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The c -Projective Symmetry Algebras of Kähler Surfaces

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Abstract

Let M be a Kähler manifold with complex structure J and Kähler metric g . A c -projective vector field is a vector field on M whose flow sends J -planar curves to J -planar curves, where J -planar curves are analogs of what (unparametrised) geodesics are for pseudo-Riemannian manifolds (without complex structure). The c -projective symmetry algebras of Kähler surfaces with essential (i.e., non-affine) c -projective vector fields are computed.

Keywords C -projective vector field · Kähler surface · J -planar curve

Mathematics Subject Classification 53A20 · 53B35 · 53B10

1 Introduction

Studying infinitesimal transformations of J -planar curves is a natural question that has received increased attention during the last two decades; a good summary is [7]. Historically, the question can be traced back to classical mathematicians like, for instance, Lie [14, 15], Painlevé [22], Beltrami [1] and Levi-Civita [13] who investigated transformations of the differential equations that describe trajectories of Hamiltonian systems up to reparametrisations.

An infinitesimal symmetry of a system of ordinary differential equations (ODEs) is a vector field on the space of dependent and independent variables whose local flow sends solutions to solutions. It is well known that the set of infinitesimal symmetries of a given ODE has the structure of a Lie algebra [21]. Studying the symmetry Lie algebras for ODEs is a powerful technique by which the equivalence of two ODEs

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can be investigated, i.e., whether one can be mapped into the other by a change of variables. Of course, the dimension of the symmetry Lie algebras of equivalent ODEs is the same. For instance, take the ODEs (we denote derivatives by subscripts)

$$y_{xx} = -e^{-2x} y_x^3 \quad (1)$$

and

$$y_{xx} = \frac{1}{2} y_x - \frac{1}{2} e^{-2x} y_x^3. \quad (2)$$

They are not equivalent as (1) has a 2-dimensional symmetry Lie algebra generated by

$$\partial_y \quad \text{and} \quad \partial_x + y \partial_y,$$

while (2) has a 3-dimensional symmetry Lie algebra generated by

$$\partial_y, \quad \partial_x + y \partial_y \quad \text{and} \quad y \partial_x + \frac{1}{2} y^2 \partial_y.$$

In other words, there is no change of variables $(x, y) \rightarrow (x_{\text{new}} = x_{\text{new}}(x, y), y_{\text{new}} = y_{\text{new}}(x, y))$ sending (1) into (2). Furthermore, neither (1) nor (2) can be mapped, by a change of variables, to

$$y_{xx} = 0 \quad (3)$$

as this latter ODE admits an 8-dimensional Lie algebra of symmetries, whose generators are

$$\partial_x, \quad \partial_y, \quad x \partial_x, \quad x \partial_y, \quad y \partial_x, \quad y \partial_y, \quad x^2 \partial_x + xy \partial_y \quad \text{and} \quad xy \partial_x + y^2 \partial_y. \quad (4)$$

These examples are, of course, not chosen randomly, but are geometrically motivated. Indeed, (1), (2) and (3) are the equations of “unparametrised geodesics” for certain Riemannian metrics. Specifically, (3) is the equation of *lines* of \mathbb{R}^2 , i.e. the geodesic curves of the Euclidean metric $dx^2 + dy^2$. Generators of the *projective algebra* of \mathbb{R}^2 are given by (4): the local flow of any generator of (4) sends lines of \mathbb{R}^2 into lines of \mathbb{R}^2 . Likewise, Eq. (1) is realised as the equation of unparametrised geodesics for the metric $e^{4x} dx^2 + e^{2x} dy^2$, Eq. (2) is realised as the equation of unparametrised geodesic of $e^{3x} dx^2 + e^x dy^2$. It follows, as a byproduct, that the metrics $e^{4x} dx^2 + e^{2x} dy^2$, $e^{3x} dx^2 + e^x dy^2$ and $dx^2 + dy^2$ are pairwise non-isometric.

Let us return to (3). By a suitable change of variables, (3) transforms into an ODE describing segments of *great circles* of the sphere. This insight is due to Liouville [16] and illustrated by Fig. 1. In fact, up to changing variables, the ODE (3) is the equation of unparametrised geodesics of any 2-dimensional metric with constant curvature.

In the present paper we ask a very similar question, yet in a different context. Before we were speaking about geodesics and (pseudo-)Riemannian metrics (i.e., of

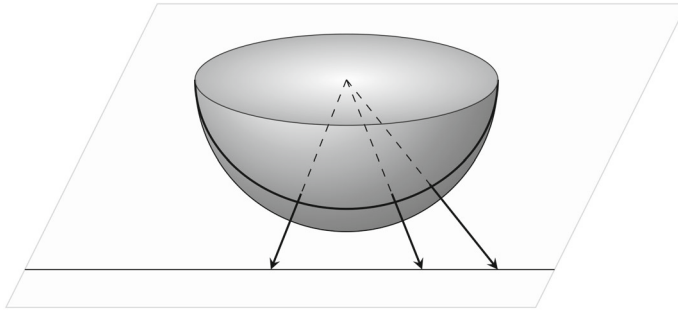


Fig. 1 The gnomonic (central) projection maps great circles to lines

arbitrary signature). Now we consider J -planar curves and Kähler metrics (of arbitrary signature) instead. The analog of projective symmetry algebras, in this context, are the so-called c-projective algebras (see below for a proper introduction). Our aim is to obtain, explicitly, the full, non-trivial¹ c-projective algebras of Kähler surfaces.

1.1 Preliminaries

Let (M, J, g) be a Kähler manifold of real dimension $2n$ and arbitrary signature. This means that M is a differentiable manifold with (pseudo-)Riemannian metric g and complex structure J , such that $\nabla J = 0$, where ∇ is the Levi-Civita connection of g , and the 2-form ω on M defined by $\omega(X, Y) = g(J(X), Y)$ is a symplectic form. In this context, the concept of unparametrised geodesics is replaced by that of J -planar curves.

Definition 1 Let (M, J, g) be a Kähler manifold. Let ∇ be its Levi-Civita connection. A curve $\gamma : I \subseteq \mathbb{R} \rightarrow \gamma(t) \in M$ such that

$$\nabla_{\dot{\gamma}} \dot{\gamma} \wedge \dot{\gamma} \wedge J\dot{\gamma} = 0$$

is called a J -planar curve.

It is easily verified that this definition is independent of parametrisation. The following definition captures the corresponding infinitesimal symmetries.

Definition 2 Let (M, J, g) be a Kähler manifold. A vector field is called *c-projective* if its local flow maps J -planar curves into J -planar curves.

It is well known that the c-projective vector fields of a Kähler manifold form a Lie algebra, see for instance [7], which we denote by $\mathfrak{p}(g)$, suppressing J in the notation. *Affine* vector fields, i.e. vector fields that preserve the Levi-Civita connection of g and the complex structure J , are simple examples of c-projective vector fields. C-projective vector fields that are not affine are called *essential*. We need to introduce two more

¹ By “non-trivial” we mean the existence of an essential (non-affine) c-projective vector field, see Definition 2.

concepts before we are able to state the main fact that serves as point of departure for the present paper.

Definition 3 Let g, \hat{g} be two Kähler metrics on the same complex manifold (M, J) . They are called *c-projectively equivalent* if they share the same J -planar curves.

The equivalence class of all Kähler surfaces that are c-projectively equivalent is called a *c-projective Kähler structure* and denoted by $(M, J, [g])$.

Provided $n \geq 2$, a c-projective vector field automatically preserves the complex structure, c.f. [4]. We also recall the definition of holomorphic sectional curvature (HSC), which is a restriction of the usual sectional curvature to J -invariant planes.

Definition 4 Let g be a Kähler metric with Riemann curvature R . The HSC of a J -invariant plane $\Pi = \langle v, Jv \rangle$ in $p \in M$ is defined by

$$K(\Pi) = \frac{R(v, Jv, v, Jv)}{\|v\|^4},$$

where $v \in T_pM$ is a vector at p . If the map $\Pi \mapsto K(\Pi)$ is constant and independent of p , then we say that g has constant HSC.

In [4] a non-sharp local description is obtained for c-projective Kähler structures admitting essential c-projective vector fields. We remark that the list does not claim to be sharp, and that indeed some parameter configurations would not lead to a metric. Such configurations are tacitly excluded from now on without further mentioning.

Fact 1 ([4]) *Let (M, J, g_0) be a Kähler surface of non-constant HSC and with an essential c-projective vector field. Then, in a neighborhood of almost every point of M , there are local coordinates such that a certain metric $g \in [g_0]$ and its Kähler form ω defined by $\omega(X, Y) = g(J(X), Y)$ are as in one of the cases below.*

(i) **Liouville type:** *In the new coordinates (x_0, x_1, s_0, s_1) ,*

$$g = (\rho_0 - \rho_1)(F_0^2 dx_0^2 + \varepsilon F_1^2 dx_1^2) + \frac{1}{\rho_0 - \rho_1} \left[\left(\frac{\rho'_0}{F_0} \right)^2 (ds_0 + \rho_1 ds_1)^2 + \varepsilon \left(\frac{\rho'_1}{F_1} \right)^2 (ds_0 + \rho_0 ds_1)^2 \right],$$

$$\omega = \rho'_0 dx_0 \wedge (ds_0 + \rho_1 ds_1) + \rho'_1 dx_1 \wedge (ds_0 + \rho_0 ds_1),$$

where $\varepsilon \in \{\pm 1\}$ and where the univariate functions ρ_0, ρ_1, F_0, F_1 are given as in one of the cases below.

- (L1) $\rho_i(x_i) = x_i, F_i = c_i \in \mathbb{R}$,
- (L2) $\rho_i(x_i) = c_i e^{(\beta-1)x_i}, F_i = d_i e^{-\frac{1}{2}(\beta+2)x_i}$ with $c_i, d_i \in \mathbb{R} (\beta \neq 1)$,
- (L3) $\rho_i(x_i) = x_i, F_i = c_i e^{-\frac{3}{2}x_i}, c_i \in \mathbb{R}$,
- (L4) $\rho_i(x_i) = -\tan(x_i), F_i = \frac{c_i e^{-\frac{3}{2}\beta x_i}}{\sqrt{|\cos(x_i)|}}, c_i \in \mathbb{R}$.

(ii) **Complex type:** *In the new coordinates (z, \bar{z}, s_0, s_1) ,*

$$\begin{aligned}
 g &= \frac{1}{4} (\bar{\rho} - \rho)(F^2 dz^2 - \bar{F}^2 d\bar{z}^2) \\
 &+ \frac{4}{\rho - \bar{\rho}} \left[\left(\frac{\bar{\rho}'}{F} \right)^2 (ds_0 + \rho ds_1)^2 - \left(\frac{\rho'}{\bar{F}} \right)^2 (ds_0 + \bar{\rho} ds_1)^2 \right], \\
 \omega &= \rho' dz \wedge (ds_0 + \bar{\rho} ds_1) + \bar{\rho}' d\bar{z} \wedge (ds_0 + \rho ds_1),
 \end{aligned} \tag{5}$$

where the holomorphic functions ρ, F are given as in one of the cases below (for brevity we denote $\rho' = \frac{\partial \rho}{\partial z}$ and $\bar{\rho}' = \frac{\partial \bar{\rho}}{\partial \bar{z}}$).

- (C1) $\rho(z) = z, F = \varsigma = \varsigma_0 + i\varsigma_1 \in \mathbb{C}$,
- (C2) $\rho(z) = e^{(\beta-1)z}, F = \varsigma e^{-\frac{1}{2}(\beta+2)z}$ with $\varsigma = \varsigma_0 + i\varsigma_1 \in \mathbb{C} (\beta \neq 1)$,
- (C3) $\rho(z) = z, F = \varsigma e^{-\frac{3}{2}z}, \varsigma = \varsigma_0 + i\varsigma_1 \in \mathbb{C}$,
- (C4) $\rho(z) = -\tan(z), F = \frac{\varsigma e^{-\frac{3}{2}\beta z}}{\sqrt{\cos(z)}}, \varsigma = \varsigma_0 + i\varsigma_1 \in \mathbb{C}$.

(iii) **Degenerate type:** In the new coordinates (x_0, x_1, s_0, s_1) ,

$$\begin{aligned}
 g &= -\rho h + \rho F^2 dx_0^2 + \frac{1}{\rho} \left(\frac{\rho'}{F} \right)^2 \theta^2, \\
 \omega &= -\rho d\tau + \rho' dx_0 \wedge \theta,
 \end{aligned}$$

where $h = \sum_{i,j} h_{ij}(s_0, s_1) ds_i ds_j$ and $\theta = dx_1 - \tau$, and where the functions ρ, F, τ are given as in one of the cases below.

- (D1) $\rho(x_0) = \frac{1}{x_0}, F(x_0) = \frac{c_1}{\sqrt{|x_0|}}$ and $\tau = s_0 ds_1$. Moreover, $h = G(s_1) ds_0^2 + \frac{1}{G(s_1)} ds_1^2$,
- (D2a) $\rho(x_0) = c_1 e^{-3x_0}, F(x_0) = d_1, c_1, d_1 \in \mathbb{R}, \tau = s_0 ds_1, h = G(s_1) ds_0^2 + \frac{1}{G(s_1)} ds_1^2$,
- (D2b) $\rho(x_0) = c_1 e^{(\beta-1)x_0}, F(x_0) = d_1 e^{-\frac{1}{2}(\beta+2)x_0} (\beta \neq 1, \beta \neq -2), c_1, d_1 \in \mathbb{R}, \tau = -\frac{1}{\beta+2} e^{-(\beta+2)s_0} G(s_1) ds_1, h = e^{-(\beta+2)s_0} G(s_1) (ds_0^2 + ds_1^2)$,
- (D3) $\rho(x_0) = \frac{1}{x_0}, F(x_0) = \frac{c_1 e^{-\frac{3}{2}x_0}}{\sqrt{|x_0|}}, \tau = -\frac{1}{3} e^{-3s_0} G(s_1) ds_1, h = e^{-3s_0} G(s_1) (ds_0^2 + ds_1^2)$.

To avoid misconceptions, we would like to point out a few particulars of the list in Fact 1:

- Not all the metrics listed in Fact 1 are of non-constant holomorphic sectional curvature. Theorem 1 is addressing this point.
- Not every metric of non-constant holomorphic curvature with an essential c-projective vector field is isometric to one explicitly stated in the list, but every such metric can be expressed by using a metric g from the list and a suitable

companion metric \hat{g} (detailed in “Appendix B”) using the formula

$$g[t_1, t_2] = \frac{\left(t_1 |\det(g)|^{\frac{1}{6}} g^{-1} + t_2 |\det(\hat{g})|^{\frac{1}{6}} \hat{g}^{-1} \right)^{-1}}{\sqrt{\det \left(t_1 |\det(g)|^{\frac{1}{6}} g^{-1} + t_2 |\det(\hat{g})|^{\frac{1}{6}} \hat{g}^{-1} \right)}}, \quad t_i \in \mathbb{R}. \quad (6)$$

The problem of describing these individual metrics, up to isometries, is not solved in the current paper, but is going to be addressed in a separate paper. For our purposes here it is sufficient to state the following: The three types in Fact 1 are disjoint, i.e., a metric satisfying the hypothesis of Fact 1 is either of Liouville, of complex or of degenerate type.

- Metrics obtained using Formula (6) have the same c-projective algebra as the metric g itself. In particular, for metrics of constant holomorphic sectional curvature, the algebra of c-projective vector fields is well understood and always isomorphic to that of the Fubini-Study metric (see Example 1). For the metrics of non-constant HSC which admit a non-affine c-projective vector field, we determine the full c-projective algebras in this paper.
- Fact 1 provides a characterisation modulo c-projective equivalence and indeed there are metrics in the list of Fact 1 that admit only c-projective vector fields that are affine (e.g., in the case L1). However, generically, the metrics that are c-projectively equivalent to one listed in Fact 1 admit essential c-projective vector fields. Note that c-projective algebras are invariant under c-projective transformations.

Before turning to our main results, let us consider the case of constant HSC more closely. Such metrics are c-projectively equivalent to the Fubini-Study metric [10, 12], and therefore their c-projective algebras are isomorphic.

Example 1 (Fubini-Study metric on $\mathbb{C}\mathbb{P}^2$) Let $z_1 = x + iy$ and $z_2 = s + it$ be complex coordinates on $\mathbb{C}\mathbb{P}^2$. The Fubini-Study metric on $\mathbb{C}\mathbb{P}^2$ is locally described by

$$g = \frac{(1 + \sum_i z_i \bar{z}_i) \sum_j dz_j d\bar{z}_j - \sum_{i,j} \bar{z}_j z_i dz_j d\bar{z}_i}{(1 + \sum_i z_i \bar{z}_i)^2}. \quad (7)$$

The complex structure J in these coordinates takes the standard form

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The c-projective algebra of (7) is isomorphic to $\mathfrak{sl}(3, \mathbb{C})$ [10, 20]. An explicit realisation of this algebra in terms of 16 generating vector fields can be found in “Appendix A”.

With this example in mind, the purpose of the present paper is twofold: we first identify the metrics in Fact 1 that have constant HSC, and then we determine the

full c-projective algebras of those metrics in Fact 1 that have non-constant HSC. In particular, we obtain the following four main theorems.

Theorem 1 *Among the metrics in Fact 1 the following have constant HSC:*

(L1) *Metrics of L1 type have constant HSC if $\varepsilon = -1$ and $c_1 = \pm c_0$.*

(L2) *Metrics of L2 type have constant HSC if*

- (i) $\beta = -\frac{1}{2}$ and $c_1 = \varepsilon c_0 \frac{d_1^2}{d_0^2}$,
- (ii) $\beta = -2$, $\varepsilon = -1$ and $d_1 = \pm d_0$.

(C1) *Metrics of C1 type have constant HSC if either $\Re(\zeta) = 0$ or $\Im(\zeta) = 0$.*

(C2) *Metrics of C2 type have constant HSC if*

- (i) $\beta = -\frac{1}{2}$ and either $\Re(\zeta) = 0$ or $\Im(\zeta) = 0$,
- (ii) $\beta = -2$ and either $\Re(\zeta) = 0$ or $\Im(\zeta) = 0$.

(D1) *Metrics of D1 type have constant HSC if the metric h has vanishing Gauß curvature.*

(D2a) *Metrics of D2a type have constant HSC if the metric h has Gauß curvature exactly $-\frac{9}{d_1^2}$.*

All other metrics in Fact 1 are of non-constant HSC.

Having answered the question which metrics of Fact 1 have constant HSC, we now turn to determining the c-projective vector fields of those metrics with non-constant HSC. We begin with Liouville type metrics.

Theorem 2 *Let g be a Kähler surface covered by Fact 1. Assume it is of non-constant HSC and that it is of Liouville type. Then its c-projective algebra is isomorphic to one of the following.*

(L1) *The c-projective algebra is 4-dimensional,*

$$\mathfrak{p}(g) = \langle \partial_{x_0} + \partial_{x_1} - s_1 \partial_{s_0}, x_0 \partial_{x_0} + x_1 \partial_{x_1} + 2s_0 \partial_{s_0} + s_1 \partial_{s_1}, \partial_{s_0}, \partial_{s_1} \rangle.$$

(L2) *The c-projective algebra is either 3-dimensional or 4-dimensional.*

(i) *If $\beta \neq 0$, then the c-projective algebra is 3-dimensional,*

$$\mathfrak{p}(g) = \langle \partial_{x_0} + \partial_{x_1} - (\beta + 2)s_0 \partial_{s_0} - (2\beta + 1)s_1 \partial_{s_1}, \partial_{s_0}, \partial_{s_1} \rangle.$$

(ii) *If $\beta = 0$, then the c-projective algebra is 4-dimensional,*

$$\mathfrak{p}(g) = \langle \partial_{x_0} + \partial_{x_1} - 2s_0 \partial_{s_0} - s_1 \partial_{s_1}, \frac{1}{c_0} e^{x_0} \partial_{x_0} + \frac{1}{c_1} e^{x_1} \partial_{x_1} + s_1 \partial_{s_0}, \partial_{s_0}, \partial_{s_1} \rangle.$$

(L3) *The c-projective algebra is 3-dimensional,*

$$\mathfrak{p}(g) = \langle \partial_{x_0} + \partial_{x_1} - (3s_0 + s_1) \partial_{s_0} - 3s_1 \partial_{s_1}, \partial_{s_0}, \partial_{s_1} \rangle.$$

(L4) The c -projective algebra is 3-dimensional,

$$p(g) = \langle \partial_{x_0} + \partial_{x_1} - (3\beta s_0 - s_1)\partial_{s_0} - (s_0 + 3\beta s_1)\partial_{s_1}, \partial_{s_0}, \partial_{s_1} \rangle.$$

Next, we consider complex type metrics.

Theorem 3 Let g be a Kähler surface covered by Fact 1. Assume it is of non-constant HSC and that it is of complex type. Then its c -projective algebra is isomorphic to one of the following.

(C1) The c -projective algebra is 4-dimensional:

$$p(g) = \langle \partial_z + \partial_{\bar{z}} - s_1\partial_{s_0}, z\partial_z + \bar{z}\partial_{\bar{z}} + 2s_0\partial_{s_0} + s_1\partial_{s_1}, \partial_{s_0}, \partial_{s_1} \rangle.$$

(C2) The c -projective algebra is either 3-dimensional or 4-dimensional,

(i) If $\beta \neq 0$, then the c -projective algebra is 3-dimensional,

$$p(g) = \langle \partial_z + \partial_{\bar{z}} - (\beta + 2)s_0\partial_{s_0} - (2\beta + 1)s_1\partial_{s_1}, \partial_{s_0}, \partial_{s_1} \rangle.$$

(ii) If $\beta = 0$, then the c -projective algebra is 4-dimensional,

$$p(g) = \langle \partial_z + \partial_{\bar{z}} - 2s_0\partial_{s_0} - s_1\partial_{s_1}, e^z\partial_z + e^{\bar{z}}\partial_{\bar{z}} + s_1\partial_{s_0}, \partial_{s_0}, \partial_{s_1} \rangle.$$

(C3) The c -projective algebra is 3-dimensional,

$$p(g) = \langle \partial_z + \partial_{\bar{z}} - (3s_0 + s_1)\partial_{s_0} - 3s_1\partial_{s_1}, \partial_{s_0}, \partial_{s_1} \rangle.$$

(C4) The c -projective algebra is 3-dimensional,

$$p(g) = \langle \partial_z + \partial_{\bar{z}} - (3\beta s_0 - s_1)\partial_{s_0} - (s_0 + 3\beta s_1)\partial_{s_1}, \partial_{s_0}, \partial_{s_1} \rangle.$$

Lastly, there are the metrics of degenerate type.

Theorem 4 Let g be a Kähler surface covered by Fact 1. Assume it is of non-constant HSC and that it is of degenerate type. Then its c -projective algebra is isomorphic to one of the following.

(D1) The c -projective algebra is of dimension 3, 4 or 5.

(i) If h has constant (non-zero) Gauß curvature,

$$h = G(s_1)ds_0^2 + \frac{ds_1^2}{G(s_1)}, \quad G(s_1) = \kappa s_1^2 + \mu_1 s_1 + \mu_2, \quad \kappa \neq 0,$$

then the c -projective algebra is 5-dimensional (see Sect. 6.3.1).

(ii) If h admits a 2-dimensional homothetic symmetry algebra, then

$$h = G(s_1)ds_0^2 + \frac{ds_1^2}{G(s_1)}, \quad G(s_1) = \kappa(\mu_1s_1 + \mu_2)^{\frac{2(\mu_1+1)}{\mu_1}}$$

with $\mathfrak{p}(h) = \langle \partial_{s_0}, (\mu_1 + 2)s_0\partial_{s_0} - (\mu_1s_1 + \mu_2)\partial_{s_1} \rangle$. Then the c -projective algebra is 4-dimensional,

$$\mathfrak{p}(g) = \langle \partial_{x_0}, \partial_{x_1}, s_1\partial_{x_1} + \partial_{s_0}, 2x_0\partial_{x_0} + 2x_1\partial_{x_1} + (\mu_1 + 2)s_0\partial_{s_0} - (\mu_1s_1 + \mu_2)\partial_{s_1} \rangle.$$

(iii) If neither of the previous two cases is assumed, then the c -projective algebra is 3-dimensional,

$$\mathfrak{p}(g) = \langle \partial_{x_0}, \partial_{x_1}, s_1\partial_{x_1} + \partial_{s_0} \rangle.$$

(D2a) The c -projective algebra is of dimension 3 or 5.

- (i) If the metric h has constant Gauß curvature, then the c -projective algebra of g is 5-dimensional (see Sect. 6.3.2).
- (ii) If h admits a 2-dimensional or 1-dimensional homothetic symmetry algebra, then the c -projective algebra is 3-dimensional,

$$\mathfrak{p}(g) = \langle \partial_{x_0}, \partial_{x_1}, s_1\partial_{x_1} + \partial_{s_0} \rangle.$$

(D2b) The c -projective algebra is of dimension 2, 3 or 5.

- (i) If h has constant Gauß curvature, then it is already flat. The c -projective algebra of g thus is 5-dimensional (see Sect. 6.3.3).
- (ii) If h admits a 2-dimensional homothetic symmetry algebra, then the c -projective algebra is 3-dimensional (see Sect. 6.3.3).
- (iii) If h has 1-dimensional homothetic algebra, then the c -projective algebra of g is 2-dimensional,

$$\mathfrak{p}(g) = \langle \partial_{x_0} + (\beta + 2)x_1\partial_{x_1} + \partial_{s_0}, \partial_{x_1} \rangle.$$

(D3) The c -projective algebra is of dimension 2, 3 or 5.

- (i) If h has constant Gauß curvature, then h is already flat. The c -projective algebra of g is thus 5-dimensional (see Sect. 6.3.4).
- (ii) If h admits a 2-dimensional homothetic symmetry algebra, then the c -projective algebra is 3-dimensional (see Sect. 6.3.4).
- (iii) If h has a 1-dimensional homothetic algebra, then the c -projective algebra of g is 2-dimensional,

$$\mathfrak{p}(g) = \langle \partial_{x_0} + 3x_1\partial_{x_1} + \partial_{s_0}, \partial_{x_1} \rangle.$$

In the remainder of the paper we shall prove Theorems 1, 2, 3 and 4. The sections are organised as follows: we begin by proving Theorem 1 in Sect. 2. Section 3 outlines the methodology we apply for the proofs of Theorems 2, 3 and 4. The proofs are then completed in Sects. 4, 5 and 6, respectively. The paper is complemented by an appendix, where we summarise some background material that will likely help readers not yet familiar with the literature, particularly [4]. However, the ‘‘Appendix’’ is not needed to follow the main text of the paper.

2 Constant Holomorphic Sectional Curvature

Theorem 1 identifies those Kähler metrics in Fact 1 that are of constant HSC. In the current section we prove this theorem. It should be noted that Kähler manifolds of constant HSC are, as such, well understood, see for example [4, 7, 10, 12]. The following fact is well-established.

Fact 2 *Let g be a Kähler surface metric ($n = 2$) of constant HSC. Then the c -projective algebra of g is (locally) isomorphic to that of the Fubini-Study metric outlined in Example 1.*

For higher dimensions, a similar statement holds true. In the theory of c -projective Kähler structures, the degree of mobility is an important attribute: informally speaking, the degree of mobility is the minimal number of parameters needed to describe the c -projective class of a given Kähler metric g . Let $[g]$ be a c -projective Kähler structure. Then it can be shown that there is a minimal number N such that $[g]$ is parametrised by N linearly independent Kähler metrics g_k ($1 \leq k \leq N$) via

$$[g] = \left\{ \frac{\left(\sum_k t_k |\det(g_k)|^{\frac{1}{6}} g_k^{-1} \right)^{-1}}{\sqrt{\det \left(\sum_k t_k |\det(g_k)|^{\frac{1}{6}} g_k^{-1} \right)}} : t_k \in \mathbb{R} \right\}.$$

This number $D([g]) = N$ is called the degree of mobility of $[g]$. We also say that $g \in [g]$ has degree of mobility $D(g) = N$.

Lemma 1 ([4, 9, 20])

- (i) *Let g be a Kähler metric of complex dimension n . It is of constant HSC if and only if $D(g) = (n + 1)^2$.*
- (ii) *Let g be a Kähler surface metric ($n = 2$) of non-constant HSC, which admits an essential c -projective vector field. Then $D(g) = 2$.*

Proof Claim (i) is proven in [9, 20] and claim (ii) in [4]. See also [18]. □

For what follows here and in the following sections, we also need a certain (1, 1)-tensor field. In order to introduce it, let g and \hat{g} be c -projectively equivalent metrics

(not necessarily of dimension $2n = 4$, but in the following we will only consider Kähler surfaces). We define

$$L[g, \hat{g}] := \left| \frac{\det(\hat{g})}{\det(g)} \right|^{\frac{1}{2(n+1)}} \hat{g}^{-1} g, \tag{8}$$

see also “Appendix B”. We proceed in two steps in order to prove Theorem 1:

- (1) In the light of Lemma 1, in the present section we are looking for metrics listed in Fact 1 whose degree of mobility is larger than two. A necessary criterion is obtained as follows: If the degree of mobility of any of the metrics in Fact 1 is larger than two, then L satisfies the (special) Sinjukov equations² of a so-called $V_4(B)$ geometry [19],

$$\nabla_k L_{ij} = g_{ik} \Lambda_j + g_{jk} \Lambda_i + \omega_{ik} \Lambda_a J^a_j + \omega_{jk} \Lambda_a J^a_i, \tag{9a}$$

$$\nabla_j \Lambda_i = \mu g_{ij} + B L_{ij}, \tag{9b}$$

$$\nabla_k \mu = 2B \Lambda_k, \tag{9c}$$

with $\Lambda_k = \nabla_k \Lambda$ and $\Lambda = \frac{1}{4} \text{tr}(L)$, and where B is a constant. We first determine necessary conditions for the parameters $\beta, \varsigma, c_i, d_i$. Indeed, if g has constant HSC, it needs to satisfy this system of partial differential equations (PDEs), where L is computed from g and a suitable companion metric as explained in “Appendix B”.

- (2) For the special parameter configurations identified in the first step, we can check the HSC of the metrics directly, deciding if g has constant HSC and thus completing the proof.

Since in the first step we merely need a necessary criterion, and since the different types of metrics in Fact 1 are qualitatively different, we split the discussion according to the type of the metrics and discuss each case separately.

2.1 Cases L1–L4

We proceed according to the outlined steps. The metrics g considered in the present section are those of Fact 1 that are within the cases L1 to L4. To obtain the necessary condition from Sinjukov’s special PDE system, we consider (9b) locally as a matrix equation. Its (2, 3)-entry³ yields a condition of the form

$$\alpha + B\beta = 0,$$

where $B \in \mathbb{R}$, and where α and β are functions depending on $\rho_0, \rho_1, F_0, F_1, \varepsilon$. Provided $\beta \neq 0$ (which is fulfilled in all cases discussed below), we may solve for B .

² We remark that the converse is not true: not every $V_n(B)$ geometry has degree of mobility larger than two.

³ Counting components starts at 0 here.

We differentiate once w.r.t. x_0 and obtain a necessary condition for g having degree of mobility larger than two, $D(g) \geq 3$.

Subsequently, in the second step, we check by a direct computation if, with the described necessary condition fulfilled, the metric has indeed constant HSC, or not. The results are detailed in the following table.

Metric	Sinjukov condition	Constant HSC?
L1	$c_0^2 \varepsilon + c_1^2 = 0$	Yes, if $\varepsilon = -1$ and $c_1 = \pm c_0$
L2	$\beta = -\frac{1}{2}, c_1 d_0^2 \varepsilon + c_0 d_1^2 = 0$	Yes, if $\varepsilon = \pm 1$ and $c_1 = -\varepsilon c_0 \frac{d_1^2}{d_0^2}$
L3	$\beta = -2, d_0^2 \varepsilon + d_1^2 = 0$ Never possible	Yes, if $\varepsilon = -1$ and $d_1 = \pm d_0$ No
L4	$c_0^2 \varepsilon + c_1^2 = 0$	No

In the last column of the table, “yes” indicates that the Liouville metrics of the respective type are of constant HSC if the stated conditions are met. “No” indicates that there do not exist such metrics for the respective case.

2.2 Cases C1–C4

The reasoning is identical to that in the cases C1–C4, with the formal replacements ($\zeta = \zeta_0 + i \zeta_1$)

- Cases C1, C3 and C4: $c_0 \rightarrow \zeta, c_1 \rightarrow \bar{\zeta}$ and $\varepsilon \rightarrow -1$,
- Case C2: $\varepsilon = -1$ and $c_0 \rightarrow 1, c_1 \rightarrow 1$ and $d_0 \rightarrow \zeta, d_1 \rightarrow \bar{\zeta}$.

We arrive at the following table (recall $\zeta = \zeta_0 + i \zeta_1$):

Metric	Sinjukov condition	Constant HSC?
C1	$\zeta^2 = \bar{\zeta}^2$	Yes, if $\zeta_0 = 0$ or $\zeta_1 = 0$
C2	$\beta = -\frac{1}{2}, \zeta^2 = \bar{\zeta}^2$ $\beta = -2, \zeta^2 = \bar{\zeta}^2$	Yes, if $\beta = -\frac{1}{2}$ and either $\zeta_0 = 0$ or $\zeta_1 = 0$ Yes, if $\beta = -2$ and either $\zeta_0 = 0$ or $\zeta_1 = 0$
C3	Never possible	No
C4	$\zeta^2 = \bar{\zeta}^2$	No

2.3 Cases D1–D3

Consider (9b) locally as a matrix equation. The (1, 3)-entry of the matrix equation gives

$$BF^3\rho^3 + 2\rho F'\rho' + 2F\rho'^2 - 2F\rho\rho'' = 0$$

for some constant $B \in \mathbb{R}$. Solving for B (note that $F\rho \neq 0$) and differentiating once w.r.t. x_0 , we obtain an explicit condition depending on ρ, F and their derivatives only:

$$3\rho^2 F'^2\rho' - F\rho^2 F''\rho' + 4F\rho F'\rho'^2 + 3F^2\rho'^3 - 3F\rho^2 F'\rho'' - 4F^2\rho\rho'\rho'' + F^2\rho^2\rho''' = 0.$$

Evaluating this condition for each of the cases D1–D3, and then examining for constant HSC, we arrive at the following table.

Metric	Sinjukov condition	Constant HSC?
D1	The constraint for higher mobility is trivial, i.e. no constraint	Yes, if h has vanishing Gauß curvature
D2a	The constraint for higher mobility is trivial, i.e. no constraint	Yes, if h has Gauß curvature $-\frac{9}{d_1^2}$
D2b	Would require $\beta = -2$ or $\beta = 1$, which are forbidden	No
D3	Never possible	No

We have therefore proven Theorem 1.

3 Method for Theorems 2, 3 and 4

We now turn to the main purpose of this paper and compute the c-projective algebras of the metrics of non-constant HSC in Fact 1. Methodologically, we use an approach similar to that employed in [4, 17, 23]. Lemma 1 ensures that the degree of mobility is precisely two if a metric from Fact 1 has non-constant HSC. Metrics with constant HSC have, of course, already been identified in Theorem 1. The method will subsequently be adapted for the individual cases of Liouville, complex and degenerate type in the following three sections. Many of the mentioned computations were performed and verified using the sagemath computer algebra software [24]. The following lemma is useful, see e.g. [3, 5].

Lemma 2 *Let g and \hat{g} be c-projectively equivalent Kähler surfaces which are not of constant HSC. Let v be a c-projective vector field of g . Then there are constants $a_{ij} \in \mathbb{R}$ ($1 \leq i, j \leq 2$) such that*

$$\mathcal{L}_v L = -a_{01}L^2 + (a_{11} - a_{00})L + a_{10}\mathbb{1}, \tag{10}$$

$$\mathcal{L}_v g = -a_{00}(n + 1)g - a_{01}\left(gL + \frac{1}{2}tr(L)g\right). \tag{11}$$

Particularly, if v is essential, then $a_{01} \neq 0$, and if v is properly homothetic (i.e. homothetic, but not Killing) for g , then $a_{01} = 0$ and $a_{00} \neq 0$. If v is Killing for g , then $a_{00} = a_{01} = 0$.

We prove the main theorems by solving the Eqs. (10) and (11). A partial solution of (10) is well known and can be exploited in the following. It concerns eigenvalues of L and is partly folklore. An explicit proof can be found in [5, 17].

Lemma 3 *Let g and \hat{g} be c -projectively equivalent Kähler surfaces which are not of constant HSC, and let v be a c -projective vector field. Any eigenvalue f of (8) satisfies*

$$v(f) = -a_{01}f^2 + (a_{11} - a_{00})f + a_{10}.$$

Our study below is going to be local, and therefore the Eqs. (10) and (11) can be represented by 4×4 matrices. In fact, for the metrics of Fact 1 these matrices have a block structure consisting of four 2×2 matrices each.

$$\text{Eq. (10)} : \quad \mathcal{L}_v L + a_{01}L^2 - (a_{11} - a_{00})L - a_{10}\mathbb{1} = 0 \quad \Leftrightarrow \quad \begin{pmatrix} (*a) & (*b) \\ (*c) & (*d) \end{pmatrix} = 0,$$

$$\text{Eq. (11)} : \quad \mathcal{L}_v g + a_{00}(n+1)g + a_{01} \left(gL + \frac{1}{2} \text{tr}(L)g \right) = 0 \quad \Leftrightarrow \quad \begin{pmatrix} (*A) & (*B) \\ (*C) & (*D) \end{pmatrix} = 0.$$

4 Proof of Theorem 2 (Liouville Type)

We proceed in two steps. First we prove the following auxiliary result, which allows us to reduce the problem to the univariate realm.

Lemma 4 *Any c -projective symmetry of a Liouville metric in the coordinates of Fact 1 is of the form*

$$v = v^0(x_0)\partial_{x_0} + v^1(x_1)\partial_{x_1} + v^2(s_0, s_1)\partial_{s_0} + v^3(s_0, s_1)\partial_{s_1}.$$

In particular, if $F_1^2 \rho_0'^2 + \varepsilon F_0^2 \rho_1'^2 \neq 0$, then even

$$\begin{aligned} v^2(s_0, s_1) &= \mu s_0 - a_{10}s_1 + v, \\ v^3(s_0, s_1) &= -a_{01}s_0 + (\mu + a_{00} - a_{11})s_1 + v_0, \end{aligned}$$

for constants μ , v and v_0 .

Using this lemma, the proof of Proposition 1 is then going to continue along the following lines: The remaining algebraic conditions are of the form

$$\sum_{\mu} \sum_{\nu} c_{\mu\nu} F_{\mu}(x_0) \hat{F}_{\nu}(x_1) = 0, \quad (12)$$

where the $c_{\mu\nu}$ depend on the parameters a_{ij} and on the parameters of the metric. Without losing generality, we may take the functions F_μ (respectively \hat{F}_ν) to be linearly independent functions [otherwise use linear dependence to replace some functions until linear independence is achieved, changing the functions F_μ and \hat{F}_ν in (12)]. We hence conclude

$$c_{\mu\nu} = 0 \quad \forall \mu, \nu,$$

and then solve all remaining conditions, arriving eventually at Theorem 2.

In the following three subsections, we first prove Lemma 4, then discuss the exceptional case

$$F_1^2 \rho_0'^2 + \varepsilon F_0^2 \rho_1'^2 = 0,$$

excluded in Lemma (4), in Sect. 4.2. The proof of Theorem 2 is then completed in Sect. 4.3.

4.1 Proof of Lemma 4

We begin the proof by computing the tensor L . For the metric g we have, in the coordinates (x_0, x_1, s_0, s_1) ,

$$L = \begin{pmatrix} \rho_0(x_0) & & & \\ & \rho_1(x_1) & & \\ & & \rho_0 + \rho_1 & \rho_0 \rho_1 \\ & & -1 & 0 \end{pmatrix}$$

and take the c-projective vector field as $v = (v^0, v^1, v^2, v^3)$ where each component is a function $v^i = v^i(x_0, x_1, s_0, s_1)$ in four variables. We first take the (0,0) and (1,1) components of (*a), i.e. which are the equations from Lemma 3. Since ρ_0 and ρ_1 are non-constant, we find ($j \in \{0, 1\}$)

$$v^j = - \frac{a_{01} \rho_j(x_j)^2 + (a_{00} - a_{11}) \rho_j(x_j) - a_{10}}{\frac{\partial \rho_j}{\partial x_j}}$$

and conclude $v^0 = v^0(x_0)$ as well as $v^1 = v^1(x_1)$. Resubstituting into (10), the block (*b) is redundant, and the block (*c) yields

$$\begin{aligned} \rho_0(x_0) \frac{\partial v^3}{\partial x_0} + \frac{\partial v^2}{\partial x_0} &= 0, \\ \rho_1(x_1) \frac{\partial v^3}{\partial x_1} + \frac{\partial v^2}{\partial x_1} &= 0. \end{aligned}$$

Moreover, then the block $(*B)$ implies

$$\begin{aligned}\frac{\partial \rho_0(x_0)}{\partial x_0} \frac{\partial v^3}{\partial x_0} &= 0, \\ \frac{\partial \rho_1(x_1)}{\partial x_1} \frac{\partial v^3}{\partial x_1} &= 0, \\ (\rho_0 - \rho_1) \frac{\partial \rho_1}{\partial x_1} \frac{\partial v^3}{\partial x_1} &= 0.\end{aligned}$$

We conclude $v^3 = v^3(s_0, s_1)$ and then $v^2 = v^2(s_0, s_1)$.

Now revisit the block $(*d)$. From the (3,3) and (3,2) components, we obtain

$$\begin{aligned}\left(a_{01} + \frac{\partial v^3}{\partial s_0}\right) \rho_0 \rho_1 + \left(a_{10} + \frac{\partial v^2}{\partial s_1}\right) &= 0, \\ \left(b + \frac{\partial v^3}{\partial s_0}\right) (\rho_0 + \rho_1) + \left(a_{00} - a_{11} + \frac{\partial v^2}{\partial s_0} - \frac{\partial v^3}{\partial s_1}\right) &= 0,\end{aligned}$$

and conclude

$$\begin{aligned}\frac{\partial v^3}{\partial s_0} &= -a_{01}, \\ \frac{\partial v^2}{\partial s_1} &= -a_{10}, \\ \frac{\partial v^3}{\partial s_1} - \frac{\partial v^2}{\partial s_0} &= a_{00} - a_{11}.\end{aligned}$$

With this additional information, we move on to the block $(*D)$. After differentiating the (2, 2) component w.r.t. s_0 and s_1 , respectively, we have

$$\begin{aligned}(F_1^2 \rho_0'^2 + \varepsilon F_0^2 \rho_1'^2) \frac{\partial^2 v^2}{\partial s_0^2} &= 0, \\ (F_1^2 \rho_0'^2 + \varepsilon F_0^2 \rho_1'^2) \frac{\partial^2 v^2}{\partial s_0 \partial s_1} &= 0.\end{aligned}$$

Assuming

$$F_1^2 \rho_0'^2 + \varepsilon F_0^2 \rho_1'^2 \neq 0, \quad (13)$$

we infer

$$\frac{\partial^2 v^2}{\partial s_0^2} = 0 \quad \text{and} \quad \frac{\partial^2 v^2}{\partial s_0 \partial s_1} = 0.$$

Therefore

$$v^2(s_0, s_1) = \mu s_0 - a_{10}s_1 + v \quad (\mu, v \in \mathbb{R})$$

and then

$$v^3(s_0, s_1) = -a_{01}s_0 + (\mu + a_{00} - a_{11})s_1 + v_0 \quad (\mu, v_0 \in \mathbb{R}).$$

For this result we have assumed (13). The next section is dedicated to the complementary case.

4.2 The Exceptional Branch of Lemma 4

According to Lemma 4, an exceptional situation occurs if the condition (13) is not satisfied. In this case we have

$$F_1^2 \rho_0'^2 + \varepsilon F_0^2 \rho_1'^2 = 0. \tag{14}$$

As F_i, ρ_i and ε are real and non-vanishing, and $\varepsilon^2 = 1$, we conclude that (14) implies

$$\varepsilon = -1.$$

We therefore have to consider the exceptional branch arising if

$$F_1^2 \rho_0'^2 - F_0^2 \rho_1'^2 = 0 \tag{15}$$

holds. We are now going to study this condition for the metrics L1–L4.

4.2.1 Metrics L1

We begin with the metrics of L1 type. We have

$$\rho_i = x_i, \quad F_i = c_i \in \mathbb{R}.$$

From the previous discussion we also infer $(t, \tau \in \mathbb{R})$

$$a_{00} = \frac{3}{5}t, \quad a_{01} = 0, \quad a_{11} = -\frac{2}{5}t, \quad a_{10} = \tau, \quad \mu = -2t.$$

Condition (15) thus becomes

$$c_0^2 = c_1^2 \Rightarrow c_1 = \pm c_0.$$

From the equations in block $(*d)$ we obtain, after substitution of the previously found equations,

$$\begin{aligned}\frac{\partial v^2}{\partial s_1} &= -a_{10}, \\ \frac{\partial v^3}{\partial s_0} &= -a_{01}, \\ \frac{\partial v^2}{\partial s_0} - \frac{\partial v^3}{\partial s_1} &= a_{00} - a_{11}.\end{aligned}$$

Next, substituting everything found so far, the block $(*A)$ yields

$$a_{01} = 0, \quad a_{11} = -\frac{2}{3}a_{00}.$$

Finally, with these solutions, we infer from the block $(*D)$ that

$$\frac{\partial v^2}{\partial s_0} = -\frac{10}{3}a_{00},$$

and we obtain

$$\begin{aligned}v^0 &= -\frac{5}{3}a_{00}x_0 + a_{10}, \\ v^1 &= -\frac{5}{3}a_{00}x_1 + a_{10}, \\ v^2 &= -\frac{10}{3}a_{00}s_0 - a_{10}s_1 + v, \\ v^3 &= -\frac{5}{3}a_{00}s_1 + v_0,\end{aligned}$$

and thus a 4-dimensional algebra of c-projective symmetries,

$$\mathfrak{p} = \langle \partial_{s_0}, \partial_{s_1}, x_0\partial_{x_0} + x_1\partial_{x_1} + s_0\partial_{s_0} + s_1\partial_{s_1}, \partial_{x_0} + \partial_{x_1} - s_1\partial_{s_0} \rangle.$$

4.2.2 Metrics L2

For the metrics of type L2, Condition (15) becomes

$$c_1^2 d_0^2 = c_0^2 d_1^2.$$

Solving the remaining equations, we find, analogously to the case L1,

$$\begin{aligned}a_{01} &= 0, \quad a_{11} = -\frac{2}{3}a_{00}, \\ \beta &= 0,\end{aligned}$$

and then, solving for the derivatives of v^j ,

$$\mathfrak{p} = \left(\partial_{s_0}, \partial_{s_1}, \partial_{x_0} + \partial_{x_1} - 2s_0\partial_{s_0} - s_1\partial_{s_1}, \frac{1}{c_0}e^{x_0}\partial_{x_0} + \frac{1}{c_1}e^{x_1}\partial_{x_1} + s_1\partial_{s_0} \right).$$

Here we have seen that, for the exceptional case of Condition (15), the algebra of c-projective symmetries is 4-dimensional and $\beta = 0$. We are going to see in Sect. 4.3.2 that, if $\beta = 0$, the algebra is 4-dimensional even if (15) is not satisfied.

4.2.3 Metrics L3

Condition (15) becomes

$$c_0^2 e^{-3x_0} = c_1^2 e^{-3x_1},$$

which cannot be identically satisfied unless $(c_0, c_1) = (0, 0)$. These values, however, are impossible as the corresponding metric of L3 type would become degenerate. We conclude that the exceptional branch is void in the case L3.

4.2.4 Metrics L4

Using that $\cos(x_0)e^{3\beta x_0}$ and $\cos(x_1)e^{3\beta x_1}$ are linearly independent functions, Condition (15) is equivalent to the system

$$c_0^2 = c_1^2 = 0,$$

which cannot be satisfied as the corresponding metric of L4 type would be degenerate. We conclude that the exceptional branch is void in the case L4.

4.3 Finalising the Proof of Theorem 2

We return to the main branch, i.e. we now assume

$$F_1^2 \rho_0^2 + \varepsilon F_0^2 \rho_1^2 \neq 0.$$

Using Lemma 4, we find that the blocks $(*a)$, $(*b)$, $(*c)$ and $(*d)$ as well as $(*B)$ and $(*C)$ become redundant. The block $(*A)$ is diagonal and the block $(*D)$ is symmetric, and these remaining conditions can be written as equations of the form

$$\sum_{\ell} c_{\ell} F_{\ell}(x_0, x_1) = 0,$$

where $F_{\ell}(x_0, x_1)$ are bivariate, linearly independent functions in x_0, x_1 . Thus $c_{\ell} = 0$ for all ℓ . We solve these remaining conditions separately for each of the metrics of Liouville type. Each of the following subsections covers one of these metrics.

4.3.1 Case L1

The remaining conditions (12) are rational functions in x_0, x_1 and thus lead to polynomial conditions, where each coefficient has to vanish independently. We arrive at

$$\mu = -\frac{10}{3}a_{00}, \quad a_{01} = 0, \quad a_{11} = -\frac{2}{3}a_{00},$$

and thus, up to a constant factor,

$$v = \begin{pmatrix} -5a_{00}x_0 + 3a_{10} \\ -5a_{00}x_1 + 3a_{10} \\ -10a_{00}s_0 - 3a_{10}s_1 + 3v \\ -5a_{00}s_1 + 3v_0 \end{pmatrix}$$

leading to the 4-dimensional algebra

$$\mathfrak{p} = \langle \partial_{s_0}, \partial_{s_1}, \partial_{x_0} + \partial_{x_1} - s_1 \partial_{s_0}, x_0 \partial_{x_0} + x_1 \partial_{x_1} + 2s_0 \partial_{s_0} + s_1 \partial_{s_1} \rangle.$$

We note that this algebra coincides with the one found in the exceptional case, see Sect. 4.2.1.

4.3.2 Case L2

Since $\beta \neq 1$, the remaining Conditions (12) are equivalent to

$$\begin{aligned} a_{01}c_0^2 e^{2\beta x_0} &= 0, \\ a_{01}c_1^2 e^{2\beta x_1} &= 0, \\ a_{10}\beta e^{2\beta x_0} &= 0, \\ a_{10}\beta e^{2\beta x_1} &= 0, \\ ((5\beta - 2)a_{00} - 3a_{11})c_0 &= 0, \\ ((5\beta - 2)a_{00} - 3a_{11})c_1 &= 0. \end{aligned}$$

We solve this system distinguishing two branches:

Case $\beta \neq 0$. We arrive at

$$\mu = -\frac{5}{3}(\beta + 2)a_{00}, \quad a_{01} = 0, \quad a_{10} = 0, \quad a_{11} = \frac{1}{3}(5\beta - 2)a_{00}.$$

This yields the 3-dimensional algebra

$$\mathfrak{p} = \langle \partial_{s_0}, \partial_{s_1}, \partial_{x_0} + \partial_{x_1} - (\beta + 2)s_0 \partial_{s_0} - (2\beta + 1)s_1 \partial_{s_1} \rangle.$$

Case $\beta = 0$. In this case we arrive at

$$\mu = -\frac{10}{3}a_{00}, \quad a_{01} = 0, \quad a_{11} = -\frac{2}{3}a_{00}.$$

This yields the 4-dimensional algebra

$$\mathfrak{p} = \langle \partial_{s_0}, \partial_{s_1}, \frac{1}{c_0}e^{x_0}\partial_{x_0} + \frac{1}{c_1}e^{x_1}\partial_{x_1} + s_1\partial_{s_0}, \partial_{x_0} + \partial_{x_1} - 2s_0\partial_{s_0} - s_1\partial_{s_1} \rangle.$$

4.3.3 Case L3

We obtain the conditions

$$\mu = -5a_{00}, \quad a_{01} = 0, \quad a_{10} = \frac{5}{3}a_{00}, \quad a_{11} = a_{00},$$

and arrive at the 3-dimensional algebra

$$\mathfrak{p} = \langle \partial_{s_0}, \partial_{s_1}, \partial_{x_0} + \partial_{x_1} - (3s_0 + s_1)\partial_{s_0} - 3s_1\partial_{s_1} \rangle.$$

4.3.4 Case L4

Solving the remaining conditions of (12), and using that $\cos(x_0)\sin(x_0)$ and $\cos^2(x_0)$ are linearly independent functions, we find the system

$$\begin{aligned} (a_{01} + a_{10})\beta - a_{00} + a_{11} &= 0, \\ (a_{00} - a_{11})\beta + a_{01} + a_{10} &= 0, \\ 3a_{01}\beta - 5a_{00} &= 0. \end{aligned}$$

We thus distinguish two cases:

Case $\beta \neq 0$. We arrive at

$$\mu = -5a_{00}a_{01} = \frac{5}{3}\frac{a_{00}}{\beta}, \quad a_{10} = -\frac{5}{3}\frac{a_{00}}{\beta}, \quad a_{11} = a_{00}$$

and thus the 3-dimensional algebra

$$\mathfrak{p} = \langle \partial_{s_0}, \partial_{s_1}, \partial_{x_0} + \partial_{x_1} - (3\beta s_0 - s_1)\partial_{s_0} - (s_0 + 3\beta s_1)\partial_{s_1} \rangle. \tag{16}$$

Case $\beta = 0$. The remaining equation of (2) can be solved for

$$a_{00} = 0, \quad a_{11} = 0, \quad a_{01} = -a_{10}$$

and the remaining conditions of (3) then yield the 3-dimensional algebra

$$\mathfrak{p} = \langle \partial_{s_0}, \partial_{s_1}, \partial_{x_0} + \partial_{x_1} + s_1\partial_{s_0} - s_0\partial_{s_1} \rangle.$$

We observe that this algebra is analogous to (16), which holds for $\beta = 0$.

5 Proof of Theorem 3 (Complex Type)

Theorem 3 is a direct corollary of Theorem 2. Performing the transformation

$$s_0^{\text{new}} = 4s_0^{\text{old}}, \quad s_1^{\text{new}} = 4s_1^{\text{old}},$$

the metric (5) becomes

$$g = -\frac{1}{4} \left((\rho - \bar{\rho})(F^2 dz^2 - \bar{F}^2 d\bar{z}^2) + \frac{1}{\rho - \bar{\rho}} \left[\left(\frac{\rho'}{F} \right)^2 (ds_0 + \bar{\rho} ds_1)^2 - \left(\frac{\bar{\rho}'}{\bar{F}} \right)^2 (ds_0 + \rho ds_1)^2 \right] \right).$$

Up to the irrelevant, constant conformal factor of $-\frac{1}{4}$, we observe a formal analogy of the complex type metrics and the Liouville type metrics where we formally replace

$$\varepsilon = -1, \quad x_0 \rightarrow z, \quad x_1 \rightarrow \bar{z} \text{ etc.},$$

as well as $c_0 \rightarrow \varsigma$ and $c_1 \rightarrow \bar{\varsigma}$ (cases C1, C3, C4) or $c_0, c_1 \rightarrow 1, d_0 \rightarrow \varsigma, d_1 \rightarrow \bar{\varsigma}$ (case C2). The statement then follows directly by comparison, in an entirely analogous manner.

6 Proof of Theorem 4 (Degenerate Type)

We proceed again by proving an auxiliary result first. It provides us with a partial solution for the c-projective vector fields, allowing us to lift 2-dimensional homotheties to the Kähler surface if certain integration conditions hold.

Proposition 1 (i) *Any c-projective vector field of a degenerate metric from Fact 1 is of the form*

$$v = v^0(x_0)\partial_{x_0} + (\eta x_1 + f(s_0, s_1))\partial_{x_1} + u,$$

where u is a homothetic vector field of h , $\eta \in \mathbb{R}$.

(ii) *Let g be a metric of type D1–D3 of Fact 1 and let a_{00}, a_{01}, a_{10} and a_{11} be constants such that*

$$\begin{aligned} \text{D1: } & a_{10} = a_{11} \text{ and } a_{01} = a_{00}, \\ \text{D2a: } & a_{10} = a_{11} - a_{00} \text{ and } a_{01} = 0, \\ \text{D2b: } & a_{10} = (\beta - 1)a_{00} \text{ and } a_{01} = 0 \text{ and } a_{11} = \beta a_{00}, \\ \text{D3: } & a_{10} = \frac{1}{2}(a_{11} - a_{00}) \text{ and } a_{01} = -\frac{1}{2}(a_{11} - a_{00}), \end{aligned} \tag{17}$$

hold. Furthermore, let $u = u^0 \partial_{s_0} + u^1 \partial_{s_1}$ be a homothetic vector field of h , $\mathcal{L}_u h = Ch$, that satisfies the respective integrability condition in the last column of Table 1. Finally, let $v^0(x_0)$, $f(s_0, s_1)$ and $\eta \in \mathbb{R}$ be as specified in the middle column of Table 1. Then

$$v = v^0(x_0) \partial_{x_0} + (\eta x_1 + f(s_0, s_1)) \partial_{x_1} + u \tag{18}$$

is a c-projective vector field for the respective metric g of type D1–D3 of Fact 1.

(iii) For a type D1–D3 metric of Fact 1, any c-projective vector field arises as in (ii).

Note that according to part (ii) of the proposition, any homothetic vector field of the 2-dimensional metric h can be extended to a c-projective vector field of g in the cases D1, D2b and D3. However, in case of metrics g of type D2a, only Killing vector fields of h can be extended to c-projective vector fields of g .

Table 1 Integrability conditions for part (ii) of Proposition 1: Integrability conditions for lifting homothetic vector fields of the 2-dimensional metric h , $\mathcal{L}_u h = Ch$, to c-projective vector fields of metrics of degenerate type of Fact 1

Case	Equations for v^0 and v^1	Integrability condition
D1	$v^0 = Cx_0 + k \ (k \in \mathbb{R})$ $\eta = C$ $\frac{\partial f}{\partial s_0} = s_0 \frac{\partial u^1}{\partial s_0}$ $\frac{\partial f}{\partial s_1} = u^0 + u^1 s_0 \frac{G'}{G} - Cs_0$	No condition
D2a	$v^0 = k \ (k \in \mathbb{R})$ $\eta = 0$ $\frac{\partial f}{\partial s_0} = s_0 \frac{\partial u^1}{\partial s_0}$ $\frac{\partial f}{\partial s_1} = u^0 + \frac{1}{2} s_0 u^1 \frac{G'}{G}$	$C = 0$
D2b	$v^0 = -\frac{C}{\beta+2}$ $\eta = C$ $\frac{\partial f}{\partial s_0} = -\frac{1}{\beta+2} G \frac{\partial u^1}{\partial s_0} e^{-(\beta+2)s_0}$ $\frac{\partial f}{\partial s_1} = \frac{CG + (\beta+2)G u^0 - G' u^1}{2(\beta+2) e^{(\beta+2)s_0}}$	No condition
D3	$v^0 = -\frac{C}{9} c_1 e^{-3x_0}$ $\eta = C$ $\frac{\partial f}{\partial s_0} = -\frac{1}{3} G e^{-3s_0} \frac{\partial u^1}{\partial s_0}$ $\frac{\partial f}{\partial s_1} = \frac{CG + 3G u^0 - G' u^1}{6 e^{3s_0}}$	No condition

Example 2 Let g be a metric of type D1–D3 of Fact 1. Then

$$v = v^0(x_0)\partial_{x_0} + c\partial_{x_1}, \quad c \in \mathbb{R},$$

with

$$\begin{aligned} v^0 &= k \in \mathbb{R} \text{ if } g \text{ is of type D1 or D2a} \\ v^0 &= 0 \text{ if } g \text{ is of type D2b or D3,} \end{aligned}$$

are c -projective vector fields of g . Indeed, this claim follows immediately from Proposition 1, integrating the systems in Table 1 for $u = 0$, i.e. $f = c \in \mathbb{R}$, and evaluating the formulas for v^0 and v^1 in Table 1 for $C = 0$.

Proposition 1 is proved in the following subsection. In Sect. 6.2 we discuss the homothetic algebra of the 2-dimensional metric h . The proof of Theorem 4 is then completed in Sect. 6.3.

6.1 Proof of Proposition 1

6.1.1 Part (i)

The c -projective vector field has the coordinate form

$$v = v^0(x_0, x_1, s_0, s_1)\partial_{x_0} + v^1(x_0, x_1, s_0, s_1)\partial_{x_1} + v^2(x_0, x_1, s_0, s_1)\partial_{s_0} + v^3(x_0, x_1, s_0, s_1)\partial_{s_1}.$$

Our aim now is to obtain necessary conditions for the vector field with components v^0, v^1, v^2, v^3 to be a c -projective vector field. We begin with the first component, i.e. v^0 . The $(0, 0)$ -component of the block $(*a)$ immediately implies the formula

$$v^0 = -\frac{a_{01}\rho^2 + (a_{00} - 2a_{01} - a_{11})\rho - a_{00} + a_{01} - a_{10} + a_{11}}{\frac{\partial \rho}{\partial x_0}}, \quad (19)$$

which in particular yields $v^0 = v^0(x_0)$ for metrics of type D1–D3.

Next, we observe from $(*a)$, specifically its $(1, 0)$ and $(1, 1)$ -components, that

$$\frac{\partial v^3}{\partial x_0} = \frac{\partial v^3}{\partial x_1} = 0.$$

This exhausts the information in $(*a)$. Writing out the blocks $(*b)$ and $(*c)$, specifically the $(2, 0)$ and the $(2, 1)$ component of (10), we furthermore obtain the equations

$$\frac{\partial v^2}{\partial x_0} = 0, \quad \frac{\partial v^2}{\partial x_1} = 0.$$

We therefore conclude that the following preliminary result holds: *The components v^2 and v^3 of the c-projective vector field depend on s_0 and s_1 only, i.e.*

$$v^2 = v^2(s_0, s_1) \quad \text{and} \quad v^3 = v^3(s_0, s_1).$$

Before analysing these components further, we first consider the component v^1 of the c-projective vector field. Recalling that we consider metrics of type D1–D3 in Fact 1, we have that $\tau = f_\tau ds_1$ with certain functions $f_\tau = f_\tau(s_0, s_1)$ specified in each case D1–D3. Considering the (0, 3)-component of (11), in the (*B) block, we conclude

$$\frac{\partial v^1}{\partial x_0} = 0,$$

and next, writing out the (1, 2)-component of (10), in block (*b),

$$\frac{\partial v^1}{\partial s_0} = f_\tau(s_0, s_1) \frac{\partial v^3}{\partial s_0}.$$

Now consider the block (*d), specifically the (2, 2) or (3, 3) component of (10). We obtain the relation

$$a_{11} - a_{00} + a_{01} - a_{10} = 0.$$

The information in (10) is now almost entirely exhausted. The last condition is drawn from block (*b) of Eq. (10), specifically its (1, 3)-component. We infer

$$\frac{\partial v^1}{\partial s_1} + f_\tau \frac{\partial v^1}{\partial x_1} = v^2 \frac{\partial f_\tau}{\partial s_0} + v^3 \frac{\partial f_\tau}{\partial s_1} + f_\tau \frac{\partial v^3}{\partial s_1}.$$

It remains to consider the further conditions arising from (11). We are going to do this individually for the specific normal forms D1–D3.

Case D1. Taking the block (*A), specifically the (0, 0) and (1, 1) components of (11), we infer

$$a_{10} = a_{11} \quad \text{and} \quad \frac{\partial v^1}{\partial x_1} = a_{00} + a_{11}. \tag{20}$$

Case D2a. Analogously, from the block (*A), we infer

$$a_{10} = -a_{00} + a_{11} \quad \text{and} \quad \frac{\partial v^1}{\partial x_1} = a_{00} + \frac{1}{2}a_{11}.$$

Case D2b. Analogously, from the block (*A), we infer

$$a_{10} = (\beta - 1)a_{00} \quad \text{and} \quad a_{11} = a_{00}\beta$$

as well as

$$\frac{\partial v^1}{\partial x_1} = a_{00}(\beta + 2).$$

Case D3. Analogously, from the block $(*A)$, we infer

$$a_{10} = \frac{1}{2}(a_{11} - a_{00}) \quad \text{and} \quad \frac{\partial v^1}{\partial x_1} = \frac{3}{2} a_{00}.$$

We thus conclude:

Lemma 5 The component v^1 is a function $v^1 = v^1(x_1, s_0, s_1)$ with $\frac{\partial v^1}{\partial x_1} =: \eta$ being constant. Thus

$$v^1 = \eta x_1 + f(s_0, s_1).$$

It remains to show that $u = v^2(s_0, s_1)\partial_{s_0} + v^3(s_0, s_1)\partial_{s_1}$ satisfies the conditions of a local homothetic vector field of the metric

$$h = f_0(s_0, s_1) ds_0^2 + f_1(s_0, s_1) ds_1^2, \quad (21)$$

where

$$f_0(s_0, s_1) = G(s_1), \quad f_1(s_0, s_1) = \frac{1}{G(s_1)} \text{ in the cases D1 and D2a,}$$

$$f_0(s_0, s_1) = f_1(s_0, s_1) = e^{Cs_0} G(s_1) \text{ in the cases D2b and D3.}$$

To this end, consider the off-diagonal part of $(*D)$, i.e., the $(3, 2)$ -component of (11). It implies the condition

$$f_0(s_0, s_1) \frac{\partial v^2}{\partial s_1} + f_1(s_0, s_1) \frac{\partial v^3}{\partial s_0} = 0. \quad (22)$$

This is indeed consistent with the equations of a homothetic vector field $u = u^0\partial_{s_0} + u^1\partial_{s_1}$ of h . To prove full equivalence, we proceed again on a case by case basis.

Case D1 Resubstituting (20) into the block $(*D)$, we find

$$\left(a_{00} + a_{11} - 2 \frac{\partial v^2}{\partial s_0} \right) G - v^3 \frac{\partial G}{\partial s_1} = 0, \quad (23a)$$

$$(a_{00} + a_{11})G^2 + v^3 \frac{\partial G}{\partial s_1} - 2G \frac{\partial v^3}{\partial s_1} = 0. \quad (23b)$$

Solve (23a) for $\frac{\partial v^2}{\partial s_0}$,

$$\frac{\partial v^2}{\partial s_0} = \frac{1}{2} \left(a_{00} + a_{11} - v^3 \frac{1}{G} \frac{\partial G}{\partial s_1} \right) \quad (24)$$

and (23b) for $\frac{\partial v^3}{\partial s_1}$,

$$\frac{\partial v^3}{\partial s_1} = \frac{1}{2} \left(a_{00} + a_{11} + v^3 \frac{1}{G} \frac{\partial G}{\partial s_1} \right). \tag{25}$$

In addition, (22) becomes

$$G^2 \frac{\partial v^2}{\partial s_1} + \frac{\partial v^3}{\partial s_0} = 0.$$

These coincide with the conditions that $u = u^0 \partial_{s_0} + u^1 \partial_{s_1}$ is a homothetic vector field of h , $\mathcal{L}_u h = Ch$,

$$\begin{aligned} \frac{\partial u^0}{\partial s_0} &= \frac{1}{2} \left(C - u^1 \frac{1}{G} \frac{\partial G}{\partial s_1} \right), \\ G^2 \frac{\partial u^0}{\partial s_1} + \frac{\partial u^1}{\partial s_0} &= 0, \\ \frac{\partial u^1}{\partial s_1} &= \frac{1}{2} \left(C + u^1 \frac{1}{G} \frac{\partial G}{\partial s_1} \right), \end{aligned}$$

after identifying $u^0 = v^2$, $u^1 = v^3$ and $C = a_{00} + a_{11}$.

It remains to consider the block (*B), which is identical to (*C), and automatically satisfied because of $a_{10} = a_{11}$. We summarise, with $\eta := a_{00} + a_{11}$

$$\begin{aligned} v^0 &= \eta x_0 - a_{00}, \\ v^1 &= \eta x_1 + f(s_0, s_1), \end{aligned}$$

where

$$\begin{aligned} \frac{\partial f}{\partial s_0} &= s_0 \frac{\partial v^3}{\partial s_0}, \\ \frac{\partial f}{\partial s_1} &= s_0 \frac{\partial v^3}{\partial s_1} - \eta s_0 + v^2. \end{aligned}$$

Case D2a Analogously to case D1, we obtain

$$\begin{aligned} \left(2a_{00} + a_{11} - 2 \frac{\partial v^2}{\partial s_0} \right) G - v^3 \frac{\partial G}{\partial s_1} &= 0, \\ \left(2a_{00} + a_{11} - 2 \frac{\partial v^3}{\partial s_1} \right) G + v^3 \frac{\partial G}{\partial s_1} &= 0, \end{aligned}$$

and then, solving for $\frac{\partial v^2}{\partial s_0}$ and $\frac{\partial v^3}{\partial s_1}$,

$$\begin{aligned}\frac{\partial v^2}{\partial s_0} &= \frac{1}{2} \left(2a_{00} + a_{11} - v^3 \frac{1}{G} \frac{\partial G}{\partial s_1} \right), \\ \frac{\partial v^3}{\partial s_1} &= \frac{1}{2} \left(2a_{00} + a_{11} + v^3 \frac{1}{G} \frac{\partial G}{\partial s_1} \right).\end{aligned}$$

Hence $u = v^2 \partial_{s_0} + v^3 \partial_{s_1}$ satisfies the conditions of a homothetic vector field of h , $\mathcal{L}_u h = Ch$, with $C = 2a_{00} + a_{11}$. We now verify that the blocks $(*B)$ and $(*C)$ are redundant and do not yield new conditions. We summarise, with $\eta = a_{00} + \frac{1}{2}a_{11}$

$$v^0 = \frac{1}{3}(a_{11} - a_{00}) \quad \text{and} \quad v^1 = \eta x_1 + f(s_0, s_1),$$

where

$$\begin{aligned}\frac{\partial f}{\partial s_0} &= s_0 \frac{\partial v^3}{\partial s_0}, \\ \frac{\partial f}{\partial s_1} &= s_0 \frac{\partial v^3}{\partial s_1} - \eta s_0 + v^2.\end{aligned}$$

Case D2b Applying a similar reasoning, we arrive at

$$\begin{aligned}\frac{\partial v^2}{\partial s_0} &= \frac{1}{2} \left((\beta + 2)(a_{00} + v^2) - v^3 \frac{1}{G} \frac{\partial G}{\partial s_1} \right), \\ \frac{\partial v^2}{\partial s_1} + \frac{\partial v^3}{\partial s_0} &= 0, \\ \frac{\partial v^3}{\partial s_1} &= \frac{1}{2} \left((\beta + 2)(a_{00} + v^2) - v^3 \frac{1}{G} \frac{\partial G}{\partial s_1} \right),\end{aligned}$$

implying that $u = v^2 \partial_{s_0} + v^3 \partial_{s_1}$ satisfies the conditions of a homothetic vector field of h , $\mathcal{L}_u h = Ch$, identifying $C = a_{00}(\beta + 2)$. We then find

$$\begin{aligned}v^0 &= -a_{00}, \\ v^1 &= a_{00}(\beta + 2)x_1 + f(s_0, s_1),\end{aligned}$$

with

$$\begin{aligned}\frac{\partial f}{\partial s_0} &= s_0 \frac{\partial v^3}{\partial s_0}, \\ \frac{\partial f}{\partial s_1} &= s_0 \frac{\partial v^3}{\partial s_1} - a_{00}(\beta + 2)s_0 + v^2,\end{aligned}$$

completing the proof.

Case D3 Analogously, we arrive at

$$\begin{aligned} \frac{\partial v^2}{\partial s_0} &= \frac{1}{2} \left(\frac{3}{2}(a_{00} + a_{11}) + 3v^2 - v^3 \frac{1}{G} \frac{\partial G}{\partial s_1} \right), \\ \frac{\partial v^3}{\partial s_0} + \frac{\partial v^2}{\partial s_1} &= 0, \\ \frac{\partial v^3}{\partial s_1} &= \frac{1}{2} \left(\frac{3}{2}(a_{00} + a_{11}) + 3v^2 - v^3 \frac{1}{G} \frac{\partial G}{\partial s_1} \right), \end{aligned}$$

implying that $u = v^2 \partial_{s_0} + v^3 \partial_{s_1}$ satisfies the conditions of a homothetic vector field of h , $\mathcal{L}_u h = Ch$, identifying $C = \frac{3}{2}(a_{00} + a_{11})$. We then find

$$\begin{aligned} v^0 &= -\frac{1}{6}(a_{00} - a_{11})c_1 e^{-3x_0}, \\ v^1 &= \frac{3}{2}a_{00}x_1 + f(s_0, s_1), \end{aligned}$$

with

$$\begin{aligned} \frac{\partial f}{\partial s_0} &= -\frac{1}{3}G e^{-3s_0} \frac{\partial v^3}{\partial s_0}, \\ \frac{\partial f}{\partial s_1} &= \frac{1}{12} \left(3(a_{00} - a_{11} + 4v^2)G - 2v^3 \frac{\partial G}{\partial s_1} \right) e^{-3s_0}, \end{aligned}$$

completing the proof.

Together with Eq. (19) and Lemma 5, the claim of part (i) follows.

6.1.2 Part (ii)

Part (i) provides necessary conditions that the components of a c-projective vector field have to satisfy. We therefore have that v has to have the form

$$v = v^0(x_0)\partial_{x_0} + (\eta x_1 + f(s_0, s_1))\partial_{x_1} + u$$

with v^0 given by (19), where moreover $u = u^0(s_0, s_1)\partial_{s_0} + u^1(s_0, s_1)\partial_{s_1}$ is a homothetic vector field of the metric h [cf. (21)]. We then choose a_{00} , a_{01} , a_{10} and a_{11} such that (17) is satisfied. Comparing to the conditions of a c-projective vector field, i.e. (10) and (11), determined as in part (i), we thus find

$$\eta = C,$$

and, moreover, that the vector field v is indeed a c-projective vector field for a metric g of a type D1–D3, if

$$\frac{\partial f}{\partial s_0} = f_\tau(s_0, s_1) \frac{\partial u^1}{\partial s_0}, \tag{26a}$$

$$\frac{\partial f}{\partial s_1} = u^0(s_0, s_1) \frac{\partial f_\tau}{\partial s_0} + u^1(s_0, s_1) + f_\tau(s_0, s_1) - C f_\tau(s_0, s_1), \quad (26b)$$

for which the integrability condition

$$\frac{\partial^2 f}{\partial s_0 \partial s_1} = \frac{\partial^2 f}{\partial s_1 \partial s_0}$$

has to be satisfied. Equation (26) are obtained analogously to the computation in (i); note that the expressions on the right hand sides of (26) depend on s_0, s_1 , the homothetic vector field u of h , and the type D1–D3 of the metric g only. The integrability condition for f is satisfied in the cases D1, D2b and D3. Yet, the integrability condition is not automatically satisfied in the case D2a; in this case, we can integrate for f if and only if

$$C = 0,$$

i.e. if u is Killing for h , $\mathcal{L}_u h = 0$. As a result, we obtain the formulas and conditions of Table 1. In particular, the integrability condition for f is specified in the second column of Table 1. This establishes (10) and (11) for the Ansatz (18) for v . It can be shown that this implies that v is a c-projective vector field. We conclude the proof of part (ii) of Proposition 1 by explicitly confirming that v is c-projective, proceeding case by case using Lie symmetry methods analogous to those outlined in “Appendix A.1”.

6.1.3 Part (iii)

The third part of the claim is easily verified by a careful inspection of the proofs of parts (i) and (ii). Indeed, in the considerations for part (i) we have exhausted all conditions for a c-projective vector field of a metric g among the types D1–D3. In part (ii) we have seen that given a proper choice of a homothetic vector field of h together with suitable constants a_{00} to a_{11} , the ansatz suggested by (i) yields a c-projective vector field of g . Exhausting the possibilities for u and a_{ij} , we therefore obtain all c-projective vector fields of a given g of type D1–D3.

6.2 Homothetic Symmetries for the 2-Dimensional Component

Proposition 1 requires homothetic vector fields of 2-dimensional metrics. These are then extended to the c-projective vector fields of g we are seeking. In the current section we are therefore going to investigate the homothetic algebras of certain real surfaces. According to Fact 1, we need to consider the metrics

$$G(s_1)ds_0^2 + \frac{1}{G(s_1)}ds_1^2 \quad (27)$$

and

$$e^{Cs_0}G(s_1)(ds_0^2 + ds_1^2). \quad (28)$$

Without loss of generality, we may suppose $G > 0$ for our purposes here, as we work locally and a constant conformal factor is irrelevant for the question under investigation. In particular, the homothetic algebra of the 2-dimensional metric h remains unaffected by a change of the sign of G . Note, however, that the sign of G becomes relevant when computing the actual c-projective vector fields using Proposition 1, cf. also Table 1; specifically, the sign of the function f depends on the sign of G in the cases D2b and D3. For the sake of legibility we are going to suppress some absolute values in some real-valued formulas that would otherwise be ill-defined, particularly in Lemmas 7 and 9, leaving them to the discretion of the reader. Due comments will be made where necessary.

Metrics (27) and (28) admit the obvious homothetic vector field ∂_{s_0} , which is Killing for (27). For (28) ∂_{s_0} is Killing if $C = 0$ and properly homothetic otherwise. Our goal is to identify those metrics of type (27) or (28) that admit at least one homothetic vector field in addition to ∂_{s_0} . We recall that, if a 2-dimensional metric has a 2-dimensional Killing algebra, it is already of constant curvature (this follows from [6]).

6.2.1 Metrics (27)

We begin by identifying metrics (27) of constant curvature. The following lemma is obtained by a direct, if cumbersome, computation.

Lemma 6 *A metric of type (27) has constant curvature if and only if*

$$\frac{\partial^3 G}{\partial s_1^3} = 0.$$

(i) *If $G(s_1) = \kappa s_1^2 + \mu_1 s_1 + \mu_2$ with $\kappa \neq 0$ then (27) has non-zero constant curvature. We distinguish three subcases according to the sign of the discriminant*

$$\Delta = \mu_1^2 - 4\kappa\mu_2$$

of G with respect to s_1 . Thus, a homothetic vector field⁴ of h is given by

$$\begin{aligned} \Delta = 0 \\ u &= \left(\xi_0 + \xi_1 \left(-\kappa s_0^2 + 4 \frac{\kappa}{\left(\frac{\partial G}{\partial s_1}\right)^2} \right) + \xi_2 (-2s_0\kappa) \right) \partial_{s_0} + \left(\xi_1 s_0 \frac{\partial G}{\partial s_1} + \xi_2 \frac{\partial G}{\partial s_1} \right) \partial_{s_1}, \\ \Delta > 0 \\ u &= \left(\xi_0 + \xi_1 \frac{\frac{\partial G}{\partial s_1}}{\sqrt{G}\sqrt{\Delta}} \cos\left(\frac{1}{2}\sqrt{\Delta}s_0\right) - \xi_2 \frac{\frac{\partial G}{\partial s_1}}{\sqrt{G}\sqrt{\Delta}} \sin\left(\frac{1}{2}\sqrt{\Delta}s_0\right) \right) \partial_{s_0} \\ &\quad + \left(\xi_1 \sin\left(\frac{1}{2}\sqrt{\Delta}s_0\right)\sqrt{G} + \xi_2 \cos\left(\frac{1}{2}\sqrt{\Delta}s_0\right)\sqrt{G} \right) \partial_{s_1}, \\ \Delta < 0 \end{aligned}$$

⁴ Note that in this particular case, the homothetic algebra coincides with the Killing algebra.

$$u = \left(\xi_0 - \xi_1 \frac{\frac{\partial G}{\partial s_1}}{\sqrt{G}\sqrt{-\Delta}} \exp\left(\frac{1}{2}\sqrt{-\Delta}s_0\right) + \xi_2 \frac{\frac{\partial G}{\partial s_1}}{\sqrt{G}\sqrt{-\Delta}} \exp\left(-\frac{1}{2}\sqrt{-\Delta}s_0\right) \right) \partial_{s_0} \\ + \left(\xi_1 \exp\left(\frac{1}{2}\sqrt{-\Delta}s_0\right)\sqrt{G} + \xi_2 \exp\left(-\frac{1}{2}\sqrt{-\Delta}s_0\right)\sqrt{G} \right) \partial_{s_1},$$

where $\xi_i \in \mathbb{R}$ are real-valued parameters.

(ii) If $G(s_1) = \mu_1 s_1 + \mu_2$ with $\mu_1 \neq 0$, then the metric is flat. Its homothetic algebra is 4-dimensional and generated by the properly homothetic vector field

$$(\mu_1 s_1 + \mu_2) \partial_{s_1}$$

and the three Killing vector fields

$$\partial_{s_0}, \frac{\sin\left(\frac{1}{2}\mu_1 s_0\right)}{\sqrt{\mu_1 s_1 + \mu_2}} \partial_{s_0} - \cos\left(\frac{1}{2}\mu_1 s_0\right) \sqrt{\mu_1 s_1 + \mu_2} \partial_{s_1}, \\ \frac{\cos\left(\frac{1}{2}\mu_1 s_0\right)}{\sqrt{\mu_1 s_1 + \mu_2}} \partial_{s_0} + \sin\left(\frac{1}{2}\mu_1 s_0\right) \sqrt{\mu_1 s_1 + \mu_2} \partial_{s_1}.$$

(iii) If $G(s_1) = \mu_2 \neq 0$, then the metric is flat. Its homothetic algebra is 4-dimensional and generated by the properly homothetic vector field

$$s_0 \partial_{s_0} + s_1 \partial_{s_1}$$

and the three Killing vector fields

$$\partial_{s_0}, \partial_{s_1}, \text{ and } \frac{s_1}{\mu_2} \partial_{s_0} - \mu_2 s_0 \partial_{s_1}.$$

Next, we turn to metrics of non-constant Gauß curvature, aiming to find those metrics (27) that admit additional homothetic vector fields, particularly such that are not Killing. Indeed, the existence of two linearly independent Killing vector fields would already imply that the metric is of constant curvature [6]. The question to be considered is for which $G = G(s_1)$ the metric (27) admits a vector field u such that

$$\mathcal{L}_u \left(G ds_0^2 \pm \frac{1}{G} ds_1^2 \right) = G ds_0^2 \pm \frac{1}{G} ds_1^2. \quad (29)$$

The following lemma provides an answer. As announced earlier, we are going to omit absolute values for the sake of conciseness and legibility. Indeed, the correct formula in Lemma 7 would read

$$G = k_3 |k_1 s_1 + k_2|^{\frac{2(k_1+1)}{k_1}}.$$

As the formula would be ill-defined for $k_1s_1 + k_2 < 0$, we may omit these absolute values without any risk of confusion. Note that the homothetic algebra remains unaffected and no solutions are lost as $k_3 \in \mathbb{R} \setminus \{0\}$ (for $k_3 = 0$ the metric would be ill-defined).

Lemma 7 *Let $G(s_1)ds_0^2 \pm \frac{1}{G(s_1)}ds_1^2$ be a metric of non-constant curvature admitting a homothetic vector field u non-proportional to ∂_{s_0} . Then*

$$G = k_3 (k_1s_1 + k_2)^{\frac{2(k_1+1)}{k_1}}$$

and u is proportional to

$$(k_1 + 2)s_0\partial_{s_0} - (k_1s_1 + k_2)\partial_{s_1}$$

where $k_i \in \mathbb{R}$, $k_1, k_3 \neq 0$, and $k_1 \neq -1, k_1 \neq -2$ (otherwise the metric is of constant curvature).

Proof Let $u = u^0\partial_{s_0} + u^1\partial_{s_1}$. Equation (29) thus yields the system

$$\begin{cases} -2G + u^1G_{s_1} + 2Gu_{s_0}^0 = 0, \\ G^2u_{s_1}^0 \pm u_{s_0}^1 = 0, \\ 2Gu_{s_1}^1 - u^1G_{s_1} - 2G = 0. \end{cases} \tag{30}$$

Solving the third equation of (30),

$$u^1 = \sqrt{G} \left(\int \frac{ds_1}{\sqrt{G}} + F(s_0) \right), \tag{31}$$

where $F = F(s_0)$ is some smooth univariate function. Substituting (31) into (30), we obtain

$$\begin{cases} \pm G^2u_{s_1}^0 + \sqrt{G}F_{s_0} = 0, \\ \sqrt{G}G_{s_1}F + \sqrt{G}G_{s_1} \int \frac{ds_1}{\sqrt{G}} + 2Gu_{s_0}^0 - 2G = 0. \end{cases} \tag{32}$$

Since $G \neq 0$, we can solve this for the derivatives $u_{s_1}^0$ and $u_{s_0}^0$, respectively. Using the symmetry of second derivatives, $\frac{\partial^2 u^0}{\partial s_1 \partial s_0} - \frac{\partial^2 u^0}{\partial s_0 \partial s_1} = 0$, we arrive at

$$F(2GG_{s_1s_1} - G_{s_1}^2) = \pm 4F_{s_0s_0} - 2\sqrt{G}G_{s_1} - 2G \int \frac{ds_1}{\sqrt{G}}G_{s_1s_1} + \int \frac{ds_1}{\sqrt{G}}G_{s_1}^2.$$

In this equation, the right-hand side is a sum of a function depending only on s_0 and a function depending only on s_1 . Therefore, differentiating the left-hand side with respect to s_0 and then with respect to s_1 yields zero, and we obtain

$$\frac{\partial^2}{\partial s_0 \partial s_1} \left(F(s_0) \left(2G \frac{\partial^2 G}{\partial s_1^2} - \left(\frac{\partial G}{\partial s_1} \right)^2 \right) \right) = 0,$$

implying

$$2G \frac{\partial F}{\partial s_0} \frac{\partial^3 G}{\partial s_1^3} = 0,$$

and thus either $G'''(s_1) = 0$ (meaning the metric is of constant curvature due to Lemma 6) or $F'(s_0) = 0$.

Note that, since $G \neq 0$, combining the first equation of (32) and the second of (30) yields $u_{s_1}^0 = 0 \Leftrightarrow u_{s_0}^1 = 0 \Leftrightarrow F_{s_0} = 0$. We have therefore found: if u is a proper homothetic vector field of the metric $G ds_0^2 \pm \frac{1}{G} ds_1^2$ of non-constant curvature, then

$$u = u^0(s_0)\partial_{s_0} + u^1(s_1)\partial_{s_1}. \quad (33)$$

The claim of the lemma is then proven by substituting (33) into (30) and then integrating. \square

The homothetic algebra in this latter case is 2-dimensional and generated by

$$\partial_{s_0} \text{ and } (k_1 + 2)s_0\partial_{s_0} - (k_1s_1 + k_2)\partial_{s_1}.$$

6.2.2 Metrics (28)

We analyse the metrics (28) in an analogous manner to the metrics (27). The following lemma (which is analogous to Lemma 6) is obtained by a direct computation.

Lemma 8 (i) *The metric (28),*

$$e^{Cs_0} G(s_1)(ds_0^2 + ds_1^2)$$

with $C \neq 0$, has constant curvature if and only if

$$G(s_1) = k_1 e^{k_2 s_1}$$

for constants $k_1, k_2 \in \mathbb{R}$, $k_1 \neq 0$. The constant curvature is then identical to zero and the homothetic algebra is 4-dimensional and generated by ∂_{s_0} and ∂_{s_1} as well as

$$\exp\left(-\frac{1}{2}Cs_0 - \frac{1}{2}k_2s_1\right) \left[\sin\left(\frac{1}{2}k_2s_0 - \frac{1}{2}Cs_1\right) \partial_{s_0} - \cos\left(\frac{1}{2}k_2s_0 - \frac{1}{2}Cs_1\right) \partial_{s_1} \right]$$

and

$$\exp\left(-\frac{1}{2}Cs_0 - \frac{1}{2}k_2s_1\right) \left[\cos\left(\frac{1}{2}k_2s_0 - \frac{1}{2}Cs_1\right) \partial_{s_0} + \sin\left(\frac{1}{2}k_2s_0 - \frac{1}{2}Cs_1\right) \partial_{s_1} \right].$$

(ii) *The metric (28) with $C = 0$,*

$$G(s_1)(ds_0^2 + ds_1^2)$$

has constant curvature if and only if

$$G(s_1) = \frac{k_3}{\cos^2(k_1 s_1 + k_2)}$$

for constants $k_1, k_2, k_3 \in \mathbb{R}, k_3 \neq 0$. In that case the Gauß curvature is $-\frac{k_1^2}{k_3}$.

Part (ii) of Lemma 8 is stated here for completeness and, for conciseness, we abstain from listing the explicit homothetic vector fields in this case as we shall not need them. Note that we will need part (i) exclusively, for the cases D2b and D3 in Sects. 6.3.2 and 6.3.4, respectively.

For the case of non-constant curvature we prove the following lemma. A comment similar to that before Lemma 7 is appropriate: the correct formula in Lemma 9 would read

$$G(s_1) = k_3 |\sin(k_1 s_1 + k_2)|^{\frac{C-2k_1}{k_1}},$$

and the sign of $\sin(k_1 s_1 + k_2)$ would appear, accordingly, in the formula for the homothetic vector field. However, this sign would lead to unnecessarily complicated formulas for the c-projective vector fields. Moreover, the sign can be absorbed by a change of the parameter k_2 , and so we are going to omit the absolute values for better readability and ease of notation.

Lemma 9 *Let $h = e^{Cs_0} G(s_1)(ds_0^2 + ds_1^2)$, with $C \neq 0$, be a metric of non-constant curvature admitting a homothetic vector field u not proportional to ∂_{s_0} . Then*

$$G(s_1) = k_3 \sin(k_1 s_1 + k_2)^{\frac{C-2k_1}{k_1}},$$

and u is a linear combination of ∂_{s_0} and

$$w = e^{-k_1 s_0} (\cos(k_1 s_1 + k_2) \partial_{s_0} - \sin(k_1 s_1 + k_2) \partial_{s_1}),$$

where $k_i \in \mathbb{R}, k_1, k_3 \neq 0$ and $k_1 \neq \frac{C}{2}$. In fact, w is Killing for h .

Proof Since $C \neq 0, \partial_{s_0}$ is properly homothetic. The existence of u as in the hypothesis, implies the existence of a Killing vector field. Without loss of generality we therefore assume u to be Killing. The components u^0, u^1 of $u = u^0 \partial_{s_0} + u^1 \partial_{s_1}$ and the function G then need to satisfy the PDE system

$$u_{s_0}^0 - u_{s_1}^1 = 0, \tag{34a}$$

$$u_{s_1}^0 + u_{s_0}^1 = 0, \tag{34b}$$

$$C u^0 G + 2G u_{s_1}^1 + u^1 G_{s_1} = 0. \tag{34c}$$

In addition, one can make use of the classification in [6, Theorem 1]. It is easily seen that only the cases (1a) and (2a) of this classification have the properties implied by the

hypothesis, particularly the commutator of any Killing vector field and any homothetic vector fields is proportional to the Killing vector field. We conclude that $[u, \partial_{s_0}] = k_1 u$ has to hold for some constant $k_1 \neq 0$, i.e.

$$u_{s_0}^0 = -k_1 u^0, \quad u_{s_0}^1 = -k_1 u^1. \tag{35}$$

The system composed of (34a), (34b) and (35) can be straightforwardly solved,

$$u^0(s_0, s_1) = \xi e^{-k_1 s_0} \cos(k_1 s_1 + k_2), \quad u^1(s_0, s_1) = -\xi e^{-k_1 s_0} \sin(k_1 s_1 + k_2),$$

where $\xi \in \mathbb{R}$. Resubstituting into (34c), we have

$$\frac{\partial \ln(G)}{\partial s_1} = (C - 2k_1) \cot(k_1 s_1 + k_2)$$

and then

$$G = k_3 \exp\left(\frac{C - 2k_1}{k_1} \ln \sin(k_1 s_1 + k_2)\right) = k_3 \sin(k_1 s_1 + k_2)^{\frac{C}{k_1} - 2}$$

with $k_3 \neq 0$ since $G \neq 0$. □

6.3 Finalising the Proof of Theorem 4

We finalise the proof by computing the explicit c-projective algebras.

6.3.1 Case D1

The 2D metric is

$$h = G(s_1) ds_0^2 + \frac{ds_1^2}{G(s_1)}.$$

We have three scenarios:

- (1) Any homothetic vector field of h is a multiple of the Killing vector field ∂_{s_0} .
- (2) The metric has a 2-dimensional homothetic algebra. In this scenario, w.l.o.g.,

$$G(s_1) = k_3 (k_1 s_1 + k_2)^{\frac{2(k_1+1)}{k_1}}$$

for $k_1, k_3 \neq 0$ (Lemma 7).

- (3) $G'''(s_1) = 0$, but $G''(s_1) \neq 0$, and the metric h is of non-zero constant curvature, see Lemma 6.

Note that if $G''(s_1) = 0$, then the metric h is flat. In this case, however, g is already of constant HSC. Indeed, recall that $\dim(\mathfrak{h}(h)) \leq 3$ in the case under consideration.

If the homothetic algebra of h would be larger, its Gauß curvature would already be zero, and thus g would have constant HSC due to Theorem 1.

Scenario 1 ($\dim(\mathfrak{h}(h)) = 1$). We have the homothetic vector fields $u = \xi \partial_{s_0}$ for h , and thus obtain

$$v^0(x_0) = -a_{00},$$

where a_{00} is a free parameter. Moreover, after a straightforward integration, we arrive at

$$v^1(x_1, s_0, s_1) = \xi s_1 + \nu,$$

where $\nu \in \mathbb{R}$. Summarising, the c-projective algebra of g is 3-dimensional and generated by

$$\partial_{x_0}, \partial_{x_1}, \text{ and } s_1 \partial_{x_1} + \partial_{s_0}.$$

We thus obtain

$$a_{00} = a_{01} = -a_{10} = -a_{11},$$

and therefore conclude that ∂_{x_1} and $s_1 \partial_{x_1} + \partial_{s_0}$ are Killing vector fields, while ∂_{x_0} is essential as $a_{01} \neq 0$.

Scenario 2 ($\dim(\mathfrak{h}(h)) = 2$). The homothetic algebra of h is parametrised by

$$\xi \left((k_1 + 2)s_0 \partial_{s_0} - (k_1 s_1 + k_2) \partial_{s_1} \right) + \xi_0 \partial_{s_0},$$

where $\xi, \xi_0 \in \mathbb{R}$. and we hence find

$$a_{11} = 2\xi - a_{00},$$

and then

$$v^1 = 2\xi x_1 - \xi_0 s_0 + \nu \quad v^0 = 2\xi x_0 - a_{00}.$$

Therefore, the c-projective vector fields of g are obtained as

$$v = (2\xi x_0 - a_{00}) \partial_{x_0} + (2\xi x_1 - \xi_0 s_1 + \nu) \partial_{x_1} + (\xi(k_1 + 2)s_0 + \xi_0) \partial_{s_0} - \xi(k_1 s_1 + k_2) \partial_{s_1}$$

with parameters $\xi, \xi_0, a_{00}, \nu \in \mathbb{R}$. Written separately, the associated generators of the c-projective symmetry algebra are

$$2x_0 \partial_{x_0} + 2x_1 \partial_{x_1} + (k_1 + 2)s_0 \partial_{s_0} - (k_1 s_1 + k_2) \partial_{s_1}, \\ \partial_{x_0}, \partial_{x_1}, \text{ and } s_1 \partial_{x_1} + \partial_{s_0}.$$

Scenario 3 ($\dim(\mathfrak{h}(h)) = 3$). The metric h admits a 3-dimensional homothetic algebra if and only if $G(s_1) = \kappa s_1^2 + \mu_1 s_1 + \mu_2$ with $\kappa \neq 0$. We obtain

$$a_{11} = -a_{00}.$$

We proceed according to the subcases obtained in Lemma 6. For each subcase, we compute the induced c -projective vector fields analogously to the previous scenarios.

Subcase $\Delta = 0$:

Integrating the PDE system, we find

$$v^1 = \xi_1 \left(\frac{1}{2} s_0^2 (2\kappa s_1 + \mu_1) - \frac{2}{2\kappa s_1 + \mu_1} \right) + \xi_0 s_1 + v$$

and then

$$v = -a_{00} \partial_{x_0} + \left(\xi_1 \left(\frac{1}{2} s_0^2 (2\kappa s_1 + \mu_1) - \frac{2}{2\kappa s_1 + \mu_1} \right) + \xi_0 s_1 + v \right) \partial_{x_1} + u,$$

where u is as in Lemma 6.

Subcase $\Delta > 0$:

We find

$$\begin{aligned} v^1 &= \xi_0 s_1 \\ &+ \xi_1 \sqrt{G} \left(\frac{2}{\sqrt{\Delta}} \cos\left(\frac{1}{2}\sqrt{\Delta}s_0\right) + s_0 \sin\left(\frac{1}{2}\sqrt{\Delta}s_0\right) \right) \\ &+ \xi_2 \sqrt{G} \left(s_0 \cos\left(\frac{1}{2}\sqrt{\Delta}s_0\right) - \frac{2}{\sqrt{\Delta}} \sin\left(\frac{1}{2}\sqrt{\Delta}s_0\right) \right) + v \end{aligned}$$

and thus

$$v = -a_{00} \partial_{x_0} + v^1 \partial_{x_1} + u,$$

where u is as in Lemma 6.

Subcase $\Delta < 0$:

We find

$$\begin{aligned} v^1 &= \xi_0 s_1 + \xi_1 \left((s_0 \Delta - 2) \sqrt{\frac{G}{-\Delta}} \exp\left(\frac{1}{2}\sqrt{-\Delta}s_0\right) \right) \\ &+ \xi_2 \left((s_0 \Delta + 2) \sqrt{\frac{G}{-\Delta}} \exp\left(-\frac{1}{2}\sqrt{-\Delta}s_0\right) \right) + v \end{aligned}$$

and then

$$v = -a_{00} \partial_{x_0} + v^1 \partial_{x_1} + u,$$

where u is as in Lemma 6.

6.3.2 Case D2a

The metric h is as in the case D1 and we continue along the analogous three cases. We have four scenarios:

- (1) Any homothetic vector field of h is a multiple of the Killing vector field ∂_{s_0} .
- (2) The metric has a 2-dimensional homothetic algebra. In this scenario, w.l.o.g., $G(s_1) = \kappa(\mu_1 s_1 + \mu_2) \frac{2(\mu_1+1)}{\mu_1}$ for $\mu_1, \kappa \neq 0$.
- (3) $G'''(s_1) = 0$, but $G''(s_1) \neq 0$ and $G''(s_1) \neq -\frac{9}{a_1^2}$, such that the metric h is of non-zero constant curvature (but g is of non-constant HSC).
- (4) $G''(s_1) = 0$ and the metric h is flat.

Scenario 1 ($\dim(\mathfrak{h}(h)) = 1$). For generic G , the only homothetic vector fields of h are ($\xi \in \mathbb{R}$)

$$u = \xi \partial_{s_0}.$$

The integrability condition for v^1 is

$$a_{11} = -2a_{00},$$

and we find

$$v = -a_{00}\partial_{x_0} + (\xi s_1 + \nu)\partial_{x_1} + \xi \partial_{s_0}$$

with parameters $a_{00}, \xi, \nu \in \mathbb{R}$.

Scenario 2 ($\dim(\mathfrak{h}(h)) = 2$). According to Lemma 7,

$$G = k_3(k_1 s_1 + k_2) \frac{2(k_1+1)}{k_1}$$

the homothetic vector fields of h are parametrised by ($\xi, \xi_0 \in \mathbb{R}$)

$$u = \xi_0 \partial_{s_0} + \xi \left((k_1 + 2)s_0 \partial_{s_0} - (k_1 s_1 + k_2) \partial_{s_1} \right).$$

From Table 1, we infer the integrability condition

$$\xi = 0,$$

as only Killing vector fields of h can be extended to c-projective vector fields of the metric g . We find ($a_{00}, \xi_0, \nu \in \mathbb{R}$)

$$v = -a_{00}\partial_{x_0} + (\xi_0 s_1 + \nu)\partial_{x_1} + \xi_0 \partial_{s_0}$$

and therefore the c-projective algebra of g is generated by

$$\partial_{x_0}, s_1 \partial_{x_1} + \partial_{s_0}, \partial_{x_1}.$$

Scenario 3 ($\dim(\mathfrak{h}(h)) = 3$). We turn to the case when h has a 3-dimensional homothetic algebra, which is analogous to Scenario 3 of case D1. Using Lemma 6, we obtain $a_{11} = -2a_{00}$.

$\Delta = 0$:

$$v^1 = \xi_0 s_1 + \xi_1 \left(\frac{1}{2} s_0^2 (2\kappa s_1 + \mu_1) - \frac{2}{2\kappa s_1 + \mu_1} \right) + v$$

and then

$$v = -3a_{00}\partial_{x_0} + v^1\partial_{x_1} + u.$$

$\Delta > 0$:

$$\begin{aligned} v^1 = & \xi_0 s_1 + \xi_1 \left(2\frac{\sqrt{G}}{\sqrt{\Delta}} \cos\left(\frac{1}{2}\sqrt{\Delta}s_0\right) + \sqrt{G}s_0 \sin\left(\frac{1}{2}s_0\sqrt{\Delta}\right) \right) \\ & + \xi_2 \left(\sqrt{G}s_0 \cos\left(\frac{1}{2}s_0\sqrt{\Delta}\right) - 2\frac{\sqrt{G}}{\sqrt{\Delta}} \sin\left(\frac{1}{2}s_0\sqrt{\Delta}\right) \right) + v \end{aligned}$$

and then

$$v = -3a_{00}\partial_{x_0} + v^1\partial_{x_1} + u.$$

$\Delta < 0$:

$$\begin{aligned} v^1 = & \xi_0 s_1 + \xi_1 \left(s_0 + \frac{2}{\sqrt{-\Delta}} \right) \sqrt{G} \exp\left(\frac{1}{2}\sqrt{-\Delta}s_0\right) \\ & + \xi_2 \left(s_0 - \frac{2}{\sqrt{-\Delta}} \right) \sqrt{G} \exp\left(-\frac{1}{2}\sqrt{-\Delta}s_0\right) + v \end{aligned}$$

and then

$$v = -3a_{00}\partial_{x_0} + v^1\partial_{x_1} + u.$$

In all three subcases, the vector field u is as in Lemma 6. The parameters are $a_{00}, \xi_0, \xi_1, \xi_2, v \in \mathbb{R}$.

Scenario 4 ($\dim(\mathfrak{h}(h)) = 4$). According to Lemma 6,

$$G = \mu_1 s_1 + \mu_2,$$

and we need to distinguish the cases $\mu_1 \neq 0$ and $\mu_1 = 0$.

Subcase $\mu_1 \neq 0$:

The homothetic vector fields of h are parametrised by

$$u = \xi_0 \partial_{s_0} + \xi_1 \left(\frac{\sin\left(\frac{1}{2}\mu_1 s_0\right)}{\sqrt{\mu_1 s_1 + \mu_2}} \partial_{s_0} - \cos\left(\frac{1}{2}\mu_1 s_0\right) \sqrt{\mu_1 s_1 + \mu_2} \partial_{s_1} \right)$$

$$+ \xi_2 \left(\frac{\cos \left(\frac{1}{2} \mu_1 s_0 \right)}{\sqrt{\mu_1 s_1 + \mu_2}} \partial_{s_0} + \sin \left(\frac{1}{2} \mu_1 s_0 \right) \sqrt{\mu_1 s_1 + \mu_2} \partial_{s_1} \right) + \xi \partial_{s_1},$$

where ξ parametrises proper homothetic vector fields and the parameters ξ_i describe the Killing vector fields. Due to the integrability condition, we have

$$\xi = 0.$$

We obtain

$$v^1 = \xi_0 s_1 + \frac{\xi_1}{\mu_1} \left(\mu_1 s_0 \sin \left(\frac{1}{2} \mu_1 s_0 \right) + 2 \cos \left(\frac{1}{2} \mu_1 s_0 \right) \right) \sqrt{\mu_1 s_1 + \mu_2} + \frac{\xi_2}{\mu_1} \left(\mu_1 s_0 \cos \left(\frac{1}{2} \mu_1 s_0 \right) - 2 \sin \left(\frac{1}{2} \mu_1 s_0 \right) \right) \sqrt{\mu_1 s_1 + \mu_2} + \nu,$$

where $\nu \in \mathbb{R}$ is an integration constant. Moreover, we have

$$v^0 = -a_{00}$$

and thus g admits the 5-dimensional c-projective algebra generated by

$$\partial_{x_0}, \partial_{x_1}, s_1 \partial_{x_1} + \partial_{s_0}$$

as well as

$$\left(\left(\mu_1 s_0 \sin \left(\frac{1}{2} \mu_1 s_0 \right) + 2 \cos \left(\frac{1}{2} \mu_1 s_0 \right) \right) \sqrt{\mu_1 s_1 + \mu_2} \right) \partial_{x_1} + \mu_1 \frac{\cos \left(\frac{1}{2} \mu_1 s_0 \right)}{\sqrt{\mu_1 s_1 + \mu_2}} \partial_{s_0} + \mu_1 \sin \left(\frac{1}{2} \mu_1 s_0 \right) \sqrt{\mu_1 s_1 + \mu_2} \partial_{s_1}$$

and

$$\left(\left(\mu_1 s_0 \cos \left(\frac{1}{2} \mu_1 s_0 \right) - 2 \sin \left(\frac{1}{2} \mu_1 s_0 \right) \right) \sqrt{\mu_1 s_1 + \mu_2} \right) \partial_{x_1} - \mu_1 \frac{\sin \left(\frac{1}{2} \mu_1 s_0 \right)}{\sqrt{\mu_1 s_1 + \mu_2}} \partial_{s_0} + \mu_1 \cos \left(\frac{1}{2} \mu_1 s_0 \right) \sqrt{\mu_1 s_1 + \mu_2} \partial_{s_1}.$$

Subcase $\mu_1 = 0$:

The homothetic vector fields of h are parametrised by

$$u = \xi (s_0 \partial_{s_0} + s_1 \partial_{s_1}) + \xi_1 \left(\frac{s_1}{\mu_2} \partial_{s_0} - \mu_2 s_0 \partial_{s_1} \right) + \xi_2 \partial_{s_0} + \xi_3 \partial_{s_1},$$

where ξ again parametrises proper homothetic vector fields; the parameters ξ_i describe the Killing vector fields. Due to the integrability condition,

$$\xi = 0,$$

and we hence obtain

$$v^1 = \xi_1 \left(-\frac{\mu_2}{2} s_0^2 + \frac{1}{2\mu_2} s_1^2 \right) + \xi_2 s_1 + v$$

($\xi_i, v \in \mathbb{R}$) and thus g admits the 5-dimensional c-projective algebra generated by

$$\partial_{x_0}, \partial_{x_1},$$

as well as

$$s_1 \partial_{x_1} + \partial_{s_0}, \partial_{s_1},$$

and

$$\left(-\frac{\mu_2}{2} s_0^2 + \frac{1}{2\mu_2} s_1^2 \right) \partial_{x_1} + \frac{s_1}{\mu_2} \partial_{s_0} - \mu_2 s_0 \partial_{s_1}.$$

6.3.3 Case D2b

The metric h is

$$h = e^{-(\beta+2)s_0} G(s_1)(ds_0^2 + ds_1^2).$$

We thus have the following three scenarios:

- (1) Any homothetic vector field of h is a multiple of the proper homothetic vector field ∂_{s_0} .
- (2) The metric h admits a 2-dimensional homothetic algebra and is as in Lemma 9 with $C = -(\beta + 2)$.
- (3) The metric h has constant curvature. Due to Lemma 8, it follows that h is flat.

Scenario 1 ($\dim(\mathfrak{h}(h)) = 1$). The metric h in this scenario only admits the proper homothetic vector fields $u = \xi \partial_{s_0}$ ($\xi \in \mathbb{R}$). Following the same steps as before, we find $v^0 = \xi$ and the PDE system for $v^1(x_1, s_0, s_1)$,

$$\frac{\partial v^1}{\partial x_1} = (\beta + 2)\xi, \quad \frac{\partial v^1}{\partial s_0} = 0, \quad \frac{\partial v^1}{\partial s_1} = 0,$$

implying

$$v^1 = \xi(\beta + 2)x_1 + v.$$

We therefore arrive at

$$v = \xi(\partial_{x_0} + (\beta + 2)x_1 \partial_{x_1} + \partial_{s_0}) + v \partial_{x_1}$$

($\xi, v \in \mathbb{R}$). Thus the c-projective algebra of g is 2-dimensional.

Scenario 2 ($\dim(\mathfrak{h}(h)) = 2$). If the homothetic algebra is 2-dimensional, we use Lemma 9 with $C = -(\beta + 2)$. We have

$$u = \xi \partial_{s_0} + \xi_1 w,$$

(w as in Lemma 9) and obtain, if $k_1 \neq -(\beta + 2)$,

$$v^1 = \xi_1 \frac{k_1 k_3}{(k_1 + \beta + 2)(\beta + 2)} \sin(k_1 s_1 + k_2)^{-\frac{k_1 + \beta + 2}{k_1}} e^{-s_0(k_1 + \beta + 2)} - \xi (\beta + 2)x_1 + \nu,$$

where $\nu \in \mathbb{R}$. If $k_1 = -(\beta + 2)$, we obtain

$$v^1 = \xi_1 \frac{k_3}{\beta + 2} \left(s_0(\beta + 2) - \ln \sin((\beta + 2)s_1 - k_2) \right) - \xi (\beta + 2)x_1 + \nu,$$

where again $\nu \in \mathbb{R}$. The c-projective algebra of g is therefore 3-dimensional. If $k_1 \neq -(\beta + 2)$, it is generated by

$$\partial_{x_1}, \partial_{x_0} - (\beta + 2)x_1 \partial_{x_1} + \partial_{s_0}$$

and

$$\begin{aligned} & \frac{k_1 k_3}{(k_1 + \beta + 2)(\beta + 2)} \sin(k_1 s_1 + k_2)^{-\frac{k_1 + \beta + 2}{k_1}} e^{-s_0(k_1 + \beta + 2)} \partial_{x_1} \\ & + e^{-k_1 s_0} \cos(k_1 s_1 + k_2) \partial_{s_0} - e^{-k_1 s_0} \sin(k_1 s_1 + k_2) \partial_{s_1}. \end{aligned}$$

If, on the other hand, $k_1 = -(\beta + 2)$, then it is generated by

$$\partial_{x_1}, \partial_{x_0} - (\beta + 2)x_1 \partial_{x_1} + \partial_{s_0}$$

and

$$\begin{aligned} & \frac{k_3}{\beta + 2} (s_0(\beta + 2) - \ln \sin((\beta + 2)s_1 - k_2)) \partial_{x_1} \\ & + e^{(\beta + 2)s_0} \cos((\beta + 2)s_1 - k_2) \partial_{s_0} + e^{(\beta + 2)s_0} \sin((\beta + 2)s_1 - k_2) \partial_{s_1}. \end{aligned}$$

Scenario 3 ($\dim(\mathfrak{h}(h)) = 4$). In this case h is flat and, due to Lemma 8,

$$G = e^{\mu_1 s_1}$$

and the homothetic algebra of h is parametrised by

$$\begin{aligned} u = & \xi_0 \partial_{s_0} + \xi_1 (\mu_1 \partial_{s_0} + (\beta + 2) \partial_{s_1}) \\ & + \xi_2 \left(\exp\left(\frac{s_0}{2}(\beta + 2) - \frac{1}{2} \mu_1 s_1\right) \sin\left(\frac{1}{2} \mu_1 s_0 + \frac{1}{2} s_1(\beta + 2)\right) \partial_{s_0} \right. \end{aligned}$$

$$\begin{aligned}
& - \exp\left(\frac{1}{2}s_0(\beta + 2) - \frac{1}{2}\mu_1s_1\right) \cos\left(\frac{1}{2}\mu_1s_0 + \frac{1}{2}s_1(\beta + 2)\right) \partial_{s_1} \\
& + \xi_3 \left(\exp\left(\frac{1}{2}s_0(\beta + 2) - \frac{1}{2}\mu_1s_1\right) \cos\left(\mu_1s_0 + \frac{1}{2}s_1(\beta + 2)\right) \partial_{s_0} \right. \\
& \left. + \exp\left(\frac{1}{2}s_0(\beta + 2) - \frac{1}{2}\mu_1s_1\right) \sin\left(\frac{1}{2}\mu_1s_0 + \frac{1}{2}s_1(\beta + 2)\right) \partial_{s_1} \right),
\end{aligned}$$

where $\xi_i \in \mathbb{R}$. We hence obtain $(\xi_i, \nu \in \mathbb{R})$

$$\begin{aligned}
v^1 &= \nu - \xi_0(\beta + 2)x_1 \\
& + \xi_2 \exp\left(\frac{1}{2}\mu_1s_1 - \frac{1}{2}s_0(\beta + 2)\right) \frac{2\mu_1 \sin(W)(\beta + 2) - \cos(W)((\beta + 2)^2 - \mu_1^2)}{((\beta + 2)^2 + \mu_1^2)(\beta + 2)} \\
& + \xi_3 \exp\left(-\frac{1}{2}(\beta + 2)s_0 + \frac{1}{2}\mu_1s_1\right) \frac{2\mu_1 \cos(W)(\beta + 2) - \sin(W)(\mu_1^2 - (\beta + 2)^2)}{((\beta + 2)^2 + \mu_1^2)(\beta + 2)}
\end{aligned}$$

with $W := \frac{1}{2}s_1(\beta + 2) + \frac{1}{2}\mu_1s_0$. Moreover, we obtain the restriction

$$a_{00} = -\xi_0$$

and thus find a 5-dimensional c-projective algebra for g , parametrised by

$$v = -\xi_0 \partial_{x_0} + v^1 \partial_{x_1} + u$$

$(\xi_0, \xi_1, \xi_2, \xi_3, \nu \in \mathbb{R})$.

6.3.4 Case D3

The metric h is

$$h = e^{-3s_0} G(s_1) (ds_0^2 + ds_1^2).$$

We thus have the following three scenarios:

- (1) Any homothetic vector field of h is a multiple of the proper homothetic vector field ∂_{s_0} .
- (2) The metric h admits a 2-dimensional homothetic algebra and is as in Lemma 9 with $C = -3$.
- (3) The metric h has constant curvature. Due to Lemma 8, it follows that h is flat.

Scenario 1 ($\dim(\mathfrak{h}(h)) = 1$). The metric h in this scenario only admits the proper homothetic vector fields $u = \xi \partial_{s_0}$ ($\xi \in \mathbb{R}$). Following the same steps as before, we find $v^0 = \xi$ and the PDE system for $v^1(x_1, s_0, s_1)$,

$$\frac{\partial v^1}{\partial x_1} + 3\xi = 0, \quad \frac{\partial v^1}{\partial s_0} = 0, \quad \frac{\partial v^1}{\partial s_1} = 0,$$

and therefore we arrive at $(\xi, \nu \in \mathbb{R})$

$$\nu = \xi(\partial_{x_0} + 3x_1\partial_{x_1} + \partial_{s_0}) + \nu\partial_{x_1}$$

and thus the c-projective algebra of g is 2-dimensional.

Scenario 2 ($\dim(\mathfrak{h}(h)) = 2$). If the homothetic algebra is 2-dimensional, due to Lemma 9,

$$u = \xi\partial_{s_0} + \xi_1 w$$

with $C = -3$. We arrive at $(\xi, \xi_1, \nu \in \mathbb{R})$

$$\nu^1 = \xi_1 \frac{k_1 k_3}{3(k_1 + 3)} e^{-(k_1+3)s_0} \sin(k_1 s_1 + k_2)^{-\frac{k_1+3}{k_1}} - 3\xi x_1 + \nu,$$

if $k_1 \neq -3$, and

$$\nu^1 = \xi_1 \frac{k_3}{3} \left(3s_0 - \ln \sin(3s_1 - k_2) \right) - \xi (\beta + 2)x_1 + \nu$$

if $k_1 = -3$. The c-projective algebra of g is therefore 3-dimensional. It is generated by

$$\partial_{x_1}, \partial_{x_0} - 3x_1\partial_{x_1} + \partial_{s_0}$$

and, if $k_1 \neq -3$,

$$\begin{aligned} & \frac{k_1 k_3}{3(k_1 + 3)} e^{-(k_1+3)s_0} \sin(k_1 s_1 + k_2)^{-\frac{k_1+3}{k_1}} \partial_{x_1} \\ & + e^{-k_1 s_0} \cos(k_1 s_1 + k_2) \partial_{s_0} - e^{-k_1 s_0} \sin(k_1 s_1 + k_2) \partial_{s_1}. \end{aligned}$$

If $k_1 = -3$, the third generator is instead

$$\begin{aligned} & \frac{k_3}{3} (3s_0 - \ln \sin(3s_1 - k_2)) \partial_{x_1} \\ & + e^{3s_0} \cos(3s_1 - k_2) \partial_{s_0} + e^{3s_0} \sin(3s_1 - k_2) \partial_{s_1}. \end{aligned}$$

Note that w.l.o.g. $\sin(k_1 s_1 + k_2) > 0$ as per the comment before Lemma 9.

Scenario 3 ($\dim(\mathfrak{h}(h)) = 4$). In this case h is flat and the homothetic algebra is analogous to that of Scenario 3 of the case D2b, where we formally replace $\beta = 1$. Proceeding as before, we find first

$$\begin{aligned} \nu^1 &= \nu - 3\xi_0 x_1 \\ &+ \xi_2 \frac{(\mu_1^2 - 9) \cos\left(\frac{1}{2}\mu_1 s_0 + \frac{3}{2}s_1\right) + 6\mu_1 \sin\left(\frac{1}{2}\mu_1 s_0 + \frac{3}{2}s_1\right)}{3(\mu_1^2 + 9)} \exp\left(\frac{1}{2}\mu_1 s_1 - \frac{3}{2}s_0\right) \end{aligned}$$

$$+ \xi_3 \frac{6\mu_1 \cos\left(\frac{1}{2}\mu_1 s_0 + \frac{3}{2}s_1\right) - (\mu_1^2 - 9) \sin\left(\frac{1}{2}\mu_1 s_0 + \frac{3}{2}s_1\right)}{3(\mu_1^2 + 9)} \exp\left(\frac{1}{2}\mu_1 s_1 - \frac{3}{2}s_0\right)$$

and then

$$v = \xi_0 \partial_{x_0} + v^1 \partial_{x_1} + u,$$

where the parameters are $\xi_0, \xi_1, \xi_2, \xi_3, v \in \mathbb{R}$. The c -projective algebra of g therefore is 5-dimensional. This concludes the proof of Theorem 4.

7 Final Remarks

In the present paper we have obtained the full c -projective algebras for all Kähler surfaces with essential c -projective vector fields, i.e. for the Kähler metrics described in [4]. We have, in particular, found that the c -projective vector fields in the case of degenerate type metrics D1–D3 arise from c -projective vector fields of a 2-dimensional metric h involved in these metrics, which likely bears some significance for practical applications. Moreover, this phenomenon should be expected to arise also in higher dimension, which might be useful for extending the results obtained here to higher dimensions.

We have seen in Theorem 1 that, while covering all metrics with essential c -projective vector fields, the list of metrics in [4] does still contain metrics different from those of interest. Moreover, one might argue that the list in [4] describes the metrics only up to a c -projective transformation (and still in a non-sharp way). On another level, one might ask for a description up to isometric transformations. The authors intend to address this problem in an upcoming paper, which will be facilitated by the results obtained here.

Appendix A Kähler Metrics with Constant HSC

This ‘‘Appendix’’ provides additional material that supplements the main body of the paper. It has two parts: we begin with an explicit realisation of the c -projective algebra of the Fubini-Study metric in complex dimension 2. The second part is dedicated to the c -projective equivalence of Kähler surfaces with constant HSC.

A.1 Fubini-Study Metric

In Example 1 the c -projective algebra of the Fubini-Study metric has been discussed. It is isomorphic to $\mathfrak{sl}(3, \mathbb{C})$. Here we obtain a realisation of $\mathfrak{sl}(3, \mathbb{C})$ in terms of vector fields on $\mathbb{C}\mathbb{P}^2$. By a straightforward computation, one obtains the c -projective connection associated to $(\mathbb{C}\mathbb{P}^2, J, g)$, given by

$$\begin{cases} y_x^2 s_{xx} - y_x s_x y_{xx} + t_x y_{xx} + s_{xx} = 0, \\ -t_x y_x y_{xx} + y_x^2 t_{xx} - s_x y_{xx} + t_{xx} = 0, \end{cases} \quad (36)$$

in the coordinates of (7). Consider the second jet space $J^2(1, 3)$ with coordinates

$$(x, y, s, t, y_x, s_x, t_x, y_{xx}, s_{xx}, t_{xx}).$$

Since we are working locally, we can think of $J^2(1, 3)$ as \mathbb{R}^{10} . Thus, we can interpret (36) as an 8-dimensional variety in $J^2(1, 3) \simeq \mathbb{R}^{10}$. It is well known that any vector field X on (an open set of) \mathbb{R}^4 can be prolonged to a vector field $X^{(2)}$ on (an open set of) $J^2(1, 3)$. The vector field X is a *point symmetry* of (36) if its local flow sends solutions of (36) into solutions: the condition for the vector field X to be a point symmetry of (36) is that $X^{(2)}$ vanishes on (36). Since c-projective vector fields are vector fields preserving J -planar curves [i.e., solutions to (36)], they coincide with the set of the point symmetries of (36). The point symmetries of this system can be studied using Lie symmetry techniques [21], and one thus obtains the generators of its c-projective algebra. These are the following 16 vector fields:

$$\begin{aligned} &(x^2 - y^2)\partial_x + 2xy\partial_y + (xs - yt)\partial_s + (xt - ys)\partial_t, \quad y\partial_x - x\partial_y, \quad x\partial_x + y\partial_y, \quad \partial_x, \\ &-2xy\partial_x + (x^2 - y^2)\partial_y - (xt + ys)\partial_s + (xs - yt)\partial_t, \quad y\partial_s - x\partial_t, \quad x\partial_s + y\partial_t, \quad \partial_y, \\ &-(xt + ys)\partial_x + (xs - yt)\partial_y - 2st\partial_s + (s^2 - t^2)\partial_t, \quad t\partial_s - s\partial_t, \quad s\partial_s + t\partial_t, \quad \partial_s, \\ &(xs - yt)\partial_x + (xt + ys)\partial_y + (s^2 - t^2)\partial_s + 2st\partial_t, \quad t\partial_x - s\partial_y, \quad s\partial_x + t\partial_y, \quad \partial_t. \end{aligned}$$

These 16 vector fields generate the c-projective algebra of (7). The c-projective algebra of any Kähler metric of constant HSC is isomorphic to it.

A.2 C-Projective Equivalence of Kähler Metrics with Constant HSC

Here we provide some background material regarding the statement of Fact 2. The following proposition seems to be considered folklore. Yet we have not been able to retrieve the original reference, and do not know if such an initial reference is in existence. According to hearsay a formal proof may exist, and might have been published in a Japanese journal in the second half of the twentieth century. In fact, the proof is straightforward if we impose the additional assumption of Riemannian signature.

Proposition 2 *Let (M, J, g) be a Kähler surface of constant HSC κ , where g is of arbitrary signature. Then, locally, it is c-projectively equivalent to $\mathbb{C}\mathbb{P}^2$ with the Fubini-Study metric.*

Proof Let us assume first that g were a Kähler metric as in the hypothesis, but specifically of Riemannian signature. Then it is a well-known fact that g is locally isometric to (see [2, 11]),

- if $\kappa > 0$, to a multiply of the Fubini-Study metric,

$$\frac{4}{\kappa} \frac{(1 + \sum_i z_i \bar{z}_i) \sum_j dz_j d\bar{z}_j - \sum_{i,j} \bar{z}_j z_i dz_j d\bar{z}_i}{(1 + \sum_i z_i \bar{z}_i)^2}, \quad (37)$$

- if $\kappa < 0$, to a multiple of the Bergman metric,

$$-\frac{4}{\kappa} \frac{(1 - \sum_i z_i \bar{z}_i) \sum_j dz_j d\bar{z}_j + \sum_{i,j} \bar{z}_j z_i dz_j d\bar{z}_i}{(1 - \sum_i z_i \bar{z}_i)^2}, \quad (38)$$

- if $\kappa = 0$, to the Euclidean metric,

$$\sum_i dz_i d\bar{z}_i. \quad (39)$$

Let us now turn to non-Riemannian signature. An inspection of the proof in [2] shows that the signature does not play a crucial role in it. One thereby verifies the following, more general statement. Let g be a Kähler metric as in the hypothesis, of arbitrary signature. Then it is locally isometric,

- if $\kappa > 0$, to

$$\frac{4}{\kappa} \frac{(1 + \sum_i \epsilon_i z_i \bar{z}_i) \sum_j \epsilon_j dz_j d\bar{z}_j - \sum_{i,j} \epsilon_i \epsilon_j \bar{z}_j z_i dz_j d\bar{z}_i}{(1 + \sum_i \epsilon_i z_i \bar{z}_i)^2}, \quad \text{for some } \epsilon_i \in \{-1, 1\}, \quad (40)$$

- if $\kappa < 0$, to

$$-\frac{4}{\kappa} \frac{(1 - \sum_i \epsilon_i z_i \bar{z}_i) \sum_j \epsilon_j dz_j d\bar{z}_j + \sum_{i,j} \epsilon_i \epsilon_j \bar{z}_j z_i dz_j d\bar{z}_i}{(1 - \sum_i \epsilon_i z_i \bar{z}_i)^2}, \quad \text{for some } \epsilon_i \in \{-1, 1\}, \quad (41)$$

- if $\kappa = 0$, to

$$\sum_i \epsilon_i dz_i d\bar{z}_i, \quad \text{for some } \epsilon_i \in \{-1, 1\}. \quad (42)$$

Note that the metrics (40), (41) and (42) indeed are generalisations of (37), (38) and (39), respectively. The signature is determined by the constants ϵ_i .

The crucial observation now is that the metrics above are c -projectively equivalent to each other. This can be confirmed by checking that they share the same c -projective connection. Indeed, in coordinates (x, y, s, t) where $z_1 = x + iy$ and $z_2 = s + it$, the c -projective connection of (40), (41) and (42) is identical to (36). This proves the claim. \square

We remark that the statement of the proposition implies that the c -projective Lie algebras of all the metrics (37)–(42) are isomorphic to $\mathfrak{sl}(3, \mathbb{C})$.

Appendix B Companion Metrics for the Metrics in Fact 1

The proofs of Theorems 1–4 do not exclusively use the metrics given in Fact 1, but rely on the tensor L introduced in Eq. (8). Such $(1, 1)$ -tensors are related to c -projectively equivalent metrics by the following proposition, see [8] and also [7, Sect. 5].

Proposition 3 ([8]) *Two Kähler metrics (of arbitrary signature) g, \hat{g} on a manifold (M, J) (of real dimension $2n$) are c -projectively equivalent if and only if the tensor*

$$L_j^i = \left| \frac{\det \hat{g}}{\det g} \right|^{\frac{1}{2(n+1)}} \hat{g}^{il} g_{lj}$$

satisfies the equation

$$\nabla_k L_{ij} = \Lambda_i g_{jk} + \Lambda_j g_{ik} + \bar{\Lambda}_i \omega_{jk} + \bar{\Lambda}_j \omega_{ik}, \tag{43}$$

where

$$\bar{\Lambda}_i = J^a_i \Lambda_a, \quad \omega_{ij} = J^a_i g_{aj}, \quad \Lambda_i = \nabla_i \Lambda, \quad \Lambda = \frac{1}{4} \text{tr}(L). \tag{44}$$

Note that the last relation in (44) is not a definition, but a consequence of (43). The prolongation equations for (43) are called *Sinjukov equations*. The (special) Sinjukov equations (9) are a special case of these.

For the proofs of Theorems 1–4, the tensor L is obtained using metrics \hat{g} c -projectively equivalent to the metrics in Fact 1. The explicit metrics \hat{g} used in each case are specified below. They have been taken from [4, Theorem 1.5]. We mention also [5], where an extension to higher dimension can be found. Note that the functions ρ, F etc. below are the same as the respective objects in Fact 1.

Via (6), a family of c -projectively equivalent metrics generated by g, \hat{g} can be constructed. In the case that the metric g is of non-constant HSC, this is the whole c -projective class $[g]$. The reason is that Lemma 1 in this case ensures that the degree of mobility is $D(g) = 2$. For an explicit, parametrised form of these families, for the metrics of Fact 1, see also [4, Theorems 1.2 and 1.5].

B.1 Liouville Type Metrics

$$\hat{g} = \frac{1}{\rho_0^2 \rho_1^2 (\rho_0 - \rho_1)} \left[(\rho_0 - \rho_1)^2 \rho_0 \rho_1 \left(\frac{F_0^2}{\rho_0} dx_0^2 + \epsilon \frac{F_1^2}{\rho_1} dx_1^2 \right) + \left(\frac{\rho'_0}{F_0} \right) \rho_1 (ds_0 + \rho_0 ds_1)^2 + \epsilon \left(\frac{\rho'_1}{F_1} \right) \rho_0 (ds_0 + \rho_0 ds_1)^2 \right].$$

B.2 Complex Type Metrics

$$\hat{g} = \frac{1}{\rho^2 \bar{\rho}^2 (\bar{\rho} - \rho)} \left[\frac{1}{4} (\bar{\rho} - \rho)^2 \rho \bar{\rho} \left(\frac{F^2}{\rho} dz^2 + \epsilon \frac{\bar{F}^2}{\bar{\rho}} d\bar{z}^2 \right) + \left(\frac{1}{F} \frac{d\bar{\rho}}{d\bar{z}} \right)^2 \rho (ds_0 + \bar{\rho} ds_1)^2 - \left(\frac{1}{F} \frac{d\rho}{dz} \right)^2 \bar{\rho} (ds_0 + \rho ds_1)^2 \right].$$

B.3 Degenerate Type Metrics

$$\hat{g} = \frac{1}{\rho - 1} \left(\rho h + \frac{\rho F^2}{\rho - 1} dx^2 + \frac{1}{\rho(\rho - 1)} \left(\frac{\rho'}{F} \right)^2 \theta^2 \right).$$

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