[r]$, and for $i \leq n$

$$\mathrm{L}[r] \vDash \bar{A}_i \in \Sigma^1_2 \text{ and } \forall \ell \in \mathcal{L}_i (\ell \cap \bar{A}_i \text{ is countable}).$$

Replacing V with $\mathrm{L}[r]$ the argument above can be repeated reaching a contradiction.

(c)⇒(a): Towards a contradiction, let $B = \{[b_0]_{\mathrm{c}}, \ldots, [b_n]_{\mathrm{c}}\}$ be a set of $n+1$ real degrees such that for every $L \in [B]^n$, $\bigoplus L \neq \bigoplus B$. Fix an $i \leq n$. Arguing as before, for all $u_{i+1}, \dots, u_{n+2} \in \mathrm{L}[b_{i+1}, \dots, b_n]$ the set

$$X_i(u_{i+1}, \dots, u_{n+2}) = \{(b_0, \dots, b_{i-1}, y, u_{i+1}, \dots, u_{n+2}) \mid y \in \mathbb{R}\} \cap A_i$$

is finite by hypothesis, and Σ^1_2-definable with parameters in $\mathrm{L}[\bigoplus_{k \neq i} b_k]$. Since $b_i \notin \mathrm{L}[\bigoplus_{k \neq i} b_k]$, it follows that

$$\forall i \leq n \, \forall u_{i+1}, \dots, u_{n+2} \in \mathrm{L}[b_{i+1}, \dots, b_n] \left((b_0, \dots, b_i, u_{i+1}, \dots, u_{n+2}) \notin \textstyle\bigcup_{k \leq i} A_k \right).$$

In particular, $\forall u_{n+1}, u_{n+2} \in \mathrm{L} \left((b_0, \dots, b_n, u_{n+1}, u_{n+2}) \notin \bigcup_{k \leq n} A_k \right)$. For each $u_{n+2} \in \mathbb{R}$, the set

$$X_{n+1}(u_{n+2}) = \{(b_0, \dots, b_n, y, u_{n+2}) \mid y \in \mathbb{R}\} \cap A_{n+1}$$

is finite by assumption. Thus, the set $\bigcup_{q \in \mathbb{Q}} X_{n+1}(q)$ is countable. As for the case (b)⇒(a), we can restrict ourselves to the case $\omega_1^{\mathrm{L}} = \omega_1$. Therefore, there exists an $\bar{x} \in \mathbb{R} \cap \mathrm{L}$ such that $(b_0, \dots, b_n, \bar{x}, q) \notin X_{n+1}(q)$ for all $q \in \mathbb{Q}$. It follows that

$$\forall q \in \mathbb{Q} \left((b_0, \dots, b_n, \bar{x}, q) \notin \textstyle\bigcup_{k \leq n+1} A_k \right)$$

and since by hypothesis the A_is cover $\mathbb{R}^{n+3}$, it follows that

$$\forall q \in \mathbb{Q} \left((b_0, \dots, b_n, \bar{x}, q) \in A_{n+2} \right),$$

but this means that the line determined by $(b_0, \dots, b_n, \bar{x})$ intersects A_{n+2} in an infinite set, against our assumption. □

Note that Theorem 4.4.1 straightforwardly relativizes to any $a \in \mathbb{R}$, with (a) being "$\mathcal{D}_{\mathrm{c}}$ has breadth at most n above $[a]_{\mathrm{c}}$" and the A_is from (b) and (c) being $\Sigma^1_2(a)$.

4.5 Open questions

4.5.1 Covering $\mathbb{R}^2$ with Σ^1_2 clouds

There are several results similar to the theorems by Sierpiński and Kuratowski asserting the equivalence between $2^{\aleph_0} \leq \aleph_n$ and the possibility of covering the plane with sets having small intersections with prescribed families of geometric objects. The following is an example, where a subset A of $\mathbb{R}^2$ is a *cloud* if there exists a point, called the *center of* A, such that each line passing through it intersects A in finitely many points.

Theorem 4.5.1 (Komjáth, Schmerl)**.** *For every $n \in \omega$, the following are equivalent:*

(a) $2^{\aleph_0} \leq \aleph_n$.

(b) $\mathbb{R}^2$ *is covered by $n+2$ clouds with distinct, non-collinear centers.*

The notion of cloud was introduced by P. Komjáth who proved in [49] the implication (a)⇒(b) for all n, and converse implication for $n = 1$, while the general case of (b)⇒(a) is from [62]. Note that the plane cannot be covered with finitely many clouds with collinear centers, so the non-collinearity assumption is essential.

Then, Törnquist and Weiss in [70] showed the following result:

Theorem 4.5.2 (Törnquist, Weiss)**.** *The following are equivalent.*

(a) $\mathbb{R} \subseteq \mathrm{L}$.

(b) $\mathbb{R}^2$ *can be covered by three Σ^1_2 clouds with constructible, non-collinear centers.*

In light of Theorem 4.4.1, it is natural to ask:

Question 4.5.3. Let $n > 0$. Is "$\mathcal{D}_c$ has breadth at most n" equivalent to $\mathbb{R}^2$ being covered by $n+3$ Σ^1_2 clouds with distinct, constructible, non-collinear centers?

4.5.2 Large continuum and small real degrees' breadth

The problem of whether the cardinal bound of Ditor's Theorem 3.2.1(a) is sharp is, to some extent, still an open problem (see Chapter 3 for a detailed discussion on the matter). On top of this, we do not know whether the cardinal bound of Ditor's Theorem is optimal when we restrict our attention to the join-semilattice of real degrees. The iterated Sacks model witnesses the optimality of the cardinal bound for the join-semilattice of real degrees when its breadth is 1. However, this is all we know. Hence the following question (see Remark 4.3.7).

Question 4.5.4. Is it consistent relative to ZF that $2^{\aleph_0} = \aleph_3$ and that the breadth of $\mathcal{D}_c$ is 2?

A seemingly easier question in this direction is the following.

Question 4.5.5. Is $2^{\aleph_0} = \aleph_3 + (\star)$ consistent relative to ZF?

4.5.3 Covering $\mathbb{R}^3$ with constructible continuous functions

Let X be a set, $n \geq 1$ and $f\colon X^n \to X$. We say that a point $(x_0, \dots, x_n) \in X^{n+1}$ is *covered* by f if there is a permutation π on $n+1$ such that

$$f\left(x_{\pi(0)}, \dots, x_{\pi(n-1)}\right) = x_{\pi(n)}.$$

A family $\mathcal{F}$ of functions from X^n to X covers $A \subseteq X^{n+1}$ if every point of A is covered by some member of $\mathcal{F}$.

Abraham and Geschke [1] have shown that, for each $n \geq 2$, it is consistent with ZFC that $2^{\aleph_0} = \aleph_n$ and that $\mathbb{R}^n$ is covered by an $\aleph_1$ subset of $C(\mathbb{R}^{n-1})$. In the iterated Sacks model (in which $2^{\aleph_0} = \aleph_2$) the following stronger property holds (see [35] and [28, Theorem 73]): $\mathbb{R}^2$ is covered by $C(\mathbb{R}) \cap \mathrm{L}$, where $C(\mathbb{R}) \cap \mathrm{L}$ is the set of all continuous real functions coded in L.

It is easy to see that, for every $n > 0$, if $\mathbb{R}^{n+1}$ is covered by $C(\mathbb{R}^n) \cap \mathrm{L}$, then $\mathcal{D}_c$ has breadth at most n: indeed, for every $n+1$ reals $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$, there would be a constructibly coded continuous function $f : \mathbb{R}^n \to \mathbb{R}$ and a permutation π on $n+1$ such that $x_{\pi(n)} = f(x_{\pi(0)}, \dots, x_{\pi(n-1)})$, and thus $x_{\pi(n)}$ would be constructible relative to $\bigoplus_{i \neq \pi(n)} x_i$. Therefore, our next and last question is a more demanding version of Question 4.5.4, and a positive answer would yield a strengthening of Abraham and Geschke's result (at least for $n = 3$).

Question 4.5.6. Is it consistent relative to ZFC that $2^{\aleph_0} = \aleph_3$ and $\mathbb{R}^3$ is covered by $C(\mathbb{R}^2) \cap \mathrm{L}$?

Chapter 5

Real degrees in the side-by-side Sacks model

5.1 Introduction

In [60], Sacks introduced the perfect-set forcing—i.e. the forcing notion consisting of perfect closed sets of reals ordered by inclusion. This forcing, also known as *Sacks forcing*, has been widely used in descriptive set theory due to its feature of adding a particularly tame generic real of minimal degree of constructibility.

In [11], Laver and Baumgartner introduced the *iterated Sacks model*. This model is obtained by forcing over a model of CH (often the constructible universe) with a countable-support iteration of ω_2-many Sacks forcings. It has been the subject of intensive study (see e.g. [12, 27, 52, 75]), mainly due to its rich combinatorial theory, well enucleated by Pawlikowski and Ciesielski's *Covering Property Axiom* (CPA) [17].

Furthermore, the *side-by-side Sacks model*, which is obtained by forcing over a model of CH with a countable-support product of infinitely many Sacks forcings, has also been studied (see e.g. [10, 34, 37, 68]).

Much is already known on the structure of the constructibility real degrees in models obtained by forcing over the constructible universe with either an iteration or a finite product of Sacks forcings: forcing with a countable-support iteration of ω_2-many Sacks forcings results in the constructibility real degrees being well-ordered with order-type ω_2 [11, 30]; forcing with a product of n Sacks forcings, for some $n \in \omega$, results in the constructibility real degrees being isomorphic to the powerset lattice of n (see e.g. [42]).

This chapter addresses the case that has been less explored: What properties do the constructibility real degrees satisfy in the side-by-side Sacks model?

The main results of this chapter are the following, which show that an infinite product of Sacks forcings behaves very differently, at least real-degrees-wise, compared to a finite product. Here, $\mathrm{L}[G]$ is the generic extension of L obtained by a countable-support product of infinitely many Sacks forcings.

Theorem 5.1.1. *In* $\mathrm{L}[G]$*,* $(\mathcal{D}_{\mathrm{c}}, \leq_c)$ *is neither a meet-semilattice, nor* σ*-complete, nor complemented.*

Theorem 5.1.2. *In* $\mathrm{L}[G]$*,* $(\mathcal{D}_{\mathrm{c}}, \leq_c)$ *is rigid, i.e. it has no non-trivial automorphisms.*

Theorem 5.1.3. *In* $\mathrm{L}[G]$*, apart from the least and greatest (if it exists) real degrees, no other real degree is definable in* $(\mathcal{D}_{\mathrm{c}}, \leq_c)$*.*

In Section 5.2, we briefly discuss some basic definitions regarding Sacks forcing and its products. In Section 5.3, we prove a representation theorem for the join-semilattice $(\mathcal{D}_{\mathrm{c}}, \leq_c)$ in $\mathrm{L}[G]$ (Theorem 5.3.1). This representation is key for the proofs of Theorems 5.1.1-5.1.3, which are presented in Section 5.4. Finally, in Section 5.5, we prove the following result in ZF showing that, in some sense, we cannot improve our representation theorem.

Theorem 5.1.4. $(\mathcal{P}(\omega), \subseteq)$ *is not isomorphic to any ideal of* $(\mathcal{D}_{\mathrm{c}}, \leq_c)$*.*

5.2 Sacks forcing and its products

We mostly adhere to the notation used in [28]. A tree $T \subseteq {}^{<\omega}2$ is a *perfect binary tree* if every node of T has two incomparable extensions in T. The poset of all perfect binary trees ordered by inclusion is known as *Sacks forcing*, and it is denoted by $\mathbb{S}$. Clearly, $\mathbf{1}_{\mathbb{S}} = {}^{<\omega}2$. The *stem* of a condition $p \in \mathbb{S}$ is the $\subseteq$-maximal node $t \in p$ such that, for every $s \in p$, either $s \subseteq t$ or $t \subseteq s$.

Given $p \in \mathbb{S}$ and $n \in \omega$ we let p^n be the set of all those $t \in p$ that are $\subseteq$-minimal in p with respect to the property of having exactly n proper initial segments that have two immediate successors in the tree p. For $p, q \in \mathbb{S}$ and $n \in \omega$, we let

$$p \leq_n q \iff p \leq q \text{ and } p^n = q^n.$$

If $p \in \mathbb{S}$ and $n \in \omega$, there is a natural way of assigning to each finite binary sequence $\sigma \in {}^n2$ an element $p(\sigma)$ of p^n. Let $p*\sigma := \{s \in p \mid s \subseteq p(\sigma) \vee p(\sigma) \subseteq s\}$.

Given a cardinal κ, we denote by $\mathbb{S}^\kappa$ the countable-support product of κ-many Sacks forcing. A condition p of $\mathbb{S}^\kappa$ is a map from κ to $\mathbb{S}$ such that $\mathrm{supp}(p) := \{\alpha \in \kappa \mid p(\alpha) \neq \mathbf{1}_{\mathbb{S}}\}$, known as the *support* of p, is countable. For any subset $D \subseteq \kappa$, we let $\mathbb{S}^\kappa \restriction D$ denote the complete subforcing of $\mathbb{S}^\kappa$ defined as the set of all the conditions of $\mathbb{S}^\kappa$ whose support is included in D. Note that $\mathbb{S}^\kappa \restriction D$ is isomorphic to $\mathbb{S}^{|D|}$. Given a condition $p \in \mathbb{S}^\kappa$, we abuse the notation and denote by $p \restriction D$ the condition of $\mathbb{S}^\kappa \restriction D$ defined in the expected way: $(p \restriction D)(\alpha) = p(\alpha)$ if $\alpha \in D$, and $(p \restriction D)(\alpha) = \mathbf{1}_{\mathbb{S}}$ otherwise.

Given $p, q \in \mathbb{S}^\kappa$, some finite $F \subseteq \kappa$ and some $n \in \omega$, we let

$$p \leq_{F,n} q \iff p \leq q \text{ and } \forall \alpha \in F \, \Big(p(\alpha) \leq_n q(\alpha)\Big).$$

A *fusion sequence* is a sequence $(p_n)_{n\in\omega}$ of elements of $\mathbb{S}^\kappa$ such that there exists a $\subseteq$-increasing sequence $(F_n)_{n\in\omega}$ of finite sets with

1. $p_{n+1} \leq_{F_n,n} p_n$ for every $n \in \omega$, and
2. $\bigcup_{n\in\omega} F_n = \bigcup_{n\in\omega} \mathrm{supp}(p_n)$.

For every fusion sequence $(p_n)_{n\in\omega}$, we let

$$\mathcal{F}((p_n)_{n\in\omega}) := \Big(\bigcap_{n\in\omega} p_n(\alpha)\Big)_{\alpha\in\kappa}$$

be its *fusion*. It is easy to check that a fusion is always an element of $\mathbb{S}^\kappa$.

If $p \in \mathbb{S}^\kappa$, $F \subseteq \kappa$ and $n \in \omega$, and $\sigma \in {}^F({}^n2)$, let $p * \sigma$ be such that for all $\alpha \in F$, $(p * \sigma)(\alpha) = p(\alpha) * \sigma(\alpha)$ and for all $\alpha \notin F$, $(p * \sigma)(\alpha) = p(\alpha)$.

If $\kappa = \omega$, we write $p \leq_n q$ instead of $p \leq_{n,n} q$, and a fusion sequence is simply a sequence $(p_n)_{n\in\omega}$ of elements of $\mathbb{S}^\omega$ such that $p_{n+1} \leq_n p_n$ for every $n \in \omega$.

5.3 Representing the real degrees in the side-by-side Sacks model

Let κ be an infinite cardinal and fix an $\mathbb{S}^\kappa$-generic filter G over L. Let $\langle s_\alpha \mid \alpha \in \kappa \rangle$ be the generic sequence of Sacks reals added by G, i.e. for each $\alpha \in \kappa$, s_α is the unique element of $\bigcap_{p\in G}[p(\alpha)]$. Let $S := \{s_\alpha \mid \alpha \in \kappa\}$ be the set of these reals.

In $\mathrm{L}[G]$, define the set $\mathcal{R}$ as follows:

$$\mathcal{R} := \left\{x \in [\kappa]^{\leq\omega} \mid \forall\alpha \in \kappa\ (s_\alpha \leq_c x \Rightarrow \alpha \in x)\right\}.$$

This section is devoted to the proof of the following theorem, which is key to proving Theorems 5.1.1-5.1.3, as it unravels much of the combinatorics of the real degrees in $\mathrm{L}[G]$.

Theorem 5.3.1. *In* $\mathrm{L}[G]$, $(\mathcal{D}_\mathrm{c}, \leq_c) \cong (\mathcal{R}, \subseteq)$.

Before carrying on with the proof, let us highlight that the choice of the constructible universe as our ground model is not due to its particular properties, which are not employed in this chapter, but instead to the fact that we are interested in studying the *constructibility degrees* of the generic extension. Indeed, all the results in this chapter also hold if we were to choose a different ground model V, but then we would need to talk about V-degrees rather than constructibility degrees.

Given a constructible $D \subseteq \kappa$, we let $G \upharpoonright D$ be the set of all the conditions in G whose support is contained in D. We denote by $\dot{G} \upharpoonright D$ its canonical name. Note that $G \upharpoonright D$ is an $\mathbb{S}^\kappa \upharpoonright D$-generic filter over L.

In order to prove Theorem 5.3.1, we first need some preliminary technical results. The first one tells us that we can often assume $\kappa = \omega$ without loss of generality. This assumption simplifies the construction of fusion sequences, at least notationally.

A constructible set is said to be *constructibly countable* if it is countable in L.

Lemma 5.3.2. *In* $\mathrm{L}[G]$, *for every* $E \in [\kappa]^{\leq\omega}$ *there exists a constructible, constructibly countable* $D \subseteq \kappa$ *such that* $E \subseteq D$ *and* $E \in \mathrm{L}[G \restriction D]$.

Proof. We work in L. Fix some $p \in \mathbb{S}^\kappa$, some $\mathbb{S}^\kappa$-name $\dot{E}$ for E and a $\mathbb{S}^\kappa$-name $\dot{f}$ such that

$$p \Vdash \dot{f} : \omega \to \kappa \text{ with } \operatorname{ran}(\dot{f}) = \dot{E}.$$

Via a simple bookkeeping argument, we can inductively define a sequence $(p_n, F_n)_{n\in\omega}$ and a family of ordinals $\langle \alpha_\sigma \mid \sigma \in {}^{F_n}({}^n2) \text{ for some } n \in \omega\rangle$ such that

i. $(p_n)_{n\in\omega}$ is a fusion sequence witnessed by $(F_n)_{n\in\omega}$,

ii. $p_0 = p$,

iii. for all $n \in \omega$ and for all $\sigma \in {}^{F_n}({}^n2)$, $p_{n+1} * \sigma \Vdash \dot{f}(n) = \alpha_\sigma$,

iv. for all $n \in \omega$ and for all $\sigma \in {}^{F_n}({}^n2)$, $\alpha_\sigma \in F_{n+1}$.

Let q be the fusion of the p_ns and let D be its support. Then it follows from our construction that q forces $\dot{E} \subseteq D$ and $\dot{E} \in \mathrm{L}[\dot{G} \restriction D]$. By density, we are done. □

Note that for every infinite countable $D \subseteq \kappa$, $\mathbb{S}^\kappa \restriction D \cong \mathbb{S}^\omega$. In particular, Lemma 5.3.2 implies that every real added by $\mathbb{S}^\kappa$ belongs to some $\mathbb{S}^\omega$-generic extension.

The next proposition tells us that any countable subset of S can construct its own enumeration induced by the generic filter G.

Proposition 5.3.3. *In* $\mathrm{L}[G]$, *for every* $A \in [S]^{\leq\omega}$, *if we let* $e_A : A \to \kappa$ *be defined by* $e(s_\alpha) = \alpha$ *for every* $s_\alpha \in A$, *then:*

1) $e_A \leq_c A$.

2) $\mathrm{L}(A) \vDash$ "*A is countable*".

Proof. By Lemma 5.3.2, there exists a constructible, constructibly countable $D \subseteq \kappa$ such that $\mathrm{ran}(e_A) \subseteq D$ and $\mathrm{ran}(e_A) \in \mathrm{L}[G \restriction D]$. Since D is constructibly countable, 2) directly follows once we prove 1). Moreover, as both A and e_A belong to $\mathrm{L}[G \restriction D]$, we can suppose without loss of generality $\kappa = \omega$ (see the remark after Lemma 5.3.2).

We now show that there must exist a $q \in G$ such that $[q(n)] \cap [q(m)] = \emptyset$ for every distinct $n, m \in \omega$. We work in L. Fix any $p \in \mathbb{S}^\omega$. It is routine to inductively define a fusion sequence $(p_n)_{n\in\omega}$ below p such that for every $n \in \omega$, for every distinct $k, m < n$, $[p_{n+1}(k)] \cap [p_{n+1}(m)] = \emptyset$. Let q be its fusion so that q extends p and satisfies the wanted property. By density, we can find such a q in G.

By the properties of the condition q and the fact that $q \in G$, we have that, in $\mathrm{L}[G]$, $e_A(s)$ is the unique $n \in \omega$ such that $s \in [q(n)]$, for every $s \in A$. Since this definition is absolute modulo the parameters A and q, and since q is constructible, we conclude that e_A is constructible relative to A, i.e. $e_A \leq_c A$. □

Corollary 5.3.4. *In* $\mathrm{L}[G]$*, for every* $A \in [S]^{\leq\omega}$*, there is a real* r *such that* $r \equiv_c A$*.*

Proof. If A is finite, then the claim is trivial. So we can assume that A is infinite. By Proposition 5.3.3, the set A is countable in $\mathrm{L}(A)$. Thus, we can fix a surjection $\psi : \omega \to A$ in $\mathrm{L}(A)$. Let $r := \bigoplus_{k\in\omega} \psi(k)$. Clearly, $A \leq_c r$. Furthermore, since ψ belongs to $\mathrm{L}(A)$, we also have $r \leq_c A$. □

Given a $\mathbb{S}^\kappa$-name $\dot{r}$ for a real, and some condition $p \in \mathbb{S}^\kappa$, we let $\dot{r}_p$ be the longest initial segment of $\dot{r}$ decided by p. Note that if p does not force $\dot{r}$ to belong to the ground model, then $\dot{r}_p$ is a finite sequence. For each $\alpha \in \kappa$, $\dot{s}_\alpha$ is the canonical $\mathbb{S}^\kappa$-name for the α-th generic Sacks real.

Proposition 5.3.5. *Let* $\alpha \in \kappa$ *and* $p \in \mathbb{S}^\kappa$ *and let* $\dot{r}$ *be a* $\mathbb{S}^\kappa$*-name for a real. The following are equivalent:*

1) $p \Vdash \dot{s}_\alpha \leq_c \dot{r}$.

2) $p \Vdash \dot{r} \notin \mathrm{L}\big[\dot{G} \restriction (\kappa \setminus \{\alpha\})\big]$.

3) *For every $q \leq p$ there exist $q_0, q_1 \leq q$ with $q_0 \restriction (\kappa \setminus \{\alpha\}) = q_1 \restriction (\kappa \setminus \{\alpha\})$ such that $\dot{r}_{q_0}$ and $\dot{r}_{q_1}$ are incomparable.*

Proof. 1) $\Rightarrow$ 2): By contraposition, suppose that p does not force $\dot{r} \notin \mathrm{L}[\dot{G} \restriction (\kappa \setminus \{\alpha\})]$. Then there exists $q \leq p$ such that q forces $\dot{r} \in \mathrm{L}[\dot{G} \restriction (\kappa \setminus \{\alpha\})]$. Then, since $\dot{s}_\alpha$ is always forced not to belong to $\mathrm{L}[\dot{G} \restriction (\kappa \setminus \{\alpha\})]$ by mutual genericity, q forces $\dot{s}_\alpha \not\leq_c \dot{r}$. Therefore $p \nVdash \dot{s}_\alpha \leq_c \dot{r}$.

2) $\Rightarrow$ 3): Again by contraposition, suppose that there exists $q \leq p$ such that for every $q_0, q_1 \leq q$, if $q_0 \restriction (\kappa \setminus \{\alpha\}) = q_1 \restriction (\kappa \setminus \{\alpha\})$, then $\dot{r}_{q*\sigma_0}$ and $\dot{r}_{q*\sigma_1}$ are comparable. Equivalently, for any $z \leq q$, any initial segment of $\dot{r}$ decided by z is already decided by $z \restriction (\kappa \setminus \{\alpha\})$. Thus, $q \Vdash \dot{r} \in \mathrm{L}[\dot{G} \restriction (\kappa \setminus \{\alpha\})]$.

3) $\Rightarrow$ 1): Suppose that p and $\dot{r}$ satisfy the hypotheses of 3). By Lemma 5.3.2 and the remark afterward, we can suppose without loss of generality that $\kappa = \omega$. Hence, we will denote α by n so to highlight that we are talking about a natural number.

By density, it suffices to show that there exists a $q \leq p$ such that $q \Vdash \dot{s}_n \leq_c \dot{r}$.

Claim 5.3.5.1. *For every $m > n$, for every $q \leq p$, for every $\sigma_0, \sigma_1 \in {}^m({}^m2)$, if $\sigma_0(n) \neq \sigma_1(n)$, then there exists a $z \leq_m q$ such that $\dot{r}_{z*\sigma_0}$ and $\dot{r}_{z*\sigma_1}$ are incomparable.*

Proof. Fix an $m > n$, a $q \leq p$ and $\sigma_0, \sigma_1 \in {}^m({}^m2)$ such that $\sigma_0(n) \neq \sigma_1(n)$. By hypothesis, there are $q_0, q_1 \leq q * \sigma_0$ such that $q_0 \restriction (\omega \setminus \{n\}) = q_1 \restriction (\omega \setminus \{n\})$, and such that $\dot{r}_{q_0}$ and $\dot{r}_{q_1}$ are incomparable.

Let
$$E := \Big\{k \in \omega \mid k \geq m \text{ or } (k < m \text{ and } \sigma_0(k) = \sigma_1(k))\Big\}.$$

Now let $p' \leq q*\sigma_1$ be defined as follows: for each k, if $k \in E$, then $p'(k) := q_0(k)$; if $k \notin E$, then $p'(k) := (q * \sigma_1)(k) = q(k) * \sigma_1(k)$.

Fix a $w \leq p'$ such that $\dot{r}_w$ is incomparable with either $\dot{r}_{q_0}$ or $\dot{r}_{q_1}$. Suppose without loss of generality that $\dot{r}_w$ is incomparable with $\dot{r}_{q_0}$ (otherwise substitute q_0 with q_1 in what follows). Then let $z \leq q$ be defined as follows: for every

$k \geq m$, $z(k) = w(k)$; for every $k < m$, let $z(k)^m = q(k)^m$ and, for every $\tau \in {}^m 2$,

$$z(k) * \tau = \begin{cases} q(k) * \tau & \text{if } \tau \notin \{\sigma_0(k), \sigma_1(k)\} \\ q_0(k) & \text{if } k \notin E \text{ and } \tau = \sigma_0(k) \\ w(k) & \text{otherwise} \end{cases} \tag{5.1}$$

Let us check that z is well-defined. More precisely, by fixing some $k < m$ and some $\tau \in {}^m 2$, we need to verify that $q(k)(\tau)$ is an initial segment of the stem of $z(k) * \tau$ as prescribed by (5.1): if τ is different from both $\sigma_0(k)$ and $\sigma_1(k)$ there is nothing to show, as by (5.1) $z(k) * \tau = q(k) * \tau$; if $k \notin E$ and $\tau = \sigma_0(k)$, then $q(k)(\sigma_0(k))$ is an initial segment of the stem of $q_0(k)$ because, by definition of q_0, $q_0(k) \leq q(k) * \sigma_0(k)$; if $k \in E$ and $\tau = \sigma_0(k) = \sigma_1(k)$, then $q(k)(\sigma_0(k))$ is an initial segment of the stem of w because, by definition of w, p' and q_0, $w(k) \leq p'(k) = q_0(k) \leq q(k) * \sigma_0(k)$; finally, if $k \notin E$ and $\tau = \sigma_1(k)$, then $q(k)(\tau)$ is an initial segment of the stem of $w(k)$ because, by definition of w and p', $w(k) \leq p'(k) = q(k) * \sigma_1(k)$.

Clearly, by definition of z, $z \leq_m q$. Moreover, it directly follows from (5.1) that $z * \sigma_1 = w$. Once we show $z * \sigma_0 \leq q_0$ we are done, as it would imply that $\dot{r}_{q_0}$ is an initial segment of $\dot{r}_{z*\sigma_0}$, and this, together with the fact that $\dot{r}_{z*\sigma_1} = \dot{r}_w$, results in $\dot{r}_{z*\sigma_0}$ and $\dot{r}_{z*\sigma_1}$ being incomparable, by our choice of w. To see this, pick any $k \in \omega$: if $k \geq m$, then $(z * \sigma_0)(k) = z(k)$, and $z(k) = w(k)$ by definition of z, and $w(k) \leq p'(k) = q_0(k)$ by the definition of w and p'; if $k < m$ and $k \notin E$, then $(z * \sigma_0)(k) = z(k) * \sigma_0(k) = q_0(k)$, by (5.1); finally, if $k < m$ and $k \in E$, then $(z * \sigma_0)(k) = z(k) * \sigma_0(k) = w(k)$ by definition of z, but, since $k \in E$, $w(k) \leq q_0(k)$.

Hence, $z * \sigma_0 \leq q_0$, and we are done. □

Claim 5.3.5.2. *There exists a $q \leq p$ such that for every $m > n$, for every $\sigma_0, \sigma_1 \in {}^m({}^m 2)$, if $\sigma_0(n) \neq \sigma_1(n)$, then $\dot{r}_{q*\sigma_0}$ and $\dot{r}_{q*\sigma_1}$ are incomparable.*

Proof. We define by induction a fusion sequence $(p_m)_{m\in\omega}$ such that: $p_m = p$ for every $m \leq n+1$; for every $m > n$, for every $\sigma_0, \sigma_1 \in {}^m({}^m 2)$, if $\sigma_0(n) \neq \sigma_1(n)$, then $\dot{r}_{p_{m+1}*\sigma_0}$ and $\dot{r}_{p_{m+1}*\sigma_1}$ are incomparable.

Suppose we have defined p_m with $m \geq n+1$, towards building p_{m+1}. Fix an enumeration $\{(\sigma_0^1, \sigma_1^1), \ldots, (\sigma_0^h, \sigma_1^h)\}$ of the couples (σ_0, σ_1) of elements of ${}^m({}^m 2)$

such that $\sigma_0(n) \neq \sigma_1(n)$. By Claim 5.3.5.1, we can define a $\leq_m$-descending sequence $(q_k)_{k \leq h}$ such that $q_0 = p_m$ and, for every $0 < k \leq h$, $\dot{r}_{q_k * \sigma_0^k}$ and $\dot{r}_{q_k * \sigma_1^k}$ are incomparable. Set $p_{m+1} = q_h$. Then $p_{m+1} \leq_m p_m$ and satisfies the desired property.

Now let q be the fusion of the p_ms. For every $m > n$, for every σ_0, σ_1 in ${}^m({}^m2)$ with $\sigma_0(n) \neq \sigma_1(n)$, we have that $\dot{r}_{q*\sigma_0}$ and $\dot{r}_{q*\sigma_1}$ are incompatible. Indeed, $q \leq_{m+1} p_{m+1}$, and since $\dot{r}_{p_{m+1}*\sigma_0}$ and $\dot{r}_{p_{m+1}*\sigma_1}$ are incomparable by construction, we are done. □

Now that we have proven Claim 5.3.5.2, fix some $q \leq p$ that satisfies its statement.

Claim 5.3.5.3. $q \Vdash \dot{s}_n \leq_c \dot{r}$.

Proof. Let us show that

$$q \Vdash \forall m > n\ \forall \sigma \in {}^m({}^m2)\ \Big(\dot{r}_{q*\sigma} \subset \dot{r} \implies \dot{s}_n \in [(q*\sigma)(n)]\Big). \tag{5.2}$$

Once (5.2) is proven, it follows that $q \Vdash \dot{s}_n \leq_c \dot{r}$. Indeed, it directly follows from (5.2) that

$$q \Vdash \dot{s}_n = \bigcup \Big\{ q(n)(\sigma(n)) \mid \exists m > n\ (\sigma \in {}^m({}^m2) \text{ and } \dot{r}_{q*\sigma} \subset \dot{r} \Big\}.$$

Suppose towards a contradiction that (5.2) does not hold. Then, there exists some $z \leq q$, some $m > n$ and $\sigma \in {}^m({}^m2)$ such that

$$z \Vdash \dot{r}_{q*\sigma} \subseteq \dot{r} \text{ and } \dot{s}_n \notin [(q*\sigma)(n)].$$

By extending z if necessary, we can assume that there exists a $\tau \in {}^m({}^m2)$ such that $z \leq q * \tau$. In particular, $\dot{r}_{q*\tau} \subseteq \dot{r}_z$. The statement $z \Vdash$ "$\dot{r}_{q*\sigma} \subseteq \dot{r}$" is equivalent to $\dot{r}_{q*\sigma} \subseteq \dot{r}_z$. Thus, $\dot{r}_{q*\sigma}$ and $\dot{r}_{q*\tau}$ are comparable. On the other hand, it follows from z forcing $\dot{s}_n \notin [(q*\sigma)(n)]$ that $\tau(n) \neq \sigma(n)$, and therefore, by the way we picked q, $\dot{r}_{q*\sigma}$ and $\dot{r}_{q*\tau}$ are incomparable. Contradiction. □

□

Corollary 5.3.6. *Let $\alpha \in \kappa$ and $p \in \mathbb{S}^\kappa$ and let $\dot{r}$ be a $\mathbb{S}^\kappa$-name for a real. The following are equivalent:*

1) $p \Vdash \dot{s}_\alpha \not\leq_c \dot{r}$.

2) $p \Vdash \dot{r} \in \mathrm{L}[\dot{G} \restriction (\kappa \setminus \{\alpha\})]$.

3) *For any $q \leq p$ there exists $z \leq q$ such that for every $z_0, z_1 \leq z$, if $z_0 \restriction (\kappa \setminus \{\alpha\}) = z_1 \restriction (\kappa \setminus \{\alpha\})$, then $\dot{r}_{z_0} = \dot{r}_{z_1}$.*

Proof. Directions 3) $\Rightarrow$ 2) $\Rightarrow$ 1) have already been shown in the proof for 1) $\Rightarrow$ 2) $\Rightarrow$ 3) of Proposition 5.3.5.

1) $\Rightarrow$ 3): Fix some $q \leq p$. By hypothesis, $q \leq p \Vdash \dot{s}_\alpha \not\leq_c \dot{r}$. Hence, by Proposition 5.3.5, q does not force $\dot{r} \notin \mathrm{L}[\dot{G} \restriction (\kappa \setminus \{\alpha\})]$. In particular, there exists a $z \leq q$ and a $\mathbb{S}^\kappa \restriction (\kappa \setminus \{\alpha\})$-name $\dot{r}'$ such that z forces $\dot{r} = \dot{r}'$. It quickly follows that $\dot{r}_{z_0} = \dot{r}_{z_1}$ for every $z_0, z_1 \leq z$ with $z_0 \restriction (\kappa\setminus\{\alpha\}) = z_1 \restriction (\kappa\setminus\{\alpha\})$. $\square$

The following result is key. It implies that any real in $\mathrm{L}[G]$ is equiconstructible with the set of Sacks reals constructible relative to it.

Proposition 5.3.7. *In $\mathrm{L}[G]$, for every real r, the following hold:*

1) *For every $A \in [S]^{\leq\omega}$, if $\{s \in S \mid s \leq_c r\} \subseteq A$, then $r \leq_c A$.*

2) $\{s \in S \mid s \leq_c r\} \leq_c r$.

In particular, $r \equiv_c \{s \in S \mid s \leq_c r\}$.

Proof. By Lemma 5.3.2 and the remark afterward, we can suppose without loss of generality that $\kappa = \omega$.

Given a $\sigma \in {}^n({}^n2)$ and some $m \leq n$, we denote by $\sigma \restriction\restriction m$ the map whose domain is m and such that $(\sigma \restriction\restriction m)(k) = \sigma(k) \restriction m$ for every $k < m$. Note that $\sigma \restriction\restriction m$ is the natural projection of σ onto ${}^m({}^m2)$.

Fix an $\mathbb{S}^\omega$-name $\dot{r}$ for a real r and fix a condition $p \in \mathbb{S}^\omega$. Before proving 1) and 2), we first need to define in L, by induction on n, a fusion sequence $(p_n)_{n\in\omega}$ along with an auxiliary map $\delta : \bigcup_{n\in\omega} {}^n({}^n2) \to 2$ that satisfy the following properties:

i) $p_0 = p$,

ii) For every $n \in \omega$ for every $\sigma \in {}^n({}^n2)$, $p_{n+1} * \sigma$ decides $\dot{r} \restriction n$,

iii) For every $n \in \omega$, for every $\sigma \in {}^n({}^n2)$, either:

iiia) $p_{n+1} * \sigma \Vdash \dot{s}_n \leq_c \dot{r}$, or

iiib) for every $q_0, q_1 \leq p_{n+1} * \sigma$, if $q_0 \restriction (\omega \setminus \{n\}) = q_1 \restriction (\omega \setminus \{n\})$, then $\dot{r}_{q_0} = \dot{r}_{q_1}$.

In the first case we set $\delta(\sigma) = 1$; otherwise we set $\delta(\sigma) = 0$,

iv) For every $m, n \in \omega$ with $m < n$, for every $\sigma_0, \sigma_1 \in {}^n({}^n2)$, if $\sigma_0(m) \neq \sigma_1(m)$ and $\sigma_0 \upharpoonleft m = \sigma_1 \upharpoonleft m$ and $\delta(\sigma_0 \upharpoonleft m) = 1$, then $\dot{r}_{p_{n+1}*\sigma_0}$ and $\dot{r}_{p_{n+1}*\sigma_1}$ are incomparable.

Note that iiia) and iiib) are mutually exclusive, as, by Corollary 5.3.6, iiib) implies $p_{n+1} * \sigma \Vdash \dot{s}_n \not\leq_c \dot{r}$. In particular, the map δ is well-defined.

Now let us proceed with the inductive construction. Suppose we have defined p_n, towards defining p_{n+1}. We do it in two steps: we first define a condition $q \leq_n p_n$ that satisfies ii) and iii). Then we define a condition $p_{n+1} \leq_n q$ to take care of iv).

Fix an enumeration $\{\sigma_0, \ldots, \sigma_h\}$ of $\sigma \in {}^n({}^n2)$. It is straightforward, using Corollary 5.3.6, to define a $\leq_n$-decreasing sequence $(q_k)_{k \leq h}$ such that $q_0 \leq_n p_n$, and, for every $k \leq h$, $q_k * \sigma_k$ decides $\dot{r} \restriction n$ and satisfies either iiia) or iiib). Given such a sequence, we let $q = q_h$.

We now define the condition $p_{n+1} \leq_n q$ that takes care of iv). Fix an enumeration $\{(\sigma_0^0, \sigma_1^0, m_0), \ldots, (\sigma_0^j, \sigma_1^j, m_j)\}$ of the triples (σ_0, σ_1, m) with $\sigma_0, \sigma_1 \in {}^n({}^n2)$ and $m < n$ such that

$$\sigma_0 \upharpoonleft m = \sigma_1 \upharpoonleft m \text{ and } \sigma_0(m) \neq \sigma_1(m) \text{ and } \delta(\sigma_0 \upharpoonleft m) = 1.$$

As before, we define a $\leq_n$-descending sequence $(z_k)_{k \leq j+1}$. Let $z_0 = q$. Fix some $k \leq j$ and suppose we have constructed z_k, towards defining z_{k+1}. Since $\delta(\sigma_0^k \upharpoonleft m_k) = 1$, we know, by δ's definition, that $p_{m_k+1} * (\sigma_0^k \upharpoonleft m_k) \Vdash \dot{s}_{m_k} \leq_c \dot{r}$. Now, since $z_k \leq_n p_n \leq_{m_k+1} p_{m_k+1}$, we have $z_k * (\sigma_0^k \upharpoonleft m_k) \leq p_{m_k+1} * (\sigma_0^k \upharpoonleft m_k)$. Therefore $z_k * (\sigma_0^k \upharpoonleft m_k) \Vdash \dot{s}_{m_k} \leq_c \dot{r}$. Finally, we let $z_{k+1} \leq_n z_k$ be such that $\dot{r}_{z_{k+1}*\sigma_0^k}$ and $\dot{r}_{z_{k+1}*\sigma_1^k}$ are incomparable—such a condition exists by Claim 5.3.5.1 of Proposition 5.3.5.

Let p_{n+1} be z_{j+1}. By construction, p_{n+1} satisfies conditions ii)-iv). This ends the inductive definition of the fusion sequence $(p_n)_{n\in\omega}$. Let w be its fusion. Going back to $\mathrm{L}[G]$, we can suppose that $w \in G$, by a density argument. Then, we claim the following (recall that we are assuming $\kappa = \omega$):

Claim 5.3.7.1. *For every $A \subseteq S$ with $\{s \in S \mid s \leq_c r\} \subseteq A$, $r \leq_c A$.*

Proof. Let $e_A : A \to \omega$ be as in the statement of Proposition 5.3.3. For each $m \in \omega$, let $\bar\sigma_m$ be the unique element of ${}^m({}^m2)$ such that $w * \bar\sigma_m \in G$, or, equivalently, such that $s_k \in [(w * \bar\sigma_m)(k)]$ for every $k \in \omega$. We want to prove the following statement:

$$\forall m \in \omega\ \forall \sigma \in {}^m({}^m2)\ \Big(\forall k < m\ (k \in \operatorname{ran}(e_A) \Rightarrow \sigma(k) = \bar\sigma_m(k)) \Rightarrow \dot r_{w*\sigma} \subset r\Big). \tag{5.3}$$

Once we show (5.3) we are done, as we would have (in $\mathrm{L}[G]$)

$$r = \bigcup\Big\{\dot r_{w*\sigma}\ \Big|\ \exists m \in \omega\ \Big(\sigma \in {}^m({}^m2) \text{ and } \forall k < m\ \Big(k \in \operatorname{ran}(e_A) \Rightarrow e_A^{-1}(k) \in [(w * \sigma)(k)]\Big)\Big)\Big\}.$$

Indeed, note that, by condition ii) of our fusion sequence, $\operatorname{length}(\dot r_{w*\sigma}) \geq m$ for every $\sigma \in {}^m({}^m2)$. Since, by Proposition 5.3.3, $e_A \in \mathrm{L}(A)$, we conclude that, by absoluteness, $r \in \mathrm{L}(A)$.

Towards showing that (5.3) holds, fix an $m \in \omega$ and a $\sigma \in {}^m({}^m2)$ such that for all $k < m$ with $k \in \operatorname{ran}(e_A)$, $\sigma(k) = \bar\sigma_m(k)$. From the definition of $\bar\sigma_m$ it directly follows that $\dot r_{w*\bar\sigma_m} \subset r$. We now build a sequence $(\tau)_{i\leq m}$ of elements of ${}^m({}^m2)$ such that $\tau_0 = \sigma$, $\tau_m = \bar\sigma_m$ and $\dot r_{w*\tau_{i+1}} = \dot r_{w*\tau_i}$ for all $i < m$. This proves that $\dot r_{w*\sigma} = \dot r_{w*\bar\sigma_m} \subset r$.

For each $i < m$, let τ_{i+1} be defined as the element of ${}^m({}^m2)$ such that $\tau_{i+1}(k) = \bar\sigma_m(k)$ for all $k \leq i$, and $\tau_{i+1}(k) = \sigma(k)$ for all $k > i$.

Fix an $i < m$, and suppose that $\tau_{i+1} \neq \tau_i$. By τ_i's definition, the only coordinate on which τ_{i+1} and τ_i can differ is the i-th. In particular, $\bar\sigma_m(i) = \tau_{i+1}(i) \neq \tau_i(i) = \sigma(i)$, and $\tau_{i+1} \restriction (m \setminus \{i\}) = \tau_i \restriction (m \setminus \{i\})$. By the way we picked σ, it must be the case that $i \notin \operatorname{ran}(e_A)$, or, equivalently, that $s_i \notin A$. In particular, by our hypothesis on A, $s_i \not\leq_c r$. Thus, $\delta(\bar{\sigma}_i) = 0$, as otherwise, by δ's definition, $p_{i+1} * \bar{\sigma}_i$ would force $\dot{s}_i \leq_c \dot{r}$, but this is impossible, as $p_{i+1} * \bar{\sigma}_i$, which is extended by $w * \bar{\sigma}_i$, belongs to the the generic filter G.

Note that both $w * \tau_i$ and $w * \tau_{i+1}$ extend $p_{i+1} * \bar{\sigma}_i$, and that $(w * \tau_i) \restriction (\omega \setminus \{i\}) = (w * \tau_{i+1}) \restriction (\omega \setminus \{i\})$. By the fact that $\delta(\bar{\sigma}_i) = 0$, we conclude that $\dot{r}_{w*\tau_i} = \dot{r}_{w*\tau_{i+1}}$ and we are done. □

Claim 5.3.7.2. *$\{s \in S \mid s \leq_c r\} \leq_c r$.*

Proof. Let the $\bar{\sigma}_m$s be defined as the beginning of proof of Claim 5.3.7.1. We want to show the following:

(‡) For every $m \in \omega$, for every $\sigma \in {}^m({}^m2)$ such that $\dot{r}_{w*\sigma} \subset r$, the following holds:

(‡)$_1$ For all $k < m$, if $s_k \leq_c r$, then $\sigma(k) = \bar{\sigma}_m(k)$.

(‡)$_2$ $s_m \leq_c r$ if and only if $\delta(\sigma) = 1$.

Once we show (‡) we are done. Indeed, it directly follows from (‡) that, in $\mathrm{L}[G]$,

$$\{s \in S \mid s \leq_c r\} = \Big\{ \bigcup \{w(k)(\tau(k)) \mid m > k \text{ and } \tau \in {}^m({}^m2) \text{ and } \dot{r}_{w*\tau} \subset r\} \\ \Big|\ k \in \omega \text{ and } \exists \sigma \in {}^k({}^k2)\ (\delta(\sigma) = 1 \text{ and } \dot{r}_{w*\sigma} \subset r) \Big\}$$

As w, $\dot{r}$ and δ are all constructible, then, by absoluteness, we conclude that $\{s \in S \mid s \leq_c r\} \in \mathrm{L}[r]$.

Towards proving (‡), fix an $m \in \omega$ and $\sigma \in {}^m({}^m2)$ such that $\dot{r}_{w*\sigma} \subset r$. We first want to prove (‡)$_1$. Suppose towards a contradiction that there exists a $k < m$ such that $s_k \leq_c r$ and $\sigma(k) \neq \bar{\sigma}_m(k)$. Let $\underline{k}$ be the least such. Let $\tau \in {}^m({}^m2)$ be defined as follows: for every $k < m$, if $k < \underline{k}$, then $\tau(k) = \sigma(k)$; otherwise, $\tau(k) = \bar{\sigma}_m(k)$.

We want to show that $\dot{r}_{w*\tau} \subset r$ and, moreover, that $\dot{r}_{w*\tau}$ and $\dot{r}_{w*\sigma}$ are incomparable, which would lead to a contradiction, as we are assuming $\dot{r}_{w*\sigma} \subset r$.

By the minimality of $\underline{k}$, we have that $\tau(k) = \bar{\sigma}_m(k)$ for every $k < m$ such that $s_k \leq_c r$. Thus, by (5.3) with $A = \{s \in S \mid s \leq_c r\}$, we conclude $\dot{r}_{w*\tau} \subset r$. So we are left to show that $\dot{r}_{w*\tau}$ and $\dot{r}_{w*\sigma}$ are incomparable in order to reach the desired contradiction. Note that, by the definition of τ, $\tau \upharpoonright\upharpoonright \underline{k} = \sigma \upharpoonright\upharpoonright \underline{k}$.

Subclaim 5.3.7.2.1. $\delta(\sigma \upharpoonright\upharpoonright \underline{k}) = 1$.

Proof. We prove the subclaim using an argument analogous to the one used to prove (5.3) and show that $\delta(\sigma \upharpoonright\upharpoonright \underline{k}) = \delta(\bar{\sigma}_{\underline{k}})$. Then, as $s_{\underline{k}} \leq_c r$, it must be that $\delta(\bar{\sigma}_{\underline{k}}) = 1$, and we conclude $\delta(\sigma \upharpoonright\upharpoonright \underline{k}) = \delta(\bar{\sigma}_{\underline{k}}) = 1$ as desired.

We build a sequence $(\rho_i)_{i \leq \underline{k}}$ of elements of ${}^{\underline{k}}({}^{\underline{k}}2)$ such that $\rho_0 = \sigma \upharpoonright\upharpoonright \underline{k}$, $\rho_{\underline{k}} = \bar{\sigma}_{\underline{k}}$ and $\delta(\rho_i) = \delta(\rho_{i+1})$ for all $i < \underline{k}$. This proves that $\delta(\sigma \upharpoonright\upharpoonright \underline{k}) = \delta(\bar{\sigma}_{\underline{k}})$.

For each $i < \underline{k}$, let ρ_{i+1} be defined as the element of ${}^{\underline{k}}({}^{\underline{k}}2)$ such that $\rho_{i+1}(k) = \bar{\sigma}_{\underline{k}}(k)$ for all $k \leq i$, and $\rho_{i+1}(k) = \sigma(k) \upharpoonright \underline{k}$ for all $k > i$.

Fix an $i < \underline{k}$, and suppose that $\rho_{i+1} \neq \rho_i$. By ρ_i's definition, the only coordinate on which ρ_{i+1} and ρ_i can differ is the i-th. In particular, $\bar{\sigma}_{\underline{k}}(i) = \rho_{i+1}(i) \neq \rho_i(i) = \sigma(i) \upharpoonright \underline{k}$, and $\rho_{i+1} \upharpoonright (\underline{k} \setminus \{i\}) = \rho_i \upharpoonright (\underline{k} \setminus \{i\})$. By the minimality of $\underline{k}$, it must be the case that $s_i \not\leq_c r$. Thus, $\delta(\bar{\sigma}_i) = 0$, as otherwise, by δ's definition, $p_{i+1} * \bar{\sigma}_i$ would force $\dot{s}_i \leq_c \dot{r}$, but this is impossible, as $p_{i+1} * \bar{\sigma}_i$, which is extended by $w * \bar{\sigma}_i$, belongs to the the generic filter G.

By the fact that $\delta(\bar{\sigma}_i) = 0$, and by δ's definition, we conclude that there exists a $\mathbb{S}^\omega \upharpoonright (\omega \setminus \{i\})$-name $\dot{r}'$ such that $p_{i+1} * \bar{\sigma}_i \Vdash \dot{r} = \dot{r}'$. In particular, for every condition $q \leq p_{i+1} * \bar{\sigma}_i$, the following holds by absoluteness of the constructibility preorder,

$$q \Vdash \dot{s}_{\underline{k}} \leq_c \dot{r}' \iff q \upharpoonright (\omega \setminus \{i\}) \Vdash \dot{s}_{\underline{k}} \leq_c \dot{r}'.$$

Thus, since both $p_{\underline{k}+1} * \rho_i$ and $p_{\underline{k}+1} * \rho_{i+1}$ extend $p_{i+1} * \bar{\sigma}_i$ and since $(p_{\underline{k}+1} * \rho_i) \upharpoonright (\omega \setminus \{i\}) = (p_{\underline{k}+1} * \rho_{i+1}) \upharpoonright (\omega \setminus \{i\})$, we conclude $\delta(\rho_i) = \delta(\rho_{i+1})$. □

By Subclaim 5.3.7.2.1 and by condition iv) of the fusion sequence, we know that $\dot{r}_{p_{m+1}*\tau}$ and $\dot{r}_{p_{m+1}*\sigma}$ are incomparable. But since $w \leq_{m+1} p_{m+1}$, we have $\dot{r}_{p_{m+1}*\tau} \subseteq \dot{r}_{w*\tau}$ and $\dot{r}_{p_{m+1}*\sigma} \subseteq \dot{r}_{w*\sigma}$. Therefore, $\dot{r}_{w*\tau}$ and $\dot{r}_{w*\sigma}$ are incomparable. We have reached the desired contradiction, and we conclude that $(\ddagger)_1$ holds.

Now we want to show $(\ddagger)_2$. Using $(\ddagger)_1$, and an argument analogous to the one used in the proof of Subclaim 5.3.7.2.1, we can show that $\delta(\sigma) = \delta(\bar{\sigma}_m)$. Indeed, note that by $(\ddagger)_1$, $\sigma(k) \neq \bar{\sigma}_m(k)$ implies $s_k \not\leq_c r$ for every $k < m$. But then, since $s_m \leq_c r$ holds if and only if $\delta(\bar{\sigma}_m) = 1$, we conclude that $s_m \leq_c r$ holds if and only if $\delta(\sigma) = 1$. □

□

Proof of Theorem 5.3.1. Consider the following map:

$$\begin{aligned} \Omega : (\mathcal{D}_c, \leq_c) &\longrightarrow (\mathcal{R}, \subseteq) \\ \mathbf{r} &\longmapsto \{\alpha \in \kappa \mid \mathbf{s}_\alpha \leq_c \mathbf{r}\}. \end{aligned}$$

We claim that Ω is an isomorphism. But first, we need to show that it is well-defined, i.e. that the range of Ω is a subset of $\mathcal{R}$. Fix an $\mathbf{r} \in \mathcal{D}_c$. It immediately follows from Lemma 5.3.2 that $\Omega(\mathbf{r}) \in [\kappa]^{\leq\omega}$. Now fix an $\alpha \in \kappa$ such that $\mathbf{s}_\alpha \leq_c \Omega(\mathbf{r})$ towards showing that $\alpha \in \Omega(\mathbf{r})$. Let $S_\mathbf{r} := \{s \in S \mid \mathbf{s} \leq_c \mathbf{r}\}$. Note that $\Omega(\mathbf{r}) = \operatorname{ran}(e_{S_\mathbf{r}})$, where the map e is the one defined in Proposition 5.3.3. By 2) of Proposition 5.3.7, $S_\mathbf{r} \leq_c \mathbf{r}$. We already noted that $S_\mathbf{r}$ must be countable. Finally, from 1) of Proposition 5.3.3 it follows that $\Omega(\mathbf{r}) = \operatorname{ran}(e_{S_\mathbf{r}}) \leq_c S_\mathbf{r}$. Thus, $\Omega(\mathbf{r}) \leq_c S_\mathbf{r} \leq_c \mathbf{r}$. Since, by assumption, $\mathbf{s}_\alpha \leq_c \Omega(\mathbf{r})$, we conclude that $\mathbf{s}_\alpha \leq_c \mathbf{r}$ and hence $\alpha \in \Omega(\mathbf{r})$ by Ω's definition. Hence, Ω is well-defined.

Claim 5.3.7.3. *Ω is injective.*

Proof. Fix two reals a, b and suppose that $\Omega(\mathbf{a}) = \Omega(\mathbf{b})$. In particular, $S_\mathbf{a} = S_\mathbf{b}$ (see the definition of $S_\mathbf{r}$ in the previous paragraph) and, by Proposition 5.3.7, $a \equiv_c S_\mathbf{a} = S_\mathbf{b} \equiv_c b$. Thus, $\mathbf{a} = \mathbf{b}$. □

Claim 5.3.7.4. *Ω is surjective.*

Proof. Fix an $x \in \mathcal{R}$. Let $A = \{s_\alpha \mid \alpha \in x\}$ and fix a real r such that $r \equiv_c A$—note that such a real exists by Corollary 5.3.4. We now prove that $\Omega(\mathbf{r}) = x$. Clearly, $x \subseteq \Omega(\mathbf{r})$. Now, fix an $\alpha \notin x$ towards showing that $\alpha \notin \Omega(\mathbf{r})$, or, equivalently, that $s_\alpha \not\leq_c A$. Since $\alpha \notin x$, we have $s_\alpha \not\leq_c x$, by $\mathcal{R}$'s definition. It follows from Corollary 5.3.6 that $x \in \mathrm{L}[G \restriction (\kappa \setminus \{\alpha\})]$. Therefore, also A belongs to $\mathrm{L}[G \restriction (\kappa \setminus \{\alpha\})]$, and we conclude that $s_\alpha \not\leq_c A$. □

With the following claim, we are done. Indeed, it implies that Ω is a join-semilattice homomorphism. Moreover, as a by-product, we get that $(\mathcal{R}, \cup)$ is a join-semilattice.

Claim 5.3.7.5. *For all* $\mathbf{a}, \mathbf{b} \in \mathcal{D}_c$, $\Omega(\mathbf{a} \oplus \mathbf{b}) = \Omega(\mathbf{a}) \cup \Omega(\mathbf{b})$.

Proof. Clearly, $\Omega(\mathbf{a}) \cup \Omega(\mathbf{b}) \subseteq \Omega(\mathbf{a} \oplus \mathbf{b})$. Now fix an $\alpha \notin \Omega(\mathbf{a}) \cup \Omega(\mathbf{b})$ towards showing that $\alpha \notin \Omega(\mathbf{a} \oplus \mathbf{b})$. By Ω's definition, $\mathbf{s}_\alpha \not\leq_c \mathbf{a}, \mathbf{b}$. By Corollary 5.3.6, both $\mathbf{a}$ and $\mathbf{b}$ belong to $\mathrm{L}[G \restriction (\kappa \setminus \{\alpha\})]$. Therefore, $\mathbf{a} \oplus \mathbf{b}$ also belong to $\mathrm{L}[G \restriction (\kappa \setminus \{\alpha\})]$, and we conclude that $\mathbf{s}_\alpha \not\leq_c \mathbf{a} \oplus \mathbf{b}$. Thus, $\alpha \notin \Omega(\mathbf{a} \oplus \mathbf{b})$. □

□

5.4 Proofs of Theorems 5.1.1, 5.1.2 and 5.1.3

In this section, we prove our main results, starting with Theorem 5.1.1. As before, we fix a generic filter G for $\mathbb{S}^\kappa$ over L.

Recall that a poset $(P, \leq)$ is said to be *complemented* if it is bounded and, for every $x \in P$, there is some (not necessarily unique) y such that $x \wedge y = \mathbf{0}_P$ and $x \vee y = \mathbf{1}_P$.

Proof of Theorem 5.1.1. By Theorem 5.3.1, we can substitute $(\mathcal{D}_c, \leq_c)$ with $(\mathcal{R}, \subseteq)$.

Let $s_0^+ := s_0 \cup \{0\}$ and $s_0^- := s_0 \setminus \{0\}$. We first claim that $s_0^+ \in \mathcal{R}$ and $s_0^- \notin \mathcal{R}$. Clearly, $s_0^+ \equiv_c s_0^- \equiv_c s_0$. Therefore, by mutual genericity of the s_α's, we have that s_0 is the only element of S which is constructible from s_0^+ and s_0^-. Since, by definition, $0 \in s_0^+$ and $0 \notin s_0^-$, we conclude that $s_0^+ \in \mathcal{R}$ and $s_0^- \notin \mathcal{R}$. Now we proceed with the proof of the theorem.

We claim that in $(\mathcal{R}, \subseteq)$ the set $\{s_0^+, \omega \setminus \{0\}\}$ does not have a greatest lower bound. Suppose towards a contradiction that such a greatest lower bound exists, and let us name it K. We want to show that the following holds:

$$s_0^- = \bigcup_{n \in s_0^-} \{n\} \subseteq K \subseteq s_0^+ \cap (\omega \setminus \{0\}) = s_0^-. \tag{5.4}$$

The first $\subseteq$ follows from having assumed K to be the greatest lower bound of $\{s_0^+, \omega \setminus \{0\}\}$, together with the fact that $\{n\} \in \mathcal{R}$ and $\{n\} \subseteq s_0^+, \omega \setminus \{0\}$ for every $n \in s_0^-$; the second $\subseteq$ follows from K being a lower bound of $\{s_0^+, \omega \setminus \{0\}\}$. It directly follows from (5.4) that $K = s_0^-$. However, s_0^- does not belong to $\mathcal{R}$, hence the contradiction.

We claim that in $(\mathcal{R}, \subseteq)$ the set $\{\{n\} \mid n \in s_0^-\} \subset \mathcal{R}$ does not have a least upper bound. If we suppose towards a contradiction that such a least upper bound exists, and we name it K, then (5.4) would still hold for reasons analogous to the ones employed in the previous paragraph. Therefore, we would reach the same contradiction, as K would coincide with s_0^-, which does not belong to $\mathcal{R}$.

We now prove that $(\mathcal{R}, \subseteq)$ is not complemented. If κ is uncountable, then the claim trivially follows since $(\mathcal{R}, \subseteq)$ is not bounded above. Hence suppose $\kappa = \omega$. Since every singleton $\{n\}$, with $n \in \omega$, belongs to $\mathcal{R}$, we must prove that there exists some $x \in \mathcal{R}$ such that $\omega \setminus x \notin \mathcal{R}$. We have already proven that $s_0^+ \in \mathcal{R}$. Moreover, $\omega \setminus s_0^+ \notin \mathcal{R}$, as $s_0 \equiv_c s_0^+ \equiv_c \omega \setminus s_0^+$, while $0 \notin \omega \setminus s_0^+$. □

We need the following couple of lemmas before proving Theorem 5.1.2.

Lemma 5.4.1. *In* $\mathrm{L}[G]$, *for every map* $f : \kappa \to \kappa$ *there exists a constructible, constructibly countable* $D \subseteq \kappa$ *closed under* f *such that* $f \restriction D \in \mathrm{L}[G \restriction D]$.

Proof. The proof is very similar to the one of Lemma 5.3.2. We work in L. Fix some $p \in \mathbb{S}^\kappa$ and some $\mathbb{S}^\kappa$-name $\dot{f}$ such that

$$p \Vdash \dot{f} : \kappa \to \kappa \text{ is a map.}$$

Via a simple bookkeeping argument, define a sequence $(p_n, F_n)_{n\in\omega}$ such that $(p_n)_{n\in\omega}$ is a fusion sequence witnessed by $(F_n)_{n\in\omega}$ with $p_0 = p$ and such that for every $n \in \omega$, for every $\sigma \in {}^{F_n}({}^n2)$, for every $\alpha \in F_n$, there exists some $\beta \in F_{n+1}$ such that $p_{n+1} * \sigma \Vdash \dot{f}(\alpha) = \beta$.

Let q be the fusion of the p_ns and let D be its support. Then, by construction, q forces D to be closed under $\dot{f}$ and $\dot{f} \restriction D \in \mathrm{L}[\dot{G} \restriction D]$. By density, we are done. □

Lemma 5.4.2. *In* L$[G]$, *for every bijection* $\psi : \kappa \to \kappa$ *there exists a constructible sequence* $(A_n)_{n\in\omega}$ *of pairwise disjoint, finite subsets of* κ, *and a constructible sequence* $(\alpha_n)_{n\in\omega}$ *of ordinals in* κ *such that* $\psi(\alpha_n) \in A_n$ *for every* n.

Proof. By Lemma 5.4.1, we can assume without loss of generality that $\kappa = \omega$. Fix some $p \in G$ and a $\mathbb{S}^\omega$-name $\dot{\psi}$ for ψ such that

$$p \Vdash \dot{\psi} : \omega \to \omega \text{ is a bijection.}$$

Working in L, we inductively define a fusion sequence $(p_n)_{n\in\omega}$ and a sequence $(A_n)_{n\in\omega}$ of finite subsets of ω and a sequence $(\alpha_n)_{n\in\omega}$ of positive integers such that:

i) $p_0 = p$.

ii) For every $n \in \omega$, for every $\sigma \in {}^n({}^n2)$, $p_{n+1} * \sigma \Vdash \dot{\psi}(\alpha_n) \in A_n$.

iii) For every $n \in \omega$, for every $\sigma \in {}^n({}^n2)$, for every $k \in A_n$, $p_{n+1} * \sigma$ decides the value of $\dot{\psi}^{-1}(k)$.

iv) For all distinct $n, m \in \omega$, $A_n \cap A_m = \emptyset$.

Now, we describe the inductive construction. Fix an $n \in \omega$ and suppose we defined p_m for all $m \leq n$ and A_m, α_m for all $m < n$ towards defining p_{n+1}, A_n and α_n. Let

$$E_n := \Big\{\alpha \in \omega \mid \exists m < n \; \exists \tau \in {}^m({}^m2) \; \exists k \in A_m \; \Big(p_{m+1} * \tau \Vdash \dot{\psi}(\alpha) = k\Big)\Big\}.$$

Let α_n be any positive integer not in E_n—note that this is possible as E_n is finite. Now, fix an enumeration $\{\sigma_0, \ldots, \sigma_h\}$ of ${}^n({}^n2)$. It is routine to define a $\leq_n$-decreasing sequence $(q_i)_{i\leq h}$ such that $q_0 \leq_n p_n$, and $q_i * \sigma_i$ decides $\dot{\psi}(\alpha_n)$ for every $i \leq h$. Given such a sequence, we let

$$A_n := \Big\{k \in \omega \mid \exists i \leq h \; \Big(q_i * \sigma_i \Vdash \dot{\psi}(\alpha_n) = k\Big)\Big\}.$$

We claim that $A_n \cap A_m = \emptyset$ for every $m < n$. Indeed, suppose towards a contradiction that there exists some $m < n$ with $A_n \cap A_m \neq \emptyset$, and fix a $k \in$

$A_n \cap A_m$. By definition of A_n, there exists an $i \leq h$ such that $q_i * \sigma_i \Vdash \dot{\psi}(\alpha_n) = k$. By construction, $q_i \leq_{m+1} p_{m+1}$, and therefore $q_i * \sigma_i \leq p_{m+1} * (\sigma_i \Uparrow m)$. By condition iii), $p_{m+1} * (\sigma_i \Uparrow m)$ already decides $\dot{\psi}^{-1}(k)$. Thus, it follows that $p_{m+1} * (\sigma_i \Uparrow m) \Vdash \dot{\psi}(\alpha_n) = k$. By definition of E_n, $\alpha_n \in E_n$, which is a contradiction, as we picked α_n outside of E_n.

Finally, defining a $p_{n+1} \leq_n q_h$ that satisfies condition iii) is routine.

The sequences defined in this way satisfy i)-iv). Let z be the fusion of the p_ns. Then z extends p and forces our sequences $(A_n)_{n\in\omega}$ and $(\alpha_n)_{n\in\omega}$ to have the desired properties. By density, we are done. □

Proof of Theorem 5.1.2. By Theorem 5.3.1, we can equivalently prove that $(\mathcal{R}, \subseteq)$ is rigid in $\mathrm{L}[G]$.

Every automorphism $f : \mathcal{R} \to \mathcal{R}$ is canonically induced by a bijection $\psi : \kappa \to \kappa$ such that, for every $x \in [\kappa]^{\leq\omega}$, $x \in \mathcal{R}$ if and only if $\psi[x] \in \mathcal{R}$. So let us assume towards a contradiction that there exists a bijection $\psi : \kappa \to \kappa$ such that $\psi \neq \mathrm{id}$ and, for every $x \in [\kappa]^{\leq\omega}$, $x \in \mathcal{R}$ if and only if $\psi[x] \in \mathcal{R}$.

Fix one such ψ, and assume $\psi(0) = 1$ just for the sake of simplicity. Fix the constructible sequences $(A_n)_{n\in\omega}$ and $(\alpha_n)_{n\in\omega}$ given by Lemma 5.4.2 for our ψ. Since the A_ns are mutually disjoint, we can assume without loss of generality that $0, 1 \notin A_n$ for any $n \in \omega$.

We now define (in $\mathrm{L}[G]$) an $r \in \mathcal{R}$ such that $1 \notin r$ and $s_0 \leq_c \psi^{-1}(r)$. To see why this leads to a contradiction, note the following: $\psi(0) = 1$ by assumption; therefore, as $1 \notin r$, we have $0 \notin \psi^{-1}(r)$; moreover, since $\psi^{-1}(r) \in \mathcal{R}$ and $s_0 \leq_c \psi^{-1}(r)$, it follows that $0 \in \psi^{-1}(r)$, and hence the contradiction.

We are now ready to define r. Let

$$r := \{0\} \cup \bigcup_{n\in s_0} A_n.$$

Note that $1 \notin r$. As the A_n's are finite, $r \in [\kappa]^{\leq\omega}$. Moreover, since the sequence $(A_n)_{n\in\omega}$ is constructible and the A_ns are mutually disjoint, we have $r \equiv_c s_0$. This implies that $r \in \mathcal{R}$, as $0 \in r$ by definition of r. Finally, note that s_0 is constructible relative to $\psi^{-1}(r)$, as the following holds for every $n \in \omega$:

$$\begin{aligned} n \in s_0 &\iff A_n \subseteq r \\ &\iff \psi^{-1}(A_n) \subseteq \psi^{-1}(r) \\ &\iff \alpha_n \in \psi^{-1}(r), \end{aligned}$$

where the first equivalence comes directly from the definition of r and the almost-disjointness of the A_n's, and the last equivalence follows from the properties of the constructible sequence $(\alpha_n)_{n\in\omega}$. Hence, we reach the contradiction described in the previous paragraph. □

A priori, it could be the case that the rigidity of $(\mathcal{D}_{\mathrm{c}}, \leq_c)$ follows from the definability of many degrees in $(\mathcal{D}_{\mathrm{c}}, \leq_c)$. However, Theorem 5.1.3 tells us that this is not the case.

Proof of Theorem 5.1.3. First note that $(\mathcal{D}_{\mathrm{c}}, \leq_c)$ has a greatest element only when $\kappa = \omega$. This fact, although being an easy consequence of Lemma 5.3.2, follows directly from Theorem 5.3.1.

In order to prove Theorem 5.1.3, we can use Theorem 5.3.1, and equivalently prove that no element of $\mathcal{R}$ other than $\emptyset$ (and ω, in case $\kappa = \omega$) is definable in the structure $(\mathcal{R}, \subseteq)$.

In $\mathrm{L}[G]$, fix some nonempty $a \in \mathcal{R}$ with $a \neq \kappa$ and a formula $\phi(x)$ without parameters such that $(\mathcal{R}, \subseteq) \vDash \phi(a)$. We want to find a $b \in \mathcal{R}$ with $a \neq b$ such that $(\mathcal{R}, \subseteq) \vDash \phi(b)$, thus showing that no parameter-free formula can pick out a unique element of $\mathcal{R}$ other than $\emptyset$ (and ω, in case $\kappa = \omega$) in $(\mathcal{R}, \subseteq)$.

Fix some $p \in G$ and a $\mathbb{S}^\kappa$-name $\dot{a}$ for a such that

$$p \Vdash (\dot{\mathcal{R}}, \subseteq) \vDash \phi(\dot{a}), \tag{5.5}$$

where $\dot{\mathcal{R}}$ is the $\mathbb{S}^\kappa$-name for $\mathcal{R}$. Moreover, since a is nonempty and different from κ, we may assume that

$$p \Vdash 0 \in \dot{a} \text{ and } 1 \notin \dot{a},$$

just for the sake of simplicity.

From now on we work in L. We want to define an automorphism $\sigma \in \mathrm{Aut}(\mathbb{S}^\kappa \downarrow p)$—i.e. an automorphism on the principal ideal of p in $\mathbb{S}^\kappa$—such that $p \Vdash$ "$\dot{s}_0 \equiv_c \sigma(\dot{s}_1)$" and $p \Vdash$ "$\dot{s}_1 \equiv_c \sigma(\dot{s}_0)$". We define it as follows: let $h : [p(0)] \to [p(1)]$ be the canonical homeomorphism between $[p(0)]$ and $[p(1)]$; given some $q \in \mathbb{S}^\kappa$ with $q \leq p$, we let $\sigma(q)$ be the condition defined by:

$$\forall \alpha \in \kappa, \quad \sigma(q)(\alpha) := \begin{cases} q(\alpha) & \text{if } \alpha \neq 0, 1 \\ \left\{ t \in {}^{<\omega}2 \mid h[N_t] \cap [q(1)] \neq \emptyset \right\} & \text{if } \alpha = 0 \\ \left\{ t \in {}^{<\omega}2 \mid h^{-1}(N_t) \cap [q(0)] \neq \emptyset \right\} & \text{if } \alpha = 1 \end{cases}$$

By construction, $\sigma(p) = p$. Moreover, it follows from the definition of σ that p forces $\sigma(\dot{s}_0) = h^{-1}(\dot{s}_1)$ and $\sigma(\dot{s}_1) = h(\dot{s}_0)$. Since the homeomorphism h is constructibly coded, we have $p \Vdash$ "$\dot{s}_0 \equiv_c \sigma(\dot{s}_1)$" and $p \Vdash$ "$\dot{s}_1 \equiv_c \sigma(\dot{s}_0)$" as desired. Moreover, $\sigma(\dot{s}_\alpha) = \dot{s}_\alpha$ for every $\alpha > 1$.

Let $\theta : \kappa \to \kappa$ be the bijection that simply swaps 0 and 1, leaving every other ordinal in κ fixed. Clearly, $\theta \circ \theta = \mathrm{id}$. Let $\dot{f}$ be the $\mathbb{S}^\kappa$-name for the function that maps every $x \in [\kappa]^{\leq\omega}$ to $\theta[x]$. Clearly, $\Vdash \dot{f} = \dot{f}^{-1}$. We claim that

$$p \Vdash \dot{f} \restriction \dot{\mathcal{R}} \text{ is an isomorphism from } (\dot{\mathcal{R}}, \subseteq) \text{ to } (\sigma(\dot{\mathcal{R}}), \subseteq). \tag{5.6}$$

In order to show that (5.6) holds, let us analyze how σ acts on the name $\dot{\mathcal{R}}$. By definition of $\mathcal{R}$, we have

$$p \Vdash \forall x \Big(x \in \dot{\mathcal{R}} \text{ iff } x \in [\kappa]^{\leq\omega} \text{ and } \forall \alpha \in \kappa \ (\dot{s}_\alpha \leq_c x \Rightarrow \alpha \in x) \Big).$$

By the Symmetry Lemma (see Chapter 0),

$$\sigma(p) = p \Vdash \forall x \Big(x \in \sigma(\dot{\mathcal{R}}) \text{ iff } x \in [\kappa]^{\leq\omega} \text{ and } \forall \alpha \in \kappa \ (\sigma(\dot{s}_\alpha) \leq_c x \Rightarrow \alpha \in x) \Big),$$

But since p forces $\sigma(\dot{s}_\alpha) \equiv_c \dot{s}_{\theta(\alpha)}$, we get

$$p \Vdash \forall x \Big(x \in \sigma(\dot{\mathcal{R}}) \text{ iff } x \in [\kappa]^{\leq\omega} \text{ and } \forall \alpha \in \kappa \ (\dot{s}_{\theta(\alpha)} \leq_c x \Rightarrow \theta(\alpha) \in \dot{f}(x)) \Big).$$

Finally, as $\Vdash \forall x \in [\kappa]^{\leq\omega} \ (\dot{f}(x) \equiv_c x)$, we conclude

$$p \Vdash \forall x (x \in \sigma(\dot{\mathcal{R}}) \text{ iff } \dot{f}(x) \in \dot{\mathcal{R}}),$$

which suffices to prove that (5.6) holds. By the Symmetry Lemma and (5.5), it follows that

$$p \Vdash (\sigma(\dot{\mathcal{R}}), \subseteq) \vDash \phi(\sigma(\dot{a})).$$

By (5.6),

$$p \Vdash (\dot{\mathcal{R}}, \subseteq) \vDash \phi\Big(\dot{f}(\sigma(\dot{a}))\Big).$$

By assumption $p \Vdash 0 \in \dot{a}$ and $1 \notin \dot{a}$. Therefore, by the Symmetry Lemma, p forces $0 \in \sigma(\dot{a})$ and $1 \notin \sigma(\dot{a})$. Thus, p forces $1 = \theta(0) \in \dot{f}(\sigma(\dot{a}))$, and, in particular, it forces $\dot{a} \neq \dot{f}(\sigma(\dot{a}))$.

Going back to $\mathrm{L}[G]$, if we let b be the evaluation $\dot{f}(\sigma(\dot{a}))$ according to the generic filter G, then $b \neq a$ and $(\mathcal{R}, \subseteq) \vDash \phi(b)$, as we wanted to show. □

Non-real degrees. The situation becomes more complicated if we are interested in the constructibility degrees of $\mathrm{L}[G]$, without focusing solely on the real ones. For instance, consider the set $\mathbf{S} := \{\mathbf{s}_n \mid n \in \omega\}$, i.e. the set of all the constructibility degrees of the generic Sacks reals. It can be shown, via an argument very similar to the one employed in Theorem 5.1.3, that the set $\mathbf{S}$ is amorphous in $\mathrm{L}(\mathbf{S})$. Recall that a set is said to be *amorphous* if it is infinite and it is not the disjoint union of two infinite subsets. This implies, in particular, that in $\mathrm{L}[G]$, the set $\mathbf{S}$ is not equiconstructible with any real. If it were, then $\mathrm{L}(\mathbf{S})$ would satisfy the axiom of choice, and thus there would be no amorphous sets in it.

5.5 Proof of Theorem 5.1.4

By Theorems 5.1.1-5.1.3, we cannot hope to improve Theorem 5.3.1 by devising, in $\mathrm{L}[G]$, an isomorphism between $(\mathcal{D}_{\mathrm{c}}, \leq_c)$ and $([\kappa]^{\leq\omega}, \subseteq)$. This is because $([\kappa]^{\leq\omega}, \subseteq)$ is a σ-complete lattice and it is far from being rigid, having 2^κ-many automorphisms. In this section, we show that this fact is not accidental, as we prove in ZF that $(\mathcal{P}(\omega), \subseteq)$ cannot be isomorphic to any ideal of $(\mathcal{D}_{\mathrm{c}}, \leq_c)$.

Lemma 5.5.1. *Suppose that $f : (\mathcal{P}(\omega), \subseteq) \to (\mathcal{D}_{\mathrm{c}}, \leq_c)$ is an order-embedding, then*

$$\Big\{f(\{n\}) \mid n \in \omega\Big\} \leq_c f(\omega) \Longrightarrow \mathbb{R} \subseteq \mathrm{L}[f(\omega)].$$

Proof. Fix an order-embedding $f : (\mathcal{P}(\omega), \subseteq) \to (\mathcal{D}_c, \leq_c)$ such that

$$A := \Big\{ f(\{n\}) \mid n \in \omega \Big\} \leq_c f(\omega).$$

Note that A is the image via f of the set of atoms of $\mathcal{P}(\omega)$. Fix a real x. Since in $\mathrm{L}[f(\omega)]$ the axiom of choice holds, we can fix an injection $g : \omega \to A$ such that $g \leq_c f(\omega)$. Let

$$y := \Big\{ n \in \omega \mid \exists m \in x \, \Big(f(\{n\}) = g(m) \Big) \Big\}.$$

Since f is an order-embedding, we must have $f(y) \leq_c f(\omega)$. But then

$$x = g^{-1}(\{\mathbf{d} \in A \mid \mathbf{d} \leq_c f(y)\}) \leq_c f(\omega),$$

where the equality follows from the fact that for any $n \in \omega$, $f(\{n\}) \leq_c f(y)$ if and only if $n \in y$, since f is an order-embedding; thus, x is constructible relative to $f(\omega)$ because A, g and $f(y)$ are constructible relative to $f(\omega)$, and because of the absoluteness of the constructibility preorder. Thus, every real is constructible relative to $f(\omega)$. □

Proof of Theorem 5.1.4. Suppose that $g : (\mathcal{P}(\omega), \subseteq) \to (\mathcal{D}_c, \leq_c)$ is an order-embedding with $\mathrm{ran}(g)$ being an ideal of $(\mathcal{D}_c, \leq_c)$, towards a contradiction. Consider the map $f : (\mathcal{P}(\omega), \subseteq) \to (\mathcal{D}_c, \leq_c)$ defined as follows: for every $x \in \mathcal{P}(\omega)$, let $f(x) = g(\{n+1 \mid n \in x\})$. Clearly, f is still an order-embedding, and its range is still an ideal of $\mathcal{D}_c$. Moreover, $g(\{0\}) \not\leq_c g(\omega \setminus \{0\}) = f(\omega)$, since g is assumed to be an embedding. Therefore, the real degree $g(\{0\})$ witnesses that not every real is constructible relative to $f(\omega)$. Furthermore,

$$\Big\{ f(\{n\}) \mid n \in \omega \Big\} = \{\mathbf{d} \leq_c f(\omega) \mid \mathbf{d} \text{ is an atom of } \mathcal{D}_c\} \leq_c f(\omega), \tag{5.7}$$

where the equality follows from $\mathrm{ran}(f)$ being an ideal of $\mathcal{D}_c$, and the second $\leq_c$ follows from the absoluteness of the constructibility preorder. However, (5.7) contradicts Lemma 5.5.1. □

We conclude with the following question, which asks to what extent Theorem 5.1.4 still holds if we weaken the assumption on the range of the isomorphism.

Question 5.5.2. Is it consistent with ZFC that $(\mathcal{P}(\omega), \subseteq)$ embeds into $(\mathcal{D}_{\mathrm{c}}, \leq_c)$ as a lattice? As a join-semilattice? As partial order?

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