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Supergravity in the geometric approach and its hidden graded Lie algebra

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Abstract

In this contribution, we present the geometric approach to supergravity. In the first part, we discuss in some detail the peculiarities of the approach and apply the formalism to the case of pure supergravity in four space-time dimensions. In the second part, we extend the discussion to theories in higher dimensions, which include antisymmetric tensors of degree higher than one, focussing on the case of eleven dimensional space–time. Here, we report the formulation first introduced by R. D'Auria and P. Fré in 1981, corresponding to a generalization of a Chevalley–Eilenberg Lie algebra, together with some more recent results, pointing out the relation of the formalism with the mathematical framework of L_∞ algebras.

Keywords: Field theory; Gravity; Supersymmetry

1. Introduction

It is more than half a century since superstring theory [43], together with its strictly related low energy description, supergravity [37,41], appeared and soon imposed themselves as some of the most investigated fields of research in High-Energy Physics. Many important results at the frontier between Physics and Mathematics have been obtained in the years in this field.

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The present contribution will concern the construction of supergravity theories through the use of geometric concepts and structures only [18,49]. The approach was first introduced in $D = 4$ space–time dimensions as a Group Manifold Approach, where the structure group of the theory is a *graded Lie group* (a Lie supergroup).[1] However, when the supergravity theory is built in a higher dimensional space–time [28], the formalism has to be generalized to what is known in Physics literature as FDA approach [30][2] since it is based on a higher algebraic structure than ordinary Lie algebras, so that an underlying structure group cannot be defined "a priori" but, as we will discuss in the last Section, can be recovered, "a posteriori", in a larger sense. What is peculiar in our approach to supergravity, is its *geometric* flavor from a mathematical point of view.

1.1. Some history

We recall that there have been actually several approaches to the construction of Supergravity theories: The well-honored Noether method was applied to construct the first instance of a supergravity theory in [37,41]; then the so-called Superspace approach appeared [60,68], which features the use of an enlarged space parametrized by Grassmann odd coordinates θ^α together with the usual space–time coordinates x^μ; and the superconformal approach of the Belgian-Dutch school [42], where the superspace theories are obtained by gauge-fixing models enjoying a larger, superconformal invariance.

Last, in order of time, is the so-called *Geometric or Rheonomic approach*. It was proposed in the year 1978 by Y. Ne'eman and T. Regge [49], as a new approach to the formulation of gauge theories acting non-trivially on space–time, specifically gravity and supergravity. It is based on the formalism introduced by E. Cartan [17] for the formulation of Riemannian geometry in a completely geometrical setting. Cartan's approach implies a geometrical and group-theoretical way of formulating General Relativity. Indeed, as the adopted formalism relies on the use of *differential forms*, Cartan's beautiful setting is independent of the choice of a given coordinate frame. At the same time, it gives a prominent role to the gauge invariance of the theory under the Lorentz group, which emerges quite naturally from the formalism. As a matter of fact, in Cartan's view, Riemannian geometry has to be seen as pertaining to finite dimensional Lie groups rather than to the infinite dimensional group of general coordinate transformations (GCTG in the following). In the latter case, it would be difficult to see how gravitation could be unified with gauge theories of other interactions, at least at the classical level, what instead seems quite natural in the geometrical formalism developed by Cartan.

Following this line of approach, Y. Ne'eman and T. Regge further developed Cartan's formalism proposing that it should be possible in principle to construct any diffeomorphic

[1] In the following, we will adopt either the physical suffix *super*- or the mathematical suffix *graded*- interchangeably.

[2] FDA is an abbreviation of *Free differential algebra*. Strictly speaking, the name could be misleading, as it is a *differential-graded algebra* (dg-algebra) which in general is free only as a graded-supercommutative superalgebra, not as a differential algebra. Several years after its introduction in the supergravity context in [30], this structure was recognized to be equivalent to a mathematical structure called L_∞ algebra. The relation between super L_∞-algebras and the "FDA"s of the supergravity literature was made explicit in [40]. We will elaborate further on this in the second part of the present contribution. Here, we will keep the name "FDA" to be easily understood in the supergravity community.

and gauge invariant theory directly on a *group manifold* G, the physical fields being defined as the Lie algebra valued gauge fields in the coadjoint representation of the group. Therefore their original formulation was denoted *Group Manifold Approach*. The above geometric formalism was then further developed in [31], where the role of the graded Lie algebra cohomology for the construction of supergravity theories was put in evidence.

Coming back from IAS to Torino University in 1978, Tullio Regge proposed to one of the authors (R D'A.) to develop the approach extensively, namely in any space–time dimension $4 < D \leq 11$ with any number allowed of supersymmetry generators[3] and in the presence of matter sources. His legacy was then further developed by his research group in the Physics Department of Torino University (mainly by R.D'A and P. Fré), and later by the Torino-Politecnico group.

Using the geometric formalism, it was possible indeed to also rewrite the pure[4] supergravity theories in five [27] space–time dimensions in a simple and elegant geometric way [33], based on the Maurer–Cartan equations satisfied, in the vacuum, by the 1-form fields dual to the generators of the structure group. Since then, the systematic use of the geometric and group-theoretical approach has been an essential tool to obtain many interesting results in supergravity. Most of the supergravity theories in every dimensions $D \leq 11$ (see [55] for a comprehensive review of the first achievements in supergravity) were reformulated or constructed from scratch within the geometric approach. Some of them are collected in [18]. Often, the use of the geometrical approach allowed to give a complete answer to problems where other approaches had given only limited answers. This was particularly fruitful when matter coupled supergravity theories were considered, in which case the geometric approach allowed to put in light all the global and local non-linear symmetries governing their interaction [6,15,16,21–25].

A typical example was the construction of the $\mathcal{N} = 2$, $D = 4$ matter-coupled supergravity [3,18,29] which was previously formulated using the superconformal approach in a coordinate dependent way [35]. The geometrical approach provided a complete Lagrangian (including all the fermions contributions) and the transformation laws leaving it invariant under supersymmetry, quite independently of the coordinates used, not only referring to the space–time frame, but also to the scalar fields description, which in these theories is generally associated with a non-linear σ model, with specific geometric features. Within the geometric approach, it was very natural to find, among the conditions for supersymmetry invariance of the theory, a set of differential and algebraic relations fully characterizing the scalar σ-models of the matter-coupled theory: regarding the scalars in the vector multiplets, these correspond to the notion of Kähler Special Geometry while, regarding those in the hypermultiplets, they were instead recognized as the defining relations of Quaternionic manifolds. (As comprehensive reviews of the subject from a physicist's perspective, we refer to Ref. [18] and, for more recent results, to the excellent review [67].)

As supergravity theories, besides their being field theories "per se", are also the low energy limit of superstring theory, the results found about the scalar manifolds of

[3] The restriction to 11 space–time dimensions is due to the fact that, for $D > 11$, supersymmetric theories necessarily include fields with helicity higher than 2.

[4] By pure (super)-gravity we mean a theory with no matter couplings.

supergravity theories also give insight and have a counterpart description in terms of Calabi–Yau compactifications of superstring's target-space description.

Another interesting point is the question of whether the geometric approach is completely equivalent to the purely space–time approach. This seems not to be the case in some *chiral* theories, like $\mathcal{N} = 1$, $D = 6$ [32], and $D = 10$, IIB [20]. What is common to these theories is their non-standard description, in terms of Hodge-duality frame, of the gauge fields involved: the geometric approach, whose frame-independence can be extended also to the electric/magnetic duality frame, allowed to obtain new results, not accessible within other approaches: As an example, in the pure, minimal $D = 6$ supergravity, the gravity multiplet contains the sechsbein, a Weyl gravitino, and a 2-form potential (that is an antisymmetric two-index tensor) with a *self-dual* 3-form field strength. Using the geometric approach in superspace, it was shown [32] that the *self-duality* of the 3-form field-strength, necessary to match the number of Bose–Fermi on-shell degrees of freedom, follows from the variational equations in *superspace*, but not from their space–time restriction. As a consequence, the theory is consistent in superspace, although its Lagrangian restricted to space–time is not supersymmetric invariant off-shell. Exactly in the same way can be treated the $D = 10$, IIB theory [20], so that also in this case the self-duality of the 5-form can be retrieved from the superspace equations of motion.

We stress that, as we are going to discuss in the following, the group manifold approach is a *superspace* approach but, differently from other superspace approaches, the (super)-fields entering the theory $\mu^A(x^\mu, \theta^\alpha)$ are never expanded in the Grassmann-odd coordinates θ^α and no Berezin integration is necessary.

However, the very real impact of the approach was realized, beginning of 1981, with the extension of the geometric method to supergravity in dimensions D higher than five, namely $5 < D \leq 11$ [30]. Indeed, in the general case, the spectrum of supergravity theories includes p-form potentials, of rank p higher than 1, associated with graded-antisymmetric tensors

$$A^{(p)} = \frac{1}{p!} A_{\mu_1 \ldots \mu_p} dx^{\mu_1} \wedge dx^{\mu_p} \, .$$

Their presence in the supergravity spectrum makes the direct construction of the Lagrangian in terms of the Maurer–Cartan 1-forms of a Lie (super)-algebra (see Appendix B) problematic. Indeed, when higher *p-forms*, with $p > 1$, are present in the physical spectrum, we cannot consider these fields as spanning the cotangent space of a (super)-group manifold and therefore the construction based on the group manifold must be modified.

In the paper [30] the authors devised how to overcome this obstacle by defining a new structure, generalizing to higher forms the Maurer–Cartan framework, in such a way as to include also p-forms, of any degree $p \geq 1$, in the set of forms spanning the Maurer–Cartan set, thus generalizing this notion. The procedure to introduce the higher p-forms was obtained by inspecting a cochain system based on an ordinary (graded) Lie algebra, following the results of the generalized Chevalley–Eilenberg cohomology group of graded Lie algebras [30]. Furthermore, mimicking the Maurer–Cartan equations of an ordinary (*super*) Lie algebra, they considered the *exterior differential*, "d", of any p-form potential and required it to be expressed as a polynomial in terms of the wedge product of all possible forms in the enlarged Maurer–Cartan set, consistently with their degree. Integrability is then obtained by the cohomological requirement $d^2 = 0$, thus providing a generalization of the dual form of the Jacobi identity.

The structure so obtained was given the name of *Cartan integrable system* (CIS) and later *free differential algebra* (FDA) (see footnote[2]). Soon after, this new formalism was also applied in [19,50].

Working on the FDA, they were also able to show that the higher p-forms can be further expressed as polynomials in terms of 1-forms, thus reconstructing, from the eleven-dimensional super-Poincaré algebra, an extended Lie algebra which can be considered as the true Lie algebra of the $D = 11$ supergravity.

The approach outlined above was undertaken in Ref. [30], where it was applied to formulate the eleven-dimensional, maximal theory of supergravity in superspace. This theory, earlier constructed on space–time [28], was the first instance where an antisymmetric tensor field, here of rank three (a 3-form potential), appeared in the spectrum of the physical fields as an essential ingredient to get a supersymmetry invariant theory. This is not a special feature of the eleven-dimensional theory only. An analogous treatment can be done to all supergravity theories where antisymmetric tensor fields appear in the spectrum, specifically to supergravity in space–time dimensions $5 < D \leq 11$, as it was performed explicitly, for example, in the minimal theory in $D = 7$ in [7].

Besides its applications in Physics, the approach of [30] turned out to be interesting also from a mathematical point of view. Indeed, some years later, in the nineties, a group of mathematicians realized [48,61] that this kind of graded algebraic structure, the CIS built in [30], being dated 1981, is actually *the first historical example of an L_∞ algebra* [40,51,52,57]. The formalism used in [30] is actually dual to the standard formalism of L_∞ algebras, which is given in terms of multi-brackets, since it is instead formulated in terms of a graded *coalgebra* of differential p-forms, namely in terms of the space of p-forms dual to the generators of the L_∞ algebra. As an example, in the original D=11 case studied in [30], the algebra is constructed in terms of generators with form-degree three and six, besides the usual 1-form generators.[5]

The mathematicians also pointed out that the name of Free Differential Algebra (FDA), given in physical literature to the CIS structures, is not fully appropriate, since such structures are not "free" but only semi-free, their underlying graded algebras being free. In the following, however, we shall not adopt the name of semi-free differential graded algebra (SFGDA), but for the sake of simplicity we will be faithful to the original name, well understood in the Physics community, and continue to call these structures FDA, the semifree character being understood.

The rest of the paper is articulated as follows:

[5] Note that this is in fact the formalism mostly used for the formulation of the extended Chevalley–Eilenberg cohomology of the Lie algebras so that we may also say that the formalism is a generalization of the Chevalley–Eilenberg cohomology of Lie algebras.

In Section 2 we give a summary of the Einstein–Cartan formulation of general relativity, putting in evidence its geometric, group theoretical formulation. This will also set the stage for the extension of the formalism to supergravity, which is then the object of Section 3, where the formulation of supergravity in the geometric approach is presented. In Section 4, we will extend the geometric formalism to supergravity theories including higher p-forms, where the theories are formulated as FDA's. In particular, we discuss in some detail the case of D=11 supergravity and how it is determined by a FDA, thus giving for the first time an explicit formulation of L_∞ algebras. We stress that the choice of the maximal theory in D=11 is motivated by the fact that supergravity in D=11 is not only the first theory where historically this new kind of structure has appeared, but even more, because D=11 supergravity is in a sense the most general supergravity theory. Indeed, from this theory, by compactification of some of the spatial dimensions, one can obtain all the $D \leq 9$ dimensional supergravity theories. Finally, in Section 5 another important consequence of the approach studied in [30] is reported, namely the possibility of trading a given FDA into an equivalent ordinary graded Lie algebra. We collected in the Appendices our notations and conventions, together with some more technical details.

2. Einstein–Cartan gravity, a short resumé

In this Section we shall first remind some of the most important properties of the Cartan formulation of the Einstein gravity in order to establish the notations and thus setting the stage for the formulation of its extension to the Poincaré group manifold. This is a preparatory discussion in view of obtaining the geometrical interpretation of supersymmetry (also called *rheonomy*) in supergravity theories. We recall the principal properties of the standard Cartan–Einstein four-dimensional Lagrangian, which is the starting point for our description, and of its extension to a theory defined on the full Poincaré group. Notations and conventions used here and in the following are given in Appendix A.

In the original Lagrangian formulation by Cartan, the field content is given by the spin connection and the vierbein, $\mu^A = \{\omega^a{}_b,\ V^a\}$, which are 1-form fields

$$\mu^A : \quad \mathcal{M}_4 \to G\,, \tag{2.1}$$

where $\mathcal{M}_4$ is the four-dimensional space–time, and G is the structure group, which in this case is the Poincaré group $G = \mathrm{ISO}(1,3)$. This means that we can identify the space–time manifold as the base space $\mathcal{M}_4$ of the principal fiber bundle structure $[\mathcal{M}_4, \mathrm{H}]$, whose fiber is the Lorentz group $H = \mathrm{SO}(1,3) \subset G$. Here, the 1-form fields μ^A locally span the cotangent space to G.

The Poincaré group is generated by the algebra $\mathfrak{iso}(1,3)$, with Lorentz generators J_{ab} and translation generators P_a, satisfying

$$[J_{ab}, J_{cd}] = -2\,\eta_{a[c}J_{d]b} + 2\,\eta_{b[c}J_{d]a}\,, \quad [J_{ab}, P_c] = -2\,P_{[a}\eta_{b]c}\,, \quad [P_a, P_b] = 0\,. \tag{2.2}$$

In the vacuum of the dynamical theory, the 1-forms ω^{ab}, V^a span the cotangent space of G, so that:

$$\omega^{ab}(J_{cd}) = 2\delta^{ab}_{cd}\,, \quad V^a(P_b) = \delta^a_b\,, \tag{2.3}$$

and they satisfy the Maurer–Cartan equations (see Appendix B):

$$d\omega^{ab} - \omega^a_c \wedge \omega^{cb} = 0 \tag{2.4}$$

$$dV^a - \omega^a{}_b \wedge V^b = 0\,. \tag{2.5}$$

This corresponds to their being left-invariant 1-forms on ISO(1,3).

Out of the vacuum, ω^{ab}, V^a are space–time valued 1-forms, corresponding to dynamical fields on space–time. They acquire curvature so that they become *non-left invariant*,[6] their curvatures being the Einstein–Lorentz curvature 2-form R^{ab} and the torsion 2-form $\mathring{T}^a$, defined as:

$$R^{ab} \equiv d\omega^{ab} - \omega^a_c \wedge \omega^{cb} \tag{2.6}$$

$$\mathring{T}^a \equiv dV^a - \omega^a{}_b \wedge V^b = \mathcal{D}V^a\,, \tag{2.7}$$

where we denoted by $\mathcal{D}V^a = dV^a - \omega^{ab} \wedge V_b$ the Lorentz-covariant differential of the vielbein. For a formal definition of the notions above, see Appendix B. A more detailed discussion of (non) left-invariant forms in gravity and supergravity is given in Section 3, while treating supergravity.

The Einstein–Cartan Action is given, in terms of the above fields, by:

$$\mathcal{A}[\omega^{ab}, V^a] = \frac{1}{4\kappa^2}\int_{\mathcal{M}_4} R^{ab} \wedge V^c \wedge V^d \epsilon_{abcd}\,, \tag{2.8}$$

where $\kappa = \sqrt{8\pi\,\mathcal{G}}$, $\mathcal{G}$ being the gravitational constant.[7] The integration is performed on the base space $\mathcal{M}_4$ of the principal bundle $[\mathcal{M}_4,\ H]$, which is identified with the physical space–time.

Let us remind some of the properties of the Einstein–Cartan Lagrangian 4-form, that is of the integrand of (2.8):

- Being written in terms of differential forms, it is completely geometrical and therefore invariant under general coordinate transformations (space–time diffeomorphisms).
- It is invariant under Lorentz gauge transformations, but *non-invariant* under gauge translations.
- Expanding the two form R^{ab} along a basis of 2-forms on $\mathcal{M}_4$, that is:

$$R^{ab} = R^{ab}{}_{cd} V^c \wedge V^d = R^{ab}{}_{\mu\nu} dx^\mu \wedge dx^\nu\,, \tag{2.9}$$

  one easily recovers the usual form of the Einstein–Cartan Lagrangian. Indeed, we can then rewrite the Lagrangian 4-form in (2.8) as:

[6] Because of this, the approach we are going to describe is sometimes named also *(soft)* group manifold approach.

[7] In the following, we will often adopt natural units, where $\kappa = \hbar = c = 1$.

$$\begin{aligned} R^{ab}\wedge V^c \wedge V^d\epsilon_{abcd} &= R^{ab}{}_{ij}V^iV^jV^cV^d\epsilon_{abcd} = \\ &= R^{ab}{}_{ij}V^i_\mu V^j_\nu V^c_\rho V^d_\sigma\, d^4x\,\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd} = \\ &= -4R^{ij}{}_{ij}\det V d^4x. \end{aligned} \tag{2.10}$$

If we denote world-indices by Greek letters, we have

$$R^{ij}{}_{ij} \equiv R^{\mu\nu}{}_{\mu\nu} = \mathcal{R}, \tag{2.11}$$

where $\mathcal{R}$ is the scalar curvature and $\det(V) = \sqrt{-g}$ is the square root of the metric determinant ($g = \det(g_{\mu\nu})$). Hence we get:

$$\int_{\mathcal{M}_4} R^{ab} \wedge V^c \wedge V^d\epsilon_{abcd} = -4\int_{\mathcal{M}_4} \mathcal{R}\sqrt{-g}\,dx^4\,. \tag{2.12}$$

Let us now observe that the formal equivalence between the Cartan and Einstein–Cartan formulations just shown does not mean that they are completely equivalent.

First of all, the Cartan formalism in terms of the vierbein 1-form, exhibiting explicit gauge invariance under Lorentz transformations, makes it possible to introduce spinors in the General Relativity framework, contrary to what happens in the usual formalism. Indeed in the world-index setting, tensors transform under $\mathrm{GL}(4,\mathbb{R})$, while spinors are in a representation of $Spin(4) \simeq \mathrm{SL}(2,\mathbb{C})$, which is the double covering of the Lorentz group $\mathrm{SO}(1,3)$ and therefore they can be naturally coupled in a formalism when Lorentz $\mathrm{SO}(1,3)$ covariance is present.

Furthermore, the Einstein–Cartan Lagrangian is a *first order Lagrangian*, that is the gauge fields ω^{ab}, V^a, being members of the same Adjoint multiplet of the Poincaré group, are off-shell independent, as it is natural in a geometric Action like (2.8). By *geometric*, we mean that it is built only in terms of differential forms, their exterior differentials, and wedge products of them.

The Einstein–Cartan Action is then the formulation of gravity where the symmetry structure of the theory, that is Poincaré group, is fully manifest and linearly realized. In this line of thought, we could look for the possible generalizations of the pure gravity Lagrangian. This can be investigated with a scaling argument, referring to the physical scale dimensions of the fields appearing in the Action, and comparing then the scale dimension of the possible extra contributions with that of the Einstein–Cartan term which, in natural units, scales as $[L^2]$. The length scale of the fields and curvatures can be immediately obtained from the Maurer–Cartan equations: as dx^μ has scale $[L^1\ ]$, then the vierbein and the torsion 2-form (2.7) must scale as $[L^1]$ in lengths units, while the connection ω^{ab} and the Riemann curvature (2.6) must scale as $[L^0]$. We then see that the Lagrangian in (2.8) scales as $[L^2]$ while products of curvatures $R^A = (R^{ab}, \mathring{T}^a)$ would have a different scaling[8] and should therefore be omitted, unless we allow some dimensional constants to enter the Lagrangian. In fact, dimensional constants, such as mass terms, naturally appear when gravity theories are coupled to matter. For pure theories described in terms of massless fields only, pure gravity being the simplest case, a dimensional constant of dimensions mass squared is allowed, the cosmological constant

[8] In principle, a 4-form term like $\mathring{T}^a \wedge \mathring{T}_a$ would have the correct length scale. However, such term would have the opposite parity with respect to the Einstein–Cartan one. We will not investigate further this case.

$\Lambda \sim [L^{-2}]$. In the Einstein–Cartan approach, this term can be included by adding to the Einstein–Cartan Lagrangian 4-form the term $\frac{1}{3}\,\Lambda\epsilon_{abcd}V^a\,V^b\,V^c\,V^d$. This gives rise to an Einstein Lagrangian with a cosmological term.[9]

Let us now write down the equations of motion derived from the action (2.8). Varying the action with respect to ω^{ab} and V^d we find, respectively:

$$\frac{\delta\mathcal{A}}{\delta\omega^{ab}} = 0 \;:\quad \mathring{T}^c \wedge V^d\epsilon_{abcd} = 0, \tag{2.13}$$

$$\frac{\delta\mathcal{A}}{\delta V^d} = 0 \;:\quad R^{ab} \wedge V^c\epsilon_{abcd} = 0\,. \tag{2.14}$$

Eq. (2.13), after expansion of the torsion 2-form along the vierbein:

$$\mathring{T}^c = \mathring{T}^c{}_{\ell m}V^\ell \wedge V^m \tag{2.15}$$

reads:

$$\mathring{T}^c{}_{\ell m}V^\ell \wedge V^m \wedge V^d\epsilon_{abcd} = 0 \tag{2.16}$$

and, writing the 3-form $V^\ell \wedge V^m \wedge V^d$ as

$$V^\ell \wedge V^m \wedge V^d = \epsilon^{\ell m d p}\,\Omega^{(3)}_p\,, \tag{2.17}$$

where $\Omega^{(3)}_p$ is a three-dimensional hypersurface element of space–time, gives:

$$\mathring{T}^c{}_{\ell m}\epsilon^{\ell m d p}\epsilon_{abcd}\,\Omega^{(3)}_p = 0\,, \tag{2.18}$$

that is:

$$\left(\mathring{T}^p{}_{ab} + 2\mathring{T}^c{}_{c[a}\delta^p_{b]}\right)\Omega^{(3)}_p = 0 \tag{2.19}$$

whose solution, since $\Omega^{(3)}_p \neq 0$, is $T^a{}_{bc} = 0$, that is $T^a = 0$. Then, the vanishing of the torsion, which allows to write the spin connection 1-form in terms of the vierbein V^a_μ and its derivatives, is a consequence of the variational principle.

With an analogous computation, from Eq. (2.14) one finds:

$$R^{ab} \wedge V^c\epsilon_{abcd} = 0 \quad\Rightarrow\quad R^{ab}{}_{\ell m}V^\ell \wedge V^m \wedge V^c\epsilon_{abcd} = 0 \tag{2.20}$$

that is, using again (2.17):

$$-6R^{ab}{}_{\ell m}\delta^{\ell m p}_{abd} = 0\,, \tag{2.21}$$

which can be rewritten, in terms of the Ricci tensor $\mathcal{R}^a{}_b \equiv R^{ac}{}_{bc}$ and of the Ricci scalar $\mathcal{R} \equiv \mathcal{R}^a{}_a$, as:

$$\mathcal{R}^a{}_b - \frac{1}{2}\delta^a_b\,\mathcal{R} = 0 \tag{2.22}$$

that is like the Einstein equations in the absence of matter sources.

It is important to stress that, besides the obvious diffeomorphism invariance, which is implicit since the Einstein–Cartan Lagrangian is coordinate-independent, being written

[9] This kind of extension, however, can be easily shown to be equivalent to starting with the group manifold of a (anti) de Sitter group instead of the Poincaré group and will not change anything in the mechanisms we are going to discuss both for gravity as for supergravity. Indeed we may note that the Poincaré group ISO(1, 3) is an Inonü-Wigner contraction of the SO(2, 3) group.

in terms of differential forms, the Lagrangian is invariant under the fiber group SO(1, 3), but not under the full Poincaré group, which is however the structure group, all the fields being valued in the co-Adjoint representation of the Poincaré group. This follows from the fact that the Lagrangian includes the tensor ϵ_{abcd} which is a Lorentz-invariant but not Poincaré-invariant tensor. This property can be easily checked by considering an infinitesimal Poincaré transformation on the gauge fields $\mu^A \equiv (\omega^{ab}, V^a)$, where $A = ([ab], a) = 1, \dots, 10$ labels the co-Adjoint representation of the Poincaré group. Defining $\epsilon^A = (\epsilon^{ab}, \epsilon^a)$, being ϵ^{ab} and ϵ^a the parameters of the infinitesimal Lorentz and translation gauge transformations, respectively, we have:

$$\delta^{(gauge)}\mu^A = (\nabla\epsilon)^A \,, \tag{2.23}$$

where we denoted by ∇ the Poincaré gauge covariant differential. Decomposing the co-Adjoint index A as indices of the Lorentz subgroup, from (2.23) it follows

$$\begin{aligned}\delta^{(gauge)}\omega^{ab} &= \mathcal{D}\epsilon^{ab},\\ \delta^{(gauge)}V^a &= \mathcal{D}\epsilon^a + \epsilon^{ab}V_b\,,\end{aligned}$$

where, we recall, $\mathcal{D} = d - \omega$ denotes the Lorentz covariant differential.

It is then easy to see that the Lagrangian (2.8) and the equations of motion are invariant under gauge Lorentz transformations, but are not invariant under a gauge translation. Indeed, performing an infinitesimal gauge transformation on the Einstein–Cartan action (2.8), we have, up to total derivative:

$$\begin{aligned}\delta^{(gauge)}\int R^{ab}\wedge V^c\wedge V^d\epsilon_{abcd} &= 2\int R^{ab}\wedge\mathcal{D}\epsilon^c\wedge V^d\epsilon_{abcd}\\ &= -2\int \epsilon^c R^{ab}\wedge\mathring{T}^d\epsilon_{abcd}\neq 0\,,\end{aligned} \tag{2.24}$$

where we used the relation

$$\delta^{(gauge)}R^{ab} = \mathcal{D}^2\epsilon^{ab} = 2R^a{}_c\epsilon^{cb}\,,$$

and we integrated by parts to get the last expression, which is not vanishing off-shell (we recall that the condition $\mathring{T}^a = 0$ is found as an equation of motion of (2.8)).

3. D=4 supergravity in the geometric approach

In this Section we will present the Ne'eman–Regge Group Manifold Approach to minimal supergravity in $D = 4$ space–time dimensions, introduced in [49] and further elaborated and applied in [18].

To this aim, we will first reconsider the Einstein–Cartan geometric description of gravity, discussed in Section 2, pointing out that it could be reformulated as a theory where the 1-form fields are defined on the "soft-group manifold" $\tilde{G}$, locally equivalent to the structure group-manifold $G = \mathrm{ISO}(1, 3)$.

Let us observe that, referring to the D=4 pure gravity theory, the full set of "generalized vierbeins" on G is given by the ten 1-forms $\mu^A = (V^a, \omega^{ab})$. They span the cotangent space of G, whose directions can be parametrized by the coordinates x^μ ($\mu, \nu, \dots = 0, 1, 2, 3$ being general coordinate indices) associated to the action of the generators P_a, and by the coordinates $y^{\mu\nu} = -y^{\nu\mu}$ associated to the one of the generators J_{ab}.

A useful observation by Ne'eman and Regge is that the domain of the 1-form fields μ^A can be safely enlarged to be the full group manifold G. The consistency of this new point of view relies on the special form of the Einstein–Cartan Lagrangian, which is shared by its extension to supergravity: As discussed in Section 2, it is indeed a *geometric* Lagrangian defined on a principal fiber bundle whose fiber is the Lorentz group and, as such, the dependence on the Lorentz parameters is *factorized*. The *factorization* is reflected in the fact that the *curvatures* R^A, which are a coadjoint multiplet of the structure group G, can in principle be expanded on a basis of 2-forms on the cotangent space to G:

$$R^A = R^A{}_{BC}\,\mu^B \wedge \mu^C = R^A{}_{bc} V^b \wedge V^c + R^A{}_{bc\,C}\,\omega^{bc} \wedge \mu^C$$

but their components in the directions of the Lorentz fiber are zero:

$$R^A{}_{ab\,C} = 0\,, \tag{3.1}$$

so that they are fully described by their parametrization on space–time:

$$R^A = R^A{}_{ab} V^a \wedge V^b\,.$$

In an analogous way, supergravity can be constructed as an extension of gravity defined on the supergroup manifold $G = \overline{\mathrm{OSp}(1|4)}$,[10] whose graded algebra is given in terms of the generators $T_A = (J_{ab}, P_a, Q_\alpha)$ as:

$$\begin{aligned}
&[J_{ab}, J_{cd}] = -2\,\eta_{a[c} J_{d]b} + 2\,\eta_{b[c} J_{d]a}\,, \quad [J_{ab}, P_c] = -2\,P_{[a}\eta_{b]c}\,, \quad [P_a, P_b] = 0\,,\\
&[J_{ab}, Q_\alpha] = \frac{1}{2}(\gamma_{ab})_\alpha{}^\beta\,Q_\beta\,, \quad [P_a, Q_\alpha] = 0\\
&\{Q_\alpha, Q_\beta\} = -i(C\gamma^a)_{\alpha\beta} P_a\,.
\end{aligned} \tag{3.2}$$

We have 10 bosonic and 4 fermionic tangent space directions in the supergroup G, and the cotangent space is spanned by the set of 1-forms $\mu^A = (\omega^{ab}, V^a, \psi^\alpha)$ where ψ^α are the 1-forms dual to the generators Q_α. In this case, the domain of the 1-form fields can be extended to be: $\mu^A = \mu^A(x^\mu, y^{\mu\nu}, \theta^\alpha)$, where θ^α are the Grassmann-odd parameters in the ψ^α directions. Therefore the 1-form fields $\mu^A = \{\omega^{ab}, V^a, \psi^\alpha\}$, and their curvatures R^A, can be thought of as superfields functions of the coordinates: $\mu^A = \mu^A(x^\mu, y^{\mu\nu}; \theta^\alpha)$. Note however that, as in the gravity case, the Lorentz group is factorized, that is it is on the fiber of a principal fiber-bundle structure. As such, the curvatures can be expanded on a basis of 2-forms in the *physical* domain, which in the supergravity theories is named *superspace*[11] and is spanned by the supervielbein $E^{\hat{a}} \equiv (V^a, \psi^\alpha)$ of the base space in the fiber bundle, with vanishing components in the ω^{ab} directions of G. We therefore

[10] The bar over OSp(1|4) means Inonü-Wigner contraction of the super Anti-de Sitter group OSp(1|4) whose bosonic subgroup is $\mathrm{Sp}(4) \simeq \mathrm{SO}(2,3)$.

[11] In the minimal $D = 4$ case *superspace* includes four Grassman-even and four Grassman-odd coordinates, and it will be denoted in the following as $\mathcal{M}_{4|4}$. In N-extended supersymmetric theories in $D = 4$, the number of Grassman-odd coordinates is extended to $4N$, so that $\mathcal{M}_{4|4} \to \mathcal{M}_{4|4N}$.

have the following parametrization:

$$R^A(x,\theta) = R^A{}_{\hat{a}\hat{b}}(x,\theta)E^{\hat{a}} \wedge E^{\hat{b}}$$
$$= R^A{}_{(2|0)ab}V^a \wedge V^b + R^A{}_{(1|1)a\alpha}V^a \wedge \psi^\alpha + R^A{}_{(0|2)\alpha\beta}\psi^\alpha \wedge \psi^\beta\,. \tag{3.3}$$

Here, we denoted by $R^A_{(p,q)}$ the components of the curvature along p bosonic vielbein V^a and q fermionic vielbein ψ. In the following, we will name as *inner* the components $R^A_{(2|0)ab}$, along the bosonic vielbein $V^a \wedge V^b$ only (and more generally, in higher dimensions, when higher forms are present, the components $R^A_{(p|0)a_1\dots a_p}$ with $p > 2$ along p bosonic vielbein only), naming instead as *outer* the components along *at least one* fermionic vielbein ψ, that is $R^A_{(1|1)a\alpha}$ and $R^A_{(0|2)\alpha\beta}$. As we will clarify in the following subsection, the role of the inner and outer components of the superspace curvatures is not symmetric. What actually happens is that the outer components of the curvature 2-forms turn out to be expressible algebraically, actually *linearly*, in terms of the inner components of the set of curvatures. As we will see in Section 3.2, this is a consequence of the geometric structure of the Lagrangian and of its field equations. This property is called *rheonomy*, and will be further discussed in Section 3.3. For the sake of brevity and simplicity we will show how this happens in the simple example of pure $\mathcal{N} = 1$, $D = 4$ supergravity . However, the relevant results hold exactly in the same way for any supergravity theory, pure or matter coupled, in any dimension $4 \le D \le 11$ and for any number $1 \le \mathcal{N} \le 8$ of supersymmetry generators in the Lie superalgebra.

Let us now discuss how to formulate a general gravity or supergravity Lagrangian in the *geometric* approach. It must respect the following requirements: *The Lagrangian should be constructed using only wedge products of p-forms and their exterior differential, d, satisfying:* $d^2 = 0$. Moreover, we require *the Hodge duality operator to be excluded from the construction of the Lagrangian*. This second requirement will be explained in a moment.[12]

The supergravity Action in a superspace with D-space–time dimensions is then obtained by integrating the Lagrangian D-form on a D-dimensional bosonic hypersurface $\mathcal{M}_D$, immersed in superspace. This in turn requires the introduction of appropriate *embedding functions*, which should then be included in the set of fields in the Action integral, resulting in a theory containing extraneous fields devoid of any physical meaning. However, using a geometric Lagrangian in superspace, this problem is automatically overcome. Indeed, being geometric, the Lagrangian is invariant under diffeomorphisms in superspace,[13] so that any variation of the embedding functions can be compensated by a diffeomorphism. This in turn implies, thinking of infinitesimal diffeomorphisms from a passive point of view, that *any surface of integration works equally well*. Therefore, the equations of motion are valid in the full superspace, since, given any hypersurface, all other hypersurfaces in superspace can be reached by diffeomorphisms. This clarifies why use of the Hodge operator should be avoided. Indeed, this condition follows from geometricity, because use of the Hodge duality operator implies choosing a given metric

[12] Note that the Cartan–Einstein Lagrangian of Section 2, and the $D = 4$, $N = 1$ Lagrangian that will be discussed in the present Section, both satisfy these requirements.

[13] We often refer as "super-diffeomorphisms", in particular, to the diffeomorphisms along the odd directions of superspace.

description (something that we would like to avoid in a geometric Lagrangian), but also because it would make problematic the extension of the field domain from $\mathcal{M}_4$ to superspace.[14]

In the following, we are going to construct explicitly the minimal, pure supergravity in $D = 4$ space–time dimensions, with structure group $G = \overline{\mathrm{OSp}(1|4)}$, and principal fiber-bundle structure $[\mathcal{M}_{4|4}, \mathrm{SO}(1,3)]$.

In this case the set of dynamical fields, defined in superspace $\mathcal{M}_{4|4}$, is given by the bosonic 1-forms ω^{ab}, V^a, but also by the fermionic 1-form vielbein ψ^α. They span the cotangent space of the structure group G, which in this case is the Super-Poincaré group. The set of curvatures R^A, which in the dynamical vacuum reduce to the Maurer–Cartan equations of G (see Appendix B), in this case turn out to be:

$$\begin{aligned}
&R^{ab} \equiv d\omega^{ab} - \omega^a{}_c \wedge \omega^{cb} \\
&T^a \equiv dV^a - \omega^a{}_b \wedge V^b - \frac{i}{2}\psi^\alpha (C\cdot\gamma^a)_{\alpha\beta} \wedge \psi^\beta = \mathcal{D}V^a - \frac{i}{2}\bar\psi\gamma^a \wedge \psi \\
&\rho^\alpha \equiv d\psi^\alpha - \frac{1}{4}(\gamma_{ab})^\alpha{}_\beta\, \omega^{ab} \wedge \psi^\beta = \mathcal{D}\psi^\alpha\,,
\end{aligned} \tag{3.4}$$

where $\mathcal{D}$ denotes the Lorentz covariant derivatives, acting differently on Lorentz vectors and Lorentz spinors. Here, the 1-form gravitino ψ is a Majorana spinor, $\bar\psi \equiv \psi^t C$ being its adjoint spinor, C is the charge-conjugation matrix and γ^a are the γ-matrices satisfying the Clifford algebra $\gamma_a\gamma_b + \gamma_b\gamma_a = 2\eta_{ab}$, see Appendix A.[15,16]

Consistency requires them to satisfy the following *Bianchi identities in superspace*:

$$\begin{aligned}
&\mathcal{D}R^{ab} = 0 \\
&\mathcal{D}T^a + R^a{}_b \wedge V^b - i\bar\psi\gamma^a \wedge \rho = 0 \\
&\mathcal{D}\rho^\alpha + \frac{1}{4}(\gamma_{ab})^\alpha{}_\beta\, R^{ab} \wedge \psi^\beta = 0\,,
\end{aligned} \tag{3.5}$$

[14] Actually, working with geometric lagrangians it is possible to formulate a *generalized action principle* where the Lagrangian is integrated on a submanifold of the full group manifold G. Factorization of coordinates belonging to gauge subgroups of G can be proven to hold as the equations of motion obtained from the variational principal imply that the curvatures are horizontal in the direction of the Lorentz generators, so that G actually becomes endowed with a fiber bundle structure. This horizontality property is not spoiled by the presence of supersymmetry. For supergroups, where the notion of superspace as base space of the fiber bundle appears, the same procedure based on the extended action principle allows also to understand why supersymmetry is not a gauge symmetry, since it shows that the field-strengths (curvatures) in superspace are not horizontal in the direction of the supersymmetry generators of G, but can instead be expressed linearly in terms of the space–time components of the field-strengths, according to the principle of *rheonomy* (see Section 3.3).

[15] To be clear, for example, $\bar\psi\gamma^a\psi = \psi^\alpha(C\gamma^a)_{\alpha\beta}\psi^\beta$, where C behaves as the metric of the spinor space, raising and lowering spinor indices.

[16] We remark that the torsion supercurvature, that we name T^a, differs from the torsion 2-form of gravity on space–time discussed in the previous Section, since it contains a bilinear current in the ψ fields. When discussing supergravity, to avoid confusion, we will reserve the symbol T^a to the supertorsion in (3.4), referring to the purely bosonic torsion of the Riemannian geometry, that is the Lorentz-covariant derivative of the vielbein, as $\mathring{T}^a \equiv \mathcal{D}V^a$.

All the terms in the definition of the curvatures and in the Bianchi identities scale homogeneously, since ω^{ab}, V^a, ψ^α and their curvatures have length scaling $[L^0]$, $[L^1]$ and $[L^{1/2}]$, respectively.

We emphasize that, as it emerges from the above discussion, in the geometric setting only the Lorentz subgroup of the (super-)Poincaré group turns out to be an actual gauge symmetry of the theory, the spin connection ω^{ab} being its gauge connection. On the other hand, the vielbein 1-form V^a and the gravitino 1-form ψ^α transform respectively as a vector and as a spinor under Lorentz transformations. We will call this property *horizontality* condition. This corresponds to the fact that, as stated in (3.3), the curvatures R^A have non-vanishing components not only along the directions dual to the bosonic vielbein V^a, analogously to what we have seen in Section 2 (see in particular the discussion after Eq. (2.22)), *but also along the fermionic vielbein* ψ^α. This reflects a general property of supersymmetric Lagrangians, when realized in superspace: Supersymmetry invariance is not a *gauge invariance* of the Lagrangian, similarly to what happens with the translations. The Lagrangian indeed is not *gauge*-invariant under the full algebra of commutators of $\overline{\mathrm{OSp}(1|4)}$, only the Lorentz subalgebra $\mathrm{SO}(1,3)$ being realized as a gauge symmetry. As we are going to see, it is however invariant under *superdiffeomorphisms* in superspace. This is analogous to what happens at the gravity level in the Einstein–Cartan formalism, where the theory is not invariant under *gauge-translations*, but under *diffeomorphisms* on space–time.

The new feature with respect to pure gravity is that the Lagrangian is invariant under local supersymmetry transformations. As we are going to show in the following, in the geometric approach the supersymmetry transformations on space–time are nothing but *diffeomorphisms in the odd directions of superspace*. This means that the geometric approach allows for a geometrical interpretation of local supersymmetry, when formulated *in superspace*, as a super-diffeomorphism, thus extending to the graded case the formulation of gravity as a theory invariant under space–time diffeomorphisms. Just as the geometric Einstein–Cartan Lagrangian is not invariant under gauge translations but instead, being geometric, is invariant under space–time diffeomorphisms, the supersymmetry invariance of the geometric Lagrangian in superspace has to be understood as an invariance under *super-diffeomorphisms in superspace*, generated by the vector fields $\epsilon^\alpha D_\alpha$, and not as a gauge invariance. To clarify this point, let us make here the following important distinction: We denote by P_a, Q_α the right-invariant (graded) generators dual to the left-invariant 1-forms of the translations and supertranslations respectively in the super-Poincaré group, while we denote by D_a, D_α their left-invariant counterpart, dual to the right-invariant 1-forms.[17] Both of them are invariant vector fields of the group manifold G (that is, symmetries of the Maurer–Cartan equations satisfied by the fields in the dynamical vacuum). However, due to the principal fiber bundle structure, out of the vacuum the vector fields D_a, D_α are not anymore invariant vectors of G. They are instead vector fields spanning the tangent space of superspace, and dual to the 1-forms V^a, ψ^α, namely:

$$D_a(V^b) = \delta^b_a; \quad D_\alpha(\psi^\beta) = \delta^\beta_\alpha; \quad D_a(\psi^\alpha) = D_\alpha(V^a) = 0\,. \tag{3.6}$$

[17] They satisfy the $\overline{\mathrm{OSp}(1|4)}$ superalgebra, with structure constants opposite to (3.2).

Correspondingly, just as diffeomorphism transformations on the fields can be expressed by *Lie derivatives* along D_a directions, superdiffeomorpism transformations in superspace, that is supersymmetry transformations, can be expressed as *Lie derivatives along odd directions* $\epsilon \equiv \epsilon^\alpha D_\alpha$ *of superspace*:

$$\delta_\epsilon \mu^A = \ell_\epsilon \mu^A \equiv d\left(\iota_\epsilon \mu^A\right) + \iota_\epsilon \left(d\mu^A\right) , \tag{3.7}$$

where we denoted by ι_ϵ the contraction of a form along the odd tangent space direction ϵ so that, in particular:

$$\iota_\epsilon \psi^\alpha = \epsilon^\alpha , \quad \iota_\epsilon V^a = 0 . \tag{3.8}$$

An alternative definition of Lie derivative, which puts into light its differences with respect to gauge transformations, is given in Appendix C.

3.1. Supersymmetry as an on-shell symmetry

We could wonder if the geometric formulation of supergravity in superspace is fully equivalent to the standard formulation of supergravity on space–time, and in particular if supersymmetry transformations on space–time are completely equivalent to *superdiffeomorphisms in superspace*. As we will see later in this Section, the answer is positive *if we require, as already mentioned (in the paragraph below Eq.* (3.3)*), that the parametrizations of the supercurvatures, Eq.* (3.4)*, should be subject to the principle of "Rheonomy"*, whose meaning and use will be clarified later, in Section 3.3. Actually, Rheonomy is an intrinsic property of all the supergravity Lagrangians formulated in the geometric approach to superspace.

There is however an important difference between diffeomorphism invariance of a geometric theory on space–time and super-diffeomorphism invariance of a geometric theory in superspace: In general, except in a few exceptional cases,[18] the supersymmetry algebra, when realized on dynamical fields, is an *on-shell* symmetry. This means that the closure of the exterior derivative operator in superspace: $d^2 = 0$, when applied on the defining fields of the theory, does not hold in general, but only on-shell, namely only if the equations of motion are satisfied. This corresponds, in the space–time description of the phenomenon, to the fact that the commutator of two supersymmetry transformations on the fields does not satisfy the Jacobi identities in general, but this property only holds on shell, namely only if the equations of motion are satisfied.

This special feature reflects the peculiarity of supersymmetry, to be such that it maps into each other bosonic and fermionic degrees of freedom (d.o.f.), so that the

[18] The exceptional cases we are referring to are the off-shell supersymmetric theories, which close supersymmetry off-shell due to the presence of a set of auxiliary fields. These fields, when added to the coadjoint supermultiplet, make the supersymmetry transformations, leaving the Lagrangian invariant, to close the supersymmetry algebra *off-shell* . This is related to the fact that the auxiliary fields allow to pair the number of off-shell degrees of freedom between boson and fermions. They are not dynamical degrees of freedom, as their equations of motion make them to vanish or to be expressed in terms of the physical fields. However, it does not seem possible to extend their introduction to theories with more than 8 supercharges nor to matter coupled supergravities (in particular when these include CPT self-conjugate matter sources, unless we extend the superspace to an infinite number of extra bosonic directions. We will not discuss further off-shell supergravity in this review.

supersymmetry representations, in general, should contain the same number of bosonic and fermionic degrees of freedom. These representations are called *supermultiplets*, and collect several fields of different spin. This clashes with the fact that, in any Lagrangian theory, the number of d.o.f. of a given field is in general different if it is counted off-shell or after imposing its equations of motion: Just as an example, spinors halve their d.o.f. on-shell, while scalar fields do not change their d.o.f. at all. Then, either we have off-shell matching of d.o.f., but then, in the general case, the classical trajectory cannot be supersymmetric, or we require on-shell matching. Since we cannot give up the validity of the theory on-shell,[19] the supergravity actions are constructed with the proviso that *consistency of supersymmetry, that is closure of superdiffeomorphism invariance, only holds on-shell.* This in turn has the consequence that the Bianchi identities (3.5), and all their extensions to more general supergravity theories, become identities only after imposing the field equations.

As we will see, this special feature will turn out to be a resource of the supergravity theories, allowing to fully characterize all the properties of the classical theory even in the absence of a Lagrangian description.

We stress again that these properties, that will be explicitly shown in the following for the particular case of minimal 4D pure supergravity, are shared by every supergravity theory in any possible number of dimensions and supersymmetry.

3.2. N=1, D=4 action in the geometric approach

Let us now proceed here in finding the supergravity Action, and the supersymmetry transformation laws leaving it invariant. The Action will be given by the integral, over a four dimensional bosonic submanifold of superspace, of a 4-form Lagrangian.

The starting point is the set of curvatures defined in Eq. (3.4), satisfying on-shell the Bianchi identities (3.5). To write down the Lagrangian, we require it to be *geometric*. Let us list here what it amounts to:

1. It must be constructed using only differential forms, wedge products among them, and the d exterior differential;
2. It must not contain the Hodge duality operator. This issue will be clarified in Section 3.2.2.

Other requirements of physical nature can be added which make easier, in more complicated cases, the search of the final form of the Lagrangian:

3. First of all, since the Einstein term, which must be always present, in natural units scales as $[L^2]$, ($[L^{D-2}]$ in D dimensions), all the terms in the Lagrangian must scale in the same way.
4. Moreover, we require the ground state of the theory, namely, the state where all the curvatures R^A vanish, to be a particular solution of the equations of motion. In this configuration, that physically corresponds to the *dynamical vacuum* of the theory, all the 1-form fields are left-invariant 1-forms of the structure group G. This last requirement is useful in constructing matter coupled or higher dimensional Lagrangians where many fermionic interaction terms are present.

[19] Supergravity is an effective field theory, extending (classical) General Relativity.

5. Finally, all the terms in the supergravity Lagrangian should have the same parity as the Einstein–Cartan term, if we want a parity preserving theory.

$N = 1$, $D = 4$ supergravity in absence of matter coupling is particularly simple, and for this theory one easily sees that the only possible term that we can add to the Einstein–Cartan term, fulfilling the requirement of being geometric, together with the other above requirements, is the Rarita–Schwinger kinetic term. Written in terms of differential forms, it reads:

$$\bar{\psi}\gamma^5\gamma_a\mathcal{D}\psi\, V^a = -i\,\bar{\psi}_\mu\gamma^{\mu\nu\rho}\mathcal{D}_\nu\psi_\rho\,\sqrt{g}\,d^4x\,. \tag{3.9}$$

Therefore the Action of $N = 1$, $D = 4$ supergravity must have the following form:

$$\mathcal{A}^{\mathcal{N}=1}_{D=4} = \frac{1}{4\kappa^2}\int_{\mathcal{M}_4\subset\mathcal{M}^{[4|4]}}\left[R^{ab}\,V^c\,V^d\epsilon_{abcd} + \alpha\overline{\psi}\gamma_5\gamma_a\mathcal{D}\psi\,V^a\right] \tag{3.10}$$

where the coefficient α between the Einstein and the Rarita Schwinger terms is related to the normalization of the gravitino 1-form ψ and will be fixed in a moment.

Note that the hypersurface on which the Lagrangian is integrated, $\mathcal{M}_4 \subset \mathcal{M}^{[4|4]}$, where the $N = 1$, $D = 4$ *superspace* $\mathcal{M}^{[4|4]}$ is the base manifold of the principal fiber bundle $[\mathcal{M}^{[4|4]}, \mathrm{SO}(1,3)]$ with fiber $\mathrm{SO}(1,3)$, can be naturally identified with physical space–time. However, as emphasized at the beginning of the present Section, any possible bosonic surface $\mathcal{M}_4$ can be equivalently chosen. Indeed, taking advantage of the fact that our Lagrangian is *geometric*, we know that the variational principle gives equations independent of the choice of the four dimensional hypersurface $\mathcal{M}_4$. Note that the fields (1-forms) (ω^{ab}, V^a, ψ) will depend on all the four bosonic and four fermionic (Grassmann) coordinates (x^μ, θ^α) parametrizing the *superspace*.

The equations of motion obtained by varying ω^{ab}, V^a and ψ, and valid on the full superspace, are, respectively:

$$\frac{\delta\mathcal{A}}{\delta\omega^{ab}} = 0\ :\quad \epsilon_{abcd}\mathcal{D}V^c\wedge V^d + \frac{\alpha}{4}\overline{\psi}\gamma_5\gamma_c\gamma_{ab}\psi\,V^c = 0\,,\ \text{that is:}$$

$$\epsilon_{abcd}\left(\mathcal{D}V^a + \frac{\mathrm{i}\alpha}{8}\,\bar{\psi}\,\gamma^a\,\psi\right)\wedge V^d = 0, \tag{3.11}$$

$$\frac{\delta\mathcal{A}}{\delta V^a} = 0\ :\quad 2R^{ab}\wedge V^c\epsilon_{abcd} - \alpha\overline{\psi}\wedge\gamma^5\gamma_d\,\rho = 0, \tag{3.12}$$

$$\frac{\delta\mathcal{A}}{\delta\overline{\psi}} = 0\ :\quad 2\gamma^5\gamma_a\,\rho\wedge V^a - \gamma_5\gamma_a\psi\wedge T^a = 0\,, \tag{3.13}$$

where we used the definition (3.4) to express the gravitino supercurvature as $\mathcal{D}\psi = \rho$.

As the equations of motion have to vanish identically when all the (super-)curvatures are zero (requirement 4.), we see that we must set on the left hand side of Eq. (3.11) $\alpha = 4$ in order to have the super-torsion 2-form T^a as defined in (3.4). With this value of α, Eq. (3.11) takes the form

$$T^c\wedge V^d\epsilon_{abcd} = 0 \tag{3.14}$$

and we see that when all the supercurvatures are zero, the equations of motion vanish identically.

To analyze the content of Eqs. (3.12), (3.13) and (3.14), which are 3-form equations valued on any bosonic hyperplane $\mathcal{M}_4$ immersed in of superspace, we expand the curvatures 2-forms along the basis $E^{\hat{a}} \wedge E^{\hat{b}}$, where $E^{\hat{a}} = (V^a\,, \psi^\alpha)$ ($a = (0, \dots, 3)$ and $\alpha = 1, \dots 4$), of 2-forms in the cotangent space of superspace, as in (3.3), that is:

$$T^a = T^a{}_{(2|0)bc} V^b\, V^c + T^a{}_{(1|1)c\,\alpha} \psi_\alpha\, V^c + \psi^\alpha T^a{}_{(0|2)\alpha\beta}\, \psi^\beta , \tag{3.15}$$

$$\rho^\alpha = \rho^\alpha_{(2|0)ab} V^a\, V^b + \rho^\alpha_{(1|1)a} \psi^\alpha\, V^a + \rho^\alpha_{(0|2)\beta\gamma} \psi^\beta\, \psi^\gamma , \tag{3.16}$$

$$R^{ab} = R^{ab}{}_{(2|0)cd} V^c\, V^d + \overline{\Theta}^{ab}_c \psi\, V^c + \bar{\psi} K^{ab} \psi\,. \tag{3.17}$$

In Eq. (3.15), $T^a{}_{(1|1)c\,\alpha}$ is a spinor vector and $T^a{}_{(0|2)\alpha\beta}$ a spinor matrix. In Eq. (3.17), we have kept for the outer components $R^{ab}{}_{(1|1)}$ and $R^{ab}{}_{(0|2)}$, the names $\overline{\Theta}^{ab}_c$ and K^{ab} respectively, that were attributed to them originally in the literature.

We warn the reader that, since we are now in superspace, the rigid indices cannot be traded with coordinate indices using the bosonic vierbein V^a_μ. Indeed, the full set of supervielbein is now given by $E^{\hat{a}} = (V^a, \psi^\alpha)$ and we should invert the matrix $(E^{\hat{a}}_\mu, E^{\hat{a}}_\alpha)$ to find the space–time components. For this reason, in the following, we will generally denote with a tilde the components of the supercurvatures along two bosonic vierbein, that is

$$R^A{}_{(2|0)ab} \equiv \tilde{R}^A{}_{ab}\,,$$

in order to distinguish them from the space–time projection of the full curvatures. They are commonly named in the literature as *supercovariant field strengths*.[20] However, a simpler way to find the space–time components is to project the equations on the space–time basis $dx^\mu \wedge dx^\nu$. For example, from Eq. (3.16), projecting on the space–time basis we obtain

$$\rho^\alpha_{\mu\nu} = \tilde{\rho}^\alpha{}_{ab} V^a_\mu\, V^b_\nu + \rho^\alpha_{(1|1)a} \psi^\alpha_{[\mu}\, V^a_{\nu]} + \rho^\alpha_{(0|2)\alpha\beta} \psi^\alpha_\mu\, \psi^\beta_\nu\,, \tag{3.18}$$

where the indices $\mu\nu$ are understood to be antisymmetric. We see that the tilded components of $\tilde{\rho}_{\mu\nu}$ differ from the real space–time components of $\rho_{\mu\nu}$ by terms in the gravitino fields, namely *outer terms*. However, as we will see in a moment, as far as the T^a and ρ curvatures of this theory are concerned, we can safely convert rigid Lorentz indices into world indices using the matrix V^a_μ, since in the present case ($N = 1$ supergravity in $D = 4$) they do not have *outer* components $(V \wedge \psi)$ and $(\psi \wedge \psi)$. Then, for the components of the aforementioned curvature 2-forms along $V^a \wedge V^b$ we can neglect the tilde symbol. Instead, as we now show and further discuss in the sequel, the distinction is relevant for the Riemann curvature in superspace, because the space-time projection of the Lorentz curvature does not coincide with its supercovariant field-strength.

Let us first work out Eq. (3.14). In (3.15), the component $T^a{}_{(1|1)c}$ is a spinor, with $\overline{T}^a{}_{(1|1)c}$ its adjoint spinor, while the $T^a{}_{(0|2)}$ component is a, possibly field-dependent, linear combination of gamma matrices.

Inspecting the 3-form Eq. (3.14), where all the components along independent elements of the basis of 3-forms in superspace should vanish independently, one easily

[20] The name *supercovariant* means that their supersymmetry transformation law does not contain derivatives of the supersymmetry parameter ϵ^α.

concludes that the components of $T^a_{(1,1)}$ must be zero,[21] while, due to the Fierz identity (A.7), $T^a_{(0,2)}$ could in principle be different from zero and take the value $T^a_{(0,2)} = \beta\gamma^a$, with β a free parameter. However, given the definition of the supertorsion, in (3.3), such contribution would only change the normalization of the gravitino, so that putting $T^a_{(0,2)} = 0$ just amounts to fixing such normalization.

In summary, we get that the supertorsion T^a has the following parametrization on a basis of 2-forms in superspace:

$$T^a = \tilde{T}^a{}_{bc} V^b \wedge V^c = T^a{}_{bc} V^b \wedge V^c \,, \tag{3.19}$$

precisely as the torsion $\mathring{T}^a$ of Einstein–Cartan gravity, discussed in Section 2 (see in particular the discussion after Eq. (2.7)). It follows that Eq. (3.14) has exactly the same form, and therefore the same solutions, as in pure gravity case, provided we replace the bosonic torsion $\mathring{T}^a$ with the supertorsion T^a defined in (3.4). In this way, with the same computations as those made for pure gravity in Section 2, one easily obtains the vanishing of the $\tilde{T}^a{}_{bc} = 0$ components and therefore that the whole super-torsion 2-form is zero:

$$T^a = 0 \,.$$

For the bosonic case, this equation is solved by expressing the components of the spin connection as functions of the vielbein and its space–time derivatives. Note, however, that in the supergravity case, solving for the spin connection ω^{ab}_μ with the usual procedure gives a spin connection that depends not only on the bosonic vielbein and their derivatives but also on gravitino bilinears (see e.g. Ref. [18]).

We can now apply the same procedure to solve Eqs. (3.12) and (3.13), by expanding the curvatures ρ and R^{ab} along a complete basis of 2-forms in superspace, according with (3.3). As $T^a = 0$, Eq. (3.13) takes the form

$$2\gamma^5 \gamma_a \, \rho \wedge V^a = 0 \,, \tag{3.20}$$

which is the superspace expression of the Rarita–Schwinger equation. We should now use in Eq. (3.20) the expansion of $\rho \equiv \mathcal{D}\psi$ on a basis of 2-forms in superspace:

$$\rho^\alpha = \rho^\alpha_{(2|0)ab} V^a \, V^b + \rho^\alpha_{(1|1)a} \psi^\alpha \, V^a + \rho^\alpha_{(0|2)\beta\gamma} \psi^\beta \, \psi^\gamma \,, \tag{3.21}$$

where, for the sake of clarity, we have made the spinor index explicit. Then, from Eq. (3.20) one easily realizes that $\rho^\alpha_{(1|1)a} = \rho^\alpha_{(0|2)\alpha\beta} = 0$, so that the 2-form ρ has only components $\rho_{(2,0)ab} \equiv \tilde{\rho}_{ab} = \rho_{ab}$, on the cotangent space of $\mathcal{M}_4$, namely

$$\rho = \rho_{ab} V^a \, V^b = \rho_{\mu\nu} \, dx^\mu \wedge dx^\nu \,. \tag{3.22}$$

If we now project Eq. (3.20) on space–time, it gives the space–time gravitino equation of motion:

$$\gamma^5 \gamma_a \, \rho \wedge V^a = 0 \quad \Rightarrow \quad \gamma^5 \gamma_a \, \rho_{bc} V^a_\sigma V^b_\mu V^c_\nu \sqrt{g} d^3x \epsilon^{\mu\nu\sigma\lambda} = 0 \tag{3.23}$$

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21 The same conclusion can be reached by observing that a spinor scaling as $[L^{-\frac{1}{2}}]$ does not exist in the theory.

that is

$$\epsilon^{\mu\nu\sigma\lambda}\gamma_5\gamma_\sigma\rho_{\mu\nu} = 0\,, \text{ or, equivalently: } \quad \epsilon^{\mu\nu\sigma\lambda}\gamma_5\gamma_\sigma\mathcal{D}_\mu\psi_\nu = 0\,. \tag{3.24}$$

This is the Rarita–Schwinger equation, in its standard formulation.

Finally, by expanding R^{ab} as in (3.17), from Eq. (3.12) we find[22]

$$\overline{\Theta}^{ab}_{c} = -\epsilon^{abrs}\bar{\rho}_{rs}\gamma_5\gamma_c - \delta^{[a}_{c}\epsilon^{b]mst}\bar{\rho}_{st}\gamma_5\gamma_m \tag{3.25}$$

while $K^{ab} = 0$, as it can be also easily checked by observing that no gamma-matrix-valued object (field or parameter) scaling, in natural units, as $[L^{-1}]$, exists in the pure theory.

In conclusion, the solution of the equations of motion (3.11), (3.12), (3.13) for the *outer* and *inner* projections of the curvature multiplet gives:

$$R^{ab} = \tilde{R}^{ab}_{cd}V^c\,V^d + \overline{\Theta}^{ab}_{c}\,\psi\,V^c, \tag{3.26}$$

$$T^a = 0, \tag{3.27}$$

$$\rho = \rho_{ab}V^a\,V^b\,, \tag{3.28}$$

where $\overline{\Theta}^{ab}_{c}$ is given by (3.25), in which the non-vanishing *outer component* of the Lorentz curvature, $R^{ab}_{(1|1)a}$, is written in terms of the *inner component* of the gravitino curvature, $\tilde{\rho}_{ab}$.

Let us emphasize here some peculiarities of the result found above for the specific model of minimal supergravity in $D = 4$, but which express general features of supergravity in the geometric approach:

- The above parametrizations, Eqs. (3.26)–(3.28), of the supercurvatures defined in (3.4), have been obtained as solutions of the field equations, that is they *hold on-shell*. As we will see later in Section 3.3, this is related to the fact, already discussed in Section 3.1, that *supersymmetry is an on-shell symmetry*.
  As a consequence, the following general rule will hold true in superspace:
  Off-shell, the supercurvatures $R^A(\mu)$ are given in terms of their definitions (3.4), while *after applying the variational principle to the Action*, that is *on-shell*, the supercurvatures will have to satisfy their parametrizations (3.3), that is in particular, for the case under study, Eqs. (3.26)–(3.28).
- As exhibited in (3.25), the non-vanishing *outer components* of the supercurvatures R^A are linearly expressed, on-shell, in terms of *inner* components of the set of supercurvatures R^A. This general property is called *rheonomy*.
  Physically, this property guarantees that the theory in superspace does not include extra on-shell degrees of freedom, besides those already present in space–time.

Finally, inserting the parametrizations (3.26), (3.27), (3.28) in the equations of motion (3.11), (3.12), (3.13), we get the components of the equations of motion along $V^a\,V^b\,V^c$, that is:

$$\tilde{R}^{ac}{}_{bc} - \frac{1}{2}\delta^a_b\tilde{R}^{cd}{}_{cd} = 0, \tag{3.29}$$

[22] To solve for $\Theta_{ab|c}$ one performs the cyclic permutations of the indices (a, b, c) and uses the same trick used in the bosonic case to compute the affine connection in terms of the metric and its derivatives.

$$\tilde{T}^a_{bc} = 0, \tag{3.30}$$

$$\epsilon^{abcd}\gamma^5\gamma_c\tilde{\rho}_{ab} = 0\,. \tag{3.31}$$

Expressing the supercovariant field-strengths in terms of the physical curvatures projected on space–time[23]:

$$R^{ab}_{\mu\nu} = \tilde{R}^{ab}{}_{cd}V^c_\mu V^d_\nu + \overline{\Theta}^{ab}_c\psi_{[\mu}V^c_{\nu]} = \tilde{R}^{ab}{}_{\mu\nu} + \overline{\Theta}^{ab}_{[\nu}\psi_{\mu]}, \tag{3.32}$$

$$T^a_{\mu\nu} = \tilde{T}^a{}_{bc}V^b_\mu V^c_\nu = \tilde{T}^a{}_{\mu\nu}, \tag{3.33}$$

$$\rho_{\mu\nu} = \tilde{\rho}_{ab}V^a_\mu V^b_\nu = \tilde{\rho}_{\mu\nu}\,, \tag{3.34}$$

we get the space–time field equations. In particular, we see that the Einstein equation of motion contains extra terms linear in the inner components $\rho_{ab}\equiv\rho_{\mu\nu}V^\mu_a V^\nu_b$. These terms give rise to the energy–momentum tensor of the gravitino field ψ_μ. Furthermore, we remark that Eq. (3.33) implies, in terms of the torsion 2-form $\mathring{T}^a$: $\mathring{T}^a = \frac{i}{2}\bar{\psi}\gamma^a\psi$.

3.2.1. Supersymmetry invariance of the action.

Let us now check the supersymmetry invariance of the Action (3.10), that we rewrite:

$$\mathcal{A}^{\mathcal{N}=1}_{D=4} = \frac{1}{4\kappa^2}\int_{\mathcal{M}_4\subset\mathcal{M}^{[4|4]}}\left[R^{ab}V^cV^d\epsilon_{abcd} + 4\overline{\psi}\gamma_5\gamma_a\mathcal{D}\psi\, V^a\right]. \tag{3.35}$$

In the geometric approach, it is expressed by the vanishing of the Lie derivative of the Lagrangian 4-form for infinitesimal diffeomorphisms in the fermionic directions of superspace. Using the Lie derivative with a tangent vector $\epsilon = \epsilon^\alpha D_\alpha$, where D_α is the tangent vector dual to ψ^β, introduced in (3.7), supersymmetry invariance requires:

$$\delta_\epsilon\mathcal{L} \equiv \ell_\epsilon\mathcal{L} = \iota_\epsilon d\mathcal{L} + d(\iota_\epsilon\mathcal{L}) = 0\,, \tag{3.36}$$

up to boundary terms. If we impose appropriate boundary conditions on $\mathcal{M}_4$, assuming that the fields vanish at radial infinity so that any exact form does not contribute to the action, then we may discard the total derivative term $d(\iota_\epsilon\mathcal{L})$ and other possible exact 4-forms on the right-hand side.[24] Note that here $d\mathcal{L}$ is not automatically zero, since the 4-form $\mathcal{L}$ is not a top form in the (4+4)-dimensional superspace. Taking into account the supercurvature definitions (3.4) and their Bianchi identities (3.5), a simple computation gives

$$\begin{aligned} d\mathcal{L} = \mathcal{D}\mathcal{L} = \frac{1}{4\kappa^2}\Big[&2R^{ab}\left(T^c + \frac{\mathrm{i}}{2}\bar{\psi}\gamma^c\psi\right)V^d\epsilon_{abcd} + 4\bar{\rho}\gamma^5\gamma_a\rho V^a + \\ &+ \bar{\psi}\gamma^5\gamma_c\gamma_{ab}\psi R^{ab}V^c - 4\bar{\psi}\gamma^5\gamma_a\rho\left(T^a + \frac{\mathrm{i}}{2}\bar{\psi}\gamma^a\psi\right)\Big]. \end{aligned} \tag{3.37}$$

Using the Fierz identity (A.7) (see also Ref. [18]) and performing some gamma matrix manipulations, one is left with:

$$d\mathcal{L} = R^{ab}T^cV^d\epsilon_{abcd} + \bar{\rho}\gamma^5\gamma_a\rho V^a - 4\bar{\psi}\gamma^5\gamma_a\rho T^a\,. \tag{3.38}$$

[23] We obtain them by projecting Eqs. (3.26), (3.27), (3.28) on the space–time 2-form differentials $dx^\mu\wedge dx^\nu$.

[24] The analysis can be extended also to theories with non-trivial boundary conditions, see [4,5,10,26], even if we will not discuss here such cases.

Finally, contracting with the tangent vector $\epsilon = \epsilon^\alpha D_\alpha$ along an odd direction of superspace, we obtain

$$\begin{aligned}\iota_\epsilon(d\mathcal{L}) &= 2(\iota_\epsilon R^{ab})\,T^c\,V^d\epsilon_{abcd} + 2R^{ab}(\iota_\epsilon T^c)\,V^d\epsilon_{abcd} + 8(\iota_\epsilon\bar\rho)\gamma^5\gamma_a\rho V^a \\ &- 4\bar\epsilon\gamma^5\gamma_a\rho\,T^a - 4\bar\psi\gamma_5\gamma_a(\iota_\epsilon\rho)T^a - 4\bar\psi\gamma^5\gamma_a\rho(\iota_\epsilon T^a).\end{aligned} \tag{3.39}$$

From (3.35) we see that we can have an invariant action if

$$\iota_\epsilon(d\mathcal{L}) = d(\text{3-form})\,, \tag{3.40}$$

that is, if we require constraints on the components of the curvatures.

This is obtained if we set

$$\iota_\epsilon T^a = 0; \quad \iota_\epsilon\rho = 0 \tag{3.41}$$

and furthermore

$$2\left(\iota_\epsilon R^{ab}\right)\,V^d\,\epsilon_{abcd} - 4\bar\epsilon\gamma_5\gamma_c\rho = 0\,, \tag{3.42}$$

in which case we find $\delta_\epsilon\mathcal{L} = 0$, that is, *invariance of the Lagrangian under supersymmetry*.

We note that the requirements (3.41) are trivially satisfied if we use, for the contraction of the supercurvatures, the constraints (3.27) and (3.28), while (3.42) is satisfied using the parametrization (3.26), which gives:

$$\iota_\epsilon R^{ab} = \overline{\Theta}{}^{ab}_{c}\,\epsilon V^c\,, \tag{3.43}$$

where Θ^{ab}_c has been defined in Eq. (3.25).

We recall that the above constraints on the supercurvatures were found from the equations of motion. In other words, requiring supersymmetry invariance of the superspace Action we retrieve exactly the same constraints on the curvatures as those found from the equations of motion.

We conclude that the *supergravity Lagrangian is invariant under (local) supersymmetry transformations, when the superspace curvatures are expressed by their on-shell parametrizations*, Eqs. (3.26)–(3.28). This in turn implies that the supersymmetry transformations leaving the Lagrangian invariant do not form a closed algebra, unless one uses the equations of motion. We remark that, to obtain the result, it was crucial the use of the Bianchi identities (3.5), expressing the closure of the supercurvatures in superspace.

Let us now explicitly work out the supersymmetry transformation laws leaving the Action invariant, and check that they close the supersymmetry algebra only on-shell. We can evaluate them by applying the Lie derivative formula (3.7), to write down the superspace diffeomorphisms of the gauge fields ω^{ab}, V^a, ψ. We should use a generic tangent vector on the full fiber-bundle. This means including, besides the tangent vectors D_a and D_α on $\mathcal{M}_{4|4}$, dual to the 1-forms V^a and ψ^α, also the tangent vector D_{ab} dual to the spin-connection ω^{ab}, that is such that $D_{ab}(\omega^{cd}) = 2\delta^{cd}_{ab}$, so that the general form of the parameter is $\vec\epsilon = \frac{1}{2}\epsilon^{ab}D_{ab} + \epsilon^a D_a + \epsilon^\alpha D_\alpha$. We find:

$$\delta_\epsilon\omega^{ab} = (\nabla\epsilon)^{ab} + \epsilon^c V^d R^{ab}_{cd} + \overline{\Theta}{}^{ab}_{c}\,\psi\epsilon^c + \overline{\Theta}{}^{ab}_{c}\,\epsilon V^c, \tag{3.44}$$

$$\delta_\epsilon V^a = (\nabla\epsilon)^a, \tag{3.45}$$

$$\delta_\epsilon\psi^\alpha = (\nabla\epsilon)^\alpha + \epsilon^a\rho^\alpha_{ab}V^b. \tag{3.46}$$

Restricting ourselves to the Lie derivative along the fermionic supersymmetry parameter ϵ^α only, that is setting $\epsilon^{ab} = \epsilon^a = 0$, we have

$$\delta_\epsilon \omega^{ab} = (\nabla\epsilon)^{ab} + \overline{\Theta}^{ab}_c \epsilon V^c, \tag{3.47}$$

$$\delta_\epsilon V^a = (\nabla\epsilon)^a, \tag{3.48}$$

$$\delta_\epsilon \psi^\alpha = (\nabla\epsilon)^\alpha. \tag{3.49}$$

Here the symbol ∇ denotes the $\overline{\mathrm{OS}p(1|4)}$ *covariant derivative* of the coadjoint multiplet $\mu^A = (\omega^{ab}, V^a, \psi)$ of $\overline{\mathrm{OS}p(1|4)}$, to be distinguished from the *Lorentz* covariant derivative, $\mathcal{D}$. Explicitly we find, for the supersymmetry transformations laws of the fields on space–time:

$$\delta_\epsilon \omega^{ab}_\mu = \overline{\Theta}^{ab}_c \epsilon V^c_\mu \tag{3.50}$$

$$\delta_\epsilon V^a_\mu = -\mathrm{i}\bar{\psi}_\mu \gamma^a \epsilon, \tag{3.51}$$

$$\delta_\epsilon \psi_\mu = \mathcal{D}_\mu \epsilon\ . \tag{3.52}$$

Now we recall that the Lie derivative along tangent vectors $\tilde{T}_A$ satisfies an algebra isomorphic to the Lie algebra of the vector fields $[\tilde{T}_A, \tilde{T}_B] = \left(C^A{}_{BC} + R^A{}_{BC}\right) \tilde{T}_C$, namely

$$[\ell_{\tilde{T}_A}, \ell_{\tilde{T}_B}] = \ell_{[\tilde{T}_A, \tilde{T}_B]}, \tag{3.53}$$

if the supercurvatures $R^A{}_{BC}$ are completely general, that is, if they do not satisfy any constraint. In our case they satisfy the constraints (3.26)–(3.28) and, in general, the Lie derivative algebra, namely, the algebra of supersymmetry transformations, *cannot close off-shell*. This can be checked explicitly by considering the commutator of two supersymmetry transformations on the fields, with parameters ϵ^α_1, ϵ^α_2. In particular, the above operation on the vielbein gives:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] V^a_\mu = -i\, \mathcal{D}_\mu \left(\bar{\epsilon}_1 \gamma^a \epsilon_2\right)\ , \tag{3.54}$$

that is it reproduces the local supersymmetry algebra (3.2), while on the gravitino the calculation, after some γ-matrix manipulation gives

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \psi_\mu = -i\ (\bar{\epsilon}_1 \gamma^\nu \epsilon_2)\, \mathcal{D}_{[\mu} \psi_{\nu]} + \alpha(x)_\mu{}^\lambda \gamma_{\lambda\nu\rho} \rho^{\nu\rho}\ , \tag{3.55}$$

where

$$\alpha(x)_\mu{}^\lambda = \frac{1}{8}\left(2\bar{\epsilon}_1 \gamma^\nu \epsilon_2 A^\lambda{}_{\mu\nu} + \bar{\epsilon}_1 \gamma^{\nu\sigma} \epsilon_2 B^\lambda{}_{\mu\nu\sigma}\right)\ , \tag{3.56}$$

$A^\lambda{}_{\mu\nu}$, $B^\lambda{}_{\mu\nu\sigma}$ being some linear combinations of γ-matrices, whose precise definition, with the details of the calculation, can be found in [18], Vol. 2, page 636. Note, in particular, that Eq. (3.55) reproduces the supersymmetry algebra only after imposing the gravitino field Eq. (3.24).

Actually, requiring that the Bianchi identities on the constrained curvatures be satisfied, one finds that their components on the bosonic cotangent plane R^A_{rs} satisfy the equations of motion of the theory. It follows that the supersymmetry algebra of the transformations leaving the Lagrangian invariant, associated to the tangent vectors $\epsilon^\alpha D_\alpha$, will in general only *close on-shell*, that is, only if the equations of motion are satisfied.

As a final comment, it is interesting to compare the supersymmetry transformation laws (3.50), (3.51), (3.52) with the $\overline{\mathrm{OS}p}(1|4)$-gauge covariant derivative of the fields in the adjoint multiplet, with parameter $\vec{\kappa} = \frac{1}{2}\kappa^{ab} J_{ab} + \kappa^a P_a + \kappa^\alpha Q_\alpha$ (for a more extended discussion on this point, see Appendix C). They read:

$$\delta^{(gauge)}\omega^{ab} = (\nabla\kappa)^{ab} = \mathcal{D}\kappa^{ab}, \tag{3.57}$$

$$\delta^{(gauge)}V^a = (\nabla\kappa)^a = \mathcal{D}\kappa^{ab} + \kappa^{ab}V_b - \mathrm{i}\bar{\psi}\gamma^a\kappa, \tag{3.58}$$

$$\delta^{(gauge)}\psi = (\nabla\kappa) = \mathcal{D}\kappa - \frac{1}{4}\kappa^{ab}\gamma_{ab}\psi, \tag{3.59}$$

and, setting again $\kappa^{ab} = \kappa^a = 0$ they reduce to the *gauge-supersymmetry transformations*, that is, gauge transformations along odd directions of the structure group. Projected on space–time, they are:

$$\delta_\kappa^{(gauge)}\omega_\mu^{ab} = 0, \tag{3.60}$$

$$\delta_\kappa^{(gauge)}V_\mu^a = -\mathrm{i}\bar{\psi}_\mu\gamma^a\kappa, \tag{3.61}$$

$$\delta_\kappa^{(gauge)}\psi_\mu = \mathcal{D}_\mu\kappa\,. \tag{3.62}$$

We note in particular the difference in the supersymmetry transformation of the spin connection, Eq. (3.50), with respect to its gauge-supersymmetry transformation, Eq. (3.60).

Let us now summarize the result obtained so far: Even if the supercurvatures T^a and ρ, whose parametrizations are given in Eqs. (3.27) and (3.28), respectively, have no components along the fermionic vielbein ψ, a non-vanishing component along $\psi \wedge V^a$ does appear in the on-shell value of the Lorentz supercurvature, that is, (3.25). *This is sufficient to exclude factorization of the odd fermionic coordinates*. Indeed its presence makes the supersymmetry transformation a *diffeomorphism* in superspace and *not a gauge transformation*.

It must also be noted that the absence of such fermionic components in the (on-shell) gravitino curvature ρ implies that the supersymmetry variation of ψ, given in Eq. (3.52), is the same as if the Lagrangian were invariant under supersymmetry *gauge transformations*. However, the supersymmetry transformations of the Lagrangian actually correspond to superdiffeomorphisms, which close the supersymmetry algebra only on-shell, and not to gauge transformations.[25] The point is that such behavior of the gravitino transformation law is due to the very simple form of the minimal $\mathcal{N} = 1$, $D = 4$ pure supergravity. Any other supergravity with $\mathcal{N} > 1$ or $D > 4$ or even the same theory $\mathcal{N} = 1$, $D = 4$ coupled to matter multiplets exhibits a gravitino curvature with components $\rho_{(1,1)} \neq 0$ so that the $\delta_\epsilon\psi$ will have, besides the Lorentz covariant derivative of the supersymmetry parameter, also terms along $\psi \wedge V^a$.

As an example, let us consider $\mathcal{N} = 2$, $D = 4$ pure supergravity. Here the supergroup is $\overline{\mathrm{OS}p}(2|4)$. The coadjoint gauge supermultiplet is now given by $\mu^A = (\omega^{ab}, V^a, \psi_i, \mathcal{A})$, where $\mathcal{A}$ is a U(1) gauge field 1-form and the index $i = 1, 2$ enumerates the gravitinos in the two-dimensional representation of U(2).

The definitions of the associated supercurvatures are obtained by starting from the Maurer–Cartan equations dual to the algebra of the structure supergroup and deforming

[25] Note that if the Lagrangian were invariant under supersymmetry gauge transformations the superfields would only depend on the x^μ coordinates.

the left-invariant 1-forms into non left-invariant ones. Without giving the derivation, for each of them we write, besides the definitions of the supercurvatures in the first line, also (in the second line) their on-shell parametrization, as found from the analysis of the equations of motion:

$$\begin{aligned} R^{ab} &\equiv d\omega^a{}_b + \omega^a{}_c\omega^c{}_b \\ &= \tilde{R}^{ab}_{cd} V^c V^d + \overline{\Theta}^{ab}_{i|c}\psi^i V^c - \bar{\psi}_i \left(\tilde{F}^{ab} + \frac{\mathrm{i}}{2}\tilde{F}_{cd}\epsilon^{abcd}\gamma_5\right)\psi_j\epsilon^{ij}\,, \end{aligned} \tag{3.63}$$

$$\begin{aligned} T^a &\equiv \mathcal{D}V^a - \frac{\mathrm{i}}{2}\bar{\psi}_i\gamma\psi^i \\ &= 0\,, \end{aligned} \tag{3.64}$$

$$\begin{aligned} F &\equiv d\mathcal{A} + \epsilon^{ij}\bar{\psi}_i\psi_j \\ &= \tilde{F}_{ab}V^a V^b\,, \end{aligned} \tag{3.65}$$

$$\begin{aligned} \rho_i &\equiv \mathcal{D}\psi_i \\ &= \tilde{\rho}_{i|ab}V^a V^b + \left(\gamma^a\,\tilde{F}_{ab} + \mathrm{i}\gamma_5\gamma^a\frac{\mathrm{i}}{2}\tilde{F}^{cd}\epsilon_{abcd}\right)\epsilon_{ij}\psi^j V^b\,. \end{aligned} \tag{3.66}$$

The important thing to note is that the parametrization of the curvature 2-forms are all given in terms of their inner components, namely, $\tilde{R}^{ab}_{cd}$, $\tilde{\rho}_{i|ab}$ and $\tilde{F}_{ab}$ (T^a_{bc} is zero).[26]

Since the on-shell value of the supercurvatures is known, the supersymmetry transformation laws of the coadjoint supermultiplet, now containing also $\mathcal{A}$, can be obtained at once from the general formula (C.4). Looking at the Lie derivative formula, we see that the transformation laws of the multiplet of fields can be simply obtained performing the contraction of the on-shell curvatures with respect to the tangent vector $\bar{\epsilon}\,D$ and adding to the gravitino transformation the Lorentz covariant derivative of the supersymmetry parameter $\epsilon^{\alpha i}$, as it happens in the $\overline{\mathrm{OS}p(1|4)}$ case. We find:

$$\delta_\epsilon\omega^{ab} = \overline{\Theta}^{ab}_{i|c}\epsilon^i V^c, \tag{3.67}$$

$$\delta_\epsilon V^a = -\mathrm{i}\bar{\psi}_i\gamma^a\epsilon^i, \tag{3.68}$$

$$\delta_\epsilon\psi_i = \mathcal{D}\epsilon_i + \mathrm{i}\,\epsilon_{ij}F^{ab}V^b\gamma^a\epsilon_j + \mathrm{i}\,\frac{1}{2}\epsilon_{ij}\epsilon_{abcd}F^{cd}V^b\gamma_5\gamma^a\epsilon^j, \tag{3.69}$$

$$\delta_\epsilon\mathcal{A} = 2\epsilon^{ij}\bar{\psi}_i\,\epsilon_j. \tag{3.70}$$

From this example we see that, in general, not only the Lorentz curvature R^{ab}, but also the other supercurvatures have non-vanishing components along the (outer) ψ-directions.

3.2.2. *Bosonic kinetic terms in the geometric approach*

Let us end this subsection on the supergravity action, with a more detailed explanation of why we cannot admit the Hodge duality operator in the construction of the Lagrangian, and how to remedy its absence when gauge potentials of internal symmetries, and more generally bosonic p-forms, are present.

This issue is not of academic interest only, since in matter coupled supergravity theories, and also in pure supergravity for theories with more than 4 supercharges ($N > 1$

[26] Note that $\tilde{F}_{ab} = F_{ab}$, since the supercurvature F has components only along $V^a V^b$.

in $D = 4$), the spectrum of the theory includes bosonic fields, whose kinetic terms are quadratic in their field-strengths.

The standard way to write a quadratic kinetic term requires the use of the duality Hodge operator. Indeed, considering a p-form potential gauge field $A^{(p)}$ in a theory in D space–time dimensions, its field-strength F is a $(p+1)$-form, ${}^*F$ its Hodge-dual, and the kinetic term in the Lagrangian D-form is (the precise coefficient depends on p and on the number of space dimensions of the specific theory considered):

$$F \wedge {}^*F \propto F_{\mu_1 \dots \mu_{p+1}} F^{\mu_1 \dots \mu_{p+1}} \sqrt{g}\, d^D x \,. \tag{3.71}$$

This expression however is background dependent and therefore introduces a dependence of the Lagrangian on the hypersurface $\mathcal{M}_D$ and its metric. As such, and also because the Hodge operator critically depends on the dimensionality of the space where it is applied, it also makes problematic the extension of the fields domain from the bosonic hypersurface $\mathcal{M}_D$ (space–time) to the superspace $\mathcal{M}_{D|N}$, and *a fortiori*, to the full structure supergroup.

Actually, the way out from this impasse is very simple. It is sufficient to write the kinetic terms of the boson fields in *first order formalism*. For example, let us consider an abelian 1-form gauge field $A = A_\mu \, dx^\mu = A_a V^a$ in $D = 4$ space–time, with field-strength $F = dA = \partial_\mu A_\nu dx^\mu \wedge dx^\nu = F_{ab} V^a \wedge V^b$. The standard expression for its kinetic term in the Action will be

$$\mathcal{A} = -\int F^{\mu\nu} \, F_{\mu\nu} \sqrt{-g}\, d^4x = \frac{1}{2} \int F \wedge {}^*F. \tag{3.72}$$

We may avoid use of the Hodge operator *, if we introduce an auxiliary 0-form antisymmetric tensor field $\hat{F}_{ab} = -\hat{F}_{ba}$ and write the new kinetic term as follows;

$$\frac{-1}{4!} \int \hat{F}_{ab} \, \hat{F}^{ab} \epsilon_{pqrs} V^{pqrs} + \alpha \int \hat{F}^{ab} F \, V^{cd} \epsilon_{abcd} . \tag{3.73}$$

Varying the Lagrangian with respect to $\hat{F}^{ab}$ we find that, choosing $\alpha = \frac{1}{2}$, we obtain

$$\hat{F}_{ab} = F_{ab} \tag{3.74}$$

where F_{ab} are the components of the 2-form F along the vierbeins. Varying next with respect to the gauge field A one finds the usual equation of motion

$$\mathcal{D}_\mu F^{\mu\nu} = \mathcal{D}_a F^{ab} = 0 \,. \tag{3.75}$$

In this way we see that, adopting a first-order formalism, we can write a *geometric* Lagrangian without using the Hodge duality operator.

Thus, quite generally, *a Lagrangian is geometric if it is constructed in terms of p-forms, wedge products, the exterior derivative d and without the use of the Hodge duality operator*.

3.3. The principle of rheonomy

We can now resume our analysis of the previous subsection in the following way:

Supersymmetry can be interpreted geometrically as the requirement that the superspace Action be invariant under diffeomorphisms along odd directions of superspace, Effectively, this corresponds to the fact that the superspace equations of motion imply that *the outer components of the super-curvatures are expressible algebraically (actually linearly) in terms of the components along two inner vielbein*. As already mentioned, this property has been called *rheonomy*. Note that *rheonomy is just a geometrical interpretation of supersymmetry originally introduced on space–time*. Explicitly, the occurrence of *rhenomy* can be written as follows:

$$R^A_{\alpha\, C} = C^{A|mn}_{\alpha\, C|B}\, R^B_{mn}, \tag{3.76}$$

where $C^{A|mn}_{\alpha\, C|B}$ are suitable invariant tensors of the supergroup G defining the basic superalgebra on which the theory is constructed, $G = \overline{\mathrm{OSp}(1|4)}$ in our case. The geometric meaning of this property can be better understood if we use the Lie derivative formula (C.5) in superspace. Inserting (3.76) in the Lie derivative formula (C.5) for a supergroup G we obtain:

$$\delta\mu^A = (\nabla\epsilon)^A + 2\bar{\epsilon}\, C^{A|mn}_{\alpha\, C|B}\, R^B_{mn}. \tag{3.77}$$

On the other hand, the Lie derivative can be interpreted either from the *passive* or from the *active* point of view. From the passive point of view, the supersymmetry transformation along the $\epsilon^\alpha = \delta\theta^\alpha$ parameter is interpreted as the lift in $\mathcal{M}_{4|4}$, from a given $\mathcal{M}_4$ to an infinitesimally close $\mathcal{M}'_4$, which does not change the physical content of the theory, since it is described by the same Lagrangian, after performing a supersymmetry transformation (and a Lorentz gauge transformation).[27] From the active point of view, however, it transforms a given configuration on $\mathcal{M}_4$, which we can take as space–time, setting $\theta^\alpha = \delta\theta^\alpha = 0$, to another physically equivalent configuration on the same space–time hypersurface. This property allows us to restrict the theory, the Lagrangian and the equations of motion, to any such arbitrarily chosen hypersurface $\mathcal{M}_4\, (\theta^\alpha = d\theta^\alpha = 0)$, embedded in superspace and identified with space–time.

One can now appreciate why we have illustrated in detail the mechanism of the Lorentz coordinate factorization in the gravity case defined on the Poincaré manifold. Actually the interpretation of the rheonomy mechanism is quite analogous to the interpretation of Lorentz transformations for gravity constructed directly on a group manifold. Indeed, in the case of pure gravity, we have seen that a transfer of information from any $\mathcal{M}_4$ to any other $\mathcal{M}'_4$ implies a $\mathrm{SO}(1,3)$ transformation or, equivalently, a change of Lorentz configuration on the fixed space–time hypersurface. On the other hand, in our example of $\mathcal{N} = 1$, $D = 4$ supergravity, besides deducing the factorization of the Lorentz coordinates exactly as in the pure gravity case, we have further illustrated that the equations of motion allow us to deduce that the transfer of information concerns not only Lorentz gauge transformations but, what is our main goal, also *supersymmetry*. Coming back to supersymmetry in the geometric approach, the supersymmetry transformations relate the fields on a given bosonic hypersurface $\mathcal{M}_4 \subset \mathcal{M}_{4|4}$, to the fields on any other bosonic submanifold $\mathcal{M}'_4 \subset \mathcal{M}_{4|4}$. However, the difference between $\mathrm{SO}(1,3)$ transformations and supersymmetry is that, due to the horizontality of the curvatures

[27] The passive interpretation of the Lie derivative explains the world *rheonomy* given to this geometrical interpretation of supersymmetry. Indeed, referring to the lift $\mathcal{M}_4 \to \mathcal{M}'_4$, in ancient Greek "rhein" means flow and "nomos" means law.

in the Lorentz directions, the supergroup G acquires the structure of the fiber bundle $[\mathcal{M}_{4|4}, \mathrm{SO}(1,3)]$. The Lie derivative along Lorentz directions in $\tilde{G}$ amounts to a Lorentz gauge transformation. On the other hand, in the case of supersymmetry, curvatures are not horizontal along the ψ gauge fields, and the Lie derivative, in this case, gives to supersymmetry the geometric interpretation of superdiffeomorphism.

Finally, we remark that the property of working on any hypersurface $\mathcal{M}_4$ immersed in superspace and identified with space–time, without the need of specifying a metric on it,[28] makes this approach quite different from the ordinary "Supergravity approach" where the fields of the Lagrangian are expressed in a given set of coordinates (x^μ, θ^α). In that approach, they are expanded in the Grassmann-odd coordinates and the integration in superspace is made using the Berezin integration on the Grassman-odd sector.

3.3.1. The role of the Bianchi identities

Until now we have described how a supergravity Action in superspace can be constructed in the geometric approach and how to find the supersymmetry transformations that leave it invariant in superspace. We have also shown that in this framework supersymmetry transformations can be given a geometrical interpretation as (super)diffeomorphisms in superspace.

A crucial role in this construction is played by the structure group G, which in the simple case of minimal pure supergravity in $D = 4$ is the super-Poincaré group $G = \overline{\mathrm{OSp}(1|4)}$, and by the adjoint multiplet of fields μ^A $(A = 1, \ldots, adj(G))$, with its G-supercurvatures

$$R^A \equiv d\mu^A + \frac{1}{2} C^A{}_{BC} \mu^B \wedge \mu^C \,, \tag{3.78}$$

where $C^A{}_{BC}$ are the structure constants of G, that in the dynamical vacuum satisfy the Maurer–Cartan equations of G.

They are, therefore, defined by symmetry principles (see the discussion in Appendix B), as an invariant set of supercurvatures of G and, as such, the consistency of the theory is encoded in the cohomology of the exterior derivative operator d (in the condition $d^2 = 0$, which is equivalent to the Jacobi Identities on the structure constants), that is, in their *Bianchi identities* which are obtained by direct application of d to (3.78):

$$dR^A + C^A{}_{BC} \mu^B \wedge R^C = 0 \,. \tag{3.79}$$

Since supersymmetry, as discussed in Section 3.1, is an *on-shell symmetry* then, when the curvatures R^A are dynamical supercurvatures of a supergroup, (3.79) is not anymore an identity, but becomes a relation that holds only upon use of the field equations. In other words, we can say that the curvatures R^A are *formally defined* on symmetry arguments, but their Bianchi "identities" (3.79) are in fact *equations* to be satisfied *on-shell* by the parametrization of the supercurvatures in superspace, Eq. (3.3), according with the principle of Rheonomy discussed above.

[28] Clearly different surfaces are just different sections of the principal fiber bundle, and are therefore *locally* equivalent. We will not discuss here global properties of the bundle.

This opens an equivalent and powerful approach to the construction of the supergravity theories in superspace (equations of motion and transformation laws), which does not rely on the existence of an Action, but is based on a systematic use of the Bianchi identities *assuming rheonomy* from the very beginning.

The Bianchi identities, then, assume the role of differential constraints among the space–time components of the supercurvature parametrizations. These differential constraints, on the other hand, can be nothing else, in disguise, than the equations of motion,[29] since Bianchi identities become identities on-shell, and cannot conflict with the differential equations obtained from the Lagrangian. Once the field equations are obtained in this way, the Lagrangian, if desired, can be easily reconstructed. In the actual computations one usually couples the two methods, namely the Lagrangian approach and the Bianchi equations, to arrive in the simplest way to the final determination of the parametrization of the curvatures in superspace (and thus to the supersymmetry transformation laws) together with the determination of all terms in the Lagrangian.

4. Higher p-Forms supergravities and their hidden supergroups

We have often stressed that the mechanism of rheonomy actually holds in all supergravities, independently of the number of supersymmetries, the dimensionality of space–time, and their matter couplings, if any. However, apart from few exceptions, most of the higher dimensional theories have a gravitational multiplet containing antisymmetric tensors of rank higher than 1. Similarly, matter supermultiplets also can have higher rank tensors. In these cases, the group manifold interpretation presented in the previous Sections as a possible starting point for supergravities, whose fields are defined on a group manifold, has to be reconsidered. Indeed the coadjoint multiplet of a (super-)group consists of 1-forms dual to the group generators, with no room for higher p-forms.

In the present Section we will show, referring mainly to the case of $D = 11$ supergravity, where this development was first presented [30], that the Maurer–Cartan equations can be generalized to more general structures, admitting in the set of Maurer–Cartan 1-forms also higher p-forms ($p > 1$). The resulting generalized Maurer–Cartan equations, satisfying the integrability requirement $d^2 = 0$, can be seen as a natural extension of Lie algebras in their dual formulation and can accommodate supermultiplets containing higher p-forms.

These results were obtained in [30] by R. D'Auria and P. Fré for maximal $D = 11$ supergravity. The space–time Lagrangian of $D = 11$ supergravity theory was previously derived using the Noether approach in Ref. [28], and includes a 3-index antisymmetric tensor, namely a 3-form gauge potential, in the gravitational multiplet.

The occurrence of a 3-form in the supergravity multiplet can be easily understood as a consistency condition for the theory to be supersymmetric, which requires the matching of the bosonic and fermionic on-shell propagating degrees of freedom of the theory.

[29] Together with all the other restrictions on the theory required by supersymmetry, among which in particular, when the theory includes scalar fields, the relations characterizing the geometry of the scalar σ-models.

In eleven-dimensional space–time, the vielbein has on-shell $\frac{1}{2}D(D-3)=44$ d.o.f., while the gravitino field has $2^{[D/2-1]}(D-3)=128$ on-shell d.o.f.. Thus we need 84 more bosonic d.o.f. in order for the bosonic and fermionic d.o.f. to match. They are in fact provided by an on-shell propagating 3-form potential. Indeed, for a propagating antisymmetric tensor gauge potential of rank three, $A_{\mu\nu\rho}$, we have $\frac{1}{3!}(D-2)(D-3)(D-4)=84$ d.o.f., so that the requirement is satisfied.

Even if this extra 3-form cannot be interpreted as a dual of a generator of a Lie algebra, nevertheless the authors of [30] tried to give a fully geometrical interpretation of the space–time formulation of the theory in such a way that all the nice properties of the geometrical group manifold approach could be extended also to that case.

In their geometrical approach, the authors of [30] introduced for the first time a generalization of the Maurer–Cartan equations in terms of an integrable systems containing higher p-forms, which they called *Cartan Integrable Systems* (CIS). In the following years, they recognized, in a paper of Sullivan [62] on Free Differential Algebras, part of the properties of the Cartan Integrable Systems and, for this reason, they changed the original name CIS into *Free Differential Algebra (FDA)*, which is the name still currently used in the supergravity literature.[30]

When, after several years, these structures went under the scrutiny of mathematicians, see e.g. [52], it was pointed out that the CIS/FDA were, historically, *the first example* of the so-called L_∞ *algebras*, which were introduced in the mathematical literature more than a decade later, albeit in a dual language (See [48] and references therein; for a comprehensive reference, see [52].)

By dual language here we mean the equivalence of two different structures: on one side we have graded Lie algebras $\mathfrak{g}$ with an operator of "derivation", D, acting as a bracket on a couple of vectors and mapping them on a linear combinations of vectors:

$$D(T_A,T_B)\equiv[T_A,T_B]=C^C{}_{AB}T_C\,; \tag{4.1}$$

on the other side we have instead the dual graded co-algebra over its dual vector space $\mathfrak{g}^*$ where the dual of the "derivation" acts on the 1-forms as the exterior derivative operator $d: d^2=0$:

$$d\sigma^A+\frac{1}{2}C^A{}_{BC}\sigma^B\wedge\sigma^C=0. \tag{4.2}$$

The equivalence is guaranteed by the equivariance relation:

$$d\sigma^A(T_B,T_C)=-\frac{1}{2}\sigma^A\left([T_B,T_C]\right)\,. \tag{4.3}$$

Further details can be found in Appendix B.

As we will discuss in the present Section, the new structure, introduced in [30], is a generalization of the Maurer–Cartan equations to a set of higher p-forms $\Theta^{A(p)}$, where the differential nilpotent operator "d" acts on a given p-form by mapping it to a polynomial of the same set. From a mathematical point of view, such construction, acting on a collection of p-forms with $1\le p\le n-1$, was understood as a *dualization* of an

[30] Actually, as pointed out in Ref. [52] the FDA denomination is a slight misnomer from the mathematical point of view, since these graded algebras are not *free* as differential algebras, but only *semi-free* (see also footnote 2).

L_n algebra. In its standard formulation, the "derivation" operator D dual to the exterior differential takes the form of a "higher bracket" structure of the L_n Lie algebra. See Refs. [48,52,56] for details on definitions of L_n algebra. Moreover, the identity $d^2 = 0$ becomes in the L_n case what is called *the strong homotopy identity*, which must be satisfied in order to have a consistent L_n algebra. Since this can be done for any n, we may say that *the new structure introduced in reference* [30] *is a dual formulation of an* L_∞ *algebra.*[31]

It must be said that models reproducing the structure of the L_∞ algebras also appeared in the physical literature at the beginning of the nineties, more or less when the mathematical definition of L_∞ algebra appeared in the literature (see Refs. [9,36,46,57,70], and [53] for an extended collection of papers on results in Physics with L_∞-algebra type structure).

In the following, we will give a short account of how to construct such "FDA"'s and how to apply the method to $D = 11$ supergravity. Moreover, we will show that, starting from the super Poincaré group in eleven dimensions from which the FDA algebra can be obtained, we can further reduce the FDA to a *hidden ordinary Lie graded algebra*.

4.1. Cartan integrable systems as a generalization of Maurer Cartan equations

The essential point of the D'Auria–Fré construction is the possibility of building higher integrable systems, introducing forms of higher degree and mimicking the structure of the Cartan–Maurer equations of a graded Lie algebra.

Indeed, suppose we introduce on a manifold $\mathcal{M}_D$, whose dimension D is not determined for the moment, a set of p-forms $\{\Theta^{A(p)}\}$ of various degrees $1 \leq p \leq p_{\max}$, where $A(p)$ is an index in a given representation of a structure group G, such that their exterior derivatives can be expressed as a polynomial in the set of $\{\Theta^{A(p)}\}$ itself, with constant coefficients:

$$d\Theta^{A(p)} + \sum_{n=1}^{N} \frac{1}{n!} C^{A(p)}{}_{B_1(p_1)B_2(p_2)\dots B_n(p_n)} \Theta^{B_1(p_1)} \wedge \Theta^{B_2(p_2)} \wedge \dots \wedge \Theta^{B_n(p_n)} = 0 \,. \tag{4.4}$$

where $N = p_{\max+1}$. The constant coefficients $C^{A(p)}{}_{B_1(p_1)B_2(p_2)\dots B_n(p_n)}$ are actually invariant tensors of G. Note that, being $\Theta^{A(p)}$ a p-form, then $p_1 + p_2 + \dots + p_n = p+1$.

It is also important to stress that the symmetry in the exchange of two lower neighboring indices of the constant C-coefficients is inherited by the exchange of two neighboring $\Theta^{B(p)}$, namely[32]:

$$B_i(p_i)\, B_{i+1}(p_{i+1}) = (-1)^{|B_i||B_{i+1}| + p_i p_{i+1}} B_{i+1}(p_{i+1}) B_i(p_i) \,, \tag{4.5}$$

where $|A(p)|$ denotes the grading of the form $\Theta^{A(p)}$. Let us now impose the integrability of Eq. (4.4) namely $d^2 = 0$:

$$d^2\Theta^{A(p)} = -\sum_{n=1}^{N} \frac{1}{(n-1)!} \sum_{m=1}^{N} \frac{1}{m!} C^{A(p)}{}_{B_{1(p_1)}B_{2(p_2)}\dots B_{n(p_n)}}\, C^{B_1(p_1)}{}_{D_{1(q_1)}D_{2(q_2)}\dots D_{m(q_m)}} \times$$

[31] Note that from this point of view the Maurer–Cartan equations are the dual of a L_2 algebra and viceversa.

[32] This change of sign in permuting two contiguous indices is called Koszul sign law in mathematics.

$$\Theta^{D_1(q_1)} \wedge \Theta^{D_2(q_2)} \wedge \cdots \wedge \Theta^{D_m(q_m)} \wedge \Theta^{B_2(p_2)} \wedge \ldots \Theta^{B_n(p_n)} = 0. \tag{4.6}$$

The above closure condition is satisfied if the set of invariant tensors $C^{A(p)}{}_{B_1(p_1)B_2(p_2)\ldots B_n(p_n)}$ satisfies the "generalized Jacobi identities":

$$\sum_{n=1}^{N} \frac{1}{(n-1)!} \sum_{m=1}^{N} \frac{1}{m!} C^{A(p)}{}_{B_{1(p_1)}B_{2(p_2)}\ldots B_{n(p_n)}} C^{B_1(p_1)}{}_{D_{1(q_1)}D_{2(q_2)}\ldots D_{m(q_m)}} = 0\,, \tag{4.7}$$

where we denoted by [...] the graded symmetrization of the indices, according to the Koszul sign-law (4.5).

Even if not evident at first sight, Eq. (4.6), first introduced in '81 in [30], reproduces, in a dual form, the condition of the *strong-homotopy Jacobi identity* satisfied by an L_∞ algebra, which is usually formulated in terms of "higher brackets". This is shown in detail, for example, in Ref. [52], where Eq. (4.6) is rewritten in a way to make it explicit. There, one can find several different but equivalent definitions of an L_∞ algebra. In particular, the definition of the L_∞ algebra, given there in terms of a **semifree differential graded algebra**, $\mathfrak{g}^*$, matches the definition, given in the original paper [30], of CIS/FDA. The equivalence is simply obtained by passing from the graded vector space $\mathfrak{g}$ of a finite dimension n, to its degree-wise dual vector space $\mathfrak{g}^*$ which we may identify with the Grassmann graded vector space of the p-forms $\Theta^{B(p)}$. In Appendix D we will report some details on the equivalent formulation of L_∞-algebras in the standard form and how our formalism can reproduce their strong homotopy identity.

Note that in the case that the coalgebra $\mathfrak{g}^*$ is an ordinary Lie algebra in dual form, namely L_2 in dual form, this is referred to as Chevalley–Eilenberg algebra. Therefore, *the FDA can be considered a generalization of the Chevalley–Eilenberg algebra from ordinary graded Lie algebras to higher graded Lie algebras of p-forms*. In particular, at least when the vector space $\mathfrak{g}$ is finite dimensional and we have an operator D with $D^2 = 0$ acting as a derivation, one can pass to the dual graded vector space $\mathfrak{g}^*$ whose Grassmann algebra is naturally equipped with the usual exterior derivative d. This gives a semifree differential graded algebra, which reproduces our approach in terms of the so-called FDA. As for the equivalence between our FDA and the extension of the Chevalley–Eilenberg to higher p-forms, see [51].

4.2. The construction of the FDA

We show in this Section how to generate a FDA starting from an ordinary Lie algebra. We start from the Maurer–Cartan equation of the Lie (co)algebra $\mathbb{G}$ of a given Lie group G, with subgroup $H \subset G$:

$$d\sigma^A + \frac{1}{2} C^A{}_{BC}\sigma^B \wedge \sigma^C = 0. \tag{4.8}$$

Next, we consider an *H-orthogonal Chevalley cochains complex*,[33] namely polynomials of p-forms of the following type:

[33] By H-orthogonal cochain we mean that if the Lie algebra has a symmetry subgroup which is a gauge symmetry of the theory, then the associated p-form cannot enter in the construction of the cochain. More precisely, a cochain $\Omega^i_{(n,p)}$ is H-orthogonal if $\iota_H \Omega^i_{(n,p)} = 0 = \iota_H \nabla^n \Omega^i_{(n,p)} = 0$. For example, if the theory we are constructing includes Lorentz transformations SO(1, 10) (which is a subgroup of the (super)-Poincaré

$$\Omega^i_{(n,p)} = C^i_{A_1\dots A_n}\,\sigma^{A_1}\wedge\cdots\wedge\sigma^{A_n} \tag{4.9}$$

where $i = 1,\dots,n$ runs in a n-dimensional representation $D^{(n)}(T_A)^i{}_j$ of the Lie algebra generators T_A of $\mathbb{G}$, $A_1,\dots,A_p$ being indices in the coadjoint representation and $C^i_{A_1\dots A_p}$ constants invariant tensors of G.

Next we introduce the $\mathbb{G}$-covariant derivative $\nabla^{(n)}$ acting on the $\Omega^i_{(n,p)}$:

$$(\nabla^{(n)})^i_j = d\delta^i_j + \sigma^A \wedge D^{(n)}(T_A)^i_j\,. \tag{4.10}$$

Actually, in the case of a Maurer–Cartan set of 1-forms, $\nabla^{(n)}$ coincides with the G-covariant derivative computed at $R^A = 0$ (see Appendix B).

Because of Eq. (4.8), we have:

$$\nabla^{(n)}\,\nabla^{(n)} = 0 \tag{4.11}$$

and as such $\nabla^{(n)}$ is named a *boundary operator*.

If the cochain is closed under $\nabla^{(n)}$, it is a *cocycle*, while a cochain is a *coboundary* if there exists a cochain $\tilde{\Omega}^i_{(n,p-1)}$ such that

$$\Omega^i_{(n,p)} = \nabla^{(n)}\tilde{\Omega}^i_{(n,p-1)}. \tag{4.12}$$

A cocycle which is not a coboundary is a representative of a higher *Chevalley–Eilenberg cohomology class* of the Lie algebra.

Now, given a cocycle $\Omega^i_{(n,p)}$, we can introduce a new form $A^i_{(n,p-1)}$ and write the generalized Maurer–Cartan equation :

$$\nabla^{(n)}\,A^i_{(n,p-1)} + \Omega^i_{(n,p)} = 0, \tag{4.13}$$

also called a *trivialization* of the cocycle.

Adding this equation to (4.8), we obtain a *higher Lie algebra*, actually a FDA (*semifree graded differential algebra*).

Of course, the process can be iterated by considering a new set of cochains containing, besides the σ^A, also the $A^i_{(n,p-1)}$, namely:

$$\hat{\Omega}^i_{(n,p)}[\sigma, A] = C^i_{A_1\dots A_r\,i_1\dots i_s}\sigma^{A_1}\dots\sigma^{A_r}\wedge A^{i_1}_{(n_1,p_1)}\dots A^{i_s}_{(n_s,p_s)}. \tag{4.14}$$

If we can find new cocycles, say Ω', in this enlarged cochain system, we then have an enlarged FDA.

The process terminates when no new cocycles can be found so that we have constructed the most general FDA derived from the Lie algebra.

In the next Section we apply this process to the construction of the general FDA of the eleven dimensional supergravity, by starting from its underlying Lie algebra, namely the Lie algebra of the super-Poincaré group $\overline{\mathrm{Osp}(1|32)}$.[34]

group), the gauge field ω^{ab} cannot enter in the construction of the cochain. In a sense, this extends to higher forms the notion of *H-factorization* introduced in Section 3 (see Eq. (3.1)). Note, however, that working with the *relative* Chevalley–Eilenberg algebra we should consider, as derivation operator, the *H-covariant derivative* $\mathcal{D}_{(H)}$, such that $(\mathcal{D}_{(H)})^2 = R^{(H)A}T_A$, $R^{(H)A}$ being the H-curvature, which vanishes in the FDA but is not zero out of the vacuum.

[34] With the overline we mean the Inonü–Wigner contraction of the OSp(1|32) to the super-Poincaré group.

4.3. The FDA associated to the super-Poincaré algebra in D=11

The Maurer–Cartan equations of the D=11 super-Poincaré graded Lie Algebra are given, in their dual form, in terms of the set of 1-forms $\sigma^A = (\omega^{ab}, V^a, \Psi^\alpha)$ (with $a, b, \dots = 0, 1, \dots, 10$, $\alpha = 1, \dots, 32$), where ω^{ab} is the SO(1, 10) spin connection and $E^{\hat{a}} = (V^a, \Psi^\alpha)$ the supervielbein of $D = 11$ superspace $\mathcal{M}_{11|32}$, Ψ being a spinor in the 32-dimensional representation of Spin(32). They read:

$$d\,\omega^{ab} - \omega^a_c \wedge \omega^{cb} = 0, \tag{4.15}$$

$$\mathcal{D}V^a - \frac{i}{2}\bar{\Psi}\,\Gamma^a \wedge \Psi = 0\,. \tag{4.16}$$

$$\mathcal{D}\Psi \equiv d\,\Psi - \frac{1}{4}\Gamma_{ab}\,\omega^{ab} \wedge \Psi = 0\,. \tag{4.17}$$

In (4.16), $\mathcal{D}V^a = dV^a - \omega^{ab} \wedge V_b$ and $\mathcal{D}\Psi$ denote the Lorentz covariant derivative of the bosonic and fermionic vielbein respectively. Because the cohomology is H-orthogonal with respect to $H = \mathrm{SO}(1, 10)$, the Chevalley cochains can be constructed using only the supervielbein V^a, Ψ.

Let us consider the trivial representation $D^{(0)}$, such that $\nabla^{(0)}$ reduces to the exterior derivative d. Constructing the Chevalley cohomology one finds that there is a non-trivial cocycle of order four, namely:

$$\Omega_{(V,\Psi)} = \frac{1}{2}\bar{\Psi} \wedge \Gamma^{ab}\,\Psi \wedge V^a \wedge V^b. \tag{4.18}$$

Indeed

$$d\Omega = \frac{i}{2}\bar{\Psi} \wedge \Gamma^{ab}\,\Psi\;\bar{\Psi} \wedge \Gamma_a\,\Psi \wedge V^b = 0 \tag{4.19}$$

where we have used Eqs. (4.16), (4.17) and the Fierz identity:

$$\bar{\Psi} \wedge \Gamma_{ab}\,\Psi \wedge \bar{\Psi}\,\Gamma^a\,\Psi = 0, \tag{4.20}$$

which was proven in [30].[35]

According to the procedure previously explained, we can therefore introduce a 3-form $A^{(3)}$ which locally "trivializes" the cocycle, writing:

$$dA^{(3)} - \frac{1}{2}\bar{\Psi} \wedge \Gamma^{ab}\,\Psi \wedge V^a \wedge V^b = 0\,, \tag{4.21}$$

where the factor 1/2 is our choice of the normalization of $A^{(3)}$. This equation, added to the Maurer–Cartan equations (4.15)–(4.17), gives a FDA suitable for a geometrical construction of the eleven-dimensional Supergravity. Indeed, recalling what was said in the preamble of the present Section, we see that the just introduced 3-form $A^{(3)}$ gives exactly the d.o.f. necessary to match bosonic and fermionic degrees of freedom of the $D = 11$ supergravity theory.

Now, after including $A^{(3)}$ in the enlarged set of MC forms, we can iterate the procedure in order to look for other non trivial cocycles. We find that there is another cohomology

[35] The Fierz identity (4.20) expresses the fact that, in the symmetric product of four Spin(32) representations, the SO(1, 10)-vector representation is absent.

class of order seven given by:

$$\begin{aligned}\Omega'(V,\Psi,A)=&\frac{i}{2}\,\bar{\Psi}\Gamma^{a_1,\dots,a_5}\wedge\Psi\wedge V^{a_1}\wedge\cdots\wedge V^{a_5}\\&+\frac{15}{2}\,\bar{\Psi}\wedge\Gamma^{ab}\Psi\wedge V^a\wedge A^{(3)}.\end{aligned}\tag{4.22}$$

This allows us to introduce a 6-form $B^{(6)}$ locally "trivializing" the new cocycle Ω':

$$\begin{aligned}dB^{(6)}=&\frac{i}{2}\bar{\Psi}\,\Gamma^{a_1\dots a_5}\wedge\Psi\wedge V^{a_1}\cdots\wedge V^{a_5}\\&+\frac{15}{2}\bar{\Psi}\wedge\Gamma^{ab}\,\Psi\wedge V^a\wedge V^b\wedge A^{(3)}.\end{aligned}\tag{4.23}$$

It can be verified that no new non trivial cocycles can be found. Therefore Eqs. (4.15)–(4.17) together with (4.21) and (4.23) define the most general FDA in superspace associated to the eleven dimensional super-Poincaré Lie Algebra, whose generators are the MC 1-forms $\sigma^A=(\omega^{ab},V^a,\Psi)$, which are 1-forms of the Lie algebra of $\overline{\mathrm{OSp}(1|32)}$, together with the 3-form $A^{(3)}$ and the 6-form $B^{(6)}$.

4.4. Geometrical construction of D=11 supergravity

Physical applications of the FDA require a generalization of the concept of left-invariant 1-forms (the set of MC forms σ^A in the above subsection) to non left-invariant "soft forms" μ^A, with their associated higher k-form curvatures $R^{A(k)}$. The set of 1-forms μ^A are dynamical fields living on the principal fiber bundle $[\mathcal{M}_{11|32},\mathrm{SO}(1,10)]$, thus extending the Maurer–Cartan equations out of the dynamical vacuum of a supergravity theory (see Appendix B). To construct the $D=11$ supergravity theory, they have to be supplemented by the dynamical 3-form field $A^{(3)}$ satisfying, in the dynamical vacuum, Eq. (4.21) and by the 6-form $B^{(6)}$ satisfying, in the vacuum, Eq. (4.23). The set of dynamical fields is then given by the forms:

$$\Pi^{A(p)}=\left(\omega^{ab},V^a,\Psi,A^{(3),}B^{(6)}\right),\tag{4.24}$$

which are in one-to-one correspondence with the left-invariant forms $\Theta^{A(p)}$. To build up the theory, all the concepts advocated for the geometrical construction of supergravity Actions based on Maurer–Cartan equations, can be straightforwardly extended to theories based on FDA's. One first introduces the (super)-curvatures $R^{A(p+1)}$ of the p-forms $\Pi^{A(p)}$, corresponding to the deviation from zero of Eqs. (4.4), when the set of $\Theta^{A(p)}$ is replaced by the "soft" forms $\Pi^{A(p)}$. Therefore instead of Eq. (4.4) we have:

$$R^{A(p+1)}\equiv d\Pi^{A(p)}+\sum_{i=1}^{N}\frac{1}{n!}C^{A(p)}{}_{B_{1(p_1)}B_{2(p_2)}\dots B_{n(p_n)}}\Pi^{B_1(p_1)}\wedge\Pi^{B_2(p_2)}\wedge\cdots\wedge\Pi^{B_n(p_n)}\tag{4.25}$$

and applying the exterior derivative to this equation we find the generalized Bianchi identity:

$$\nabla R^{A(p+1)} = dR^{A(p+1)} - \sum_{i=1}^{N} \frac{1}{(n-1)!} C^{A(p)}{}_{B_{1(p_1)} B_{2(p_2)} \dots B_{n(p_n)}} \times \\ \times\ R^{B_1(p_1+1)} \wedge \Pi^{B_2(p_2)} \wedge \dots \wedge \Pi^{B_n(p_n)} = 0. \tag{4.26}$$

In complete analogy to what one does for 1-forms, Eq. (4.26) defines the *coadjoint covariant derivative* of the set of $(p+1)$-form field-strengths.

Let us now write down the complete set of differential equations defining the FDA in D=11:

$$R^a{}_b \equiv d\omega^a{}_b - \omega^a{}_c \wedge \omega^c{}_b \,, \tag{4.27}$$

$$T^a \equiv \mathcal{D}V^a - \frac{\mathrm{i}}{2} \overline{\Psi} \Gamma^a \wedge \Psi \,, \tag{4.28}$$

$$\rho \equiv \mathcal{D}\Psi = d\Psi - \frac{1}{4} \omega^{ab} \wedge \Gamma_{ab} \Psi \,, \tag{4.29}$$

$$F^{(4)} \equiv dA^{(3)} - \frac{1}{2} \overline{\Psi} \Gamma^{ab} \wedge \Psi \wedge V^a \wedge V^b \,, \tag{4.30}$$

$$F^{(7)} \equiv dB^{(6)} - \frac{i}{2} \overline{\Psi} \Gamma^{a_1 \dots a_5} \wedge \Psi \wedge V^{a_1, \dots, \wedge V^{a_5}} - \frac{15}{2} \overline{\Psi} \wedge \Gamma^{ab} \Psi \wedge V^a \wedge V^b \wedge A + \\ -\ 15 F^{(4)} \wedge A^{(3)} \,. \tag{4.31}$$

The last term in (4.31) (which is obviously zero in the vacuum) has been added to the right-hand side of Eq. (4.31) in order to have gauge invariance of the curvatures under the higher-form transformations:

$$A^{(3)} \to A^{(3)} + d\phi^{(2)} \,, \tag{4.32}$$

$$B^{(6)} \to B^{(6)} + d\lambda^{(5)} \,, \tag{4.33}$$

where $\phi^{(2)}$, $\lambda^{(5)}$ are general 2-forms and 5-forms respectively.

As previously done for the theories based on ordinary Lie algebras, to construct a supergravity Action, given the definitions above, one then requires that:

- The Action is given in terms of a 11-form Lagrangian integrated over an eleven dimensional bosonic submanifold $\mathcal{M}_{11}$, immersed in the full superspace $\mathcal{M}_{11|32}$ parametrized by 11 bosonic and 32 fermionic coordinates, $(x^\mu; \theta^\alpha)$, respectively.
- The Lagrangian is completely *geometric* that is it is constructed in terms of p-forms and wedge products only, without the use of the Hodge-duality operator. In this case it is easily seen that the fundamental properties of geometric Lagrangians in superspace based on ordinary Lie algebras, discussed in some detail in Section 3 for the case of pure $D = 4$ supergravity, still hold for more general theories: In particular, even if the Lagrangian is integrated on a bosonic eleven dimensional hypersurface of superspace (space–time), its being *geometric* gives equations of motion valid on the full superspace.
- We also add, for physical reasons, some symmetry conditions: the requirement that the Lagrangian be *gauge invariant* under the gauge symmetries of the theory, which include the Lorentz SO(1, 10) gauge symmetry, together with the higher-form gauge

invariances, Eqs. (4.32) and (4.33). Moreover, we add the obvious requirement that all terms scale and have the same parity properties as the Einstein–Cartan term.[36]

- It is also useful, as a consistency check on the superspace geometric Lagrangian, to verify that its equations of motion admit, among their solutions, the vacuum solution, namely all the curvatures $R^{A(p+1)} = 0$.

We notice that the presence of the 6-form $B^{(6)}$, and of the associated curvature $F^{(7)}$ in the FDA, seems to violate the matching between bosonic and fermionic on-shell propagating d.o.f. However, once the supersymmetric and gauge invariant Lagrangian have been written down, one finds that all the terms involving the 6-form $B^{(6)}$ sum up to a total differential and therefore the field $B^{(6)}$ is not propagating. Furthermore, from the analysis of the Bianchi "identities" in superspace, it also follows that the components along the bosonic vielbein of the two field strengths $F^{(7)}_{a_1\dots a_7}$ and $F^{(4)}_{a_1\dots a_4}$ are actually Hodge dual to each other and therefore dynamically the degrees of freedom of $F^{(7)}_{a_1\dots a_7}$ are not independent from the ones of $F^{(4)}_{a_1\dots a_4}$. Physically, this means that once projected on space–time through V^a_μ, the 7-form field strength $F^{(7)}_{\mu_1\dots\mu_7}$, is the "magnetic" Hodge dual of the "electric" field strength $F^{(4)}_{\mu_1\dots\mu_4}$.[37]

The obvious conclusion of this Section would be now to construct the D=11 theory. However, we do not report in this contribution the explicit construction of the $D = 11$ Lagrangian and/or the associated rheonomic parametrizations of the graded curvatures, satisfying on-shell the Bianchi identities in superspace. The explicit construction of the D=11 Supergravity, using these geometric tools, can be found in Refs. [18,30] (Vol 2, pag 861).

Our interest, in this contribution, has been instead to show the basic geometric structures for its actual construction, namely, the structure of its FDA which, as we have discussed (see Appendix D), has an equivalent description in terms of L_∞ algebras.

5. FDA and hidden Lie algebra of D=11 supergravity

An interesting development of the geometric approach in terms of FDA is the following:

It is possible to reduce the FDA of $D = 11$ supergravity, constructed by starting from the super-Poincaré Lie algebra, in terms of an ordinary graded Lie algebra of which the Poincaré algebra is a contraction.

This was shown in the same paper [30]. There, the authors asked themselves whether one could trade the FDA structure on which the theory is based with a new ordinary Lie superalgebra, written in its dual Cartan form, that is in terms of 1-form gauge fields valued in non-trivial tensor and spinor representations of the Lorentz group

[36] The scaling of each field is immediately obtained from the Maurer–Cartan equations from which also the scaling of the cocycles is derived.

[37] The above remark shows that the information on the Bianchi "identities" in superspace is in general richer than the one available at the Lagrangian level. This holds in particular when the theory includes mutually Hodge-dual fields. In this case, it is possible to write a dual Lagrangian, where $B^{(6)}$, but not $A^{(3)}$, is among the dynamical propagating "electric" fields, its space–time Hodge-dual $A^{(3)}$ being in that case "magnetic".

SO(1, 10). This would allow to disclose the fully extended Lie superalgebra hidden in the supersymmetric FDA.

This was proven to be true, and the hidden superalgebra underlying the FDA of $D = 11$ supergravity was presented for the first time.

To arrive at the desired result, it was shown that it is possible to associate, to the 3-forms $A^{(3)}$ and the 6-form $B^{(6)}$, a set of bosonic 1-forms B_{ab} and $B_{a_1\cdots a_5}$ valued in the antisymmetric representations of SO(1, 10), and furthermore an extra *spinor* 1-form η, in the same spinor representation as Ψ. The Maurer–Cartan equations satisfied by the new 1-forms are:

$$\mathcal{D}B_{a_1a_2} = \frac{1}{2}\overline{\Psi} \wedge \Gamma_{a_1a_2}\Psi\,, \tag{5.1}$$

$$\mathcal{D}B_{a_1\ldots a_5} = \frac{i}{2}\overline{\Psi} \wedge \Gamma_{a_1\ldots a_5}\Psi\,, \tag{5.2}$$

$$\mathcal{D}\eta = iE_1\Gamma_a\Psi \wedge V^a + E_2\Gamma^{ab}\Psi \wedge B_{ab} + iE_3\Gamma^{a_1\ldots a_5}\Psi \wedge B_{a_1\ldots a_5}\,, \tag{5.3}$$

$\mathcal{D}$ being the Lorentz-covariant derivatives, and E_1, E_2, E_3 some constant coefficients.

The whole consistency of this approach requires:

- The d^2 closure of the newly introduced 1-form fields B_{ab}, $B_{a_1\cdots a_5}$ and η, which are thus included in the Maurer–Cartan set:

$$(\omega^{ab}, V^a, \Psi, B_{ab}, B_{a_1\cdots a_5}, \eta)\,. \tag{5.4}$$

  Given the Lorentz-horizontality, in this case it is convenient to consider the relative SO(1, 10) Chevalley–Eilenberg cohomology, using as derivation operator, instead of d, the Lorentz-covariant derivative $\mathcal{D}$, which at zero curvatures (and in particular, for $R^{ab} = 0$) satisfies $\mathcal{D}^2 = 0$ (see footnote[33]). For the two bosonic 1-form fields B_{ab} and $B_{a_1\cdots a_5}$, the $\mathcal{D}^2$ closure is obvious in the vacuum state, because of the vanishing of the curvatures R^{ab} and $\rho = \mathcal{D}\Psi$, see (5.4), while $\mathcal{D}^2\eta = 0$ requires the further condition:

$$E_1 + 10E_2 - 720E_3 = 0\,, \tag{5.5}$$

  which can be derived by differentiation and use of the Fierz identities of the wedge product of three gravitino 1-forms in superspace.
- An appropriate *parametrization* of the 3-form $A^{(3)}$ on the set of 1-forms spanning the hidden algebra. The most general decomposition of the 3-form in terms of product of the 1-forms (5.4) is[38]:

$$\begin{aligned} A^{(3)}_{par} =& T_0 B_{ab} \wedge V^a \wedge V^b + T_1 B_{ab} \wedge B^b{}_c \wedge B^{ca} + T_2 B_{b_1a_1\ldots a_4} \wedge B^{b_1}{}_{b_2} \wedge B^{b_2a_1\ldots a_4} + \\ &+ T_3\,\epsilon_{a_1\ldots a_5b_1\ldots b_5m} B^{a_1\ldots a_5} \wedge B^{b_1\ldots b_5} \wedge V^m + \\ &+ T_4\epsilon_{m_1\ldots m_6n_1\ldots n_5} B^{m_1m_2m_3p_1p_2} \wedge B^{m_4m_5m_6p_1p_2} \wedge B^{n_1\ldots n_5} + \\ &+ iS_1\overline{\Psi}\Gamma_a\eta \wedge V^a + S_2\overline{\Psi}\Gamma^{ab}\eta \wedge B_{ab} + iS_3\overline{\Psi}\Gamma^{a_1\ldots a_5}\eta \wedge B_{a_1\ldots a_5}\,, \end{aligned} \tag{5.6}$$

  where T_i $(i = 1, \ldots, 5)$ and S_j $(j = 1, 2, 3)$ are numerical coefficients.
- To show the equivalence of the FDA with this new ordinary super-Lie algebra (in dual form), it is then required that differentiation of Eq. (5.6) gives the same result

[38] We do not include the ω^{ab} connection since we are using the relative Chevalley–Eilenberg cohomology.

as the differentiation of $\mathcal{A}^{(3)}$, Eq. (4.21), namely:

$$d A^{(3)}_{par} - \frac{1}{2}\overline{\Psi} \wedge \Gamma_{ab} \Psi \wedge V^a \wedge V^b = 0\,. \tag{5.7}$$

Performing the differentiation and using Poincaré algebra, together with the differentials of B_{ab}, $B_{a_1 a_2 \dots a_5}$ and η, one finds that the requirement is satisfied provided the T_i and S_i coefficients are fixed as given in Appendix E. Note that they can be all written in terms of the ratio E_3/E_2.[39]

We remark that the parametrization (5.6) provides a *trivialization* of the 3-form $A^{(3)}$ of the FDA in terms of the 1-forms defining, in the dual basis, the hidden superalgebra of the theory. We stress that to obtain such consistent solutions, *the extra terms involving the spinor 1-form η turn out to be necessary:* The Ansatz (5.6), if the set extra 1-forms is restricted to the bosonic 1-forms only, does not work. In other words, the inclusion of the spinor 1-form field η enters in the decomposition of the 3-form $A^{(3)}_{par}$ in an essential way, to properly give to $A^{(3)}_{par}$ a decomposition compatible with Eq. (4.21), which describes the FDA on ordinary superspace.

In this way, one arrives at the following set of Maurer–Cartan equations for the left-invariant 1-forms (ω^{ab} , V^a , Ψ , B_{ab} , $B_{a_1 \dots a_5}$, η):

$$d\omega^{ab} = \omega^{ac} \wedge \omega_c^{\ b}, \tag{5.8}$$

$$\mathcal{D}V^a = \frac{\mathrm{i}}{2}\overline{\Psi} \wedge \Gamma^a \Psi, \tag{5.9}$$

$$\mathcal{D}\Psi = 0, \tag{5.10}$$

$$\mathcal{D}B_{a_1 a_2} = \frac{1}{2}\overline{\Psi} \wedge \Gamma_{a_1 a_2} \Psi, \tag{5.11}$$

$$\mathcal{D}B_{a_1 \dots a_5} = \frac{\mathrm{i}}{2}\overline{\Psi} \wedge \Gamma_{a_1 \dots a_5} \Psi, \tag{5.12}$$

$$\mathcal{D}\eta = \mathrm{i}E_1 \Gamma_a \Psi \wedge V^a + E_2 \Gamma^{ab} \Psi \wedge B_{ab} + \mathrm{i}E_3 \Gamma^{a_1 \dots a_5} \Psi \wedge B_{a_1 \dots a_5}\,. \tag{5.13}$$

This set of Maurer–Cartan equations identifies (in dual form) the hidden super-Lie algebra underlying the FDA of $D = 11$ supergravity, when the set of MC forms is extended to include the 3-form $A^{(3)}$ (but disregarding $B^{(6)}$), that is:

$$R^a{}_b \equiv d\omega^a{}_b - \omega^a{}_c \wedge \omega^c{}_b, \tag{5.14}$$

$$T^a \equiv \mathcal{D}V^a - \frac{\mathrm{i}}{2}\overline{\Psi}\Gamma^a \wedge \Psi, \tag{5.15}$$

$$\rho \equiv \mathcal{D}\Psi = d\Psi - \frac{1}{4}\omega^{ab} \wedge \Gamma_{ab}\Psi \tag{5.16}$$

$$F^{(4)} = dA^{(3)} - \frac{1}{2}\,\overline{\Psi}\Gamma^{ab} \wedge \Psi \wedge V^a \wedge V^b\,. \tag{5.17}$$

[39] In [30], the first coefficient T_0 was arbitrarily fixed to $T_0 = 1$ giving only 2 possible solutions for the set of parameters $\{T_i, S_j, E_k\}$. It was pointed out later in [11,12] that this restriction can be relaxed thus giving a more general solution in terms of one parameter. Indeed, as observed in the quoted reference, one of the E_i can be reabsorbed in the normalization of η, so that, owing to the relation (5.3), we are left with one free parameter, say E_3/E_2 Then, in [8], a physical interpretation to the free parameter was given.

Let us finally write down the hidden superalgebra in terms of its generators

$$T_A \equiv \{P_a, Q, J_{ab}, Z^{ab}, Z^{a_1\ldots a_5}, Q'\}, \tag{5.18}$$

closing a set of graded commutation relations.

They are dual to the 1-forms $\left(V^a,\ \Psi,\ \omega^{ab}, B_{ab},\ B_{a_1\ldots a_5},\ \eta\right)$ respectively. In particular:

$$\omega^{ab}(J_{cd}) = 2\delta^{ab}_{cd}, \quad V^a(P_b) = \delta^a_b, \quad \Psi^\alpha(Q_\beta) = \delta^\alpha_\beta, \tag{5.19}$$

as in $D = 4$ supergravity, and furthermore:

$$B^{ab}(Z_{cd}) = 2\delta^{ab}_{cd}, \quad B^{a_1\cdots a_5}(Z_{b_1\cdots b_5}) = 5!\delta^{a_1\cdots a_5}_{b_1\cdots b_5}, \quad \eta^\alpha(Q'_\beta) = \delta^\alpha_\beta. \tag{5.20}$$

One then finds, as shown in [30] that the $D = 11$ FDA which includes the 3-form $A^{(3)}$ among the set of MC forms, corresponds to the following hidden superalgebra, which can be referred to, after the authors, as *DF-algebra*:

$$\begin{aligned}
[J_{ab}, J_{cd}] &= -2\,\eta_{a[c}J_{d]b} + 2\,\eta_{b[c}J_{d]a}, \\
[J_{ab}, P_c] &= -2\,P_{[a}\eta_{b]c}, \\
\left\{Q, \overline{Q}\right\} &= -\left(\mathrm{i}\Gamma^a P_a + \frac{1}{2}\Gamma^{ab} Z_{ab} + \frac{\mathrm{i}}{5!}\Gamma^{a_1\ldots a_5} Z_{a_1\ldots a_5}\right), \\
\left[Q', \overline{Q}'\right] &= 0, \\
[Q, P_a] &= -2\mathrm{i}E_1\Gamma_a Q', \\
[Q, Z^{ab}] &= -4E_2\Gamma^{ab}Q', \\
[Q, Z^{a_1\ldots a_5}] &= -2\,(5!)\mathrm{i}E_3\Gamma^{a_1\ldots a_5}Q', \\
[J_{ab}, Z^{cd}] &= -8\delta^{[c}_{[a}Z_{b]}{}^{d]}, \\
[J_{ab}, Z^{c_1\ldots c_5}] &= -20\,\delta^{[c_1}_{[a}Z_{b]}{}^{c_2\ldots c_5]}, \\
[J_{ab}, Q] &= -\Gamma_{ab}Q, \\
[J_{ab}, Q'] &= -\Gamma_{ab}Q'.
\end{aligned} \tag{5.21}$$

All the other graded commutators vanish.

In the Lie algebra (5.21), the generators Q', $Z^{a_1\ldots a_5}$ and Z^{ab} are "quasi-central", in the sense that they commute with all the algebra but Lorentz generators. They provide a (quasi)-central extension of the supersymmetry algebra.

Actually, as shown in Ref. [8], from a cohomological point of view, to reproduce the integrability of $dA^{(3)}$ the presence of the 1-form $B_{a_1\ldots a_5}$ in the decomposition (4.21) is not necessary, since all the terms where it appears sum up to give an exact 3-form.[40] More precisely, in Ref. [8] it was shown that, once formulated in terms of its hidden superalgebra of 1-forms, $A^{(3)}$ can be actually decomposed into the sum of two parts having different group-theoretical meaning:

$$A^{(3)}_{par} = A^{(3)}_{(0)} + \alpha A^{(3)}_{(e)}, \tag{5.22}$$

[40] This feature pairs an analogous result for $B^{(6)}$ in the supergravity Lagrangian in superspace, where all the contributions in $B^{(6)}$ sum up to a topological term, as it was shortly discussed at the end of Section 4.4.

where α is a free parameter, and:

$$dA^{(3)}_{(0)} = \frac{1}{2}\overline{\psi} \wedge \Gamma_{ab}\psi \wedge V^a \wedge V^b, \tag{5.23}$$

$$dA^{(3)}_{(e)} = 0\,. \tag{5.24}$$

The part $A^{(3)}_{(0)} = A^{(3)}_{(0)}(V^a, B_{ab}, \Psi, \eta)$, which gives the non-trivial contribution to the 4-form cohomology in superspace, does not depend on $B_{a_1\cdots a_5}$, while $A^{(3)}_{(e)} = A^{(3)}_{(e)}(V^a, B_{ab}, B_{a_1\cdots a_5}, \Psi)$ does not contribute to the 4-form cohomology, being a 3-cocycle of the FDA; however, it enjoys invariance under a symmetry algebra which is a parallelization of the (uncontracted) superalgebra $\mathfrak{osp}(1|32)$. It is actually the only contribution in $A^{(3)}_{par}$ depending on the 1-form $B^{a_1\cdots a_5}$.

This provides a physical meaning to the free parameter in the solution to Eq. (5.7). More details on this point can be found in [8].

5.1. D=11 supergravity and M-theory

Several years after the publication of [30], on the basis of different considerations the same algebra, but *without the inclusion of the nilpotent generator* Q', was rediscovered. This superalgebra, actually a contraction of the hidden superalgebra (5.21), was named M-algebra [34,44,45,59,66]. The crucial commutation relation in the M-algebra is the third of (5.21):

$$\{Q, \overline{Q}\} = -\left(\mathrm{i}\Gamma^a P_a + \frac{1}{2}\Gamma^{ab}Z_{ab} + \frac{\mathrm{i}}{5!}\Gamma^{a_1\ldots a_5}\, Z_{a_1\ldots a_5}\right), \tag{5.25}$$

which expresses the anticommutator of two supersymmetry generators, and includes on its right-hand side, besides the translation generator P_a, also the quasi-central generators $Z_{ab}, Z_{a_1\ldots a_5}$. It is indeed the natural extension to $D = 11$ supergravity of the centrally extended supersymmetry algebra of [69] (where the central generators were associated with electric and magnetic topological charges) and, as such, has in fact a topological meaning. The important role of the quasi central generators $Z_{ab}, Z_{a_1\ldots a_5}$ was in fact understood in several papers from the mid eighties on, see in particular [1,2,34], where it was clarified that they should be associated with extended objects (2-brane and 5-brane charges, respectively), topological defects in eleven dimensional superspace. After the discovery of Dp-branes as non-perturbative sources for the R-R gauge potentials [54], and the following *second string revolution*, where the role of duality relations, and in particular the one between eleven dimensional supergravity and Type IIA string theory in ten dimensions, was clarified, Eq. (5.25) was revived once more. Indeed, the bosonic generators $Z^{ab}, Z^{a_1\cdots a_5}$ were interpreted as p-brane charges, sources of the dual potentials $A_{(3)}$ and $B_{(6)}$ [47,64], and analogous extended algebras governing the different perturbative descriptions, in space–time dimensions $D \le 10$, of non-perturbative superstring theory were given. To this structure was then given the name of *M-theory*, explaining why Eq. (5.25) was then referred to as *M-algebra*.

The M-algebra is now commonly considered as the super-Lie algebra underlying M-theory [38,58,65] in its low energy limit corresponding to $D = 11$ supergravity in the presence of non-trivial M-brane sources [2,13,14,39,63,64]. Together with its lower dimensional versions, it is understood as the natural generalization of the supersymmetry algebra in higher dimensions, in the presence of non-trivial topological extended sources (black p-branes).

However, if we hold on the idea that the low energy limit of M-theory, and then the M-algebra, should be based on the ordinary superspace spanned by the supervielbein (V^a, Ψ), as in the original formulation of $D = 11$ supergravity [28], then the M-algebra cannot be the final answer, since it does not contain the extra 1-form η dual to the nilpotent generator fermionic generator Q'. Indeed, as shown in [7], a field theory based on the M-algebra (but excluding η, that is setting to zero Q' in (5.21)) is naturally described on a domain corresponding to an *enlarged superspace* whose cotangent space is spanned, besides the supervielbein (V^a, Ψ), also by the bosonic fields $\{B_{ab}, B_{a_1 \ldots a_5}\}$, that is in a theory different from 11-D supergravity. In order to reproduce the FDA (5.14)–(5.17) on which $D = 11$ supergravity is based, the presence of η among the 1-form generators is necessary. Actually, the DF-algebra (5.21) differs from its contraction, the M-algebra, precisely because it also *includes the nilpotent fermionic generator* Q', *(*$Q'^2 = 0$*)*, dual to the spinor 1-form η. Indeed, as we have seen in the previous subsection, such spinor 1-form is crucially introduced in the trivialization of the 3-form, Eq. (5.6), in order for Eqs. (5.7) to hold. Its contribution to the Maurer–Cartan equations of the DF-algebra (5.8)–(5.13) is given in equation (5.13). As it was shown in [7], Eq. (5.7) in turn implies that the group manifold generated by the DF-algebra gets a fiber bundle structure $[\mathcal{M}_{11|32}, \mathcal{H}]$, whose base space is ordinary superspace $\mathcal{M}_{11|32}$, while the fiber $\mathcal{H} \supset \mathrm{SO}(1, 10)$ is generated by the subalgebra $\mathfrak{h}$ of the DF-algebra spanned by $(J_{ab}, Z_{ab}, Z_{a_1 \cdots a_5})$. Its cotangent space is then spanned, besides the Lorentz spin connection ω^{ab} of $\mathrm{SO}(1, 10)$, also by the bosonic 1-form generators $B_{ab}, B_{a_1,\ldots a_5}$. Considering the group manifold generated by the DF-algebra, whose coadjoint multiplet is $\mu^A = (\omega^{ab}, B_{ab}, B_{a_1 \ldots a_5}, V^a, \Psi, \eta)$, allows to think of the 1-forms B_{ab} and $B_{a_1 \ldots a_5}$ as gauge fields in ordinary superspace instead of additional vielbeins of an enlarged superspace, that is, their curvatures on the fiber are *horizontal*. This is due to the dynamical cancellation of their unphysical contributions to the supersymmetry and gauge transformations with the supersymmetry and gauge transformations of η.[41] The same then should apply to the field equations, where the dynamics of all the unphysical contributions is expected to be decoupled from the physical one.

Let us conclude with a final remark. We wonder if the DF-algebra does reproduce the full hidden symmetry of the low-energy, supergravity limit of M-theory, or if some extra generators (maybe an infinite number) have to be included. To our knowledge, the general answer is still an open problem. We expect, however, that the DF algebra has to be further extended if one wants to take into account the full non perturbative description of the theory, including the dual Lagrangian description where the 6-form $B^{(6)}$ is electric,

[41] As observed in [7], all the above procedure of enlarging the field space to recover a well defined description of the physical degrees of freedom is strongly reminiscent of the BRST-procedure, and the behavior of η is such that it can be actually thought of as a ghost for the 3-form gauge symmetry, when the 3-form is parametrized in terms of 1-forms.

$A^{(3)}$ being instead magnetic.[42] To disclose the full algebra, in the same spirit of the way opened in [30], one should find a trivialization $B^{(6)}_{par}$ also for the 6-form $B^{(6)}$, in terms of 1-form fields, such that the FDA relation (4.23) be satisfied by it:

$$\begin{aligned} dB^{(6)}_{par} &= \frac{i}{2}\overline{\Psi}\,\Gamma^{a_1\dots a_5}\wedge\Psi\wedge V^{a_1}\cdots\wedge V^{a_5} \\ &+ \frac{15}{2}\overline{\Psi}\wedge\Gamma^{ab}\,\Psi\wedge V^a\wedge V^b\wedge A^{(3)}_{par}, \end{aligned} \tag{5.26}$$

analogously to the prescription (5.7) for $A^{(3)}$.

This is still an unsolved issue, also due to the technical complexity of the calculation in the expansion of a 7-form in superspace. However, a partial answer was given in [7], where a dimensional reduction of $D = 11$ supergravity on the orbifold $T^4/\mathbb{Z}_2$ to the minimal $D = 7$ supergravity was considered. In this case, the theory has a rich FDA structure which includes, besides the supervielbein and spin-connection, also a 3-form $B^{(3)}$, with its Hodge-dual form $B^{(2)}$, together with a triplet of 1-forms A^x, with their Hodge duals $A^{(4)|x}$. The hidden algebra trivializing the mutually Hodge-dual forms $B^{(3)}$ and $B^{(2)}$ was explored in detail in [7], showing that in this case *two inequivalent nilpotent spinor charges are required* to get the hidden algebra, with a fiber-bundle structure and the superspace $\mathcal{M}_{7|8}$ as base space. However, it was also found that two subalgebras of the hidden algebra exist, each of them including only one nilpotent spinor charge. One of the two subalgebras is the relevant one to fully describe the trivialization of $B^{(3)}$, the other, instead, gives the parametrization in terms of hidden 1-forms of its Hodge-dual $B^{(2)}$. For this reason they were named *Lagrangian subalgebras*.

This analysis suggests that the full hidden algebra of the FDA underlying $D = 11$ supergravity should at least include one more spinor charge, playing a role in the hidden description of $B^{(6)}$. This is left to future investigations.

CRediT authorship contribution statement

L. Andrianopoli: Writing – review & editing, Writing – original draft, Supervision, Project administration, Methodology, Investigation, Formal analysis, Conceptualization. **R. D'Auria:** Writing – review & editing, Writing – original draft, Supervision, Project administration, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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[42] We recall that the space–time projections of the corresponding field-strengths, Eqs. (4.31) and (4.30), are related by Hodge-duality, as discussed at the end of Section 4.4.

Prof. Veeravalli Seshadri Varadarajan — A memory from one of the athors, Riccardo D'Auria:

I am particularly grateful to the editors R. Fioresi and M. A. LLedó for the chance of honoring the figure of the eminent Mathemathician V.S. Varadarajan to whom I was related by scientific admiration and a lasting friendship.

I met him for the first time during my frequent visits in the nineties to UCLA University, as a consultant and as a teacher in some PHD topics in group theory and string theory. There, I also had the opportunity to follow some of his lectures and I immediately appreciated his way of presenting some issues relevant to physicists. I could appreciate in particular his work on the mathematical aspects of theoretical physics, as also testified by his interesting book on supersymmetry. We had, actually, an important collaboration, together with S. Ferrara and M. A. Lledó, on Spinors Algebras, a topic very useful in supersymmetric theories.

However, the human side of his personality is not less important than his excellent achievements in Mathematics. Immediately after we met for the first time, a friendship was born between us as an effect of our discussions on classical music, to which we shared the same passion. Particularly, we shared a particular admiration for Mozart music. Actually, he was able to reproduce, being an excellent clarinet executor, some Mozart pieces of music for this instrument. Our common interest was the beginning of a lasting and deep friendship, which was strengthened every time I was in UCLA and any time he visited my Department at Torino Politecnico. On such occasions, we had several dinners together discussing, besides scientifical topics, also musical events and literature. He was indeed a man of excellent culture, and I owe to him, among others, the discovery of excellent indian writers of english language.

Even when we were separated by the ocean, we had frequent E-mails regarding our opinion about some events concerning new Mozart discographic executions.

This was the kind of our friendship that, even if initially born from common interests, revealed through the years his gentle character and deep humanity. Therefore I was much troubled when some years ago he passed away.

This contribution, in collaboration with Laura Andrianopoli, is dedicated to his memory.

Appendix A. Notations and conventions

All over the paper, we adopted a "mostly minus" signature for the space–time metric, that is:

$$\eta_{ab} = \mathrm{diag}(+, -, \ldots, -)\,.$$

A.1. Fierz identities in D=4 minimal theory

The four-dimensional Dirac matrices are defined as

$$\gamma^a \equiv \begin{pmatrix} \sigma^a & 0 \\ 0 & \overline{\sigma}^a \end{pmatrix}, \quad \gamma_5 \equiv -\frac{i}{4!}\epsilon_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d\,, \quad \gamma^{a_1\ldots a_k} \equiv \gamma^{[a_1}\ldots\gamma^{a_k]} \tag{A.1}$$

and fulfill

$$\gamma_0^\dagger = \gamma_0, \qquad \gamma_0\gamma^a\gamma_0 = (\gamma^a)^\dagger, \tag{A.2}$$

$$\gamma_5^\dagger = \gamma_5, \qquad \gamma_5^* = \gamma_5, \qquad (\gamma_5)^2 = \mathbb{I}, \tag{A.3}$$

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}, \qquad [\gamma^a, \gamma^b] = 2\gamma^{ab}, \qquad \gamma^a\gamma^b = \eta^{ab} + \gamma^{ab}\,. \tag{A.4}$$

The charge-conjugation matrix C has the following properties:

$$\begin{gathered} C^2 = -1\,, \quad C^T = -C\,, \quad (C\gamma_a)^T = C\gamma_a\,, \quad (C\gamma_5)^T = -C\gamma_5\,, \\ (C\gamma_5\gamma_a)^T = -C\gamma_5\gamma_a\,, \quad (C\gamma_{ab})^T = C\gamma_{ab} \end{gathered} \tag{A.5}$$

The gravitino 1-form in $D = 4$ is a Majorana spinor, satisfying the condition:

$$\bar{\psi} \equiv \psi^\dagger\gamma^0 = \psi^t C \tag{A.6}$$

where $\bar{\psi}$ denotes the adjoint of the spinor ψ.

The following 3-gravitini Fierz identity holds on $D = 4$, $N = 1$ superspace:

$$\gamma_a\psi\bar{\psi}\gamma^a\psi = 0\,. \tag{A.7}$$

A.2. Fierz identities in D=11

The content of this appendix is taken from [7,30].

The gravitino 1-form Ψ_α, $(\alpha = 1, \ldots, 32)$, of eleven dimensional supergravity is a Majorana spinor belonging to the spinor representation of SO(1, 10), $Spin(32)$.

The symmetric product $(\alpha, \beta, \gamma) \equiv \Psi_{(\alpha} \wedge \Psi_\beta \wedge \Psi_{\gamma)}$, whose dimension is **5984**, belongs to the three-times symmetric, reducible representation of $Spin(32)$. Its decomposition into irreducible representations of $Spin(32)$ gives the 3-Ψ Fierz identities. One obtains:

$$\mathbf{5984} \to \mathbf{32} + \mathbf{320} + \mathbf{1408} + \mathbf{4224} \tag{A.8}$$

and the corresponding irreducible spinor representations of the Lorentz group $SO(1, 10)$ will be denoted as follows:

$$\Xi^{(32)} \in \mathbf{32}\,, \quad \Xi_a^{(320)} \in \mathbf{320}\,, \quad \Xi_{a_1a_2}^{(1408)} \in \mathbf{1408}\,, \quad \Xi_{a_1\ldots a_5}^{(4224)} \in \mathbf{4224}\,, \tag{A.9}$$

where the indices $a_1 \cdots a_n$ are antisymmetrized, and each of them satisfies $\Gamma^a\Xi_{ab_1\cdots b_n} = 0$. One can easily compute the coefficients of the explicit decomposition into the irreducible basis, obtaining: [18], [30]:

$$\Psi \wedge \overline{\Psi} \wedge \Gamma_a\Psi = \Xi_a^{(320)} + \frac{1}{11}\Gamma_a\Xi^{(32)}, \tag{A.10}$$

$$\Psi \wedge \overline{\Psi}\Gamma_{a_1a_2}\Psi = \Xi_{a_1a_2}^{(1408)} - \frac{2}{9}\Gamma_{[a_2}\Xi_{a_2]}^{(320)} + \frac{1}{11}\Gamma_{a_1a_2}\Xi^{(32)}, \tag{A.11}$$

$$\begin{aligned} \Psi \wedge \overline{\Psi} \wedge \Gamma_{a_1\ldots a_5}\Psi = {} & \Xi_{a_1\ldots a_5}^{(4224)} + 2\Gamma_{[a_1a_2a_3}\Xi_{a_4a_5]}^{(1408)} + \frac{5}{9}\Gamma_{[a_1\ldots a_4}\Xi_{a_5]}^{(320)} \\ & -\frac{1}{77}\Gamma_{a_1\ldots a_5}\Xi^{(32)}. \end{aligned} \tag{A.12}$$

Appendix B. Maurer–Cartan equations and curvatures

Let us consider a (possibly graded) Lie group G, with tangent space spanned by the Lie algebra $\mathfrak{G}$.[43] Let $\{T_A\}$ be the generators of $\mathfrak{G}$, with $A = 1, \ldots, \dim \mathfrak{G}$, and commutation relations

$$[T_A, T_B] = C^C{}_{AB} T_C \,, \tag{B.1}$$

whose consistency relies on the Jacobi identities

$$[[T_A, T_B], T_C] + [[T_B, T_C], T_A] + [[T_C, T_A], T_B] = 0 \tag{B.2}$$

implying:

$$C^L{}_{[AB} C^D{}_{C]L} = 0 \,. \tag{B.3}$$

The same algebra can be expressed, in a dual way, in terms of left-invariant 1-forms σ^A, spanning a basis of the cotangent space of the group manifold G (also called Maurer–Cartan 1-forms, to be referred to, in the following, also as MC 1-forms) so that

$$\sigma^A(T_B) = \delta^A_B \tag{B.4}$$

and satisfying the Maurer–Cartan equations:

$$d\sigma^C + \frac{1}{2} C^C{}_{AB} \sigma^A \wedge \sigma^B = 0 \,. \tag{B.5}$$

Here, d denotes the exterior differential operator on G, which carries 1-form degree. In the dual form of the algebra, the consistency condition is encoded in the cohomological condition $d^2 = 0$, indeed:

$$\begin{aligned} 0 =& d^2\sigma^C = -\frac{1}{2} C^C{}_{AB} d\left(\sigma^A \wedge \sigma^B\right) \\ =& - C^C{}_{AB} d\sigma^A \wedge \sigma^B = -\frac{1}{2} C^C{}_{AB} C^A{}_{LM} \sigma^L \wedge \sigma^M \wedge \sigma^B \end{aligned} \tag{B.6}$$

whose validity implies (B.3). The equivalent description of the algebra in the Maurer–Cartan formulation with the one in the standard form, Eq. (B.1), further requires to define the action of the 2-form $d\sigma^C$ on a couple of tangent vectors, as follows:

$$d\sigma^C\,(T_L, T_M) = -\frac{1}{2}\sigma^C\,([T_L, T_M]) = -\frac{1}{2} C^C{}_{LM} \,. \tag{B.7}$$

The 1-forms σ^A can be thought of as the components of the algebra-valued 1-form (pure gauge):

$$\sigma \equiv g^{-1} dg = \sigma^A T_A \in \mathfrak{G} \,, \qquad g = \exp \alpha^A T_A \in G \,, \tag{B.8}$$

α_A being group parameters. Indeed, then:

$$\begin{aligned} d\sigma =& dg^{-1} \wedge dg = -g^{-1} dg \wedge g^{-1} dg \\ =& - \sigma \wedge \sigma \end{aligned} \tag{B.9}$$

[43] In the discussion here we will consider explicitly the case of a bosonic Lie algebra, but the generalization to the case of a superalgebra is conceptually straightforward.

that is, in components:

$$d\sigma^C T_C = -\sigma^A \wedge \sigma^B T_A \cdot T_B = -\frac{1}{2}\sigma^A \wedge \sigma^B [T_A, T_B]$$
$$= -\frac{1}{2} C^C{}_{AB} \sigma^A \wedge \sigma^B T_C \tag{B.10}$$

Locally, close to the origin in G (where $g \approx \mathbb{I} + \alpha^A T_A$), σ is approximated by an exact form: $\sigma \approx d\alpha^A T_A$.

In physical applications, it is often useful to generalize the notion of MC 1-forms to non left-invariant 1-forms, μ, behaving as *G-connections* on a given base manifold $\mathcal{M}(x)$, interpreted as space–time, of the fiber-bundle structure $[\mathcal{M}, G]$. We will sometimes refer to them in the text as "soft forms". They can be defined as:

$$\mu(g, x) = g^{-1} \mathring{\mu}(x) g + g^{-1} dg \,, \tag{B.11}$$

where $\mathring{\mu}$ is a $\mathfrak{G}$-valued 1-form on $\mathcal{M}(x)$, and they do not satisfy the Maurer–Cartan equations (B.5), but instead:

$$R(x, g) \equiv d\mu + \mu \wedge \mu \tag{B.12}$$
$$= g^{-1} \left[d\mathring{\mu}(x) + \mathring{\mu}(x) \wedge \mathring{\mu}(x) \right] g = g^{-1} \mathring{R}(x) g = R^A(x, g) T_A \,.$$

The quantity

$$\mathring{R}(x) \equiv d\mathring{\mu}(x) + \mathring{\mu}(x) \wedge \mathring{\mu}(x) \,, \tag{B.13}$$

expressing the failure of the 1-forms $\mathring{\mu} = \mathring{\mu}^A(x) T_A$ to satisfy the MC equations, is an algebra-valued 2-form on $\mathcal{M}$, the curvature (or field-strength), and it is a tensor in the co-adjoint representation of G. In components, it reads:

$$\mathring{R}(x) = \mathring{R}^C(x) T_C = \left(d\mathring{\mu}^C + \frac{1}{2} C_{AB}{}^C \mathring{\mu}^A \wedge \mathring{\mu}^B \right) T_C \,. \tag{B.14}$$

If we now expand the 2-form (B.12) on a basis of 1-forms in G, that is:

$$R^A(x, g) = R^A{}_{BC}(x, g) \mu^B \wedge \mu^C \,, \tag{B.15}$$

then the expression (B.12) can be rewritten in the suggestive, equivalent form:

$$d\mu^C + \frac{1}{2} \left[C^C{}_{AB} - 2R^C{}_{AB}(x, g) \right] \mu^A \wedge \mu^B = 0 \,, \tag{B.16}$$

which shows that the non left-invariant 1-forms μ^A satisfy a would-be MC equation, but in terms of *structure functions* on space–time:

$$\mathcal{C}^C{}_{AB}(x) \equiv C^C{}_{AB} - 2R^C{}_{AB}(x, g) \,, \tag{B.17}$$

instead of the *structure constants* $C^C{}_{AB}$.

Appendix C. Gauge transformations versus diffeomorphisms

We would like to show here, in an explicit way, how a diffeomorphism reduces to a gauge transformation when the curvatures are horizontal, while it differs by curvature terms in the general case. We perform the derivation in a general group-theoretical setting so that it may apply to any (softened) group or supergroup $\tilde{G}$, locally equivalent to a Lie group G, that is to any fiber bundle with the group G as its fiber.

An infinitesimal element of $\tilde{G}$ is given by a tangent vector on $\tilde{G}$, $\vec{t} = \epsilon^M T_M$, with $\epsilon^M = \delta x^M$, where the middle alphabet Latin capital indices are coordinate indices on $\tilde{G}$. Using the vielbein μ^A of the whole (soft) group $\tilde{G}$ we can rewrite a tangent vector $\vec{t}$ as follows:

$$\epsilon = \epsilon^A \tilde{T}_A, \tag{C.1}$$

where $\epsilon^A = \epsilon^M \mu^A_M$, and $\tilde{T}_A = T_M\, \mu^M_A$. Here $\tilde{T}_A$ is the vector field generator dual to the non left-invariant 1-form μ^A, $\mu^A(\tilde{T}_B) = \delta^A_B$, and $\epsilon^A = \delta x^A$ is the infinitesimal parameter associated to the shift. An infinitesimal diffeomorphisms generated by ϵ^A is given by the Lie derivative

$$\begin{aligned} \ell_\epsilon \mu^A &= (\iota_\epsilon d + d\iota_\epsilon)\, \mu^A = \\ &= \iota_\epsilon d\mu^A + d\left(\iota_{(\epsilon)}\mu^A\right) \\ &= \iota_\epsilon d\mu^A + d\epsilon^A\ . \end{aligned} \tag{C.2}$$

where ι_ϵ is the contraction operator along $\epsilon = \epsilon^B \tilde{T}_B$.

Adding and subtracting $C^A{}_{BC}\mu^B \wedge \mu^C$ to $d\mu^A$ and using the definition of the covariant derivative

$$\nabla\epsilon^A = d\epsilon^A + C^A{}_{BC}\mu^B\, \epsilon^C, \tag{C.3}$$

we find :

$$\ell_\epsilon \mu^A = \iota_\epsilon \left(d\mu^A + \frac{1}{2} C^A{}_{BC}\mu^B \wedge \mu^C \right) - \epsilon^B C^A{}_{BC}\mu^C + d\epsilon^A. \tag{C.4}$$

where we have used the antisymmetry of $C^A{}_{BC}$ in the lower indices. The terms in brackets define the curvature R^A while the other two terms, using the antisymmetry of the structure constants in (B, C) define the gauge covariant differential of ϵ^A. Therefore, using the *anholonomized* parameter[44] ϵ^A, the Lie derivative can be written as follows:

$$\ell_\epsilon \mu^A = (\nabla\epsilon)^A + \iota_\epsilon R^A\ . \tag{C.5}$$

Hence *an infinitesimal diffeomorphism on the manifold* $\tilde{G}$ *is a* G-*gauge transformation plus curvature correction terms*.

In particular if the curvature R^A has vanishing projection along the — vector $\epsilon^B \tilde{T}_B$, where B is an adjoint index of the subgroup $H \subset \tilde{G}$ so that

$$\iota_\epsilon R^A \equiv \epsilon^B R^A{}_{BC}\mu^C = 0, \tag{C.6}$$

then *the action of the Lie derivative* ℓ_ϵ *coincides with a gauge transformation*. In this case we recover the result that the curvatures are *horizontal* along the $H \subset G$ directions, in which case the group manifold itself acquires the structure of a principal fiber bundle whose base manifold, (super)space, can be identified with $\tilde{G}/H$, H being the gauge group.

[44] By anholonomized parameter we mean that we are using the rigid group index of the vielbein μ^A.

We stress that the derivation of the formula in Eq. (C.5), makes no explicit reference to the specific group $\tilde{\mathrm{G}}$. It holds for any group, including supergroups, as we can see in the supergravity case.

Appendix D. On the equivalence of FDA with the classical definition of L-infinity algebra

We report in this Appendix part of the content of Ref. [52] showing that the definition of CIS/FDA structures for the extension of Lie algebras to higher p-forms structures[45] gives a dual formulation of an L_∞-algebra.

To show this, let us first shortly remind the definition of an L_∞ algebra.

An L_∞ algebra is defined as:

- a $\mathbb{Z}$-graded vector space g;
- For each $n \in \mathbb{N}$, a multilinear map l_n, called the *n-ary bracket*, of the form

$$l_n(\dots) = [-,-,\dots,-] : g \bigotimes \cdots \bigotimes g \to g \tag{D.1}$$

and of degree $n-1$, such that the following conditions hold:

1. (**graded skew symmetry**) : each l_n is graded antisymmetric, in that for every permutation σ and for every n-tuple of homogeneously graded $v_i \in g$ then:

$$l_n\left(v_{\sigma_1}, v_{\sigma_2} \dots v_{\sigma_n}\right) = \chi(\sigma, v_1, \dots v_n) l_n\left(v_1, v_2 \dots v_n\right) \tag{D.2}$$

where the graded signature $\chi(\sigma, v_1, \dots v_n)$ is defined as the product of the signature of the permutation times a factor $(-1)^{|v^i||v^j|}$ for each interchange of neighbors $\left(\dots v_i v_j \dots\right)$ to $\left(\dots v_j v_i \dots\right)$ involved in the decomposition of the permutation as a sequence of swapping neighboring pairs. Note that this definition of χ matches our law sign of Eq. (4.5)namely the Koszul sign law.
2. (**strong homotopy Jacobi identity**):
For all $n \in \mathbb{N}$ and for all n-tuples $\left(v_1, v_2 \dots v_n\right)$ of homogeneously graded elements v_i, the following equation holds:

$$\sum_{i,j(i+j=n+1)} \left[\sum_{\sigma\, \varepsilon Unsh(i,j-1)} \chi(\sigma, v_1, \dots v_n) (-1)^{i(j-1)} l_j\left(l_i\left(v_{\sigma_1}, v_{\sigma_2} \dots v_{\sigma_i}\right), v_{\sigma_{i+1}}, \dots v_{\sigma_n}\right) \right] = 0 \tag{D.3}$$

where the sum "Unsh"= "Unshuffled" means that we must sum over all the permutations of $(1, 2, \dots, n)$ that keep $i_1, \dots, i_j$ and $i_{j+1}, \dots, i_n$ in the same relative order.

Actually, one equivalent definition can be obtained passing to the degreewise finite dimensional *dual graded vector space* of the Grassmann algebra of the p-forms, that is to the dual FDA. In this case one has a *semifree differential graded algebra* and the

[45] Actually, they are extended Chevalley–Eilenberg Lie algebras, "Chevalley-Eilenberg algebra".

Grassmann algebra is naturally equipped with an ordinary differential, namely the exterior derivative $d : d^2 = 0$.

The action of the differential d on a set of basic elements t^a of the Grassman algebra is written in the following way:

$$dt^a = -\sum_{k=1}^{\infty} \frac{1}{k!}[t_{a_1} \dots t_{a_k}]^a \, t^{a_1} \wedge \dots \wedge t^{a_k} \, , \tag{D.4}$$

where the multiple bracket $[t_{a_1} \dots t_{a_k}]^a$ is introduced. Comparing with (4.4) we see that we can identify $[t_{a_1} \dots t_{a_k}]^a$ with the generalized structure constants $C^{A(p)}_{B_{1(p_1)}B_{2(p_2)}\dots B_{n(p_n)}}$ of (4.4), and that the t^{a_i} span a basis of p-forms of the Grassmann algebra, so that they can be identified with our $\Theta^{B_i(p_i)}$ (the p_i are the form degree of Θ^{a_i}). The d^2 operator gives:

$$\begin{aligned} d\, dt^a &= -d \sum_{k=1}^{\infty} \frac{1}{k!}[t_{a_1} \dots t_{a_k}]^a \, t^{a_1} \wedge \dots \wedge t^{a_k} = \\ &= \sum_{k,l}^{\infty} \frac{1}{(k-1)!\, l!}[[t_{b_1} \dots t_{b_l}]\, t_{a_2} \dots t_{a_k}]^a \, t^{b_1} \wedge \dots \wedge t^{b_l} \wedge t^{a_2} \wedge \dots \wedge t^{a_k} = 0 \end{aligned} \tag{D.5}$$

which of course, given the previous identifications, coincides with (4.6). The important observation now is that the wedge products of the t^{a_i} (as for the equivalent $\Theta^{B_i(p_i)}$ forms) project the nested brackets onto their graded symmetric components. This occurs because one can sum over all permutations σ of the $k + l - 1$ indices weighted with the Koszul phase of the permutation, which was identified, in the FDA formalism, with the phase $(-1)^{B_i B_{i+1} + p_i p_{i+1}} = (-1)^\sigma$ as it is shown in (4.5).

It follows that we can rewrite the right-hand side of (D.5) as follows:

$$\sum_{k,l}^{\infty} \frac{1}{(k+l-1)!} \sum_{\sigma \varepsilon Unsh(l,k-1)} (-1)^\sigma \frac{1}{(k-1)!\, l!} \, [[t_{b_1} \dots t_{b_l}]\, t_{a_2} \dots t_{a_k}]^a \wedge \dots \wedge t_{b_l} \wedge t^{a_2} \wedge \dots t^{a_k} = 0 \, . \tag{D.6}$$

Now the sum over all permutations can be decomposed into a sum over the $(l, k-1)$ *unshuffled*, and a sum over permutations inside the first l and the last $k-1$ indices. These latter permutations do not change the graded symmetry of the nested brackets, since the same permutation acts on the t^{a_i} forms. As there are $(k-1)!\, l!$ of them, Eq. (D.6) can be rewritten as follows:

$$\sum_{k,l=1}^{\infty} \frac{1}{(k+l-1)!} \sum_{\sigma \varepsilon Unsh\,(l,k-1)} (-1)^\sigma \, [[t_{a_1} \dots t_{a_l}]\, t_{a_{l+1}} \dots t_{a_{k+l-1}}] t^{a_1} \wedge \dots \wedge t^{a_{k+l-1}} = 0 \, . \tag{D.7}$$

Therefore the condition $d^2 = 0$ is equivalent to the condition

$$\sum_{k+l=n-1} \sum_{\sigma \varepsilon Unsh\,(l,k-1)} (-1)^\sigma \, [[t_{a_1} \dots t_{a_l}]\, t_{a_{l+1}} \dots t_{a_{k+l-1}}] = 0 \tag{D.8}$$

that reproduces the condition of strong homotopy identity (D.3), and therefore defines an L_∞ algebra.

Appendix E. Coefficients in the hidden-algebra description of D=11 FDA

To satisfy Eq. (5.7), the coefficients T_i, S_j, E_j, with $i = 1, \dots, 5$, $j = 1, 2, 3$, should satisfy the following set of algebraic equations (from [7,8]):

$$\begin{cases} T_0 - 2S_1E_1 - 1 = 0\,, \\ T_0 - 2S_1E_2 - 2S_2E_1 = 0\,, \\ 3T_1 - 8S_2E_2 = 0\,, \\ T_2 + 10S_2E_3 + 10S_3E_2 = 0\,, \\ 120T_3 - S_3E_1 - S_1E_3 = 0\,, \\ T_2 + 1200S_3E_3 = 0\,, \\ T_3 - 2S_3E_3 = 0\,, \\ 9T_4 + 10S_3E_3 = 0\,, \\ S_1 + 10S_2 - 720S_3 = 0\,, \\ E_1 + 10E_2 - 720E_3 = 0\,, \end{cases} \tag{E.1}$$

which are solved by:

$$\begin{cases} T_0 &= \frac{1}{6} + \alpha, \\ T_1 &= -\frac{1}{90} + \frac{1}{3}\alpha, \\ T_2 &= -\frac{1}{4!}\alpha, \\ T_3 &= \frac{1}{(5!)^2}\alpha, \\ T_4 &= -\frac{1}{3[2!\cdot(3!)^2\cdot 5!]}\alpha, \end{cases} \tag{E.2}$$

$$\begin{cases} S_1 &= \frac{1}{2C}\left(\frac{10}{5!} + \sqrt{\frac{\alpha}{5!}}\right)\,, \\ S_2 &= \frac{1}{2C}\left(-\frac{1}{5!} + \frac{1}{2}\sqrt{\frac{\alpha}{5!}}\right)\,, \\ S_3 &= \frac{1}{2C}\frac{1}{5!}\sqrt{\frac{\alpha}{5!}} \end{cases} \tag{E.3}$$

$$\begin{cases} E_1 &= 5!C\left(-\frac{10}{5!} + \sqrt{\frac{\alpha}{5!}}\right)\,, \\ E_2 &= 5!C\left(\frac{1}{5!} + \frac{1}{2}\sqrt{\frac{\alpha}{5!}}\right)\,, \\ E_3 &= 5!C\left(\frac{1}{5!}\sqrt{\frac{\alpha}{5!}}\right)\,, \end{cases} \tag{E.4}$$

α being a free parameter on which the hidden algebra discussed in Section 5 depends, while C is a spurious coefficient due to the fact that Eqs. (E.1) contain the parameters S_i and E_j (with $i, j = 1, 2, 3$) always homogeneously, or in the combination S_iE_j. For the same reason, given the solutions (E.3) and (E.4), also the set of coefficients with interchanged values $2C\,S_i \leftrightarrow -\frac{1}{5!C}\,E_i$ is an equivalent solution to (E.1). A particularly symmetric choice of C is $C = i\frac{1}{\sqrt{5!2!}}$. Finally, we note that the relations presented here, and in particular (E.3) and (E.4), look different, and are in fact more explicit, from the equivalent formulas in [8].

Data availability

Data will be made available on request.

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