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Advances in Mathematics 461 (2025) 110079

Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim

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ARTICLE INFO

Article history:
Received 2 March 2024
Received in revised form 18 November 2024
Accepted 3 December 2024
Available online xxxx
Communicated by Karen Smith

To our dear friend and colleague Gianfranco Casnati

MSC:
05E14
05E40

Keywords:
Permanents
Dimension
Torus actions

ABSTRACT

In this article, we study *permanental varieties*, i.e., varieties defined by the vanishing of permanents of fixed size of a generic matrix. Permanents and their varieties play an important, and sometimes poorly understood, role in combinatorics. However, there are essentially no geometric results about them in the literature, in very sharp contrast to the well-behaved and ubiquitous case of determinants and minors. Motivated by the study of the singular locus of the permanental hypersurface, we focus on the codimension of these varieties. We introduce a $\mathbb{C}^*$-action on matrices and prove a number of results. In particular, we improve a lower bound on the codimension of the aforementioned singular locus established by von zur Gathen in 1987.

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https://doi.org/10.1016/j.aim.2024.110079

1. Introduction

Despite their structural similarity, the determinant and the permanent are worlds apart.

Avi Wigderson [26]

The most important polynomial associated to a square matrix is its determinant. In hindsight, its ubiquitous presence in mathematics might be related to its very large isotropic group, whose description dates back to Frobenius. Arguably, the second most important polynomial is the *permanent*. Let $M = (x_{i,j})$ be a $k \times k$ matrix with entries in a field F. Let $\mathfrak{S}_k$ be the symmetric group of permutations of the set $\{1, \ldots, k\}$. The *permanent* of M is the polynomial

$$\operatorname{perm}(M) = \sum_{\sigma \in \mathfrak{S}_k} x_{1,\sigma(1)} \cdots x_{k,\sigma(k)}.$$

The permanent has a much smaller isotropic group than the determinant, namely the product of the normalizers of two algebraic tori. Permanents and determinants are famously related by a generating function, the content of the *MacMahon's master theorem* [18, vol. I, §3, Chapter II, 63-66]. The striking tension between these two polynomials is at the heart of *geometric complexity theory*. Indeed, while the determinant may be computed in polynomial time using Gaussian elimination, the permanent is *not known* to be exactly computable in polynomial time. The mere existence of such an algorithm would imply **P**=**NP**. The grand idea of geometric complexity theory is to approach fundamental problems in complexity theory using tools and techniques from algebraic geometry and representation theory. For instance, the **VP** versus **VNP** problem, that may be considered the polynomial cousin of the well-known **P** versus **NP** problem, concerns finding a sequence $(p_k)_{k\in\mathbb{N}}$ of polynomials whose algebraic circuit size grows faster than any polynomial in k; see [15, §1.2] and [26, §12.4] for details. Valiant conjectured that the $k \times k$ permanents perm_k form such a sequence [24]. A possibly weaker but more concrete version of Valiant's conjecture deals with the complexity measure of the permanent, as opposed to the determinant. In detail, the *determinantal complexity* of a polynomial p is the smallest number $\operatorname{dc}(p)$ such that p is an affine linear projection of a determinant of that size. Valiant conjectured that $\operatorname{dc}(\operatorname{perm}_k)$ grows faster than any polynomial in k [24]. The best result known so far is due to Mignon and Ressayre [20]: $\operatorname{dc}(\operatorname{perm}_k) \geq O(k^2)$; this first super-linear lower bound is still far from the full conjecture. On the algebraic geometry side, a study to compare the structure of Fano schemes of determinantal and permanental hypersurfaces was conducted by Chan and Ilten [6].

It is worth noticing that permanents naturally arise in combinatorics and especially in graph theory [21]. Given a bipartite graph G, one naturally associates to G its adjacency square 0/1-matrix M_G; then the permanent of M_G is the number of perfect matchings of G. One difficult problem about 0/1-matrices was posed by Minc in 1967, who asked for an upper bound on the value of the permanent. This was solved in 1973 by Brégman [4] and later by Radhakrishnan using *entropy* from quantum information theory [22]. Permanents were also the subject of the Van der Waerden's conjecture for doubly stochastic matrices, that asked for a lower bound on the value of the permanent for such matrices. This was solved by Egorychev and Falikman in 1981 [8,9], and later also by Gurvits in 2007 [12], with a shorter argument involving stability of real polynomials. All these results have brilliant proofs and are nicely featured in the beautiful book by Aigner and Ziegler [1]. Other interesting appearances of permanents in the sciences include applications to order statistics (the Bapat-Beg Theorem) [2,13] and quantum mechanics, see [5] and references therein.

Determinant and permanent share a common historically important generalization in representation theory. Given a $k \times k$ matrix M and a partition $\lambda = (\lambda_1, \dots, \lambda_s)$ of k, the *immanant* of M is

$$\mathrm{Imm}_\lambda(M) = \sum_{\sigma \in \mathfrak{S}_k} \chi_\lambda(\sigma) x_{1,\sigma(1)} \cdots x_{k,\sigma(k)},$$

where χ_λ is the character of the irreducible representation of $\mathfrak{S}_k$ corresponding to λ. The determinant and permanent are special immanants corresponding respectively to the alternating ($\lambda = (1, \dots, 1)$) and trivial ($\lambda = (k)$) representations. Immanants were introduced by Littlewood and Richardson [17]. The problems we will tackle for permanents are interesting for every immanant, as not much is known about their geometry. It would be interesting to study their structural properties as λ varies.

Going back to permanents, it is apparent from the above discussion that they have a tendency to be extremely difficult objects to study. The perspective from which we look at them once again confirms this. Given a generic $k \times k$ matrix M over a field F, with $k \geq 3$, the *permanental hypersurface* is $P = \{\mathrm{perm}(M) = 0\} \subset F^{k \times k}$. A folklore question asks for a description of the singular locus of this hypersurface [15,25]. Similarly, one may ask this for any *immanantal hypersurface* and its singular locus. However this problem seems very elusive and much more involved than the corresponding one for the determinant already for the permanent. We expect that many of the techniques introduced in this paper for permanents carry over to study immanants, but the conclusions will depend on λ.

Definition 1.1 *(Permanental rank [27]).* Let M be a matrix. Its *permanental rank* is the largest integer k such that there is a $k \times k$ submatrix of M whose permanent is nonzero. The permanental rank of M is denoted $\mathrm{prk}(M)$.

Terminology. In this paper, by a *variety* over a field F, we mean a separated scheme of finite type that is reduced but not necessarily irreducible over F. Our varieties are affine cones over projective varieties that are typically reducible; we neglect the scheme structure of their components, as we shall be concerned with their codimension.

Given the affine cone $P_{k,n} = \{\mathrm{prk}(M) \leq k-1\} \subset F^{k\times n}$, we sometimes look at $\mathbb{P}(P_{k,n}) \subset \mathbb{P}^{kn-1}$, its corresponding projective variety. Since we are interested in the codimension of these varieties, we shall jump back and forth between affine and projective spaces, according to the convenience of the approach at hand.

Main results.
We first study the codimension of the variety of maximal permanents of a generic matrix, in some ranges.

Theorem *(Theorem 3.18). Let F be a field of characteristic zero, and M a generic $k \times n$ matrix of linear forms, with $n \geq k+1$. Then, for $2 \leq k \leq 5$, the codimension of the variety $P_{k,n} = \{\mathrm{prk}(M) \leq k-1\} \subset F^{k\times n}$ is n. In particular, when $2 \leq k \leq 4$, $P_{k,k+1}$ is a complete intersection.*

We speculate (Conjecture 3.4) that the previous result holds in much more generality. In fact, the core of the proof of Theorem 3.18 for those special values of k is based on the following.

Theorem *(Theorem 3.19). Let $k \geq 1$. If $P_{h,h+1} \subset F^{h\times(h+1)}$ has codimension $h+1$ for any $h \leq k$, then $P_{k,n}$ has codimension n, for any $n \geq k+1$. In particular, the validity of Conjecture 3.4 for every $k \in \mathbb{N}$ implies that $P_{k,n}$ has codimension n, for every $k \in \mathbb{N}$ and $n \geq k+1$.*

Note that the sequence of ideals $I(P_{k,n})$ for $n \in \mathbb{N}_{\geq k+1}$ is an example of *symmetric wide-matrix variety* of Draisma-Eggermont-Farooq [7]. They show that the number of components up to the action of the symmetric group $\mathfrak{S}_n$ is a quasi-polynomial in n [7, Theorem 1.1.1].

We introduce a $T = \mathbb{C}^*$-action on matrices which unravels a subtle geometric structure of $P_{k,k+1}$. We establish a *correspondence* between certain vector bundles coming from the tangent bundle and irreducible components of $P_{k,k+1}$; see §4.1 and §4.2.

Theorem *(Theorem 4.5). Let X be any irreducible component of $Y = P_{k,k+1}$. Then X coincides with $T^1_{X^T,Y}$, the closure of the total space of the weight one subbundle (under the torus action) of the tangent bundle over some open set in X^T.*

In a second part of the paper we improve a lower bound due to von zur Gathen, established in 1987, again with the help of a $\mathbb{C}^*$-action. This allows us to have a description for the irreducible components of the singular locus $\mathrm{Sing}(P)$ of the permanental hypersurface as subvarieties of total spaces of the aforementioned vector bundles; see §4.5.

Our result in this direction reads as follows.

Theorem *(Theorem 4.23). Let $k \geq 6$, and let $P = \{\mathrm{perm}(M) = 0\} \subset \mathbb{C}^{k\times k}$ be the permanental hypersurface. The codimension of the singular locus* $\mathrm{Sing}(P) = \{\mathrm{prk}(M) \leq k-2\}$ *satisfies the inequality* $6 \leq \mathrm{codim}\,\mathrm{Sing}(P) \leq 2k$.

Organization of the paper.
In §2, we discuss the case of permanents of $2 \times n$ matrices, where we also underline the differences between our case and the case of minors. In §3, we initiate the study of the variety of maximal permanents of $k \times (k+1)$ matrices. Its analysis will be developed further in §4, where we employ $\mathbb{C}^*$-actions that are useful to organize irreducible components of the aforementioned variety. In these two sections we prove the main results showcased in this introduction. Finally, in §5, we include the scripts used to deal with the description of irreducible components in §4.3 and §4.4.

Acknowledgments.
A.B., E.C. and E.V. are members of GNSAGA group of INdAM (Italy). M.M. was supported by the DFG grant 467575307. E.V. was partially supported by the INdAM-GNSAGA project "Classification Problems in Algebraic Geometry: Lefschetz Properties and Moduli Spaces" (CUP_E55F22000270001). E.V. also thanks The Department of Mathematics of Universität Konstanz, for warm hospitality and financial support, where part of this project was conducted. This study was carried out within the '0-Dimensional Schemes, Tensor Theory, and Applications' project 2022E2Z4AK – funded by European Union – Next Generation EU within the PRIN 2022 program (D.D. 104 - 02/02/2022 Ministero dell'Università e della Ricerca) CUP E53D23005680006. M.M. thanks the Institute for Advanced Study for a great working environment and support through the Charles Simonyi Endowment. We thank an anonymous reviewer for useful suggestions improving the readability, and for providing the majority of the newer code in §5, which simplifies that in an earlier version of this paper. We acknowledge the invaluable help of the algebra software `Macaulay2` [11].

2. Permanents of $2 \times n$ matrices of linear forms

In this section F will be a field of characteristic zero, unless explicitly stated otherwise. The ideal generated by 2×2 permanents of a generic matrix is by now very much understood. A Gröbner basis and a complete description of its minimal primes were obtained in [16]. More recently, in [10], the authors determined the minimal free resolution of the 2×2 permanents of a $2 \times n$ matrix. The results and observations in this section are most naturally stated in projective space.

Theorem 2.1. *Let M be a generic $2 \times n$ matrix of linear forms with $n \geq 3$. The variety $\mathbb{P}(P_{2,n}) = \mathbb{P}(\{\mathrm{prk}(M) \leq 1\}) \subset \mathbb{P}^{2n-1}$ has codimension n. Its singular locus has dimension 1 and consists of n^2 lines.*

Proof. We first assume that the entries of M are the coordinates of $\mathbb{P}^{2n-1}$ denoted by x_{ij}, where $1 \leq i \leq 2$ and $1 \leq j \leq n$. By [16, Theorem 4.1], the matrices in $\mathbb{P}(P_{2,n})$ are such that either $(n-2)$ of their columns vanish and the smooth quadric (on the left two columns) vanishes, or one of their rows vanishes. Thus, the variety has codimension n. The irreducible components of $\mathbb{P}(P_{2,n})$ are two $\mathbb{P}^{n-1}$'s and $\binom{n}{2}$ quadrics in $\mathbb{P}^3$. They are all smooth.

We introduce n^2 lines as follows: for each x_{ij}, consider the n lines whose local coordinates are x_{ij} and $x_{\ell k}$ where either $\ell = i$ and $k \in [n] \setminus \{j\}$ or $\ell \neq i$ and $k = j$.

We show that these n^2 lines are in the singular locus $S = \mathrm{Sing}(\mathbb{P}(P_{2,n}))$. To see this, up to permuting rows or columns, we have two types of lines: r_1 with local coordinates x_{11}, x_{12}, and r_2 with local coordinates x_{11}, x_{21}.

Let $J = [n] \setminus \{1, 2\}$. The line r_1 is contained in one of the two $\mathbb{P}^{n-1}$'s and the smooth quadric in a $\mathbb{P}^3$ defined by the ideal $(x_{1,j\in J}, x_{2,j\in J}, x_{11}x_{22} + x_{12}x_{21})$. Therefore r_1 is in S. The line r_2 is contained in $(n-1)$ of the smooth quadrics above whose equation is $x_{11}x_{2j} + x_{21}x_{1j} = 0$ for $j = 2, \ldots, n$. Hence r_2 is in S. There are $n^2 - n$ lines of the same type as r_1, and n of the same type as r_2. In conclusion, S contains the n^2 lines just described.

Now we look at all the set-theoretic intersections of the irreducible components. The two copies of $\mathbb{P}^{n-1}$ are disjoint. Each $\mathbb{P}^{n-1}$ intersects all the $\binom{n}{2}$ smooth quadrics in $(n^2 - n)/2$ lines (all these lines are of the same type as r_1). Two quadrics intersect at most along one of the lines of the same type as r_2. Thus S is contained in the n^2 lines described above.

Any matrix M' (regarded as a vector) in the orbit under the linear action of $\mathrm{GL}(2n, \mathbb{C})$ of the vector $M \in \mathbb{C}^{2n}$ is a matrix with $2n$ linearly independent linear forms. The induced action on projective space preserves the invariants of the irreducible components of $\mathbb{P}(P_{2,n})$ and S. $\square$

Remark 2.2. Let M be a $2 \times n$ Hankel matrix, i.e. a matrix of the form

$$M = \begin{pmatrix} x_0 & x_1 & \cdots & x_{n-1} \\ x_1 & \cdots & x_{n-1} & x_n \end{pmatrix}. \tag{1}$$

While $\mathbb{P}(\{\mathrm{rk}(M) \leq 1\}) \subset \mathbb{P}^n$ is a rational normal curve of degree n, the permanental version (in characteristic different than 2) is different and surprisingly small, as shown in the next result.

In the next proposition, we are interested in the scheme structure.

Proposition 2.3. *Let F be a field,* $\mathrm{char}(F) \neq 2$*, and let M be a $2 \times n$ Hankel matrix of the form* (1)*. The scheme $\mathbb{P}(P_{2,n}) = \mathbb{P}(\{\mathrm{prk}(M) \leq 1\}) \subset \mathbb{P}^n$ is zero-dimensional and of degree 8, supported at two points. In particular, the degree of this zero-dimensional scheme does not depend on n. If* $\mathrm{char}(F) = 2$*, then $\mathbb{P}(P_{2,n}) = \mathbb{P}(\{\mathrm{rk}(M) \leq 1\})$ is a rational normal curve of degree n.*

Proof. From the description given in [16], it is easy to see that the scheme is supported on the two points $p_0 = [1 : 0 : \cdots : 0]$ and $p_n = [0 : \cdots : 0 : 1]$. We work on the affine chart $\{x_n \neq 0\}$. In this chart, call the resulting ideal J_n. Then, for $2 \leq i \leq n$, we have $x_{n-i} + x_{n-i+1}x_{n-1} \in J_n$.

Let

$$g_1 = -x_{n-2}^2 - x_{n-3}x_{n-1} - x_{n-2}x_{n-1} + x_{n-1}^2 - x_{n-3} - \frac{1}{2}x_{n-2},$$
$$g_2 = -x_{n-2}^2 - x_{n-3}x_{n-1} + x_{n-1}^2 + x_{n-2} - \frac{1}{2}x_{n-1},$$
$$g_3 = x_{n-2}x_{n-1} + x_{n-1}^2 + x_{n-3} + x_{n-2} + \frac{1}{2}.$$

Then $x_{n-1}^4 = g_1(x_{n-1}^2 + x_{n-2}) + g_2(x_{n-1}x_{n-2} + x_{n-3}) + g_3(x_{n-2}^2 + x_{n-1}x_{n-3}) \in J_n$. Hence, for $4 \leq i \leq n$, the variable x_{n-i} is zero in the quotient $K[x_1, \ldots, x_n]/J_n$. Thus $1, x_{n-3}, x_{n-2}, x_{n-1}$ generate the quotient and form a basis. It follows that the scheme has degree 4 at p_n. By symmetry, it has degree 4 at p_0, as well. □

3. Permanents of $k \times (k+1)$ matrices of linear forms

Proposition 3.1. *Let F be an arbitrary field, $k \geq 2$, and M a generic $k \times n$ matrix of linear forms, with $n \geq k$. Then the codimension of $P_{k,n} := \{\mathrm{prk}(M) \leq k-1\} \subset F^{k\times n}$ satisfies the inequality $n - k + 1 \leq \mathrm{codim}(P_{k,n}) \leq n$.*

Proof. The upper bound $\mathrm{codim}(P_{k,n}) \leq n$ holds for any $n \geq k \geq 2$, because we have linear spaces of codimension n inside $P_{k,n}$.

For the lower bound, we proceed by induction on $k \geq 2$. The case $k = 2$ is settled in Theorem 2.1. Let $k \geq 3$, and assume that the statement is true for $k-1$. Let C be an irreducible component of $P_{k,n}$. We have two cases: either C is a cone over an irreducible component of $P_{k-1,n}(M')$ where M' (up to permuting rows and columns) is a $(k-1) \times n$ matrix of linear forms, or it is not. In the first case, by induction C has codimension at least $n - (k-1) + 1 = n - k + 2 > n - k + 1$ in $F^{k\times n}$. In the second case, let A be the generic point of C and set $J = \{1, \ldots, k-1\}$. We may assume $\mathrm{perm}(M_{J,J})(A) \neq 0$, i.e. the $(k-1) \times (k-1)$ permanent of the upper-left corner of M is nonzero at A. Thus, for all $k \leq j \leq n$, $x_{k,j_{|C}}$ is a rational function on C in the rational functions $x_{i,\ell}$ where $1 \leq i \leq k-1$ and $\ell \in [n]$ or $i = k$ and $1 \leq \ell \leq k-1$. So we have an inclusion of function fields $F(C) \subset F(x_{i,\ell})$, where $x_{i,\ell}$ are the $(k-1)(n+1)$ coordinates above. Hence $\dim_{F^{k\times n}} C = \mathrm{transdeg}_F(F(C)) - 1 \leq (k-1)(n+1) - 1$ and then $\mathrm{codim}(C) \geq (kn-1) - (k-1)(n+1) - 1 = n - k + 1$. □

Corollary 3.2. *Let F be an arbitrary field, $k \geq 2$, and M a generic $k\times k$ square matrix over F. Let $P = \{\mathrm{perm}(M) = 0\}$ be the permanental hypersurface, and denote by $\mathrm{Sing}(P)$ its singular locus. Then $\mathrm{codim}\ \mathrm{Sing}(P) \geq 4$. In particular, $\mathrm{perm}(M)$ is an irreducible polynomial over F.*

Proof. The singular locus of the permanental hypersurface is

$$\mathrm{Sing}(P) = P_{k-1,k} = \{\mathrm{prk}(M) \leq k-2\}.$$

A similar strategy as in the proof of Proposition 3.1 shows that $\mathrm{codim}(P_{k-1,k}) \geq 4$. If $\mathrm{perm}(M)$ were reducible over F, then the codimension of $\mathrm{Sing}(P)$ would be at most 2. Thus $\mathrm{perm}(M)$ is irreducible over F. □

Lemma 3.3. *Let M be a generic $k \times n$ matrix and let $h \leq \min\{k, n\}$. Then the $h \times h$ permanents of M are linearly independent.*

Proof. Every such permanent is of the form $\mathrm{perm}(M')$ for some $h \times h$ submatrix M' of M. Hence it is uniquely determined by the monomial given by the product of the elements in the main diagonal of M'. This monomial does not appear in any other $\mathrm{perm}(M'')$ for $M'' \neq M'$. □

Proposition 3.1 applies to ideals of maximal minors as well. In fact, it is very weak when $n = k + 1$. In contrast, we propose the following

Conjecture 3.4. *Let M be a generic $k \times (k+1)$ matrix with $k \geq 2$. Then $P_{k,k+1}$ is a complete intersection. In particular,* $\mathrm{codim}(P_{k,k+1}) = k + 1$.

This conjecture holds true for $k = 2$. One can show the following result.

Proposition 3.5. *Let M be a generic $k \times (k+1)$ matrix with $k \geq 1$. Then the $k \times k$ permanents of M are algebraically independent.*

Proof. The statement for $k = 1$ is obvious. Fix $k \geq 2$, let $M = (x_{ij})$ and N be the $(k+1) \times (k+1)$ matrix obtained from M by adding a row of $(k+1)$ extra variables $y_1, \ldots, y_{k+1}$. Let $P_N = \mathrm{perm}(N)$ be the permanent polynomial of N in the $(k+1)^2$ variables x_{ij}, y_ℓ. Then the $k \times k$ permanents of M are the $(k+1)$ partial derivatives $\partial P_N / \partial_{y_\ell}$ of P_N. The main result in [20] is proven showing that the Hessian matrix of P_N has nonzero determinant. This is equivalent (see [23, §7]) to saying that the first partial derivatives of P_N are algebraically independent. Hence any subset of first partial derivatives of P_N consists of algebraically independent elements. □

Remark 3.6. By Proposition 3.5, there is no analogue in the permanental case of Plücker coordinates of minors of a $k \times (k+1)$ matrix. However, permanents are generally not algebraically independent. For instance, one can check that the 10 permanents of a 2×5 generic matrix are algebraically dependent.

Next, we prove that the linear spaces given by the vanishing of a row are indeed irreducible components of $P_{k,k+1}$. Due to unmixedness, to prove Conjecture 3.4 it would be enough to show that $P_{k,k+1}$ is arithmetically Cohen-Macaulay.

Remark 3.7. In the case of ideals of minors of fixed size k of any generic matrix, one way to show that they are Cohen-Macaulay ideals is to show that the corresponding quotient ring is of the form $S^G \subset S$, where S is a polynomial ring and S^G is the subring of invariants under the action of the group $G = \mathrm{GL}(k, F)$. This approach cannot work in the case of permanents, because we know that $P_{k,k+1}$ must be reducible by the next proposition.

Proposition 3.8. *The variety $P_{k,k+1} \subset F^{k\times(k+1)}$ has at least k linear spaces among its irreducible components in codimension $k+1$.*

Proof. Let M be a $k \times (k+1)$ of the form

$$M = \begin{pmatrix} x_{1,1} & x_{1,2} & \dots & x_{1,k+1} \\ x_{2,1} & x_{2,2} & \dots & x_{2,k+1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{k,1} & x_{k,2} & \dots & x_{k,k+1} \end{pmatrix}.$$

Consider the projective variety $\mathbb{P}(P_{k,k+1}) \subset \mathbb{P}^{k(k+1)-1}$. We show that the k linear spaces in $\mathbb{P}(P_{k,k+1})$ defined by the vanishing of one row of M are irreducible components of $\mathbb{P}(P_{k,k+1})$.

Up to the action of the symmetric group permuting rows, it is enough to show that the linear space L_k whose defining ideal is $(x_{k,1}, \dots, x_{k,k+1})$ is an irreducible component of $P_{k,k+1}$.

We fix a point $p \in \mathbb{P}(P_{k,k+1})$ whose coordinates are $x_{i,j}(p) = 1$ for all $1 \le i \le k-1$ and $1 \le j \le k+1$, and zero otherwise. We work inside the affine chart $U = \{x_{1,1} \neq 0\} \cong F^{k(k+1)-1}$ of $\mathbb{P}^{k(k+1)-1}$. Inside U, we change coordinates so that p is the origin of U. In this coordinates $\widetilde{x}_{i,j}$, the variety $P_{k,k+1} \cap U$ is defined by ideal $\widetilde{I}$ generated the $k \times k$ permanents of the following $k \times (k+1)$ matrix

$$\widetilde{M} = \begin{pmatrix} 1 & \widetilde{x}_{1,2}+1 & \dots & \widetilde{x}_{1,k+1}+1 \\ \widetilde{x}_{2,1}+1 & \widetilde{x}_{2,2}+1 & \dots & \widetilde{x}_{2,k+1}+1 \\ \vdots & \vdots & \vdots & \vdots \\ \widetilde{x}_{k,1} & \widetilde{x}_{k,2} & \dots & \widetilde{x}_{k,k+1} \end{pmatrix}.$$

Let $\widetilde{I}^{\mathrm{lin}}$ be the ideal generated by the linear part of all $g \in \widetilde{I}$. The affine tangent space $T_p(\mathbb{P}(P_{k,k+1}))$ to p at $\mathbb{P}(P_{k,k+1})$ is given by $\mathrm{Spec}(F[\widetilde{x}_{i,j}, (i,j) \neq (1,1)]/\widetilde{I}^{\mathrm{lin}})$. Let g_j for $j = 1, \dots, k+1$ be the permanents of $\widetilde{M}$. The linear part of g_j is

$$\widetilde{x}_{k,1} + \widetilde{x}_{k,2} + \dots + \widehat{\widetilde{x}_{k,i}} + \dots + \widetilde{x}_{k,k+1},$$

where $\widehat{\widetilde{x}_{k,i}}$ means that we are omitting that summand. Since we are in characteristic zero, these linear forms are linearly independent.

Hence, the tangent space $T_p(\mathbb{P}(P_{k,k+1}))$ is of dimension at most the dimension of L_k. As $L_k \subset \mathbb{P}(P_{k,k+1})$ the tangent space has to be of the same dimension as L_k. Thus, p is a smooth point of $\mathbb{P}(P_{k,k+1})$ belonging to a unique component of dimension equal to the dimension of L_k. It follows that L_k is an irreducible component of $\mathbb{P}(P_{k,k+1})$. $\square$

Remark 3.9. Other types of irreducible components arise from the vanishing of the generic linear forms in a column. One can check that they all have codimension $k+1$ as well.

Proposition 3.10. *For $k = 3$ or 4, let M be a generic $k \times (k+1)$ matrix of linear forms. Then the codimension of $P_{k,k+1} = \{\mathrm{prk}(M) \leq k-1\} \subset F^{k\times(k+1)}$ is $k+1$. Equivalently, Conjecture 3.4 holds for $2 \leq k \leq 4$ and $P_{k,k+1}$ is a complete intersection.*

Proof. Let $k = 3$. Define L to be the linear space transforming the matrix $M = (x_{ij})$ into the following *circulant Hankel matrix*

$$H_3 = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} \\ x_{12} & x_{13} & x_{14} & x_{15} \\ x_{13} & x_{14} & x_{15} & x_{11} \end{pmatrix}.$$

Let $V_3 = P_{3,4} \cap L$. The ideal I_{V_3} of V_3 is the ideal of 3×3 permanents of H_3. Using `Macaulay2`, we check that $\mathrm{ht}(I_{V_3}) = 4$. Thus $\dim(Z) = 1$ for all irreducible components $Z \subset V_3$. Therefore, for any irreducible component $X \subset P_{3,4}$, we have

$$1 = \dim(Z) \geq \dim(X) + \dim(L) - 12 = \dim(X) - 7.$$

Thus $\dim(X) \leq 8$ and hence $\mathrm{codim}(P_{3,4}) \geq 4$.

Let $k = 4$. Define L to be the linear space transforming the matrix $M = (x_{ij})$ into the following *circulant Hankel matrix*

$$H_4 = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} \\ x_{12} & x_{13} & x_{14} & x_{15} & x_{11} \\ x_{13} & x_{14} & x_{15} & x_{11} & x_{12} \\ x_{14} & x_{15} & x_{11} & x_{12} & x_{13} \end{pmatrix}.$$

Let $V_4 = P_{4,5} \cap L$. The ideal I_{V_4} of V_4 is the ideal of 4×4 permanents of H_4. Using `Macaulay2`, we check that $\mathrm{ht}(I_{V_4}) = 5$. Thus $\dim(V_4) = 0$. On the other hand, for any irreducible component $X \subset P_{4,5}$, we have

$$0 = \dim(V_4) \geq \dim(X) + \dim(L) - 20 = \dim(X) - 15.$$

Thus $\dim(X) \leq 15$ and hence $\mathrm{codim}(P_{4,5}) \geq 5$. $\square$

Although we could have employed `Macaulay2` to compute directly the codimension of $P_{3,4}$ and $P_{4,5}$, we believe that restricting to a suitable linear space (or, more generally, to a variety) might be a strategy to give the desired lower bound on the codimension. In fact, the approach pursued in Proposition 3.10 comes from the observation that the dimension of the ideal of permanents of a circulant Hankel matrix tends to be small. This could certify the codimension of the original permanental ideal. The behavior is clear for 2×2 permanents, as shown in the following lemma.

Lemma 3.11. *Let $k \geq 2$ and let $S = F[x_j]$ be the polynomial ring in the $k+1$ variables x_j with $1 \leq j \leq k+1$. Let $M = (x_j)$ be a $k \times (k+1)$ circulant Hankel matrix. Let $Q_1 = \{\mathrm{prk}(M) \leq 1\}$ be the variety whose ideal is generated by the 2×2 permanents of M. Then $\mathrm{codim}(Q_1) = k+1$.*

Proof. For $k = 2$, the statement can be checked similarly as in the proof of Proposition 2.3. Let $k \geq 3$. It is enough to check that some power of each x_j is in the ideal $I(Q_1) \subset S$. First note that, by definition, all the generators of $I(Q_1)$ are of the form

$$x_{\ell'}x_{m'} + x_\ell x_m, \text{ where } \ell' + m' \equiv \ell + m \mod k+1.$$

Vice versa, any equality of the form $\ell' + m' \equiv \ell + m \mod k+1$ gives rise to a generator of $I(Q_1)$. To see these statements, note that giving a generator of $I(Q_1)$ is equivalent to giving an arbitrary choice of two rows and two columns. Fix a row r, pick two elements on r, say x_ℓ and $x_{\ell'}$ where $\ell' \equiv \ell + h \mod k+1$; we have just selected two columns. Now, choose a second row r', pick the elements $x_{m'}$, where $m' \equiv \ell + s \mod k+1$, and x_m where $m \equiv \ell + s + h \mod k+1$; these last two choices are forced as we have already selected the two columns. Notice that the indices of the variables satisfy the desired equation in modular arithmetic.

For each $1 \leq j \leq k+1$, we have distinct generators of the form

$$x_j^2 + x_\ell x_m, \tag{2}$$

$$x_j^2 + x_{\ell'}x_{m'}, \text{ and} \tag{3}$$

$$x_\ell x_m + x_{\ell'}x_{m'}, \tag{4}$$

where $\ell, m \neq j$ are possibly the same index; similarly $\ell', m' \neq j$ are possibly the same index. Thus $1/2 \cdot (2) + 1/2 \cdot (3) - 1/2 \cdot (4) = x_j^2 \in I(Q_1)$ for every $1 \leq j \leq k+1$. $\square$

Remark 3.12. In comparison, the codimension of the ideal of 2×2 permanents of a generic $k \times (k+1)$ matrix is $k^2 - 1$ [16, Theorem 4.1].

3.1. The nondegenerate permanental ideal

Kirkup shows that $\mathrm{codim}(P_{3,4}) = 4$ [14, §7]. For each k, he looks at the colon ideal $J_k = I(P_{k,k+1}) : (\prod_{i,j} x_{i,j})^\infty \supset I(P_{k,k+1})$ [14, Corollary 10]. This ideal unravels a lot of the structure of $I(P_{k,k+1})$, therefore it deserves a definition on its own.

Definition 3.13. The ideal $J_k = I(P_{k,k+1}) : (\prod_{i,j} x_{i,j})^\infty \supset I(P_{k,k+1})$ is called the *nondegenerate permanental ideal*. The corresponding variety $\mathcal{V}_k = V(J_k)$ is the *nondegenerate permanental variety*.

Proposition 3.14 *(Kirkup). The nondegenerate permanental variety $\mathcal{V}_3$ is irreducible of codimension* 4 *and degree* 66*.*

The next lemma, which is implicit in [14, §6], shows that the nondegenerate permanental variety is non-empty and distinct from the linear spaces of Proposition 3.8 or from the irreducible components arising from the vanishing of a single column (Remark 3.9).

Lemma 3.15 *(Kirkup). Let $k \geq 3$. The Kirkup matrix*

$$\mathcal{K}_{k,k+1} := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 2-3k \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & 1 & 1 & 2-3k \\ 1 & 1 & \dots & 1 & (2-2k) & (k-2)(k-1) \\ 1 & 1 & \dots & 1 & k & (2k-1)(k-2) \end{pmatrix} \in \mathbb{Z}^{k\times(k+1)}$$

is an element of the nondegenerate permanental variety: $\mathcal{K}_{k,k+1} \in \mathcal{V}_k$.

It is indeed immediate to show that the Kirkup matrix has all $k \times k$ permanents vanishing.

Definition 3.16 *(Kirkup components).* A *Kirkup component* is an irreducible component of $\mathcal{V}_k$ containing the Kirkup matrix $\mathcal{K}_{k,k+1}$.

When $k = 3$, $\mathcal{V}_3$ is also the unique Kirkup component. Kirkup found equations for $\mathcal{V}_k$ which we recall here. Let $M = (x_{i,j})$ be a $k \times (k+1)$ generic matrix and let $\mathrm{perm}_j(M)$ be the permanent of the matrix obtained from M by removing the jth column. Define

$$M_{\ell,i,j} = \frac{\partial}{\partial x_{\ell,i}} \mathrm{perm}_j(M).$$

Then $M_{\ell,i,j}$ is the $(k-1)\times(k-1)$ permanent of the matrix obtained from M by omitting row ℓ and columns i and j (when $i = j$ the value is zero). We define two types of matrices:

$$A_j = \begin{pmatrix} M_{1,1,j} & \dots & M_{1,k+1,j} \\ \vdots & \vdots & \vdots \\ M_{k,1,j} & \dots & M_{k,k+1,j} \end{pmatrix}$$

and

$$B_\ell = \begin{pmatrix} M_{\ell,1,1} & \dots & M_{\ell,1,k+1} \\ \vdots & \vdots & \vdots \\ M_{\ell,k+1,1} & \dots & M_{\ell,k+1,k+1} \end{pmatrix}.$$

The matrix A_j is $k \times (k+1)$ and its jth column is zero; call C_j the $k \times k$ submatrix obtained from A_j by omitting its jth column of zeros. The matrix B_ℓ is symmetric of format $(k+1) \times (k+1)$.

The next result is [14, Proposition 9]; we include a proof for completeness.

Proposition 3.17 *(Kirkup). The determinants $f_j = \det(C_j)$ and $g_\ell = \det(B_\ell)$ are in the nondegenerate permanental ideal J_k.*

Proof. Thanks to the action of the symmetric group, it is enough to show the statement for f_1 and for g_1. For $j \neq 1$, by the Laplace expansion we have $\mathrm{perm}_j = \sum_{i=1}^{k} x_{i,1} M_{i,1,j}$. Let e_j be the determinant of the $(k-1) \times (k-1)$ submatrix of C_1 obtained by omitting the first column and the jth row. Then

$$I(P_{k,k+1}) \ni \sum_{j=1}^{k} (-1)^j e_j \cdot \mathrm{perm}_j(M) = \sum_{i=1}^{k} \sum_{j=1}^{k} (-1)^j M_{i,1,j} \cdot e_j.$$

For $i = 1$, the interior sum in the right-most side is $\det(C_1) = f_1$. For $i \neq 1$, the interior sum is the Laplace expansion of the determinant of the matrix obtained from C_1 replacing the first column with its ith column; so this sum is zero. Hence $x_{1,1} \cdot f_1 \in I(P_{k,k+1})$. Thus, by definition, $f_1 \in J_k$. The proof for the g_ℓ's is similar, using the relation $\mathrm{perm}_j(M) = \sum_{i=1}^{k} x_{1,i} M_{1,i,j}$. $\square$

3.2. Permanents of $k \times n$ matrices of linear forms

Theorem 3.18. *Let F be a field of characteristic zero and M a generic $k \times n$ matrix of linear forms, with $n \geq k+1$. The codimension of the variety $P_{k,n} = \{\mathrm{prk}(M) \leq k-1\} \subset F^{k \times n}$ is n for $2 \leq k \leq 5$.*

Proof. We perform induction on the number of columns n and on the number of rows k. The base cases 2×3, 3×4, 4×5, 5×6 are proven in Theorem 2.1, Propositions 3.10 and 4.13.

Let $n \geq k+2$ and suppose that the statement is proven for $n-1$ and any number of rows $\leq k-1$. Let C be an irreducible and reduced component of $P_{k,n}$. We have two cases: either C is a cone over an irreducible component of $P_{k-1,n}(M')$ where M' is the $(k-1) \times n$ submatrix of M consisting of the first $(k-1)$ rows, or it is not. In the first case, C has codimension $\geq n$, because the only irreducible components of $P_{k-1,n}(M')$ have codimension $\geq n$ by inductive hypothesis.

In the second case, up to permuting columns, we may assume that in C the Zariski principal open set $U_C = C \cap \{f \neq 0\}$, where $f = \mathrm{perm}(M_{[1,\dots,k-1],[1,\dots,k-1]})$, is nonempty. Let $B = M_{[1,\dots,k],[1 \leq j \leq n-1]}$ be the $k \times (n-1)$ matrix consisting of the first $n-1$ columns of M. We denote the linear forms in B by x_{ij}. Let z be the linear form in the (k,n)-th entry of M. For $1 \leq i \leq k-1$ and $j = n$, let y_{ij} be the linear form in the (i,j)-th entry of M.

Define $R = F[x_{ij}, y_{ij}, z]$. Let $I \subset R$ be the ideal generated by the $k \times k$ permanents in B, and let $J \subset R$ be the ideal generated by the $k\times k$ permanent $\mathrm{perm}(M_{[1,\dots,k],[1,\dots,k-1,n]})$. The latter may be expressed as

$$\mathrm{perm}(M_{[1,\dots,k],[1,\dots,k-1,n]}) = f \cdot z + g(x_{ij}, y_{ij}),$$

where $g(x_{ij}, y_{ij})$ is some polynomial in the variables x_{ij}, y_{ij} only. Note that $I + J \subset I(P_{k,n}) \subset I(C)$, where the latter is the prime ideal of C. Let Y be the affine variety defined by the ideal $I + J$. Therefore $\emptyset \neq U_C = C \cap \{f \neq 0\} \subset Y \cap \{f \neq 0\} = U_Y$. The inclusion implies $\dim(U_C) \leq \dim(U_Y)$. The coordinate ring of the principal Zariski open set U_Y is the localization of the coordinate ring of Y at the element $f \in F[x_{ij}]$, i.e. $F[U_Y] = R[f^{-1}]/(I' + J')$, where I' and J' are the ideals I and J defined above after localizing at f.

Let $S = F[x_{ij}, y_{ij}]$. Note that the rings $R[f^{-1}]$ and $S[f^{-1}]$ are domains and $J' = (z + h)$, for $h \in S[f^{-1}]$.

We show that $F[U_Y]$ is isomorphic to the ring $S[f^{-1}]/\widetilde{I}$, where $\widetilde{I}$ is the ideal in $S[f^{-1}]$ generated by the $k \times k$ permanents of B. An element of $F[U_Y]$ is an equivalence class $\overline{g} = g+(z+h)g_1+\sum_{i=1}^{\ell} p_i q_i$, where the p_i's are the generators of I and $q_i \in R[f^{-1}]$. Since the latter ring is a domain, we may perform Euclidean division of g and of each q_i by the element $z+h$. So each equivalence class is of the form $\overline{g} = g+(z+h)g_2+\sum_{i=1}^{\ell} p_i r_i$, where $\deg_z(g) = 0$ and $\deg_z(r_i) = 0$ for each i. This condition means that the $g, r_i \in S[f^{-1}]$ for each i. Define the map

$$\varphi : F[U_Y] \longrightarrow S[f^{-1}]/\widetilde{I}$$

where $\varphi(\overline{g}) = g + \sum_{i=1}^{\ell} p_i r_i$. This is a ring isomorphism.

Now, regard the ideal I above as an ideal in S. By induction, $\mathrm{ht}(I) = n-1$. As height can only go up after localization with respect to an element not in the ideal, one has $\mathrm{ht}(\widetilde{I}) \geq n - 1$.

Since $S[f^{-1}]$ is a finitely generated domain, one has the equality

$$\begin{aligned}\dim(F[U_Y]) &= \dim(S[f^{-1}]/\widetilde{I}) = \\ &= \dim(S[f^{-1}]) - \mathrm{ht}(\widetilde{I}) \leq (kn - 1) - (n - 1) = (k - 1)n.\end{aligned}$$

Thus $\dim(C) = \dim(U_C) \leq \dim(U_Y) \leq (k - 1)n$ and hence $\mathrm{codim}(C) \geq n$. □

In this last result we explicitly employed the knowledge of the codimension of $P_{k,k+1} \subset F^{k\times(k+1)}$ for $2 \leq k \leq 5$. Notice that the core of the proof of Theorem 3.18 is the following.

Theorem 3.19. *Let $k \geq 1$. If $P_{h,h+1} \subset F^{h\times(h+1)}$ has codimension $h+1$ for any $h \leq k$, then $P_{k,n}$ has codimension n, for any $n \geq k + 1$. In particular, the validity of Conjecture 3.4 for every $k \in \mathbb{N}$ implies that $P_{k,n}$ has codimension n, for every $k \in \mathbb{N}$ and $n \geq k + 1$.*

4. Torus actions

In this section, we work over the field of complex numbers. Let $T = \mathbb{C}^*$ act on a vector space V with weights 0 and 1. This means that $V = V_0 \oplus V_1$ and for every $v \in V_i$ and $t \in T$, we have $t \cdot v = t^i v$. The action induces naturally an action on $\mathbb{P}(V)$.

Let $Y \subset V$ be a T-invariant variety. Then, since the torus is irreducible, any irreducible component X is invariant under the action of T. Let X^T be the locus of fixed points under the T-action. It is not difficult to check that X^T is smooth if X is so. Then there exists a morphism

$$\varphi_{t\to 0} : X \longrightarrow X^T,$$

defined by $\varphi_{t\to 0}(x) = \lim_{t\to 0} t \cdot x \in X^T$. This is the restriction of the projection $V \to V_0$ with kernel V_1. Suppose X is an affine cone over $X' \subset \mathbb{P}(V)$. Then $\varphi_{t\to 0}$ induces a map

$$\psi_{t\to 0} : U \longrightarrow X'^T,$$

where $U \subset X'$ is the set of all $[x] \in X'$ such that $\lim_{t\to 0} t \cdot [x] \neq 0$.

For any $x \in X^T$, the tangent space $T_{Y,x}$ to $Y \supset X$ splits into two summands of weight 0 and 1, which we call $T^0_{Y,x}$ and $T^1_{Y,x}$, respectively.

Definition 4.1. Let $Y \subset V$ be a variety, where $T = \mathbb{C}^*$ acts on V as above. Let Z be an irreducible subvariety of Y. There exists a nonempty, Zariski open subset $U \subset Z$, such that the restriction of the tangent sheaf of Y to U is a vector bundle $T_{Y|U}$. If T acts on U then we obtain $T_{Y|U} = T^1_{Y|U} \oplus T^0_{Y|U}$. The total spaces of all three bundles map naturally to V and we identify $T^1_{Y|U}$ with its image.

We define $T^1_{Z,Y}$ as the Zariski closure in V of $T^1_{Y|U}$. This irreducible variety does not depend on the choice of U.

Proposition 4.2. *Any irreducible component X of a variety $Y \subset V$ as above is contained in $T^1_{X^T,Y}$.*

Proof. Note that X^T is irreducible, as it is the image $\varphi_{t\to 0}(X)$. If $X = X^T$, then the statement is true as the total space $T^1_{X^T,Y}$ is canonically identified with X^T, as each fiber is zero. If $X^T \neq X$, the general point $x \in X$ will map to a point $\varphi_{t\to 0}(x) \in X^T$ belonging to an open set $U \subset X^T$ over which T_Y is a vector bundle. We have to prove that $x \in T^1_{Y,\varphi_{t\to 0}(x)}$. Indeed, the whole orbit $T \cdot x$ consists of vectors in the fiber $T^1_{Y,\varphi_{t\to 0}(x)}$, as the closure of this orbit is a line contained in Y, passing through $\varphi_{t\to 0}(x)$ and on which T acts with weight one. This proves the desired inclusion. $\square$

Proposition 4.2 is a simpler version of the Białynicki-Birula decomposition theorem [3, Chapter II, Theorem 4.2], but in a possibly singular context.

Definition 4.3 *(Type).* The *type* of X is the rank of the bundle $T^1_{Y|U}$ in Proposition 4.2.

4.1. Irreducible components of $P_{k,k+1}$ and torus actions

Let W be the linear component of $Y = P_{k,k+1}$ given by the vanishing of the first row. If $p \in W$, then the Jacobian $J(Y)_p$ of Y at p is a $(k(k+1)) \times (k+1)$-matrix with two blocks:

$$J(Y)_p = \begin{pmatrix} B_1(A_p) \\ \mathbf{0} \end{pmatrix},$$

where A_p is the $(k-1) \times (k+1)$ nonzero submatrix of p and $B_1(A_p)$ is the matrix introduced in §3.1 evaluated at the entries of A_p. The matrix $B_1(A_p)$ is a symmetric $(k+1) \times (k+1)$-matrix whose main diagonal consists of zeros. The zero-block $\mathbf{0}$ has size $((k-1)(k+1)) \times (k+1)$.

Let $T = \mathbb{C}^*$ act on $V = \mathbb{C}^{k\times(k+1)}$ scaling by t the first row of a matrix in V and preserving the other entries.

Corollary 4.4. *For any irreducible component X of $Y = P_{k,k+1} \subset V$, we have a map*

$$\varphi_{t\to 0} : X \longrightarrow X^T \subset W = V^T.$$

If $X \neq W$, then any point $p \in X^T$ is in the singular locus of Y. Let A_p be the corresponding $(k-1) \times (k+1)$ nonzero submatrix of p. For $p \in \varphi_{t\to 0}(X)$, its tangent space to Y is

$$T_{Y,p} = V^T \oplus \ker(B_1(A_p)),$$

where $T^0_{Y,p} = V^T$ and $T^1_{Y,p} = \ker(B_1(A_p))$. In particular, $\dim_{\mathbb{C}} T^1_{Y,p} = \dim_{\mathbb{C}} \ker(B_1(A_p))$.

Proof. The point p has to be singular as it belongs to two components: X and W. The tangent space $T_{Y,p}$ is the kernel of transpose of $J(Y)_p$. The kernel of the matrix $B_1(A_p)$ sits inside the span of the variables $x_{1,h}$ for $1 \leq h \leq k+1$, corresponding to the first row, which is a complement to the subspace V^T in V. $\square$

Let $\mathcal{X}_i = \{p \in V^T \mid \operatorname{crk}(B_1(A_p)) = i\} \subset V^T$ for $0 \leq i \leq k+1$, where crk denotes the corank of $B_1(A_p)$, regarded as a matrix in $\mathbb{C}^{(k+1)\times(k+1)}$. This is a constructible set.

Theorem 4.5. *Let X be any irreducible component of $Y = P_{k,k+1}$. Then X coincides with $T^1_{X^T,Y}$. Note that there is a unique i such that the generic point of X^T sits inside $\mathcal{X}_i$, i.e. X is of type i.*

Proof. By Proposition 4.2, X is a subvariety of $T^1_{X^T,Y}$. There is a unique such index i, by the irreducibility of X^T and by semi-continuity of matrix rank.

We have to show the opposite inclusion. By the irreducibility of $T^1_{X^T,Y}$ and since X is closed, it is enough to show that $T^1_{Y|X^T\cap\mathcal{X}_i} \subset X$. Let $(q,p) \in T^1_{Y|X^T}$ for $p \in X^T \cap \mathcal{X}_i$. Hence $q \in \ker(B_1(A_p))$. Now, regard $q = (q_1, \ldots, q_{k+1}) \in \mathbb{C}^{k+1}$. By definition, one has

$$B_1(A_p) \cdot \begin{pmatrix} q_1 \\ \vdots \\ q_{k+1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The j-th linear condition $B_1(A_p)_j \cdot q^t = 0$ is equivalent to the vanishing of the $k \times k$ permanent of the submatrix of M obtained by removing the j-th column of M and evaluating at (q,p). Hence $(q,p) \in Y$ and thus $(tq,p) \in Y$ for every $t \in T$. Thus $T^1_{Y|X^T\cap\mathcal{X}_i}$ is irreducible, contained in Y and intersecting X in a Zariski dense set. As X is a component of Y it follows that $X = T^1_{X^T,Y}$. $\square$

Corollary 4.6. *Let X be any irreducible component of Y. Let $p \in X^T$ be general. Then*

$$\dim X = \dim X^T + \dim_{\mathbb{C}} \ker B_1(A_p).$$

Equivalently, one has

$$\text{codim } X = \text{codim}_{V^T} X^T + \text{rk}(B_1(A_p)).$$

4.2. Correspondence between vector bundles and components of $Y = P_{k,k+1}$

Theorem 4.5 shows a geometric feature lurking behind the variety Y, that is a hierarchy of irreducible components: each irreducible component corresponds to a vector bundle of rank $\text{crk}(B_1(A_p))$. The irreducible component $W = V^T \subset Y$ is of type 0: $\text{codim}_{V^T} \varphi_{t\to 0}(W) = 0$ and $B_1(A_p)$ is full-rank for a general $p \in V^T$. The irreducible components that are cones over irreducible components of $P_{k-1,k+1}$ correspond to $\text{rk}(B_1(A_p)) = 0$, and hence they are of type $k+1$.

Proposition 4.7. *There is no irreducible component X of Y such that the general point of X^T belongs to $\mathcal{X}_k$, i.e. there is no irreducible component of type k.*

Proof. A necessary condition for the existence of such an irreducible component is that $\mathcal{X}_k \neq \emptyset$. This condition means that $\text{rk}(B_1(A_p)) = 1$ for some $p \in V^T$. However, $B_1(A_p)$ is a symmetric matrix with zeros on the diagonal. Therefore, if it is nonzero, then it has a 2×2 submatrix N of the form

$$N = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix},$$

with $a \neq 0$. Thus $\text{rk}(B_1(A_p)) \geq 2$, whenever $B_1(A_p)$ is nonzero. $\square$

Proposition 4.8. *The irreducible components described in Remark 3.9, i.e. those given by choosing a vanishing column and the vanishing complementary permanent, are of type $k-1$.*

Proof. Given such an irreducible component, it is immediate to see that $B_1(A_p)$ has nonzero only one row and one column and so $\mathrm{rk}(B_1(A_p)) = 2$. □

The locus $\mathcal{X}_1 \subset V^T$ is given by the vanishing locus of $\det(B_1)$ from which one removes the locus of larger corank. Hence this is a codimension-one constructible set inside V^T, which is possibly reducible.

Corollary 4.9. *Let X be an irreducible component of Y such that the general point of X^T belongs to $\mathcal{X}_1$, i.e. X is of type one. Then* codim $X \geq k+1$.

Proof. By Corollary 4.6, we have codim $X = \mathrm{codim}_{V^T} X^T + k$. Note that we have $\mathrm{codim}_{V^T} X^T \geq \mathrm{codim}_{V^T} \mathcal{X}_1 \geq 1$, which shows the inequality. □

Proposition 4.10. *Every irreducible component containing the Kirkup matrix $\mathcal{K}_{k,k+1}$ is of type one and so it has codimension at least $k+1$.*

Proof. Let X be an irreducible component of Y containing $\mathcal{K}_{k,k+1}$. Let p_k be the corresponding point of $\mathcal{K}_{k,k+1}$. To prove the statement it is enough to show that $X^T \cap \mathcal{X}_1 \neq \emptyset$ by Corollary 4.9. To check the validity of the latter statement, it is sufficient to prove that $\mathrm{rk}(B_1(A_{p_k})) = k$. The upper-left $k \times k$ submatrix of $B_1(A_{p_k})$ is of the form

$$E = \begin{pmatrix} 0 & a & \cdots & a & b \\ a & 0 & \cdots & a & b \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a & a & \cdots & 0 & b \\ b & b & \cdots & b & 0 \end{pmatrix},$$

where $a, b \neq 0$. The main diagonal of E consists of zeros. One has $\mathrm{rk}(E) = k$. To see this, first note that the $(k-1) \times (k-1)$ submatrix A of E only consisting of a's and zeros on the main diagonal has full-rank $k-1$: indeed the vector $\mathbf{1} = (1, \ldots, 1) \in \mathbb{C}^{k-1}$ is in the row span of A (add up all the rows and scale), and so every standard vector in $\mathbb{C}^{k-1}$ is in its row span (subtract from $\mathbf{1}$ each scaled row). Now, one can easily check that the last row of E cannot be linearly dependent from the first $k-1$ rows. Thus $\mathrm{rk}(B_1(A_{p_k})) = k$. □

Remark 4.11. To understand dimensions of irreducible components of $P_{k,k+1}$ one simply needs to understand dimensions of irreducible components of each $\mathcal{X}_i$. Indeed, every irreducible component of $P_{k,k+1}$ comes from a rank i vector bundle over a component of $\mathcal{X}_i$. However, it is not easy to conclude that components of $\mathcal{X}_i$ have codimension i in

Table 1
Irreducible components of $P_{4,5}$.

Type	Irreducible components
0	V^T
1	Kirkup component
2	No components
3	There exist such components
4	No components
5	Cones over $P_{3,5}$

W. As we have seen some $\mathcal{X}_i$ are empty, thus it is not true that $\mathcal{X}_{i+1}$ is contained in the closure of $\mathcal{X}_i$.

4.3. Codimension of $P_{4,5}$

We use the correspondence with vector bundles, to prove that the codimension of all irreducible components of $P_{4,5}$ is 5. This gives an alternate proof of the case $k = 4$ in Proposition 3.10.

Proposition 4.12. *All the irreducible components of $P_{4,5}$ have codimension 5. In Table 1, we organize them according to their type.*

Proof. We employ `Macaulay2` to perform the required computations. In §5, we provide a script to check some of the cases reported in the table. For instance, the script checks that $\det B_1(A_p)$ is smooth in codimension one after intersecting scheme-theoretically with a subspace. As singular points remain singular after such intersection, this implies that $\det B_1(A_p)$ is smooth in codimension one. If it had several components, then each one would be of codimension one in W, and as all varieties we deal with are cones over projective varieties, the components would need to intersect in codimension one inside $\det B_1(A_p)$ [19, Theorem 2.22]. In particular, the variety would have to be singular in codimension one. Thus, we conclude that $\det B_1(A_p)$ is irreducible. This implies that there is a unique Kirkup component.

In a similar way, by intersecting $\mathcal{X}_2$ with a fixed linear subspace, the script verifies that there are no type 2 irreducible components in codimension ≤ 5. Since $I(P_{4,5})$ is generated by five polynomials, Krull's principal ideal theorem implies that there are no type 2 components. $\square$

4.4. Components of $P_{5,6}$

We use the correspondence with vector bundles to prove that the codimension of all the irreducible components of $P_{5,6}$ is 6.

Proposition 4.13. *All the irreducible components of $P_{5,6}$ have codimension 6. In Table 2, we organize them according to their type.*

Table 2
Irreducible components of $P_{5,6}$.

Type	Irreducible components
0	V^T
1	Kirkup component
2	No components
3	Potential components in codimension 6
4	There exist such components
5	No components
6	Cones over $P_{4,6}$

Proof. We employ `Macaulay2` to perform the required computations. In §5, we provide a script to check the case where $\mathrm{rk}(B_1(A_p)) = 2$, i.e. the irreducible components of type 4. Let X be an irreducible component of this type so we have $X^T \subset \mathcal{X}_4$, where $\mathcal{X}_4$ is defined by the 3×3 minors of the matrix $B_1(A)$, A being a generic matrix in V^T. By Corollary 4.6, $\operatorname{codim} X = \operatorname{codim}_{V^T} X^T + \mathrm{rk}(B_1(A)) = \operatorname{codim}_{V^T} X^T + 2$. So $\operatorname{codim} X \geq 6$ is equivalent to verifying that $\operatorname{codim}_{V^T} X^T \geq 4$. Since $X^T \subset \overline{\mathcal{X}_4}$, it is sufficient to check that $\operatorname{codim}_{V^T} \mathcal{X}_4 \geq 4$. Let $\mathbb{P}(\overline{\mathcal{X}_4}) \subset \mathbb{P}(V^T)$ be the corresponding projective variety. Then it is enough to find a $L = \mathbb{P}^3 \subset \mathbb{P}(V^T)$ such that their intersection $\mathbb{P}(\mathcal{X}_4) \cap L$ is empty [19, Theorem 2.22]. The choice of such a suitable L is reported on the script. The output of the script reads: `[gb]12(400)13(420)14(840)number of (nonminimal) gb elements = 455, number of monomials = 49455, used 33.8193 seconds`. With a similar code, we also check all the other cases. For instance, in type 3, we find a $\mathbb{P}^2$ such that $\mathbb{P}(\mathcal{X}_3) \cap \mathbb{P}^2 \subset \mathbb{P}(V^T)$ is empty. To check that in type 1 we have a unique Kirkup irreducible component, we confirm that the singular locus of the set defined by $\det(B_1(A)) = 0$, for $A \in V^T$, has codimension higher than two in V^T. $\square$

4.5. Singular locus of the permanental hypersurface: von zur Gathen's problem

Let $k \geq 3$, M be a generic $k \times k$ square matrix, and let $P = \{\mathrm{perm}(M) = 0\}$ be the $k \times k$ permanental hypersurface. A folklore question asks for a description of the singular locus of this hypersurface in terms of numerical invariants of various kinds. This is a challenging and poorly understood question, in sharp contrast with the singular locus of the determinantal hypersurface that has natural interpretation in terms of rank of matrices.

The codimension of this set is currently unknown for $k \geq 5$. A first result towards determining its codimension, which was so far the strongest in this direction, is due to von zur Gathen:

Theorem 4.14 *(von zur Gathen [25]). Let $k \geq 3$. The singular locus* $\mathrm{Sing}(P) = \{\mathrm{prk}(M) \leq k-2\}$ *has codimension between* 5 *and* $2k$.

Note that $Y = \mathrm{Sing}(P)$ is defined by the $(k-1) \times (k-1)$ permanents of M. Let V be the vector space of $k \times k$ complex matrices and let X be an irreducible component of Y.

We fix the following $T = \mathbb{C}^*$-action on V: given $p \in V$, let $t \cdot p$ be the matrix where the first two rows are those of p scaled by t, while the other entries are unchanged. Hence V^T is the linear space given by the matrices of the form

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 0 \\ * & * & * & * \\ \vdots & \cdots & \cdots & \vdots \\ * & * & * & * \end{pmatrix}.$$

Recall that, as at the beginning of §4, we have a surjective map $\varphi_{t\to 0} : X \to X^T \subset V^T$.

Remark 4.15. Let $p \in V^T$ be the matrix $\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ A_p \end{pmatrix}$, where A_p is a $(k-2) \times k$ matrix. Then the $k^2 \times k^2$ Jacobian $J(Y)_p$ of Y at p has the following form

$$J(Y)_p = \begin{array}{c} \\ x_{1,h} \\ x_{2,h} \\ x_{i,j} \end{array} \!\!\begin{array}{c} \begin{array}{ccc} S_1 & S_2 & S_3 \end{array} \\ \begin{pmatrix} \mathbf{0} & L_p & \mathbf{0} \\ L_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array}.$$

Here S_1 is the set of permanents that do not use the first row, S_2 is the set of permanents that do not use the second row, and S_3 is the set of permanents that use the first and second rows. Moreover, the $k \times k$ matrix $L_p = (\ell_{ij})$ is such that ℓ_{ij} is the $(k-2)\times(k-2)$ permanent of A_p which does not use columns i and j. Note that L_p is symmetric with zeros on the main diagonal.

Corollary 4.16. *With the same notation as in Remark 4.15, whenever $p \in \varphi_{t\to 0}(X) = X^T$ with $X \neq V^T$, the tangent space to Y at p is:*

$$T_{Y,p} = V^T \oplus \ker(L_p)^{\oplus 2},$$

where $T^0_{Y,p} = V^T$ and $T^1_{Y,p} = \ker(L_p)^{\oplus 2}$. In particular, $\dim_{\mathbb{C}} T^1_{Y,p} = 2\dim_{\mathbb{C}} \ker(L_p)$.

Proof. The tangent space $T_{Y,p}$ is the kernel of the transpose of $J(Y)_p$. The two copies of $\ker(L_p)$ live in the span of the first and second rows, respectively. □

Corollary 4.17. *Let $p \in X$ be any irreducible component of $Y = \mathrm{Sing}(P)$. Then the following upper bound holds:*

$$\dim X \leq \dim X^T + 2\dim_{\mathbb{C}} \ker(L_p).$$

Equivalently, one has $\mathrm{codim}_V X \geq \mathrm{codim}_{V^T} X^T + 2\mathrm{rk}(L_p)$.

Proof. By Proposition 4.2, X is contained in the closure of the vector bundle $T^1_{Y|X^T}$ over X^T. Its rank is $2\dim_{\mathbb{C}} \ker(L_p)$ by Corollary 4.16. □

A direct consequence of Corollary 4.17 is as follows.

Corollary 4.18. *Let $p \in X^T$ be general. If $\mathrm{rk}(L_p) \geq 3$, then $\mathrm{codim}_V X \geq 6$.*

Lemma 4.19. *Let $p \in X^T$ be general. Then $\mathrm{rk}(L_p) \neq 1$.*

Proof. This is analogous to Proposition 4.7. □

Proposition 4.20. *Let $k \geq 4$. Let $p \in X^T$ be general and suppose $\mathrm{rk}(L_p) = 2$. Then:*

$$\mathrm{codim}_{V^T} X^T \geq 2.$$

Proof. By assumption all the 3×3 minors of L_p vanish. Any such principal submatrix L has the form

$$L = \begin{pmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{pmatrix}.$$

Hence $\det(L) = 2abc$. We regard the point p as a $(k-2) \times k$ matrix. For any choice of columns i_1, i_2, i_3, there must be two indices i_n, i_m such that the $(k-2)\times(k-2)$ permanent not involving i_n and i_m is zero. Since $k \geq 4$, up to permuting columns, we may assume that X^T is inside the locus C defined by the vanishing of the $(k-2) \times (k-2)$ permanent perm_{12} not involving columns $1, 2$ and of the $(k-2) \times (k-2)$ permanent perm_{34} not involving columns $3, 4$. Since each of these permanents is irreducible by Corollary 3.2, and since perm_{12} and perm_{34} are linearly independent, C is a complete intersection of codimension two in V^T. Hence $\mathrm{codim}_{V^T} X^T \geq \mathrm{codim}_{V^T} C = 2$. □

Theorem 4.21. *Let $k \geq 6$. Let $p \in X^T$ be general and suppose $\mathrm{rk}(L_p) = 0$. Then one has*

$$\mathrm{codim}_{V^T} \varphi_{t\to 0}(X) \geq 6.$$

Proof. The assumption implies that $\varphi_{t\to 0}(X)$ is inside the locus C defined by the vanishing of all $(k-2) \times (k-2)$ permanents of any $(k-2) \times k$ matrix in V^T.

We have two cases:

(i) For a general $p \in X^T$, all the $(k-3) \times (k-3)$ permanents vanish. Hence X^T is inside a cone over an irreducible component of the locus C, given by the vanishing of all the $(k-3) \times (k-3)$ of a $(k-3) \times k$ generic matrix. In this case one has $\mathrm{codim}_{V^T} X^T \geq 6$, by Proposition 4.22 below for $h = 3$ and $\ell = k-3$. We postpone its proof because it involves a more technical analysis.

(ii) For a general $p \in X^T$, there exists a $(k-3)\times(k-3)$ permanent that does not vanish at p.
We claim that in this case the inequality in Corollary 4.17 is strict, i.e. X is *strictly* contained in the closure of the vector bundle $T^1_{Y|X^T}$ over X^T. Otherwise, if equality holds, then for a general $p \in X^T$ any extension q to a $k\times k$ matrix satisfies $q \in X$. Put free variables z_{ij} on the first two rows of q. Any $(k-1)\times(k-1)$ permanent of q vanishes, because $q \in Y$. Consider a $(k-1)\times(k-1)$ permanent $\mathrm{perm}_{k-1,k-1}$ of q involving the first two rows consisting of z_{ij} and containing a $(k-3)\times(k-3)$ nonvanishing subpermanent; the latter exists because of the assumption on p. The condition $\mathrm{perm}_{k-1,k-1}(q) = 0$ gives a linear relation among the 2×2 permanents of the $2\times k$ submatrix of q whose entries are the z_{ij}. However, permanents of fixed arbitrary size of a generic matrix are linearly independent, by Lemma 3.3. Therefore we reached a contradiction.
Thus $\mathrm{codim}_V X \geq \mathrm{codim}_{V^T} X^T + 1$. To conclude it is enough to show that $\mathrm{codim}_{V^T} X^T \geq 5$. This is proven in Proposition 4.22, where $h = 2$ and $\ell = k-2$.

This concludes the proof. □

The previous proof relies on the following result, which in turn improves the easier lower bound of Proposition 3.1.

Proposition 4.22. *Let M be a generic $\ell\times(\ell+h)$ complex matrix for $h \geq 1$ and let $V = \mathbb{C}^{\ell\times(\ell+h)}$. Let $P_{\ell,\ell+h}$ be the variety defined by all the $\ell\times\ell$ permanents of M. Then* $\mathrm{codim}_V P_{\ell,\ell+h} \geq h+3$ *for* $\ell \geq 3$.

Proof. The proof is by induction on ℓ, with $\ell = 3$ as base case, which is implied by Theorem 3.18. Let $Y = P_{\ell,\ell+h}$. We fix the $T = \mathbb{C}^*$-action scaling by t the first row of M. Hence V^T is a linear subspace of $(\ell-1)\times(\ell+h)$ matrices. Let X be an irreducible component of Y. As before, we have a surjective map $\varphi_{t\to 0} : X \to X^T \subset V^T$. The Jacobian $J(Y)$ calculated at a point $p \in V^T$ has the form

$$J(Y)_p = \begin{pmatrix} N_p \\ \mathbf{0} \end{pmatrix},$$

where the entries of N_p are $(\ell-1)\times(\ell-1)$ permanents of the $(\ell-1)\times(\ell+h)$ matrix p. Here the columns of $J(Y)_p$ are indexed by subsets of ℓ elements of the $\ell+h$ column set of M. The rows of N_p correspond to the $\ell+h$ variables on the first row of M. From the description of the Jacobian, as in Corollary 4.17 and using Proposition 4.2, we find that $\mathrm{codim}_V X \geq \mathrm{codim}_{V^T} X^T + \mathrm{rk}(N_p)$.

We claim that $\mathrm{rk}(N_p) \neq 1, \dots, h$. Indeed, assume that there is a nonzero entry in N_p. This corresponds to a nonvanishing $(\ell-1)\times(\ell-1)$ permanent $\mathrm{perm}_{\ell-1,\ell-1}$. Up to permuting columns, we may assume that $\mathrm{perm}_{\ell-1,\ell-1}$ involves the first $\ell-1$ columns and set $\mathrm{perm}_{\ell-1,\ell-1}(p) = a \neq 0$.

Consider the submatrix N of N_p with $\ell+h-(\ell-1)=h+1$ columns, each corresponding to a subset ℓ columns of M. Thus

$$N = \begin{array}{c} \\ x_{1,\ell} \\ x_{1,\ell+1} \\ \vdots \\ x_{1,\ell+h} \end{array} \begin{array}{c} \begin{array}{cccc} \{1,\dots,\ell\} & \{1,\cdots,\ell-1,\ell+1\} & \dots & \{1,\dots,\ell-1,\ell+h\} \end{array} \\ \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & a \end{pmatrix} \end{array},$$

where the variables $x_{1,\ell+j}$ are the last $h+1$ variables in the first row of M. The matrix N is a diagonal $(h+1)\times(h+1)$ matrix with the evaluated permanent $\mathrm{perm}_{\ell-1,\ell-1}(p)=a$ on the main diagonal. Hence $\det(N)\neq 0$ and so either $\mathrm{rk}(N_p)=0$ or $h+1\leq \mathrm{rk}(N_p)\leq \ell+h$.

We shall be done if we prove that $\mathrm{rk}(N_p)\geq h+3$. To this aim, we have to deal with the cases:

(i) $\mathrm{rk}(N_p)=0$;
(ii) $\mathrm{rk}(N_p)=h+1$;
(iii) $\mathrm{rk}(N_p)=h+2$.

Suppose (i) holds true. Then all the $(\ell-1)\times(\ell-1)$ permanents of the $(\ell-1)\times(\ell+h)$ matrix p vanish. This implies that X^T is inside an irreducible component of $P_{\ell-1,\ell-1+h}$. So, by induction on $\ell\geq 3$, we have $\mathrm{codim}_{V^T}X^T\geq h+3$.

Suppose (ii) holds true. Then it is enough to find two $(h+2)\times(h+2)$ minors, without common factors to prove that $\mathrm{codim}_{V^T}X^T\geq 2$. Consider the following $(h+2)\times(h+2)$ submatrix of N_p:

$$S = \begin{array}{c} \\ x_{1,\ell} \\ x_{1,\ell+1} \\ \vdots \\ x_{1,\ell+h} \\ x_{1,\ell-1} \end{array} \begin{array}{c} \begin{array}{ccccc} \{1,\dots,\ell\} & \{1,\cdots,\ell-1,\ell+1\} & \dots & \{1,\dots,\ell-1,\ell+h\} & \{1,\dots,\ell-2,\ell,\ell+1\} \end{array} \\ \begin{pmatrix} a & 0 & \cdots & 0 & b_2 \\ 0 & a & \cdots & 0 & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a & 0 \\ b_1 & b_2 & \cdots & b_{\ell+1} & 0 \end{pmatrix} \end{array},$$

where a is the permanent on columns $\{1,\dots,\ell-1\}$ of p, b_1 is the permanent on columns $\{1,\dots,\ell-2,\ell\}$ of p, and b_2 is the permanent on columns $\{1,\dots,\ell-2,\ell+1\}$ of p. Note that $\det(S)=-2a^hb_1b_2$. Since $\det(S)=0$ and $a\neq 0$, we have either $b_1=0$ or $b_2=0$. Picking a different minor from the one above, we find another irreducible vanishing permanent. Since X^T must be contained in the vanishing of two irreducible and linearly independent permanents, we find that $\mathrm{codim}_{V^T}X^T\geq 2$.

To conclude in case (iii), it is enough to find a point $p \in V^T$ and a $(h+3) \times (h+3)$ minor of N_p that is nonzero. Indeed, then $\operatorname{codim}_{V^T} \varphi_{t \to 0}(X) \geq 1$. Let $q \in V^T$ be a matrix of the form

$$q = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & \cdots & \ell-1 & \ell & \ell+1 & \ell+2 & \cdots \end{array} \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & a & b & 0 & 0 \\ 0 & 0 & 0 & 1 & d & c & 1 & 0 \end{pmatrix} \end{array}.$$

Then consider the following $(h+3) \times (h+3)$ minor of N_q, the submatrix of $J(Y)_q$:

$$Q = \begin{array}{c|cccccc} & \{1,\dots,\ell\} & \{1,\cdots,\ell-1,\ell+1\} & \dots & \{1,\dots,\ell-1,\ell+h\} & \{1,\dots,\ell-2,\ell,\ell+1\} & \{1,\dots,\ell-2,\ell,\ell+2\} \\ x_{1,\ell-2} & a & d & \cdots & 0 & ac+bd & a \\ x_{1,\ell-1} & d & c & \cdots & 1 & 0 & 1 \\ x_{1,\ell} & 1 & 0 & \cdots & 0 & c & 0 \\ x_{1,\ell+1} & 0 & 1 & \cdots & 0 & d & 0 \\ x_{1,\ell+2} & 0 & 0 & 1 & 0 & 0 & d \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1,\ell+h} & 0 & 0 & 0 & 1 & 0 & 0 \end{array}.$$

The lower-left $(h+1) \times (h+1)$ corner is the identity matrix. The matrix Q is divided into two linearly independent blocks: an $(h-2) \times (h-2)$ identity matrix (inside the lower-left $(h+1) \times (h+1)$ identity corner) and the following 5×5 matrix

$$Q' = \begin{pmatrix} a & d & 0 & ac+bd & a \\ d & c & 1 & 0 & 1 \\ 1 & 0 & 0 & c & 0 \\ 0 & 1 & 0 & d & 0 \\ 0 & 0 & 1 & 0 & d \end{pmatrix}.$$

Hence $\operatorname{rk}(Q) = h - 2 + \operatorname{rk}(Q')$. Now $\det(Q') = d(d^2 - db + 2ac - d + b)$, which is nonzero for generic choices of a, b, c, d. For such choices, $\operatorname{rk}(Q') = 5$ and the proof is complete. □

We are ready to improve von zur Gathen's Theorem 4.14.

Theorem 4.23. *Let $k \geq 6$. The singular locus $\operatorname{Sing}(P) = \{\operatorname{prk}(M) \leq k-2\}$ has codimension between 6 and $2k$.*

Proof. By Corollary 4.17, it is enough to show that for any irreducible component X of $Y = \operatorname{Sing}(P)$, we have

$$\operatorname{codim}_{V^T} X^T + 2\operatorname{rk}(N_p) \geq 6.$$

By Corollary 4.18 and Corollary 4.19, we have two cases to deal with: either $\mathrm{rk}(N_p) = 2$ or $\mathrm{rk}(N_p) = 0$. The first case is achieved by Proposition 4.20. The second case is implied by Theorem 4.21. $\square$

Definition 4.24. For any subset $R \subset [k]$ of rows, let J_R be the ideal generated by all the $|R| \times |R|$ permanents of the $|R| \times k$ submatrix $M_{R,[k]}$, i.e. the submatrix of M whose rows are indexed by R. For any subset of columns $C \subset [k]$, one similarly defines J_C.

We omit the proof of the following straightforward result.

Lemma 4.25. *For any partition of rows $R_1 \sqcup R_2 = [k]$ or of columns $C_1 \sqcup C_2 = [k]$, we have the inclusions of ideals*

$$I(\mathrm{Sing}(P)) \subset J_{R_1} + J_{R_2} \text{ and } I(\mathrm{Sing}(P)) \subset J_{C_1} + J_{C_2}.$$

Corollary 4.26. *Suppose Conjecture 3.4 holds. Then any irreducible component C of* $\mathrm{Sing}(P)$ *whose prime ideal $I(C)$ contains $J_{R_1} + J_{R_2}$ for a partition $R_1 \sqcup R_2 = [k]$ has codimension $\geq 2k$.*

Proof. By Theorem 3.19, the assumption implies that the codimension of $J_{R_1} + J_{R_2}$ is $2k$. Since $I(C)$ contains $J_{R_1} + J_{R_2}$, the statement follows. $\square$

Conjecture 4.27. *We have the following equality of radical ideals:*

$$\mathrm{rad}(I(\mathrm{Sing}(P)) = \bigcap_{(S_1,S_2)\in\Pi} \mathrm{rad}(J_{S_1} + J_{S_2}), \tag{5}$$

where Π is the set of partitions (S_1, S_2) of the k rows or the k columns.

Remark 4.28. If both Conjectures 4.27 and 3.4 were true for each k, the codimension of $\mathrm{Sing}(P)$ would be $2k$, i.e. the upper bound in von zur Gathen's Theorem 4.14 would be sharp for each k. Equality (5) has been computationally checked for $k = 3$ in `Macaulay2`. We do not know whether it is true even for $k = 4$.

5. Code

The majority of the following code, which simplifies that in an earlier version of this paper, was provided by an anonymous reviewer, to whom we are grateful.

```
K = QQ;
-- k x (k+1) matrices
k = 4 -- or k = 5
R = K[x_(1,1)..x_(k,k+1)];
M = matrix for i in 1..k list for j in 1..k+1 list x_(i,j);
P = permanents(k,M);
B1 = diff(matrix{{x_(1,1)..x_(1,k+1)}}, transpose gens P);
v = flatten entries transpose M_{1..k}^{1..k-1};

-- random A (for k = 4)
A = random(K^(k-1),K^(#v));

-- special A (for k = 5)
A = matrix {{3, 3, 2, 1, -1, 0, -3, 3, 2, -3, 2, 0, -3, 2, 3, -2, 2, 2, -3, -3},
{-2, -2, -1, 1, -1, 0, -2, -2, -1, -3, 2, -2, -1, 3, -2, -2, 2, -1, -1, -1},
{-2, -2, 1, 2, 3, 0, 0, -3, 2, 2, -3, -3, -1, 2, -3, 2, -2, 3, -2, 2},
{-3, 0, -3, -1, 1, 2, -1, 2, -3, 2, 1, 0, -3, -1, -1, -3, -2, 3, -1, -3}};

F = first entries (matrix{{x_(2,1)..x_(k,1)}}*A);
L = apply(#v, i-> v#i => F#i);
BB = sub(B1, L);


-- k = 4 case
P = det(BB);
S = K[x_(2,1),x_(3,1),x_(4,1)];
PP = sub(P,S);
Sing = ideal diff(vars S,PP);
time codim Sing

use R
J = time minors(4,BB);
gbTrace=1
time codim J

-- k = 5 case
J = time minors(3,BB);
gbTrace=1
time codim J
```

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