Counting numbers that are divisible by the product of their digits

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Abstract

Let $b \geq 3$ be a positive integer. A natural number is said to be a *base-b Zuckerman number* if it is divisible by the product of its base-b digits. Let $\mathcal{Z}_b(x)$ be the set of base-b Zuckerman numbers that do not exceed x, and assume that $x \to +\infty$.

First, we prove an upper bound of the form $|\mathcal{Z}_b(x)| < x^{z_b^+ + o(1)}$, where $z_b^+ \in (0,1)$ is an effectively computable constant. In particular, we have that $z_{10}^+ = 0.665\ldots$, which improves upon the previous upper bound $|\mathcal{Z}_{10}(x)| < x^{0.717}$ due to Sanna. Moreover, we prove that $|\mathcal{Z}_{10}(x)| > x^{0.204}$, which improves upon the previous lower bound $|\mathcal{Z}_{10}(x)| > x^{0.122}$, due to De Koninck and Luca.

Second, we provide a heuristic suggesting that $|\mathcal{Z}_b(x)| = x^{z_b + o(1)}$, where $z_b \in (0,1)$ is an effectively computable constant. In particular, we have that $z_{10} = 0.419\ldots$.

Third, we provide algorithms to count, respectively enumerate, the elements of $\mathcal{Z}_b(x)$, and we determine their complexities. Implementing one of such counting algorithms, we computed $|\mathcal{Z}_b(x)|$ for $b = 3, \ldots, 12$ and large values of x (depending on b), and we showed that the results are consistent with our heuristic.

Keywords: Base-b representation, digits, prodigious number, Zuckerman number.
MSC 2020: 11A63, 11N25, 11Y16, 11Y55.

1 Introduction

Let $b \geq 2$ be a positive integer. Natural numbers that satisfy special arithmetic constraints in terms of their base-b digits have been studied by several authors.

For instance, a natural number is said to be a *base-b Niven number* if it is divisible by the sum of its base-b digits. Cooper and Kennedy proved that the set of base-10 Niven numbers has natural density equal to zero [2], provided a lower bound for its counting function [3], and gave an asymptotic formula for the counting function of base-10 Niven numbers with a fixed sum of digits [4]. Later, Vardi [18, Sec. 2.3] gave stronger upper and lower bounds for the counting function of base-10 Niven numbers. Then, De Koninck, Doyon, and Kátai [7], and (independently) Mauduit, Pomerance, and Sárközy [14], proved that the number of base-b Niven numbers not exceeding x is asymptotic to $c_b x / \log x$, as $x \to +\infty$, where $c_b > 0$ is an explicit constant (see [5] for a generalization). Furthermore, De Koninck and Doyon [6] studied large gaps between base-b Niven numbers, and De Koninck, Doyon, and Kátai [8] provided an asymptotic formula for the number of r-tuples of consecutive base-b Niven numbers not exceeding x. Conditionally to Hooley's Riemann hypothesis, Sanna [17] proved that every sufficiently large positive integer is the sum of a bounded number (depending only on b) of base-b Niven numbers.

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A natural number is said to be a *base-b Zuckerman number* if it is divisible by the product of its base-b digits. (If one requires divisibility by the product of nonzero base-b digits, then one gets the *base-b prodigious numbers* [11].) Note that the base-2 Zuckerman numbers are simply the *Mersenne numbers* $2^k - 1$, where $k = 1, 2, \dots$, which are also called the *base-2 repunits*. Hence, hereafter, we assume that $b \geq 3$. Zuckerman numbers are less studied than their additive counterpart of Niven numbers, but there are some results. For all $x \geq 1$, let $\mathcal{Z}_b(x)$ be the sets of base-b Zuckerman numbers not exceeding x. De Koninck and Luca [9, 10] proved upper and lower bounds for the cardinality of $\mathcal{Z}_{10}(x)$. Then Sanna [16] gave an upper bound for the cardinality of $\mathcal{Z}_b(x)$, which for $b = 10$ improves the result of De Koninck and Luca. More precisely, putting together the bounds of De Koninck–Luca and Sanna, we have that

$$x^{0.122} < |\mathcal{Z}_{10}(x)| < x^{0.717} \tag{1}$$

for all sufficiently large x.

Our contribution is the following. Assume that $x \to +\infty$. In Section 4, we prove an upper bound of the form $|\mathcal{Z}_b(x)| < x^{z_b^+ + o(1)}$, where $z_b^+ \in (0, 1)$ is an effectively computable constant. In particular, we have that $z_{10}^+ = 0.665...$, which improves upon the upper bound of (1). In Section 5, we prove the lower bound $|\mathcal{Z}_b(x)| > x^{0.204}$, which improves upon the lower bound of (1). In Section 6, we provide a heuristic suggesting that $|\mathcal{Z}_b(x)| = x^{z_b + o(1)}$, where $z_b \in (0, 1)$ is an effectively computable constant. In Section 7, we provide algorithms to count, respectively to enumerate, the elements of $\mathcal{Z}_b(x)$. Our best (general) counting algorithm has complexity $x^{z_b^* + o(1)}$, where $z_b^* \in (0, 1)$ is an effectively computable constant. We collected the values of z_b, z_b^+, z_b^*, for $b = 3, \dots, 12$, in Table 1. For $b = 10$, we also provide a counting, respectively enumeration, algorithm with complexity $x^{0.3794}$, respectively $x^{0.3794} + |\mathcal{Z}_{10}(x)|$. In particular, assuming the previous heuristic, this enumeration algorithm is (asymptotically) optimal. Finally, Qizheng He implemented one of the counting algorithms in `C++` (the implementation is freely available on GitHub [12]). Using such implementation, we computed the number of base-b Zuckerman numbers with exactly N digits, for $b = 3, \dots, 12$ and large values of N (about 80 hours on a consumer laptop). The results are collected in Table 2 and support our heuristic, see Table 3 (and Lemma 3.7 for the justification of the comparison). (The terms in Table 2 with $b = 10$ and $N \leq 16$ were already computed by Giovanni Resta, see his comment to sequence A007602 of OEIS [15].)

b	3	4	5	6	7	8	9	10	11	12
z_b	.3690...	.2075...	.4560...	.3727...	.4604...	.2483...	.3625...	.4197...	.4481...	.3537...
z_b^+	.5257...	.4024...	.6634...	.5948...	.6885...	.4988...	.6081...	.6657...	.6977...	.6130...
z_b^*	.4318...	.3018...	.4361...	.3866...	.4559...	.3304...	.4017...	.4416...	.4653...	.4068...

Table 1: Exponents z_b, z_b^+, z_b^* for $b = 3, \dots, 12$.

2 Notation

Throughout the rest of the paper, let $b \geq 3$ be a fixed integer. For the sake of brevity, we say "digits", "representation", and "Zuckerman numbers" instead of "base-b digits", "base-b representation", and "base-b Zuckerman numbers", respectively. We employ the Landau–Bachmann "Big Oh" and "little oh" notations O and o with their usual meanings. In particular, the implied constants in O, and how fast expressions like $o(1)$ go to zero, may depend on b. We let $\mathbb{N} := \{0, 1, \dots\}$ be the set of nonnegative integers, and we let $|\mathcal{S}|$ denote the cardinality of every finite set $\mathcal{S}$. For every integer $n \geq 1$, let $\mathcal{D}_b(n)$ be the set of the digits of n, and let $p_b(n) := \prod_{d=0}^{b-1} d^{w_{b,d}(n)}$ be the product of the digits of n, where $w_{b,d}(n)$ denotes the number of times that the digit d appears in the representation of n. Let $\mathcal{Z}_b$ be the set of Zuckerman numbers, and let $\mathcal{Z}_{b,N}$ be the set of Zuckerman numbers with exactly N digits.

N \ b	3	4	5	6	7	8	9	10	11	12
1	2	3	4	5	6	7	8	9	10	11
2	2	2	3	4	3	4	5	5	5	6
3	4	4	14	8	23	15	18	20	33	21
4	6	7	10	20	29	9	33	40	63	43
5	9	6	42	27	96	38	107	117	224	107
6	10	8	78	55	203	49	191	285	645	222
7	14	16	184	109	533	78	518	747	2000	544
8	33	18	385	188	1295	163	914	1951	5411	1213
9	46	22	795	364	3299	294	1959	5229	16532	2718
10	43	36	1570	653	7630	376	4903	13493	45464	6267
11	72	38	3208	1095	19130	631	11129	35009	135967	13738
12	171	53	6411	2076	43687	1246	22161	91792	393596	31483
13	211	77	13741	3866	111255	1966	50391	239791	1161371	71482
14	252	96	29200	7373	276967	3408	116777	628412	3406099	160109
15	428	129	60864	14622	690189	6038	261725	1643144	10012223	366977
16	728	177	126080	27972	1710625	8291	578324	4314987	29355933	845908
17	986	237	263060	53201	4124693	13470	1276433	11319722	86022519	
18	1400	317	545025	103132	10097943	28419	2851060	29713692	251993074	
19	2214	425	1137646	203051	24765215	46596	6310957	78042616	737799286	
20	3450	558	2371769	398775	60708268	71497	13886129	204939760		
21	5007	772	4946854	774024	148622249	126490	30753950	538453205		
22	7370	997	10296601	1506714	364176274	198722	68293912	1414773364		
23	11234	1305	21454503	2915442	894674969	320763	151573306			
24	16981	1817	44678532	5658200	2204890644	603722				
25	25324	2305	93110027	10999574	5390633926	1015093				
26	37716	3096	193971630	21369791		1585495				
27	56757	4164	404103162	41626279		2717026				
28	85493	5495	841843065	81172184						
29	127774	7402	1753948967	158009860						
30	191665	9936	3653927956	307539610						
31	287481	13013	7612395846	598683507						
32	431622	17308								
33	646816	23372								
34	970475	31037								
35	1455724	41399								
36	2183782	55034								
37	3275092	73086								
38	4914274	98142								
39	7371941	130591								
40	11057697	173916								
41	16586242	232253								
42	24880345	309102								
43	37318948	412940								
44	55979205	549336								
45	83963507	733783								
46	125950398	978893								
47	188921345	1305037								
48	283385733	1738126								
49	425085179	2319219								
50	637608602	3091664								
51	956428288	4119790								
52	1434628060									
53	2152013870									
54	3227959147									
55	4841970543									
56	7262855061									
57	10894279904									
58	16341567376									
59	24512322843									

Table 2: The number of base-b Zuckerman numbers with N digits in base b.

b	N	$\lvert\mathcal{Z}_{b,N}\rvert$	$\widetilde{z}_{b,N}$	z_b	error
3	59	24512322843	.36907025	.36907024	$1.7...\times 10^{-8}$
4	51	4119790	.21543273	.20751874	$3.8...\times 10^{-2}$
5	31	7612395846	.45604067	.45604068	$2.8...\times 10^{-8}$
6	31	598683507	.36385650	.37272266	$2.3...\times 10^{-2}$
7	25	5390633926	.46061589	.46049815	$2.5...\times 10^{-4}$
8	27	2717026	.26387156	.24839536	$6.2...\times 10^{-2}$
9	23	151573306	.37273465	.36252164	$2.8...\times 10^{-2}$
10	22	1414773364	.41594031	.41978534	$9.1...\times 10^{-3}$
11	19	737799286	.44818212	.44816395	$4.0...\times 10^{-5}$
12	16	845908	.34327662	.35378177	$2.9...\times 10^{-2}$

Table 3: Here $\mathcal{Z}_{b,N}$ is the set of base-b Zuckerman numbers with N digits, $\widetilde{z}_{b,N} := \log\lvert\mathcal{Z}_{b,N}\rvert/\log(b^N)$, and the error is equal to $\lvert\widetilde{z}_{b,N} - z_b\rvert/z_b$.

3 Preliminaries

This section collects some preliminary lemmas needed in subsequent proofs.

3.1 Zuckerman numbers

The next lemma regards two simple, but useful, facts.

Lemma 3.1. *Let $n \in \mathcal{Z}_b$. Then*

(i) $0 \notin \mathcal{D}_b(n)$*;*

(ii) b *does not divide* $p_b(n)$*.*

Proof. Since $n \in \mathcal{Z}_b$, we have that $p_b(n)$ divides n. If $0 \in \mathcal{D}_b(n)$, then $p_b(n) = 0$, and so $n = 0$, which is impossible. Thus (i) follows. If b divides $p_b(n)$, then b divides n, so that the least significant digit of n is equal to zero, which is impossible by (i). Thus (ii) follows. $\square$

We need to define some families of sets of digits and some related quantities. Let Ω_b be the family of all subsets $\mathcal{D} \subseteq \{1, \dots, b-1\}$ such that b does not divide $d_1 \cdots d_k$ for all $k \geq 1$ and $d_1, \dots, d_k \in \mathcal{D}$. Moreover, let Ω_b^* be the family of all $\mathcal{D} \in \Omega_b$ such that there exists no $\mathcal{D}' \in \Omega_b$ with $\mathcal{D} \subsetneq \mathcal{D}'$. (A more explicit description of Ω_b^* is given by Remark 3.1.) Note that every $\mathcal{D} \in \Omega_b^*$ contains each $d \in \{1, \dots, b-1\}$ with $\gcd(b, d) = 1$. In particular, since $b \geq 3$, we have that $|\mathcal{D}| \geq 2$.

For every k pairwise distinct digits $d_1, \dots, d_k \in \{1, \dots, b-1\}$, if there exist integers $v_1, \dots, v_k \geq 1$ such that b divides $d_1^{v_1} \cdots d_k^{v_k}$ then let $V_b(d_1, \dots, d_k)$ be the minimum possible value of $\max(v_1, \dots, v_k)$, otherwise let $V_b(d_1, \dots, d_k) := 0$. Furthermore, let W_b be the maximum of $V_b(d_1, \dots, d_k)$ as $d_1, \dots, d_k$ range over all the possible k pairwise distinct digits in $\{1, \dots, b-1\}$. For each $\mathcal{D} \in \Omega_b^*$, let $\mathcal{Z}_{b,\mathcal{D}}$ be the set of all $n \in \mathcal{Z}_b$ such that $d \in \mathcal{D}_b(n) \setminus \mathcal{D}$ implies $w_{b,d}(n) < W_b$. Also, for every integer $N \geq 1$, put $\mathcal{Z}_{b,\mathcal{D},N} := \mathcal{Z}_{b,\mathcal{D}} \cap [b^{N-1}, b^N)$.

Lemma 3.2. *We have that* $\mathcal{Z}_b = \bigcup_{\mathcal{D} \in \Omega_b^*} \mathcal{Z}_{b,\mathcal{D}}$.

Proof. Since $\mathcal{Z}_{b,\mathcal{D}} \subseteq \mathcal{Z}_b$ for every $\mathcal{D} \in \Omega_b^*$, it suffices to prove that $\mathcal{Z}_b \subseteq \bigcup_{\mathcal{D} \in \Omega_b^*} \mathcal{Z}_{b,\mathcal{D}}$. Let $n \in \mathcal{Z}_b$. We have to prove that there exists $\mathcal{D} \in \Omega_b^*$ such that for each $d \in \mathcal{D}_b(n)$ we have that either $d \in \mathcal{D}$ or $w_{b,d}(n) < W_b$. Note that it suffices to prove the existence of $\mathcal{D} \in \Omega_b$, since every set in Ω_b is a subset of a set in Ω_b^*.

If $\mathcal{D}_b(n) \in \Omega_b$ then the claim is obvious. Hence, assume that $\mathcal{D}_b(n) \notin \Omega_b$. Thus there exist k pairwise distinct digits $d_1, \dots, d_k \in \mathcal{D}_b(n)$ and integers $v_1, \dots, v_k \geq 1$ such that b divides $d_1^{v_1} \cdots d_k^{v_k}$. In particular, without loss of generality, we can assume that $\max(v_1, \dots, v_k) = V_b(d_1, \dots, d_k)$. Consequently, we have that $\max(v_1, \dots, v_k) \leq W_b$.

If $w_{b,d_i}(n) \geq W_b$ for each $i \in \{1, \dots, k\}$, then we get that b divides $d_1^{w_{b,d_1}(n)} \cdots d_k^{w_{b,d_k}(n)}$, which in turn divides $p_b(n)$. But by Lemma 3.1(ii) this is impossible, since $n \in \mathcal{Z}_b$.

Therefore, there exists $i_1 \in \{1, \dots, k\}$ such that $w_{b,d_{i_1}}(n) < W_b$. At this point, we can repeat the previous reasonings with $\mathcal{D}' := \mathcal{D}_b(n) \setminus \{d_{i_1}\}$ in place of $\mathcal{D}_b(n)$. If $\mathcal{D}' \in \Omega_b$ then the claim follows. If $\mathcal{D}' \notin \Omega_b$, then there exist k' pairwise distinct digits in $\mathcal{D}'$ such that a certain product of them is divisible by b, ...and so on.

It is clear that this procedure terminates after at most $|\mathcal{D}_b(n)|$ steps (note that $\varnothing \in \Omega_b$), thus producing the desidered $\mathcal{D} \in \Omega_b$. $\square$

Remark 3.1. It follows easily that the elements of Ω_b^* are the sets

$$\big\{d \in \{1, \dots, b-1\} : p \nmid d\big\},$$

where p runs over the prime factors of b. In particular, if b is a prime number then the only element of Ω_b^* is $\{1, \dots, b-1\}$.

3.2 Entropy and multinomial sums

For all real numbers $x_1, \dots, x_k \in [0, 1]$ such that $\sum_{i=1}^k x_i = 1$, define the *entropy*

$$H(x_1, \dots, x_k) := -\sum_{i=1}^k x_i \log x_i,$$

with the usual convention that $0 \log 0 := 0$.

Lemma 3.3. *Let $N_1, \dots, N_k \geq 0$ be integers and put $N := N_1 + \cdots + N_k$. Then*

$$\frac{N!}{N_1! \cdots N_k!} = \exp\left(H\left(\frac{N_1}{N}, \dots, \frac{N_k}{N}\right) N + O_k(\log N)\right),$$

as $N \to +\infty$.

Proof. The claim follows easily from Stirling's formula in the form

$$\log n! = n \log n - n + O(\log n),$$

as $n \to +\infty$. □

Lemma 3.4. *Let $a_1, \dots, a_k \geq 0$ and c be real numbers such that*

$$\min\{a_1, \dots, a_k\} < c < \frac{1}{k} \sum_{i=1}^k a_i. \tag{2}$$

Then the equation

$$\sum_{i=1}^k (a_i - c) e^{a_i \lambda} = 0 \tag{3}$$

has a unique solution $\lambda \in (-\infty, 0)$. Moreover, the maximum of $H(x_1, \dots, x_k)$ under the constraints $\sum_{i=1}^k x_i = 1$ and $\sum_{i=1}^k a_i x_i \leq c$ is equal to

$$H_{\max} := -c\lambda + \log\left(\sum_{i=1}^k e^{a_i \lambda}\right),$$

and it is achieved if and only if $x_i = e^{a_i \lambda} / \sum_{j=1}^k e^{a_j \lambda}$ for $i = 1, \dots, k$.

Proof. First, note that $a_1, \dots, a_k$ are not all equal, otherwise (2) would not be satisfied. For every real number t, define

$$F(t) := \frac{\sum_{i=1}^k a_i e^{a_i t}}{\sum_{i=1}^k e^{a_i t}} - c.$$

We have that

$$F'(t) = \frac{\left(\sum_{i=1}^k a_i^2 e^{a_i t}\right)\left(\sum_{i=1}^k e^{a_i t}\right) - \left(\sum_{i=1}^k a_i e^{a_i t}\right)^2}{\left(\sum_{i=1}^k e^{a_i t}\right)^2} > 0,$$

where we applied the Cauchy–Schwarz inequality, which is strict since $a_1, \dots, a_k$ are not all equal. Furthermore, by (2) we have that

$$\lim_{t \to -\infty} F(t) = \min\{a_1, \dots, a_k\} - c < 0 \quad \text{and} \quad F(0) = \frac{1}{k} \sum_{i=1}^k a_i - c > 0.$$

Therefore, there exists a unique $\lambda \in (-\infty, 0)$ such that $F(\lambda) = 0$, which is equivalent to (3).

The rest of the lemma is a standard application of the method of Lagrange multipliers (cf. [13, p. 328, Example 3]). We only sketch the proof. After introducing the slack variable x_{k+1}, the problem becomes to maximize $H = -\sum_{i=1}^k x_i \log x_i$ under the constrains $G_1 := \sum_{i=1}^k x_i - 1 = 0$ and $G_2 := \sum_{i=1}^k a_i x_i + x_{k+1}^2 - c = 0$. Let $L := H + \lambda_1 G_1 + \lambda_2 G_2$, where $\lambda_1, \lambda_2 \in \mathbb{R}$ are the Lagrange multipliers. Note that the $2 \times (k+1)$ matrix whose entry of the ith row and jth colum is equal to $\partial G_i / \partial x_j$ has full rank. Hence, by Lagrange's theorem [13, p. 326, Theorem], the constrained local extrema of H are obtained by solving the system of $k+2$ equations $\partial L/\partial x_i = 0$ $(i = 1, \dots, k)$ and $\partial L / \partial \lambda_j = 0$ $(j = 1, 2)$. This system has the unique solution $x_i = e^{\lambda_1 + a_i \lambda_2 - 1}$ $(i = 1, \dots, k)$, $x_{k+1} = 0$, $\lambda_1 = 1 - \log\big(\sum_{i=1}^k e^{a_i \lambda_2}\big)$, and $\lambda_2 = \lambda$. We point out that in the proof of this last claim the condition $c < \sum_{i=1}^k a_i / k$ is used to show that $\lambda_2 \neq 0$, which in turn implies that $x_{k+1} = 0$. Finally, that the local extremum is a maximum follows easily from the study of the Hessian matrix of H. $\square$

Remark 3.2. Somehow conversely to the first part of Lemma 3.4, it is not difficult to prove that if $a_1, \dots, a_k \geq 0$, c, and λ are real numbers satifying (3), then (2) holds.

Lemma 3.5. *Let $a_1, \dots, a_k, c$ and $H_{\max}$ be as in Lemma 3.4, let $h \geq k$ be an integer, and let $C \geq 0$ be a real number. Then we have that*

$$\sum \frac{N!}{N_1! \cdots N_h!} = \exp\big((H_{\max} + o(1))N\big), \tag{4}$$

as $N \to +\infty$, where $N := N_1 + \cdots + N_h$, the sum is over all integers $N_1, \dots, N_h \geq 0$ such that $\sum_{i=1}^k a_i N_i \leq cN$ and $N_i \leq C$ for each integer $i \in (k, h]$, and the $o(1)$ depends only on $a_1, \dots, a_k, c, h, C$.

Proof. Let $N_- := \sum_{i=1}^k N_i$ and $N_+ := \sum_{i=k+1}^h N_i$. Then we have that

$$\begin{aligned}\sum \frac{N!}{N_1! \cdots N_h!} &= \sum_{\sum_{i=1}^k a_i N_i \leq cN} \frac{N_-!}{\prod_{i=1}^k (N_i!)} \sum_{\substack{N_i \leq C \\ i = k+1, \dots, h}} \frac{\prod_{n=1}^{N_+} (N_- + n)}{\prod_{i=k+1}^h (N_i!)} \\ &= \sum_{\sum_{i=1}^k a_i N_i \leq cN} \frac{N_-!}{\prod_{i=1}^k (N_i!)} \exp\big(O_{C,h}(\log N)\big), \end{aligned}\tag{5}$$

as $N \to +\infty$, where we employed the facts that $N_i \leq C$ for each integer $i \in (k, h]$, and consequently $N_+ \leq Ch$. Furthermore, letting

$$M := \max_{\sum_{i=1}^k a_i N_i \leq cN} \frac{N_-!}{\prod_{i=1}^k (N_i!)},$$

we have that

$$\sum_{\sum_{i=1}^k a_i N_i \leq cN} \frac{N_-!}{\prod_{i=1}^k (N_i!)} = M \exp\big(O_k(\log N)\big), \tag{6}$$

as $N \to +\infty$, where we exploited the fact that the sum in (6) has at most $(N+1)^k$ terms.

For each $t \geq 0$ that is sufficiently small so that (2) holds if c is replaced by $c + t$, let $H_{\max}(t)$ be the quantity corresponding to $H_{\max}$ if c is replaced by $c + t$, and let

$$M_-(t) := \max_{\sum_{i=1}^k a_i N_i \leq (c+t) N_-} \frac{N_-!}{\prod_{i=1}^k (N_i!)}.$$

Since $N_- \leq N \leq N_- + Ch$, we have that $cN_- \leq cN \leq (c+t)N_-$ for $t > 0$ and $N \geq Ch(c/t + 1)$. Hence, we get that

$$M_-(0) \leq M \leq M_-(t), \tag{7}$$

if $t > 0$ and $N \geq Ch(c/t + 1)$. Moreover, by Lemma 3.3 and Lemma 3.4, we get that, uniformly for bounded $t \geq 0$, it holds

$$M_-(t) = \exp((H_{\max}(t) + o(1))N_-) = \exp((H_{\max}(t) + o(1))N)\,, \tag{8}$$

and $N \to +\infty$. Note that to deduce (8) we have to approximate $H(x_1, \dots, x_k)$, where $(x_1, \dots, x_k)$ is the point of maximum, with $H(N_1/N_-, \dots, N_k/N_-)$; and this can be done with an error of $O(1/N_-)$, since H has bounded partial derivatives in a neighborhood of $(x_1, \dots, x_k)$. At this point, from (7) and (8), and noticing that $H_{\max}(t) \to H_{\max}$ as $t \to 0$, we get that

$$M = \exp((H_{\max} + o(1))N)\,, \tag{9}$$

as $N \to +\infty$.

Finally, putting together (5), (6), and (9), we get (4), as desired. □

3.3 Further lemmas

Lemma 3.6. *Let $\mathcal{D} \subseteq \{1, \dots, b-1\}$ be nonempty, and let $(a_d)_{d\in\mathcal{D}}$ and c be nonnegative real numbers such that*

$$\min_{d\in\mathcal{D}} a_d < c < \frac{1}{|\mathcal{D}|} \sum_{d\in\mathcal{D}} a_d.$$

Then the equation

$$\sum_{d\in\mathcal{D}} (a_d - c)e^{a_d\lambda} = 0$$

has a unique solution $\lambda \in (-\infty, 0)$. Moreover, for each $C \geq 0$, we have that

$$\sum \frac{N!}{N_1! \cdots N_{b-1}!} = \exp\big((H_{\max} + o(1))N\big),$$

as $N \to +\infty$, where $N := \sum_{d=1}^{b-1} N_d$, the sum is over all integers $N_1, \dots, N_{b-1} \geq 0$ such that $\sum_{d\in\mathcal{D}} a_d N_d \leq cN$ and $N_d \leq C$ for each integer $d \in \{1, \dots, b-1\} \setminus \mathcal{D}$,

$$H_{\max} := -c\lambda + \log\left(\sum_{d\in\mathcal{D}} e^{a_d\lambda}\right),$$

and the $o(1)$ depends only on $(a_d)_{d\in\mathcal{D}}, c, h, C$.

Proof. The claim is a direct consequence of Lemma 3.4 and Lemma 3.5, after an appropriate change of the indices of the N_i's. □

Lemma 3.7. *Let $\mathcal{S}$ be a set of positive integers, let $\mathcal{S}(x) := \mathcal{S} \cap [1, x]$ for every real number $x \geq 1$, let $\mathcal{S}_N := \mathcal{S} \cap [b^{N-1}, b^N)$ for each integer $N \geq 1$, and let $t \in (0, 1)$. If $|\mathcal{S}_N| \leq b^{(t+o(1))N}$ as $N \to +\infty$, then $|\mathcal{S}(x)| \leq x^{t+o(1)}$ as $x \to +\infty$. Moreover, the previous claim holds if the two inequalities are reversed, or replaced by equalities.*

Proof. For every real number $x \geq 1$, let $N_x := \lfloor \log x / \log b \rfloor + 1$ and $M_x := \frac{1}{2}t\lfloor \log x/\log b \rfloor$. On the one hand, if $|\mathcal{S}_N| \leq b^{(t+o(1))N}$ as $N \to +\infty$, then we get that

$$|\mathcal{S}(x)| \leq x^{t/2} + \sum_{N=M_x+1}^{N_x} |\mathcal{S}_N| \leq x^{t/2} + N_x b^{(t+o(1))N_x} = x^{t+o(1)},$$

as $x \to +\infty$. On the other hand, if $|\mathcal{S}_N| \geq b^{(t+o(1))N}$ as $N \to +\infty$, then we get that

$$|\mathcal{S}(x)| \geq |\mathcal{S}_{N_x - 1}| \geq b^{(t+o(1))(N_x-1)} = x^{t+o(1)},$$

as $x \to +\infty$. □

Lemma 3.8. *If $b \geq 6$ then*
$$\sum_{d \in \mathcal{D}} \log d > \frac{|\mathcal{D}| \log b}{2} \tag{10}$$
for each $\mathcal{D} \in \Omega_b^*$.

Proof. By Remark 3.1, if b is prime then $\mathcal{D} = \{1, \dots, b-1\}$, and the claim follows easily. Hence, assume that b is composite. Again by Remark 3.1, we have that
$$\mathcal{D} = \big\{d \in \{1, \dots, b-1\} : p \nmid d\big\}, \tag{11}$$
where p is a prime factor of b. With the aid of a computer, we can verify the claim for each composite $b \in [6, 84]$. Hence, assume that $b > 84$. Let $m := \lfloor (b-1)/p \rfloor$. From (11) it follows that $|\mathcal{D}| = b - 1 - m$ and
$$\sum_{d \in \mathcal{D}} \log d = \log\big((b-1)!\big) - \sum_{k=1}^{m} \log(pk) = \log\big((b-1)!\big) - m \log p - \log(m!).$$
Therefore, we have that (10) is equivalent to
$$\log\big((b-1)!\big) - \tfrac{1}{2}(b-1)\log b > \log(m!) + m \log p - \tfrac{1}{2} m \log b. \tag{12}$$
Employing the bounds $n \log n - n \leq \log(n!) \leq n \log n$, holding for every integer $n \geq 1$, we get that
$$\begin{aligned} \log\big((b-1)!\big) - \tfrac{1}{2}(b-1)\log b &\geq (b-1)\left(\tfrac{1}{2}\log(b-1) - 1 - \tfrac{1}{2}\log\big(b/(b-1)\big)\right) \\ &> (b-1)\left(\tfrac{1}{2}\log(b-1) - 1.1\right) \end{aligned}$$
while
$$\begin{aligned} \log(m!) + m\log p - \tfrac{1}{2} m \log b &\leq m \log(mp) - \tfrac{1}{2} m \log(b-1) \\ &\leq \tfrac{1}{2} m \log(b-1) \\ &\leq \tfrac{1}{4}(b-1)\log(b-1) \end{aligned}$$
since $mp \leq b-1$ and $m \leq (b-1)/2$. Hence, recalling that $b > 84$, we get that (12) is satisfied. ☐

Remark 3.3. One can check that (10) does not hold if $b \in \{3, 4, 5\}$ and $\mathcal{D} \in \Omega_b^*$.

For all real numbers $s \geq 0$ and for every $\mathcal{D} \in \Omega_b^*$, define
$$\zeta_{\mathcal{D}}(s) := \sum_{d \in \mathcal{D}} \frac{1}{d^s}.$$
The next lemma regards an equation involving $\zeta_{\mathcal{D}}(s)$.

Lemma 3.9. *Let $\mathcal{D} \in \Omega_b^*$. If $v = 1$, or $v = 2$ and $b \geq 6$, then the equation*
$$\left(s + \frac{v \log|\mathcal{D}|}{\log b}\right) \frac{\zeta_{\mathcal{D}}'(s)}{\zeta_{\mathcal{D}}(s)} - \log\left(\frac{\zeta_{\mathcal{D}}(s)}{|\mathcal{D}|^v}\right) = 0. \tag{13}$$
has a unique solution $s \in (0, +\infty)$.

Proof. Let $G_{b,\mathcal{D},v}(s)$ denote the left-hand side of (13). A quick computation yields that
$$G_{b,\mathcal{D},v}'(s) = \left(s + \frac{v \log|\mathcal{D}|}{\log b}\right) \frac{\zeta_{\mathcal{D}}(s)\,\zeta_{\mathcal{D}}''(s) - \big(\zeta_{\mathcal{D}}'(s)\big)^2}{\big(\zeta_{\mathcal{D}}(s)\big)^2} > 0,$$

where we used the Cauchy–Schwarz inequality, which is strict since $|\mathcal{D}| \geq 2$. Moreover, we have that

$$G_{b,\mathcal{D},v}(0) = -\left(1 + v\left(\frac{\sum_{d\in\mathcal{D}} \log d}{|\mathcal{D}| \log b} - 1\right)\right) \log|\mathcal{D}|,$$

and

$$\lim_{s\to+\infty} G_{b,\mathcal{D},v}(s) = v \log|\mathcal{D}| \left(1 - \frac{\log \min(\mathcal{D})}{\log b}\right) > 0.$$

Hence, if

$$1 + v\left(\frac{\sum_{d\in\mathcal{D}} \log d}{|\mathcal{D}| \log b} - 1\right) > 0 \tag{14}$$

then $G_{b,\mathcal{D},v}(s)$ has a unique zero in $(0, +\infty)$ (otherwise $G_{b,\mathcal{D},v}(s)$ has no zero).

If $v = 1$ then it is clear that (14) holds. If $v = 2$ and $b \geq 6$, then (14) is equivalent to (10), which in turn is true by Lemma 3.8. □

4 Upper bound

In light of Lemma 3.9, for every $\mathcal{D} \in \Omega_b^*$ let $s_{b,\mathcal{D},1}$ be the unique solution to (13) with $v = 1$ and define

$$z_{b,\mathcal{D}}^+ := \frac{\log|\mathcal{D}|}{\log b}\left(1 + \frac{\zeta'_{\mathcal{D}}(s_{b,\mathcal{D},1})}{\zeta_{\mathcal{D}}(s_{b,\mathcal{D},1}) \log b}\right) \quad \text{and} \quad z_b^+ := \max_{\mathcal{D}\in\Omega_b^*} z_{b,\mathcal{D}}^+.$$

It follows easily that $z_b^+ \in (0, 1)$.

We have the following upper bound for the counting function of Zuckerman numbers.

Theorem 4.1. *We have that*

$$|\mathcal{Z}_b(x)| < x^{z_b^+ + o(1)},$$

as $x \to +\infty$.

Proof. In light of Lemma 3.2 and Lemma 3.7, it suffices to prove that

$$|\mathcal{Z}_{b,\mathcal{D},N}| < b^{(z_{b,\mathcal{D}}^+ + o(1))N}, \tag{15}$$

as $N \to +\infty$, for every $\mathcal{D} \in \Omega_b^*$. Pick an arbitrary $\mathcal{D} \in \Omega_b^*$. Let $s > 0$ be a constant to be defined later (depending only on b and $\mathcal{D}$), and let

$$\alpha := -\frac{\zeta'_{\mathcal{D}}(s)}{\zeta_{\mathcal{D}}(s) \log b}, \quad \beta := \frac{\log|\mathcal{D}|}{\log b}(1 - \alpha), \quad \text{and} \quad \gamma := \alpha s + \frac{\log(\zeta_{\mathcal{D}}(s))}{\log b}. \tag{16}$$

For every integer $N \geq 1$, define the sets

$$\mathcal{Z}'_{b,\mathcal{D},N} := \{n \in \mathcal{Z}_{b,\mathcal{D},N} : p_b(n) > b^{\alpha N}\}$$

and $\mathcal{Z}''_{b,\mathcal{D},N} := \mathcal{Z}_{b,\mathcal{D},N} \setminus \mathcal{Z}'_{b,\mathcal{D},N}$. Thus $\mathcal{Z}'_{b,\mathcal{D},N}$, $\mathcal{Z}''_{b,\mathcal{D},N}$ is a partition of $\mathcal{Z}_{b,\mathcal{D},N}$.

Pick $n \in \mathcal{Z}_{b,\mathcal{D},N}$. For the sake of brevity, put $N_d := w_{b,d}(n)$ for each $d \in \{1, \dots, b-1\}$. Hence, we have that n has exactly N digits, and $N_1 + \cdots + N_{b-1} = N$ (recall Lemma 3.1(i)). Moreover, recalling the definition of $\mathcal{Z}_{b,\mathcal{D}}$, we get that $N_d < W_b$ for every $d \in \mathcal{D}^c$, where $\mathcal{D}^c := \{1, \dots, b-1\} \setminus \mathcal{D}$.

First, suppose that $n \in \mathcal{Z}'_{b,\mathcal{D},N}$. Let ℓ be the unique integer such that $b^\ell \leq p_b(n) < b^{\ell+1}$. Since $p_b(n) > b^{\alpha N}$, it follows that $\ell > \alpha N - 1$. Moreover, since $p_b(n)$ divides n, we have that $b^\ell \leq p_b(n) \leq n < b^N$, so that $\ell < N$. Recalling that $N_d < W_b$ for every $d \in \mathcal{D}^c$, we get that the number of possible choices for the $N - \ell$ most significant digits of n is at most

$$\sum_{j=0}^{(W_b-1)|\mathcal{D}^c|} \binom{N-\ell}{j} |\mathcal{D}^c|^j \, |\mathcal{D}|^{N-\ell-j} = N^{O(1)} |\mathcal{D}|^{N-\ell} < b^{(\beta+o(1))N},$$

as $N \to +\infty$. Furthermore, since $b^\ell \leq p_b(n)$ and $p_b(n)$ divides n, we get that for each of the previous choices there is at most one choice for the ℓ least significant digits of n (a similar idea is used in the proof of [1, Lemma 2]). Therefore, we obtain that

$$|\mathcal{Z}'_{b,\mathcal{D},N}| < b^{(\beta+o(1))N}, \tag{17}$$

as $N \to +\infty$.

Suppose that $n \in \mathcal{Z}''_{b,\mathcal{D},N}$. Since $p_b(n) \leq b^{\alpha N}$, we have that $\sum_{d\in\mathcal{D}}(\log d)N_d \leq (\alpha \log b)N$. Moreover, by elementary combinatorics, for fixed values of $N_1, \dots, N_{b-1}$ there are at most

$$\frac{N!}{N_1! \cdots N_{b-1}!}$$

possible values of n. Therefore, we can apply Lemma 3.6 with $a_d = \log d$ for $d \in \mathcal{D}$, $c = \alpha \log b$, $C = W_b - 1$, and $\lambda = -s$ (as a consequence of the definition of α, and also recalling Remark 3.2). This yields

$$|\mathcal{Z}''_{b,\mathcal{D},N}| < b^{(\gamma+o(1))N}, \tag{18}$$

as $N \to +\infty$.

At this point, by choosing $s = s_{b,\mathcal{D},1}$, from (13) we get that $\beta = \gamma = z^+_{b,\mathcal{D}}$. Therefore, putting together (17) and (18), we obtain (15), as desired. □

5 Lower bound for base 10

We have the following lower bound for the counting function of base-10 Zuckerman numbers.

Theorem 5.1. *We have that*

$$|\mathcal{Z}_{10}(x)| > x^{0.204}$$

as $x \to +\infty$.

Proof. In light of Lemma 3.7, it suffices to prove that

$$|\mathcal{Z}_{10,N}| > 10^{0.204N},$$

as $N \to +\infty$. For each integer $n \geq 0$, let $\nu(n)$ be the maximum integer v such that 2^v divides $p_{10}(n)$ (where $p_{10}(0) := 1$), and let $\mathcal{A}_n$ be the set of n-digit numbers whose digits belong to $\{1, 2, 4, 8\}$. Let ℓ be a positive integer to be determined later, and put

$$\delta := \frac{1}{\ell} \max_{0 \leq x < 2^\ell} \min_{\substack{y \in \mathcal{A}_\ell \\ y \equiv x \pmod{2^\ell}}} \nu(y), \tag{19}$$

where $\min \varnothing := +\infty$. Assume that $\delta < 1$. Let us prove that for each integer $k \geq 0$ there exists $n_k \in \mathcal{A}_{k\ell}$ such that $2^{k\ell}$ divides n_k and $\nu(n_k) \leq \delta k \ell$. We proceed by induction. For $k = 0$ it suffices to put $n_0 := 0$. Suppose that we proved the existence of n_k for some integer $k \geq 0$. Let us construct n_{k+1}. By the induction hypothesis, we have that $m_k := n_k / 2^{k\ell}$ is an integer. Moreover, by definition (19), there exists $y_k \in \mathcal{A}_\ell$ such that $y_k \equiv -5^{-k\ell} m_k \pmod{2^\ell}$ and $\nu(y_k) \leq \delta \ell$. Put $n_{k+1} := 10^{k\ell} y_k + n_k$. It is clear that $n_{k+1} \in \mathcal{A}_{(k+1)\ell}$. Furthermore, we have that $n_{k+1} \equiv 2^{k\ell}(5^{k\ell} y_k + m_k) \equiv 0 \pmod{2^{(k+1)\ell}}$ and $\nu(n_{k+1}) = \nu(y_k) + \nu(n_k) \leq \delta\ell + \delta k \ell = \delta(k+1)\ell$, as desired. The proof of the claim is complete.

Let N be a sufficiently large integer, and put $k := \lceil \alpha N / \ell \rceil$, $N_{\mathsf{lo}} := k\ell$, and $N_{\mathsf{hi}} := N - N_{\mathsf{lo}}$, where $\alpha \in (0, 1)$ is a constant to be determined later. Suppose that $n = 10^{N_{\mathsf{lo}}} n_{\mathsf{hi}} + n_k$, where $n_{\mathsf{hi}} \in \mathcal{A}_{N_{\mathsf{hi}}}$ satisfies

$$\nu(n_{\mathsf{hi}}) \leq \frac{\alpha(1-\delta)}{1-\alpha} N_{\mathsf{hi}}.$$

Then we have that $n \in \mathcal{A}_N$ and

$$\nu(n) = \nu(n_{\mathsf{hi}}) + \nu(n_k) \le \frac{\alpha(1-\delta)}{1-\alpha} N_{\mathsf{hi}} + \delta N_{\mathsf{lo}} \le (1-\delta)N_{\mathsf{lo}} + \delta N_{\mathsf{lo}} = N_{\mathsf{lo}}.$$

Consequently, recalling that $2^{N_{\mathsf{lo}}}$ divides n_k, and since $p_{10}(n) = 2^{\nu(n)}$, we get that $p_{10}(n)$ divides n, so that $n \in \mathcal{Z}_{10,N}$.

At this point, setting $\mathcal{D} = \{1, 2, 4, 8\}$, $a_1 = 0$, $a_2 = 1$, $a_4 = 2$, $a_8 = 3$, $c = \alpha(1-\delta)/(1-\alpha)$, and $C = 0$, the number of possible choices for n_{hi} can be estimated by Lemma 3.6 (if the hypothesis on a_1, a_2, a_4, a_8, c are satified), and it is equal to, say,

$$\exp\big((H_{\alpha,\delta} + o(1))N_{\mathsf{hi}}\big) = \exp\big(((1-\alpha)H_{\alpha,\delta} + o(1))N\big) = 10^{((1-\alpha)H_{\alpha,\delta}/\log 10 + o(1))N}.$$

Finally, we run the numbers. Picking $\ell = 24$, a computation [12] (see Remark 5.2) shows that $\delta = 1/2$. Then the choice $\alpha = 1/2$ yields that $H_{\alpha,\delta} > 0.940$ and

$$\frac{(1-\alpha)H_{\alpha,\delta}}{\log 10} > 0.204,$$

as desired. □

Remark 5.1. The proof of Theorem 5.1 is similar to the proof of the lower bound of [9, Theorem 1(i)], but with a significant difference. Instead of the max-min approach employing (19), in [9] the sequence $(n_k)_{k\ge0}$ is constructed by an inductive process that allows one to make some choices. Then it is shown that the choices leading to the worst-case scenario do not happen too often. This provided a worse parameter $\delta = 3/4$. Also, as a minor difference, we constructed $(n_k)_{k\ge0}$ by employing the digits $1, 2, 4, 8$, while [9] only uses the digits $1, 2, 4$.

Remark 5.2. We have that 24 is the smallest value of ℓ for which $\delta \le 1/2$, and we did not find a value of ℓ for which $\delta < 1/2$ (we tried up to $\ell = 28$). Note that the search space to compute δ has $|\mathcal{A}_\ell| = 4^\ell$ elements, which for $\ell = 24$ is already impractical for a brute-force search. However, employing some basic pruning techniques allowed us to complete the computation within a few seconds.

Remark 5.3. The strategy of the proof of Theorem 5.1 might generalize to other composite values of b. The idea would be to fix a prime factor q of b, let $\mathcal{A}_n$ be the set of n-digit numbers whose digits are all powers of q, let $\nu(n)$ be the maximum integer v such that q^v divides $p_b(n)$, and let δ be defined as in (19), but with 2^ℓ replaced by q^ℓ. If $\delta < 1$ for a sufficiently large ℓ, then one gets a lower bound for $|\mathcal{Z}_b(x)|$.

6 Heuristic

For every $\mathcal{D} \in \Omega_b^*$, let

$$z_{b,\mathcal{D}} := \frac{\log\big(\zeta_{\mathcal{D}}(1)\big)}{\log b}, \quad \text{and let} \quad z_b := \max_{\mathcal{D}\in\Omega_b^*} z_{b,\mathcal{D}}.$$

In fact, in light of Remark 3.1, it follows easily that

$$z_b = \frac{1}{\log b} \log\Big(\sum_{1 \le d < b,\ P^+(b) \nmid d} \frac{1}{d}\Big),$$

where $P^+(b)$ denotes the greatest prime factor of b.

In this section, we describe a heuristic suggesting that, for each $\mathcal{D} \in \Omega_b^*$, we have that

$$|\mathcal{Z}_{b,\mathcal{D},N}| = b^{(z_{b,\mathcal{D}} + o(1))N}, \tag{20}$$

as $N \to +\infty$, and consequently, by Lemma 3.2 and Lemma 3.7, that

$$|\mathcal{Z}_b(x)| = x^{z_b + o(1)},$$

as $x \to +\infty$.

The heuristic suggesting (20) works as follows. Fix $\mathcal{D} \in \Omega_b^*$, let $N \geq 1$ be an integer, and let n be a positive integer whose representation has exactly N digits. In light of Lemma 3.1(i), suppose that $0 \notin \mathcal{D}_b(n)$, and put $N_d := w_{b,d}(n)$ for each $d \in \{1, \dots, b-1\}$. Hence, we have that $N = \sum_{d=1}^{b-1} N_d$. For fixed values of $N_1, \dots, N_{b-1}$, the number of possible values of n is equal to

$$\frac{N!}{\prod_{d=1}^{b-1}(N_d!)}.$$

Each of these n's is a Zuckerman number if and only if it is divisible by $p_b(n) = \prod_{d=1}^{b-1} d^{N_d}$. Heuristically, the probability that this occurs is equal to $1/p_b(n)$. Hence, the expected number of Zuckerman numbers among the aforementioned n's is equal to

$$\frac{N!}{\prod_{d=1}^{b-1}\big((N_d!)d^{N_d}\big)}. \tag{21}$$

At this point, to (heuristically) compute $|\mathcal{Z}_{b,\mathcal{D},N}|$, we sum (21) over all the values of $N_1, \dots, N_{b-1}$ with $N_d < W_b$ for each $d \in \mathcal{D}^c := \{1, \dots, b-1\} \setminus \mathcal{D}$. Letting $N_{\mathcal{D}} := \sum_{d \in \mathcal{D}} N_d$, this yields that

$$\begin{aligned}
|\mathcal{Z}_{b,\mathcal{D},N}| &= \sum \frac{N!}{\prod_{d=1}^{b-1}\big((N_d!)d^{N_d}\big)} \\
&= \sum_{N_{\mathcal{D}}=1}^{N} \sum_{\sum_{d\in\mathcal{D}} N_d = N_{\mathcal{D}}} \frac{N_{\mathcal{D}}!}{\prod_{d\in\mathcal{D}}\big((N_d!)d^{N_d}\big)} \cdot \sum_{\substack{N_d < W_b \\ d \in \mathcal{D}^c}} \frac{\prod_{k=1}^{N_{\mathcal{D}^c}}(N_{\mathcal{D}}+k)}{\prod_{d\in\mathcal{D}^c}\big((N_d!)d^{N_d}\big)} \\
&= \sum_{N_{\mathcal{D}}=1}^{N} \big(\zeta_{\mathcal{D}}(1)\big)^{N_{\mathcal{D}}} \cdot b^{o(1)N} = b^{(z_{b,\mathcal{D}}+o(1))N},
\end{aligned}$$

as $N \to +\infty$, where in the last equality we employed the multinomial theorem. Thus (20) is proved.

7 Algorithms

In this section, we describe some algorithms to count, or to enumerate, the Zuckerman numbers that have exactly N digits. The first three algorithms (Sections 7.1, 7.2, and 7.3) are asymptotically worse than the last two (Sections 7.4 and 7.5). Hence, we only give a brief description of them, without diving into the details, and we omit possible improvements coming from Lemma 3.2.

7.1 Brute force

Of course, the simplest algorithm proceeds by brute force. Each integer with N digits, which are all nonzero (Lemma 3.1(i)), is tested to determine if it is a Zuckerman number or not. The complexity of this algorithm is of the order of $(b-1)^N$.

7.2 Enumerating multiples

First, the algorithm runs over the possible values P of the product of N digits such that $P > b^{\alpha N}$, where $\alpha > 0$ is a constant. For each P, the algorithm runs over the multiples of P that have N digits, checking if each of them has digits consistent with the product P, and thus is a Zuckerman number. Second, the algorithms runs over the N-digit numbers n such that $p_b(n) \leq b^{\alpha N}$, and determines which of them are Zuckerman numbers.

The complexity of the first part if $b^{(1-\alpha+o(1))N}$, while (with reasonings similar to those leading to (18)), the complexity of the second part is $b^{(\delta+o(1))N}$, where

$$\delta := \alpha s + \frac{\log\big(\zeta_b(s)\big)}{\log b}, \quad \text{with} \quad \zeta_b(s) := \zeta_{\{1,\dots,b-1\}}(s),$$

and $s > 0$ is arbitrary. The optimal choice for α and s is taking s to be the unique solution of the equation

$$\frac{(s+1)\zeta_b'(s)}{\zeta_b(s)} - \log\left(\frac{\zeta_b(s)}{b}\right) = 0$$

(see [16, Theorem 2.2]) and setting

$$\alpha = \frac{\log\big(b/\zeta_b(s)\big)}{(s+1)\log b}.$$

In particular, for $b = 10$ the algorithm has complexity of the order of $10^{0.717N}$.

This algorithm can also be easily modified to enumerate Zuckerman numbers, which increases the complexity by an additive term $|\mathcal{Z}_{b,N}|$.

7.2.1 Dynamic programming

It is also possible to count the Zuckerman numbers n such that $p_b(n) \leq b^{\alpha N}$ by using a dynamic programming approach. Let $f[p][i][\mathcal{S}]$ denote the number of n such that $n \equiv i \pmod{p}$ and the multiset of digits of n is equal to $\mathcal{S}$. Then $f[p][\cdots][\mathcal{T}]$, with $|\mathcal{T}| = k+1$, can be computed from $f[p][\cdots][\mathcal{S}]$, with $|\mathcal{S}| = k$, by initializing it with zeros and then adding $f[p][i][\mathcal{S}]$ to

$$f[p][(ib+d) \bmod p][\mathcal{S} \cup \{d\}]$$

for all the possible values of i, d, and $\mathcal{S}$. (This corresponds to adding a new least significant digit d.) Finally, we only need to compute

$$\sum_{p \leq b^{\alpha N},\, |\mathcal{S}| = N,\, \prod_{v \in \mathcal{S}} v = p} f[p][0][\mathcal{S}]. \tag{22}$$

Since the number of $\mathcal{S}$'s that we have to consider is negligible (of the order of $b^{o(N)}$), the complexity of computing (22) is of the order of $b^{(\alpha+o(1))N}$. By setting $\alpha = 1/2$, we get an overall complexity of $b^{(1/2+o(1))N}$.

7.3 Meet in the middle

This algorithm follows a meet-in-the-middle approach. Letting $N_{\mathsf{lo}} := \lfloor N/2 \rfloor$ and $N_{\mathsf{hi}} := N - N_{\mathsf{lo}}$, each N-digit number n is written as $n = n_{\mathsf{hi}}\, b^{N_{\mathsf{lo}}} + n_{\mathsf{lo}}$, where n_{hi} and n_{lo} are N_{hi}-digit and N_{lo}-digit numbers, respectively. Hence, we have that n is a Zuckerman number if and only if $n_{\mathsf{lo}} \equiv -n_{\mathsf{hi}}\, b^{N_{\mathsf{lo}}} \pmod{P}$, where $P := P_{\mathsf{hi}} P_{\mathsf{lo}}$, $P_{\mathsf{hi}} := p_b(n_{\mathsf{hi}})$, and $P_{\mathsf{lo}} := p_b(n_{\mathsf{lo}})$.

The algorithm runs over the possible values P_{hi} and P_{lo} of the product of N_{hi} digits and N_{lo} digits, respectively, and for each pair $(P_{\mathsf{hi}}, P_{\mathsf{lo}})$ does the following computation. First, it builds a table $T[r]$, with $r = 0, \dots, P-1$, such that $T[r]$ is equal to the number of n_{hi} satisfying $p_b(n_{\mathsf{hi}}) = P_{\mathsf{hi}}$ and $r \equiv -n_{\mathsf{hi}}\, b^{N_{\mathsf{lo}}} \pmod{P}$. Second, for each n_{lo} with $p_b(n_{\mathsf{lo}}) = P_{\mathsf{lo}}$ it increases the counter of Zuckerman numbers by $T[n_{\mathsf{lo}} \bmod P]$.

The complexity of this algorithm is of the order of $b^{(1/2+o(1))N}$. Also, by storing n_{hi} in the table T, the algorithm can be easily modified to enumerate Zuckerman numbers. This increases the complexity to $b^{(1/2+o(1))N} + |\mathcal{Z}_{b,N}|$.

7.4 An improved algorithm

The strategy of this algorithm, which we call $\mathsf{ZuckermanCount}$, is the following. Let n be an integer with N digits, and let $N_d := w_{b,d}(n)$ for $d = 0, \dots, b-1$. We assume that $N_0 = 0$, by Lemma 3.1(i). The first part of the algorithm (Figure 1) runs over the possible values of $N_0, \dots, N_{b-1}$, taking into account the restriction given by Lemma 3.2, and computes the product of digits $P := \prod_{d=1}^{b-1} d^{N_d}$. Then, depending if $P > b^{\alpha N}$ or not, where $\alpha > 0$ is a constant defined in the proof of Theorem 7.1, the subroutine $\mathsf{LargeProduct}$ (Figure 2) or $\mathsf{SmallProduct}$ (Figure 3) is called, respectively.

The subroutine $\mathsf{LargeProduct}$ counts the number of n's that are divisible by P by following the same strategy of the first part of the proof of Theorem 4.1. More precisely, $\mathsf{LargeProduct}$ runs over the possible values of the most significant $N - \ell$ digits of n, where $\ell := \lfloor \log P / \log b \rfloor$, and uniquely determines the remaining ℓ digits of n by the condition that P divides n.

The subroutine $\mathsf{SmallProduct}$ counts the number of n's that are divisible by P by using a meet-in-the-middle approach similar to that of Section 7.3. The idea is the following. Let n_{hi} and n_{lo} be the unique integers such that $n = n_{\mathsf{hi}}\, b^{N_{\mathsf{lo}}} + n_{\mathsf{lo}}$, $1 \leq n_{\mathsf{hi}} < b^{N_{\mathsf{hi}}}$, and $1 \leq n_{\mathsf{lo}} < b^{N_{\mathsf{lo}}}$, where N_{hi} is an integer to be defined later and $N_{\mathsf{lo}} := N - N_{\mathsf{hi}}$. Then P divides n if and only if $n_{\mathsf{lo}} \equiv -n_{\mathsf{hi}}\, b^{N_{\mathsf{lo}}} \pmod{P}$. Hence, $\mathsf{SmallProduct}$ first builds a table $T[r]$, with $r = 0, \dots, P-1$, such that $T[r]$ is equal to the number of n_{hi}'s satisfying $r \equiv -n_{\mathsf{hi}}\, b^{N_{\mathsf{lo}}} \pmod{P}$. Then, $\mathsf{SmallProduct}$ runs over the possible values of n_{lo}, and increases the counter of Zuckerman numbers by $T[n_{\mathsf{lo}} \bmod P]$.

The choice of N_{hi} is made so that the number of values of n_{lo} and the number of values of n_{hi} are both of the order of $M^{1/2}$, where $M := N!/(N_0! \cdots N_{b-1}!)$. This requires some considerations on what we call the *separating index* $\sigma_b(n)$. For each integer n, whose digits are all nonzero, let $\pi_b(n)$ be the number of integers that can be obtained by permuting the digits of n. For every positive integer $i \leq N$, let $n[:i]$ denote the i-digit integer consisting of the i most significant digits of n. We define $\sigma_b(n)$ as the minimum positive integer i such that $\pi_b\big(n[:i]\big) \geq M^{1/2}$. It follows easily that $\sigma_b(n)$ is well defined and satisfies $M^{1/2} \leq \pi_b\big(n[:\sigma_b(n)]\big) < N M^{1/2}$, since adding one digit increases the number of permutations by a factor at most equal to N. The algorithm constructs N_{hi} so that, a posteriori, we have that $N_{\mathsf{hi}} = \sigma_b(n)$. This requires to discard the integers n such that $\pi_b\big(n[:(N_{\mathsf{hi}} - 1)]\big) \geq M^{1/2}$. In this way, the number of possible values for n_{hi} and n_{lo} are not exceeding $N M^{1/2}$ and $M^{1/2}$, respectively.

By storing n_{hi} in the table T, the algorithm can be easily modified to enumerate Zuckerman numbers. This increases the complexity by an additive term $|\mathcal{Z}_{b,N}|$.

7.4.1 Complexity analysis

For every $\mathcal{D} \in \Omega_b^*$, if $b \in \{3,4,5\}$ then let $s_{b,\mathcal{D},2} := 0$, while if $b \geq 6$ then, in light of Lemma 3.9, let $s_{b,\mathcal{D},2}$ be the unique solution to (13) with $v = 2$. Then define

$$z_{b,\mathcal{D}}^* := \frac{\log|\mathcal{D}|}{\log b}\left(1 + \frac{\zeta_{\mathcal{D}}'(s_{b,\mathcal{D},2})}{\zeta_{\mathcal{D}}(s_{b,\mathcal{D},2})\log b}\right) \quad \text{and} \quad z_b^* := \max_{\mathcal{D}\in\Omega_b^*} z_{b,\mathcal{D}}^*.$$

It follows easily that $z_b^* \in (0,1)$.

Theorem 7.1. *The algorithm $\mathsf{ZuckermanCount}$ (Figure 1) has complexity $b^{(z_b^*+o(1))N}$, as $N \to +\infty$.*

Proof. First, note that the loop in $\mathsf{ZuckermanCount}$ has at most $N^b = b^{o(N)}$ steps, which is negligible for our estimate. Hence, it suffices to estimate the complexities of $\mathsf{LargeProduct}$ and $\mathsf{SmallProduct}$. Furthermore, in light of Lemma 3.2, we can compute the complexities of $\mathsf{LargeProduct}$ and $\mathsf{SmallProduct}$ when they count the Zuckerman numbers in $\mathcal{Z}_{b,\mathcal{D},N}$, for a fixed $\mathcal{D} \in \Omega_b^*$, and then consider the worst-case $\mathcal{D}$.

Let $s > 0$ be a constant to be defined later (depending only on b and $\mathcal{D}$), and let α, β, γ be defined as in (16).

For $\mathsf{LargeProduct}$, the complexity can be estimated exactly as the upper bound (17), hence it is at most $b^{(\beta+o(1))N}$.

For $\mathsf{SmallProduct}$, the number of steps in the outer loop is negligible, and consequently the complexity is at most the cost of generating all the n_{hi}'s and n_{lo}'s, which by construction is at most $NM^{1/2}$. Thus, ignoring negligible factors, the complexity of $\mathsf{SmallProduct}$ is at most

$$\sum_{\prod_{d\in\mathcal{D}} d^{N_d} \le b^{\alpha N}} \left(\frac{N!}{N_1!\cdots N_{b-1}!}\right)^{1/2} \le \left(\sum_{\prod_{d\in\mathcal{D}} d^{N_d} \le b^{\alpha N}} 1\right)^{1/2} \left(\sum_{\prod_{d\in\mathcal{D}} d^{N_d} \le b^{\alpha N}} \frac{N!}{N_1!\cdots N_{b-1}!}\right)^{1/2}$$
$$\le N^{b/2}\left(\sum_{\prod_{d\in\mathcal{D}} d^{N_d} \le b^{\alpha N}} \frac{N!}{N_1!\cdots N_{b-1}!}\right)^{1/2} < b^{(\gamma/2+o(1))N},$$

where we employed the Cauchy–Schwarz inequality and Lemma 3.6 with $a_d = \log d$ for $d \in \mathcal{D}$, $c = \alpha \log b$, $C = W_b - 1$, and $\lambda = -s$ (recalling also Remark 3.2).

It remains to choose s so that $\max(\beta, \gamma/2)$ is minimal. If $b \ge 6$ then, in light of Lemma 3.9, we choose $s = s_{b,\mathcal{D},2}$, so that $\beta = \gamma/2 = z^*_{b,\mathcal{D}}$. Hence, we obtain that the complexities of $\mathsf{LargeProduct}$ and $\mathsf{SmallProduct}$ are both equal to $b^{(z^*_{b,\mathcal{D}}+o(1))N}$. Finally, considering the worst-case scenario for $\mathcal{D}$, we get the complexity $\mathsf{ZuckermanCount}$ is equal to $b^{(z^*_b+o(1))N}$. If $b \in \{3,4,5\}$, then by Remark 3.1 we have that $|\Omega_b| = 1$. Moreover, it can be verified that $\max(\beta,\gamma/2) = \beta$ and that the optimal choice is taking s arbitrary small, as it is done in the definition of z^*_b for $b \in \{3,4,5\}$. $\square$

7.5 A further improved algorithm for base 10

We now describe an ad hoc algorithm for $b = 10$, which we call $\mathsf{ZuckermanCount10}$. It is defined as $\mathsf{ZuckermanCount}$ (with $b = 10$) but the subroutine $\mathsf{SmallProduct}$ is replaced by $\mathsf{SmallProduct10}$, see Figure 4.

The idea on which $\mathsf{SmallProduct10}$ is based is the following. Let N_{lo}, N_{mi}, and N_{hi} be nonnegative integers such that $N_{\mathsf{lo}} + N_{\mathsf{mi}} + N_{\mathsf{hi}} = N$. We write each N-digit integer as

$$n = 10^{N_{\mathsf{mi}}+N_{\mathsf{lo}}} n_{\mathsf{hi}} + 10^{N_{\mathsf{lo}}} n_{\mathsf{mi}} + n_{\mathsf{lo}}, \tag{23}$$

where n_{hi}, n_{mi}, and n_{lo} are N_{hi}-digit, N_{mi}-digit, and N_{lo}-digit integers, respectively. Let v be the greatest integer such that 2^v divides $p_{10}(n)$. Suppose that

$$N_{\mathsf{lo}} \le \min\left\{\frac{\log 2}{\log 10}\, v, \frac{\log 2}{\log 5}\, N_{\mathsf{mi}}\right\}. \tag{24}$$

Let $u := \lceil N_{\mathsf{lo}} \log 10/\log 2\rceil$, so that $n_{\mathsf{lo}} < 10^{N_{\mathsf{lo}}} \le 2^u$. From (24), we get that $N_{\mathsf{lo}} \le v \log 2/\log 10$ and so $u \le v$. Moreover, from (24) we have that $N_{\mathsf{lo}} \le N_{\mathsf{mi}} \log 2/\log 5$, which implies that $N_{\mathsf{mi}} + N_{\mathsf{lo}} \ge N_{\mathsf{lo}} \log 10/\log 2$, and so $N_{\mathsf{mi}} + N_{\mathsf{lo}} \ge u$.

If n is a Zuckerman number, then 2^v divides n. Hence, by the previous considerations, we obtain that 2^u divides both n and $10^{N_{\mathsf{mi}}+N_{\mathsf{lo}}}$. Thus we have that $n_{\mathsf{lo}} \equiv -10^{N_{\mathsf{lo}}} n_{\mathsf{mi}} \pmod{2^u}$ and $0 \le n_{\mathsf{lo}} < 2^u$. Consequently, we get that n_{lo} is uniquely determined by n_{mi}, N_{lo}, and v.

At this point, the idea is to follow a meet-in-the-middle approach on the possible permutations of the digits of n_{hi} and n_{mi}, similarly to how it is done in $\mathsf{SmallProduct}$, but with the advantage that we are considering permutations of only $N_{\mathsf{hi}} + N_{\mathsf{mi}}$ digits, instead of N digits.

More precisely, we put $\alpha := 0.57992$, $\gamma := 0.37938$, and

$$N_{\mathsf{lo}} := \left\lfloor \min\left\{\frac{\log 2}{\log 10}\, v, \frac{\log 10 \log 2}{\log 8 \log 5}\, \gamma N\right\}\right\rfloor.$$

Then we compute N^*_{hi} and N^*_{mi} to perform a balanced meet-in-the-middle for the permutations of the most significant $N - N_{\mathsf{lo}}$ digits of n, and we set

$$N_{\mathsf{mi}} := \left\lfloor \max\left\{ \frac{\log 10}{\log 8}\,\gamma N, N^*_{\mathsf{mi}} \right\} \right\rfloor,$$

which ensures (24), and $N_{\mathsf{hi}} := N - N_{\mathsf{lo}} - N_{\mathsf{mi}}$.

Theorem 7.2. *The algorithm* ZuckermanCount10 *has complexity at most* $10^{\gamma N}$*, as* $N \to +\infty$.

Proof. The proof is similar to that of Theorem 7.1, thus we only sketch it. First, the number of steps of the outer loops is negligible, and the complexity of LargeProduct is $10^{(1-\alpha+o(1))N}$. Hence, it remains only to estimate the complexity of SmallProduct10.

Note that $\Omega^*_{10} = \{\mathcal{D}_2, \mathcal{D}_5\}$, where $\mathcal{D}_d := \{1, 2, \dots, 9\} \setminus \{d\}$ for $d \in \{2, 5\}$. We consider only when SmallProduct10 counts the Zuckerman numbers in $\mathcal{Z}_{10,\mathcal{D}_5,N}$. For $\mathcal{Z}_{10,\mathcal{D}_2,N}$ one applies a similar reasoning, and can verify that $\mathcal{D}_5$ is indeed the worst-case scenario.

If $N_{\mathsf{mi}} = N^*_{\mathsf{mi}}$ then, with a reasoning similar to that of the proof of Theorem 7.1, the complexity of SmallProduct10 is of the order of

$$\left(\sum \frac{(N - N_{\mathsf{lo}})!}{N_1! \cdots N_9!} \right)^{1/2}$$

where the sum is over all integers $N_1, \dots, N_9 \geq 0$ such that $\sum_{d=1}^{9} N_d = N - N_{\mathsf{lo}}$, $N_5 = 0$, $\sum_{d=1}^{9} (\log d) N_d \leq \alpha \log 10$, while

$$N_{\mathsf{lo}} := \left\lfloor \min\left\{ \frac{\log 2}{\log 10}(N_2 + 2N_4 + N_6 + 3N_8), \frac{\log 10 \log 2}{\log 8 \log 5}\,\gamma N \right\} \right\rfloor.$$

Therefore, using Lemma 3.3, and a strategy similar to that of the proof of Lemma 3.6, we get that SmallProduct10 has complexity $10^{\gamma N}$, since γ is (slightly) greater than the maximum of

$$-\frac{1}{2\log 10}\,(1-y) \sum_{d=1}^{9} \frac{x_d}{1-y} \log\left(\frac{x_d}{1-y}\right), \tag{25}$$

where

$$y := \min\left\{ \frac{\log 2}{\log 10}(x_2 + 2x_4 + x_6 + 3x_8), \frac{\log 10 \log 2}{\log 8 \log 5}\gamma \right\},$$

and $x_1, \dots, x_9 \geq 0$ satisfy the constraints $\sum_{d=1}^{9} x_d = 1 - y$, $x_5 = 0$, and $\sum_{d=1}^{9} (\log d) x_d \leq \alpha \log 10$. (Essentially, $x_d = N_d/N$ and $y = N_{\mathsf{lo}}/N$.) See Remark 7.2 for more details about the computation of such maximum.

If $N_{\mathsf{mi}} > N^*_{\mathsf{mi}}$ (and so $N_{\mathsf{hi}} < N^*_{\mathsf{mi}}$) then the meet-in-the-middle is unbalanced, with the complexity of computing the permutations of the digits of n_{mi} dominating the overall computation. However, since $N_{\mathsf{mi}} \leq N\gamma \log 10/\log 8$, we have that the number of permutations of the digits of n_{mi} is at most $8^{N_{\mathsf{mi}}} \leq 10^{\gamma N}$. Hence, in any case, the complexity of SmallProduct10 is at most $10^{(\gamma+o(1))N}$, as desired. □

Remark 7.1. The ideas of the algorithm ZuckermanCount10 and of the proof of Theorem 7.2 might generalize to other composite values of b.

Remark 7.2. Let $\gamma' := \gamma \log 10 \log 2/\log 8 \log 5$. To compute the maximum of (25) under the aforementioned constraints, one can reason as follows. First, assuming that $y = \gamma'$, one can compute the maximum under the constraints $\sum_{d=1}^{9} (\log d) x_d \leq \alpha \log 10$ and $x_5 = 0$. This can be done by using Lemma 3.6 after the change of variables $x'_d := x_d/(1 - \gamma')$. This maximum is 0.359..., which is less than γ. Then it remains to compute the maximum of

$$F := -\frac{1}{2\log 10} \sum_{d=1}^{9} x_d \log\left(\frac{x_d}{S}\right),$$

where $S := \sum_{d=1}^{9} x_d$, under the constraints $x_5 = 0$,

$$\frac{\log 2}{\log 10}(x_2 + 2x_4 + x_6 + 3x_8) = 1 - S,$$

and $\sum_{d=1}^{9} (\log d)x_d \leq \alpha \log 10$. This amount to solving the nonlinear system of equations given by the method of Lagrange multipliers, and can be done with arbitrary precision using numerical methods. The maximum is 0.3793709..., which is less than γ. Note that the domain determined by the constraints is convex and F is concave, since the Hessian matrix of $-F$ is positive semidefined. Thus each local maximum of F is in fact a global maximum [13, p. 192, Theorem 1].

$\mathsf{ZuckermanCount}(N)$

$c \leftarrow 0$
for $N_0, \ldots, N_{b-1} \in \mathbb{N}$ s.t. $N_0 + \cdots + N_{b-1} = N,\ N_0 = 0,$
  and for some $\mathcal{D} \in \Omega_b^*$ we have $N_d < W_b$ for each $d \notin \mathcal{D}$ **do**
  $P \leftarrow \prod_{d=1}^{b-1} d^{N_d}$
  if $P > b^{\alpha N}$ **then**
    $c \leftarrow c + \mathsf{LargeProduct}(N_0, \ldots, N_{b-1}, N, P)$
    else
    $c \leftarrow c + \mathsf{SmallProduct}(N_0, \ldots, N_{b-1}, N, P)$
  end if
end for
return c

Figure 1: Algorithm to count the number of Zuckerman numbers with N digits.

$\mathsf{LargeProduct}(N_0, \ldots, N_{b-1}, N, P)$

$c \leftarrow 0$
$\ell \leftarrow \lfloor \log P / \log b \rfloor$
for $n_{\mathsf{hi}} \in \mathbb{N}$ s.t. $1 \leq n_{\mathsf{hi}} < b^{N-\ell}$
        and $w_{b,d}(n_{\mathsf{hi}}) \leq N_d$ for $d = 0, \ldots, b-1$ **do**
  if exists $n_{\mathsf{lo}} \in \mathbb{N}$ s.t. $1 \leq n_{\mathsf{lo}} < b^{\ell}$
            and $n_{\mathsf{lo}} \equiv -n_{\mathsf{hi}}\, b^{\ell} \pmod{P}$ **then**
    $n \leftarrow n_{\mathsf{hi}}\, b^{\ell} + n_{\mathsf{lo}}$
    if $w_{b,d}(n) = N_d$ for $d = 0, \ldots, b-1$ **then**
      $c \leftarrow c + 1$
    end if
  end if
end for
return c

Figure 2: Subroutine $\mathsf{LargeProduct}$ of $\mathsf{ZuckermanCount}$.

$\mathsf{SmallProduct}(N_0, \ldots, N_{b-1}, N, P)$

$c \leftarrow 0$
for $N_{\mathsf{hi},0}, \ldots, N_{\mathsf{hi},b-1} \in \mathbb{N}$ s.t. $N_{\mathsf{hi},d} \le N_d$ for $d = 0, \ldots, b-1$
    and $M^{1/2} \le M_{\mathsf{hi}} < NM^{1/2}$, where $M := N!/(N_0! \cdots N_{b-1}!)$,
    $M_{\mathsf{hi}} := N_{\mathsf{hi}}!/(N_{\mathsf{hi},0}! \cdots N_{\mathsf{hi},b-1}!)$, $N_{\mathsf{hi}} := N_{\mathsf{hi},0} + \cdots + N_{\mathsf{hi},b-1}$ **do**
  $T \leftarrow$ Table with keys $0, \ldots, P-1$ and initialized with zeros
  for $n_{\mathsf{hi}} \in \mathbb{N}$ s.t. $1 \le n_{\mathsf{hi}} < b^{N_{\mathsf{hi}}}$, $\pi_b\big(n_{\mathsf{hi}}[:(N_{\mathsf{hi}}-1)]\big) < M^{1/2}$
      and $w_{b,d}(n_{\mathsf{hi}}) = N_{\mathsf{hi},d}$ for $d = 0, \ldots, b-1$ **do**
    $i \leftarrow (-n_{\mathsf{hi}}\, b^{N_{\mathsf{hi}}} \bmod P)$
    $T[i] \leftarrow T[i] + 1$
  end for
  for $n_{\mathsf{lo}} \in \mathbb{N}$ s.t. $1 \le n_{\mathsf{lo}} < b^{N-N_{\mathsf{hi}}}$
      and $w_{b,d}(n_{\mathsf{lo}}) = N_d - N_{\mathsf{hi},d}$ for $d = 0, \ldots, b-1$ **do**
    $j \leftarrow (n_{\mathsf{lo}} \bmod P)$
    $c \leftarrow c + T[j]$
  end for
end for
return c

Figure 3: Subroutine SmallProduct of ZuckermanCount.

$\mathsf{SmallProduct10}(N_0, \ldots, N_9, N, P)$

$c \leftarrow 0$
$v \leftarrow N_2 + 2N_4 + N_6 + 3N_8$
$N_{\mathsf{lo}} \leftarrow \left\lfloor \min\left\{ \dfrac{\log 2}{\log 10} v, \dfrac{\log 10 \log 2}{\log 8 \log 5} \gamma N \right\} \right\rfloor$
for $N^*_{\mathsf{hi},0}, \ldots, N^*_{\mathsf{hi},9} \in \mathbb{N}$ s.t. $N^*_{\mathsf{hi},d} \le N_d$ for $d = 0, \ldots, 9$
    and $M^{1/2} \le M^*_{\mathsf{hi}} < NM^{1/2}$, where $M := N!/(N_0! \cdots N_9!)$,
    $M^*_{\mathsf{hi}} := N^*_{\mathsf{hi}}!/(N^*_{\mathsf{hi},0}! \cdots N^*_{\mathsf{hi},9}!)$, $N^*_{\mathsf{hi}} := N^*_{\mathsf{hi},0} + \cdots + N^*_{\mathsf{hi},9}$ **do**
  $N^*_{\mathsf{mi}} \leftarrow N - N^*_{\mathsf{hi}} - N_{\mathsf{lo}}$
  $N_{\mathsf{mi}} \leftarrow \left\lfloor \max\left\{ \dfrac{\log 10}{\log 8} \gamma N, N^*_{\mathsf{mi}} \right\} \right\rfloor$
  $N_{\mathsf{hi}} = N - N_{\mathsf{mi}} - N_{\mathsf{lo}}$
  for $N_{\mathsf{hi},0}, \ldots, N_{\mathsf{hi},9} \in \mathbb{N}$ s.t. $N_{\mathsf{hi},d} \le N^*_{\mathsf{hi},d}$ for $d = 0, \ldots, 9$
      and $N_{\mathsf{hi},0} + \cdots + N_{\mathsf{hi},9} = N_{\mathsf{hi}}$ **do**
    $T \leftarrow$ Table with keys $0, \ldots, P-1$ and initialized with zeros
    for $n_{\mathsf{hi}} \in \mathbb{N}$ s.t. $1 \le n_{\mathsf{hi}} < 10^{N_{\mathsf{hi}}}$
          and $w_{10,d}(n_{\mathsf{hi}}) = N_{\mathsf{hi},d}$ for $d = 0, \ldots, 9$ **do**
      $i \leftarrow (-n_{\mathsf{hi}} 10^{N_{\mathsf{mi}} + N_{\mathsf{lo}}} \bmod P)$
      $T[i] \leftarrow T[i] + 1$
    end for
    for $n_{\mathsf{mi}} \in \mathbb{N}$ s.t. $1 \le n_{\mathsf{mi}} < 10^{N_{\mathsf{mi}}}$
          and $w_{10,d}(n_{\mathsf{mi}}) \le N_d - N_{\mathsf{hi},d}$ for $d = 0, \ldots, b-1$ **do**
      $u \leftarrow \lceil N_{\mathsf{lo}} \log 10 / \log 2 \rceil$
      $n_{\mathsf{lo}} \leftarrow$ unique x s.t. $x \equiv -10^{N_{\mathsf{lo}}} n_{\mathsf{mi}} \pmod{2^u}$ and $0 \le x < 2^u$
      if $w_{10,d}(n_{\mathsf{mi}}) + w_{10,d}(n_{\mathsf{lo}}) = N_d - N_{\mathsf{hi},d}$ for $d = 0, \ldots, 9$ **then**
        $j \leftarrow (10^{N_{\mathsf{lo}}} n_{\mathsf{mi}} + n_{\mathsf{lo}} \bmod P)$
        $c \leftarrow c + T[j]$
      end if
    end for
  end for
end for
return c

Figure 4: Subroutine SmallProduct10 of ZuckermanCount10.

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