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***Abstract*—We describe a model for multi degree-of-freedom energy harvesters for ambient mechanical vibrations, based on coupled resonators that form a networked structure. We derive the governing equations of motions for the model, and we use stochastic calculus and numerical simulations for their analysis. The power balance equation that we derive obeys a fluctuation-dissipation theorem. We show that at steady state, the power injected in the harvester by random vibrations is partially dissipated by internal friction, and partially transferred to an electrical load. We show that all these quantities can be calculated by solving a Lyapunov matrix equation. We describe an optimization algorithm based on the steepest ascent method. Because an analytic formula for the harvested power is not available, we calculate numerically the derivatives for the gradient estimation. The algorithm converges fast, and is very efficient, giving a significant advantage in optimization problems where a large parameter space must be explored. Finally, we show that the networked structure harvests significant more power with respect to a single degree-of-freedom system.**

I. Introduction

Energy harvesting is the process of capturing ambient energy dispersed in the environment, that would otherwise be wasted, and convert it into usable electrical power. It can involve tapping energy from sources like solar radiation, wind, heat, vibrations, or radio frequency signals. Solar and wind energy harvesting are mature technologies that are being continuously improved, but they are restricted to outdoor applications, and typically require large infrastructures. In many practical applications, harvesting a feeble amount of energy is enough to sustain small devices such as wireless sensors, wearable electronics, and other low-power systems. By eliminating the need for batteries, or at least limiting the requirement of external power sources, energy harvesting enables more sustainable and maintenance-free technologies, particularly useful in miniaturized, remote or inaccessible locations [1]–[3].

In low-power applications, even sources characterized by a limited power density such as small heat gradients, dispersed electromagnetic waves and parasitic mechanical vibrations, are potentially useful sources. In particular, parasitic mechanical vibrations have been thoroughly considered, both for their relatively high power density, and for the possibility to implement different mechanical-to-electrical energy conversion mechanisms which can be miniaturized with relative easiness [4], [5].

Ambient dispersed parasitic mechanical vibrations can result from many different and independent sources, such as household or industrial machinery, vehicle motion, human motion or environmental forces, like wind. The typical frequency interval of this sources ranges from few hertz for human motion, up to some kilo-hertz for acoustic (sound) waves [**?**], [6]–[8].

The basic scheme of an energy harvester collecting mechanical vibrations includes an oscillating structure, designed to capture vibration kinetic energy, and a transducer, to convert the kinetic energy into electrical power. The oscillating structure can be schematically modeled as a mass-spring system, possibly complemented by a dampener accounting for internal friction. The mass-spring system represents a mechanical resonator characterized by its resonant frequency, where the oscillating amplitude, and consequently the harvested power, is maximum. At neighbor frequencies, the power performances degrade rapidly, implying that, if the energy of ambient vibrations is distributed over a wide frequency interval, only a limited portion of the available energy is actually harvested.

A possible solution to increase the efficiency of energy harvesters is to use nonlinear resonators. A nonlinear resonator is typically characterized by hysteresis, offering a potentially

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wider bandwidth (although with a lower maximum amplitude) in the frequency response [9]–[11]. Nonlinear harvesters may also exhibit multi-stability, meaning that two or more stable operating points coexist. The resonator may jump from one operating point to the other, this transitions corresponding to "large" oscillations with an increase in the harvested power [12]–[14]. The analysis of nonlinear systems is significantly more complex, making the effective design and optimization of nonlinear energy harvesters problematic.

Recently, we have proposed an alternative solution based on impedance matching [15], [16]. The idea is to interpose a matching network between the transducer and the load, to decrease the impedance mismatch between the mechanical and the electrical part. The solution permits to increase the harvested power and the power conversion efficiency by a significant amount, and can be applied to both linear and nonlinear, including multi-stable, energy harvesters [17], [18]. The main problem with the aforementioned solution is that a relatively large inductance is required to achieve the impedance matching, a well known problem in mechanical-to-electrical impedance matching applications [19]. Different architectures are possible for the matching network circuit. Interestingly, such circuits often rely on resonators, possibly coupled together to obtain a broadband impedance matching.

Broadband impedance matching is a classical topic in electronic and telecommunication engineering. The fundamental theory of impedance matching is well developed, at least for AC problems. A relatively simple and quite efficient solution to broadband matching consists in coupling together more than one resonator, obtaining a networked multi degree-of-freedom (DOF) structure [20]. Inspired by the application of broadband impedance matching in electrical engineering problems, in this work we deal with the analysis and optimization of a networked, multi degree-of-freedom energy harvester for mechanical vibrations based on coupled resonators. In the proposed setup, the structure responsible for capturing random vibrations is a chain of coupled mechanical resonators, connected together to form a networked structure, and effectively acting as a mechanical filter. The mechanical filter can be treated similarly to an electrical filter, and it is designed to pass a range of frequencies, while blocking others. In particular, our filter will be optimized to harvested maximum power.

We model random vibrations as a stochastic process, and we use stochastic calculus and numerical simulations to assess the power performances of the harvester. Finally, we apply the steepest ascent algorithm for the optimization of the harvester parameters, with the goal of maximizing the available electrical power for the load. Results are presented for 2 and 3 mechanical DOF devices, showing a significant improvement in the harvested power.

Section II is devoted to introduce the schematic representation of the networked energy harvester, and to the derivation of the governing equations. A description of the stochastic model for random vibration is also given. The model solution for what concerns the relevant moments (average and variance) is the subject of Section III, while the parametric optimization of the device is introduced in Section IV. The application of the steepest ascent optimization is discussed in Section V. Finally, conclusions are drawn in Section VI.

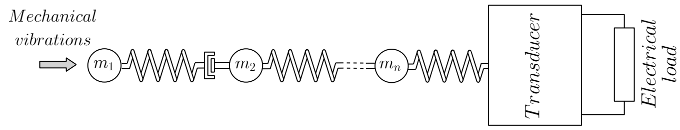

Fig. 1. Schematic representation of a networked multi-DOF resonator for energy harvesting applications.

II. Networked multi DOF energy harvester modeling

The schematic representation of a networked, multi-DOF resonator for energy harvesting applications introduced in this work is shown in figure 1.

The mechanical resonator is composed of n masses m_j, coupled pairwise through springs with elastic potentials

$$U_j(q_j, q_{j+1}) = \frac{1}{2}\, k_j \,(q_j - q_{j+1})^2 \qquad j = 1, \ldots, n \tag{1}$$

where q_j is the displacement of the j-th mass from the rest position, and k_j is the elastic constant of the j-th spring. Each mass-spring couple corresponds to a single DOF, so that the networked resonator has n-DOF. Notice that since the last spring is connected to the transducer, we define $q_{n+1} = 0$.

The equation of motion for the j-th resonator is

$$m_j \ddot{q}_j + \gamma \dot{q}_j \delta_{1j} + k_{j-1}(q_{j-1} - q_j) + k_j(q_j - q_{j+1}) = f(t) \tag{2}$$

where γ is a damping constant, δ_{ij} is the Kronecker delta function, and $f(t)$ is the resultant mechanical force applied to the networked resonators. Boundary conditions are imposed in the form $k_0 = 0$, $q_0 = q_{n+1} = 0$.

The mechanical force is the superposition of two effects. First, the force exerted on the resonator by the electrical load, through the action of transducer, and second, the external force representing ambient dispersed mechanical vibrations. Different physical mechanisms can be considered for energy conversion, including piezoelectric effect, magnetic induction and electrostatic transduction. Irrespective of the principle, the mechanical force is proportional to the electrical voltage applied to the load, and because the electro-mechanical coupling is usually very small, a simple linear relationship can be used. In particular, in our analysis we shall consider piezoelectric transduction, but the result for a different conversion mechanism is analogous [15].

Ambient dispersed mechanical vibrations are typically random in nature, and thus they are best described as stochastic processes. We consider here a simple mathematical model, in the form of a white Gaussian noise. Although more realistic models, which take into account that most energy of mechanical vibration is concentrated at low frequencies, could be considered [16], a white Gaussian noise has the advantage of being mathematically more tractable, yet providing an accurate model for most practical problems.

Under this assumption, the mathematical model for the networked energy harvester subject to white Gaussian noise can be rewritten as the system of stochastic differential equations (SDEs)

$$dQ_{1t} = \frac{1}{m_1} P_{1t}\, dt \tag{3a}$$

$$dP_{1t} = \left[-k_1(Q_{1t} - Q_{2t}) - \frac{\gamma}{m_1} P_{1t} \right] dt + D\, dW_t \tag{3b}$$

$$dQ_{2t} = \frac{1}{m_2} P_{2t}\, dt \tag{3c}$$

$$dP_{2t} = \left[-k_2(Q_{2t} - Q_{3t}) - k_1(Q_{2t} - Q_{1t}) \right] dt \tag{3d}$$

$$\vdots$$

$$dQ_{nt} = \frac{1}{m_n} P_{nt}\, dt \tag{3e}$$

$$dP_{nt} = \left[-k_n Q_{nt} - k_{n-1}(Q_{nt} - Q_{n-1t}) - \alpha V \right] dt \tag{3f}$$

$$dV = \left(\frac{\alpha}{C_{\text{pz}}\, m_n} P_n - \frac{G}{C_{\text{pz}}} V \right) dt \tag{3g}$$

where $P_{jt} = \dot{Q}_{jt}/m_j$ is the j-th conjugate momentum, V is the output voltage (the voltage across the electrical load), $D\, dW_t$ is the random mechanical force, W_t is a Wiener process, characterized by $\mathrm{E}[W_t] = 0$ and $\mathrm{E}[W_t\, W_s] = \min(t, s)$, α is the electro-mechanical coupling constant (in N/V or As/m), and C_{pz} is the electrical capacitance of the piezoelectric transducer.

System (3) can be rewritten in the compact, vector form

$$d\mathbf{X}_t = \mathbf{A}\mathbf{X}_t\, dt + \mathbf{B}\, dW_t \tag{4}$$

where $\mathbf{X}_t = [Q_{1t}, P_{1t}, \ldots, Q_{nt}, P_{nt}, V]^T$ is the vector of the stochastic processes, $\mathbf{A}\mathbf{X}_t$ is the drift vector and $\mathbf{B}$ is the diffusion matrix. Because noise in the SDEs system (4) is additive, the results are independent in the interpretation adopted. To perform our calculations, we shall always interpret (4) as an Itô SDE.

III. Analysis

The SDE system (4) is a $2n + 1$-dimensional Ornstein-Uhlenbeck process [21], [22], therefore $\mathbf{X}_t$ is distributed according to a multivariate normal distribution. The expectation vector and the covariance matrix can be easily calculated as follows. For the expectation vector, we consider the total energy stored in the harvester

$$E = \sum_{j=1}^{n} \left(\frac{p_j^2}{2m_j} + \frac{1}{2} k_j (q_j - q_{j+1})^2 \right) + \frac{1}{2} C_{\text{pz}} v^2 \tag{5}$$

where the first term is the sum of the kinetic energies of the masses, the second is the sum of the elastic potential energies of the springs, and the last is the electrical energy stored in the piezo-electric transducer due to its capacitive behavior.

Applying Itô formula [21], [22] to (5) and using (3) we obtain:

$$\begin{aligned} dE &= \sum_{j=1}^{n} \left(\frac{\partial E}{\partial p_j} dP_j + \frac{\partial E}{\partial q_j} dQ_j \right) + \frac{1}{2} \frac{\partial^2 E}{\partial p_1^2} dP_1^2 + \frac{\partial E}{\partial V} dV \\ &= \left(-\frac{\gamma}{m_1^2} P_1^2 - G V^2 + \frac{D^2}{2m_1} \right) dt + \frac{D}{m_1} P_1\, dW_t \end{aligned} \tag{6}$$

For the deterministic system (obtained setting $D = 0$), we have that $E(\mathbf{x} = 0) = 0$, and $E(\mathbf{x}) \geq 0$, $dE/dt < 0$ for all $\mathbf{x} \neq 0$. Thus the energy is a Lyapunov function and $\mathbf{x} = 0$ is an asymptotically stable equilibrium point for (3) under condition $D = 0$. Because for linear systems asymptotic stability implies hyperbolicity, $\mathbf{x} = 0$ is also hyperbolic, and thus $\mathbf{A}$ is a stable matrix (all its eigenvalues are bounded to the left half of the complex plane).

Taking expectations in equation (3) and using the property $\mathrm{E}[dW_t] = 0$ yields

$$\frac{d}{dt} \mathrm{E}[\mathbf{X}_t] = \mathbf{A}\, \mathrm{E}[\mathbf{X}_t] \tag{7}$$

which implies

$$\lim_{t \to +\infty} \mathrm{E}[\mathbf{X}_t] = 0 \tag{8}$$

because $\mathbf{A}$ is stable.

For the covariance matrix, using Itô formula we have:

$$\begin{aligned} d(\mathbf{X}_t \mathbf{X}_t^T) = &\left(\mathbf{A}\, \mathbf{X}_t \mathbf{X}_t^T + \mathbf{X}_t \mathbf{X}_t^T \mathbf{A}^T + \mathbf{B}\mathbf{B}^T \right) dt \\ &+ \left(\mathbf{B}\mathbf{X}_t^T + \mathbf{X}_t \mathbf{B}^T \right) dW_t \end{aligned} \tag{9}$$

Again, taking the expectations

$$\frac{d}{dt} \mathrm{E}[\mathbf{X}_t \mathbf{X}_t^T] = \mathbf{A}\, \mathrm{E}[\mathbf{X}_t \mathbf{X}_t^T] + \mathrm{E}[\mathbf{X}_t \mathbf{X}_t^T]\, \mathbf{A}^T + \mathbf{B}\mathbf{B}^T \tag{10}$$

Equation (10) is a Lyapunov equation. Since matrix $\mathbf{A}$ is stable, the steady state solution of the Lyapunov equation is unique, and it can be found exploiting Gaussian elimination, with computational complexity $t_1(n) = \mathcal{O}(n^6)$, or using the Bartels-Stewart algorithm with computational complexity roughly estimated as $t_2(n) = \mathcal{O}(n^3)$ [23].

Taking expectations in the energy equation (6) yields the power balance equation

$$\mathrm{E}\left[\frac{dE}{dt} \right] = \frac{D^2}{2m_1} - \frac{\gamma}{m_1^2} \mathrm{E}\left[P_1^2 \right] - G\, \mathrm{E}\left[V^2 \right] \tag{11}$$

Using the passive sign convention, the first term on the right hand side: $P_{\text{in}} = D^2/(2m_1)$ represents the average power injected by the noise into the harvester. The second term: $P_{\text{dis}} = \gamma \mathrm{E}[P_1^2]/m_1^2$ represents the average power dissipated by the internal friction, while the last term: $P_{\text{out}} = G\mathrm{E}\left[V^2\right]$ is the average output power that is harvested and transferred to the electrical load. As a consequence of the fluctuation-dissipation theorem, we expect that asymptotically the system reaches a steady state, where the power injected by the noise perfectly balances the power dissipated by the internal friction and the power absorbed by the load.

The power efficiency of the harvester is therefore:

$$\eta = \frac{P_{\text{out}}}{P_{\text{in}}} = \frac{2\, m_1\, G}{D^2}\, \mathrm{E}\left[V^2\right] = \frac{G\,\mathrm{E}[V^2]}{G\,\mathrm{E}[V^2] + \dfrac{\gamma}{m_1}\mathrm{E}[P_1^2]} \quad (12)$$

According to (11) and (12), the harvested power and the power efficiency are calculated from the components of the covariance matrix $\mathrm{E}[\mathbf{X}_t \mathbf{X}_t^T]$.

IV. Energy harvester optimization

The goal here is to present a methodology for the optimization of the networked energy harvester. In particular, we aim at finding the values of the masses and the spring's elastic constants that maximize the average harvested power, or the power efficiency. Such a problem would require to solve the Lyapunov equation for all possible parameters' values, a task that becomes very time consuming when a large parameter space must be explored.

The method presented here is based on the application of the gradient ascent algorithm. Because an analytical formula for the objective function is not available, derivative-free methods could in principle be better suited for our problem, but they rely on heuristic or empirical search techniques [24]. Instead, we prefer a gradient based method, because it has proven reliable and robust, implementing a numerical approach to calculate the objective function derivatives.

We consider the last mass and the spring elastic constant to be fixed parameters, so as to make comparisons with a single DOF energy harvester fair. These parameters are fixed at $m_n = 10$ g and $k_n = 1$ kN/m.

To better formulate the problem, we define the parameter vector $\boldsymbol{\mu} = [m_1, k_1, \ldots, m_{n-1}, k_{n-1}]^T$, and the objective function $\sigma^* : P \subseteq \mathbb{R}^{2(n-1)} \mapsto \mathbb{R}$, representing the component of the covariance matrix $\boldsymbol{\sigma} = \mathrm{E}[\mathbf{X}_t \mathbf{X}_t^T]$ corresponding to $\mathrm{E}[V^2]$, which permits to calculate the average output power and the power efficiency.

The problem is formulated as follows: find

$$\boldsymbol{\mu}^* = \arg\max_{\boldsymbol{\mu} \in P} \sigma^*(\boldsymbol{\mu}) \quad (13)$$

where P is a predetermined, suitable subset of $\mathbb{R}^{2(n-1)}$, representing reasonable values for the masses and the elastic constants.

The optimization procedure starts with the initial guess $\boldsymbol{\mu} = \boldsymbol{\mu}_1$. The corresponding initial value for the objective function is calculated solving the associated stationary Lyapunov equation

$$\sigma^*(\boldsymbol{\mu}_1) \rightarrow \mathbf{A}(\boldsymbol{\mu}_1)\boldsymbol{\sigma} + \boldsymbol{\sigma}\,\mathbf{A}^T(\boldsymbol{\mu}_1) + \mathbf{B}\,\mathbf{B}^T = 0 \quad (14)$$

To calculate the derivatives, the i-th component of $\boldsymbol{\mu}_1$ is changed by a small amount to $\boldsymbol{\mu}_{1,i} \rightarrow \boldsymbol{\mu}_{1,i} + \delta$, then (14) is solved again finding $\sigma^*(\boldsymbol{\mu}_{1,i} + \delta)$, and the derivative is approximated by the difference quotient:

$$\frac{\partial \sigma^*}{\partial \boldsymbol{\mu}_{1,i}} \approx \frac{\sigma^*(\boldsymbol{\mu}_{1,i} + \delta) - \sigma^*(\boldsymbol{\mu}_{1,i})}{\delta} \quad (15)$$

The procedure is repeated for each component of $\boldsymbol{\mu}$, and the parameter vector is updated in the direction spanned by the gradient according to:

$$\boldsymbol{\mu_2} = \boldsymbol{\mu}_1 + \theta\,\nabla\sigma^* \quad (16)$$

where the hyperparameter $\theta > 0$ is the leaning rate. Eventually, a small random quantity is added to the right hand side of (16), changing $\theta\,\nabla\sigma^* \rightarrow (\theta + \xi)\,\nabla\sigma^*$ where ξ is a small random coefficient with uniform distribution. This accelerates convergence to the maximum, preventing the gradient ascent from getting stuck in a local maximum or in a saddle point.

The algorithm is iterated until each component of the variation vector $\boldsymbol{\Delta} = |\boldsymbol{\mu}_{k+1} - \boldsymbol{\mu}_k|$ is less than a fixed threshold, meaning that the objective function is close to a stationary point, or until a maximum number of iterations is reached.

V. Results

We applied the gradient based algorithm to the optimization of the networked energy harvester described in section II. The objective function to be maximized was the root mean square output voltage $V_{\text{rms}} = \sqrt{\mathrm{E}[V^2]}$, proportional to the average output power

$$P_{\text{out}} = G\,V_{\text{rms}}^2 \quad (17)$$

For reference and validation, the SDE system (3) has been integrated numerically using a stochastic Runge-Kutta strong order one numerical integration scheme. Each simulation had a time length of 100 s, with time integration step of 1 μs. Initial conditions were set at random. The root mean square output voltage was calculated averaging over the whole simulation, after removing the first 3 s, to get rid of the transient.

Figure 2 shows the average output voltage as a function of the mass m_1 and the compliance $1/k_1$ (inverse of the elastic constant), for the networked energy harvester with two DOFs. This simple networked system was considered because it enables a clear visualization of the harvested power as a function of the parameters.

The surface represents the average output power calculated with a grid search, using a 100×100 grid of equally spaced points with $0 < m_{1,i} \leq 0.1$ kg, and $0 < 1/k_{1,i} \leq 0.01$ m/N. Red and blue markers correspond to values of $\boldsymbol{\mu}_k$ touched during the the gradient ascent optimization algorithm procedure, for two different choices of the initial parameters' values $\boldsymbol{\mu}_1 = [m_1, k_1]^T$. The maximum of the objective function is reached in about 230 iterations for both initial conditions, showing the efficiency of the algorithm. It is well known that the convergence of gradient based algorithms strongly depends on the learning rate θ and on the tolerance Δ. There are no established criteria to chose the hyperparameters, besides relying on intuition and experience.

Interestingly, addition of the small random coefficient ξ, with magnitude linearly decreasing with the iteration number, significantly accelerate the converge, decreasing the number of iterations needed to reach the maximum by about 25%. The values of the parameters used in the simulations are summarized in table I.

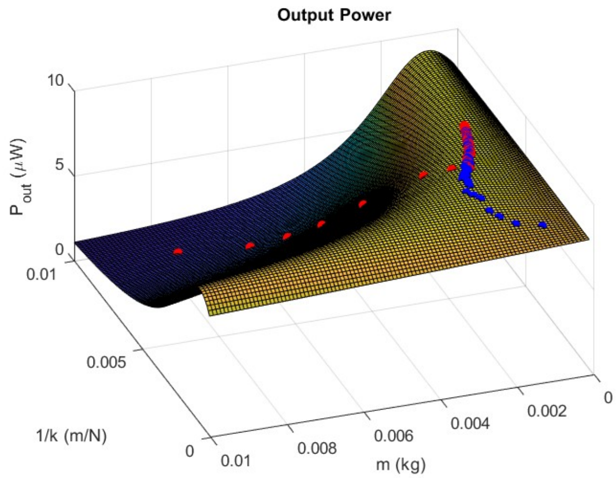

Fig. 2. Application of the optimization algorithm for maximization of the average output power.

Parameter	Value
γ	0.12 kg/s
D	10^{-3} N
α	0.0042 N/V (Am/s)
m_n, m	10 g
k_n, k	1 kN/m
C_{pz}	80 nF
R	1 MΩ

TABLE I
PARAMETERS' VALUES USED IN THE SIMULATIONS.

Finally, table II reports a comparison of the average output power, and the values of the parameters after the optimization for the single, two and three-DOF energy harvester.

VI. CONCLUSIONS

We have presented the schematic representation of a multi-DOF energy harvester for random ambient mechanical vibrations. The harvester includes a mechanical resonator, composed by coupled mass-spring systems, connected together to form the network. Such a structure could be implemented using different technologies and processes, and is suitable to be miniaturized to create a MEMS device.

We have derived the equations of motion for the harvester, under the assumption that random vibrations can be modeled as a white Gaussian noise. Using stochastic calculus, we found that the average output power and power efficiency can be calculated from the covariance matrix of the stochastic processes, which in turn is found solving a matrix Lyapunov equation. The results have been confirmed by numerical solutions of the SDEs.

The optimization of the networked energy harvester has been presented, based on the application of the steepest ascent algorithm. Because an analytic formula for the objective function (the average output power or the power efficiency) is not available, the derivatives cannot be calculated as well. Therefore we resorted to a numerical calculation of the gradient components, calculating the difference quotients solving the Lyapunov matrix equation for two very close values of the parameters.

The algorithm proves very robust and efficient in the optimization, and the optimized network clearly shows the advantage of adding few mechanical DOF to the resonator.

DOF	P_{out}	Parameters
1	7.89 μW	$m = 10$ g $k = 1$ kN
2	9.05 μW	$m_1 = 1.2$ g $k_1 = 0.16$ kN/m $m_2 = 10$ g $k_2 = 1$ kN
3	9.16 μW	$m_1 = 1.1$ g $m_2 = 3.1$ g $m_3 = 10$ g $k_1 = 0.14$ kN/m $k_2 = 0.48$ kN/m $k_3 = 1$ kN/m

TABLE II
AVERAGE OUTPUT POWER AND VALUES OF THE FREE PARAMETERS OBTAINED FROM THE OPTIMIZATION ALGORITHM.

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