Geometric relational framework for general-relativistic gauge field theories

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Geometric Relational Framework for General-Relativistic Gauge Field Theories

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In loving memory of Olivia, who nurtured her young nephew's nascent passion for the universe, and whose curiosity, creative energy, and dedication continue to shine upon us.

It is recalled how relationality arises as the core insight of general-relativistic gauge field theories from the articulation of the generalized hole and point-coincidence arguments. Hence, a compelling case for a manifestly relational framework ensues naturally. A formulation for such a framework is proposed, based on a significant development of the dressing field method of symmetry reduction. A version for the group Aut(P) of automorphisms of a principal bundle P over a manifold M is first developed, as it is the most natural and elegant, and as P hosts all the mathematical structures relevant to general-relativistic gauge field theory. However, as the standard formulation is local, on M, the relational framework for local field theory is then developed. The generalized point-coincidence argument is manifestly implemented, whereby the physical field-theoretical degrees of freedoms co-define each other and define, coordinatize, the physical spacetime itself. Applying the framework to General Relativity, relational Einstein equations are obtained, encompassing various notions of "scalar coordinatization" à la Kretschmann–Komar and Brown–Kuchař.

1. Introduction

A key moment in the development of General Relativity (GR) was the realization by Einstein of the meaning of diffeomorphism covariance in the theory: the fact that spacetime points, or regions, are defined relationally, via field values coincidences, and so are, by extension, the physical degrees of freedoms (d.o.f.) of those fields. The conclusion results from the articulation of the famous "hole argument" and "point-coincidence argument". This essential physical insight, *relationality*, at the heart of general-relativistic physics is curiously often overlooked. Which unfortunately leads to misleading statements spreading frictionlessly in the modern technical literature.

In Section 2, we therefore propose to briefly recapitulate the logic behind the relational picture in general-relativistic gauge field theory (gRGFT), detailed in ref. [1]. We start with reminding the logic as it arises in GR, and then also argue that a similar line of arguments makes a strong case for the notion that the gauge principle (GP) in gauge field theory (GFT) is a way to encode the relational character of gauge physics. For this, one has to admit that the internal d.o.f. of gauge fields are probing an enriched spacetime whose points are not structureless: such a space is described by the geometry of a fiber bundle P (bundle geometry being widely recognized as the foundation of classical GFT). We articulate the dialectics arising from requiring "gauge invariance" under both passive and active gauge transformations on a bundle. In such a space, we argue that relationality results from the conjunction of an "internal hole argument" and an "internal point-coincidence argument". Finally, bringing GR and GFT together, we thus conclude that

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the paradigmatic core of the framework of gRGFT is the relational character of the physics it describes, and that it is encoded in the requirement of covariance under the diffeomorphism and gauge groups, its local covariance groups.

Therefore, the current default formalism of gRGFT is manifestly covariant and tacitly relational. It is certainly a worthy endeavor to search for an invariant and manifestly relational formulation of gRGFT, which would have many advantages: its observables and fundamental d.o.f. would be easily identified, its field equations would have a well-posed Cauchy problem, its quantization would possibly be more easily achieved (an idea going back at least to Dirac[2,3]), etc. In this paper, we propose such a formulation based on the dressing field method (DFM) of symmetry reduction[4–8] – see also ref. [9]. The latter being best understood in terms of the geometry of field space, we will give our account of this geometry, *twice over*.

First, in Section 3, we shall consider what we call the "global field space", i.e., the space of fields (differential forms) on a finite-dimensional principal bundle P over a base manifold M, with structure Lie group H. This global field space will be described as an infinite-dimensional principal bundle with structure group the automorphism group $\mathrm{Aut}(P)$ of P which contains (so to speak) both diffeomorphisms of its base manifold M and the subgroup of vertical automorphisms, also known as (a.k.a.) the gauge group $\mathcal{H}$ of P. We further frame integration on P as an operation on what we call the "associated bundle of regions" of P. Then, in Section 4, we develop the DFM for $\mathrm{Aut}(P)$: in a nutshell, the method amounts to a systematic algorithm to build *basic* objects on the global field space, and on the bundle of regions. In the latter case, a notion of field-dependent regions of P arises naturally.

The standard formulation of field theory is not done on P, but rather on its base manifold M, in what one may call "bundle coordinate patches": field theory is thus done non-intrinsically, up-to bundle coordinate changes (a.k.a. "gauge choices"). Yet the bundle P is the natural space hosting the mathematical structures relevant to field theory. In that respect, it may seem that a more fundamental formulation of gRGFT on P is lacking. It is as if the only available, state of the art, formulation of GR was its coordinate tensor calculus version, rather than its formulation via intrinsic differential geometric methods. When such a bundle formulation of gRGFT is available, our formalism will allow its relational formulation. This is one of the motivation for pursuing it to the extent we do.

Another motivation is that it gives a (much simpler) template for the formalism as it applies to local field theory. The latter requires to elaborate on the bundle geometry of what we call the "local field space", i.e., fields on M that are the local representatives of global objects living on P. The structure group of the local field space is the group $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$ – with $\mathcal{H}_{\mathrm{loc}}$ the local gauge group of internal gauge theory – usually understood to be the covariance group of gRGFT. We develop this bundle geometry in Section 5, also framing integration as an operation on the associated bundle of regions of M. The DFM then implies to build basic objects on the local field space and on its associated bundle of regions. The physical relational spacetime arises from the notion of dressed integrals. The formalism is applied to obtain a manifestly relational and invariant reformulation of a gRGFT. We illustrate it on GR and GR coupled to (scalar) electromagnetism (EM). In the conclusion 6, we take stock of our results and sketch further developments of our program. Appendices complete the main text.

2. Relationality in General-Relativistic Gauge Field Theory

The statement that physics is relational may appear too obvious to mention. At an elementary level it may be understood as a simple kinematical proposition, meaning that physical objects evolve with respect to (w.r.t.) each other. General-relativistic physics adds two fundamental refinements. The first is that the very definition of physical objects has to be relational, so that they *co-define*, and evolve w.r.t., each others. Secondly, it takes this refinement so seriously so as to insist that there are no physical entities that can influence others without being influenced in return: as it is often phrased "nothing can act upon without being acted upon". This implies not only that no physical structure may constitute an absolute reference (i.e., a fixed background) but also that these are unnecessary. They arise only as limit cases of more fundamental dynamical entities. This is known as background independence.

It is usually much less widely appreciated that the physics of GFT shares this fundamental relational character, so that it is a core insight of the union of the two: gRGFT. In this global framework, relationality emerges from the requirement of heuristic symmetry principles, the general covariance (GC) and GPs, understood as principles of "democratic epistemic access" to Nature, according to which there are no privileged situated viewpoints to contemplate the law of Physics. The dialectics between the generalized hole argument and point-coincidence argument is key to the outcome. The precise logic is laid out in detail in ref. [1]. As it serves as motivational background for this work, we briefly review it below.

2.1. General-Relativistic Physics

The principle of GC of field equations requires that laws of physics be indifferent to the choice of coordinates, which implies that the equations describing these laws, the field equations, be tensorial. In a first step of analysis, it means that the physics described is that of a spacetime and its (field) content faithfully modeled by a differentiable manifold M and fields ϕ defined on it, satisfying tensorial field equations $\boldsymbol{E}(\phi)$: The geometrical entities (M, ϕ) are coordinate-invariant, and model the objective – i.e., viewpoint independent – structure of spacetime and its field content.

But a second step of analysis is required. As the automorphism group of M, diffeomorphisms $\mathrm{Diff}(M)$, are also automorphisms of any *natural bundles* over M and their spaces of sections, i.e., of all geometric objects on M, in particular of the space of tensors. The field equations are then $\mathrm{Diff}(M)$-covariant, so that $\mathrm{Diff}(M)$ is an automorphism group of the solution space $\mathcal{S} \coloneqq \{\phi \mid \boldsymbol{E}(\phi) = 0\}$, which is furthermore *foliated into orbits*: any solution $\phi \in \mathcal{S}$ has a $\mathrm{Diff}(M)$-orbit $\mathcal{O}_\phi \subset \mathcal{S}$.[1] This has fundamental consequences, stemming from articulating Einstein's famous hole argument and point-coincidence argument.[10–13]

1 The action of $\mathrm{Diff}(M)$ is a priori not free, a solution may have Killing symmetries, $\mathsf{K}_\phi \coloneqq \{\psi \in \mathrm{Diff}(M) \mid \psi^*\phi = \phi\} \neq \mathrm{id}_M$.

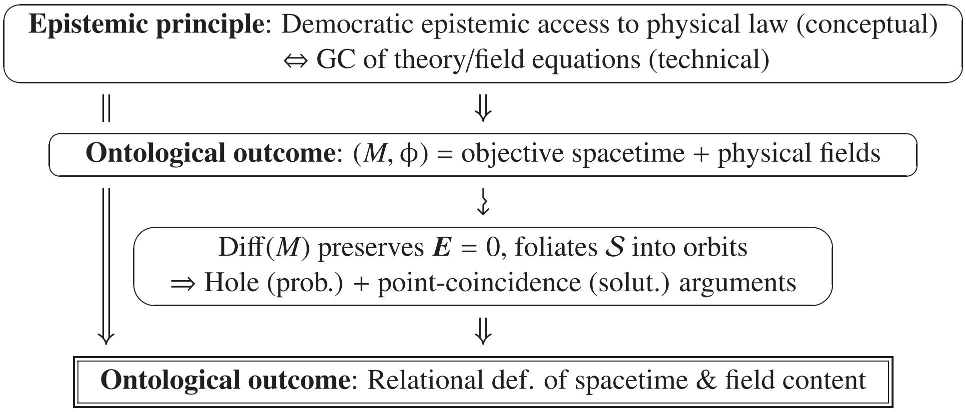

Figure 1. Relationality in the general-relativistic framework.

The hole argument highlights the consequence of the collision between Diff(M)-covariance of $E(\phi) = 0$ and the view that (M, ϕ) faithfully represent a physical state of affair. It is standard to phrase it in terms of solutions $\phi, \phi' \in \mathcal{O}_\phi$, i.e., $\phi' = \psi^*\phi$, such that (s.t.) ψ is a compactly supported diffeomorphism whose support $D_\psi \subset M$ is the "hole": Manifestly, such a situation raises an issue with the Cauchy problem, and in particular with the initial value problem, i.e., with determinism. To avoid these, two options are available: One may either renounce GC of the field equations – which was envisaged by Einstein in the 1913-1915 period[10] – or one may conclude that all solutions within the same Diff(M)-orbit $\mathcal{O}_\phi$ represent the *same* physical state. This one-to-many correspondence between a physical state and its mathematical descriptions in general/Diff(M)-covariant theories means they are unable to physically distinguish between Diff(M)-related solutions of $\boldsymbol{E} = 0$, and consequently make no physical distinction between Diff(M)-related points of M. In other words, spacetime and its field content are not described by (M, ϕ) only, but by its Diff(M)-class.

This fact, far from being a drawback of the formalism, encodes the essential insight of GR physics, which Einstein identified through his famous point-coincidence argument: it is the apparently obvious observation that physical interactions – thus all measurements – happen as spacetime coincidences of the objects involved, and that the description of such coincidences is Diff(M)-invariant. We may write this statement of the Diff(M)-invariance of point-wise mutual *relations* $\mathcal{R}$ among the fields in the collection $\{\phi\}$ symbolically as

$$\begin{aligned}
\mathcal{R} : \mathcal{S} \times M &\to \mathcal{S} \times M/\!\sim \\
&\xrightarrow{\simeq} \text{Relational spatiotemporal physical d.o.f.}, \\
(\phi, x) &\mapsto (\phi, x) \sim \left(\psi^*\phi, \psi^{-1}(x)\right) \\
&\mapsto \mathcal{R}(\phi; x) = \mathcal{R}\left(\psi^*\phi; \psi^{-1}(x)\right),
\end{aligned} \tag{1}$$

where $\mathcal{S} \times M/\!\sim$ is the quotient of $\mathcal{S} \times M$ by the equivalence relation $(\phi, x) \sim \left(\psi^*\phi, \psi^{-1}(x)\right)$, i.e., it is the space of these equivalence classes.[2] The point-coincidence argument not only dissolves the apparent indeterminism issue raised by the hole argument, but taken to its logical conclusion it may be read both ways: (1) can be understood to mean first that physical spacetime points are *defined*, or *individuated*, as relational coincidences of distinct physical field-theoretical d.o.f., and then, second, that these d.o.f. are not instantiated within the individual, mathematical, fields $\{\phi\}$ but by the *relations* among them.

In summary, the ultimate (ontological) consequence of the (epistemic) GC principle, resulting from the conjunction of the hole and point-coincidence arguments, is the relationality of general-relativistic physics, which we may state as follows: Spacetime is relationally defined via its field content, and fields are relationally defined, and evolve, w.r.t. each other. Relationality of physics is thus what is *tacitly* encoded by the Diff(M)-covariance of a general-relativistic theory.[3] The diagram **Figure 1** summarizes the logic leading to the relational picture in GR.

An all but similar analysis can be carried through for GFT.

2.2. Gauge Field Physics

The GP requires that the laws of physics be indifferent to the choice of "gauge" representatives of the fields under consideration, which implies that the field equations describing these laws be gauge-tensorial. Classical GFT being based on the geometry of fiber bundles, one easily identifies changes of gauge representatives as changes of principal bundle coordinates, a.k.a. *local gluings*, or *passive gauge transformations*. The GP can thus be understood, in first analysis, to imply that gauge field physics describes the structure and dynamics of an *enriched spacetime*

[2] We shall meet such spaces later, in Section 3.5.1, understood as "associated bundles" to the field space Φ seen as an infinite-dimensional principal bundle.

[3] We may add that relationality is also a priori a feature of solutions of $\boldsymbol{E}(\phi) = 0$ with Killing symmetries $\mathsf{K}_\phi \subset$ Diff(M), the subgroup K_ϕ encoding further physical properties of a solution: e.g., signaling that ϕ has a privileged class of observers. This holds in particular for homogeneous metric solutions, $\phi = g$, among which Minkowski solution $g = \eta$. Thus, as Kretschmann hinted at, Special Relativity (SR) in fact enjoys Diff(M)-covariance. What makes it special is that its field equation $\boldsymbol{E}(g) = \text{Riem}(g) = 0$ implies that the metric decouples from other fields and has frozen dynamics (no d.o.f.), making it a background structure. The solution $g = \eta$ has a Killing group, the Poincaré group $\mathsf{K}_\eta = ISO(1, 3)$, distinguishing geodesic (inertial) observers as privileged.

and its field content, faithfully represented by a smooth principal fiber bundle P and fields ϕ defined on it, satisfying gauge-tensorial equations $\boldsymbol{E}(\phi) = 0$: The geometrical objects (P, ϕ) are bundle-coordinate invariant, and model the objective – i.e., viewpoint independent – structure of the enriched spacetime and its gauge-field content.

Now, the group of *vertical automorphisms* $\text{Aut}_v(P)$ of the bundle, isomorphic to its gauge group $\mathcal{H}$, is an automorphism group of the geometric objects defined on P, notably connections (i.e., gauge potentials) and tensorial forms (field strengths, matter fields and their covariant derivatives): it defines their *active gauge transformations*. The field equations are then $\text{Aut}_v(P)$-covariant, and $\text{Aut}_v(P)$ is thus an automorphism group of the space of solutions $\mathcal{S} := \{\phi \,|\, \boldsymbol{E}(\phi) = 0\}$, which it foliates into orbits: any solution $\phi \in \mathcal{S}$ has a $\text{Aut}_v(P)$-orbit $\mathcal{O}_\phi \subset \mathcal{S}$.[4] From this fact, one may articulate an *internal* hole argument and an *internal* point-coincidence argument.

The internal hole argument stresses the incompatibility between $\text{Aut}_v(P)$-covariance of $\boldsymbol{E}(\phi) = 0$ and the view that (P, ϕ) faithfully represent the physical structure of an enriched spacetime and its field content. It can be expressed thus: The existence of two solutions $\phi, \phi' \in \mathcal{O}_\phi$, i.e., $\phi' = \psi^*\phi$, s.t. ψ is a compactly supported vertical automorphism whose support $D_\psi \subset P$ is the "internal" hole, manifestly raises an issue with the Cauchy problem and determinism. There are again two possibilities to deal with this: One may either drop the requirement of gauge-covariance of the theory (abandon the GP), which is unadvisable given the empirical success of the GFT framework, or conclude that all solutions within the same $\text{Aut}_v(P)$-orbit $\mathcal{O}_\phi$ represent a single physical state. This is the well-know fact that in GFT there is a one-to-many correspondence between a physical state and its mathematical descriptions. A GFT is thus unable to physically distinguish between $\text{Aut}_v(P)$-related solutions of the field equations $\boldsymbol{E}(\phi) = 0$, and consequently cannot distinguish $\text{Aut}_v(P)$-related points within fibers of P either. One must then conclude that the enriched spacetime and its field content is not described by (P, ϕ), but by its $\text{Aut}_v(P)$-class.

The *active* gauge-covariance under $\text{Aut}_v(P)$ is not a mere redundancy, it encodes a fundamental physical insight, established by the internal point-coincidence argument, which is this: Only the *relative* values of the internal d.o.f. of the fields at a point $p \in P$ have physical meaning, and the description of these point-wise coincidences is $\text{Aut}_v(P)$-invariant.[5] The internal point-coincidence argument dissolves any appearance of indeterminism arising from (active) gauge symmetry of the theory and the associated internal hole argument. Taking it a step further, it can be understood to mean that points of the physical internal structure of spacetime are *defined* via coincidences of distinct internal physical field-theoretical d.o.f., and that these d.o.f. do not belong to the individual mathematical fields $\{\phi\}$ per se, but are instantiated as *internal relations* among them.

Summarizing, the ontological consequence of the GP, resulting from the internal hole and point-coincidence arguments, is the relationality of GFT physics, which one may state thus: The internal structure of spacetime is relationally defined via its field content, and internal d.o.f. are relationally defined and evolve w.r.t. each other. Relationality of gauge field physics is thus *tacitly* encoded by the $\text{Aut}_v(P)$-covariance of a GFT.[6]

The lessons from the general-relativistic and gauge field-theoretic frameworks obviously carry over to their union, which we now consider.

2.3. General-Relativistic Gauge Field Theory

Taken together, the GC principle and the GP are the starting points of the framework of gRGFT: They must be understood as principles of democratic epistemic access to Nature, technically implemented as the requirement that the field equations representing the laws of physics must be general covariant and gauge-covariant, i.e., tensorial and gauge-tensorial.

In first analysis, this means that the general-relativistic gauge physics describes the structure and dynamics of an enriched spacetime, with structureful points, and its field content. The enriched spacetime is described by a principal bundle P with structure group H, acting freely (and transitively on fibers, i.e., H-orbits), so that the space of fibers is a manifold $P/H = M$ (the base manifold) and one has the projection map $\pi : P \to M$. Thus, the geometric structure that is to describe the spatiotemporal d.o.f. is seen to emerge as a quotient space of the geometric structure describing the totality of the elementary d.o.f. (spatiotemporal + internal). Physical fields are described by geometric objects ϕ on P, satisfying tensorial and gauge-tensorial field equations $\boldsymbol{E}(\phi) = 0$. The geometric structure (P, ϕ) is coordinate and bundle-coordinate invariant, and models the objective, viewpoint independent, structure of the enriched relativistic spacetime and its gauge field content.

Now, the maximal group of transformations of a bundle P is its group of automorphisms $\text{Aut}(P)$: the subgroup of H-equivariant diffeomorphisms, preserving the fibration structure. It thus induces smooth diffeomorphisms of the space of fibers $P/H = M$: there is a surjection $\tilde{\pi} : \text{Aut}(P) \to \text{Diff}(M)$. The group of vertical automorphisms induces the identity transformation on M – i.e., it is the kernel of $\tilde{\pi}$ – and is contained as a normal subgroup, $\text{Aut}_v(P) \triangleleft \text{Aut}(P)$.[7] We have the short exact sequence (SES) of groups

$$\text{id}_P \to \text{Aut}_v(P) \simeq \mathcal{H} \xrightarrow{\triangleleft} \text{Aut}(P) \xrightarrow{\tilde{\pi}} \text{Diff}(M) \to \text{id}_M. \tag{2}$$

Naturally, $\text{Aut}(P)$ acts as an automorphism group on the space of geometric objects on P, notably connections and tensorial forms: its action on objects of those spaces defines their combined

[4] There again, the action of $\text{Aut}_v(P)$ is a priori not free: some solutions may have gauge Killing symmetries, ${}^{gt}\mathsf{K}_\phi \neq \text{id}_P$.

[5] For example, in QED, only the *relative phase* of the electromagnetic (EM) potential and charged field is meaningful. A stark illustration of this is the Aharonov–Bohm (AB) effect.

[6] Notice that relationality is also enjoyed in particular by solutions with gauge Killing symmetries ${}^{gt}\mathsf{K}_\phi \subset \text{Aut}_v(P)$ – the latter encoding further physical properties of the solution.

[7] One may define vertical diffeomorphisms of P, $\text{Diff}_v(P) := \{\psi \in \text{Diff}(P) \,|\, \pi \circ \psi = \pi\}$, so that $\text{Aut}_v(P) \subset \text{Diff}_v(P)$, which also induce $\text{id}_M \in \text{Diff}(M)$ on M. But, since these are not H-equivariant, they are not natural morphisms (arrows) in the category of principal bundles. Still, they induce *generalized* gauge transformations, so are relevant for GFT. See refs. [8, 14]. We will encounter this structure again in the next section.

Epistemic principle: Democratic epistemic access to physical law (conceptual)
$\Leftrightarrow$ General Covariance + Gauge Principle (technical)

$\Downarrow$

Ontological outcome: (P, ϕ) = objective enriched spacetime + physical fields

$\downarrow$

Aut(P) preserves $\mathcal{S}$, foliates it into orbits
$\Rightarrow$ Generalised hole (prob.) + point-coincidence (solut.) arguments

$\Downarrow$

Ontological outcome: Relational def. of the enriched spacetime & gauge field content

Figure 2. Relationality in general-relativistic gauge field theory.

active gauge transformations and diffeomorphism transformations. The field equations of a general-relativistic gauge theory are thus Aut(P)-covariant, and Aut(P), as an automorphism group of the space of solutions $\mathcal{S} \coloneqq \{\phi \,|\, \boldsymbol{E}(\phi) = 0\}$, foliates it into orbits so that a solution $\phi \in \mathcal{S}$ has an Aut(P)-orbit $\mathcal{O}_\phi \subset \mathcal{S}$. This set the stage for the *generalized* hole and point-coincidence arguments.

The generalized hole argument establishes the clash between Aut(P)-covariance of the field equations $\boldsymbol{E}(\phi) = 0$ and the initial notion that (P, ϕ) faithfully represents an enriched general-relativistic spacetime and its gauge field content. Indeed, the possibility of having two solutions $\phi, \phi' \in \mathcal{O}_\phi$, i.e., $\phi' = \psi^*\phi$, s.t. ψ is a compactly supported automorphism whose support $D_\psi \subset P$ is the "bundle hole", is an issue for the Cauchy problem and determinism. The empirical success of the framework, preventing us to abandon GC and gauge-covariance, forces to admit that physical d.o.f. have to be Aut(P)-invariant, yielding the revision: The structure and dynamics of the enriched general-relativistic spacetime and its gauge field content are modeled by the Aut(P)-class of (P, ϕ).

Again, Aut(P)-covariance, far from being a mere redundancy, encodes the fundamental physical insight brought forth by the generalized point-coincidence argument: Only the relative values of the fields at a point $p \in P$ have a physical meaning, and the description of these point-wise coincidences is invariant under automorphisms of P. The Aut(P)-invariance of point-wise mutual relations $\mathcal{R}$ between the fields $\{\phi\}$ we shall write

$$
\begin{aligned}
\mathcal{R} : \mathcal{S} \times P \to \mathcal{S} \times P/\!\sim\; &\xrightarrow{\simeq} \text{Relational physical d.o.f.},\\
(\phi, p) \mapsto (\phi, p) &\sim \left(\psi^*\phi, \psi^{-1}(p)\right) \qquad (3)\\
&\mapsto \mathcal{R}(\phi; p) = \mathcal{R}\left(\psi^*\phi; \psi^{-1}(p)\right),
\end{aligned}
$$

where $\mathcal{S} \times P/\!\sim$ is the quotient of $\mathcal{S} \times P$ by the equivalence relation $(\phi, p) \sim \left(\psi^*\phi, \psi^{-1}(p)\right)$, for $\psi \in Aut(P)$ – See Section 3.5.1. Taking the generalized point-coincidence argument to its logical conclusion implies that (3) can be understood to mean both that points of the physical enriched spacetime are *defined* via coincidences of distinct physical field-theoretical d.o.f., and that the latter are not the individual mathematical fields $\{\phi\}$ per se, but the *relational* d.o.f. established between them.

The key ontological consequence of the epistemic GC and GPs, reached via the articulation of the generalized hole and point-coincidence arguments, is the *relationality* of general-relativistic gauge physics: The enriched spacetime is relationally defined via its field content, and all physical d.o.f. are relationally defined and evolve w.r.t. each other. The diagram **Figure 2** summarizes the logic.

The above logic, establishing relationality as the conceptual core of gRGFT, would naturally extended to any (empirically successful) field theory resting on covariance/invariance under local symmetries, as these may point to an underlying geometric structure modeling objective physical entities. For example, it could apply to models of supersymmetric theory, such as supergravity or strings models, as well as higher gauge theory – relying respectively on supergeometry, higher geometry, or higher supergeometry – if ever these were to make relevant empirical contact with fundamental physics.

In the standard mathematical formulation of gRGFT, relationality of physics is tacit, encoded in the manifest invariance, or covariance, of the theory under its symmetries (Diff(M), $\text{Aut}_v(P) \simeq \mathcal{H}$, and Aut($P$)). It can thus be easily overlooked, which may lead to a number of unfortunate misconceptions. For instance, the notion that "boundaries" break Diff(M) or $\mathcal{H}$-gauge symmetries, which has the same conceptual structure as a hole argument. This notion evaporates once it is recognized that a physical boundary is relationally defined, and is invariant (under Diff(M), $\text{Aut}_v(P) \simeq \mathcal{H}$, or Aut($P$)). This and similar misconceptions, together with the various countermeasures put forward to solve the alleged issue, would be avoided had one a framework in which both relationality and strict invariance are manifest. Such a reformulation of gRGFT would also give ready access to the observables of a theory.

In the following, we propose such a framework, whose technical touchstone is the DFM. It is best formulated within the bundle geometry of the space of fields of a general-relativistic GFT. In the next Section 3, we give our account of this bundle geometry, before describing the DFM for the gRGFT framework in Section 4. Its adaptation to field theory is the object of Section 5.

3. Geometry of Global Field Space

In this section we elaborate on the geometry of field space as an infinite-dimensional fiber bundle, extending standard notions defined in the finite-dimensional context to the infinite-dimensional setting.[15,16]

3.1. Field Space as a Principal Bundle

We are interested in gauge field theories based on a principal bundle $P(M, H)$ with structure group H. For a realistic description of spinorial matter fields, the principal bundles considered must be of the form $P = Q \times OM$ with $H = G \times SO(r,s)$, where $Q(M, G)$ is a G-bundle over M and OM is the orthonormal frame bundle of M.[8] Therefore, our field space Φ is made of tensors or differential forms taking values in various representations V of H, $\Omega^\bullet(P) \otimes V$. For example (Ehresmann or Cartan) connection 1-forms and their curvature 2-forms ($V = \text{Lie}H$), or V-valued tensorial 0-forms (describing e.g., charged spinors) and their covariant derivative 1-forms. A collection of such objects will be noted as $\phi \in \Phi$, i.e., it is a single point in field space.

The group $\text{Aut}(P)$ has a natural action by pullback on Φ: Given $\phi \in \Phi$ and $\psi \in \text{Aut}(P)$ we have $\phi^\psi := \psi^*\phi$, where the left-hand side is a notation for the pullback on the right-hand side. This is a right-action since it is well-known that the pullback satisfies $(f \circ g)^* = g^* \circ f^*$ for any two smooth maps $M \xrightarrow{g} N \xrightarrow{f} Q$.[9] We therefore write

$$\begin{aligned} \Phi \times \text{Aut}(P) &\to \Phi, \\ (\phi, \psi) &\mapsto R_\psi \phi := \psi^*\phi, \end{aligned} \tag{4}$$

with indeed, for another $\psi' \in \text{Aut}(P)$: $R_{\psi'} R_\psi \phi := \psi'^* \psi^* \phi = (\psi \circ \psi')^* \phi =: R_{\psi \circ \psi'} \phi$. The field space is fibered by the action of $\text{Aut}(P)$, the fiber through a point ϕ being its orbit $\mathcal{O}(\phi)$ under automorphisms. We denote the set of orbits, or moduli space, by $\Phi/\text{Aut}(P) =: \mathcal{M}$. Under adequate restrictions of either of Φ or $\text{Aut}(P)$, the field space Φ can be understood as an infinite-dimensional principal fiber bundle over the base $\mathcal{M}$ with structure group $\text{Aut}(P)$:[10]

$$\begin{aligned} &\Phi \xrightarrow{\pi} \mathcal{M}, \\ &\phi \mapsto \pi(\phi) =: [\phi]. \end{aligned} \tag{5}$$

The projection π is s.t. $\pi \circ R_\psi = \pi$. The fiber over a point $[\phi] \in \mathcal{M}$, $\pi^{-1}([\phi]) = \mathcal{O}(\phi)$, is diffeomorphic to the structure group $\text{Aut}(P)$ as a manifold. We pass on the local bundle structure of Φ: since the space of orbits is not manageable operationally, one usually does not work on it. For details we refer the reader to ref. [8].

3.1.1. Natural Transformation Groups

As an infinite-dimensional manifold, Φ has a diffeomorphism group $\boldsymbol{Diff}(\Phi)$, but *as a principal bundle* its maximal transformation group is its group of *automorphisms*

$$\boldsymbol{Aut}(\Phi) := \left\{ \Xi \in \boldsymbol{Diff}(\Phi) \,|\, \Xi \circ R_\psi = R_\psi \circ \Xi \right\}, \tag{6}$$

whose elements preserve the fibration structure and thus project naturally as elements of $\boldsymbol{Diff}(\mathcal{M})$.

The subgroup of *vertical diffeomorphisms*

$$\boldsymbol{Diff}_v(\Phi) := \{ \Xi \in \boldsymbol{Diff}(\Phi) \,|\, \pi \circ \Xi = \pi \} \tag{7}$$

induces the identity transformation on $\mathcal{M}$: Since these are motions along fibers, to $\Xi \in \boldsymbol{Diff}_v(\Phi)$ must correspond a unique $\boldsymbol{\psi} : \Phi \to \text{Aut}(P)$ s.t. $\Xi(\phi) = R_{\boldsymbol{\psi}(\phi)}\phi := [\boldsymbol{\psi}(\phi)]^*\phi$: i.e., $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \text{Aut}(P)\right)$. Notice that the map composition law for $\boldsymbol{Diff}_v(\Phi)$ gives rise to a peculiar composition operation for $C^\infty\left(\Phi, \text{Aut}(P)\right)$: For $\Xi, \Xi' \in \boldsymbol{Diff}_v(\Phi)$ to which correspond $\boldsymbol{\psi}, \boldsymbol{\psi}' \in C^\infty\left(\Phi, \text{Aut}(P)\right)$, one finds $\Xi' \circ \Xi\,(\phi) = R_{\boldsymbol{\psi}'(\Xi(\phi))}\, \Xi(\phi) = R_{\boldsymbol{\psi}'(\Xi(\phi))}\, R_{\boldsymbol{\psi}(\phi)} \phi = R_{\boldsymbol{\psi}(\phi) \circ \boldsymbol{\psi}'(\Xi(\phi))} \phi$. Thus we have,

$$\begin{aligned} \Xi' \circ \Xi \in \boldsymbol{Diff}_v(\Phi) &\text{ corresponds to } \boldsymbol{\psi} \circ \left(\boldsymbol{\psi}' \circ R_{\boldsymbol{\psi}}\right) \\ &\in C^\infty\left(\Phi, \text{Aut}(P)\right). \end{aligned} \tag{8}$$

Remark the distinction between the composition law $\circ$ of maps on Φ, and the composition law $\circ$ of maps on P.

This is an example of the composition law of the infinite-dimensional group of bisections of a Lie groupoid.[17–19] As a mater of fact, what we have been discussing above can be reframed in the groupoid framework as follows: One defines the generalized action groupoid $\boldsymbol{\Gamma} \rightrightarrows \Phi$ with $\boldsymbol{\Gamma} = \Phi \rtimes \left(C^\infty\left(\Phi, \text{Aut}(P)\right) \simeq \boldsymbol{Diff}_v(\Phi)\right)$, and with source and target maps $s : \boldsymbol{\Gamma} \to \Phi$, $(\phi, \boldsymbol{\psi} \simeq \Xi) \mapsto \phi$, and $t : \boldsymbol{\Gamma} \to \Phi$, $(\phi, \boldsymbol{\psi} \simeq \Xi) \mapsto \Xi(\phi) = \boldsymbol{\psi}(\phi)^*\phi$. The associative composition law: $g \circ f \in \boldsymbol{\Gamma}$ for $f, g \in \boldsymbol{\Gamma}$ is defined whenever $t(f) = s(g)$. It generalizes the action groupoid $\bar{\boldsymbol{\Gamma}} = \Phi \rtimes \text{Aut}(P) \rightrightarrows \Phi$ associated with the right action of $\text{Aut}(P)$ on Φ. The group of bisections $\mathcal{B}(\boldsymbol{\Gamma})$ of $\boldsymbol{\Gamma}$ is the set of sections of s – maps $\sigma : \Phi \to \boldsymbol{\Gamma}$, $\phi \mapsto \sigma(\phi) = (\phi, \boldsymbol{\psi} \sim \Xi)$ s.t. $s \circ \sigma = \text{id}_\Phi$ – s.t. $t \circ \sigma : \Phi \to \Phi$ is invertible: thus $t \circ \sigma \in \boldsymbol{Diff}(\Phi)$. We have indeed, for $\sigma \in \mathcal{B}(\boldsymbol{\Gamma})$, that $t \circ \sigma = \Xi\,(= \boldsymbol{\psi}^*) \in \boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \text{Aut}(P)\right)$. The group law of bisections is defined as $\left(\sigma_2 \star \sigma_1\right)(\phi) := \sigma_2\left(\left(t \circ \sigma_1(\phi)\right) \circ \sigma_1(\phi)\right)$. After composition with the target map t on the left, this reproduces precisely the peculiar group law (8). Then, slightly abusing the terminology, we may refer to $C^\infty\left(\Phi, \text{Aut}(P)\right)$ as the group of bisections of the generalized action groupoid $\boldsymbol{\Gamma}$ associated to Φ. In the rest of this paper though, we will simply, and accurately, refer to it as the group of generating maps of $\boldsymbol{Diff}_v(\Phi)$.

This generalizes the subgroup of *vertical automorphisms*

$$\boldsymbol{Aut}_v(\Phi) := \left\{ \Xi \in \boldsymbol{Aut}(\Phi) \,|\, \pi \circ \Xi = \pi \right\} = \boldsymbol{Diff}_v(\Phi) \cap \boldsymbol{Aut}(\Phi), \tag{9}$$

isomorphic to the *gauge group*

$$\boldsymbol{Aut}(P) := \left\{ \boldsymbol{\psi} : \Phi \to \text{Aut}(P) \,|\, \boldsymbol{\psi}(\phi^\psi) = \psi^{-1} \circ \boldsymbol{\psi}(\phi) \circ \psi \right\} \tag{10}$$

via $\Xi(\phi) = R_{\boldsymbol{\psi}(\phi)}\,\phi$ still. The equivariance of elements $\boldsymbol{\psi}$ of $\boldsymbol{Aut}(P)$ implies that to $\Xi' \circ \Xi \in \boldsymbol{Aut}_v(\Phi)$ corresponds $\boldsymbol{\psi}' \circ \boldsymbol{\psi} \in \boldsymbol{Aut}(P)$: i.e., the composition operation $\circ$ in $\boldsymbol{Aut}_v(\Phi)$ translates to the usual composition operation $\circ$ of the group $\boldsymbol{Aut}(P)$.

We have that $\boldsymbol{Aut}(\Phi)$ is in the normalizer of $\boldsymbol{Diff}_v(\Phi)$ and, since a group is a subgroup of its normalizer, we get: $N_{\boldsymbol{Diff}(\Phi)}\left(\boldsymbol{Diff}_v(\Phi)\right) \supset \langle \boldsymbol{Diff}_v(\Phi) \cup \boldsymbol{Aut}(\Phi) \rangle$. We have the special

8 $P = Q \times OM$ is bundle over M, *not* $M \times M$: it is a *pullback*, or *fibered product* in the category of principal bundles.

9 This is clear on forms, a.k.a. contravariant tensors. On vector fields and more general covariant tensors, the pullback action is the pushforward by the inverse: $\psi^* := {\psi^{-1}}_*$. So, on general (mixed) tensors on P, ψ^* is indeed a well-defined right action of $\text{Aut}(P)$.

10 In particular, this requires that all points ϕ have trivial stability groups, meaning that we a priori reject Killing symmetries of ϕ.

case $N_{\boldsymbol{Diff}(\Phi)}\left(\boldsymbol{Aut}_v(\Phi)\right) = \boldsymbol{Aut}(\Phi)$, i.e., $\boldsymbol{Aut}_v(\Phi) \lhd \boldsymbol{Aut}(\Phi)$, which gives the SES

$$\mathrm{id}_\Phi \to \boldsymbol{Aut}(P) \simeq \boldsymbol{Aut}_v(\Phi) \xrightarrow{\lhd} \boldsymbol{Aut}(\Phi) \longrightarrow \boldsymbol{Diff}(\mathcal{M}) \to \mathrm{id}_{\mathcal{M}}, \tag{11}$$

where the image of each arrow is in the kernel of the next. One often encounters in the literature the notion of "*field-dependent*" gauge transformations and diffeomorphisms. The proper mathematical embodiment of this notion is the group $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \mathrm{Aut}(P)\right)$. Nonetheless, stricto sensu, the gauge transformations on Φ are defined via the action of the subgroup $\boldsymbol{Aut}_v(\Phi) \simeq \boldsymbol{Aut}(P)$. The structure group $\mathrm{Aut}(P)$ supplies the notion of "*field-independent*" gauge transformations and diffeomorphisms. The linearization of (11) gives a SES of Lie algebras defining the Atiyah Lie algebroid of the principal bundle Φ.

To study the differential structure of Φ, let us first recall the following general results. Consider the manifolds M, N and their tangent bundles TM, TN, together with a diffeomorphism $\psi : M \to N$ and the flow $\phi_\tau : N \to N$ of a vector field $X \in \Gamma(TN)$, s.t. $X_{|\phi_0} = \frac{d}{d\tau}\phi_\tau|_{\tau=0} \in T_{\phi_0}N$. One defines the flow

$$\varphi_\tau := \psi^{-1} \circ \phi_\tau \circ \psi : M \to M \tag{12}$$

of a vector field $Y \in \Gamma(TM)$ related to X as

$$Y := \frac{d}{d\tau}\left(\psi^{-1} \circ \phi_\tau \circ \psi\right)\Big|_{\tau=0} = (\psi^{-1})_* X \circ \psi. \tag{13}$$

We have the composition of maps $M \xrightarrow{\psi} N \xrightarrow{X} TN \xrightarrow{(\psi^{-1})_*} TM$ resulting in the above vector field $Y : M \to TM$. So, $X \in \Gamma(TN)$ and $Y \in \Gamma(TM)$ are ψ-related (13) when their flows are ψ-*conjugated* (12). Furthermore, ψ-relatedness is a morphism of Lie algebras, that is $[Y, Y'] = (\psi^{-1})_*[X, X'] \circ \psi$.

As a variation of the above, suppose ϕ is some tensor field on M and $\mathfrak{L}_X\phi$ its Lie derivative along $X \in \Gamma(TM)$ with flow φ_τ, for $\psi \in \mathrm{Diff}(M)$ we have

$$\begin{aligned}
\psi^*\left(\mathfrak{L}_X\phi\right) &= \psi^* \frac{d}{d\tau}\varphi_\tau^*\phi\Big|_{\tau=0} = \frac{d}{d\tau}(\varphi_\tau \circ \psi)^*\phi\Big|_{\tau=0} \\
&= \frac{d}{d\tau}(\varphi_\tau \circ \psi)^*\ (\psi^{-1})^*\psi^*\phi\Big|_{\tau=0} \\
&= \frac{d}{d\tau}(\psi^{-1} \circ \varphi_\tau \circ \psi)^*\ \psi^*\phi\Big|_{\tau=0} \\
&=: \mathfrak{L}_{[(\psi^{-1})_* X \circ \psi]}\,(\psi^*\phi).
\end{aligned} \tag{14}$$

3.2. Differential Structure

As a manifold, Φ has a tangent bundle $T\Phi$, a cotangent bundle $T^\star\Phi$, or more generally a space of forms $\Omega^\bullet(\Phi)$. Considering these structures in turn, it will be important to distinguish the pushforward and pullback on P and Φ: we reserve $*$ to denote these operations on P, and $\star$ for their counterparts on Φ.

3.2.1. Tangent Bundle and Subbundles

Sections of the tangent bundle $\mathfrak{X} : \Phi \to T\Phi$ are vector fields on Φ, we note $\mathfrak{X} \in \Gamma(T\Phi)$. They form a Lie algebra under the bracket of vector field $[\ ,\ ] : \Gamma(T\Phi) \times \Gamma(T\Phi) \to \Gamma(T\Phi)$. We may write a vector field at $\phi \in \Phi$ as $\mathfrak{X}_{|\phi} = \frac{d}{d\tau}\Psi_\tau(\phi)|_{\tau=0}$, with $\Psi_\tau \in \boldsymbol{Diff}(\Phi)$ its flow s.t. $\Psi_{\tau=0}(\phi) = \phi$. As derivations of the algebra of functions $C^\infty(\Phi)$ we write: $\mathfrak{X} = \mathfrak{X}(\phi)\frac{\delta}{\delta\phi}$, where $\frac{\delta}{\delta\phi}$ denotes the functional differentiation w.r.t. ϕ, and $\mathfrak{X}(\phi)$ are the functional components. The Lie bracket in the Lie algebra of $\boldsymbol{Diff}(\Phi)$ is *minus* the bracket in $\Gamma(T\Phi)$: $\mathfrak{diff}(\Phi) := \left(\Gamma(T\Phi), -[\ ,\ ]_{\Gamma(T\Phi)}\right)$.

The pushforward by the projection is $\pi_\star : T_\phi\Phi \to T_{\pi(\phi)}\mathcal{M} = T_{[\phi]}\mathcal{M}$. The pushforward by the right action of $\psi \in \mathrm{Aut}(P)$ is $R_{\psi\star} : T_\phi\Phi \to T_{\psi^*\phi}\Phi$. In general $R_{\psi\star}\mathfrak{X}_{|\phi} \neq \mathfrak{X}_{|\psi^*\phi}$, meaning that a generic vector field "rotates" as it is pushed vertically along fibers. So, in general, $\pi_\star\mathfrak{X}$ is not a well-defined vector field on the base $\mathcal{M}$: at $[\phi] \in \mathcal{M}$ the vector obtained would vary depending on where on the fiber over $[\phi]$ the projection is taken.

The Lie subalgebra of *right-invariant* vector fields, which do not rotate as they are pushed vertically, is

$$\Gamma_{\mathrm{inv}}(T\Phi) := \left\{\mathfrak{X} \in \Gamma(T\Phi)\ |\ R_{\psi\star}\mathfrak{X}_{|\phi} = \mathfrak{X}_{|\psi^*\phi}\right\}. \tag{15}$$

They have well-defined projections on $\mathcal{M}$: For $\mathfrak{X} \in \Gamma_{\mathrm{inv}}(T\Phi)$, we have $\pi_\star\mathfrak{X}_{|\psi^*\phi} = \pi_\star R_{\psi\star}\mathfrak{X}_{|\phi} = (\pi \circ R_\psi)_\star\mathfrak{X}_{|\phi} = \pi_\star\mathfrak{X}_{|\phi} =: \mathfrak{Y}_{|[\phi]} \in T_{[\phi]}\mathcal{M}$. Then, $\pi_\star\mathfrak{X} =: \mathfrak{Y} \in \Gamma(T\mathcal{M})$ is a well-defined vector field. The defining property of invariant vector fields implies that their flows are automorphisms of Φ: we have

$$R_{\psi\star}\mathfrak{X}_{|\phi} = \frac{d}{d\tau}R_\psi\Psi_\tau(\phi)\Big|_{\tau=0} \quad \text{and} \quad \mathfrak{X}_{|R_\psi\phi} = \frac{d}{d\tau}\Psi_\tau(R_\psi\phi)\Big|_{\tau=0}, \tag{16}$$

which implies $R_\psi \circ \Psi_\tau = \Psi_\tau \circ R_\psi$. The Lie subalgebra $\Gamma_{\mathrm{inv}}(T\Phi)$ is thus the Lie algebra of $\boldsymbol{Aut}(\Phi)$:

$$\boldsymbol{\mathfrak{aut}}(\Phi) = \left(\Gamma_{\mathrm{inv}}(T\Phi); -[\ ,\ ]_{\Gamma(T\Phi)}\right). \tag{17}$$

The latter should not be confused with the Lie algebra $\mathfrak{aut}(P)$ of the structure group $\mathrm{Aut}(P)$:

$$\mathfrak{aut}(P) = \left(\Gamma_{\mathrm{inv}}(TP); -[\ ,\ ]_{\Gamma(TP)}\right). \tag{18}$$

We may use the notation $[X, Y]_{\mathfrak{aut}} := -[X, Y]_{\Gamma(TP)}$ when useful.

The *vertical tangent bundle* $V\Phi := \ker \pi_\star$ is a canonical subbundle of the tangent bundle $T\Phi$. Vertical vector fields are elements of $\Gamma(V\Phi) := \{\mathfrak{X} \in \Gamma(T\Phi)\ |\ \pi_\star\mathfrak{X} = 0\}$. Since $V\Phi$ is a subbundle, $\Gamma(V\Phi)$ is a Lie ideal of $\Gamma(T\Phi)$. Indeed, since $\pi_\star : \Gamma(T\Phi) \to \Gamma(T\mathcal{M})$ is a Lie algebra morphism, we have, for $\mathfrak{X} \in \Gamma(V\Phi)$ and $\mathfrak{Y} \in \Gamma(T\Phi)$: $\pi_\star[\mathfrak{X}, \mathfrak{Y}] = [\pi_\star\mathfrak{X}, \pi_\star\mathfrak{Y}] = [0, \pi_\star\mathfrak{Y}] = 0$, i.e. $[\mathfrak{X}, \mathfrak{Y}] \in \Gamma(V\Phi)$.

We now consider the vertical vector fields induced by the respective actions of $\mathfrak{aut}(P)$, $\boldsymbol{\mathfrak{aut}}(P)$ and $\boldsymbol{\mathfrak{diff}}_v(\Phi)$. A *fundamental* vertical vector field at $\phi \in \Phi$ generated by $X = \frac{d}{d\tau}\psi_\tau|_{\tau=0} \in \mathfrak{aut}(P)$ with flow $\psi_\tau \in \mathrm{Aut}(P)$ is:

$$X^v_{|\phi} := \frac{d}{d\tau}R_{\psi_\tau}\phi\Big|_{\tau=0} = \frac{d}{d\tau}\psi_\tau^*\phi\Big|_{\tau=0} =: \mathfrak{L}_X\phi, \tag{19}$$

The Lie derivative on P is also given by the Cartan formula $\mathfrak{L}_X = [\iota_X, d] = \iota_X d + d\iota_X$, with d the de Rham exterior derivative on P. It is a degree 0 derivation of the algebra $\Omega^\bullet(P)$ of forms on P, since ι_X is of degree -1 and d is of degree 1. Manifestly, fundamental vector fields satisfy $\pi_\star X^v \equiv 0$, since $\pi_\star X^v_{|\phi} = \frac{d}{d\tau}\pi \circ R_{\psi_\tau}\phi|_{\tau=0} = \frac{d}{d\tau}\pi(\phi)|_{\tau=0}$. One shows (see Appendix B) that the map $|^v : \mathfrak{aut}(P) \to \Gamma(V\Phi)$, $X \mapsto X^v$, is a Lie algebra morphism: i.e., $([X,Y]_{\mathfrak{aut}})^v = (-[X,Y]_{\Gamma(TP)})^v = [X^v, Y^v]$. The pushforward by the right-action of Aut(P) on a fundamental vertical vector field is:

$$\begin{aligned} R_{\psi\star}X^v_{|\phi} &:= \tfrac{d}{d\tau} R_\psi \circ R_{\psi_\tau}\phi\,\Big|_{\tau=0} = \tfrac{d}{d\tau} R_{\psi_\tau\circ\psi}\phi\,\Big|_{\tau=0} \\ &= \tfrac{d}{d\tau} R_{\psi_\tau\circ\psi}\, R_{\psi^{-1}\circ\psi}\,\phi\,\Big|_{\tau=0} \\ &= \tfrac{d}{d\tau} R_{(\psi^{-1}\circ\psi_\tau\circ\psi)}\, R_\psi\,\phi\,\Big|_{\tau=0} = \tfrac{d}{d\tau} R_{(\psi^{-1}\circ\psi_\tau\circ\psi)}\,\psi^*\phi\,\Big|_{\tau=0} \\ &=: \left((\psi^{-1})_* X\circ\psi\right)^v_{|\psi^*\phi}. \end{aligned} \tag{20}$$

Therefore, fundamental vector fields generated by $\mathfrak{aut}(P)$ are not right-invariant.

On the other hand, the fundamental vector fields induced by $\boldsymbol{\mathfrak{aut}}(P)$, the Lie algebra of the gauge group $\boldsymbol{Aut}(P)$, are right-invariant. To $\boldsymbol{\psi}_\tau \in \boldsymbol{Aut}(P)$ corresponds $\boldsymbol{X} = \tfrac{d}{d\tau}\boldsymbol{\psi}_\tau|_{\tau=0} \in \boldsymbol{\mathfrak{aut}}(P)$. Given the definition (10) of the gauge group, whose elements transformation property is $R^\star_\psi \boldsymbol{\psi} = \psi^{-1}\circ\boldsymbol{\psi}\circ\psi$, by (12)–(13) we have

$$\boldsymbol{\mathfrak{aut}}(P) := \left\{ \boldsymbol{X} : \Phi \to \mathfrak{aut}(P) \mid R^\star_\psi \boldsymbol{X} = (\psi^{-1})_* \boldsymbol{X}\circ\psi \right\}. \tag{21}$$

This transformation property can also be written as: $\boldsymbol{X}(\phi^\psi) = \boldsymbol{X}(\psi^*\phi) = (\psi^{-1})_*\boldsymbol{X}(\phi)\circ\psi$. Observe that the infinitesimal version is given by the Lie derivative on Φ along the corresponding fundamental vector field:

$$\begin{aligned} \boldsymbol{L}_{X^v}\boldsymbol{X} = X^v(\boldsymbol{X}) &= \tfrac{d}{d\tau}\, R^\star_{\psi_\tau}\boldsymbol{X}\,\Big|_{\tau=0} = \tfrac{d}{d\tau}\,(\psi_\tau^{-1})_*\boldsymbol{X}\circ\psi_\tau\,\Big|_{\tau=0} \\ &=: \mathfrak{L}_X\boldsymbol{X} = [X,\boldsymbol{X}]_{\Gamma(TP)} = [\boldsymbol{X}, X]_{\mathfrak{aut}}. \end{aligned} \tag{22}$$

A fundamental vector field generated by $\boldsymbol{X} \in \boldsymbol{\mathfrak{aut}}(P)$ is

$$\boldsymbol{X}^v_{|\phi} := \tfrac{d}{d\tau} R_{\boldsymbol{\psi}_\tau(\phi)}\phi\,\Big|_{\tau=0} = \tfrac{d}{d\tau}(\boldsymbol{\psi}_\tau(\phi))^*\phi\,\Big|_{\tau=0} =: \mathfrak{L}_{\boldsymbol{X}}\phi. \tag{23}$$

Its pushforward by the right-action of Aut(P) is

$$\begin{aligned} R_{\psi\star}\boldsymbol{X}^v_{|\phi} &:= \tfrac{d}{d\tau} R_\psi \circ R_{\boldsymbol{\psi}_\tau(\phi)}\phi\,\Big|_{\tau=0} = \tfrac{d}{d\tau} R_{(\psi^{-1}\circ\boldsymbol{\psi}_\tau(\phi)\circ\psi)}\, R_\psi\,\phi\,\Big|_{\tau=0} \\ &= \tfrac{d}{d\tau} R_{\boldsymbol{\psi}_\tau(\psi^*\phi)}\,\psi^*\phi\,\Big|_{\tau=0} =: \boldsymbol{X}^v_{|\psi^*\phi}. \end{aligned} \tag{24}$$

Furthermore, one shows (see Appendix B) that the "verticality map" $|^v : \boldsymbol{\mathfrak{aut}}(P) \to \Gamma_{\text{inv}}(V\Phi)$, $\boldsymbol{X} \mapsto \boldsymbol{X}^v$, is a Lie algebra *anti*-morphism: i.e., $([\boldsymbol{X},\boldsymbol{Y}]_{\mathfrak{aut}})^v = (-[\boldsymbol{X},\boldsymbol{Y}]_{\Gamma(TP)})^v = -[\boldsymbol{X}^v, \boldsymbol{Y}^v]$. Therefore, since the Lie subalgebra of right-invariant vertical vector fields is the Lie algebra of the group $\boldsymbol{Aut}_v(\Phi)$, we have

$$\boldsymbol{\mathfrak{aut}}(P) \simeq \boldsymbol{\mathfrak{aut}}_v(\Phi) = \left(\Gamma_{\text{inv}}(V\Phi); -[\ ,\ ]_{\Gamma(T\Phi)}\right). \tag{25}$$

We can thus write the infinitesimal version of (11), i.e., the SES describing the Atiyah Lie algebroid of the bundle Φ:

$$0 \to \boldsymbol{\mathfrak{aut}}(P) \simeq \boldsymbol{\mathfrak{aut}}_v(\Phi) \xrightarrow{|^v} \boldsymbol{\mathfrak{aut}}(\Phi) \xrightarrow{\pi_\star} \boldsymbol{\mathfrak{diff}}(\mathcal{M}) \to 0, \tag{26}$$

A splitting of this SES, i.e., the datum of a map $\boldsymbol{\mathfrak{aut}}(\Phi) \to \boldsymbol{\mathfrak{aut}}(P)$ – or equivalently of a map $\boldsymbol{\mathfrak{diff}}(\mathcal{M}) \to \boldsymbol{\mathfrak{aut}}(\Phi)$ – which would allow to decompose a (right-invariant) vector field on Φ as a sum of a gauge element and a vector field on $\mathcal{M}$, is supplied by a choice of Ehresmann connection 1-form on Φ.

Finally, consider the Lie algebra of the group of vertical diffeomorphisms $\boldsymbol{Diff}_v(\Phi)$:

$$\boldsymbol{\mathfrak{diff}}_v(\Phi) := \left\{ \boldsymbol{X}^v_{|\phi} = \tfrac{d}{d\tau}\Xi_\tau(\phi)\Big|_{\tau=0} = \tfrac{d}{d\tau} R_{\boldsymbol{\psi}_\tau(\phi)}\phi\Big|_{\tau=0} \in \Gamma(V\Phi) \right\}, \tag{27}$$

where $\Xi_\tau \in \boldsymbol{Diff}_v(\Phi)$, and $\boldsymbol{\psi}_\tau \in C^\infty(\Phi, \text{Aut}(P))$ is the flow of $\boldsymbol{X} : \Phi \to \Gamma_{\text{inv}}(TP) \simeq \mathfrak{aut}(P)$. Therefore, we have that $\boldsymbol{\mathfrak{diff}}_v(\Phi) \simeq C^\infty(\Phi, \mathfrak{aut}(P))$. The pushforward of $\boldsymbol{X}^v$ by the action of Aut(P) is the same as for a fundamental vector field (20),

$$R_{\psi\star}\boldsymbol{X}^v_{|\phi} = \left((\psi^{-1})_*\boldsymbol{X}\circ\psi\right)^v_{|\psi^*\phi}. \tag{28}$$

The map $|^v : C^\infty(\Phi, \mathfrak{aut}(P)) \to \Gamma(V\Phi)$ is a Lie algebra morphism, yet the bracket on $C^\infty(\Phi, \mathfrak{aut}(P))$ extends the bracket in $\mathfrak{aut}(P)$ taking into account the Φ-dependence of its elements. Indeed, for $\boldsymbol{X}^v, \boldsymbol{Y}^v \in \boldsymbol{\mathfrak{diff}}_v(\Phi)$ one has

$$\begin{aligned} [\boldsymbol{X}^v, \boldsymbol{Y}^v]_{\Gamma(T\Phi)} &= \{-[\boldsymbol{X},\boldsymbol{Y}]_{\Gamma(TP)}\}^v + [\boldsymbol{X}^v(\boldsymbol{Y})]^v - [\boldsymbol{Y}^v(\boldsymbol{X})]^v \\ &= \left\{[\boldsymbol{X},\boldsymbol{Y}]_{\mathfrak{aut}} + \boldsymbol{X}^v(\boldsymbol{Y}) - \boldsymbol{Y}^v(\boldsymbol{X})\right\}^v =: \{\boldsymbol{X},\boldsymbol{Y}\}^v. \end{aligned} \tag{29}$$

The result is proven in ref. [14] for the finite-dimensional case. The bracket on $C^\infty(\Phi, \mathfrak{aut}(P))$ is

$$\{\boldsymbol{X},\boldsymbol{Y}\} := [\boldsymbol{X},\boldsymbol{Y}]_{\mathfrak{aut}} + \boldsymbol{X}^v(\boldsymbol{Y}) - \boldsymbol{Y}^v(\boldsymbol{X}). \tag{30}$$

From this follows naturally that,

$$[\boldsymbol{L}_{\boldsymbol{X}^v}, \boldsymbol{L}_{\boldsymbol{Y}^v}] = \boldsymbol{L}_{[\boldsymbol{X}^v,\boldsymbol{Y}^v]} = \boldsymbol{L}_{\{\boldsymbol{X},\boldsymbol{Y}\}^v}. \tag{31}$$

The relation (29) is the infinitesimal version of (8), (30) reflecting the bisection composition law in $C^\infty(\Phi, \text{Aut}(P))$.

Echoing our discussion of the action groupoid $\boldsymbol{\Gamma}$, let us briefly recall the Lie algebroid picture of the above.[17,20] Associated to the action of $\mathfrak{aut}(P)$ on Φ, i.e., to the Lie algebra morphism $\alpha = |^v : \mathfrak{aut}(P) \to \Gamma(V\Phi) \subset \Gamma(T\Phi)$, is the action (or transformation) Lie algebroid $\boldsymbol{A} = \Phi \rtimes \mathfrak{aut}(P) \to \Phi$ with anchor $\rho : \boldsymbol{A} \to V\Phi \subset T\Phi$, $(\phi, X) \mapsto \alpha(X)_{|\phi} = X^v_{|\phi}$, inducing the Lie algebra morphism $\tilde\rho : \Gamma(\boldsymbol{A}) \to \Gamma(V\Phi) \subset \Gamma(T\Phi)$. The space of sections $\Gamma(\boldsymbol{A}) = \{\Phi \to \boldsymbol{A}, \phi \mapsto (\phi, \boldsymbol{X}(\phi))\}$ is naturally identified with the space $C^\infty(\Phi, \mathfrak{aut}(P))$,[11] so that $\tilde\rho = \alpha = |^v$. The Lie algebroid bracket $[\ ,\ ]_{\Gamma(A)}$ is uniquely determined by the Leibniz condition $[\boldsymbol{X}, f\boldsymbol{Y}]_{\Gamma(A)} = f[\boldsymbol{X},\boldsymbol{Y}]_{\Gamma(A)} + \tilde\rho(\boldsymbol{X})f\cdot\boldsymbol{Y}$, for $f \in C^\infty(\Phi)$, and the requirement that $[\ ,\ ]_{\Gamma(A)} = [\ ,\ ]_{\mathfrak{aut}(P)}$ on constant sections. It is found to be: $[\boldsymbol{X},\boldsymbol{Y}]_{\Gamma(A)} = [\boldsymbol{X},\boldsymbol{Y}]_{\mathfrak{aut}(P)} + \tilde\rho(\boldsymbol{X})\boldsymbol{Y} - \tilde\rho(\boldsymbol{Y})\boldsymbol{X}$. The action Lie algebroid bracket indeed reproduces (30) above.

Quite intuitively, the action Lie algebroid $\boldsymbol{A}$ is the limit of the action Lie groupoid $\boldsymbol{\Gamma}$ when the target and source maps are infinitesimally close. Hence $C^\infty(\Phi, \mathfrak{aut}(P)) \simeq \Gamma(A)$ is the Lie algebra of the group (of bisections) $C^\infty(\Phi, \text{Aut}(P))$, as we have shown explicitly above.

Remark that Lie algebroid brackets appear at two levels in the geometry just exposed. First, as previously mentioned, the SES

[11] Sections of $\Gamma(\boldsymbol{A})$ are just the graphs of elements of $C^\infty(\Phi, \mathfrak{aut}(P))$.

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(26) defines the (transitive) Atiyah Lie algebroid of Φ as a principal bundle. So, the bracket in $\mathfrak{aut}(\Phi)$ is a Lie algebroid bracket. This would be true for any choice of structure group for Φ – i.e., an internal gauge group $\text{Aut}_v(P) \simeq \mathcal{H}$, or $\text{Diff}(M)$ as done in ref. [8]. Its restriction to $\mathfrak{aut}_v(\Phi)$ trivializes as the Lie bracket of the Lie algebra of the gauge group of Φ. In the case at hand, with structure group $\text{Aut}(P)$ for Φ, this Lie algebra is $\mathfrak{aut}(P)$: so the Lie algebra bracket is the Lie algebroid bracket of P. The (action Lie algebroid) bracket (30) in $C^\infty\big(\Phi, \mathfrak{aut}(P)\big) \simeq \boldsymbol{Diff}_v(\Phi) \supset \mathfrak{aut}(P) \simeq \mathfrak{aut}_v(\Phi)$ *extends* the bracket of the Lie algebra of the gauge group of Φ; in our case, it extends the Atiyah Lie algebroid bracket $[\ ,\ ]_{\mathfrak{aut}}$ of P.

To the best of our knowledge, the first introduction of a bracket of the type (30) is to be found in Bergmann & Komar[21] (for $\phi = g_{\mu\nu}$ and $C^\infty\big(\Phi, \text{Diff}(M)\big)$) and Salisbury & Sundermeyer[22] – one may look up equations (3.1)–(3.2) in ref. [21] and equation (2.1) in ref. [22]. It was later reintroduced by Barnich & Troessaert in ref. [23], Equation (8), for the study of asymptotic symmetries of GR. This bracket also appears more recently in the covariant phase space literature, e.g., in refs. [24–28]. It has been interpreted as an action Lie algebroid bracket first in ref. [29]. In the next section we will show that it is also a special case of the Frölicher-Nijenhuis (FN) bracket of vector-valued forms.

Finally, we state the following result (proven in Appendix B), key to the geometric definition of general vertical and gauge transformations on field space. The pushforward by a vertical diffeomorphism $\Xi \in \boldsymbol{Diff}_v(\Phi)$, to which corresponds $\boldsymbol{\psi} \in C^\infty\big(\Phi, \text{Aut}(P)\big)$, is a map $\Xi_\star : T_\phi\Phi \to T_{\Xi(\phi)}\Phi = T_{\psi^*\phi}\Phi$. For a generic $\mathfrak{X} \in \Gamma(T\Phi)$ it is

$$
\begin{aligned}
\Xi_\star \mathfrak{X}_{|\phi} &= R_{\psi(\phi)\star}\mathfrak{X}_{|\phi} + \left\{\boldsymbol{\psi}(\phi)_*^{-1} \boldsymbol{d\psi}_{|\phi}(\mathfrak{X}_{|\phi})\right\}^v_{|\,\Xi(\phi)} \\
&= R_{\psi(\phi)\star}\Big(\mathfrak{X}_{|\phi} + \left\{\boldsymbol{d\psi}_{|\phi}(\mathfrak{X}_{|\phi}) \circ \boldsymbol{\psi}(\phi)^{-1}\right\}^v_{|\phi}\Big).
\end{aligned}
\tag{32}
$$

The proof holds the same for $\Xi \in \boldsymbol{Aut}_v(\Phi) \sim \boldsymbol{\psi} \in \boldsymbol{Aut}(P)$. This relation can be used to obtain the formula for repeated pushforwards: e.g., to obtain the result for $(\Xi'\circ\Xi)_\star \mathfrak{X}_{|\phi}$, per (8), one only needs to substitute $\boldsymbol{\psi} \to \boldsymbol{\psi} \circ \big(\boldsymbol{\psi}' \circ R_\psi\big)$. In case $\Xi, \Xi' \in \boldsymbol{Aut}_v(\Phi)$, one substitutes $\boldsymbol{\psi} \to \boldsymbol{\psi}' \circ \boldsymbol{\psi}$.

3.2.2. Differential Forms and Their Derivations

Consider the space of forms $\Omega^\bullet(\Phi)$ with the graded Lie algebra of its derivations $\text{Der}_\bullet\big(\Omega^\bullet(\Phi)\big) = \bigoplus_k \text{Der}_k\big(\Omega^\bullet(\Phi)\big)$ whose graded bracket is $[D_k, D_l] = D_k \circ D_l - (-)^{kl} D_l \circ D_k$, with $D_i \in \text{Der}_i\big(\Omega^\bullet(\Phi)\big)$.

The de Rham complex of Φ is $\big(\Omega^\bullet(\Phi); \boldsymbol{d}\big)$ with $\boldsymbol{d} \in \text{Der}_1$ the de Rham (exterior) derivative, which is nilpotent – $\boldsymbol{d}^2 = 0 = 1/2[\boldsymbol{d}, \boldsymbol{d}]$ – and defined via the Koszul formula. The exterior product $\wedge$ is defined on scalar-valued forms as usual, so that $\big(\Omega^\bullet(\Phi, \mathbb{K}), \wedge, \boldsymbol{d}\big)$ is a differential graded algebra. The exterior product can also be defined on the space $\Omega^\bullet(\Phi, \mathsf{A})$ of forms with values in an algebra $(\mathsf{A}, \cdot)$, using the product in A instead of the product in $\mathbb{K}$. So $\big(\Omega^\bullet(\Phi, \mathsf{A}), \wedge, \boldsymbol{d}\big)$ is a again a differential graded algebra.[12]

One may define vector field-valued differential forms $\Omega^\bullet(\Phi, T\Phi) = \Omega^\bullet(\Phi) \otimes T\Phi$. Then, the subalgebra of *algebraic* derivations is defined as $D_{|\Omega^0(\Phi)} = 0$; they have the form $\iota_K \in \text{Der}_{k-1}$ for $K \in \Omega^k(\Phi, T\Phi)$, with ι the inner product. It generalizes the inner contraction of a form on a vector field: For $\omega \otimes \mathfrak{X} \in \Omega^\bullet(\Phi, T\Phi)$ we have $\iota_K(\omega \otimes \mathfrak{X}) := \iota_K\omega \otimes \mathfrak{X} = \boldsymbol{\omega} \circ \boldsymbol{K} \otimes \mathfrak{X}$. The *Nijenhuis-Richardson bracket* (or *algebraic* bracket) is defined by

$$
[\boldsymbol{K}, \boldsymbol{L}]_{\text{NR}} := \iota_K \boldsymbol{L} - (-)^{(k-1)(l-1)}\, \iota_L \boldsymbol{K}
\tag{33}
$$

and makes the map $\iota : \Omega^\bullet(\Phi, T\Phi) \to \text{Der}_\bullet\big(\Omega^\bullet(\Phi)\big)$, $\boldsymbol{K} \mapsto \iota_K$, a graded Lie algebra morphism:

$$
[\iota_K, \iota_L] = \iota_{[K,L]_{\text{NR}}}.
\tag{34}
$$

The *Nijenhuis-Lie derivative* is the map

$$
\begin{aligned}
\boldsymbol{L} := [\iota, \boldsymbol{d}] : \Omega^\bullet(\Phi, T\Phi) &\to \text{Der}_\bullet\big(\Omega^\bullet(\Phi)\big) \\
\boldsymbol{K} &\mapsto \boldsymbol{L}_K := \iota_K \boldsymbol{d} - (-)^{k-1} \boldsymbol{d}\iota_K.
\end{aligned}
\tag{35}
$$

We have $\boldsymbol{L}_K \in \text{Der}_k$ for $\boldsymbol{K} \in \Omega^k(\Phi, T\Phi)$. It generalizes the Lie derivative along vector fields, $\boldsymbol{L}_{\mathfrak{X}} \in \text{Der}_0$. It is s.t. $[\boldsymbol{L}_K, \boldsymbol{d}] = 0$. Given $\boldsymbol{K} = K \otimes \mathfrak{X} \in \Omega^k(\Phi, T\Phi)$ and $\boldsymbol{J} = J \otimes \mathfrak{Y} \in \Omega^l(\Phi, T\Phi)$, the *FN bracket* is

$$
\begin{aligned}
[\boldsymbol{K}, \boldsymbol{J}]_{\text{FN}} = &\ K \wedge J \otimes [\mathfrak{X}, \mathfrak{Y}] + K \wedge \boldsymbol{L}_{\mathfrak{X}} J \otimes \mathfrak{Y} - \boldsymbol{L}_{\mathfrak{Y}} K \wedge J \otimes \mathfrak{X} \\
&+ (-)^k \big(dK \wedge \iota_{\mathfrak{X}} J \otimes \mathfrak{Y} + \iota_{\mathfrak{Y}} K \wedge dJ \otimes \mathfrak{X}\big).
\end{aligned}
\tag{36}
$$

It makes the Nijenhuis–Lie derivative a morphism of graded Lie algebras:

$$
[\boldsymbol{L}_K, \boldsymbol{L}_J] = \boldsymbol{L}_{[K,J]_{\text{FN}}}.
\tag{37}
$$

The following relations hold:

$$
\begin{aligned}
&[\boldsymbol{L}_K, \iota_J] = \iota_{[K,J]_{\text{FN}}} - (-)^{k(l-1)} \boldsymbol{L}_{(\iota_K J)}, \\
&[\iota_J, \boldsymbol{L}_K] = \boldsymbol{L}_{(\iota_K J)} + (-)^k\, \iota_{[J,K]_{\text{FN}}}.
\end{aligned}
\tag{38}
$$

For a systematic exposition of the above notions (in the finite-dimensional setting), see ref. [30] Chapter II, Section 8.

The FN bracket reproduces the bracket (29) as a special case. Indeed, specializing (36) in degree 0, for $\boldsymbol{f} = f \otimes \mathfrak{X}$ and $\boldsymbol{g} = g \otimes \mathfrak{Y} \in \Omega^0(\Phi, T\Phi)$ we get

$$
\begin{aligned}
[\boldsymbol{f}, \boldsymbol{g}]_{\text{FN}} &= f \wedge g \otimes [\mathfrak{X}, \mathfrak{Y}] + f \wedge \boldsymbol{L}_{\mathfrak{X}} g \otimes \mathfrak{Y} - \boldsymbol{L}_{\mathfrak{Y}} f \wedge g \otimes \mathfrak{X}, \\
&= f \wedge g \otimes [\mathfrak{X}, \mathfrak{Y}] + [\boldsymbol{f}, \boldsymbol{dg}]_{\text{NR}} - [\boldsymbol{g}, \boldsymbol{df}]_{\text{NR}},
\end{aligned}
\tag{39}
$$

and

$$
[\boldsymbol{L}_f, \iota_g] = \iota_{[f,g]_{\text{FN}}}.
\tag{40}
$$

The map $|^v : C^\infty\big(\Phi, \mathfrak{aut}(P)\big) \to \Gamma(V\Phi)$, $X \mapsto X^v$, allows to see $X^v \in \mathfrak{diff}_v(\Phi)$ as a (vertical) vector-valued 0-form on Φ: $X^v \in$

[12] On the other hand, an exterior product cannot be defined on $\Omega^\bullet(\Phi, V)$ where V is merely a vector space.

$\Omega^0(\Phi, V\Phi) \subset \Omega^\bullet(\Phi, T\Phi)$. The Nijenhuis-Richardson and FN brackets thus apply: for $\boldsymbol{X}^v, \boldsymbol{Y}^v \in \Omega^0(\Phi, V\Phi)$ we find

$$[\boldsymbol{X}^v, \boldsymbol{dY}^v]_{\text{NR}} = \{\iota_{X^v}\boldsymbol{dY}\}^v = \{\boldsymbol{X}^v(\boldsymbol{Y})\}^v,$$
$$[\boldsymbol{dX}^v, \boldsymbol{Y}^v]_{\text{NR}} = -[\boldsymbol{Y}^v, \boldsymbol{dX}^v]_{\text{NR}} = -\{\iota_{Y^v}\boldsymbol{dX}\}^v = -\{\boldsymbol{Y}^v(\boldsymbol{X})\}^v, \tag{41}$$

so that the FN bracket for 0-forms (39) is

$$[\boldsymbol{X}^v, \boldsymbol{Y}^v]_{\text{FN}} = \big([\boldsymbol{X}, \boldsymbol{Y}]_{\mathfrak{aut}} + \boldsymbol{X}^v(\boldsymbol{Y}) - \boldsymbol{Y}^v(\boldsymbol{X})\,\big)^v = \{\boldsymbol{X}, \boldsymbol{Y}\}^v. \tag{42}$$

Then we have the following special cases of identities (34), (38), and (37) among derivations in $\text{Der}^\bullet$:

$$[\iota_{X^v}, \iota_{\{dY\}^v}] = \iota_{[X^v, dY^v]_{\text{NR}}} = \iota_{\{\iota_{X^v}dY\}^v}, \tag{43}$$

$$[\boldsymbol{L}_{X^v}, \iota_{Y^v}] = \iota_{[X^v, Y^v]_{\text{FN}}}, \tag{44}$$

$$[\boldsymbol{L}_{X^v}, \boldsymbol{L}_{Y^v}] = \boldsymbol{L}_{[X^v, Y^v]_{\text{FN}}} = \boldsymbol{L}_{\{X,Y\}^v}. \tag{45}$$

As expected, (45) reproduces (31). The above reproduces as special cases various identities derived heuristically in the covariant phase space literature, e.g., refs. [24, 25, 27].

Remarkable Forms: The action by pullback of $\text{Aut}(P)$ on a form $\boldsymbol{\alpha} \in \Omega^\bullet(\Phi)$ defines its *equivariance*, $R^\star_\psi \alpha$. The action by pullback of $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty(\Phi, \text{Aut}(P))$, which we write $\boldsymbol{\alpha}^{\boldsymbol{\psi}} := \Xi^\star \boldsymbol{\alpha}$, defines *vertical transformations*, while the action by pullback of $\boldsymbol{Aut}_v(\Phi) \simeq \boldsymbol{Aut}(P)$ defines *gauge transformations*.

We write a generic form at $\phi \in \Phi$ as

$$\boldsymbol{\alpha}_{|\phi} = \alpha\left(\wedge^\bullet \boldsymbol{d}\phi_{|\phi}; \phi\right), \tag{46}$$

where $\boldsymbol{d}\phi \in \Omega^1(\Phi)$ is the basis 1-form on Φ and $\alpha(\;;\;)$ is the functional expression of $\boldsymbol{\alpha}$, alternating multilinear in the first arguments and with arbitrary ϕ-dependence in the second argument (in physics, often polynomial). The equivariance and vertical transformation of $\boldsymbol{\alpha}$ are:

$$\begin{aligned} R^\star_\psi \boldsymbol{\alpha}_{|\phi^\psi} &= \alpha\left(\wedge^\bullet R^\star_\psi \boldsymbol{d}\phi_{|\phi^\psi};\, R_\psi \phi\right) = \alpha\left(\wedge^\bullet R^\star_\psi \boldsymbol{d}\phi_{|\phi^\psi};\, \phi^\psi\right), \\ &\quad \text{for } \psi \in \text{Aut}(P), \\ \boldsymbol{\alpha}^{\boldsymbol{\psi}}_{|\phi} := \Xi^\star \boldsymbol{\alpha}_{|\Xi(\phi)} &= \alpha\left(\wedge^\bullet \Xi^\star \boldsymbol{d}\phi_{|\Xi(\phi)};\, \Xi(\phi)\right) = \alpha\left(\wedge^\bullet \Xi^\star \boldsymbol{d}\phi_{|\phi^{\boldsymbol{\psi}}};\, \phi^{\boldsymbol{\psi}}\right), \\ &\quad \text{for } \Xi \in \boldsymbol{Diff}_v(\Phi) \sim \boldsymbol{\psi} \in C^\infty\left(\Phi, \text{Aut}(P)\right). \end{aligned} \tag{47}$$

The infinitesimal equivariance and vertical transformations are given by the (Nijenhuis-)Lie derivative along the elements of $\Gamma(V\Phi)$ generated respectively by $\mathfrak{aut}(P)$ and $C^\infty\left(\Phi, \mathfrak{aut}(P)\right)$:

$$\begin{aligned} \boldsymbol{L}_{X^v}\boldsymbol{\alpha} &= \tfrac{d}{d\tau} R^\star_{\psi_\tau} \boldsymbol{\alpha}\Big|_{\tau=0} \quad \text{with } X \in \mathfrak{aut}(P), \\ \boldsymbol{L}_{\boldsymbol{X}^v}\boldsymbol{\alpha} &= \tfrac{d}{d\tau} \Xi^\star_\tau \boldsymbol{\alpha}\Big|_{\tau=0} \quad \text{with } \boldsymbol{X} \in C^\infty\left(\Phi, \mathfrak{aut}(P)\right). \end{aligned} \tag{48}$$

There are forms of particular interest whose gauge transformations need not be computed explicitly, but rather are read from the forms special properties.

First, *equivariant* forms are those whose equivariance is controlled by either representations of the structure group, or by 1-cocycles for the action of the structure group. *Standard equivariant* forms are valued in representations $(\rho, \boldsymbol{V})$ of the structure group $\text{Aut}(P)$ and s.t.:

$$\Omega^\bullet_{\text{eq}}(\Phi, \rho) := \left\{ \boldsymbol{\alpha} \in \Omega^\bullet(\Phi, \boldsymbol{V}) \;|\; R^\star_\psi \boldsymbol{\alpha}_{|\phi^\psi} = \rho(\psi)^{-1}\boldsymbol{\alpha}_{|\phi} \right\}. \tag{49}$$

The infinitesimal version of the equivariance property is $\boldsymbol{L}_{X^v}\boldsymbol{\alpha} = -\rho_*(X)\boldsymbol{\alpha}$ for $X \in \mathfrak{aut}(P)$.

The *twisted equivariant* forms[31] have equivariance controlled by a 1-cocycle for the action of $\text{Aut}(P)$ on Φ, i.e., a map:

$$\begin{aligned} &C : \Phi \times \text{Aut}(P) \to G, \\ &\qquad G \text{ some Lie group (possibly infinite-dimensional).} \\ &\qquad (\phi, \psi) \mapsto C(\phi; \psi) \qquad \text{s.t.} \\ &\qquad\qquad C(\phi; \psi' \circ \psi) = C(\phi; \psi') \cdot C(\phi^{\psi'}; \psi). \end{aligned} \tag{50}$$

Manifestly, ϕ-independent 1-cocycles are group morphisms, i.e., 1-cocycles are generalizations of representations. From the 1-cocycle property (50) follows that $C(\phi; \text{id}_M) = \text{id}_G = C(\phi^\psi; \text{id}_M)$, thus that $C(\phi; \psi)^{-1} = C(\phi^\psi; \psi^{-1})$. Given a G-space $\boldsymbol{V}$, one defines twisted equivariant forms as

$$\Omega^\bullet_{\text{eq}}(\Phi, C) := \left\{ \boldsymbol{\alpha} \in \Omega^\bullet(\Phi, \boldsymbol{V}) \;|\; R^\star_\psi \boldsymbol{\alpha}_{|\phi^\psi} = C(\phi; \psi)^{-1}\boldsymbol{\alpha}_{|\phi} \right\}. \tag{51}$$

The property (50) ensures compatibility with the right action: $R^\star_{\psi'} R^\star_\psi = R^\star_{\psi' \circ \psi}$. The infinitesimal equivariance is $\boldsymbol{L}_{X^v}\boldsymbol{\alpha} = -a(X; \phi)\boldsymbol{\alpha}$, where $a(X, \phi) := \frac{d}{d\tau}\, C(\phi, \psi_\tau)|_{\tau=0}$ is a 1-cocycle for the action of $\mathfrak{aut}(P)$ on Φ:

$$\begin{aligned} &a : \Phi \times \mathfrak{aut}(P) \to \mathfrak{g}, \quad \mathfrak{g} \text{ the Lie algebra of } G. \\ &\qquad (\phi, X) \mapsto a(X; \phi) \qquad \text{s.t.} \\ &\qquad\qquad X^v \cdot a(Y; \phi) - Y^v \cdot a(X; \phi) + \big[a(X; \phi),\; a(Y; \phi)\big]_{\mathfrak{g}} \\ &\qquad\qquad = a([X, Y]_{\mathfrak{aut}}; \phi). \end{aligned} \tag{52}$$

The infinitesimal relation (52) ensures compatibility with the right action: $[\boldsymbol{L}_{X^v}, \boldsymbol{L}_{X^v}] = \boldsymbol{L}_{[X^v, Y^v]} = \boldsymbol{L}_{([X,Y]_{\mathfrak{aut}})^v}$. Observe that it is a non-Abelian generalization of the Wess-Zumino (WZ) consistency condition for anomalies $a(X; \phi)$. The WZ consistency condition being reproduced for G Abelian.

The subspace of *invariant* forms are those whose equivariance is trivial, and *horizontal* forms are those vanishing on vertical vector field:

$$\begin{aligned} \Omega^\bullet_{\text{inv}}(\Phi) &= \left\{ \boldsymbol{\alpha} \in \Omega^\bullet(\Phi) \;|\; R^\star_\psi \boldsymbol{\alpha} = \boldsymbol{\alpha} \right\}, \text{ infinitesimally } \boldsymbol{L}_{X^v}\boldsymbol{\alpha} = 0, \\ \Omega^\bullet_{\text{hor}}(\Phi) &= \{ \boldsymbol{\alpha} \in \Omega^\bullet(\Phi) \;|\; \iota_{X^v}\boldsymbol{\alpha} = 0 \}. \end{aligned} \tag{53}$$

A form which is both equivariant and horizontal is said *tensorial*. We have thus standard tensorial forms

$$\Omega^\bullet_{\text{tens}}(\Phi,\rho) \coloneqq \left\{ \boldsymbol{\alpha} \in \Omega^\bullet(\Phi, V) \mid R^\star_\psi \boldsymbol{\alpha} = \rho(\psi)^{-1}\boldsymbol{\alpha},\ \&\ \iota_{X^v}\boldsymbol{\alpha} = 0 \right\}. \tag{54}$$

Similarly, the space of *twisted tensorial* forms is

$$\Omega^\bullet_{\text{tens}}(\Phi,C) \coloneqq \left\{ \boldsymbol{\alpha} \in \Omega^\bullet(\Phi, V) \mid R^\star_\psi \boldsymbol{\alpha} = C(\phi;\psi)^{-1}\boldsymbol{\alpha},\ \&\ \iota_{X^v}\boldsymbol{\alpha} = 0 \right\}. \tag{55}$$

In either case, we have $\Omega^0_{\text{tens}}(\Phi) = \Omega^0_{\text{eq}}(\Phi)$.

Let us recall the well-known fact that the de Rham derivative $\boldsymbol{d}$ does not preserve the space of tensorial forms (horizontality is lost). This is a reason for the introduction of a notion of *connection* on Φ so as to define a *covariant derivative* on the space of tensorial forms. As we will review in Section 3.4 below, for standard tensorial forms one needs an Ehresmann connection 1-form, while for twisted tensorial forms one needs a generalization called *twisted connection*.[31]

Finally, forms that are both invariant and horizontal are called *basic*:

$$\Omega^\bullet_{\text{basic}}(\Phi) \coloneqq \left\{ \boldsymbol{\alpha} \in \Omega^\bullet(\Phi) \mid R^\star_\psi \boldsymbol{\alpha} = \boldsymbol{\alpha}\ \&\ \iota_{X^v}\boldsymbol{\alpha} = 0 \right\}. \tag{56}$$

This space is preserved by $\boldsymbol{d}$, so $\left(\Omega^\bullet_{\text{basic}}(\Phi), \boldsymbol{d}\right)$ is a subcomplex of the de Rham complex of Φ: the *basic subcomplex*. Therefore, basic forms can also be defined as $\text{Im}(\pi^\star)$ (hence their name):

$$\Omega^\bullet_{\text{basic}}(\Phi) \coloneqq \{\boldsymbol{\alpha} \in \Omega^\bullet(\Phi) \mid \exists\, \boldsymbol{\beta} \in \Omega^\bullet(\mathcal{M}) \text{ s.t. } \boldsymbol{\alpha} = \pi^\star \boldsymbol{\beta}\}. \tag{57}$$

The cohomology of $\left(\Omega^\bullet_{\text{basic}}(\Phi), \boldsymbol{d}\right)$ is the *equivariant cohomology* of Φ. As $[\boldsymbol{d}, \pi^\star] = 0$, it is isomorphic to the cohomology $\left(\Omega^\bullet(\mathcal{M}), \boldsymbol{d}\right)$ of the base moduli space. Hence its importance, especially when it is unpractical (or impossible) to work concretely on $\mathcal{M}$, as it is the case in GFT.

We stress that the analogue of $\Gamma_{\text{inv}}(T\Phi)$ for forms is not $\Omega^\bullet_{\text{inv}}(\Phi)$ but $\Omega^\bullet_{\text{basic}}(\Phi)$. Only basic forms project to well-defined forms in $\Omega^\bullet(\mathcal{M})$, containing only physical d.o.f. In Section 4, we will detail a systematic method to build the basic version $\boldsymbol{\alpha}^b \in \Omega^\bullet_{\text{basic}}(\Phi)$ of a form $\boldsymbol{\alpha} \in \Omega^\bullet(\Phi)$.

3.3. Vertical Transformations and Gauge Transformations

As previously seen, the vertical transformation a form $\boldsymbol{\alpha} \in \Omega^\bullet(\Phi)$ is its pullback by $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \text{Aut}(P)\right)$: $\boldsymbol{\alpha}^{\boldsymbol{\psi}} \coloneqq \Xi^\star\boldsymbol{\alpha}$. The notation on the left-hand side is justified by the fact that the vertical transformation is expressed in term of the generating element $\boldsymbol{\psi} \in C^\infty\left(\Phi, \text{Aut}(P)\right)$ associated to $\Xi \in \boldsymbol{Diff}_v(\Phi)$. Performing two vertical transformations, using (8), one has

$$(\boldsymbol{\alpha}^{\boldsymbol{\psi}})^{\boldsymbol{\psi}'} \coloneqq \Xi'^\star\Xi^\star\boldsymbol{\alpha} = \left(\Xi\circ\Xi'\right)^\star\boldsymbol{\alpha} \eqqcolon \boldsymbol{\alpha}^{\boldsymbol{\psi}'\circ(\boldsymbol{\psi}\circ R_{\boldsymbol{\psi}'})}. \tag{58}$$

For gauge transformations, defined by the action of $\boldsymbol{Aut}_v(\Phi) \simeq \boldsymbol{Aut}(P)$, whose elements have specific equivariance $R^\star_\psi\boldsymbol{\psi} = \boldsymbol{\psi}\circ R_\psi = \psi^{-1}\circ\boldsymbol{\psi}\circ\psi$, the previous expression simplifies:

$$(\boldsymbol{\alpha}^{\boldsymbol{\psi}})^{\boldsymbol{\psi}'} = \boldsymbol{\alpha}^{\boldsymbol{\psi}\circ\boldsymbol{\psi}'}. \tag{59}$$

Infinitesimal vertical transformations by $\mathfrak{diff}_v(\Phi) \simeq C^\infty(\Phi, \mathfrak{aut}(P))$ are given by the Nijenhuis–Lie derivative:

$$\boldsymbol{L}_{X^v}\boldsymbol{\alpha} = \begin{cases} \frac{d}{d\tau}\,\Xi^\star_\tau\boldsymbol{\alpha}\,\Big|_{\tau=0} \\ [\iota_{X^v}, \boldsymbol{d}]\,\boldsymbol{\alpha} \end{cases} \quad \text{so} \quad [\boldsymbol{L}_{X^v}, \boldsymbol{L}_{Y^v}]\,\boldsymbol{\alpha} = \boldsymbol{L}_{[X^v,Y^v]_{\text{FN}}}\,\boldsymbol{\alpha} = \boldsymbol{L}_{\{X,Y\}^v}\,\boldsymbol{\alpha}. \tag{60}$$

The second equation uses (31)/(45). It is the infinitesimal version of (58). For infinitesimal gauge transformations, defined by the action of $\mathfrak{aut}_v(\Phi) \simeq \mathfrak{aut}(P)$, this reduces to $[\boldsymbol{L}_{X^v}, \boldsymbol{L}_{Y^v}]\,\boldsymbol{\alpha} = \boldsymbol{L}_{(-[X,Y]_{\mathfrak{aut}})^v}\,\boldsymbol{\alpha} = \boldsymbol{L}_{([X,Y]_{\Gamma(TP)})^v}\,\boldsymbol{\alpha}$.

To get concrete expressions, one uses the duality between pullback and pushforward together with (32): For any $\mathfrak{X}, \mathfrak{X}', \ldots \in \Gamma(T\Phi)$ one has

$$\begin{aligned}
\boldsymbol{\alpha}^{\boldsymbol{\psi}}_{|\phi}(\mathfrak{X}_{|\phi}, \ldots) &= \Xi^\star\boldsymbol{\alpha}_{|\Xi(\phi)}(\mathfrak{X}_{|\phi}, \ldots) = \boldsymbol{\alpha}_{|\Xi(\phi)}(\Xi_\star\mathfrak{X}_{|\phi}, \ldots) \\
&= \boldsymbol{\alpha}_{|\phi^{\boldsymbol{\psi}(\phi)}}\left(R_{\boldsymbol{\psi}(\phi)\star}\left(\mathfrak{X}_{|\phi} + \left\{\boldsymbol{d\psi}_{|\phi}(\mathfrak{X}_{|\phi})\circ\boldsymbol{\psi}(\phi)^{-1}\right\}^v_{|\phi}\right), \ldots\right) \\
&= R^\star_{\boldsymbol{\psi}(\phi)}\,\boldsymbol{\alpha}_{|\phi^{\boldsymbol{\psi}(\phi)}}\left(\mathfrak{X}_{|\phi} + \left\{\boldsymbol{d\psi}_{|\phi}(\mathfrak{X}_{|\phi})\circ\boldsymbol{\psi}(\phi)^{-1}\right\}^v_{|\phi}, \ldots\right).
\end{aligned} \tag{61}$$

It is clear from (61) that the vertical transformation of a form is controlled by its equivariance and verticality properties. In particular, the vertical transformation of a tensorial form is simply given by its equivariance:

$$\begin{aligned}
&\text{For}\ \ \boldsymbol{\alpha} \in \Omega^\bullet_{\text{tens}}(\Phi,\rho), \quad \boldsymbol{\alpha}^{\boldsymbol{\psi}} = \rho(\boldsymbol{\psi})^{-1}\boldsymbol{\alpha}. \\
&\text{For}\ \ \boldsymbol{\alpha} \in \Omega^\bullet_{\text{tens}}(\Phi,C), \quad \boldsymbol{\alpha}^{\boldsymbol{\psi}} = C(\boldsymbol{\psi})^{-1}\boldsymbol{\alpha}.
\end{aligned} \tag{62}$$

In the second line we introduce the simplified notation $[C(\boldsymbol{\psi})](\phi) \coloneqq C\left(\phi; \boldsymbol{\psi}(\phi)\right)$. The map $C(\boldsymbol{\psi}) : \Phi \to G$ is twice dependent on the point $\phi \in \Phi$. We stress that $\boldsymbol{\alpha}^{\boldsymbol{\psi}} = \Xi^\star\boldsymbol{\alpha} \notin \Omega^\bullet_{\text{tens}}(\Phi,\rho)$ unless $\boldsymbol{\psi} \in \boldsymbol{Aut}(P) \sim \Xi \in \boldsymbol{Aut}_v(\Phi)$ – see ref. [14] – making *gauge transformations* special indeed: they preserve the space of tensorial forms, while $\boldsymbol{Diff}_v(\Phi)$ does not.

The infinitesimal versions of (61) is, by definition (60),

$$\boldsymbol{L}_{X^v}\boldsymbol{\alpha} = \frac{d}{d\tau}\,R^\star_{\boldsymbol{\psi}_\tau}\,\boldsymbol{\alpha}\,\Big|_{\tau=0} + \iota_{\{\boldsymbol{d}X\}^v}\boldsymbol{\alpha}, \tag{63}$$

with $\boldsymbol{X} = \frac{d}{d\tau}\boldsymbol{\psi}_\tau\Big|_{\tau=0}$. Notice that $\{\boldsymbol{d}X\}^v$ can be seen as an element of $\Omega^1(\Phi, V\Phi)$, so $\iota_{\{dX\}^v}$ is an algebraic derivation (of degree 0) as discussed in Section 3.2.2. For $\boldsymbol{\alpha} \in \Omega^\bullet_{\text{eq}}(\phi)$, depending if it is standard or twisted equivariant, (63) specializes to:

$$\boldsymbol{L}_{X^v}\boldsymbol{\alpha} = \begin{cases} -\rho_*(\boldsymbol{X})\,\boldsymbol{\alpha} + \iota_{\{dX\}^v}\boldsymbol{\alpha}, \\ -a(\boldsymbol{X})\,\boldsymbol{\alpha} + \iota_{\{dX\}^v}\boldsymbol{\alpha}, \end{cases} \tag{64}$$

where we introduce the notation $[a(\boldsymbol{X})](\phi) \coloneqq a\left(\boldsymbol{X}(\phi); \phi\right)$ for the linearized 1-cocycle. In particular, for the pullback representation $\rho(\psi)^{-1} = \psi^*$, i.e., for $\Omega^\bullet(P)$-valued (or tensor-valued) forms $\boldsymbol{\alpha}$, (63) gives naturally:

$$\boldsymbol{L}_{X^v}\boldsymbol{\alpha} = \mathfrak{L}_X\boldsymbol{\alpha} + \iota_{\{dX\}^v}\boldsymbol{\alpha}. \tag{65}$$

This formula clarifies the geometrical meaning of the so-called "anomaly operator", Δ_X, featuring in the covariant phase space literature:[24,25,32–34] in our notations $\Delta_X := \boldsymbol{L}_{X^v} - \mathfrak{L}_X - \iota_{\{dX\}^v}$. This operator can only be non-zero on Φ in theories admitting background non-dynamical structures or fields "breaking" Aut(P)-covariance. Those fundamentally fail to comply with the core physical (symmetry) principles of general-relativistic GFT. We further elaborate on this point in Section 4.

The infinitesimal versions of (62), for tensorial forms, are:

$$\boldsymbol{L}_{X^v}\boldsymbol{\alpha} = -\rho_*(X)\boldsymbol{\alpha} \quad \text{and} \quad \boldsymbol{L}_{X^v}\boldsymbol{\alpha} = -a(X)\boldsymbol{\alpha}. \tag{66}$$

From the commutativity property (31)/(45) of the Nijenhuis–Lie derivative applied to a twisted tensorial form $\boldsymbol{\alpha}$, $[\boldsymbol{L}_{X^v}, \boldsymbol{L}_{Y^v}]\,\boldsymbol{\alpha} = \boldsymbol{L}_{[X^v,Y^v]_{\text{FN}}}\,\boldsymbol{\alpha} = \boldsymbol{L}_{\{X,Y\}^v}\,\boldsymbol{\alpha}$, follows the relation for the infinitesimal 1-cocycle:

$$\begin{aligned}
&\boldsymbol{X}^v\big(a(\boldsymbol{Y};\phi)\big) - \boldsymbol{Y}^v\big(a(\boldsymbol{X};\phi)\big) - a(\{\boldsymbol{X},\boldsymbol{Y}\};\phi) + [a(\boldsymbol{X};\phi), a(\boldsymbol{Y};\phi)]_{\mathfrak{g}} = 0,\\
&\boldsymbol{X}^v\big(a(\underline{\boldsymbol{Y}};\phi)\big) - \boldsymbol{Y}^v\big(a(\underline{\boldsymbol{X}};\phi)\big) - a([\boldsymbol{X},\boldsymbol{Y}]_{\mathfrak{aut}(P)};\phi) + [a(\boldsymbol{X};\phi), a(\boldsymbol{Y};\phi)]_{\mathfrak{g}} = 0.
\end{aligned} \tag{67}$$

The second equation is obtained from the first using the FN bracket (42)/(29): The notation $\underline{\boldsymbol{Y}}, \underline{\boldsymbol{X}}$ means that the elements $\boldsymbol{Y}, \boldsymbol{X}$ are considered ϕ-independent, so $\boldsymbol{X}^v, \boldsymbol{Y}^v$ pass through. Therefore, (67) reproduces the defining infinitesimal 1-cocycle property (52).

To illustrate, let us consider the case of elements of the gauge group, $\boldsymbol{\eta} \in \boldsymbol{Aut}(P)$, and its Lie algebra, $\boldsymbol{Y} \in \boldsymbol{\mathfrak{aut}}(P)$. As 0-forms they are trivially horizontal, and their equivariance are specified by definition (10)–(21): they are thus tensorial, so we have

$$\boldsymbol{\eta}^{\boldsymbol{\psi}} = \boldsymbol{\psi}^{-1} \circ \boldsymbol{\eta} \circ \boldsymbol{\psi}, \quad \text{and} \quad \boldsymbol{Y}^{\boldsymbol{\psi}} = (\boldsymbol{\psi}^{-1})_* \boldsymbol{Y} \circ \boldsymbol{\psi}. \tag{68}$$

The infinitesimal gauge transformation of $\boldsymbol{Y}$ is then $\boldsymbol{L}_{X^v}\boldsymbol{Y} = \mathfrak{L}_X \boldsymbol{Y} = [\boldsymbol{Y}, X]_{\mathfrak{aut}(P)}$, as expected from its infinitesimal equivariance (22).

As a special case of (62), or given their definition, basic forms are strictly gauge invariant:

$$\text{For} \quad \boldsymbol{\alpha} \in \Omega^\bullet_{\text{basic}}(\Phi): \quad \boldsymbol{\alpha}^{\boldsymbol{\psi}} = \boldsymbol{\alpha}, \quad \text{so} \quad L_{X^v}\boldsymbol{\alpha} = 0. \tag{69}$$

Another important example is that of the basis 1-form $\boldsymbol{d}\phi \in \Omega^1(\Phi)$, since its vertical transformation $\boldsymbol{d}\phi^{\boldsymbol{\psi}} := \Xi^\star \boldsymbol{d}\phi$ features in the general formulae (47) for the vertical transformation of a generic form. The equivariance and verticality properties of $\boldsymbol{d}\phi$ are given by definition:

$$R^\star_\psi \boldsymbol{d}\phi := \psi^* \boldsymbol{d}\phi, \quad \text{and} \quad \boldsymbol{\iota}_{X^v}\boldsymbol{d}\phi := \mathfrak{L}_X \phi. \tag{70}$$

The verticality must reproduce the $\mathfrak{aut}(P)$-transformation of the field ϕ. It is then immediate that

$$\boldsymbol{d}\phi^{\boldsymbol{\psi}} := \Xi^\star \boldsymbol{d}\phi = \boldsymbol{\psi}^*\big(\boldsymbol{d}\phi + \mathfrak{L}_{\boldsymbol{d}\boldsymbol{\psi}\circ\boldsymbol{\psi}^{-1}}\phi\big). \tag{71}$$

This generalizes a standard result of the covariant phase space literature (see e.g., ref. [35]). From (65), the linear version is

$$\boldsymbol{L}_{X^v}\boldsymbol{d}\phi = \mathfrak{L}_X \boldsymbol{d}\phi + \mathfrak{L}_{\boldsymbol{d}X}\phi. \tag{72}$$

The same may be obtained via $\boldsymbol{L}_{X^v}\boldsymbol{d}\phi = \boldsymbol{d}(\iota_{X^v}\boldsymbol{d}\phi) = \boldsymbol{d}\left(\mathfrak{L}_X\phi\right)$.

3.4. Connections on Field Space

As previously observed, the exterior derivative $\boldsymbol{d}$ does not preserve $\Omega^\bullet_{\text{tens}}(\Phi)$ of standard/twisted tensorial forms. To build a first order linear differential operator that does, the *covariant derivative*, one needs to endow Φ with an adequate notion of connection 1-form.

3.4.1. Ehresmann Connections

A Ehresmann connection 1-form $\boldsymbol{\omega} \in \Omega^1_{\text{eq}}\big(\Phi, \mathfrak{aut}(P)\big)$ on field space Φ is defined by the following two properties:

$$\begin{aligned}
&\boldsymbol{\omega}_{|\phi}\Big(X^v_{|\phi}\Big) = X, \quad \text{for } X \in \mathfrak{aut}(P),\\
&R^\star_\psi \boldsymbol{\omega}_{|\phi^\psi} = \psi^{-1}_*\,\boldsymbol{\omega}_{|\phi} \circ \psi.
\end{aligned} \tag{73}$$

Infinitesimally, the equivariance of the connection under $\mathfrak{aut}(P)$ is

$$\boldsymbol{L}_{X^v}\boldsymbol{\omega} = \tfrac{d}{d\tau}R^\star_{\psi_\tau}\boldsymbol{\omega}\Big|_{\tau=0} = \tfrac{d}{d\tau}{\psi_\tau^{-1}}_*\boldsymbol{\omega} \circ \psi_\tau\Big|_{\tau=0} = [X, \boldsymbol{\omega}]_{\Gamma(TP)} = [\boldsymbol{\omega}, X]_{\mathfrak{aut}(P)}. \tag{74}$$

The space of connection C is an affine space modeled on the vector space $\Omega^1_{\text{tens}}\big(\Phi, \mathfrak{aut}(P)\big)$: For $\boldsymbol{\omega}, \boldsymbol{\omega}' \in C$, we have that $\boldsymbol{\beta} := \boldsymbol{\omega}' - \boldsymbol{\omega} \in \Omega^1_{\text{tens}}\big(\Phi, \mathfrak{aut}(P)\big)$. Or, given $\boldsymbol{\omega} \in C$ and $\boldsymbol{\beta} \in \Omega^1_{\text{tens}}\big(\Phi, \mathfrak{aut}(P)\big)$, we have that $\boldsymbol{\omega}' = \boldsymbol{\omega} + \boldsymbol{\beta} \in C$.

A connection allows to define the horizontal subbundle $H\Phi := \ker\boldsymbol{\omega}$ complementary to the vertical subbundle, $T\Phi = V\Phi \oplus H\Phi$. The horizontal projection is the map $|^h : T\Phi \to H\Phi$, $\mathfrak{X} \mapsto \mathfrak{X}^h := \mathfrak{X} - [\boldsymbol{\omega}(\mathfrak{X})]^v$, as clearly $\boldsymbol{\omega}(\mathfrak{X}^h) = 0$.

A covariant derivative associated to $\boldsymbol{\omega}$ is defined as $\boldsymbol{D} := \boldsymbol{d}\circ|^h : \Omega^\bullet_{\text{eq}}(\Phi, \rho) \to \Omega^{\bullet+1}_{\text{tens}}(\Phi, \rho)$. On tensorial forms it has the algebraic expression $\boldsymbol{D} : \Omega^\bullet_{\text{tens}}(\Phi, \rho) \to \Omega^{\bullet+1}_{\text{tens}}(\Phi, \rho)$, $\boldsymbol{\alpha} \mapsto \boldsymbol{D\alpha} = \boldsymbol{d\alpha} + \rho_*(\boldsymbol{\omega})\boldsymbol{\alpha}$, the sought after first order linear operator.

The curvature 2-form is defined as $\boldsymbol{\Omega} := \boldsymbol{d\omega}\circ|^h$, which implies $\boldsymbol{\Omega} \in \Omega^2_{\text{tens}}\big(\Phi, \mathfrak{aut}(P)\big)$. Algebraically, it is also given by Cartan structure equation

$$\boldsymbol{\Omega} = \boldsymbol{d\omega} + \tfrac{1}{2}[\boldsymbol{\omega}, \boldsymbol{\omega}]_{\mathfrak{aut}(P)}. \tag{75}$$

The Bianchi identity $\boldsymbol{D\Omega} = \boldsymbol{d\Omega} + [\boldsymbol{\omega}, \boldsymbol{\Omega}]_{\mathfrak{aut}(P)} \equiv 0$ is an algebraic consequence. On tensorial forms, $\boldsymbol{D}\circ\boldsymbol{D} = \rho_*(\boldsymbol{\Omega})$. To prove (75), the FN bracket (42)/(29) plays a key role. Indeed, one needs to show that both sides of the equality vanish on $\boldsymbol{X}^v, \boldsymbol{Y}^v \in \mathfrak{diff}_v(\Phi)$ with $\boldsymbol{X}, \boldsymbol{Y} \in C^\infty\big(\Phi, \mathfrak{aut}(P)\big)$. It is the case of the left-hand side since $(\boldsymbol{X}^v)^h \equiv 0$; the right-hand side, using Koszul formula for $\boldsymbol{d}$ and (29), yields:

$$\begin{aligned}
&\boldsymbol{d\omega}(\boldsymbol{X}^v, \boldsymbol{Y}^v) + [\boldsymbol{\omega}(\boldsymbol{X}^v), \boldsymbol{\omega}(\boldsymbol{Y}^v)]_{\mathfrak{aut}(P)}\\
&\quad = \boldsymbol{X}^v\big(\omega(\boldsymbol{Y}^v)\big) - \boldsymbol{Y}^v\big(\omega(\boldsymbol{X}^v)\big) - \boldsymbol{\omega}([\boldsymbol{X}^v, \boldsymbol{Y}^v]) + [\boldsymbol{X}, \boldsymbol{Y}]_{\mathfrak{aut}(P)}\\
&\quad = \boldsymbol{X}^v(\boldsymbol{Y}) - \boldsymbol{Y}^v(\boldsymbol{X}) - \boldsymbol{\omega}\big(\{\boldsymbol{X}, \boldsymbol{Y}\}^v\big) + [\boldsymbol{X}, \boldsymbol{Y}]_{\mathfrak{aut}(P)}\\
&\quad = [\boldsymbol{X}, \boldsymbol{Y}]_{\mathfrak{aut}(P)} + \boldsymbol{X}^v(\boldsymbol{Y}) - \boldsymbol{Y}^v(\boldsymbol{X}) - \{\boldsymbol{X}, \boldsymbol{Y}\} \equiv 0.
\end{aligned} \tag{76}$$

Given the defining equivariance and verticality properties (73) of a connection, using (32)/(61) one shows that its vertical transformation under $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\big(\Phi, \mathrm{Aut}(P)\big)$ is

$$\boldsymbol{\omega}^{\boldsymbol{\psi}} := \Xi^\star \boldsymbol{\omega} = \boldsymbol{\psi}_*^{-1}\,\boldsymbol{\omega} \circ \boldsymbol{\psi} + \boldsymbol{\psi}_*^{-1} \boldsymbol{d}\boldsymbol{\psi}. \tag{77}$$

The formula is the same for its gauge transformation under $\boldsymbol{Aut}_v(\Phi) \simeq \boldsymbol{Aut}(P)$. The difference between the two is seen only upon repeated transformations of each type, as stressed in Section 3.3, see (58)–(59). Here again, we note that $\boldsymbol{\omega}^{\boldsymbol{\psi}} = \Xi^\star \boldsymbol{\omega} \notin \mathcal{C}$ unless $\boldsymbol{\psi} \in \boldsymbol{Diff}(M) \sim \Xi \in \boldsymbol{Aut}_v(\Phi)$ – see ref. [14] – only $\boldsymbol{Aut}_v(\Phi) \simeq \boldsymbol{Aut}(P)$ preserves $\mathcal{C}$. By (60), the $\boldsymbol{\mathfrak{diff}}_v(\Phi) \simeq C^\infty\big(\Phi, \mathfrak{aut}(P)\big)$ transformations of a connection are given by the Nijenhuis–Lie derivative,

$$\boldsymbol{L}_{\boldsymbol{X}^v} \boldsymbol{\omega} = \boldsymbol{d}\boldsymbol{X} + [\boldsymbol{\omega}, \boldsymbol{X}]_{\mathfrak{aut}(P)}. \tag{78}$$

Infinitesimal gauge transformations, under $\boldsymbol{\mathfrak{aut}}_v(\Phi) \simeq \boldsymbol{\mathfrak{aut}}(P)$, are given by the same relation, but can be written $\boldsymbol{L}_{\boldsymbol{X}^v}\boldsymbol{\omega} = \boldsymbol{D}\boldsymbol{X}$ as $\boldsymbol{X} \in \boldsymbol{\mathfrak{aut}}(P)$ is a tensorial 0-form. Similarly, the finite and infinitesimal vertical transformations of the curvature (and gauge transformations, with the above caveat) are given, as special cases of (62) and (66), by:

$$\boldsymbol{\Omega}^{\boldsymbol{\psi}} := \Xi^\star \boldsymbol{\Omega} = \boldsymbol{\psi}_*^{-1}\,\boldsymbol{\Omega} \circ \boldsymbol{\psi}, \quad \text{so} \quad \boldsymbol{L}_{\boldsymbol{X}^v}\boldsymbol{\Omega} = [\boldsymbol{\Omega}, \boldsymbol{X}]_{\mathfrak{aut}(P)}. \tag{79}$$

Equation (77) allows to write the following useful lemma: For $\boldsymbol{\alpha}, \boldsymbol{D\alpha} \in \Omega^\bullet_{\text{tens}}(\Phi, \rho)$, we have on the one hand $\boldsymbol{d}\Xi^\star\boldsymbol{\alpha} = \boldsymbol{d}\big(\rho(\boldsymbol{\psi})^{-1}\boldsymbol{\alpha}\big)$. On the other hand, by $\Xi^\star \boldsymbol{D\alpha} = \rho(\boldsymbol{\psi})^{-1}\boldsymbol{D\alpha}$,

$$\begin{aligned}
\Xi^\star \boldsymbol{d\alpha} &= \rho(\boldsymbol{\psi})^{-1}\boldsymbol{D\alpha} - \Xi^\star\big(\rho_*(\boldsymbol{\omega})\boldsymbol{\alpha}\big)\\
&= \rho(\boldsymbol{\psi})^{-1}\boldsymbol{d\alpha} + \rho(\boldsymbol{\psi})^{-1}\,\rho_*(\boldsymbol{\omega})\boldsymbol{\alpha} - \rho_*(\boldsymbol{\omega}^{\boldsymbol{\psi}})\boldsymbol{\alpha}^{\boldsymbol{\psi}}\\
&= \rho(\boldsymbol{\psi})^{-1}\boldsymbol{d\alpha} - \rho_*(\boldsymbol{\psi}_*^{-1}\boldsymbol{d\psi})\rho(\boldsymbol{\psi})^{-1}\boldsymbol{\alpha}\\
&= \rho(\boldsymbol{\psi})^{-1}\big(\boldsymbol{d\alpha} - \rho_*(\boldsymbol{d\psi} \circ \boldsymbol{\psi}^{-1})\boldsymbol{\alpha}\big).
\end{aligned}$$

By naturality of the exterior derivative, $[\Xi^\star, \boldsymbol{d}] = 0$, we obtain the identity:

$$\boldsymbol{d}\big(\rho(\boldsymbol{\psi})^{-1}\boldsymbol{\alpha}\big) = \rho(\boldsymbol{\psi})^{-1}\big(\boldsymbol{d\alpha} - \rho_*(\boldsymbol{d\psi} \circ \boldsymbol{\psi}^{-1})\,\boldsymbol{\alpha}\big). \tag{80}$$

In particular, for the pullback representation, $\rho(\psi)^{-1} = \psi^*$ and $-\rho_*(X) = \mathfrak{L}_X$, this is:

$$\boldsymbol{d}(\boldsymbol{\psi}^*\boldsymbol{\alpha}) = \boldsymbol{\psi}^*\big(\boldsymbol{d\alpha} + \mathfrak{L}_{\boldsymbol{d}\psi \circ \psi^{-1}}\boldsymbol{\alpha}\big). \tag{81}$$

The latter appears in the covariant phase space literature, e.g., in refs. [35, 36].[13]

3.4.2. Twisted Connections

Twisted equivariant/tensorial forms $\boldsymbol{\alpha}$ have values in a G-space $\boldsymbol{V}$, G a (possibly infinite-dimensional) Lie group, and their equivariance is given by a 1-cocycle (50) for the action of $\mathrm{Aut}(P)$ on Φ,

$C : \Phi \times \mathrm{Aut}(P) \to G$, $(\phi, \psi) \mapsto C(\phi; \psi)$, s.t.

$$R^\star_{\psi}\boldsymbol{\alpha} = C(\phi;\psi)^{-1}\boldsymbol{\alpha}, \quad \text{with} \quad C(\phi;\psi' \circ \psi) = C(\phi;\psi')\cdot C(\phi^{\psi'};\psi). \tag{82}$$

Their infinitesimal equivariance is given by $\boldsymbol{L}_{X^v}\boldsymbol{\alpha} = -a(X,\phi)\boldsymbol{\alpha}$, with $a(X,\phi) := \frac{d}{d\tau}\,C(\phi,\psi_\tau)|_{\tau=0}$ a 1-cocycle (52) for the action of $\mathfrak{aut}(P)$ on Φ. If the target group of the 1-cocycle is the automorphism group itself, $G = \mathrm{Aut}(P)$, or a subgroup thereof, and $\boldsymbol{V}$ is a space of tensors of P, (82) specializes to

$$R^\star_{\psi}\boldsymbol{\alpha} = C(\phi;\psi)^*\boldsymbol{\alpha}, \quad \text{with} \quad C(\phi;\psi' \circ \psi) = C(\phi;\psi') \circ C(\phi^{\psi'};\psi). \tag{83}$$

A *twisted connection* 1-form $\boldsymbol{\varpi} \in \Omega^1_{\text{eq}}(\Phi, \mathfrak{g})$ is defined by the two properties:

$$\begin{aligned}
\boldsymbol{\varpi}_{|\phi}\Big(X^v_{|\phi}\Big) &= \tfrac{d}{d\tau}C(\phi;\psi_\tau)\Big|_{\tau=0} = a(X,\phi) \in \mathfrak{g}, \quad \text{for } X \in \mathfrak{aut}(P),\\
R^\star_{\psi}\boldsymbol{\varpi}_{|\phi^\psi} &= \mathrm{Ad}_{C(\phi;\psi)^{-1}}\,\boldsymbol{\varpi}_{|\phi} + C(\phi;\psi)^{-1}\boldsymbol{d}C(\ ;\psi)_{|\phi}.
\end{aligned} \tag{84}$$

In the special case $G = \mathrm{Aut}(P)$, the equivariance of $\boldsymbol{\varpi} \in \Omega^1_{\text{eq}}\big(\Phi, \mathfrak{aut}(P)\big)$ is

$$R^\star_{\psi}\boldsymbol{\varpi}_{|\phi^\psi} = [C(\phi;\psi))^{-1}]_*\,\boldsymbol{\varpi}_{|\phi} \circ C(\phi;\psi) + [C(\phi;\psi))^{-1}]_*\boldsymbol{d}C(\ ;\psi)_{|\phi}, \tag{85}$$

The infinitesimal equivariance under $\mathfrak{aut}(P)$ is

$$\boldsymbol{L}_{X^v}\boldsymbol{\varpi} = \tfrac{d}{d\tau}\,R^\star_{\psi_\tau}\,\boldsymbol{\varpi}\,\Big|_{\tau=0} = \boldsymbol{d}a(X;\ ) + [\boldsymbol{\varpi}, a(X;\ )]_{\mathfrak{g}}. \tag{86}$$

Or, in the case $G = \mathrm{Aut}(P)$, $\boldsymbol{L}_{X^v}\boldsymbol{\varpi} = \boldsymbol{d}a(X;\ ) + [\boldsymbol{\varpi}, a(X;\ )]_{\mathfrak{aut}(P)}$.

The space of twisted connections $\bar{\mathcal{C}}$ is an affine space modeled on the vector space $\Omega^1_{\text{tens}}(\Phi, \mathfrak{g})$: For $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathcal{C}$, we have $\boldsymbol{\beta} := \boldsymbol{\omega}' - \boldsymbol{\omega} \in \Omega^1_{\text{tens}}(\Phi, \mathfrak{g})$. Or, given $\boldsymbol{\omega} \in \bar{\mathcal{C}}$ and $\boldsymbol{\beta} \in \Omega^1_{\text{tens}}(\Phi, \mathfrak{g})$, we have that $\boldsymbol{\omega}' = \boldsymbol{\omega} + \boldsymbol{\beta} \in \bar{\mathcal{C}}$.

A *twisted covariant derivative* is defined as $\bar{\boldsymbol{D}} : \Omega^\bullet_{\text{eq}}(\Phi, C) \to \Omega^{\bullet+1}_{\text{tens}}(\Phi, C)$, $\boldsymbol{\alpha} \mapsto \bar{\boldsymbol{D}}\boldsymbol{\alpha} := \boldsymbol{d\alpha} + \rho_*(\boldsymbol{\varpi})\boldsymbol{\alpha}$. This is the first order linear operator adapted to twisted equivariant/tensorial forms.

The curvature 2-form of $\boldsymbol{\varpi}$ is defined by the Cartan structure equation:

$$\bar{\boldsymbol{\Omega}} := \boldsymbol{d\varpi} + \tfrac{1}{2}[\boldsymbol{\varpi}, \boldsymbol{\varpi}]_{\mathfrak{g}} \ \in \Omega^2_{\text{tens}}(\Phi, \mathfrak{g}). \tag{87}$$

It thus satisfies the Bianchi identity, $\bar{\boldsymbol{D}}\bar{\boldsymbol{\Omega}} = \boldsymbol{d}\bar{\boldsymbol{\Omega}} + [\boldsymbol{\varpi}, \bar{\boldsymbol{\Omega}}]_{\mathfrak{g}} = 0$. And it holds that $\bar{\boldsymbol{D}} \circ \bar{\boldsymbol{D}} = \rho_*(\bar{\boldsymbol{\Omega}})$. For $X^v, Y^v \in \Gamma(V\Phi)$ with $X, Y \in \mathfrak{diff}(M)$, we have

$$\begin{aligned}
\bar{\boldsymbol{\Omega}}(X^v, Y^v) &= X^v\big(\omega(Y^v)\big) - Y^v\big(\omega(X^v)\big) - \boldsymbol{\omega}([X^v, Y^v])\\
&\quad + [\boldsymbol{\omega}(X^v), \boldsymbol{\omega}(Y^v)]_{\mathfrak{g}}\\
0 &= X^v\big(a(Y;\phi)\big) - Y^v\big(a(X;\phi)\big) - a([X,Y]_{\mathfrak{aut}(P)};\phi)\\
&\quad + [a(X;\phi), a(Y;\phi)]_{\mathfrak{g}},
\end{aligned} \tag{88}$$

which reproduces the infinitesimal 1-cocycle property (52).

[13] It ought not to be confused with with (71) as, despite the superficial similarity, the two results are distinct geometric statements.

The vertical transformation under $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \text{Aut}(P)\right)$ of a twisted connection is found to be, using (84) and (32)/(61),

$$\boldsymbol{\varpi}^{\boldsymbol{\psi}} := \Xi^\star \boldsymbol{\varpi} = \text{Ad}_{C(\boldsymbol{\psi})^{-1}}\, \boldsymbol{\varpi} + C(\boldsymbol{\psi})^{-1} \boldsymbol{d} C(\boldsymbol{\psi}). \tag{89}$$

In case $G = \text{Aut}(P)$ this specializes to: $\boldsymbol{\varpi}^{\boldsymbol{\psi}} = C(\boldsymbol{\psi})_*^{-1}\, \boldsymbol{\varpi} \circ C(\boldsymbol{\psi}) + C(\boldsymbol{\psi})_*^{-1} \boldsymbol{d} C(\boldsymbol{\psi})$. Gauge transformations under $\boldsymbol{Aut}_v(\Phi) \simeq \boldsymbol{Aut}(P)$ are given by the same formula, the difference showing upon repeated transformations of each type, see (58)–(59) in Section 3.3. Transformations of a twisted connection under $\mathfrak{diff}_v(\Phi) \simeq C^\infty\left(\Phi, \mathfrak{aut}(P)\right)$ are given by the Nijenhuis–Lie derivative,

$$\boldsymbol{L}_{\boldsymbol{X}^v} \boldsymbol{\varpi} = \boldsymbol{d} a(\boldsymbol{X}) + [\boldsymbol{\varpi}, a(\boldsymbol{X})]_{\mathfrak{g}}. \tag{90}$$

Infinitesimal gauge transformations, under $\mathfrak{aut}_v(\Phi) \simeq \mathfrak{aut}(P)$, are given by the same relation. The difference being seen upon iteration, as reflected by the commutation property of the Nijenhuis–Lie derivative (31)/(45).

Finite and infinitesimal general vertical transformations of the curvature are given by,

$$\bar{\boldsymbol{\Omega}}^{\boldsymbol{\psi}} := \Xi^\star \bar{\boldsymbol{\Omega}} = \text{Ad}_{C(\boldsymbol{\psi})^{-1}}\, \bar{\boldsymbol{\Omega}}, \quad \text{so} \quad \boldsymbol{L}_{\boldsymbol{X}^v} \bar{\boldsymbol{\Omega}} = [\bar{\boldsymbol{\Omega}}, a(\boldsymbol{X})]_{\mathfrak{g}}. \tag{91}$$

For $G = \text{Aut}(P)$, this is $\bar{\boldsymbol{\Omega}}^{\boldsymbol{\psi}} := \Xi^\star \bar{\boldsymbol{\Omega}} = \boldsymbol{\psi}_*^{-1}\, \bar{\boldsymbol{\Omega}} \circ \boldsymbol{\psi}$ and $\boldsymbol{L}_{\boldsymbol{X}^v} \bar{\boldsymbol{\Omega}} = [\bar{\boldsymbol{\Omega}}, \boldsymbol{X}]_{\mathfrak{aut}(P)}$. These illustrate (62) and (66). The same relations hold for its gauge transformations, with the usual caveat. When $\boldsymbol{X}^v, \boldsymbol{Y}^v \in \mathfrak{diff}_v(\Phi)$ with $\boldsymbol{X}, \boldsymbol{Y} \in C^\infty\left(\Phi, \mathfrak{aut}(P)\right)$, using the definition (42)/(29) of the FN bracket, we have

$$\begin{aligned}
\bar{\boldsymbol{\Omega}}(\boldsymbol{X}^v, \boldsymbol{Y}^v) &= \boldsymbol{X}^v\left(\omega(\boldsymbol{Y}^v)\right) - \boldsymbol{Y}^v\left(\omega(\boldsymbol{X}^v)\right) - \boldsymbol{\omega}\left([\boldsymbol{X}^v, \boldsymbol{Y}^v]\right) + [\boldsymbol{\omega}(\boldsymbol{X}^v), \boldsymbol{\omega}(\boldsymbol{Y}^v)]_{\mathfrak{g}}, \\
0 &= \boldsymbol{X}^v\left(a(\boldsymbol{Y}; \phi)\right) - \boldsymbol{Y}^v\left(a(\boldsymbol{X}; \phi)\right) - a(\{\boldsymbol{X}, \boldsymbol{Y}\}; \phi) + [a(\boldsymbol{X}; \phi), a(\boldsymbol{Y}; \phi)]_{\mathfrak{g}}, \\
0 &= \boldsymbol{X}^v\left(a(\underline{\boldsymbol{Y}}; \phi)\right) - \boldsymbol{Y}^v\left(a(\underline{\boldsymbol{X}}; \phi)\right) - a([\boldsymbol{X}, \boldsymbol{Y}]_{\mathfrak{diff}(M)}; \phi) + [a(\boldsymbol{X}; \phi), a(\boldsymbol{Y}; \phi)]_{\mathfrak{g}}.
\end{aligned} \tag{92}$$

This reproduces (67), where the notation of the last line was first used.

3.5. Associated Bundles, Bundle of Regions of P and Integration

Given a principal fiber bundle, it is standard that one can build an associate bundle (over the same base) via each representation of the structure group. This can be generalized by replacing representation by 1-cocycles for the action of the structure group.[31] Below we review these constructions in our case, where the principal bundle is Φ with structure group $\text{Aut}(P)$.

Given a representation space (ρ, V) of $\text{Aut}(P)$, consider the direct product space $\Phi \times \boldsymbol{V}$, with the two natural projections: $\pi_\Phi : \Phi \times \boldsymbol{V} \to \Phi$ and $\pi_{\boldsymbol{V}} : \Phi \times \boldsymbol{V} \to \boldsymbol{V}$. One defines a right action of $\text{Aut}(P)$ on $\Phi \times \boldsymbol{V}$ by:

$$\begin{aligned}
&(\Phi \times V) \times \text{Aut}(P) \to \Phi \times V, \\
&\qquad \left((\phi, \nu), \psi\right) \mapsto \left(\psi^* \phi,\ \rho(\psi)^{-1} \nu\right) = \left(R_\psi \phi,\ \rho(\psi)^{-1} \nu\right) \\
&\qquad\qquad =: \bar{R}_\psi(\phi, \nu).
\end{aligned} \tag{93}$$

The bundle $\boldsymbol{E}$ associated to Φ via the representation ρ is the quotient of $\Phi \times \boldsymbol{V}$ by $\bar{R}$:

$$\boldsymbol{E} = \Phi \times_\rho \boldsymbol{V} := \Phi \times \boldsymbol{V}/\sim \tag{94}$$

where $(\phi', \nu') \sim (\phi, \nu)$ when $\exists\, \psi \in \text{Aut}(P)$ s.t. $(\phi', \nu') = \bar{R}_\psi(\phi, \nu)$. We write $\bar{\pi}_{\boldsymbol{E}} : \Phi \times \boldsymbol{V} \to \boldsymbol{E}$. A point in $\boldsymbol{E}$ is an equivalence class $e = [\phi, \nu]$. The projection of $\boldsymbol{E} \xrightarrow{\pi_E} \mathcal{M}$ is $\pi_{\boldsymbol{E}}([\phi, \nu]) := \pi(\phi) = [\phi]$. It is a well-known result that there is a bijection between sections of $\boldsymbol{E}$ and V-valued ρ-equivariant functions on Φ:

$$\begin{aligned}
\Gamma(\boldsymbol{E}) &:= \left\{ \boldsymbol{s} : \mathcal{M} \to \boldsymbol{E} \right\} \ \simeq\ \Omega^0_{\text{eq}}(\Phi, \rho) \\
&:= \left\{ \boldsymbol{\varphi} : \Phi \to V \mid R^\star_\psi \boldsymbol{\varphi} = \rho(\psi)^{-1} \boldsymbol{\varphi} \right\},
\end{aligned} \tag{95}$$

the isomorphism being $\boldsymbol{s}([\phi]) = [\phi, \boldsymbol{\varphi}(\phi)]$. Equivariant functions are tensorial 0-forms, so their vertical (gauge) transformations are given by (62)–(66): $\boldsymbol{\varphi}^{\boldsymbol{\psi}} = \rho(\boldsymbol{\psi})^{-1} \boldsymbol{\varphi}$ and $\boldsymbol{L}_{\boldsymbol{X}^v} \boldsymbol{\varphi} = -\rho_*(\boldsymbol{X}) \boldsymbol{\varphi}$, for $\boldsymbol{\psi} \in C^\infty\left(\Phi, \text{Aut}(P)\right) \simeq \boldsymbol{Diff}_v(\Phi)$ and $\boldsymbol{X} \in C^\infty\left(\Phi, \mathfrak{aut}(P)\right) \simeq \mathfrak{diff}_v(\Phi)$. A Ehresmann connection is needed for their covariant differentiation, see Section 3.4.1.

The construction holds the same replacing representations ρ by 1-cocycles $C : \Phi \times \text{Aut}(P) \to G$ as defined by (50) in Section 3.2.2. Given a G-space $\boldsymbol{V}$, one defines the right action of $\text{Aut}(P)$ on $\Phi \times \boldsymbol{V}$: $(\Phi \times \boldsymbol{V}) \times \text{Aut}(P) \to \Phi \times \boldsymbol{V}$, $\left((\phi, \nu), \psi\right) \mapsto \bar{R}_\psi(\phi, \nu) = \left(\psi^* \phi,\ C(\phi; \psi)^{-1} \nu\right)$. The *twisted* bundle $\widetilde{\boldsymbol{E}} \to \mathcal{M}$ associated to Φ via the 1-cocycle C is then: $\widetilde{\boldsymbol{E}} = \Phi \times_C \boldsymbol{V} := \Phi \times \boldsymbol{V}/\sim$, with $\left(\psi^* \phi,\ C(\phi; \psi)^{-1} \nu\right) \sim (\phi, \nu)$. As above, its space of sections is isomorphic to the space of *twisted* equivariant function on Φ:

$$\begin{aligned}
\Gamma\left(\widetilde{\boldsymbol{E}}\right) &:= \left\{ \tilde{s} : \mathcal{M} \to \widetilde{\boldsymbol{E}} \right\} \ \simeq\ \Omega^0_{\text{eq}}(\Phi, C) \\
&:= \left\{ \tilde{\boldsymbol{\varphi}} : \Phi \to V \mid R^\star_\psi \tilde{\boldsymbol{\varphi}} = C(\phi; \psi)^{-1} \tilde{\boldsymbol{\varphi}} \right\}.
\end{aligned} \tag{96}$$

The vertical transformation of such twisted equivariant functions is given by (62)–(66). A twisted connection as discussed in Section 3.4.2 is needed for their covariant differentiation.

3.5.1. Associated Bundle of Regions

Let $\boldsymbol{E} = \bar{\boldsymbol{V}}(P)$ be the bundle canonically associated to Φ via the *defining representation* of $\text{Aut}(P)$: the σ-algebra of open sets of P, $\boldsymbol{V}(P) := \left\{ V \subset P \mid V \text{ open set} \right\}$.[14] The right action of $\text{Aut}(P)$ on the product space $\Phi \times \boldsymbol{V}(P)$ is

$$\begin{aligned}
&\left(\Phi \times \boldsymbol{V}(P)\right) \times \text{Aut}(P) \to \Phi \times \boldsymbol{V}(P), \\
&\qquad \left((\phi, V), \psi\right) \mapsto \bar{R}_\psi(\phi, V) := \left(\psi^* \phi, \psi^{-1}(V)\right).
\end{aligned} \tag{97}$$

The *associated bundle of regions* of P is thus:

$$\bar{\boldsymbol{V}}(P) = \Phi \times_{\text{Aut}(P)} \boldsymbol{V}(P) := \Phi \times \boldsymbol{V}(P)/\sim. \tag{98}$$

Its space of sections $\Gamma\left(\bar{\boldsymbol{V}}(P)\right) := \left\{ \bar{s} : \mathcal{M} \to \bar{\boldsymbol{V}}(P) \right\}$ is isomorphic to

$$\Omega^0_{\text{eq}}\left(\Phi, \boldsymbol{V}(P)\right) := \left\{ \boldsymbol{V} : \Phi \to \boldsymbol{V}(P) \mid R^\star_\psi \boldsymbol{V} = \psi^{-1}(\boldsymbol{V}) \right\}. \tag{99}$$

[14] It is actually the defining representation of $\text{Diff}(P)$, as a (Lie) pseudogroup[37] and restricts naturally to $\text{Aut}(P)$.

A map $\phi \to V(\phi)$ may be seen as a "field-dependent" open set of P, a region of P defined in a "*ϕ-relative*" and $\mathrm{Aut}(P)$-equivariant way. By (62)–(66), its transformations under $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty(\Phi, \mathrm{Aut}(P))$ and $\mathfrak{diff}_v(\Phi) \simeq C^\infty(\Phi, \mathfrak{aut}(P))$ are respectively:

$$\boldsymbol{V}^{\boldsymbol{\psi}} = \boldsymbol{\psi}^{-1}(\boldsymbol{V}), \quad \text{and} \quad \boldsymbol{L}_{\boldsymbol{X}^v}\boldsymbol{V} = -\boldsymbol{X}(\boldsymbol{V}). \tag{100}$$

Integration on P provides just such an example of equivariant function, as can be shown by framing it as a natural construction over $\Phi \times \boldsymbol{V}(P)$.

3.5.2. Integration Map

Associated bundles are defined via the action of the structure group $\mathrm{Aut}(P)$ on $\Phi \times \boldsymbol{V}$: $\bar{R}_\psi(\phi, v) := (\psi^*\phi,\ \rho(\psi)^{-1}v)$. The corresponding action of $C^\infty(\Phi, \mathrm{Aut}(P)) \simeq \boldsymbol{Diff}_v(\Phi)$ is

$$\begin{aligned}(\Phi \times \boldsymbol{V}) \times C^\infty(\Phi, \mathrm{Aut}(P)) &\to \Phi \times \boldsymbol{V},\\ \big((\phi, v), \boldsymbol{\psi}\big) &\mapsto \big(\boldsymbol{\psi}^*\phi,\ \rho(\boldsymbol{\psi})^{-1}v\big)\\ &= \big(\Xi(\phi),\ \rho(\boldsymbol{\psi})^{-1}v\big) =: \bar{\Xi}(\phi, v).\end{aligned} \tag{101}$$

In particular, for $\boldsymbol{\psi} \in \boldsymbol{Aut}(P)$ we have $\bar{\Xi} \circ \bar{R}_\psi = \bar{R}_\psi \circ \bar{\Xi}$. The action of $C^\infty(\Phi, \mathfrak{aut}(P)) \simeq \mathfrak{diff}_v(\Phi)$ is the linearization $(\Phi \times \boldsymbol{V}) \times C^\infty(\Phi, \mathfrak{aut}(P)) \to V(\Phi \times \boldsymbol{V}) \simeq V\Phi \oplus V\boldsymbol{V} \subset T(P \times \boldsymbol{V})$.

The induced actions of $\mathrm{Aut}(P)$ and $C^\infty(\Phi, \mathrm{Aut}(P)) \simeq \boldsymbol{Diff}_v(\Phi)$ on $\Omega^\bullet(\Phi) \times \boldsymbol{V}$ are

$$\begin{aligned}\big(\Omega^\bullet(\Phi) \times \boldsymbol{V}\big) \times \mathrm{Aut}(P) &\to \Omega^\bullet(\Phi) \times \boldsymbol{V},\\ \big((\boldsymbol{\alpha}, v), \psi\big) &\mapsto \Big(R^\star_\psi \boldsymbol{\alpha},\ \rho(\psi)^{-1}v\Big) =: \tilde{R}_\psi(\boldsymbol{\alpha}, v)\end{aligned} \tag{102}$$

and

$$\begin{aligned}\big(\Omega^\bullet(\Phi) \times \boldsymbol{V}\big) \times C^\infty(\Phi, \mathrm{Aut}(P)) &\to \Omega^\bullet(\Phi) \times \boldsymbol{V},\\ \big((\boldsymbol{\alpha}, v), \boldsymbol{\psi}\big) &\mapsto \big(\Xi^\star \boldsymbol{\alpha}, \rho(\boldsymbol{\psi})^{-1}v\big) =: \tilde{\Xi}(\boldsymbol{\alpha}, v).\end{aligned} \tag{103}$$

Correspondingly, the induced actions of $\mathfrak{aut}(P)$ and $C^\infty(\Phi, \mathfrak{aut}(P)) \simeq \mathfrak{diff}_v(\Phi)$ are the linearizations:

$$\begin{aligned}\big((\boldsymbol{\alpha}, v), X\big) &\mapsto \tfrac{d}{d\tau}\, \tilde{R}_{\psi_\tau}(\boldsymbol{\alpha}, v)\Big|_{\tau=0} = \big(\boldsymbol{L}_{X^v}\boldsymbol{\alpha},\ v\big) \oplus \big(\boldsymbol{\alpha},\ -\rho_*(X)v\big),\\ \big((\boldsymbol{\alpha}, v), \boldsymbol{X}\big) &\mapsto \tfrac{d}{d\tau}\, \tilde{\Xi}_\tau(\boldsymbol{\alpha}, v)\Big|_{\tau=0} = \big(\boldsymbol{L}_{\boldsymbol{X}^v}\boldsymbol{\alpha},\ v\big) \oplus \big(\boldsymbol{\alpha},\ -\rho_*(\boldsymbol{X})v\big).\end{aligned} \tag{104}$$

Furthermore, given a representation $(\tilde{\rho}, \boldsymbol{W})$ of $\mathrm{Aut}(P)$,

$$\begin{aligned}\text{if } \boldsymbol{\alpha} \in \Omega^\bullet_{\text{eq}}(\Phi, \boldsymbol{W}) \text{ then } \tilde{R}_\psi(\boldsymbol{\alpha}, v) &= \Big(R^\star_\psi \boldsymbol{\alpha},\ \rho(\psi)^{-1}v\Big)\\ &= \big(\tilde{\rho}(\psi)^{-1}\boldsymbol{\alpha},\ \rho(\psi)^{-1}v\big),\\ \text{if } \boldsymbol{\alpha} \in \Omega^\bullet_{\text{tens}}(\Phi, \boldsymbol{W}) \text{ then } \tilde{\Xi}(\boldsymbol{\alpha}, v) &= \big(\Xi^\star \boldsymbol{\alpha},\ \rho(\boldsymbol{\psi})^{-1}v\big)\\ &= \big(\tilde{\rho}(\boldsymbol{\psi})^{-1}\boldsymbol{\alpha},\ \rho(\boldsymbol{\psi})^{-1}v\big),\end{aligned} \tag{105}$$

with linearizations read from (104). The exterior derivative $\boldsymbol{d}$ on Φ extends to $\Phi \times \boldsymbol{V}$ as $\boldsymbol{d} \to \boldsymbol{d} \times \mathrm{id}$. Yet, after the action of $C^\infty(\Phi, \mathfrak{aut}(P)) \simeq \mathfrak{diff}_v(\Phi)$ and due to the ϕ-dependence of $\boldsymbol{\psi}$, it will also act on the second factor $\rho(\boldsymbol{\psi})^{-1}v$.

Let $(\bar{\rho}, \boldsymbol{V}^*)$ be a representation of $\mathrm{Aut}(P)$ *dual* to $(\rho, \boldsymbol{V})$ w.r.t. a non-degenerate $\mathrm{Aut}(P)$-invariant *pairing*

$$\begin{aligned}\langle\ ,\ \rangle : \boldsymbol{V}^* \times \boldsymbol{V} &\to \mathbb{R},\\ (w, v) &\mapsto \langle w, v\rangle, \quad \text{s.t.} \quad \langle\, \bar{\rho}(\psi)w, \rho(\psi)v\rangle = \langle w, v\rangle.\end{aligned} \tag{106}$$

Under the action of $\mathfrak{aut}(P)$, with induced representation $\bar{\rho}_*$ and ρ_*, it holds that

$$\langle\, \bar{\rho}_*(X)w, v\rangle + \langle\, w, \rho_*(X)v\rangle = 0. \tag{107}$$

For $\boldsymbol{\alpha} \in \Omega^\bullet(\Phi, \boldsymbol{V}^*)$, let us define the operation $\mathcal{I}$ on $\Omega^\bullet(\Phi, \boldsymbol{V}^*) \times \boldsymbol{V}$ by,

$$\begin{aligned}\mathcal{I} : \Omega^\bullet(\Phi, \boldsymbol{V}^*) \times \boldsymbol{V} &\to \Omega^\bullet(\Phi),\\ (\boldsymbol{\alpha}, v) &\mapsto \mathcal{I}(\boldsymbol{\alpha}, v) := \langle \boldsymbol{\alpha}, v\rangle.\end{aligned} \tag{108}$$

This can be seen as an object on $\Phi \times \boldsymbol{V}$:

$$\begin{aligned}\mathcal{I}(\boldsymbol{\alpha},\ ) : \Phi \times \boldsymbol{V} &\to \Lambda^\bullet(\Phi),\\ (\phi, v) &\mapsto \mathcal{I}(\boldsymbol{\alpha}_{|\phi}, v) := \langle \boldsymbol{\alpha}_{|\phi}, v\rangle.\end{aligned} \tag{109}$$

We thus have:

$$\boldsymbol{d}\mathcal{I}(\boldsymbol{\alpha},\ ) = \mathcal{I}(\boldsymbol{d\alpha},\ ), \quad \text{and} \quad \iota_{\mathfrak{X}}\mathcal{I}(\boldsymbol{\alpha},\ ) = \mathcal{I}(\iota_{\mathfrak{X}}\boldsymbol{\alpha},\ ) \quad \text{for } \mathfrak{X} \in \Gamma(T\Phi). \tag{110}$$

The induced actions of $\mathrm{Aut}(P)$ and $C^\infty(\Phi, \mathrm{Aut}(P)) \simeq \boldsymbol{Diff}_v(\Phi)$ on such objects are:

$$\begin{aligned}\tilde{R}^\star_\psi \mathcal{I}(\boldsymbol{\alpha},\ )_{|(\psi^*\phi,\, \rho(\psi)^{-1}v)} &:= \langle\ ,\ \rangle \circ \tilde{R}_\psi(\boldsymbol{\alpha}, v) = \langle R^\star_\psi \boldsymbol{\alpha}_{|\psi^*\phi},\ \rho(\psi)^{-1}v\rangle,\\ \tilde{\Xi}^\star \mathcal{I}(\boldsymbol{\alpha},\ )_{|(\Xi(\phi),\, \rho(\boldsymbol{\psi})^{-1}v)} &:= \langle\ ,\ \rangle \circ \tilde{\Xi}(\boldsymbol{\alpha}, v) = \langle \Xi^\star \boldsymbol{\alpha}_{|\Xi(\phi)},\ \rho(\boldsymbol{\psi})^{-1}v\rangle.\end{aligned} \tag{111}$$

The actions of $\mathfrak{aut}(P)$ and $C^\infty(\Phi, \mathfrak{aut}(P)) \simeq \mathfrak{diff}_v(\Phi)$ are thus:

$$\begin{aligned}\tfrac{d}{d\tau}\, \tilde{R}^\star_{\psi_\tau} \mathcal{I}(\boldsymbol{\alpha}, v)\Big|_{\tau=0} &= \langle \boldsymbol{L}_{X^v}\boldsymbol{\alpha},\ v\rangle + \langle \boldsymbol{\alpha},\ -\rho_*(X)v\rangle,\\ \tfrac{d}{d\tau}\, \tilde{\Xi}^\star_\tau \mathcal{I}(\boldsymbol{\alpha}, v)\Big|_{\tau=0} &= \langle \boldsymbol{L}_{\boldsymbol{X}^v}\boldsymbol{\alpha},\ v\rangle + \langle \boldsymbol{\alpha},\ -\rho_*(\boldsymbol{X})v\rangle.\end{aligned} \tag{112}$$

Observe that for $\boldsymbol{\alpha} \in \Omega^\bullet_{\text{eq}}(\Phi, \boldsymbol{V}^*)$:

$$\begin{aligned}\tilde{R}^\star_\psi \mathcal{I}(\boldsymbol{\alpha},\ )_{|(\psi^*\phi,\ \rho(\psi)^{-1}v)} &:= \langle R^\star_\psi \boldsymbol{\alpha}_{|\psi^*\phi},\ \rho(\psi)^{-1}v\rangle\\ &= \langle \bar{\rho}(\psi)^{-1}\boldsymbol{\alpha}_{|\phi},\ \rho(\psi)^{-1}v\rangle = \langle \boldsymbol{\alpha}_{|\phi}, v\rangle\\ &=: \mathcal{I}(\boldsymbol{\alpha},\ )_{|(\phi, v)}.\end{aligned} \tag{113}$$

From this follows, by (112):

$$\begin{aligned}\langle \boldsymbol{L}_{X^v}\boldsymbol{\alpha},\ v\rangle + \langle \boldsymbol{\alpha},\ -\rho_*(X)v\rangle &= 0, \qquad X \in \mathfrak{aut}(P).\\ \langle -\bar{\rho}_*(X)\,\boldsymbol{\alpha},\ v\rangle + \langle \boldsymbol{\alpha},\ -\rho_*(X)v\rangle &= 0.\end{aligned} \tag{114}$$

If $\boldsymbol{\alpha} \in \Omega^\bullet_{\text{tens}}(\Phi, \boldsymbol{V}^*)$:

$$
\begin{aligned}
\tilde{\Xi}^\star \mathcal{I}(\boldsymbol{\alpha},\ )_{|(\Xi(\phi),\ \rho(\psi)^{-1}\nu)} &\coloneqq \langle \Xi^\star \boldsymbol{\alpha}_{|\Xi(\phi)},\ \rho(\boldsymbol{\psi})^{-1}\nu\rangle \\
&= \langle \bar\rho(\boldsymbol{\psi})^{-1}\boldsymbol{\alpha}_{|\phi},\ \rho(\boldsymbol{\psi})^{-1}\nu\rangle = \langle \boldsymbol{\alpha}_{|\phi}, \nu\rangle \\
&\eqqcolon \mathcal{I}(\boldsymbol{\alpha},\ )_{|(\phi,\nu)}.
\end{aligned} \tag{115}
$$

And it follows by (112):

$$
\begin{aligned}
&\langle L_{X^v}\boldsymbol{\alpha},\ \nu\rangle + \langle \boldsymbol{\alpha},\ -\rho_*(\boldsymbol{X})\nu\rangle = 0, \quad \boldsymbol{X} \in C^\infty(\Phi, \mathfrak{aut}(P)). \\
&\langle -\bar\rho_*(\boldsymbol{X})\,\boldsymbol{\alpha},\ \nu\rangle + \langle \boldsymbol{\alpha},\ -\rho_*(\boldsymbol{X})\nu\rangle = 0.
\end{aligned} \tag{116}
$$

When $\boldsymbol{\alpha}$ is tensorial, $\mathcal{I}(\boldsymbol{\alpha},\ )$ is "basic" on $\Phi \times \boldsymbol{V}$, meaning it is well-defined on $\boldsymbol{E} = \Phi \times \boldsymbol{V}/\sim$. Since $\mathcal{I}(\boldsymbol{\alpha},\ )$ is constant along an Aut(P)-orbit in $\Phi \times \boldsymbol{V}$, it allows to define $\boldsymbol{\varphi}_{\mathcal{I}(\alpha)} \in \Omega^0_{\text{eq}}(\Phi, \rho)$ via:

$$
\begin{aligned}
&\boldsymbol{\varphi}_{\mathcal{I}(\alpha)}(\phi) \coloneqq \pi_V(\phi, \nu)_{|\mathcal{I}(\alpha_{|\phi},\nu)=\text{cst}} \equiv \nu, \\
&\boldsymbol{\varphi}_{\mathcal{I}(\alpha)}(\psi^*\phi) \coloneqq \pi_V(\psi^*\phi, \rho(\psi)^{-1}\nu)_{|\mathcal{I}(\alpha_{|\phi},\nu)=\text{cst}} \equiv \rho(\psi)^{-1}\nu.
\end{aligned} \tag{117}
$$

By (95), the latter is equivalent to a section $\boldsymbol{s}_{\mathcal{I}(\alpha)} : \mathcal{M} \to \boldsymbol{E}$.

Observe that (115) implies that for $\boldsymbol{\alpha} \in \Omega^\bullet_{\text{tens}}(\Phi, \boldsymbol{V}^*)$ one has $\boldsymbol{d}\tilde{\Xi}^\star \mathcal{I}(\boldsymbol{\alpha},\ ) = \boldsymbol{d}\mathcal{I}(\boldsymbol{\alpha},\ ) = \mathcal{I}(\boldsymbol{d\alpha},\ )$. Also, we derive the following lemma:

$$
\begin{aligned}
\tilde{\Xi}^\star\langle \boldsymbol{d\alpha}, \nu\rangle &\coloneqq \langle \Xi^\star \boldsymbol{d\alpha},\ \rho(\boldsymbol{\psi})^{-1}\nu\rangle = \langle \boldsymbol{d}\,\Xi^\star\boldsymbol{\alpha},\ \rho(\boldsymbol{\psi})^{-1}\nu\rangle \\
&= \langle \boldsymbol{d}\,\bar\rho(\boldsymbol{\psi})^{-1}\boldsymbol{\alpha},\ \rho(\boldsymbol{\psi})^{-1}\nu\rangle \\
&= \langle \bar\rho(\boldsymbol{\psi})^{-1}\big(\boldsymbol{d\alpha} - \bar\rho_*(\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1})\,\boldsymbol{\alpha}\big),\ \rho(\boldsymbol{\psi})^{-1}\nu\rangle \\
&= \langle \boldsymbol{d\alpha} - \bar\rho_*(\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1})\boldsymbol{\alpha},\ \nu\rangle, \\
\hookrightarrow \quad \tilde{\Xi}^\star\langle \boldsymbol{d\alpha}, \nu\rangle &= \langle \boldsymbol{d\alpha}, \nu\rangle + \langle -\bar\rho_*(\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1})\boldsymbol{\alpha},\ \nu\rangle.
\end{aligned} \tag{118}
$$

We now specialize the above construction to the fundamental representation $\boldsymbol{V} = \boldsymbol{V}(P)$ and the representation $\boldsymbol{V}^* = \Omega^{\text{top}}(V)$ of volume forms on $V \in \boldsymbol{V}(P)$ on which Aut(P) acts by pullback. These are dual under the invariant *integration pairing*:

$$
\begin{aligned}
&\langle\ ,\ \rangle : \boldsymbol{\Omega}^{\text{top}}(V) \times \boldsymbol{V}(P) \to \mathbb{R}, \\
&\qquad (\omega, V) \mapsto \langle \omega, V\rangle \coloneqq \int_V \omega.
\end{aligned} \tag{119}
$$

The invariance property is the familiar identity

$$
\langle \psi^*\omega, \psi^{-1}(V)\rangle = \langle \omega, V\rangle \quad \to \quad \int_{\psi^{-1}(V)} \psi^*\omega = \int_V \omega. \tag{120}
$$

This, as a special case of (107) with $-\bar\rho_*(X) = \mathfrak{L}_X$ and $-\rho_*(X) = -X$, gives:

$$
\langle \mathfrak{L}_X\omega, V\rangle + \langle \omega, -X(V)\rangle = 0 \quad \to \quad \int_V \mathfrak{L}_X\omega + \int_{-X(V)} \omega = 0. \tag{121}
$$

This can be read as a *continuity equation* for the action of $\mathfrak{aut}(P)$. By Stokes theorem, the de Rham derivative d on $\Omega^\bullet(V)$ and the boundary operator ∂ on $\boldsymbol{V}(P)$ are adjoint operators w.r.t. to the integration pairing:

$$
\langle d\omega, V\rangle = \langle \omega, \partial V\rangle \quad \to \quad \int_V d\omega = \int_{\partial V} \omega. \tag{122}
$$

Considering $\boldsymbol{\alpha} \in \Omega^\bullet\left(\Phi, \Omega^{\text{top}}(V)\right)$, the field-dependent volume forms, we define the integration map on $\Phi \times \boldsymbol{V}(P)$:

$$
\mathcal{I}(\boldsymbol{\alpha}_{|\phi}, V) = \langle \boldsymbol{\alpha}_{|\phi}, V\rangle \coloneqq \int_V \boldsymbol{\alpha}_{|\phi}. \tag{123}
$$

The simplified notation $\boldsymbol{\alpha}_V$ may be used when more convenient. As a special case of (110), we have $\boldsymbol{d}\mathcal{I}(\boldsymbol{\alpha}, V) = \mathcal{I}(\boldsymbol{d\alpha}, V)$ and $\iota_{\mathfrak{X}}\mathcal{I}(\boldsymbol{\alpha}, V) = \mathcal{I}(\iota_{\mathfrak{X}}\boldsymbol{\alpha}, V)$ for $\mathfrak{X} \in \Gamma(T\Phi)$. The induced actions of Aut(P) and $C^\infty\left(\Phi, \text{Aut}(P)\right) \simeq \boldsymbol{Diff}_v(\Phi)$ on integrals are:

$$
\begin{aligned}
&\tilde{R}^\star_\psi \mathcal{I}(\boldsymbol{\alpha},\ )_{|\left(\psi^*\phi,\ \psi^{-1}(V)\right)} \coloneqq \langle R^\star_\psi \boldsymbol{\alpha}_{|\psi^*\phi},\ \psi^{-1}(V)\rangle = \int_{\psi^{-1}(V)} R^\star_\psi \boldsymbol{\alpha}_{|\psi^*\phi}, \\
&\tilde{\Xi}^\star \mathcal{I}(\boldsymbol{\alpha},\ )_{|\left(\Xi(\phi),\ \boldsymbol{\psi}^{-1}(V)\right)} \coloneqq \langle \Xi^\star \boldsymbol{\alpha}_{|\Xi(\phi)},\ \boldsymbol{\psi}^{-1}(V)\rangle = \int_{\boldsymbol{\psi}^{-1}(V)} \Xi^\star \boldsymbol{\alpha}_{|\Xi(\phi)}.
\end{aligned} \tag{124}
$$

In the latter case, $\boldsymbol{d}$ acts also on the transformed region $\boldsymbol{\psi}^{-1}(V)$ due to the ϕ-dependence of $\boldsymbol{\psi}$. We may write the above as $\boldsymbol{\alpha}_V{}^\psi$ and $\boldsymbol{\alpha}_V{}^{\boldsymbol{\psi}}$, respectively. The actions of $\mathfrak{aut}(P)$ and $C^\infty\left(\Phi, \mathfrak{aut}(P)\right) \simeq \mathfrak{diff}_v(\Phi)$ on integrals are

$$
\begin{aligned}
&\tfrac{d}{d\tau}\tilde{R}^\star_{\psi_\tau}\mathcal{I}(\boldsymbol{\alpha}, V)\Big|_{\tau=0} = \langle L_{X^v}\boldsymbol{\alpha}, V\rangle + \langle \boldsymbol{\alpha}, -X(V)\rangle = \int_V L_{X^v}\boldsymbol{\alpha} + \int_{-X(V)} \boldsymbol{\alpha}, \\
&\tfrac{d}{d\tau}\tilde{\Xi}^\star_\tau\mathcal{I}(\boldsymbol{\alpha}, V)\Big|_{\tau=0} = \langle L_{\boldsymbol{X}^v}\boldsymbol{\alpha}, V\rangle + \langle \boldsymbol{\alpha}, -\boldsymbol{X}(V)\rangle = \int_V L_{\boldsymbol{X}^v}\boldsymbol{\alpha} + \int_{-\boldsymbol{X}(V)} \boldsymbol{\alpha}.
\end{aligned} \tag{125}
$$

When convenient, we may write the above as $\delta_X\boldsymbol{\alpha}_V$ and $\delta_{\boldsymbol{X}}\boldsymbol{\alpha}_V$ respectively.

When $\boldsymbol{\alpha}$ is tensorial on Φ, as a special case of the results (115) above, we have that $\boldsymbol{\alpha}_V = \mathcal{I}(\boldsymbol{\alpha}_{|\phi}, V)$ is $C^\infty\left(\Phi, \text{Aut}(P)\right)$-invariant: $\boldsymbol{\alpha}_V{}^{\boldsymbol{\psi}} = \boldsymbol{\alpha}_V$. From which follows

$$
\begin{aligned}
&\delta_{\boldsymbol{X}}\boldsymbol{\alpha}_V = 0 \Rightarrow \langle L_{\boldsymbol{X}^v}\boldsymbol{\alpha}, V\rangle + \langle \boldsymbol{\alpha}, -\boldsymbol{X}(V)\rangle = 0, \quad \boldsymbol{X} \in C^\infty(\Phi, \mathfrak{aut}(P)). \\
&\langle \mathfrak{L}_{\boldsymbol{X}}\alpha, V\rangle + \langle \boldsymbol{\alpha}, -\boldsymbol{X}(V)\rangle = 0 \to \int_V \mathfrak{L}_{\boldsymbol{X}}\alpha + \int_{-\boldsymbol{X}(V)} \boldsymbol{\alpha} = 0,
\end{aligned} \tag{126}
$$

as a special case of (116) and (125). This can be interpreted as a continuity equation. For $\boldsymbol{\alpha}$ equivariant, we have Aut(P)-invariance of its integral: $\boldsymbol{\alpha}_V{}^\psi = \boldsymbol{\alpha}_V$, and (126) holds mutatis mutandis ($\boldsymbol{X} \to X$). For $\boldsymbol{\alpha}$ tensorial, $\boldsymbol{\alpha}_V = \mathcal{I}(\boldsymbol{\alpha}, V)$ is thus well-defined on the bundle of regions $\bar{\boldsymbol{V}}(P) = \Phi \times \boldsymbol{V}(P)/\sim$, and one may define an equivariant $\boldsymbol{V}(P)$-valued function on Φ as in (117). It also means that $\boldsymbol{d}(\boldsymbol{\alpha}_V{}^{\boldsymbol{\psi}}) = \boldsymbol{d\alpha}_V$, i.e.,

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$$
d\langle \psi^*\alpha, \psi^{-1}(V)\rangle = \langle d\alpha, V\rangle = d\langle \alpha, V\rangle \to d\int_{\psi^{-1}(V)} \psi^*\alpha
= \int_V d\alpha = d\int_V \alpha. \tag{127}
$$

Also, specializing (118), we get the identity

$$
\begin{aligned}
&(d\alpha_V)^\psi = d\alpha_V + \langle \mathfrak{L}_{d\psi\circ\psi^{-1}}\alpha, V\rangle,\\
&\langle d\alpha, V\rangle^\psi = \langle d\alpha, V\rangle + \langle \mathfrak{L}_{d\psi\circ\psi^{-1}}\alpha, V\rangle.
\end{aligned} \tag{128}
$$

As we will see in 5.1.2, the *local* version of the above are relevant to the standard formulation of field theory, i.e., its formulation on $(U \subset)\, M$. The results as presented in this section are relevant to a programmatic *global* formulation of gRGFT, on P. As indicated in conclusion 6, this part of our program is to be pursued in future work.

4. The Dressing Field Method

The DFM is a formal but systematic, algorithm to build basic forms on a bundle. In the context of GFT, it allows to construct gauge-invariants, generalizing *Dirac variables*. First developed for Yang–Mills theories and gauge gravity theories formulated via Cartan geometry, i.e., for internal gauge groups,[4–7,38–42] it is then extended to general-relativistic theories, i.e., for diffeomorphism symmetry in ref. [8]. Its first (field-theoretic) application to supersymmetric field theory (and supergravity) is given in ref. [43].

It can be understood as the geometric framework underlying various notions encountered in recent literature on gauge theories and gravity: notably that of *edge modes* as introduced in ref. [35] and further developed e.g., in refs. [27, 28, 33, 36], and that of *gravitational dressings*[44–50] – see also ref. [51] – as well as "dynamical reference frames",[52,53] or "embedding fields".[28,54–56]

In ref. [8], dealing with Diff(M), it was argued that in the most natural (favourable) situations the DFM renders manifest the *relational* character of general-relativistic physics: It does so by systematically implementing a notion of *relational observables* for Diff(M)-theories.[57] In the following, we unify the internal and Diff(M) versions of the DFM, treating them both at the same time by considering the extension of the method to the full automorphism group Aut(P) of a principal bundle.

4.1. Building Basic Forms via Dressing

Let $Q \to N$ be a reference fiber bundle with base N. The space of Aut(P)-dressing fields is defined as the set of bundle morphisms s.t.

$$
\mathcal{D}r[Q,P] := \left\{\, u : Q \to P \mid u^\psi := \psi^{-1}\circ u, \;\text{ with }\; \psi \in \mathrm{Aut}(P)\right\}
\subset \mathrm{Hom}(Q,P). \tag{129}
$$

Which means that dressing fields are special morphisms in the category of fiber bundles. The linearization of their defining property is then $\delta_X u := -X\circ u$ for $X \in \mathfrak{aut}(P)$. The dressing map is defined as:

$$
\begin{aligned}
|^u : \Phi &\to \Phi^u,\\
\phi &\mapsto \phi^u := u^*\phi.
\end{aligned} \tag{130}
$$

The object ϕ^u, called the *dressing* of ϕ, is Aut(P)-invariant: explicitly, $(u^*\phi)^\psi := (u^\psi)^*(\phi^\psi) = (\psi^{-1}\circ u)^*(\psi^*\phi) = u^*\phi$. The space Φ^u of dressed fields is a subset of fields living on Q.

We define *field-dependent* Aut(P)*-dressing fields* as

$$
\begin{aligned}
\boldsymbol{u} : \Phi &\to \mathcal{D}r[Q,P],\\
\phi &\mapsto \boldsymbol{u}(\phi), \quad \text{s.t.} \quad R^\star_\psi \boldsymbol{u} = \psi^{-1}\circ \boldsymbol{u} \quad \text{i.e.,} \quad \boldsymbol{u}(\phi^\psi) = \psi^{-1}\circ \boldsymbol{u}(\phi).
\end{aligned} \tag{131}
$$

Said otherwise, $\boldsymbol{u}$ is an equivariant 0-form on Φ with value in the representation $\mathcal{D}r[Q,P]$ of Aut(P). Its infinitesimal equivariance, i.e., $\mathfrak{aut}(P)$-transformation, is then $\boldsymbol{L}_{X^v}\boldsymbol{u} = -X\circ\boldsymbol{u}$. Therefore, its $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \mathrm{Aut}(P)\right)$ and $\boldsymbol{\mathfrak{diff}}_v(\Phi) \simeq C^\infty\left(\Phi, \mathfrak{aut}(P)\right)$ transformations are, respectively:

$$
\boldsymbol{u}^{\boldsymbol{\psi}} := \Xi^\star \boldsymbol{u} = \boldsymbol{\psi}^{-1}\circ\boldsymbol{u}, \qquad \text{and} \qquad \boldsymbol{L}_{\boldsymbol{X}^v}\boldsymbol{u} = -\boldsymbol{X}\circ\boldsymbol{u}. \tag{132}
$$

A ϕ-dependent dressing field induces a map

$$
\begin{aligned}
F_{\boldsymbol{u}} : \Phi &\to \mathcal{M},\\
\phi &\mapsto F_{\boldsymbol{u}}(\phi) := \boldsymbol{u}(\phi)^*\phi \sim [\phi], \quad \text{s.t.} \quad F_{\boldsymbol{u}}\circ R_\psi = F_{\boldsymbol{u}}.
\end{aligned} \tag{133}
$$

It is constant along Aut(P)-orbits: $F_{\boldsymbol{u}}(\phi^\psi) = \boldsymbol{u}(\phi^\psi)^*(\phi^\psi) = \left(\psi^{-1}\circ\boldsymbol{u}(\phi)\right)^*\psi^*\phi = \boldsymbol{u}(\phi)^*\phi =: F_{\boldsymbol{u}}(\phi)$. As Aut$(P)$-orbits $[\phi] \in \mathcal{M}$ of $\phi \in \Phi$ are represented by fields $\boldsymbol{u}(\phi)^*\phi$ on Q, not on P, the image of $F_{\boldsymbol{u}}$ can be understood as a "coordinatization" of $\mathcal{M}$. As an Aut(P)-invariant, the dressed field $\phi^{\boldsymbol{u}}$ represents physical d.o.f. in a manifestly *relational* way: the expression $\boldsymbol{u}(\phi)^*\phi$ is an explicit field-dependent coordinatization of the physical d.o.f., meaning that physical d.o.f. are defined w.r.t. to each other. Dressed fields $\phi^{\boldsymbol{u}}$ are relational Dirac variables, also called "complete observables" in refs. [57, 58]. Applying the DFM then amounts to reformulate a field theory in a manifestly Aut(P)-invariant and relational way.

The map (133) realizes the bundle projection, $F_{\boldsymbol{u}} \sim \pi$. It therefore allows to build basic forms on Φ, since $\Omega^\bullet_{\text{basic}}(\Phi) = \mathrm{Im}\,\pi^\star \simeq \mathrm{Im}\, F^\star_{\boldsymbol{u}}$. To build the basic counterpart of a form $\boldsymbol{\alpha} = \alpha\left(\wedge^\bullet \boldsymbol{d}\phi;\phi\right) \in \Omega^\bullet(\Phi)$, one must first consider its formal analogue on the base space $\bar{\boldsymbol{\alpha}} = \alpha\left(\wedge^\bullet \boldsymbol{d}[\phi];[\phi]\right) \in \Omega^\bullet(\mathcal{M})$, then define

$$
\boldsymbol{\alpha}^{\boldsymbol{u}} := F^\star_{\boldsymbol{u}}\bar{\boldsymbol{\alpha}} = \alpha\left(\wedge^\bullet F^\star_{\boldsymbol{u}}\boldsymbol{d}[\phi];F_{\boldsymbol{u}}(\phi)\right) \quad \in \Omega^\bullet_{\text{basic}}(\Phi). \tag{134}
$$

We call this object the *dressing* of $\boldsymbol{\alpha}$. It is basic by construction, so it is invariant under $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \mathrm{Aut}(P)\right)$ – and under $\boldsymbol{Aut}_v(\Phi) \simeq \boldsymbol{Aut}(P)$: $(\boldsymbol{\alpha}^{\boldsymbol{u}})^{\boldsymbol{\psi}} = \boldsymbol{\alpha}^{\boldsymbol{u}}$.

To get a final expression for $\boldsymbol{\alpha}^{\boldsymbol{u}}$, one needs $F^\star_{\boldsymbol{u}}\boldsymbol{d}[\phi]$, a basis for basic forms, expressed in terms of $\boldsymbol{d}\phi$ and $\boldsymbol{u}$. We may find it via the pushforward $F_{\boldsymbol{u}\star} : T_\phi\Phi \to T_{F_{\boldsymbol{u}}(\phi)}\mathcal{M}$, $\mathfrak{X}_{|\phi} \mapsto F_{\boldsymbol{u}\star}\mathfrak{X}_{|\phi}$. For a generic $\mathfrak{X} \in \Gamma(T\Phi)$ with flow $\varphi_\tau : \Phi \to \Phi$, s.t. $\mathfrak{X}_{|\phi} = \frac{d}{d\tau}\varphi_\tau(\phi)|_{\tau=0} = \mathfrak{X}(\phi)\frac{\delta}{\delta\phi}$, one has $F^\star_{\boldsymbol{u}}\boldsymbol{d}[\phi]_{|F_{\boldsymbol{u}}(\phi)}\left(\mathfrak{X}_{|\phi}\right) = \boldsymbol{d}[\phi]_{|F_{\boldsymbol{u}}(\phi)}\left(F_{\boldsymbol{u}\star}\mathfrak{X}_{|\phi}\right)$. So,

$$F_{u\star}\mathfrak{X}_{|\phi} := F_{u\star}\tfrac{d}{d\tau}\,\varphi_\tau(\phi)\Big|_{\tau=0} = \tfrac{d}{d\tau}\,F_u\big(\varphi_\tau(\phi)\big)\Big|_{\tau=0}$$
$$= \tfrac{d}{d\tau}\,u\big(\varphi_\tau(\phi)\big)^*\big(\varphi_\tau(\phi)\big)\Big|_{\tau=0}$$
$$= \tfrac{d}{d\tau}u\big(\varphi_\tau(\phi)\big)^*\phi\Big|_{\tau=0} + \tfrac{d}{d\tau}\,u(\phi)^*\big(\varphi_\tau(\phi)\big)\Big|_{\tau=0}.$$

Inserting $\mathrm{id}_Q = u(\phi)^{-1}\circ u(\phi)$ in the first term: $\tfrac{d}{d\tau}u(\phi)^* u(\phi)^{-1*}u(\varphi_\tau(\phi))^*\phi|_{\tau=0} = u(\phi)^*\tfrac{d}{d\tau}(u(\varphi_\tau(\phi))\circ u(\phi)^{-1})^*\phi|_{\tau=0}$. The term $u(\varphi_\tau(\phi))\circ u(\phi)^{-1}$ is a curve in P, so $\tfrac{d}{d\tau}u(\varphi_\tau(\phi))\circ u(\phi)^{-1}|_{\tau=0} = du_{|\phi}(\mathfrak{X}_{|\phi})\circ u(\phi)^{-1}\in\Gamma(TP)$. Therefore,

$$\tfrac{d}{d\tau}u\big(\varphi_\tau(\phi)\big)^*\phi\Big|_{\tau=0} = u(\phi)^*\tfrac{d}{d\tau}\,\big(u\big(\varphi_\tau(\phi)\big)\circ u(\phi)^{-1}\big)^*\,\phi\Big|_{\tau=0}$$
$$= u(\phi)^*\mathfrak{L}_{du_{|\phi}(\mathfrak{X}_{|\phi})\circ u(\phi)^{-1}}\phi. \tag{135}$$

The second term is $\tfrac{d}{d\tau}u(\phi)^*(\varphi_\tau(\phi))|_{\tau=0} =: \tfrac{d}{d\tau}F_{u(\phi)}(\varphi_\tau(\phi))|_{\tau=0} = F_{u(\phi)\star}\mathfrak{X}_{|\phi}$. As a vector on $\mathcal{M}$, we find its expression as a derivation by applying it to $g\in C^\infty(\mathcal{M})$:

$$\big[F_{u(\phi)\star}\mathfrak{X}\,(g)\big]([\phi]) = \tfrac{d}{d\tau}\,g\big(F_{u(\phi)}\big(\varphi_\tau(\phi)\big)\big)\Big|_{\tau=0}$$
$$= \Big(\tfrac{\delta}{\delta[\phi]}g\Big)\big(F_{u(\phi)}(\phi)\big)\;\underbrace{\tfrac{d}{d\tau}\,F_{u(\phi)}\big(\varphi_\tau(\phi)\big)\Big|_{\tau=0}}_{[\mathfrak{X}(F_{u(\phi)})](\phi)}$$
$$= \Big(\tfrac{\delta}{\delta[\phi]}g\Big)\left(\underbrace{u(\phi)^*\phi}_{\sim[\phi]}\right)\;\underbrace{\Big(\tfrac{\delta}{\delta\phi}F_{u(\phi)}\Big)(\phi)}_{\tfrac{\delta}{\delta\phi}\,u(\phi)^*\phi = u(\phi)^*}\;\mathfrak{X}(\phi)$$
$$= \Big(\tfrac{\delta}{\delta[\phi]}g\Big)([\phi])\;u(\phi)^*\mathfrak{X}(\phi)$$
$$= \Big[u(\phi)^*\mathfrak{X}(\phi)\tfrac{\delta}{\delta[\phi]}(g)\Big]([\phi]).$$

We have finally

$$F_{u\star}\mathfrak{X}_{|\phi} = u(\phi)^*\Big(\mathfrak{X}(\phi) + \mathfrak{L}_{du_{|\phi}(\mathfrak{X}_{|\phi})\circ u(\phi)^{-1}}\phi\Big)\,\frac{\delta}{\delta[\phi]}_{\,|F_{u(\phi)}(\phi)}. \tag{136}$$

From which we find, for any $\mathfrak{X}\in\Gamma(T\Phi)$:

$$F_u^\star d[\phi]_{|F_{u(\phi)(\phi)}}\big(\mathfrak{X}_{|\phi}\big) = d[\phi]_{|F_{u(\phi)(\phi)}}\big(F_{u\star}\mathfrak{X}_{|\phi}\big)$$
$$= d[\phi]_{|F_{u(\phi)(\phi)}}$$
$$\times\left(u(\phi)^*\Big(\mathfrak{X}(\phi) + \mathfrak{L}_{du_{|\phi}(\mathfrak{X}_{|\phi})\circ u(\phi)^{-1}}\phi\Big)\tfrac{\delta}{\delta[\phi]}_{\,|F_{u(\phi)(\phi)}}\right)$$
$$= u(\phi)^*\Big(\mathfrak{X}(\phi) + \mathfrak{L}_{du_{|\phi}(\mathfrak{X}_{|\phi})\circ u(\phi)^{-1}}\phi\Big)$$
$$= u(\phi)^*\Big(d\phi_{|\phi}\,\big(\mathfrak{X}_{|\phi}\big) + \mathfrak{L}_{du_{|\phi}(\mathfrak{X}_{|\phi})\circ u(\phi)^{-1}}\phi\Big)$$
$$= \Big(u(\phi)^*\Big(d\phi_{|\phi} + \mathfrak{L}_{du_{|\phi}\circ u(\phi)^{-1}}\phi\Big)\Big)(\mathfrak{X}_{|\phi}).$$

We get the dressing of $d\phi$, the basic 1-form basis:

$$d\phi^u := F_u^\star d[\phi] = u^*\big(d\phi + \mathfrak{L}_{du\circ u^{-1}}\phi\big)\quad\in\ \Omega^1_{\text{basic}}(\Phi). \tag{137}$$

Inserted into (134), this yields the dressing of α:

$$\alpha^u = \alpha\,(\wedge^\bullet d\phi^u;\ \phi^u)\ \in\ \Omega^\bullet_{\text{basic}}(\Phi). \tag{138}$$

On account of the formal similarity between $\Xi(\phi) = \psi(\phi)^*\phi$ and $F_u(\phi) = u(\phi)^*\phi$, and $d\phi^\psi$ (71) and $d\phi^u$ (137), resulting into the close formal expressions of α^ψ (47) and α^u (134)–(138), the following rule of thumb to obtain the dressing of any form α holds: First compute the $\textbf{\textit{Diff}}_v(\Phi)\simeq C^\infty\big(\Phi,\mathrm{Aut}(P)\big)$ transformation α^ψ, then substitute $\psi\to u$ in the resulting expression to obtain α^u. This rule may be used be systematically.

For example, for $\alpha\in\Omega^\bullet_{\text{tens}}(\Phi,\rho)$ the rule gives us $\alpha^u = \rho(u)^{-1}\alpha$. For an Ehresmann connection, in view of (77), the rule ensures that $\omega^u = u_*^{-1}\omega\circ u + u_*^{-1}du\in\Omega^1_{\text{basic}}(\Phi)$. These two results allows to write a lemma analogous to (80)–(81): For $\alpha, D\alpha\in\Omega^\bullet_{\text{tens}}(\Phi,\rho)$, we have $d\,F_u^\star\bar\alpha = d\,\big(\rho(u)^{-1}\alpha\big)$. On the other hand, since $F_u^\star\bar D\bar\alpha = d\alpha^u + \rho_*(\omega^u)\alpha^u = \rho(u)^{-1}D\alpha$, we have

$$F_u^\star d\bar\alpha = \rho(u)^{-1}D\alpha - \rho_*(\omega^u)\alpha^u$$
$$= \rho(u)^{-1}d\alpha + \rho(u)^{-1}\,\rho_*(\omega)\alpha - \rho_*(\omega^u)\alpha^u$$
$$= \rho(u)^{-1}d\alpha - \rho_*(u_*^{-1}du)\rho(u)^{-1}\alpha$$
$$= \rho(u)^{-1}\big(d\alpha - \rho_*(du\circ u^{-1})\alpha\big).$$

By $[F_u^\star, d] = 0$ (remark that the exterior derivatives belong to different spaces) we obtain:

$$d\big(\rho(u)^{-1}\alpha\big) = \rho(u)^{-1}\big(d\alpha - \rho_*(du\circ u^{-1})\,\alpha\big). \tag{139}$$

For the pullback representation $\rho(u)^{-1} = u^*$, this specializes to

$$d(u^*\alpha) = u^*\big(d\alpha + \mathfrak{L}_{du\circ u^{-1}}\alpha\big). \tag{140}$$

This identity, generalizing the one appearing e.g., in refs. [35, 36, 51], must not be conflated with (137). Despite their formal similarity, these two results have distinct geometric origin.

4.1.1. Dressing Field and Flat Connections

- A field-dependent dressing field u, induces a *flat Ehresmann connection* $\omega_0 := -du\circ u^{-1}\in\Omega^1(\Phi)$. By (132) we have

$$\omega_{0|\phi}(X^v_{|\phi}) = -du_{|\phi}(X^v_{|\phi})\circ u(\phi)^{-1} = -L_{X^v}u\circ u(\phi)^{-1}$$
$$= X\circ u(\phi)\circ u(\phi)^{-1} = X\in\mathfrak{aut}(P), \tag{141}$$

and by (131), using the naturality of d, we get

$$R^\star_\psi\omega_0 = -R^\star_\psi du\circ(R^\star_\psi u)^{-1} = -d(R^\star_\psi u)\circ(R^\star_\psi u)^{-1}$$
$$= -d(\psi^{-1}\circ u)\circ(\psi^{-1}\circ u)^{-1} = \psi_*^{-1}du\circ u^{-1}\circ\psi \tag{142}$$
$$= \psi_*^{-1}\,\omega_0\circ\psi.$$

These are indeed the defining properties (73) of an Ehresmann connection, and $\Omega_0 = \boldsymbol{d}\boldsymbol{\omega}_0 + \tfrac{1}{2}[\boldsymbol{\omega}_0, \boldsymbol{\omega}_0]_{\mathfrak{aut}(P)} \equiv 0$ is immediate. It follows that, as a special case of (77) and (78),

$$\boldsymbol{\omega}_0^{\boldsymbol{\psi}} = \boldsymbol{\psi}_*^{-1}\,\boldsymbol{\omega}_0 \circ \boldsymbol{\psi} + \boldsymbol{\psi}_*^{-1}\boldsymbol{d\psi}, \quad \text{and} \quad L_{\boldsymbol{X}^v}\boldsymbol{\omega}_0 = \boldsymbol{dX} + [\boldsymbol{\omega}_0, \boldsymbol{X}]_{\mathfrak{aut}(P)}. \tag{143}$$

Thus, (137) can also be written as $\boldsymbol{d}\phi^{\boldsymbol{u}} = \boldsymbol{u}^*\left(\boldsymbol{d}\phi - \mathfrak{L}_{\boldsymbol{\omega}_0}\phi\right) \in \Omega^1_{\text{basic}}(\Phi)$.

The existence of a flat connection is a strong topological constraint on a bundle, then so is that of a global dressing field. A global dressing field may indeed give a global trivialization of the bundle Φ. However the field space may not be trivial in general, and Gribov-like obstructions may exclude the existence of global dressing fields. Local dressing fields are always compatible with the local triviality of Φ.

- A dressing field may also induce a *flat twisted connection* $\boldsymbol{\varpi}_0 := -\boldsymbol{d}C(\boldsymbol{u}) \cdot C(\boldsymbol{u})^{-1}$. To see this, one only has to assume that the 1-cocycle (50) $C : \Phi \times \mathrm{Aut}(P) \to G$ controlling the equivariance (51) of twisted forms has a functional expression that can be extended to $C : \Phi \times \mathrm{Hom}(Q, P) \to G$. Then, we have the well-defined map

$$\begin{aligned} &C(\boldsymbol{u}) : \Phi \to G, \\ &\qquad \phi \mapsto [C(\boldsymbol{u})](\phi) := C\big(\phi; \boldsymbol{u}(\phi)\big). \end{aligned} \tag{144}$$

By the cocycle property (50), it is a twisted equivariant 0-form:[15]

$$\begin{aligned} \left[R^\star_{\boldsymbol{\psi}} C(\boldsymbol{u})\right](\phi) &= \left[C(\boldsymbol{u}) \circ R_{\boldsymbol{\psi}}\right](\phi) := C\big(\phi^{\boldsymbol{\psi}}; \boldsymbol{u}(\phi^{\boldsymbol{\psi}})\big) \\ &= C\big(\phi^{\boldsymbol{\psi}}; \boldsymbol{\psi}^{-1} \circ \boldsymbol{u}(\phi)\big) \\ &= C\big(\phi^{\boldsymbol{\psi}}; \boldsymbol{\psi}^{-1}\big) \cdot C\big(\phi; \boldsymbol{u}(\phi)\big) \\ &= C(\phi; \boldsymbol{\psi})^{-1} \cdot C\big(\phi; \boldsymbol{u}(\phi)\big) \\ &=: \left[C(\,;\boldsymbol{\psi})^{-1} \cdot C(\boldsymbol{u})\right](\phi). \end{aligned} \tag{145}$$

Its infinitesimal equivariance is $L_{\boldsymbol{X}^v} C(\boldsymbol{u}) = \iota_{\boldsymbol{X}^v}\boldsymbol{d}C(\boldsymbol{u}) = -a(X;\,) \cdot C(\boldsymbol{u})$. From these, one easily checks that $\boldsymbol{\varpi}_0$ satisfies the defining properties (84) of a twisted connection. As a special case of (89)–(90), we have

$$\begin{aligned} &\boldsymbol{\varpi}_0^{\boldsymbol{\psi}} = \mathrm{Ad}\big(C(\boldsymbol{\psi})^{-1}\big)\,\boldsymbol{\varpi}_0 + C(\boldsymbol{\psi})^{-1}\boldsymbol{d}C(\boldsymbol{\psi}), \quad \text{and} \\ &L_{\boldsymbol{X}^v}\boldsymbol{\varpi}_0 = \boldsymbol{d}a(\boldsymbol{X}) + [\boldsymbol{\varpi}_0, a(\boldsymbol{X})]_{\mathfrak{g}}. \end{aligned} \tag{146}$$

As we will see in the next section, twisted dressings $C(\boldsymbol{u})$ and associated $\boldsymbol{\varpi}_0$ underly the construction of WZ counterterms in field theory.

4.2. Composition of Dressing Operations

As stated in introduction, the DFM has been first developed for "internal symmetry groups" – for either Yang-Mills type theory, or for gauge theories of gravity based on Cartan geometry – i.e., for $\mathrm{Aut}_v(P)$. It has been developed for general-relativistic theories, i.e., for $\mathrm{Diff}(M)$, in ref. [8]. To indicate how to recover both cases from the above construction, we may rely on the decomposition (A8) in Appendix A of bundle morphisms: an $\mathrm{Aut}(P)$-dressing can be written as the composition $\boldsymbol{u} = f_{\mathbf{u}} \circ \bar{\boldsymbol{\upsilon}}$, as in the following diagram

$$\begin{tikzcd} Q \arrow[r, "\bar{\upsilon}"'] \arrow[rr, bend left, "\boldsymbol{u}"] \arrow[d] & P \arrow[r, "f_{\mathbf{u}}"'] \arrow[d] & P \arrow[d] \\ N \arrow[r, "\upsilon"'] & M \arrow[r, equal] & M \end{tikzcd} \tag{147}$$

where $\bar{\boldsymbol{\upsilon}} : \Phi \to \mathcal{D}r[Q, P]$ is covering $\boldsymbol{\upsilon} : \Phi \to \mathcal{D}r[N, M]$, $\phi \mapsto (\boldsymbol{\upsilon}[\phi] : N \to M)$ which is a $\mathrm{Diff}(M)$-dressing as defined in ref. [8]. The map $f_{\mathbf{u}} : \Phi \to \mathcal{D}r[P, P]$, $\phi \mapsto \left(f_{\mathbf{u}[\phi]} : P \to P\right)$, covers id_M. The latter is thus, omitting the ϕ-dependence for notational simplicity, of the form $f_{\mathbf{u}}(p) := p\mathbf{u}(p)$, with $\mathbf{u} : P \to H$. By definition of an $\mathrm{Aut}_v(P)$-dressing field, it satisfies $f_{\mathbf{u}}^{\psi} = \psi^{-1} \circ f_{\mathbf{u}}$, for $\psi \in \mathrm{Aut}_v(P)$, so that

$$f_{\mathbf{u}}^{\psi}(p) := \psi^{-1} \circ f_{\mathbf{u}}(p) = \psi^{-1}\big(p\mathbf{u}(p)\big) = \psi^{-1}(p)\mathbf{u}(p) = p\gamma(p)^{-1}\mathbf{u}(p), \tag{148}$$

where $\gamma^{-1} \in \mathcal{H}$ is the gauge group element corresponding to $\psi^{-1} \in \mathrm{Aut}_v(P)$. One has thus that $f_{\mathbf{u}}^{\psi} = f_{\mathbf{u}^{\gamma}} = f_{\gamma^{-1}\mathbf{u}}$, i.e., one finds that the generating element $\mathbf{u}$ of the $\mathrm{Aut}_v(P)$-dressing $f_{\mathbf{u}}$ is

$$\mathbf{u} : P \to H, \quad \text{s.t.} \quad \mathbf{u}^{\gamma} = \gamma^{-1}\mathbf{u}. \tag{149}$$

This is the bundle version of the original field-theoretic definition of an $\mathcal{H}$-dressing field as given e.g., in refs. [4, 41].[16]

The decomposition $\boldsymbol{u} = f_{\mathbf{u}} \circ \bar{\boldsymbol{\upsilon}}$ is suggested by the semi-direct product structure of $\mathrm{Aut}(P) = \overline{\mathrm{Diff}} \ltimes \mathrm{Aut}_v(P)$, discussed in Appendix A. As illustrated by the diagram (147), it shows that one may perform dressings in a stepwise manner: first dressing for $\mathrm{Aut}_v(P)$ via $f_{\mathbf{u}}$ and then for $\overline{\mathrm{Diff}}$ via $\bar{\boldsymbol{\upsilon}}$. This works if, again as suggested by the semi-direct structure of $\mathrm{Aut}(P)$, the two dressing maps $f_{\mathbf{u}}$ and $\bar{\boldsymbol{\upsilon}}$ satisfy, in addition to their defining relations (left column below), the compatibility conditions (right column) under $\psi = (\bar{\psi}, \eta) \in \mathrm{Aut}(P) = \overline{\mathrm{Diff}} \ltimes \mathrm{Aut}_v(P)$:

$$f_{\mathbf{u}}^{\eta} := \eta^{-1} \circ f_{\mathbf{u}}, \tag{150}$$

$$\bar{\boldsymbol{\upsilon}}^{\bar{\psi}} := \bar{\psi}^{-1} \circ \bar{\boldsymbol{\upsilon}}, \tag{151}$$

$$f_{\mathbf{u}}^{\bar{\psi}} := \bar{\psi}^{-1} \circ f_{\mathbf{u}} \circ \bar{\psi}, \tag{152}$$

$$\bar{\boldsymbol{\upsilon}}^{\eta} := \bar{\boldsymbol{\upsilon}}. \tag{153}$$

[15] Remark that it is then a dressing field for twisted forms.

[16] One may define a dressing for a subgroup of $\mathrm{Aut}_v(P)$, leading to a partial symmetry reduction, to which corresponds a $\mathcal{K}$-dressing field $\mathbf{u} : P \to K \subset H$ s.t. $\mathbf{u}^{\kappa} = \kappa^{-1}\mathbf{u}$, for $\kappa \in \mathcal{K} \subset \mathcal{H}$. See e.g., refs. [6, 7] for details.

The condition (152) ensures that the first dressed fields $f_u^*\phi$ retains well-defined $\overline{\text{Diff}}$-transformations, that can be dressed for via (151). While condition (153) ensures that upon dressing by $\bar{\boldsymbol{v}}$ the $\text{Aut}_v(P)$-invariance is preserved. From these, one checks explicitly the invariance of the dressed fields $\phi^{\boldsymbol{u}} := \boldsymbol{u}^*\phi = \bar{\boldsymbol{v}}^* f_u^*\phi$:

$$
\begin{aligned}
(\phi^{\boldsymbol{u}})^{\eta} &= \left(\bar{\boldsymbol{v}}^* f_u^*\phi\right)^{\eta} = (\bar{\boldsymbol{v}}^{\eta}) * (f_u^*\phi)^{\eta} = \bar{\boldsymbol{v}} * f_u^*\phi = \phi^{\boldsymbol{u}},\\
(\phi^{\boldsymbol{u}})^{\bar{\psi}} &= (\bar{\boldsymbol{v}}^* f_u^*\phi)^{\bar{\psi}} = (\bar{\boldsymbol{v}}^{\bar{\psi}}) * (f_u^{\bar{\psi}}) * \phi^{\bar{\psi}}\\
&= (\bar{\psi}^{-1}\circ\bar{\boldsymbol{v}}) * (\bar{\psi}^{-1}\circ f_u\circ\bar{\psi}) * \bar{\psi}^*\phi\\
&= \bar{\boldsymbol{v}}^*(\bar{\psi}^{-1}) * \bar{\psi}^* f_u^*(\bar{\psi}^{-1}) * \bar{\psi}^*\phi = \phi^{\boldsymbol{u}},
\end{aligned}
\tag{154}
$$

by (150) and (153) in the first line, and by (151) and (152) in the second line. From this immediately follows that $[(\phi^{\boldsymbol{u}})^{\eta}]^{\bar{\psi}} = [(\phi^{\boldsymbol{u}})^{\bar{\psi}}]^{\eta}$. The consecutive dressing operations can be summarized via the following sequence:

$$
\begin{aligned}
&\left(\text{Aut}(P) = \overline{\text{Diff}} \ltimes \text{Aut}_v(P); \phi\right) \xrightarrow{f_u} \left(\overline{\text{Diff}}; f_u^*\phi\right)\\
&\xrightarrow{\bar{v}} \left(\emptyset; \bar{\boldsymbol{v}}^* f_u^*\phi = (f_u\circ\bar{\boldsymbol{v}})^*\phi =: \mathbf{u}^*\phi = \phi^{\mathbf{u}}\right).
\end{aligned}
\tag{155}
$$

A similar pattern of stepwise dressings may also appear for $\text{Aut}_v(P)$ if it has a semi-direct structure. Then dressing compatibility conditions such as (152)–(153) must hold, for the same reasons. Such is notably the case in conformal Cartan geometry, where the DFM can be used to build conformal tractors and twistors,[39,40] and applied to conformal gravity.[59] One may then specialize the results of Section 4.1 to the distinct cases of $\text{Aut}_v(P)$ and $\overline{\text{Diff}}$ dressings. As we shall see in Section 5, this is especially relevant when dealing with (local) field theory, where the semi-direct structure of the local version of $\text{Aut}(P)$ is unavoidable.

4.3. Residual Symmetries

We ought to discuss two kinds of *residual* symmetries in the DFM. First, the genuine residual symmetry coming from the elimination of a subgroup of the original symmetry group. Secondly, a new symmetry replacing the eliminated one, parametrising the choice of dressing fields. Both may be present simultaneously, yet in the following we discuss them in turn for the sake of clarity.

4.3.1. Residual Symmetries of the First Kind

Consider a dressing field $\boldsymbol{u}:\Phi \to \mathcal{D}r[Q,P]$ whose defining equivariance is controlled by a subgroup $\text{Aut}_0(P)$:

$$
R^\star_{\varphi}\boldsymbol{u} = \varphi^{-1}\circ\boldsymbol{u}, \quad \text{for } \varphi\in\text{Aut}_0(P)\subset\text{Aut}(P). \tag{156}
$$

Applying the DFM is then equivalent to building a bundle projection $\Phi \to \Phi/\text{Aut}_0(P) =: \Phi^{\boldsymbol{u}}$. For $\Phi^{\boldsymbol{u}}$ to be principal (sub)bundle in its own right, the quotient $\text{Aut}(P)/\text{Aut}_0(P)$ must be a group, so $\text{Aut}_0(P)$ has to be a *normal* subgroup of $\text{Aut}(P)$: $\text{Aut}_0(P) \triangleleft \text{Aut}(P)$. We assume that it is and denote $\text{Aut}_r(P) := \text{Aut}(P)/\text{Aut}_0(P)$ the residual structure group of $\Phi^{\boldsymbol{u}}$.

For instance, one may consider the subgroup of automorphisms $\text{Aut}_0(P) = \text{Aut}_c(P)$ whose domains $P_{|D}\subset P$ have π-projections $D\subset M$ that are compact subsets of M.[17] Another noteworthy example, as seen in the previous section, is the subgroup of vertical automorphisms $\text{Aut}_0(P) = \text{Aut}_v(P)$, in which case $\text{Aut}_r(P) = \overline{\text{Diff}} \simeq \text{Diff}(M)$. These are relevant examples for local field theory applications, see Section 5.

Dressed objects $\boldsymbol{\alpha}^{\boldsymbol{u}}$ are then $\text{Aut}_0(P)$-basic on Φ, i.e., by construction invariant under $C^\infty\left(\Phi, \text{Aut}_0(P)\right) \subset C^\infty\left(\Phi, \text{Aut}(P)\right) \simeq \boldsymbol{Diff}_v(\Phi)$, so in particular invariant under $\boldsymbol{Aut}_0(P)\subset \boldsymbol{Aut}(P)$. One expects them to exhibit *residual transformations* under $C^\infty\left(\Phi, \text{Aut}_r(P)\right) \subset C^\infty\left(\Phi, \text{Aut}(P)\right)$. The transformation of $\boldsymbol{\alpha}$ under $C^\infty\left(\Phi, \text{Aut}_r(P)\right)$ is known, so determining that of $\boldsymbol{\alpha}^{\boldsymbol{u}}$ boils down to finding the residual transformation of the dressing field $\boldsymbol{u}$ under $C^\infty\left(\Phi, \text{Aut}_r(P)\right)$, which is given by its equivariance $R^\star_{\psi}\boldsymbol{u}$ for $\psi\in\text{Aut}_r(P)$.

We illustrate this in the following two propositions dealing with two interesting cases that can be systematically treated. Let us consider $\boldsymbol{\alpha}\in\Omega^\bullet_{\text{tens}}(\Phi,\rho)$ and $\boldsymbol{\omega}\in\mathcal{C}$, whose dressing by $\boldsymbol{u}$ as in (156) above are $\boldsymbol{\alpha}^{\boldsymbol{u}}$ and $\boldsymbol{\omega}^{\boldsymbol{u}}$, both $C^\infty\left(\Phi, \text{Aut}_0(P)\right)$-invariant.

Proposition 1. *If $\boldsymbol{u}:\Phi \to \mathcal{D}r[Q,P]$, with $Q\subseteq P$, is s.t.*

$$
R^\star_{\psi}\boldsymbol{u} = \psi^{-1}\circ\boldsymbol{u}\circ\psi, \quad \textit{for } \psi\in\text{Aut}_r(P), \tag{157}
$$

then $\boldsymbol{\alpha}^{\boldsymbol{u}}\in\Omega_{tens}(\Phi,\rho)$ and $\boldsymbol{\omega}^{\boldsymbol{u}}\in\mathcal{C}$. Then, their residual $C^\infty(\Phi;\text{Aut}_r(P))$-transformations are

$$
(\boldsymbol{\alpha}^{\boldsymbol{u}})^{\boldsymbol{\psi}} = \rho(\boldsymbol{\psi}^{-1})\boldsymbol{\alpha}^{\boldsymbol{u}}, \quad \textit{and} \quad (\boldsymbol{\omega}^{\boldsymbol{u}})^{\boldsymbol{\psi}} = \boldsymbol{\psi}^{-1}_{\ *}\,\boldsymbol{\omega}^{\boldsymbol{u}}\circ\boldsymbol{\psi} + \boldsymbol{\psi}^{-1}_{\ *}\boldsymbol{d\psi}. \tag{158}
$$

Indeed, when $\boldsymbol{\alpha}$ is horizontal so is $\boldsymbol{\alpha}^{\boldsymbol{u}} = \rho(\boldsymbol{u})^{-1}\boldsymbol{\alpha}$, and $R^\star_{\psi}\boldsymbol{\alpha}^{\boldsymbol{u}} = \rho(R^\star_{\psi}\boldsymbol{u})^{-1}R^\star_{\psi}\boldsymbol{\alpha} = \rho(\psi^{-1}\circ\boldsymbol{u}\circ\psi)^{-1}\rho(\psi)^{-1}\boldsymbol{\alpha} = \rho(\psi^{-1})\boldsymbol{\alpha}^{\boldsymbol{u}}$. Also, given the linearization of (157),

$$
\begin{aligned}
&\boldsymbol{L}_{X^v}\boldsymbol{u} = \iota_{X^v}\boldsymbol{du} = \tfrac{d}{d\tau}\, R^\star_{\psi_\tau}\boldsymbol{u}\Big|_{\tau=0} = -X\circ\boldsymbol{u} + \boldsymbol{u}_*X, \quad \text{for}\\
&X := \tfrac{d}{d\tau}\psi_\tau\Big|_{\tau=0}\in\mathfrak{aut}_r(P),
\end{aligned}
\tag{159}
$$

one finds that $\boldsymbol{\omega}^{\boldsymbol{u}}(X^v) = \boldsymbol{u}^{-1}_*\boldsymbol{\omega}(X^v)\circ\boldsymbol{u} + \boldsymbol{u}^{-1}_*\boldsymbol{du}(X^v) = \boldsymbol{u}^{-1}_*X\circ\boldsymbol{u} + \boldsymbol{u}^{-1}_*\left(-X\circ\boldsymbol{u} + \boldsymbol{u}_*X\right) = X$. Then, by (157) and (73), we have $R^\star_{\psi}\boldsymbol{\omega}^{\boldsymbol{u}} = \psi^{-1}_{\ *}\,\boldsymbol{\omega}^{\boldsymbol{u}}\circ\psi$, for $\psi\in\text{Aut}_r(P)$. The dressed connection $\boldsymbol{\omega}^{\boldsymbol{u}}$ satisfies the defining properties (73) of an $\text{Aut}_r(P)$-principal connection. Thus, (158) follows from (62) and (77). Remark that $\boldsymbol{\omega}^{\boldsymbol{u}}$, being $\mathfrak{aut}(P)$-valued, splits as an $\mathfrak{aut}_r(P)$-connection and an $\mathfrak{aut}_0(P)$-valued tensorial 1-form. In particular, from either results in (158), one derives that $\boldsymbol{\Omega}^{\boldsymbol{u}}\in\Omega^2_{\text{tens}}(\Phi,\mathfrak{aut}(P))$, so $(\boldsymbol{\Omega}^{\boldsymbol{u}})^{\boldsymbol{\psi}} = \boldsymbol{\psi}^{-1}_{\ *}\,\boldsymbol{\Omega}^{\boldsymbol{u}}\circ\boldsymbol{\psi}$.

Proposition 2. *If $\boldsymbol{u}:\Phi\to\mathcal{D}r[Q,P]$ is s.t.*

$$
R^\star_{\psi}\boldsymbol{u} = \psi^{-1}\circ\boldsymbol{u}\circ C(\,;\psi), \quad \textit{for } \psi\in\text{Aut}_r(P), \tag{160}
$$

where $C:\Phi\times\text{Aut}_r(P)\to\text{Aut}(Q)$ is a special case of 1-cocycle (50), satisfying $C(\phi;\psi'\circ\psi) = C(\phi;\psi')\circ C(\phi^{\psi'};\psi)$, then

17 In case the structure group H of P is compact, their domains are compact and these are indeed compactly supported automorphims.

$\boldsymbol{\alpha}^u \in \Omega^\bullet_{tens}(\Phi, C)$ and $\boldsymbol{\omega}^u \in \bar{\mathcal{C}}$. Their residual $C^\infty\left(\Phi, \mathrm{Aut}_r(P)\right)$-transformations are then

$$
\begin{aligned}
(\boldsymbol{\alpha}^u)^{\boldsymbol{\psi}} &= \rho\big(C(\boldsymbol{\psi})\big)^{-1}\boldsymbol{\alpha}^u, \quad \textit{and}\\
(\boldsymbol{\omega}^u)^{\boldsymbol{\psi}} &= C(\boldsymbol{\psi})_*^{-1}\boldsymbol{\omega}^u \circ C(\boldsymbol{\psi}) + C(\boldsymbol{\psi})_*^{-1}\boldsymbol{d}C(\boldsymbol{\psi}).
\end{aligned} \tag{161}
$$

As above, if $\boldsymbol{\alpha}$ is horizontal, so is $\boldsymbol{\alpha}^u = \rho(\boldsymbol{u})^{-1}\boldsymbol{\alpha}$, and $R^\star_\psi \boldsymbol{\alpha}^u = \rho(R^\star_\psi \boldsymbol{u})^{-1} R^\star_\psi \boldsymbol{\alpha} = \rho(\psi^{-1}\circ \boldsymbol{u} \circ C(\ \ ;\psi))^{-1}\rho(\psi)^{-1}\boldsymbol{\alpha} = \rho(C(\ \ ;\psi))^{-1}\boldsymbol{\alpha}^u$. For ρ the pullback action, $\rho(\psi)^{-1} = \psi^*$, we have $R^\star_\psi \boldsymbol{\alpha}^u = C(\,;\psi)^*\boldsymbol{\alpha}^u$ and $(\boldsymbol{\alpha}^u)^{\boldsymbol{\psi}} = C(\boldsymbol{\psi}))^*\boldsymbol{\alpha}^u$. The linear version of (160) reads

$$
\begin{aligned}
\boldsymbol{L}_{X^v}\boldsymbol{u} = \iota_{X^v}\boldsymbol{d}\boldsymbol{u} = \tfrac{d}{d\tau} R^\star_{\psi_\tau}\boldsymbol{u}\Big|_{\tau=0} &= -X\circ\boldsymbol{u} + \boldsymbol{u}_*\tfrac{d}{d\tau}C(\,;\psi_\tau)\Big|_{\tau=0}\\
&= -X\circ\boldsymbol{u} + \boldsymbol{u}_*\, a(X;\ \ ), \qquad \text{for } X\in \mathfrak{aut}_{\mathrm{r}}(P).
\end{aligned} \tag{162}
$$

So one has $\boldsymbol{\omega}^u(X^v) = \boldsymbol{u}_*^{-1}\boldsymbol{\omega}(X^v)\circ\boldsymbol{u} + \boldsymbol{u}_*^{-1}\boldsymbol{d}\boldsymbol{u}(X^v) = \boldsymbol{u}_*^{-1}X\circ\boldsymbol{u} + \boldsymbol{u}_*^{-1}\left(-X\circ\boldsymbol{u} + \boldsymbol{u}_*\,a(X;\ \ )\right) = a(X;\ \ )\ \in \mathfrak{aut}(Q)$. Also, given (160) and (73), it is easily shown that: $R^\star_\psi\boldsymbol{\omega}^u = C(\ \ ;\psi)_*^{-1}\,\boldsymbol{\omega}^u\circ C(\ \ ;\psi) + C(\ \ ;\psi)_*^{-1}\boldsymbol{d}C(\ \ ;\psi)$, for $\psi\in\mathrm{Aut}_{\mathrm{r}}(P)$. The dressed connection $\boldsymbol{\omega}^u$ satisfies the defining properties (84)–(85) of a $\mathrm{Aut}_{\mathrm{r}}(P)$-twisted connection. The transformations (161) follow from (62) and (89). From either follows that $\boldsymbol{\Omega}^u \in \Omega^2_{\text{tens}}(\Phi, C)$, so $(\boldsymbol{\Omega}^u)^{\boldsymbol{\psi}} = C(\boldsymbol{\psi})_*^{-1}\,\boldsymbol{\Omega}^u\circ C(\boldsymbol{\psi})$.

4.3.2. Residual Symmetries of the Second Kind

The defining equivariance $R^\star_\psi\boldsymbol{u} = \psi^{-1}\circ\boldsymbol{u}$ of a dressing field $\boldsymbol{u}: \Phi \rightarrow \mathcal{D}r[Q,P]$ means that, as a bundle map $\boldsymbol{u}(\phi): Q\rightarrow P$, $\mathrm{Aut}(P)$ acts on its target space. But then a priori there is a natural right action of $\mathrm{Aut}(Q)$ on its source space, which we write:

$$
\begin{aligned}
&\mathcal{D}r[Q,P]\times\mathrm{Aut}(Q) \rightarrow \mathcal{D}r[Q,P]\\
&\qquad\qquad (\boldsymbol{u},\varphi) \mapsto \varphi^*\boldsymbol{u} = \boldsymbol{u}\circ\varphi =: \boldsymbol{u}^\varphi.
\end{aligned} \tag{163}
$$

One way to interpret this, is that two candidates dressing fields $\boldsymbol{u}'$ and $\boldsymbol{u}$ may a priori be related by an element $\varphi\in\mathrm{Aut}(Q)$: $\boldsymbol{u}' = \boldsymbol{u}\circ\varphi = \boldsymbol{u}^\varphi$. The group $\mathrm{Aut}(Q)$ does not act on Φ, so we write $\phi^\varphi = \phi$, but it acts on $\phi^u = F_{\boldsymbol{u}}(\phi) := \boldsymbol{u}^*\phi$ as a field on Q:

$$
\phi^u \mapsto \phi^{u^\varphi} := (\boldsymbol{u}^\varphi)^*\phi = (\boldsymbol{u}\circ\varphi)^*\phi = \varphi^*(\boldsymbol{u}^*\phi) = \varphi^*\phi^u. \tag{164}
$$

The space $\boldsymbol{\Phi}^u$ of dressed fields is thus fibered by the right action of $\mathrm{Aut}(Q)$:

$$
\begin{aligned}
&\Phi^u\times\mathrm{Aut}(Q)\rightarrow\Phi^u,\\
&\qquad (\phi^u,\varphi)\mapsto R_\varphi\phi^u := \varphi^*\phi^u.
\end{aligned} \tag{165}
$$

So, in analogy with the original field space Φ, the space of dressed fields is a principal bundle $\Phi^u \xrightarrow{\pi} \Phi^u/\,\mathrm{Aut}(Q) =: \mathcal{M}^u$.[18] We have the SES of groups

$$
\boldsymbol{Aut}(Q) \simeq \boldsymbol{Aut}_v(\Phi^u) \rightarrow \boldsymbol{Aut}(\Phi^u) \rightarrow \boldsymbol{Diff}(\mathcal{M}^u), \tag{166}
$$

where $\boldsymbol{Aut}(Q) := \left\{\boldsymbol{\varphi}: \Phi^u \rightarrow \mathrm{Aut}(Q) \mid R^\star_\varphi\boldsymbol{\varphi} = \varphi^{-1}\circ\boldsymbol{\varphi}\circ\varphi\right\}$ is the gauge group of Φ^u. The latter is a subgroup of $C^\infty\left(\Phi^u, \mathrm{Aut}(Q)\right) \simeq \boldsymbol{Diff}_v(\Phi^u)$, acting on $\Gamma(T\Phi^u)$ and $\Omega^\bullet(\Phi^u)$ as previously described. In particular, in exact analogy with (47) and (71), a dressed form $\boldsymbol{\alpha}^u = \alpha\left(\wedge^\bullet \boldsymbol{d}\phi^u; \phi^u\right) \in \Omega^\bullet(\Phi^u)$ transforms under $\boldsymbol{\varphi} \in C^\infty\left(\Phi^u, \mathrm{Aut}(Q)\right)$ as:

$$
(\boldsymbol{\alpha}^u)\boldsymbol{\varphi} = \alpha\left(\wedge^\bullet(\boldsymbol{d}\phi^u)\boldsymbol{\varphi};\ (\phi^u)\boldsymbol{\varphi}\right), \tag{167}
$$

$$
\text{with}\quad (\boldsymbol{d}\phi^u)^{\boldsymbol{\varphi}} = \boldsymbol{\varphi}^*\left(\boldsymbol{d}\phi^u + \mathfrak{L}_{\{\boldsymbol{d}\boldsymbol{\varphi}\circ\boldsymbol{\varphi}^{-1}\}}\phi^u\right), \tag{168}
$$

from which follows that $(\boldsymbol{\alpha}^u)^{\boldsymbol{\varphi}} = R^\star_{\boldsymbol{\varphi}}\boldsymbol{\alpha}^u + \iota_{\{\boldsymbol{d}\boldsymbol{\varphi}\circ\boldsymbol{\varphi}^{-1}\}^v}\boldsymbol{\alpha}^u$.

We notice that $(\phi^u)^\varphi$ is $\mathrm{Aut}(P)$-invariant for all $\varphi\in\mathrm{Aut}(Q)$, meaning that all representatives in the $\mathrm{Aut}(Q)$-orbit $\mathcal{O}_{\mathrm{Aut}(Q)}[\phi^u]$ of ϕ^u are valid coordinatizations of $[\phi]\in\mathcal{M}$. So, a priori $\mathcal{O}_{\mathrm{Aut}(Q)}[\phi^u] \simeq \mathcal{O}_{\mathrm{Aut}(P)}[\phi]$, and $\mathcal{M}^u\simeq\mathcal{M}$. By (166), it is clear that $\boldsymbol{Aut}(Q)\simeq\boldsymbol{Aut}_v(\Phi^u)$ is isomorphic to the original gauge group $\boldsymbol{Aut}(P)\simeq\boldsymbol{Aut}_v(\Phi)$, and that $\boldsymbol{Diff}(\mathcal{M}^u)\simeq\boldsymbol{Diff}(\mathcal{M})$, which are "physical symmetries". A priori, the new symmetry $\boldsymbol{Diff}_v(\Phi^u) \supset \boldsymbol{Aut}_v(\Phi^u)$ does not enjoy a more direct physical interpretation than the original symmetry it replaces.

The group $C^\infty\left(\Phi^u, \mathrm{Aut}(Q)\right)\simeq\boldsymbol{Diff}_v(\Phi^u)$, and transformations like (163), (165) and (168), encompass what is known, in the covariant phase space literature on edge modes, as "surface symmetries" or "corner symmetries" – or more rarely as "physical symmetries", which is misleading, per our previous remark.

We observe that, in concrete situations, the process by which one constructs a dressing field $\boldsymbol{u}(\phi)$ out of the field content of a theory may be s.t. the choice reflected in the relation (163) is parametrized by only a subgroup of $\mathrm{Aut}(Q)$, possibly even a discrete one. A possibility forfeited for dressing fields introduced by hand in a theory.

4.4. Dressed Regions and Integrals

In Section 3.5.2, we observed that for $V\in\boldsymbol{V}(P)$ and $\boldsymbol{\alpha}\in\Omega^\bullet\left(\Phi,\Omega^{\text{top}}(V)\right)$, integrals $\boldsymbol{\alpha}_V = \langle\boldsymbol{\alpha}, V\rangle = \int_V\boldsymbol{\alpha}$ are objects on $\Phi\times\boldsymbol{V}(P)$, with values in $\Omega^\bullet(\Phi)$. Integrals of tensorial integrand are invariant under the action of $C^\infty\left(\Phi,\mathrm{Aut}(P)\right)\simeq\boldsymbol{Diff}_v(\Phi)$, defined by (124). Their projection along $\bar{\pi}:\Phi\times\boldsymbol{V}(P)\rightarrow\bar{\boldsymbol{V}}(P)$, $(\phi, V)\mapsto[\phi, V] = [\psi^*\phi, \psi^{-1}(V)]$ is well-defined, and they can be said "basic" w.r.t. $\bar{\pi}$. So, they descend on the associated bundle of regions $\bar{\boldsymbol{V}}(P) := \Phi\times\boldsymbol{V}(P)/\!\sim$, which is the quotient of the product space by the action of $\mathrm{Aut}(P)$ (97).

In Section 4.1, dressed objects were defined as being in $\mathrm{Im}\,\pi^\star$ (i.e., basic on Φ) with the projection realized via a dressing field, $F_{\boldsymbol{u}}\sim\pi$. Relying on the formal similarity of the actions of $F_{\boldsymbol{u}}$ and $\Xi\in\boldsymbol{Diff}_v(\Phi)\simeq C^\infty\left(\Phi,\mathrm{Aut}(P)\right)$, the rule of thumb to obtain the dressing $\boldsymbol{\alpha}^u$ of a form $\boldsymbol{\alpha}$ is to replace the field-dependent parameter $\boldsymbol{\psi}$ in $\boldsymbol{\alpha}^{\boldsymbol{\psi}} := \Xi^\star\boldsymbol{\alpha}$ by the dressing field $\boldsymbol{u}$. In the same way,

[18] Provided that the action of $\mathrm{Aut}(Q)$ is free and transitive, which requires in general some restriction on either $\mathrm{Aut}(Q)$ or Φ^u. See refs. [60, 61] in the case of the action of a gauge group $\mathcal{H}\simeq\mathrm{Aut}_v(P)$ on the connection space $\Phi = \mathcal{C}$ of P.

we define dressed integrals as being basic on $\Phi \times \boldsymbol{V}(P)$, i.e., in $\operatorname{Im} \bar{\pi}^\star$, with the projection realized as:

$$\begin{aligned}\bar{F}_{\boldsymbol{u}} : \Phi \times \boldsymbol{V}(P) &\to \bar{V}(P) \simeq \Phi^{\boldsymbol{u}} \times \boldsymbol{V}(Q),\\ (\phi, V) &\mapsto \bar{F}_{\boldsymbol{u}}(\phi, V) := \big(F_{\boldsymbol{u}}(\phi), \boldsymbol{u}^{-1}(V)\big) = \big(\phi^{\boldsymbol{u}}, \boldsymbol{u}^{-1}(V)\big).\end{aligned} \tag{169}$$

We highlight the fact that the region $V^{\boldsymbol{u}} := \boldsymbol{u}^{-1}(V) \in \boldsymbol{V}(Q)$ is a map $V^{\boldsymbol{u}} : \Phi \times \boldsymbol{V}(P) \to \boldsymbol{V}(Q)$ s.t. for $\psi \in \operatorname{Aut}(P)$:

$$\begin{aligned}(V^{\boldsymbol{u}})^{\psi} &:= \bar{R}^\star_{\psi}\, V^{\boldsymbol{u}} = (R^\star_{\psi}\boldsymbol{u})^{-1} \circ \psi^{-1}(V)\\ &= (\psi^{-1} \circ \boldsymbol{u})^{-1} \circ \psi^{-1}(V) = \boldsymbol{u}^{-1}(V) =: V^{\boldsymbol{u}}.\end{aligned} \tag{170}$$

The same then holds for $\boldsymbol{\psi} \in C^\infty\big(\Phi, \operatorname{Aut}(P)\big) \simeq \boldsymbol{Diff}_v(\Phi)$: $(V^{\boldsymbol{u}})^{\boldsymbol{\psi}} := \bar{\Xi}^\star V^{\boldsymbol{u}} = V^{\boldsymbol{u}}$. Therefore, $V^{\boldsymbol{u}}$ is a ϕ-dependent $\operatorname{Aut}(P)$-invariant region of what we may call, as we did in Section 2, the physical *relationally defined enriched spacetime*. As we have seen there, the generalized hole argument and the generalized point-coincidence argument establish that the physical enriched spacetime is defined *relationally*, in an $\operatorname{Aut}(P)$-invariant way, by its physical gauge field content. A fact that is tacitly encoded by the $\operatorname{Aut}(P)$-covariance/invariance of general-relativistic gauge field theories, and made manifest via the DFM: $V^{\boldsymbol{u}}$ are manifestly $\operatorname{Aut}(P)$-invariant and manifestly ϕ-relationally defined regions, faithfully representing regions of the physical enriched spacetime, on which relationally defined and $\operatorname{Aut}(P)$-invariant fields $\phi^{\boldsymbol{u}} := \boldsymbol{u}^*\phi$ live – and might be integrated over. From this follows that the physical, *relational boundary* $\partial V^{\boldsymbol{u}}$ of a true enriched spacetime region $V^{\boldsymbol{u}}$ is necessarily $\operatorname{Aut}(P)$-invariant. This observation is key to dissolve the so-called "boundary problems" often discussed in field theory, as we will see in Section 5.2.4.

So, on the space $\Omega^\bullet\big(\Phi, \Omega^{\text{top}}(V)\big) \times \boldsymbol{V}(P)$ we define:

$$\begin{aligned}\tilde{F}_{\boldsymbol{u}} : \Omega^\bullet\big(\Phi, \Omega^{\text{top}}(V)\big) \times \boldsymbol{V}(P) &\to \Omega^\bullet_{\text{basic}}\big(\Phi, \Omega^{\text{top}}(Q)\big) \times \boldsymbol{V}(Q)\\ (\boldsymbol{\alpha}, V) &\mapsto \tilde{F}^\star_{\boldsymbol{u}}(\boldsymbol{\alpha}, V) := \big(\boldsymbol{\alpha}^{\boldsymbol{u}}, \boldsymbol{u}^{-1}(V)\big).\end{aligned} \tag{171}$$

With indeed $\Omega^\bullet_{\text{basic}}\big(\Phi, \Omega^{\text{top}}(Q)\big) \simeq \Omega^\bullet\big(\Phi^{\boldsymbol{u}}, \Omega^{\text{top}}(Q)\big)$, as dressed fields $\phi^{\boldsymbol{u}}$ live on Q. Then, in formal analogy with (124), the dressing of an integral $\boldsymbol{\alpha}_V := \langle \boldsymbol{\alpha}, V\rangle = \int_V \boldsymbol{\alpha}$ is

$$(\boldsymbol{\alpha}_V)^{\boldsymbol{u}} = \langle\ ,\ \rangle \circ \tilde{F}_{\boldsymbol{u}}(\boldsymbol{\alpha}, V) = \langle \boldsymbol{\alpha}^{\boldsymbol{u}}, \boldsymbol{u}^{-1}(V)\rangle = \int_{\boldsymbol{u}^{-1}(V)} \boldsymbol{\alpha}^{\boldsymbol{u}}. \tag{172}$$

We remark that the residual symmetry $C^\infty\big(\Phi^{\boldsymbol{u}}, \operatorname{Aut}(Q)\big) \simeq \boldsymbol{Diff}_v(\Phi^{\boldsymbol{u}})$ may act on dressed integrals, analogously to (124), as

$$\big((\boldsymbol{\alpha}_V)^{\boldsymbol{u}}\big)^{\boldsymbol{\varphi}} = \int_{\boldsymbol{\varphi}^{-1}(V^{\boldsymbol{u}})} (\boldsymbol{\alpha}^{\boldsymbol{u}})^{\boldsymbol{\varphi}}. \tag{173}$$

Now, for $\boldsymbol{\alpha} \in \Omega^\bullet_{\text{tens}}\big(\Phi, \Omega^{\text{top}}(V)\big)$ one has

$${\boldsymbol{\alpha}_V}^{\boldsymbol{u}} = \langle \boldsymbol{\alpha}^{\boldsymbol{u}}, \boldsymbol{u}^{-1}(V)\rangle = \langle \boldsymbol{u}^*\boldsymbol{\alpha}, \boldsymbol{u}^{-1}(V)\rangle = \langle \boldsymbol{\alpha}, V\rangle = \boldsymbol{\alpha}_V, \tag{174}$$

by the invariance property (120) of the integration pairing. Therefore, $\boldsymbol{d\alpha}_V = \langle \boldsymbol{d\alpha}, V\rangle = \boldsymbol{d}\langle \boldsymbol{u}^*\boldsymbol{\alpha}, \boldsymbol{u}^{-1}(V)\rangle = \boldsymbol{d}({\boldsymbol{\alpha}_V}^{\boldsymbol{u}})$. The latter result can also be proven using the lemma (139)–(140) and concluding by the invariance property (121). This calculation also supplies the analogue of (118)/(128):

$$\begin{aligned}(\boldsymbol{d\alpha}_V)^{\boldsymbol{u}} &= \boldsymbol{d\alpha}_V + \langle \mathfrak{L}_{\boldsymbol{du}\circ \boldsymbol{u}^{-1}}\boldsymbol{\alpha}, V\rangle,\\ \langle \boldsymbol{d\alpha}, V\rangle^{\boldsymbol{u}} &= \langle \boldsymbol{d\alpha}, V\rangle + \langle \mathfrak{L}_{\boldsymbol{du}\circ \boldsymbol{u}^{-1}}\boldsymbol{\alpha}, V\rangle.\end{aligned} \tag{175}$$

The local version is key to the interplay between the DFM and the variational principle, as shown in Section 5.2.4.

5. Local Field Theory

Local field theory is usually expressed not directly via fields on a principal bundle P, but via local representatives of these global objects on (open sets of) the base space M. In this section, we thus consider the local counterpart of the above formalism to obtain applications to field theory in its standard formulation. As we shall see, the local case features several subtleties. We thus define the elementary notions and provide the most important relations necessary for field theory.

5.1. Local Field Space

For local field theory, the field space Φ is now the space of local representatives $\phi = \{A, F, \upphi, D\upphi, \dots\}$ on M of global object defined on P: e.g., A is a gauge potential over $U \subset M$ representing the connection ω over $P_{|U} \subset P$, while F is the field strength representing the curvature Ω.

Locally, over $U \subset M$, the group $\operatorname{Aut}(P)$ of automorphisms – the structure group of Φ seen as a principal bundle – is represented as the semi-direct product group $\operatorname{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$, where $\mathcal{H}_{\text{loc}}$ is the local representative of the gauge group $\mathcal{H}$ of P (isomorphic to $\operatorname{Aut}_v(P)$):

$$\mathcal{H}_{\text{loc}} = \big\{\upgamma, \upeta : U \to H \mid \upeta^{\upgamma} := \upgamma^{-1}\upeta\upgamma\big\}. \tag{176}$$

This field-theoretic characterization of the local (active) gauge group flows from the geometric definition of the gauge group of P. Similarly, the action of $\mathcal{H}_{\text{loc}}$ on a (local) field ϕ is *defined* as the local version of the action of $\mathcal{H}$ on the corresponding global object: For example, the $\mathcal{H}$-gauge transformation of ω is $\omega^\gamma := \psi^*\omega = \gamma^{-1}\omega\gamma + \gamma^{-1}d\gamma$, for $\psi \in \operatorname{Aut}_v(P) \sim \gamma \in \mathcal{H}$ – remind the isomorphism $\psi(p) = p\gamma(p)$ – which locally gives $A^{\upgamma} := \upgamma^{-1}A\upgamma + \upgamma^{-1}d\upgamma$, for $\upgamma \in \mathcal{H}_{\text{loc}}$. Likewise, for $\beta \in \Omega^\bullet_{\text{tens}}(P, \rho)$, one has $\beta^\gamma := \psi^*\beta = \rho(\gamma)^{-1}\beta$, which gives locally $b^{\upgamma} := \rho(\upgamma)^{-1}b$. Take $b = \{F, \upphi, D\upphi, \dots\}$ with their respective representation ρ. The right action of the structure group on Φ, the local version of (4), is then

$$\begin{aligned}\Phi \times \big(\operatorname{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\big) &\to \Phi,\\ (\phi, (\uppsi, \upgamma)) &\mapsto R_{(\uppsi,\upgamma)}\,\phi = (\uppsi, \upgamma)^*\phi := \uppsi^*(\phi^{\upgamma}).\end{aligned} \tag{177}$$

Remark that it does not matter which of $\operatorname{Diff}(M)$ or $\mathcal{H}_{\text{loc}}$ transformation applies first. This follows from the fact that $(\uppsi, \upgamma)$ is the local representative of a given automorphism $\psi = (\bar{\psi}, \eta) \in \operatorname{Aut}(P) = \overline{\operatorname{Diff}} \ltimes \operatorname{Aut}_v(P)$ which, by (A2), is written as $\psi(p) = (\eta \circ \bar{\psi})\,(p) = \bar{\psi}(p)\gamma\,\big(\bar{\psi}(p)\big) = \bar{\psi}(p)\,(\bar{\psi}^*\gamma)\,(p)$. The action (177) is the local version of the pullback action ψ^*. The semi-direct structure (A12) of the local group $\operatorname{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$ is inherited from the semi-direct structure of Aut(P) as detailed in Appendix A. It can also be derived from field-theoretic considerations, by iterating twice the transformation (177):

$$\begin{aligned}\phi &\mapsto \psi'^*(\phi^{\gamma'}) = (\psi'^*\phi)^{\psi'^*\gamma'}\\ &\mapsto \psi^*\Big(\Big[(\psi'^*\phi)^{\psi'^*\gamma'}\Big]^{\gamma}\Big) = \Big[(\psi^*\psi'^*\phi)^{\psi^*\psi'^*\gamma'}\Big]^{\psi^*\gamma}\\ &= \Big[(\psi^*\psi'^*\phi)^{\psi^*\psi'^*\gamma'}\Big]^{\psi^*\psi'^*\psi'^{-1*}\gamma}\\ &= \psi^*\psi'^*\Big(\big[\phi^{\gamma'}\big]^{\psi'^{-1*}\gamma}\Big)\\ &= (\psi'\circ\psi)^*\Big(\phi^{\gamma'\cdot(\psi'^{-1*}\gamma)}\Big).\end{aligned} \tag{178}$$

One may test this more explicitly for the simple case $\phi = b$. Given the right action of $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$ on Φ, $R_{(\psi,\gamma)}\, R_{(\psi',\gamma')}\,\phi = R_{(\psi',\gamma')\cdot(\psi,\gamma)}\,\phi$, its *semi-direct product* is found to be

$$\big(\psi',\gamma'\big)\cdot(\psi,\gamma) = \big(\psi'\circ\psi,\ \gamma'\cdot(\psi'^{-1*}\gamma)\big). \tag{179}$$

The corresponding Lie algebra $\mathfrak{diff}(M) \oplus \mathrm{Lie}\mathcal{H}_{\mathrm{loc}}$ is a semi-direct sum with Lie bracket

$$\big[(X',\lambda'),(X,\lambda)\big]_{\mathrm{Lie}} = \big([X',X]_{\mathfrak{diff}(M)}, [\lambda',\lambda]_{\mathrm{Lie}H} - X'(\lambda) + X(\lambda')\big). \tag{180}$$

Remark that $[X',X]_{\mathfrak{diff}(M)} := -[X',X]_{\Gamma(TM)}$. The bracket (180) can also be understood as the bracket on sections of the (locally trivialized) Atiyah Lie algebroid associated to P – see e.g., ref. [62] – described by the SES

$$0 \to \mathrm{Lie}\mathcal{H}_{\mathrm{loc}} \to \mathfrak{diff}(M) \oplus \mathrm{Lie}\mathcal{H}_{\mathrm{loc}} \to \mathfrak{diff}(M) \to 0. \tag{181}$$

The vertical subbundle of field space $\Phi \xrightarrow{\pi} \Phi/\big(\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\big) =: \mathcal{M}$, with $\mathcal{M}$ the moduli space of gauge orbits, is $V\Phi := \ker \pi_*$. The fundamental vertical vector field are

$$\begin{aligned}(X,\lambda)^v_{|\phi} &:= \tfrac{d^2}{d\tau ds} R_{(\psi_\tau,\gamma_s)}\phi\Big|_{\tau=0,s=0} = \tfrac{d^2}{d\tau ds}\psi^*_\tau(\phi^{\gamma_s})\Big|_{\tau=0,s=0}\\ &= \mathfrak{L}_X\phi + \delta_\lambda\phi = X^v_{|\phi} + \lambda^v_{|\phi},\end{aligned} \tag{182}$$

with $\mathfrak{L}_X\phi$ is the Lie derivative of ϕ along the vector field $X \in \Gamma(TM)$ generating the diffeomorphism $\psi_\tau \in \mathrm{Diff}(M)$, and $\delta_\lambda\phi$ is the infinitesimal (active) gauge transformation of ϕ by the element $\lambda : U \to \mathrm{Lie}H \in \mathrm{Lie}\,\mathcal{H}_{\mathrm{loc}}$. For example, for $\phi = A$ this is $\delta_\lambda A = D\lambda$, for $\phi = b$ this is $\delta_\lambda b = -\rho_*(\lambda)b$. The pushforward by the $\big(\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\big)$-right action of a fundamental vector field is

$$R_{(\psi,\gamma)\star}\,(X,\lambda)^v_{|\phi} := \big(\mathrm{Ad}_{(\psi,\gamma)^{-1}}(X,\lambda)\big)^v_{|\,R_{(\psi,\gamma)}\phi} \tag{183}$$

where $(\psi,\gamma)^{-1} = \big(\psi^{-1},\psi^*\gamma^{-1}\big)$ and the adjoint action is given by

$$\mathrm{Ad}_{(\psi,\gamma)^{-1}}(X,\lambda) = \big(\psi^{-1}_*X\circ\psi,\ \psi^*\big(\mathrm{Ad}_{\gamma^{-1}}\lambda - \gamma^{-1}\mathfrak{L}_X\gamma\big)\big), \tag{184}$$

as proven in (A14) of Appendix A.

The maximal group of transformations of the *local* field space Φ is again

$$\boldsymbol{Aut}(\Phi) := \Big\{\Xi \in \boldsymbol{Diff}(\Phi)\ |\ \Xi\circ R_{(\psi,\gamma)} = R_{(\psi,\gamma)}\circ\Xi\Big\}, \tag{185}$$

whose elements cover those of $\boldsymbol{Diff}(\mathcal{M})$. The vertical diffeomorphisms

$$\boldsymbol{Diff}_v(\Phi) := \Big\{\Xi \in \boldsymbol{Diff}(\Phi)\ |\ \pi\circ\Xi = \pi\Big\} \tag{186}$$

cover the identity $\mathrm{id}_{\mathcal{M}}$. Therefore, a vertical diffeomorphism is given by $\Xi(\phi) = R_{(\boldsymbol{\psi}(\phi),\boldsymbol{\gamma}(\phi))}\,\phi := \big(\boldsymbol{\psi}(\phi),\boldsymbol{\gamma}(\phi)\big)^*\phi$, with the maps $(\boldsymbol{\psi},\boldsymbol{\gamma}) : \Phi \to \mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$, i.e., one has $\boldsymbol{Diff}(\Phi) \simeq C^\infty\big(\Phi, \mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\big)$. The former is the action Lie groupoid of Φ, the latter its group of sections. We have that, as the local version of (8),

$$\begin{aligned}\Xi'\circ\Xi \in \boldsymbol{Diff}(\Phi)\quad &\text{corresponds to}\quad (\boldsymbol{\psi},\boldsymbol{\gamma})\cdot\big((\boldsymbol{\psi}',\boldsymbol{\gamma}')\circ R_{(\boldsymbol{\psi},\boldsymbol{\gamma})}\big)\\ &\in C^\infty\big(\Phi, \mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\big).\end{aligned} \tag{187}$$

This extends the normal subgroup of vertical automorphisms,

$$\boldsymbol{Aut}_v(\Phi) := \big\{\Xi \in \boldsymbol{Aut}(\Phi)\ |\ \pi\circ\Xi = \pi\big\} \lhd \boldsymbol{Aut}(\Phi), \tag{188}$$

which is isomorphic, still via $\Xi(\phi) = R_{(\boldsymbol{\psi}(\phi),\boldsymbol{\gamma}(\phi))}\,\phi := \big(\boldsymbol{\psi}(\phi),\boldsymbol{\gamma}(\phi)\big)^*\phi$, to the gauge group

$$\begin{aligned}\boldsymbol{Diff}(M) \ltimes \boldsymbol{\mathcal{H}}_{\mathrm{loc}} := \Big\{&(\boldsymbol{\psi},\boldsymbol{\gamma}) : \Phi \to \mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\ |\ R^\star_{(\psi,\gamma)}(\boldsymbol{\psi},\boldsymbol{\gamma})\\ &= \mathrm{Conj}(\psi,\gamma)^{-1}(\boldsymbol{\psi},\boldsymbol{\gamma})\Big\}.\end{aligned} \tag{189}$$

By the equivariance of gauge group elements, one has as a special case of (187) that to $\Xi'\circ\Xi \in \mathrm{Aut}_v(\Phi)$ corresponds the gauge group element $(\boldsymbol{\psi}',\boldsymbol{\gamma}')\cdot(\boldsymbol{\psi},\boldsymbol{\gamma}) \in \boldsymbol{Diff}(M) \ltimes \boldsymbol{\mathcal{H}}_{\mathrm{loc}}$. The SES associated to the local field space is

$$\mathrm{id}_\Phi \to \boldsymbol{Diff}(M) \ltimes \boldsymbol{\mathcal{H}}_{\mathrm{loc}} \simeq \boldsymbol{Aut}_v(\Phi) \xrightarrow{\lhd} \boldsymbol{Aut}(\Phi) \longrightarrow \boldsymbol{Diff}(\mathcal{M}) \to \mathrm{id}_{\mathcal{M}}. \tag{190}$$

We have the Lie algebra morphism $\mathfrak{diff}_v(\Phi) \simeq C^\infty\big(\Phi, \mathfrak{diff}(M) \oplus \mathrm{Lie}\mathcal{H}_{\mathrm{loc}}\big)$ induced by the "verticality map" (182) $|^v : C^\infty\big(\Phi, \mathfrak{diff}(M) \oplus \mathrm{Lie}\mathcal{H}_{\mathrm{loc}}\big) \to \mathfrak{diff}_v(\Phi)$, $(\boldsymbol{X},\boldsymbol{\lambda}) \mapsto (\boldsymbol{X},\boldsymbol{\lambda})^v$. The bracket is

$$\begin{aligned}&\big[(\boldsymbol{X},\boldsymbol{\lambda})^v, (\boldsymbol{X}',\boldsymbol{\lambda}')^v\big]_{\Gamma(T\Phi)}\\ &= \Big\{\big[(\boldsymbol{X},\boldsymbol{\lambda}),(\boldsymbol{X}',\boldsymbol{\lambda}')\big]_{\mathrm{Lie}} + (\boldsymbol{X},\boldsymbol{\lambda})^v\big((\boldsymbol{X}',\boldsymbol{\lambda}')\big) - (\boldsymbol{X}',\boldsymbol{\lambda}')^v\big((\boldsymbol{X},\boldsymbol{\lambda})\big)\Big\}^v\\ &=: \big\{(\boldsymbol{X},\boldsymbol{\lambda}),(\boldsymbol{X}',\boldsymbol{\lambda}')\big\}^v.\end{aligned} \tag{191}$$

It is the local version of (29). The *extended bracket* $\{\ ,\ \}$ on $C^\infty\big(\Phi, \mathfrak{diff}(M) \oplus \mathrm{Lie}\mathcal{H}_{\mathrm{loc}}\big)$ and can be understood both as the action Lie algebroid bracket $A = \Phi \rtimes \big(\mathfrak{diff}(M) \oplus \mathrm{Lie}\mathcal{H}_{\mathrm{loc}}\big) \to \Phi$ bracket, and as the FN bracket on $\Omega^0(\Phi, V\Phi)$. Observe that, as already mentioned, $[\ ,\ ]_{\mathrm{Lie}}$ is the trivial Atiyah Lie algebroid bracket (180). So the (action Lie algebroid, FN) bracket $\{\ ,\ \}$ extends the trivial Atiyah Lie algebroid bracket $[\ ,\ ]_{\text{Lie}}$ to field-dependent parameters *with unspecified equivariance*.

The right-invariant vector fields $\Gamma_{\text{inv}}(T\Phi)$ constitute the Lie algebra $\mathfrak{aut}(\Phi)$ of $\boldsymbol{Aut}(\Phi)$, with Lie ideal $\mathfrak{aut}_v(\Phi)$. As a special case of the above, we have the isomorphism $\mathfrak{aut}_v(\Phi) \simeq \mathfrak{diff}(M) \oplus \mathbf{Lie}\boldsymbol{\mathcal{H}}_{\text{loc}}$, with the gauge Lie algebra

$$\mathfrak{diff}(M) \oplus \mathbf{Lie}\boldsymbol{\mathcal{H}}_{\text{loc}} := \Big\{(\boldsymbol{X}, \boldsymbol{\lambda}) : \Phi \to \mathfrak{diff}(M) \oplus \text{Lie}\mathcal{H}_{\text{loc}} \mid R^*_{(\psi,\gamma)}(\boldsymbol{X}, \boldsymbol{\lambda}) = \text{Conj}(\psi, \gamma)^{-1}(\boldsymbol{X}, \boldsymbol{\lambda})\Big\}. \tag{192}$$

So, by the equivariance of gauge Lie algebra elements, linearly given by the adjoint action $\text{ad}_{(X,\lambda)} = [(X, \lambda),\ ]_{\text{Lie}}$, (191) specializes to

$$\left[(\boldsymbol{X}, \boldsymbol{\lambda})^v, (\boldsymbol{X}', \boldsymbol{\lambda}')^v\right]_{\Gamma(T\Phi)} = \left\{-\left[(\boldsymbol{X}, \boldsymbol{\lambda}), (\boldsymbol{X}', \boldsymbol{\lambda}')\right]_{\text{Lie}}\right\}^v. \tag{193}$$

We have the SES of Lie algebras

$$0 \to \mathfrak{diff}(M) \oplus \mathbf{Lie}\boldsymbol{\mathcal{H}}_{\text{loc}} \simeq \mathfrak{aut}_v(\Phi) \xrightarrow{|^v} \mathfrak{aut}(\Phi) \xrightarrow{\pi_\star} \mathfrak{diff}(\mathcal{M}) \to 0, \tag{194}$$

describing the Atiyah Lie algebroid associated to the local field space.

The pushforward of $\mathfrak{X} \in \Gamma(T\Phi)$ by $\Xi \in \boldsymbol{Diff}_v(\Phi) \sim (\boldsymbol{\psi}, \boldsymbol{\gamma}) \in C^\infty\left(\Phi, \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\right)$ – a local version of (32) – is

$$\begin{aligned}\Xi_\star \mathfrak{X}_{|\phi} &= R_{(\boldsymbol{\psi},\boldsymbol{\gamma})\star}\mathfrak{X}_{|\phi} + \left\{(\boldsymbol{\psi}, \boldsymbol{\gamma})^{-1} \cdot (\boldsymbol{d\psi}, \boldsymbol{d\gamma})_{|\phi}(\mathfrak{X}_{|\phi})\right\}^v_{|\Xi(\phi)} \\ &= R_{(\boldsymbol{\psi},\boldsymbol{\gamma})\star}\mathfrak{X}_{|\phi} + \left\{\left(\boldsymbol{\psi}_*^{-1}\boldsymbol{d\psi}, \boldsymbol{\psi}^*\left(\boldsymbol{\gamma}^{-1}\boldsymbol{d\gamma}\right)\right)_{|\phi}(\mathfrak{X}_{|\phi})\right\}^v_{|\Xi(\phi)}\end{aligned}$$

$$\begin{aligned}\Xi_\star \mathfrak{X}_{|\phi} &= R_{(\boldsymbol{\psi},\boldsymbol{\gamma})\star}\left(\mathfrak{X}_{|\phi} + \left\{(\boldsymbol{d\psi}, \boldsymbol{d\gamma})_{|\phi}(\mathfrak{X}_{|\phi}) \cdot (\boldsymbol{\psi}, \boldsymbol{\gamma})^{-1}\right\}^v_{|\phi}\right) \\ &= R_{(\boldsymbol{\psi},\boldsymbol{\gamma})\star}\left(\mathfrak{X}_{|\phi} + \left\{\left(\boldsymbol{d\psi} \circ \boldsymbol{\psi}^{-1},\ \boldsymbol{d\gamma}\,\boldsymbol{\gamma}^{-1} - \boldsymbol{\gamma}\,\mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\gamma}^{-1}\right)_{|\phi}(\mathfrak{X}_{|\phi})\right\}^v_{|\phi}\right)\end{aligned} \tag{195}$$

as proven in (B16)–(B22) in Appendix B. It is necessary to compute geometrically vertical and gauge transformations on the local field space Φ.

5.1.1. Forms on Local Field Space and Their Transformations

Forms $\boldsymbol{\alpha} = \alpha\left(\wedge^\bullet \boldsymbol{d}\phi; \phi\right) \in \Omega^\bullet(\Phi)$ on the local field space, together with the basic operations $\boldsymbol{d}$, $\iota_{\mathfrak{X}}$, $\boldsymbol{L}_{\mathfrak{X}}$, for $\mathfrak{X} \in \Gamma(T\Phi)$ – as well as the extensions via the Nijenhuis-Richardson and FN brackets – are defined as in section 3.2.2. Their equivariance is defined by (the result of) $R^*_{(\psi,\gamma)}\boldsymbol{\alpha}$, with infinitesimal version given by $\boldsymbol{L}_{(X,\lambda)^v}\boldsymbol{\alpha}$. Of special interest are tensorial forms: For (ρ, V) a representation for $\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$, one defines

$$\Omega^\bullet_{\text{tens}}(\Phi, \rho) := \left\{\boldsymbol{\alpha} \in \Omega^\bullet(\Phi, V) \mid R^\star_{(\psi,\gamma)}\boldsymbol{\alpha} = \rho(\psi, \gamma)^{-1}\boldsymbol{\alpha},\ \&\ \iota_{(X,\lambda)^v}\boldsymbol{\alpha} = 0\right\}. \tag{196}$$

The infinitesimal equivariance is then $\boldsymbol{L}_{(X,\lambda)^v}\boldsymbol{\alpha} = -\rho_*\left(X, \lambda\right)\boldsymbol{\alpha}$. Given a 1-cocycle

$$\begin{aligned}C : \Phi \times \left(\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\right) &\to G, \\ (\phi; (\psi, \gamma)) &\mapsto C(\phi; (\psi, \gamma)),\end{aligned} \tag{197}$$

$$\text{s.t.} \quad C(\phi; (\psi', \gamma') \cdot (\psi, \gamma)) = C(\phi; (\psi', \gamma'))C\left(R_{(\psi',\gamma')}\phi; (\psi, \gamma)\right),$$

and $\boldsymbol{V}$ a G-space, one extends the above to *twisted tensorial* forms

$$\begin{aligned}&\Omega^\bullet_{\text{tens}}(\Phi, C) \\ &:= \left\{\boldsymbol{\alpha} \in \Omega^\bullet(\Phi, \boldsymbol{V}) \mid R^\star_{(\psi,\gamma)}\boldsymbol{\alpha} = C\left(\phi; (\psi, \gamma)\right)^{-1}\boldsymbol{\alpha},\ \&\ \iota_{(X,\lambda)^v}\boldsymbol{\alpha} = 0\right\}.\end{aligned} \tag{198}$$

The infinitesimal equivariance is then $\boldsymbol{L}_{(X,\lambda)^v}\boldsymbol{\alpha} = -a\left((X, \lambda); \phi\right)\boldsymbol{\alpha}$, where $a : \Phi \times \left(\mathfrak{diff}(M) \oplus \text{Lie}\mathcal{H}_{\text{loc}}\right) \to \mathfrak{g}$ is the infinitesimal 1-cocycle associated to C. Acting via $[\boldsymbol{L}_{(X,\lambda)^v}, \boldsymbol{L}_{(X',\lambda')^v}] = \boldsymbol{L}_{[(X,\lambda)^v,(X',\lambda')^v]} = \boldsymbol{L}_{\left([(X,\lambda),(X',\lambda')]_{\text{Lie}}\right)^v}$ reproduces the defining $\left(\mathfrak{diff}(M) \oplus \text{Lie}\mathcal{H}_{\text{loc}}\right)$-1-cocycle property

$$\begin{aligned}&(X, \lambda)^v a\left((X', \lambda'); \phi\right) - \left(X', \lambda'\right)^v a\left((X, \lambda); \phi\right) \\ &\quad + \left[a\left((X, \lambda); \phi\right), a\left((X', \lambda'); \phi\right)\right]_{\mathfrak{g}} = a\left(\left[(X, \lambda),\ (X', \lambda')\right]_{\text{Lie}}; \phi\right).\end{aligned} \tag{199}$$

This, as we will see, generalizes the WZ consistency condition for anomalies. Fundamental to our discussions are the *basic forms*,

$$\begin{aligned}\Omega^\bullet_{\text{basic}}(\Phi, C) &:= \left\{\ \boldsymbol{\alpha} \in \Omega^\bullet(\Phi) \mid R^\star_{(\psi,\gamma)}\boldsymbol{\alpha} = \boldsymbol{\alpha},\ \&\ \iota_{(X,\lambda)^v}\boldsymbol{\alpha} = 0\right\} \\ &= \left\{\ \boldsymbol{\alpha} \in \Omega^\bullet(\Phi) \mid \boldsymbol{\alpha} = \pi^\star\boldsymbol{\beta},\ \text{for}\ \boldsymbol{\beta} \in \Omega^\bullet(\mathcal{M})\right\}.\end{aligned} \tag{200}$$

Our aim in the next section will be to write basic counterparts of forms on Φ via the DFM.

Vertical and Gauge Transformations: The finite vertical and gauge transformations of forms are defined by the pullback actions of $\boldsymbol{Diff}_v(\Phi)$ and $\boldsymbol{Aut}_v(\Phi)$ respectively: $\boldsymbol{\alpha} \mapsto \boldsymbol{\alpha}^{(\psi,\gamma)} := \Xi^\star\boldsymbol{\alpha}$, with Ξ generated by $(\boldsymbol{\psi}, \boldsymbol{\gamma})$, element of $C^\infty\left(\Phi, \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\right)$ or $\boldsymbol{Diff}(M) \ltimes \boldsymbol{\mathcal{H}}_{\text{loc}}$, and computed geometrically via (195):

$$\begin{aligned}\boldsymbol{\alpha}^{(\psi,\gamma)}{}_{|\phi}\left(\mathfrak{X}_{|\phi}, \cdots\right) &:= \Xi^\star\boldsymbol{\alpha}_{|\Xi(\phi)}\left(\mathfrak{X}_{|\phi}, \cdots\right) = R^\star_{(\boldsymbol{\psi},\boldsymbol{\gamma})}\boldsymbol{\alpha}_{|\Xi(\phi)} \\ &\quad \times \left(\mathfrak{X}_{|\phi} + \left\{(\boldsymbol{d\psi}, \boldsymbol{d\gamma})_{|\phi}(\mathfrak{X}_{|\phi}) \cdot (\boldsymbol{\psi}, \boldsymbol{\gamma})^{-1}\right\}^v_{|\phi}, \cdots\right).\end{aligned} \tag{201}$$

Clearly, the transformation of a form is controlled by its equivariance and verticality properties. The infinitesimal transformation is given

$$\boldsymbol{L}_{(X,\lambda)^v}\boldsymbol{\alpha} = \tfrac{d}{d\tau}\,R^\star_{(\boldsymbol{\psi}_\tau,\boldsymbol{\gamma}_\tau)}\,\boldsymbol{\alpha}\,\Big|_{\tau=0} + \iota_{(\boldsymbol{d}X,\boldsymbol{d}\lambda)^v}\,\boldsymbol{\alpha}, \tag{202}$$

with $(\boldsymbol{X}, \boldsymbol{\lambda}) := \frac{d}{d\tau}\left(\boldsymbol{\psi}_\tau, \boldsymbol{\gamma}_\tau\right)\Big|_{\tau=0}$ in $C^\infty\left(\Phi, \mathfrak{diff}(M) \oplus \text{Lie}\mathcal{H}_{\text{loc}}\right)$ or $\mathfrak{diff}(M) \oplus \mathbf{Lie}\boldsymbol{\mathcal{H}}_{\text{loc}}$. One may observe that $(\boldsymbol{dX}, \boldsymbol{d\lambda})^v$ can be seen as an element of $\Omega^1(\Phi, V\Phi)$, so $\iota_{(\boldsymbol{d}X,\boldsymbol{d}\lambda)^v}$ is a degree 0 algebraic derivation, as discussed in Section 3.2.2. For standard and twisted equivariant forms respectively, the above gives

$$L_{(X,\lambda)^v}\boldsymbol{\alpha} = \begin{cases} -\rho_*(X,\lambda)\,\boldsymbol{\alpha} + \iota_{(dX,\,d\lambda)^v}\,\boldsymbol{\alpha}, \\ -a(X,\lambda)\,\boldsymbol{\alpha} + \iota_{(dX,\,d\lambda)^v}\,\boldsymbol{\alpha}, \end{cases} \tag{203}$$

where we introduce the notation $\left[a(X,\lambda)\right](\phi) := a\left(X(\phi),\lambda(\phi);\phi\right)$ for the linearized 1-cocycle. In particular, for $\Omega^\bullet(M)$-valued forms or tensor-valued forms, i.e., for the natural representation $\rho(\psi,\gamma)^{-1} = (\psi^*,\{-\}^\gamma)$ the above gives

$$L_{(X,\lambda)^v}\boldsymbol{\alpha} = \left(\mathfrak{L}_X,\delta_\lambda\right)\boldsymbol{\alpha} + \iota_{(dX,\,d\lambda)^v}\,\boldsymbol{\alpha}. \tag{204}$$

This both generalizes and clarifies the geometrical meaning of the "anomaly operators", Δ_X and Δ_λ, featuring in refs. [24, 25, 32–34]: in our notation $\left(\Delta_X,\Delta_\lambda\right) := L_{(X,\lambda)^v} - \left(\mathfrak{L}_X,\delta_\lambda\right) - \iota_{(dX,\,d\lambda)^v}$. As we signaled in the global case, this operator is non-zero only in theories admitting background non-dynamical structures or fields "breaking" $\left(\mathrm{Diff}(M)\ltimes\mathcal{H}_{\rm loc}\right)$-covariance. Such theories thus fail to comply with the core physical principles of gRGFT.

Observe that the field-dependent gauge algebra is given by the commutator of Lie derivatives and involves the FN brackets (191) of the field-dependent parameters

$$\left[L_{(\boldsymbol{\psi},\boldsymbol{\gamma})^v}, L_{(\boldsymbol{\psi}',\boldsymbol{\gamma}')^v}\right] = L_{\left[(\boldsymbol{\psi},\boldsymbol{\gamma})^v,(\boldsymbol{\psi}',\boldsymbol{\gamma}')^v\right]_{\rm FN}} = L_{\left\{(\boldsymbol{\psi},\boldsymbol{\gamma}),(\boldsymbol{\psi}',\boldsymbol{\gamma}')\right\}^v}. \tag{205}$$

As a special case of (201) and (203), a tensorial form $\boldsymbol{\alpha}\in\Omega^\bullet_{\rm tens}(\Phi,\rho)$ transforms as

$$\boldsymbol{\alpha}^{(\boldsymbol{\psi},\boldsymbol{\gamma})} = \rho(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\boldsymbol{\alpha}, \quad \text{so} \quad L_{(X,\lambda)^v}\boldsymbol{\alpha} = -\rho_*(X,\lambda)\,\boldsymbol{\alpha}. \tag{206}$$

For a twisted tensorial form $\boldsymbol{\alpha}\in\Omega^\bullet_{\rm tens}(\Phi,C)$, instead, the transformation reads

$$\boldsymbol{\alpha}^{(\boldsymbol{\psi},\boldsymbol{\gamma})} = C(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\boldsymbol{\alpha}, \quad \text{so} \quad L_{(X,\lambda)^v}\boldsymbol{\alpha} = -a(X,\lambda)\,\boldsymbol{\alpha}, \tag{207}$$

where we introduce the notation $[C(\boldsymbol{\psi},\boldsymbol{\gamma})](\phi) := C(\phi;(\boldsymbol{\psi}(\phi),\boldsymbol{\gamma}(\phi)))$ and $[a(X,\lambda)](\phi) := a((X(\phi),\lambda(\phi));\phi)$. Acting via (205) reproduces the $(\mathfrak{diff}(M)\oplus\mathbf{Lie}\mathcal{H}_{\rm loc})$-1-cocycle property

$$\begin{aligned}&(X,\lambda)^v a\left((X',\lambda');\phi\right) - \left(X',\lambda'\right)^v a\left((X,\lambda);\phi\right)\\ &\quad + \left[a\left((X,\lambda);\phi\right), a\left((X',\lambda');\phi\right)\right]_{\mathfrak{g}} = a\left(\left\{(X,\lambda),(X',\lambda')\right\};\phi\right).\end{aligned} \tag{208}$$

This actually reduces to (199), all computation done. As we know, basic forms $\boldsymbol{\alpha}\in\Omega^\bullet_{\rm basic}(\Phi)$ are invariant:

$$\boldsymbol{\alpha}^{(\boldsymbol{\psi},\boldsymbol{\gamma})} = \boldsymbol{\alpha}, \quad \text{so} \quad L_{(X,\lambda)^v}\boldsymbol{\alpha} = 0. \tag{209}$$

A crucial example is the basis 1-form $\boldsymbol{d}\phi\in\Omega^1(\Phi)$, whose equivariance and verticality properties are defined as

$$R^\star_{(\psi,\gamma)}\boldsymbol{d}\phi = (\psi,\gamma)^*\boldsymbol{d}\phi := \psi^*\left(\boldsymbol{d}\phi^\gamma\right), \qquad \text{and} \qquad \iota_{(X,\lambda)^v}\boldsymbol{d}\phi = \boldsymbol{d}\phi_{|\underline{\phi}}\left[(X,\lambda)^v_{|\underline{\phi}}\right] := \mathfrak{L}_X\underline{\phi} + \delta_\lambda\underline{\phi}. \tag{210}$$

Then, by (195) and (210), the $\boldsymbol{\mathcal{D}iff}_v(\Phi)\simeq C^\infty\left(\Phi,\mathrm{Diff}(M)\ltimes\mathcal{H}_{\rm loc}\right)$-transformation of the basis 1-form is

$$\begin{aligned}
\boldsymbol{d}\phi^{(\boldsymbol{\psi},\boldsymbol{\gamma})}{}_{|\phi}\left(\mathfrak{X}_{|\phi}\right)
&:= \Xi^\star\boldsymbol{d}\phi_{|\Xi(\phi)}\left(\mathfrak{X}_{|\phi}\right)\\
&= \boldsymbol{d}\phi_{|\Xi(\phi)}\left(\Xi_\star\mathfrak{X}_{|\phi}\right)\\
&= R^\star_{(\boldsymbol{\psi},\boldsymbol{\gamma})}\boldsymbol{d}\phi_{|\Xi(\phi)}\left(\mathfrak{X}_{|\phi} + \left\{\left[(\boldsymbol{d\psi},\boldsymbol{d\gamma})\cdot(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\right]_{|\phi}\left(\mathfrak{X}_{|\phi}\right)_{|\phi}\right\}^v_{|\phi}\right)\\
&= (\boldsymbol{\psi},\boldsymbol{\gamma})^*\boldsymbol{d}\phi_{|\phi}\\
&\qquad\times\left(\mathfrak{X}_{|\phi} + \left\{\left[\left(\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1},\boldsymbol{d\gamma}\boldsymbol{\gamma}^{-1} - \boldsymbol{\gamma}\mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\gamma}^{-1}\right)\right]_{|\phi}\left(\mathfrak{X}_{|\phi}\right)_{|\phi}\right\}^v_{|\phi}\right)\\
&= (\boldsymbol{\psi},\boldsymbol{\gamma})^*\left[\boldsymbol{d}\phi_{|\phi} + \mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\phi + \delta_{\boldsymbol{d\gamma\gamma}^{-1}-\boldsymbol{\gamma}\mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\gamma}^{-1}}\phi\right]\left(\mathfrak{X}_{|\phi}\right)\\
&= (\boldsymbol{\psi},\boldsymbol{\gamma})^*\left[\boldsymbol{d}\phi_{|\phi} + \mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\phi + \delta_{\boldsymbol{d\gamma\gamma}^{-1}}\phi + \delta_{(\mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\gamma})\boldsymbol{\gamma}^{-1}}\phi\right]\left(\mathfrak{X}_{|\phi}\right).
\end{aligned} \tag{211}$$

Hence the final result,

$$\boldsymbol{d}\phi^{(\boldsymbol{\psi},\boldsymbol{\gamma})} = \boldsymbol{\psi}^*\left[\left(\boldsymbol{d}\phi + \mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\phi + \delta_{(\mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\gamma})\boldsymbol{\gamma}^{-1}}\phi + \delta_{\boldsymbol{d\gamma\gamma}^{-1}}\phi\right)^{\boldsymbol{\gamma}}\right], \tag{212}$$

To crosscheck this result one may compute it in another, less straightforward but still geometrical, way. From (210) one can derive

$$\begin{aligned}
R^\star_{(\psi,\gamma)}\boldsymbol{d}\phi_{|R_{(\psi,\gamma)}\phi}\left[(X,\lambda)^v_{|\phi}\right] &= \psi^*\left(\mathfrak{L}_X(\phi^\gamma) + \delta_{(\mathrm{Ad}_{\gamma^{-1}}\lambda - \gamma^{-1}\mathfrak{L}_X\gamma)}\phi^\gamma\right)\\
\hookrightarrow (\psi,\gamma)^*\boldsymbol{d}\phi_{|\phi}\left[(X,\lambda)^v_{|\phi}\right] &= \psi^*\left(\left(\mathfrak{L}_X\phi\right)^\gamma\right) + \psi^*\left(\left(\delta_\lambda\phi\right)^\gamma\right)\\
&= \psi^*\left(\left(\mathfrak{L}_X\phi\right)^\gamma\right) + \psi^*\left(\delta_{\mathrm{Ad}_{\gamma^{-1}}\lambda}\,\phi^\gamma\right),
\end{aligned} \tag{213}$$

where, in the second line, we used the well-known identity

$$\left(\delta_\lambda\phi\right)^\gamma = \delta_{\mathrm{Ad}_{\gamma^{-1}}\lambda}\,\phi^\gamma. \tag{214}$$

From this follows the useful identity

$$\mathfrak{L}_X(\phi^\gamma) = \left(\mathfrak{L}_X\phi + \delta_{\mathfrak{L}_X\gamma\cdot\gamma^{-1}}\phi\right)^\gamma. \tag{215}$$

Then, the alternative way to compute the $\boldsymbol{\mathcal{D}iff}_v(\Phi)\simeq C^\infty(\Phi,\mathrm{Diff}(M)\ltimes\mathcal{H}_{\rm loc})$-transformation of the basis 1-form is

$$
\begin{aligned}
&d\phi^{(\psi,\gamma)}{}_{|\phi}\left(\mathfrak{X}_{|\phi}\right)\\
&:= \Xi^\star d\phi_{|\Xi(\phi)}\left(\mathfrak{X}_{|\phi}\right)\\
&= d\phi_{|\Xi(\phi)}\ \left(\Xi_\star \mathfrak{X}_{|\phi}\right)\\
&= d\phi_{|\Xi(\phi)}\ \left(R_{(\psi,\gamma)\star}\mathfrak{X}_{|\phi} + \left[\left(\psi_*^{-1}d\psi, \psi^*(\gamma^{-1}d\gamma)\right)_{|\phi}\left(\mathfrak{X}_{|\phi}\right)\right]^v_{|\Xi(\phi)}\right)\\
&= R^\star_{(\psi,\gamma)}\, d\phi_{|\Xi(\phi)}\left(\mathfrak{X}_{|\phi}\right) + d\phi_{|\Xi(\phi)}\\
&\quad\times\left(\left[\left(\psi_*^{-1}d\psi, \psi^*(\gamma^{-1}d\gamma)\right)_{|\phi}\left(\mathfrak{X}_{|\phi}\right)\right]^v_{|\Xi(\phi)}\right)\\
&= (\psi,\gamma)^* d\phi_{|\phi}\ \left(\mathfrak{X}_{|\phi}\right) + \mathfrak{L}_{\psi_*^{-1}d\psi_{|\phi}(\mathfrak{X}_{|\phi})}\,\Xi(\phi) + \delta_{[\psi^*(\gamma^{-1}d\gamma)]_{|\phi}(\mathfrak{X}_{|\phi})}\,\Xi(\phi)\\
&= \psi^*\left(\left[d\phi_{|\phi}\right]^\gamma\left(\mathfrak{X}_{|\phi}\right)\right) + \mathfrak{L}_{\psi_*^{-1}d\psi_{|\phi}(\mathfrak{X}_{|\phi})}\,\psi^*\left(\phi^\gamma\right)\\
&\quad + \delta_{[\psi^*(\gamma^{-1}d\gamma)]_{|\phi}(\mathfrak{X}_{|\phi})}\,\psi^*\left(\phi^\gamma\right)\\
&= \psi^*\left(\left[d\phi_{|\phi}\right]^\gamma\left(\mathfrak{X}_{|\phi}\right)\right) + \psi^*\left(\mathfrak{L}_{d\psi_{|\phi}(\mathfrak{X}_{|\phi})\circ\psi^{-1}}\phi^\gamma\right)\\
&\quad + \psi^*\left(\delta_{(\gamma^{-1}d\gamma)_{|\phi}(\mathfrak{X}_{|\phi})}\,\phi^\gamma\right),
\end{aligned}
\tag{216}
$$

where (14) is used for the second term of the last line. Using (214) for the third term, we get the result

$$
d\phi^{(\psi,\gamma)} = \psi^*\left(d\phi^\gamma + \mathfrak{L}_{d\psi\circ\psi^{-1}}\,\phi^\gamma + \left(\delta_{d\gamma\gamma^{-1}}\,\phi\right)^\gamma\right). \tag{217}
$$

Using the identity (215) to rewrite the second term, we get the alternative form

$$
d\phi^{(\psi,\gamma)} = \psi^*\left[\left(d\phi + \mathfrak{L}_{d\psi\circ\psi^{-1}}\,\phi + \delta_{(\mathfrak{L}_{d\psi\circ\psi^{-1}}\gamma)\gamma^{-1}}\,\phi + \delta_{d\gamma\gamma^{-1}}\,\phi\right)^\gamma\right], \tag{218}
$$

which matches (212). It is interesting to see the semi-direct structure of $\mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}$ manifest. These results encompass both the pure $\mathrm{Diff}(M)$ case – see ref. [8] Equation (67) – and the pure $\mathcal{H}_{\mathrm{loc}}$ case (i.e., "internal", Yang-Mills and/or Cartan geometric) – see ref. [42] Equation (7), also ref. [6].[19]

Connections on Local Field Space: We now briefly cover the two notions of connections adapted to the covariant derivation of tensorial and twisted tensorial forms on the local field space Φ.

- *Ehresmann connection* 1-forms on Φ are $\omega = \left(\omega_{\mathrm{Diff}}, \omega_{\mathcal{H}_{\mathrm{loc}}}\right) \in \Omega^1_{\mathrm{eq}}\left(\Phi, \mathfrak{diff}(M)\oplus\mathrm{Lie}\mathcal{H}_{\mathrm{loc}}\right)$ s.t.

$$
\begin{aligned}
&\omega_{|\phi}\left((X,\lambda)^v_{|\phi}\right) = (X,\lambda) \qquad \in \mathfrak{diff}(M)\oplus\mathrm{Lie}\mathcal{H}_{\mathrm{loc}},\\
&R^\star_{(\psi,\gamma)}\omega_{\,|R_{((\psi,\gamma))}\phi} = \mathrm{Ad}_{(\psi,\gamma)^{-1}}\,\omega_{|\phi}.
\end{aligned}
\tag{219}
$$

The equivariance property can be written more explicitly as

$$
R^\star_{(\psi,\gamma)}\left(\omega_{\mathrm{Diff}}, \omega_{\mathcal{H}_{\mathrm{loc}}}\right) = \left(\psi_*^{-1}\omega_{\mathrm{Diff}}\circ\psi, \psi^*\left(\mathrm{Ad}_{\gamma^{-1}}\omega_{\mathcal{H}_{\mathrm{loc}}} - \gamma^{-1}\mathfrak{L}_{\omega_{\mathrm{Diff}}}\gamma\right)\right). \tag{220}
$$

The infinitesimal equivariance is thus given by

$$
\begin{aligned}
L_{(X,\lambda)^v}\omega &= \left[\omega, (X,\lambda)\right]_{\mathrm{Lie}},\\
L_{(X,\lambda)^v}\left(\omega_{\mathrm{Diff}}, \omega_{\mathcal{H}_{\mathrm{loc}}}\right) &= \left[\left(\omega_{\mathrm{Diff}}, \omega_{\mathcal{H}_{\mathrm{loc}}}\right), (X,\lambda)\right]_{\mathrm{Lie}}\\
&= \left([\omega_{\mathrm{Diff}}, X]_{\mathfrak{diff}(M)}, [\omega_{\mathcal{H}_{\mathrm{loc}}}, \lambda]_{\mathrm{Lie}H}\right.\\
&\qquad \left. - \omega_{\mathrm{Diff}}(\lambda) + X\left(\omega_{\mathcal{H}_{\mathrm{loc}}}\right)\right),
\end{aligned}
\tag{221}
$$

as is also checked via the formula (180) for the Lie bracket. The associated curvature 2-form $\Omega = \left(\Omega_{\mathrm{Diff}}, \Omega_{\mathcal{H}_{\mathrm{loc}}}\right) \in \Omega^2_{\mathrm{tens}}(\Phi, \mathrm{Ad})$ is given by the Cartan structure equation

$$
\begin{aligned}
&\Omega = d\omega + \tfrac{1}{2}[\omega,\omega]_{\mathrm{Lie}},\\
&\hookrightarrow\ \left(\Omega_{\mathrm{Diff}}, \Omega_{\mathcal{H}_{\mathrm{loc}}}\right) = \left(d\omega_{\mathrm{Diff}} + \tfrac{1}{2}\left[\omega_{\mathrm{Diff}},\ \omega_{\mathrm{Diff}}\right]_{\mathfrak{diff}(M)},\ d\omega_{\mathcal{H}_{\mathrm{loc}}}\right.\\
&\qquad\left. + \tfrac{1}{2}\left[\omega_{\mathcal{H}_{\mathrm{loc}}}, \omega_{\mathcal{H}_{\mathrm{loc}}}\right]_{\mathrm{Lie}} - \omega_{\mathrm{Diff}}\left(\omega_{\mathcal{H}_{\mathrm{loc}}}\right)\right).
\end{aligned}
\tag{222}
$$

The connection allows to define a covariant derivative on tensorial forms $D: \Omega^\bullet_{\mathrm{tens}}(\Phi,\rho) \to \Omega^{\bullet+1}_{\mathrm{tens}}(\Phi,\rho)$, $\alpha \mapsto D\alpha = d\alpha + \rho_*(\omega)\,\alpha = d\alpha + \rho_*\left(\omega_{\mathrm{Diff}}, \omega_{\mathcal{H}_{\mathrm{loc}}}\right)\alpha$. In particular, Ω satisfies the Bianchi identity $D\Omega = d\Omega + [\omega,\Omega]_{\mathrm{Lie}} \equiv 0$.

The $\mathit{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}\right)$-transformation of a connection are easily found via by (219) and (195):

$$
\begin{aligned}
&\omega^{(\psi,\gamma)}{}_{|\phi}\left(\mathfrak{X}_{|\phi}\right)\\
&:= \Xi^\star\omega_{|\Xi(\phi)}\left(\mathfrak{X}_{|\phi}\right)\\
&= \omega_{|\Xi(\phi)}\left(\Xi_\star\mathfrak{X}_{|\phi}\right)\\
&= \omega_{|\Xi(\phi)}\left(R_{(\psi,\gamma)\star}\mathfrak{X}_{|\phi} + \left[\left(\psi_*^{-1}d\psi,\ \psi^*(\gamma^{-1}d\gamma)\right)_{|\phi}\left(\mathfrak{X}_{|\phi}\right)\right]^v_{|\Xi(\phi)}\right)\\
&= \mathrm{Ad}_{(\psi,\gamma)^{-1}}\,\omega_{|\phi}\left(\mathfrak{X}_{|\phi}\right) + \left(\psi_*^{-1}d\psi,\ \psi^*(\gamma^{-1}d\gamma)\right)_{|\phi}\left(\mathfrak{X}_{|\phi}\right).
\end{aligned}
\tag{223}
$$

Which, by the explicit form of the adjoint action, gives

$$
\begin{aligned}
&\omega^{(\psi,\gamma)} := \Xi^\star\omega = \mathrm{Ad}_{(\psi,\gamma)^{-1}}\omega + \left(\psi_*^{-1}d\psi, \psi^*(\gamma^{-1}d\gamma)\right),\\
&\hookrightarrow\ \left(\omega^{(\psi,\gamma)}_{\mathrm{Diff}},\ \omega^{(\psi,\gamma)}_{\mathcal{H}_{\mathrm{loc}}}\right) = \left(\psi_*^{-1}\omega_{\mathrm{Diff}}\circ\psi + \psi_*^{-1}d\psi, \psi^*\right.\\
&\qquad\left.\times\left(\mathrm{Ad}_{\gamma^{-1}}\omega_{\mathcal{H}_{\mathrm{loc}}} + \gamma^{-1}d\gamma - \gamma^{-1}\mathfrak{L}_{\omega_{\mathrm{Diff}}}\gamma\right)\right).
\end{aligned}
\tag{224}
$$

The transformations under $C^\infty\left(\Phi, \mathfrak{diff}(M)\oplus\mathrm{Lie}\mathcal{H}_{\mathrm{loc}}\right)$ – and $\mathfrak{diff}(M)\oplus\mathbf{Lie}\mathcal{H}_{\mathrm{loc}}$ – are, by (203),

[19] In the last case, in ref. [42], the notation $R^\star_\gamma d\phi = \rho(\gamma)^{-1}d\phi$ is used, where $\rho = (\mathrm{Ad}, \rho)$ on $d\phi = (dA, d\varphi)$, where φ is a matter field transforming via the representation ρ. Indeed, dA transforms *Ad*-tensorially under $\mathcal{H}_{\mathrm{loc}}$ because the space of gauge potentials (connections) is affine modeled on *Ad*-tensorial forms.

$$
\begin{aligned}
L_{(X,\lambda)^v}\boldsymbol{\omega} &= -\mathrm{ad}_{(X,\lambda)}\boldsymbol{\omega} + (\boldsymbol{d}X,\, \boldsymbol{d}\lambda)\\
&= \boldsymbol{d}(X,\lambda) + \big[\boldsymbol{\omega}, (X,\lambda)\big]_{\text{Lie}},\\
\hookrightarrow\ L_{(X,\lambda)^v}\left(\boldsymbol{\omega}_{\text{Diff}}, \boldsymbol{\omega}_{\mathcal{H}_{\text{loc}}}\right) &= \big(\boldsymbol{d}X + [\omega_{\text{Diff}}, X]_{\mathfrak{diff}(M)}, \boldsymbol{d}\lambda\\
&\quad + [\boldsymbol{\omega}_{\mathcal{H}_{\text{loc}}}, \lambda]_{\text{Lie}H} - \boldsymbol{\omega}_{\text{Diff}}(\lambda) + X\left(\boldsymbol{\omega}_{\mathcal{H}_{\text{loc}}}\right)\big).
\end{aligned}
\tag{225}
$$

For $(X,\lambda) \in \mathfrak{diff}(M) \oplus \textbf{Lie}\mathcal{H}_{\text{loc}}$, i.e., tensorial 0-forms, one may legitimately write the result as their (geometric) covariant derivative: $L_{(X,\lambda)^v}\boldsymbol{\omega} = D(X,\lambda)$.

Similarly, the $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty(\Phi, \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}})$-transformation of the curvature $\boldsymbol{\Omega} \in \Omega^\bullet_{\text{tens}}(\Phi, \text{Ad})$ is given by

$$
\begin{aligned}
&\boldsymbol{\Omega}^{(\boldsymbol{\psi},\boldsymbol{\gamma})} := \Xi^\star\boldsymbol{\Omega} = \text{Ad}_{(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}}\boldsymbol{\Omega},\\
&\hookrightarrow \left(\boldsymbol{\Omega}^{(\boldsymbol{\psi},\boldsymbol{\gamma})}_{\text{Diff}}, \boldsymbol{\Omega}^{(\boldsymbol{\psi},\boldsymbol{\gamma})}_{\mathcal{H}_{\text{loc}}}\right) = \left(\boldsymbol{\psi}^{-1}_*\boldsymbol{\Omega}_{\text{Diff}} \circ \boldsymbol{\psi}, \boldsymbol{\psi}^*\left(\text{Ad}_{\boldsymbol{\gamma}^{-1}}\boldsymbol{\Omega}_{\mathcal{H}_{\text{loc}}} - \boldsymbol{\gamma}^{-1}\mathfrak{L}_{\boldsymbol{\Omega}_{\text{Diff}}}\boldsymbol{\gamma}\right)\right),
\end{aligned}
\tag{226}
$$

with linear version

$$
\begin{aligned}
L_{(X,\lambda)^v}\boldsymbol{\Omega} &= -\mathrm{ad}_{(X,\lambda)}\boldsymbol{\Omega} = \big[\boldsymbol{\Omega}, (X,\lambda)\big]_{\text{Lie}},\\
\hookrightarrow\ L_{(X,\lambda)^v}\left(\boldsymbol{\Omega}_{\text{Diff}}, \boldsymbol{\Omega}_{\mathcal{H}_{\text{loc}}}\right) &= \big([\boldsymbol{\Omega}_{\text{Diff}}, X]_{\mathfrak{diff}(M)}, [\boldsymbol{\Omega}_{\mathcal{H}_{\text{loc}}}, \lambda]_{\text{Lie}H}\\
&\quad - \boldsymbol{\Omega}_{\text{Diff}}(\lambda) + X\left(\boldsymbol{\Omega}_{\mathcal{H}_{\text{loc}}}\right)\big).
\end{aligned}
\tag{227}
$$

More could be said, but we refrain from pursuing too much tangent observations.

- *Twisted connection* 1-forms on Φ are $\boldsymbol{\varpi} \in \Omega^1_{\text{eq}}(\Phi, \mathfrak{g})$ s.t.

$$
\begin{aligned}
\boldsymbol{\varpi}_{|\phi}\left((X,\lambda)^v_{|\phi}\right) &= \tfrac{d}{d\tau}C\big(\phi; (\psi_\tau, \gamma_\tau)\big)\Big|_{\tau=0} = a\left((X,\lambda);\phi\right) \in \mathfrak{g},\\
R^\star_{(\psi,\gamma)}\boldsymbol{\varpi}_{|R_{((\psi,\gamma))}\phi} &= \text{Ad}_{C(\phi;(\psi,\gamma))^{-1}}\boldsymbol{\varpi}_{|\phi} + C(\phi;(\psi,\gamma))^{-1}\boldsymbol{d}C(\ ;(\psi,\gamma))_{|\phi}.
\end{aligned}
\tag{228}
$$

The infinitesimal equivariance is

$$
L_{(X,\lambda)^v}\,\boldsymbol{\varpi} = \boldsymbol{d}a\big((X,\lambda);\ \ \big) + \big[\boldsymbol{\varpi}, a\big(\phi;(X,\lambda)\big)\big]_{\mathfrak{g}}. \tag{229}
$$

The twisted curvature 2-form $\bar{\boldsymbol{\Omega}} \in \Omega^2_{\text{tens}}(\Phi, C)$ is *defined* by the Cartan structure equation

$$
\bar{\boldsymbol{\Omega}} := \boldsymbol{d\varpi} + \tfrac{1}{2}[\boldsymbol{\varpi}, \boldsymbol{\varpi}]_{\mathfrak{g}}. \tag{230}
$$

The twisted connection allows to define a covariant derivative on twisted forms $\bar{\boldsymbol{D}} : \Omega^\bullet_{\text{eq}}(\Phi, C) \to \Omega^{\bullet+1}_{\text{tens}}(\Phi, C)$, $\alpha \mapsto \bar{\boldsymbol{D}}\boldsymbol{\alpha} = \boldsymbol{d\alpha} + \boldsymbol{\varpi}\,\boldsymbol{\alpha}$. In particular, $\bar{\boldsymbol{\Omega}}$ satisfies the Bianchi identity $\bar{\boldsymbol{D}}\bar{\boldsymbol{\Omega}} = \boldsymbol{d}\bar{\boldsymbol{\Omega}} + [\boldsymbol{\varpi}, \boldsymbol{\Omega}]_{\mathfrak{g}} = 0$.

The vertical transformation under $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty(\phi, \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}})$ of a twisted connection is

$$
\boldsymbol{\varpi}^{(\boldsymbol{\psi},\boldsymbol{\gamma})} := \Xi^\star\boldsymbol{\varpi} = \text{Ad}_{C(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}}\,\boldsymbol{\varpi} + C(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\boldsymbol{d}C(\boldsymbol{\psi},\boldsymbol{\gamma}). \tag{231}
$$

The infinitesimal version, under $\mathfrak{diff}_v(\Phi) \simeq C^\infty(\Phi, \mathfrak{diff}(M) \oplus \text{Lie}\mathcal{H}_{\text{loc}})$, is given by

$$
L_{(X,\lambda)^v}\,\boldsymbol{\varpi} = \boldsymbol{d}a(X,\lambda) + [\boldsymbol{\varpi}, a(X,\lambda)]_{\mathfrak{g}}. \tag{232}
$$

Finite and infinitesimal general vertical transformations of the curvature are given by,

$$
\bar{\boldsymbol{\Omega}}^{(\boldsymbol{\psi},\boldsymbol{\gamma})} := \Xi^\star\bar{\boldsymbol{\Omega}} = \text{Ad}_{C(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}}\,\bar{\boldsymbol{\Omega}}, \quad \text{so} \quad L_{(X,\lambda)^v}\bar{\boldsymbol{\Omega}} = [\bar{\boldsymbol{\Omega}}, a(X,\lambda)]_{\mathfrak{g}}. \tag{233}
$$

Twisted geometry features in an essential way into gRGFT, as it controls the phenomenon of $\mathcal{H}_{\text{loc}}$ and $\text{Diff}(M)$ *anomalies*.[63,64]

5.1.2. Associated Bundle of Regions and Integration Theory for Local Field Theory

Associated Bundle of Regions for Local Field Space: Given a representation space (ρ, V) for $\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$, and defining the right action

$$
\begin{aligned}
&(\Phi \times V) \times \big(\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\big) \to \Phi \times V,\\
&\qquad \big((\phi, v), (\psi,\gamma)\big) \mapsto \big(R_{(\psi,\gamma)}\phi, \rho(\psi,\gamma)^{-1}v\big) =: \bar{R}_{(\psi,\gamma)}(\phi, v),
\end{aligned}
\tag{234}
$$

one defines the associated bundle $\boldsymbol{E} := \Phi \times_\rho \boldsymbol{V} := \Phi \times \boldsymbol{V}/\sim$, with $\sim$ the equivalence relation under the right action $\bar{R}$. As usual one has the isomorphism of spaces

$$
\begin{aligned}
\Gamma(\boldsymbol{E}) := \big\{\boldsymbol{s} : \mathcal{M} \to \boldsymbol{E}\big\} \ &\simeq\ \Omega^0_{\text{eq}}(\Phi, \rho)\\
&:= \left\{\boldsymbol{\varphi} : \Phi \to \boldsymbol{V} \mid R^\star_{(\psi,\gamma)}\boldsymbol{\varphi} = \rho(\psi,\gamma)^{-1}\boldsymbol{\varphi}\right\},
\end{aligned}
\tag{235}
$$

The same construction holds when $\boldsymbol{V}$ is a G-space and ρ is replaced by a 1-cocycle $C : \Phi \times \big(\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\big) \to G$, $\big(\phi, (\psi,\gamma)\big) \mapsto C\left(\phi; (\psi,\gamma)\right)$, so that one may build twisted associated bundles $\tilde{E} := \Phi \times_C \boldsymbol{V}$ whose sections are twisted tensorial 0-forms.

One considers $\boldsymbol{V} = \boldsymbol{U}(M) := \{U \subset M \mid U \text{ open set}\}$, the σ-algebra of open sets of M, and the corresponding *associated bundle of regions* of M,

$$
\bar{\boldsymbol{U}}(M) = \Phi \times_{\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}} \boldsymbol{U}(M) := \Phi \times \boldsymbol{U}(M)/\sim \tag{236}
$$

where one the equivalence relation is under the right action

$$
\begin{aligned}
&(\Phi \times \boldsymbol{U}(M)) \times \big(\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\big) \to \Phi \times \boldsymbol{U}(M),\\
&\qquad \big((\phi, U), (\psi,\gamma)\big) \mapsto \big(R_{(\psi,\gamma)}\phi, \psi^{-1}(U)\big)\\
&\qquad\qquad = \big(\psi^*(\phi^\gamma), \psi^{-1}(U)\big)\\
&\qquad\qquad =: \bar{R}_{(\psi,\gamma)}(\phi, U).
\end{aligned}
\tag{237}
$$

As a special case of the above, we have

$$\Gamma(\bar{U}(M)) \coloneqq \left\{s : \mathcal{M} \to \bar{U}(M)\right\} \simeq \Omega^0_{\text{eq}}\big(\Phi, \boldsymbol{U}(M)\big) \coloneqq \left\{ \boldsymbol{U} : \Phi \to \boldsymbol{U}(M) \mid R^\star_{(\psi,\gamma)}\boldsymbol{U} = \psi^{-1}(\boldsymbol{U}) \right\}. \tag{238}$$

More explicitly, the equivariance can be written as

$$\boldsymbol{U}\big(\psi^*(\phi^\gamma)\big) = \psi^{-1}\big(\boldsymbol{U}(\phi)\big). \tag{239}$$

Such $\boldsymbol{U}$'s can be understood as field-dependent, or ϕ-relative, regions of M. Notice that, naturally, $\mathcal{H}_{\text{loc}}$ has trivial action on $\boldsymbol{U}(M)$ since M is defined as the space of fibers of P. So, $\boldsymbol{U}$'s are $\mathcal{H}_{\text{loc}}$-basic 0-forms on Φ, and project onto the $\text{Diff}(M)$-subbundle $\Phi' \subset \Phi$.

The transformations of these equivariant 0-forms under $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\big(\Phi, \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\big)$ and $\boldsymbol{\mathfrak{diff}}_v(\Phi) \simeq C^\infty\big(\Phi, \mathfrak{diff}(M) \oplus \text{Lie}\mathcal{H}_{\text{loc}}\big)$ are

$$\boldsymbol{U}^{(\boldsymbol{\psi},\boldsymbol{\gamma})} = \boldsymbol{\psi}^{-1}(\boldsymbol{U}), \quad \text{and} \quad \boldsymbol{L}_{(\boldsymbol{X},\boldsymbol{\lambda})^v} = -\boldsymbol{X}(\boldsymbol{U}). \tag{240}$$

In the following, we formulate integration on M as a natural operation involving the above objects.

Integration in Local Field Theory: For pedagogical benefit, let us start from general considerations before specializing to our case of interest. The action (234) induces the action

$$\begin{aligned}(\Phi \times V) \times C^\infty\big(\Phi, \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\big) &\to \Phi \times V,\\ ((\phi, v), (\boldsymbol{\psi}, \boldsymbol{\gamma})) &\mapsto \big(R_{(\boldsymbol{\psi},\boldsymbol{\gamma})}\phi,\ \rho(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}v\big)\\ &= \big(\Xi(\phi), \rho(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}v\big) \eqqcolon \bar{\Xi}(\phi, v).\end{aligned} \tag{241}$$

The corresponding linearization is: $(\Phi \times V) \times C^\infty\big(\Phi, \mathfrak{diff}(M) \oplus \text{Lie}\mathcal{H}_{\text{loc}}\big) \to V(\Phi \times V) \simeq V\Phi \oplus VV \subset T(P \times V)$. The induced actions of $\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$ and $C^\infty\big(\Phi, \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\big) \simeq \boldsymbol{Diff}_v(\Phi)$ on $\Omega^\bullet(\Phi) \times V$ are

$$\begin{aligned}(\Omega^\bullet(\Phi) \times V) \times \big(\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\big) &\to \Omega^\bullet(\Phi) \times V,\\ ((\boldsymbol{\alpha}, v), (\psi, \gamma)) &\mapsto \Big(R^\star_{(\psi,\gamma)}\boldsymbol{\alpha}, \rho(\psi,\gamma)^{-1}v\Big) \eqqcolon \tilde{R}_\psi(\boldsymbol{\alpha}, v)\end{aligned} \tag{242}$$

and

$$\begin{aligned}\big(\Omega^\bullet(\Phi) \times V\big) \times C^\infty\big(\Phi, \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\big) &\to \Omega^\bullet(\Phi) \times V,\\ \big((\boldsymbol{\alpha}, v), (\boldsymbol{\psi}, \boldsymbol{\gamma})\big) &\mapsto \big(\Xi^\star\boldsymbol{\alpha},\ \rho(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}v\big)\\ &\eqqcolon \tilde{\Xi}(\boldsymbol{\alpha}, v).\end{aligned} \tag{243}$$

The induced actions of $\mathfrak{diff}(M) \oplus \text{Lie}\mathcal{H}_{\text{loc}}$ and $C^\infty\big(\Phi, \mathfrak{diff}(M) \oplus \text{Lie}\mathcal{H}_{\text{loc}}\big) \simeq \boldsymbol{\mathfrak{diff}}_v(\Phi)$ are the linearizations:

$$\begin{aligned}\big((\boldsymbol{\alpha}, v), (X, \lambda)\big) &\mapsto \tfrac{d}{d\tau}\tilde{R}_{(\psi_\tau,\gamma_\tau)}(\boldsymbol{\alpha}, v)\Big|_{\tau=0}\\ &= \big(L_{(X,\lambda)^v}\boldsymbol{\alpha}, v\big) \oplus \big(\boldsymbol{\alpha}, -\rho_*(X,\lambda)v\big),\\ \big((\boldsymbol{\alpha}, v), (\boldsymbol{X}, \boldsymbol{\lambda})\big) &\mapsto \tfrac{d}{d\tau}\tilde{\Xi}_\tau(\boldsymbol{\alpha}, v)\Big|_{\tau=0}\\ &= \big(\boldsymbol{L}_{(\boldsymbol{X},\boldsymbol{\lambda})^v}\boldsymbol{\alpha}, v\big) \oplus \big(\boldsymbol{\alpha}, -\rho_*(\boldsymbol{X},\boldsymbol{\lambda})v\big).\end{aligned} \tag{244}$$

Given a representation space $(\tilde{\rho}, \boldsymbol{W})$ of $\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$,

$$\begin{aligned}&\text{if } \boldsymbol{\alpha} \in \Omega^\bullet_{\text{eq}}(\Phi, \boldsymbol{W}) \text{ then } \tilde{R}_\psi(\boldsymbol{\alpha}, v) = \Big(R^\star_{(\psi,\gamma)}\boldsymbol{\alpha}, \rho(\psi,\gamma)^{-1}v\Big)\\ &\qquad\qquad = \big(\tilde{\rho}(\psi,\gamma)^{-1}\boldsymbol{\alpha}, \rho(\psi,\gamma)^{-1}v\big),\\ &\text{if } \boldsymbol{\alpha} \in \Omega^\bullet_{\text{tens}}(\Phi, \boldsymbol{W}) \text{ then } \tilde{\Xi}(\boldsymbol{\alpha}, v) = \big(\Xi^\star\boldsymbol{\alpha}, \rho(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}v\big)\\ &\qquad\qquad = \big(\tilde{\rho}(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\boldsymbol{\alpha}, \rho(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}v\big),\end{aligned} \tag{245}$$

with linearizations being read from (244). The exterior derivative $\boldsymbol{d}$ on Φ extends to $\Phi \times V$ as $\boldsymbol{d} \to \boldsymbol{d} \times \text{id}$. But applying after the action of $C^\infty\big(\Phi, \mathfrak{diff}(M) \oplus \text{Lie}\mathcal{H}_{\text{loc}}\big) \simeq \boldsymbol{\mathfrak{diff}}_v(\Phi)$, due to the ϕ-dependence of $(\boldsymbol{\psi}, \boldsymbol{\gamma})$, it will act on the factor $\rho(\boldsymbol{\psi}, \boldsymbol{\gamma})^{-1}v$.

Consider $(\bar{\rho}, V^*)$ a representation of $\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$ *dual* to (ρ, V) w.r.t. a non-degenerate *pairing*

$$\begin{aligned}\langle\ ,\ \rangle : V^* \times V &\to \mathbb{R},\\ (w, v) &\mapsto \langle w, v\rangle.\end{aligned} \tag{246}$$

If it is $\big(\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\big)$-invariant, it satisfies

$$\begin{aligned}&\langle\, \bar{\rho}(\psi,\gamma)w, \rho(\psi,\gamma)v\rangle = \langle w, v\rangle,\\ \hookrightarrow\ &\langle\, \bar{\rho}_*(X,\lambda)w, v\rangle + \langle\, w, \rho_*(X,\lambda)v\rangle = 0,\end{aligned} \tag{247}$$

where the second line features the induced representation $\bar{\rho}_*$ and ρ_* for the action of $\mathfrak{diff}(M) \oplus \text{Lie}\mathcal{H}_{\text{loc}}$. For $\boldsymbol{\alpha} \in \Omega^\bullet(\Phi, V^*)$, we define an operation $\mathcal{I}$ on $\Omega^\bullet(\Phi, V^*) \times V$ by

$$\begin{aligned}\mathcal{I} : \Omega^\bullet(\Phi, V^*) \times V &\to \Omega^\bullet(\Phi),\\ (\boldsymbol{\alpha}, v) &\mapsto \mathcal{I}(\boldsymbol{\alpha}, v) \coloneqq \langle \boldsymbol{\alpha}, v\rangle,\end{aligned} \tag{248}$$

which can be understood as an object on $\Phi \times V$ by

$$\begin{aligned}\mathcal{I}(\boldsymbol{\alpha},\ ) : \Phi \times V &\to \Lambda^\bullet(\Phi),\\ (\phi, v) &\mapsto \mathcal{I}(\boldsymbol{\alpha}_{|\phi}, v) \coloneqq \langle \boldsymbol{\alpha}_{|\phi}, v\rangle.\end{aligned} \tag{249}$$

It is therefore the case that

$$\boldsymbol{d}\mathcal{I}(\boldsymbol{\alpha},\ ) = \mathcal{I}(\boldsymbol{d}\boldsymbol{\alpha},\ ) \quad \text{and} \quad \iota_{\mathfrak{X}}\mathcal{I}(\boldsymbol{\alpha},\ ) = \mathcal{I}(\iota_{\mathfrak{X}}\boldsymbol{\alpha},\ ), \quad \text{for } \mathfrak{X} \in \Gamma(T\Phi). \tag{250}$$

The induced actions of $\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$ and $C^\infty\big(\Phi, \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\big) \simeq \boldsymbol{Diff}_v(\Phi)$ on such objects are:

$$\begin{aligned}\tilde{R}^\star_\psi\mathcal{I}(\boldsymbol{\alpha},\ )_{|(R_{(\psi,\gamma)}\phi, \rho(\psi,\gamma)^{-1}v)} &\coloneqq \langle\ ,\ \rangle \circ \tilde{R}_{(\psi,\gamma)}(\boldsymbol{\alpha}, v)\\ &= \langle R^\star_{(\psi,\gamma)}\boldsymbol{\alpha}_{|\,R_{(\psi,\gamma)}\phi},\ \rho(\psi,\gamma)^{-1}v\rangle,\\ \tilde{\Xi}^\star\mathcal{I}(\boldsymbol{\alpha},\ )_{|(\Xi(\phi), \rho(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}v)} &\coloneqq \langle\ ,\ \rangle \circ \tilde{\Xi}(\boldsymbol{\alpha}, v) = \langle \Xi^\star\boldsymbol{\alpha}_{|\Xi(\phi)}, \rho(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}v\rangle.\end{aligned} \tag{251}$$

The actions of $\mathfrak{diff}(M) \oplus \mathrm{Lie}\mathcal{H}_{\mathrm{loc}}$ and $C^\infty(\Phi, \mathfrak{diff}(M) \oplus \mathrm{Lie}\mathcal{H}_{\mathrm{loc}}) \simeq \boldsymbol{\mathfrak{diff}}_v(\Phi)$ are

$$
\begin{aligned}
&\tfrac{d}{d\tau}\, \tilde{R}^\star_{(\psi_\tau, \gamma_\tau)} \mathcal{I}(\boldsymbol{\alpha}, \nu)\Big|_{\tau=0} = \langle L_{(X,\lambda)^v} \boldsymbol{\alpha},\, \nu\rangle + \langle \boldsymbol{\alpha},\, -\rho_*(X,\lambda)\nu\rangle, \\
&\tfrac{d}{d\tau}\, \tilde{\Xi}^\star_\tau \mathcal{I}(\boldsymbol{\alpha}, \nu)\Big|_{\tau=0} = \langle L_{(\boldsymbol{X},\boldsymbol{\lambda})^v} \boldsymbol{\alpha},\, \nu\rangle + \langle \boldsymbol{\alpha},\, -\rho_*(\boldsymbol{X},\boldsymbol{\lambda})\nu\rangle.
\end{aligned}
\tag{252}
$$

For $\boldsymbol{\alpha} \in \Omega^\bullet_{\mathrm{eq}}(\Phi, \boldsymbol{V}^*)$ we have

$$
\begin{aligned}
\tilde{R}^\star_{(\psi,\gamma)} \mathcal{I}(\boldsymbol{\alpha},\ )_{|\,(R_{(\psi,\gamma)}\phi,\, \rho(\psi,\gamma)^{-1}\nu)} &:= \langle R^\star_{(\psi,\gamma)} \boldsymbol{\alpha}_{|R_{(\psi,\gamma)}\phi},\, \rho(\psi,\gamma)^{-1}\nu\rangle \\
&= \langle \bar{\rho}(\psi,\gamma)^{-1} \boldsymbol{\alpha}_{|\phi},\, \rho(\psi,\gamma)^{-1}\nu\rangle, \\
\hookrightarrow\ \langle L_{(X,\lambda)^v}\boldsymbol{\alpha}, \nu\rangle + \langle \boldsymbol{\alpha}, -\rho_*(X,\lambda)\nu\rangle &= \langle -\bar{\rho}_*(X,\lambda)\boldsymbol{\alpha}, \nu\rangle \\
&\quad + \langle \boldsymbol{\alpha}, -\rho_*(X,\lambda)\nu\rangle.
\end{aligned}
\tag{253}
$$

If the pairing is invariant, by (247) we then get

$$
\begin{aligned}
&\tilde{R}^\star_{(\psi,\gamma)} \mathcal{I}(\boldsymbol{\alpha},\ \ )_{|\,(R_{(\psi,\gamma)}\phi,\ \rho(\psi,\gamma)^{-1}\nu)} = \langle \boldsymbol{\alpha}_{|\phi}, \nu\rangle =: \mathcal{I}(\boldsymbol{\alpha},\ \ )_{|(\phi,\nu)}, \\
&\hookrightarrow\quad \langle -\bar{\rho}_*(X,\lambda)\, \boldsymbol{\alpha},\ \nu\rangle + \langle \boldsymbol{\alpha},\ -\rho_*(X,\lambda)\nu\rangle = 0.
\end{aligned}
\tag{254}
$$

If $\boldsymbol{\alpha} \in \Omega^\bullet_{\mathrm{tens}}(\Phi, \boldsymbol{V}^*)$:

$$
\begin{aligned}
\tilde{\Xi}^\star \mathcal{I}(\boldsymbol{\alpha},\ )_{|(\Xi(\phi),\rho(\psi,\gamma)^{-1}\nu)} &:= \langle \Xi^\star \boldsymbol{\alpha}_{|\Xi(\phi)}, \rho(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\nu\rangle \\
&= \langle \bar{\rho}(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\boldsymbol{\alpha}_{|\phi}, \rho(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\nu\rangle, \\
\hookrightarrow\ \langle L_{(X,\lambda)^v}\boldsymbol{\alpha}, \nu\rangle + \langle \boldsymbol{\alpha}, -\rho_*(\boldsymbol{X},\boldsymbol{\lambda})\nu\rangle &= \langle -\bar{\rho}_*(\boldsymbol{X},\boldsymbol{\lambda})\boldsymbol{\alpha}, \nu\rangle \\
&\quad + \langle \boldsymbol{\alpha}, -\rho_*(\boldsymbol{X},\boldsymbol{\lambda})\nu\rangle.
\end{aligned}
\tag{255}
$$

If the pairing is invariant, by (247) we get

$$
\begin{aligned}
&\tilde{\Xi}^\star \mathcal{I}(\boldsymbol{\alpha},\ )_{|(\Xi(\phi),\ \rho(\psi,\gamma)^{-1}\nu)} = \langle \boldsymbol{\alpha}_{|\phi}, \nu\rangle =: \mathcal{I}(\boldsymbol{\alpha},\ )_{|(\phi,\nu)}, \\
&\hookrightarrow\quad \langle -\bar{\rho}_*(\boldsymbol{X},\boldsymbol{\lambda})\, \alpha,\ \nu\rangle + \langle \boldsymbol{\alpha},\ -\rho_*(\boldsymbol{X},\boldsymbol{\lambda})\nu\rangle = 0.
\end{aligned}
\tag{256}
$$

Whenever $\boldsymbol{\alpha} \in \Omega^\bullet_{\mathrm{tens}}(\Phi, \boldsymbol{V}^*)$ and the pairing is invariant, $\mathcal{I}(\boldsymbol{\alpha},\ )$ is then "basic" on $\Phi \times \boldsymbol{V}$, inducing a well-defined object on $\boldsymbol{E} = \Phi \times \boldsymbol{V}/\sim$. Being constant along a $(\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}})$-orbit in $\Phi \times \boldsymbol{V}$, $\mathcal{I}(\boldsymbol{\alpha},\ )$ allows to define $\boldsymbol{\varphi}_{\mathcal{I}(\alpha)} \in \Omega^0_{\mathrm{eq}}(\Phi, \rho)$ via

$$
\begin{aligned}
&\boldsymbol{\varphi}_{\mathcal{I}(\alpha)}(\phi) := \pi_V(\phi, \nu)_{|\mathcal{I}(\alpha_{|\phi},\nu)=\mathrm{cst}} \equiv \nu, \\
&\boldsymbol{\varphi}_{\mathcal{I}(\alpha)}\big(R_{(\psi,\gamma)}\phi\big) := \pi_V\big(R_{(\psi,\gamma)}\phi, \rho(\psi)^{-1}\nu\big)_{|\mathcal{I}(\alpha_{|\phi},\nu)=\mathrm{cst}} \equiv \rho(\psi,\gamma)^{-1}\nu.
\end{aligned}
\tag{257}
$$

As expressed by (235), the latter is equivalent to a section $\boldsymbol{s}_{\mathcal{I}(\alpha)} : \mathcal{M} \to \boldsymbol{E}$. By Equation (256) one can write $\boldsymbol{d}\tilde{\Xi}^\star \mathcal{I}(\boldsymbol{\alpha},\ ) = \boldsymbol{d}\mathcal{I}(\boldsymbol{\alpha},\ ) = \mathcal{I}(\boldsymbol{d\alpha},\ )$. In that case, one also finds the following lemma to hold:

$$
\begin{aligned}
\tilde{\Xi}^\star\langle \boldsymbol{d\alpha}, \nu\rangle &:= \langle \Xi^\star \boldsymbol{d\alpha}, \rho(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\nu\rangle = \langle \boldsymbol{d}\Xi^\star\boldsymbol{\alpha}, \rho(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\nu\rangle \\
&= \langle \boldsymbol{d}\big(\bar{\rho}(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\boldsymbol{\alpha}\big), \rho(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\nu\rangle \\
&= \big\langle \bar{\rho}(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\big(\boldsymbol{d\alpha} - \bar{\rho}_*\big(\boldsymbol{d}(\boldsymbol{\psi},\boldsymbol{\gamma}) \cdot (\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\big)\boldsymbol{\alpha}\big), \rho(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\nu\big\rangle \\
&= \langle \boldsymbol{d\alpha} - \bar{\rho}_*\big((\boldsymbol{d\psi},\boldsymbol{d\gamma}) \cdot (\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\big)\boldsymbol{\alpha}, \nu\rangle, \\
\hookrightarrow\ \tilde{\Xi}^\star\langle \boldsymbol{d\alpha}, \nu\rangle &= \langle \boldsymbol{d\alpha}, \nu\rangle + \langle -\bar{\rho}_*\big((\boldsymbol{d\psi},\boldsymbol{d\gamma}) \cdot (\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\big)\boldsymbol{\alpha}, \nu\rangle \\
&= \langle \boldsymbol{d\alpha}, \nu\rangle + \big\langle -\bar{\rho}_*\big(\boldsymbol{d\psi} \circ \boldsymbol{\psi}^{-1}, \boldsymbol{d\gamma\gamma}^{-1} - \boldsymbol{\gamma}\mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\gamma}^{-1}\big)\boldsymbol{\alpha}, \nu\big\rangle,
\end{aligned}
\tag{258}
$$

using the result (B19) in the last line. We will see shortly a useful application of this result.

The above construction specializes to the fundamental representation spaces $\boldsymbol{V} = \boldsymbol{U}(M)$ and $\boldsymbol{V}^* = \Omega^{\mathrm{top}}(U)$ (top forms on $U \in \boldsymbol{U}(M)$) for the action of $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$. They are dual under the *integration pairing*:

$$
\begin{aligned}
&\langle\ ,\ \rangle : \boldsymbol{\Omega}^{\mathrm{top}}(U) \times \boldsymbol{U}(M) \to \mathbb{R}, \\
&\qquad (\omega, U) \mapsto \langle \omega, U\rangle := \int_U \omega.
\end{aligned}
\tag{259}
$$

This pairing is invariant if and only if ω is $\mathcal{H}_{\mathrm{loc}}$-invariant, $\omega^\gamma = \omega$, since $\mathcal{H}_{\mathrm{loc}}$ does not act on $U \in \boldsymbol{U}(M)$. We have then the special case of (247), for $\bar{\rho}(\psi,\gamma)^{-1} = \psi^*$ and $\rho(\psi,\gamma)^{-1}(U) = \psi^{-1}(U)$, yielding the identity:

$$
\langle \psi^*\omega, \psi^{-1}(U)\rangle = \langle \omega, U\rangle \to \int_{\psi^{-1}(U)} \psi^*\omega = \int_U \omega,
\tag{260}
$$

reproducing the well-known $\mathrm{Diff}(M)$-invariance of integration as an intrinsic operation on M. This implies, for $-\bar{\rho}_*(X,\lambda) = \mathfrak{L}_X$ and $-\rho_*(X,\lambda)(U) = -X(U)$:

$$
\langle \mathfrak{L}_X\omega, U\rangle + \langle \omega, -X(U)\rangle = 0 \to \int_U \mathfrak{L}_X\omega + \int_{-X(U)} \omega = 0,
\tag{261}
$$

which can be understood as a sort of *continuity equation* for the action of $\mathfrak{diff}(M)$. By Stokes theorem, the de Rham derivative d on $\Omega^\bullet(U)$ and the boundary operator ∂ on $\boldsymbol{U}(M)$ are adjoint operators w.r.t. to the integration pairing:

$$
\langle d\omega, U\rangle = \langle \omega, \partial U\rangle \to \int_U d\omega = \int_{\partial U} \omega.
\tag{262}
$$

Considering $\boldsymbol{\alpha} \in \Omega^\bullet\big(\Phi, \Omega^{\mathrm{top}}(U)\big)$, the field-dependent top forms, we define the integration map on $\Phi \times \boldsymbol{U}(M)$:

$$
\mathcal{I}(\boldsymbol{\alpha}_{|\phi}, U) = \langle \boldsymbol{\alpha}_{|\phi}, U\rangle := \int_U \boldsymbol{\alpha}_{|\phi}.
\tag{263}
$$

We may use the notation $\boldsymbol{\alpha}_U$ when convenient. Equation (250) holds here as a special case. The induced actions of $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$ and $C^\infty\big(\Phi, \mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\big) \simeq \boldsymbol{Diff}_v(\Phi)$ on integrals are, respectively,

$$
\begin{aligned}
\tilde{R}^\star_{(\psi,\gamma)}\mathcal{I}(\boldsymbol{\alpha},\ )_{|(R_{(\psi,\gamma)}\phi,\psi^{-1}(U))} &:= \langle R^\star_{(\psi,\gamma)}\boldsymbol{\alpha}_{|R_{(\psi,\gamma)}\phi}, \psi^{-1}(U)\rangle \\
&= \int_{\psi^{-1}(U)} R^\star_{(\psi,\gamma)}\boldsymbol{\alpha}_{|\psi^*(\phi^\gamma)}, \\
\tilde{\Xi}^\star\mathcal{I}(\boldsymbol{\alpha},\ )_{|(\Xi(\phi),\boldsymbol{\psi}^{-1}(U))} &:= \langle \Xi^\star\boldsymbol{\alpha}_{|\Xi(\phi)}, \boldsymbol{\psi}^{-1}(U)\rangle = \int_{\boldsymbol{\psi}^{-1}(U)} \Xi^\star\boldsymbol{\alpha}_{|\boldsymbol{\psi}^*(\phi^\gamma)}.
\end{aligned}
\tag{264}
$$

We may write the above as $\boldsymbol{\alpha}_U{}^{(\psi,\gamma)}$ and $\boldsymbol{\alpha}_U{}^{(\boldsymbol{\psi},\boldsymbol{\gamma})}$, respectively. Applying $\boldsymbol{d}$ on the second line, it will also act on the transformed region $\boldsymbol{\psi}^{-1}(U)$ due to the ϕ-dependence of $\boldsymbol{\psi}$. One shows indeed that

$$\begin{aligned}\boldsymbol{d}\left(\bar{\Xi}^\star\langle\boldsymbol{\alpha},U\rangle\right)&=\langle\boldsymbol{d}\Xi^\star\boldsymbol{\alpha},\boldsymbol{\psi}^{-1}(U)\rangle+\langle\Xi^\star\boldsymbol{\alpha},\boldsymbol{d}\boldsymbol{\psi}^{-1}(U)\rangle\\&=\langle\Xi^\star\boldsymbol{d\alpha},\boldsymbol{\psi}^{-1}(U)\rangle+\langle\Xi^\star\boldsymbol{\alpha},-\boldsymbol{\psi}_*^{-1}\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}(U)\rangle\\&=\bar{\Xi}^\star\langle\boldsymbol{d\alpha},U\rangle-\langle\mathfrak{L}_{\boldsymbol{\psi}_*^{-1}\boldsymbol{d\psi}}\Xi^\star\boldsymbol{\alpha},\boldsymbol{\psi}^{-1}(U)\rangle,\end{aligned}$$

$$\text{or}\quad \boldsymbol{d}\left(\boldsymbol{\alpha}_U{}^{(\boldsymbol{\psi},\boldsymbol{\gamma})}\right)=\left(\boldsymbol{d\alpha}_U\right)^{(\boldsymbol{\psi},\boldsymbol{\gamma})}-\langle\mathfrak{L}_{\boldsymbol{\psi}_*^{-1}\boldsymbol{d\psi}}\Xi^\star\boldsymbol{\alpha},\boldsymbol{\psi}^{-1}(U)\rangle. \tag{265}$$

The identity (261) has been used to conclude. This means that on $\Phi\times\boldsymbol{U}(M)$ we have $[\bar{\boldsymbol{\Xi}}^\star,\boldsymbol{d}]\neq 0$ and the commutator is a boundary term, $\langle\iota_{\boldsymbol{\psi}_*^{-1}\boldsymbol{d\psi}}\Xi^\star\boldsymbol{\alpha},\partial\left(\boldsymbol{\psi}^{-1}(U)\right)\rangle$, since $\boldsymbol{\alpha}$ is a top form on U and by (262). This should be contrasted to the standard relation $[\boldsymbol{\Xi}^\star,\boldsymbol{d}]=0$ holding on Φ.

From (264) the induced actions of $\mathfrak{diff}(M)\oplus\text{Lie}\mathcal{H}_{\text{loc}}$ and $C^\infty\left(\Phi,\mathfrak{diff}(M)\oplus\text{Lie}\mathcal{H}_{\text{loc}}\right)\simeq\boldsymbol{\mathfrak{diff}}_v(\Phi)$ on integrals are

$$\begin{aligned}\tfrac{d}{d\tau}\tilde{R}^\star_{(\psi_\tau,\gamma_\tau)}\mathcal{I}(\boldsymbol{\alpha},U)\Big|_{\tau=0}&=\langle L_{(X,\lambda)^v}\boldsymbol{\alpha},U\rangle+\langle\boldsymbol{\alpha},-X(U)\rangle\\&=\int_U L_{(X,\lambda)^v}\boldsymbol{\alpha}+\int_{-X(U)}\boldsymbol{\alpha},\\\tfrac{d}{d\tau}\tilde{\Xi}^\star_\tau\mathcal{I}(\boldsymbol{\alpha},U)\Big|_{\tau=0}&=\langle\boldsymbol{L}_{(\boldsymbol{X},\boldsymbol{\lambda})^v}\boldsymbol{\alpha},U\rangle+\langle\boldsymbol{\alpha},-\boldsymbol{X}(U)\rangle\\&=\int_U\boldsymbol{L}_{(\boldsymbol{X},\boldsymbol{\lambda})^v}\boldsymbol{\alpha}+\int_{-\boldsymbol{X}(U)}\boldsymbol{\alpha}.\end{aligned}\tag{266}$$

If convenient, we may use the notation $\delta_{(X,\lambda)}\boldsymbol{\alpha}_U$ and $\delta_{(\boldsymbol{X},\boldsymbol{\lambda})}\boldsymbol{\alpha}_U$ for the above results.

We shall consider in particular equivariant and tensorial forms for the natural action of $\text{Diff}(M)\ltimes\mathcal{H}_{\text{loc}}$, satisfying respectively

$$\begin{aligned}&R^\star_{(\psi,\gamma)}\boldsymbol{\alpha}=\boldsymbol{\alpha}^{(\psi,\gamma)}=\bar{\rho}(\psi,\gamma)^{-1}\boldsymbol{\alpha}=\psi^*(\boldsymbol{\alpha}^\gamma),\\&L_{(X,\lambda)^v}\boldsymbol{\alpha}=-\bar{\rho}_*(X,\lambda)\boldsymbol{\alpha}=\mathfrak{L}_X\boldsymbol{\alpha}+\delta_\lambda\boldsymbol{\alpha},\\&\Xi^\star_{(\boldsymbol{\psi},\boldsymbol{\gamma})}\boldsymbol{\alpha}=\boldsymbol{\alpha}^{(\boldsymbol{\psi},\boldsymbol{\gamma})}=\bar{\rho}(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\boldsymbol{\alpha}=\boldsymbol{\psi}^*(\boldsymbol{\alpha}^{\boldsymbol{\gamma}}),\\&\boldsymbol{L}_{(\boldsymbol{X},\boldsymbol{\lambda})^v}\boldsymbol{\alpha}=-\bar{\rho}_*(\boldsymbol{X},\boldsymbol{\lambda})\boldsymbol{\alpha}=\mathfrak{L}_{\boldsymbol{X}}\boldsymbol{\alpha}+\delta_{\boldsymbol{\lambda}}\boldsymbol{\alpha}.\end{aligned}\tag{267}$$

Then we have that

$$d\left(\boldsymbol{\alpha}_U{}^{(\boldsymbol{\psi},\boldsymbol{\gamma})}\right)=\boldsymbol{d}\langle\boldsymbol{\psi}^*(\boldsymbol{\alpha}^{\boldsymbol{\gamma}}),\boldsymbol{\psi}^{-1}(U)\rangle=\boldsymbol{d}\langle\boldsymbol{\alpha}^{\boldsymbol{\gamma}},U\rangle=\langle\boldsymbol{d}(\boldsymbol{\alpha}^{\boldsymbol{\gamma}}),U\rangle. \tag{268}$$

And the relation (265) specializes to

$$\begin{aligned}\boldsymbol{d}\left(\boldsymbol{\alpha}_U{}^{(\boldsymbol{\psi},\boldsymbol{\gamma})}\right)&=\left(\boldsymbol{d\alpha}_U\right)^{(\boldsymbol{\psi},\boldsymbol{\gamma})}-\langle\mathfrak{L}_{\boldsymbol{\psi}_*^{-1}\boldsymbol{d\psi}}\,\boldsymbol{\psi}^*(\boldsymbol{\alpha}^{\boldsymbol{\gamma}}),\boldsymbol{\psi}^{-1}(U)\rangle\\&=\left(\boldsymbol{d\alpha}_U\right)^{(\boldsymbol{\psi},\boldsymbol{\gamma})}-\langle\boldsymbol{\psi}^*\mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}(\boldsymbol{\alpha}^{\boldsymbol{\gamma}}),\boldsymbol{\psi}^{-1}(U)\rangle\\&=\left(\boldsymbol{d\alpha}_U\right)^{(\boldsymbol{\psi},\boldsymbol{\gamma})}-\langle\mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}(\boldsymbol{\alpha}^{\boldsymbol{\gamma}}),U\rangle.\end{aligned}\tag{269}$$

Taken together the last two results imply to derive the following lemma:

$$\boldsymbol{d}(\boldsymbol{\psi}^*\boldsymbol{\alpha})=\boldsymbol{\psi}^*\left(\boldsymbol{d\alpha}+\mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\alpha}\right). \tag{270}$$

When $\boldsymbol{\alpha}$ is $\text{Diff}(M)$-tensorial *and* $\mathcal{H}_{\text{loc}}$-basic on Φ – i.e., $\boldsymbol{\alpha}^{\boldsymbol{\gamma}}=\boldsymbol{\alpha}$, and $\delta_{\boldsymbol{\lambda}}\boldsymbol{\alpha}=0$ – Equation (256) holds and we have that $\boldsymbol{\alpha}_U=\mathcal{I}(\boldsymbol{\alpha}_{|\phi},U)$ is $C^\infty\left(\Phi,\text{Diff}(M)\ltimes\mathcal{H}_{\text{loc}}\right)$-invariant: $\boldsymbol{\alpha}_U{}^{(\boldsymbol{\psi},\boldsymbol{\gamma})}=\boldsymbol{\alpha}_U$, and

$$\begin{aligned}\delta_{(\boldsymbol{X},\boldsymbol{\lambda})}\boldsymbol{\alpha}_U=0\Rightarrow&\langle\boldsymbol{L}_{(\boldsymbol{X},\boldsymbol{\lambda})^v}\boldsymbol{\alpha},U\rangle+\langle\boldsymbol{\alpha},-\boldsymbol{X}(U)\rangle=0,\\&\langle\mathfrak{L}_{\boldsymbol{X}}\alpha,U\rangle+\langle\boldsymbol{\alpha},-\boldsymbol{X}(U)\rangle=0\\&\to\int_U\mathfrak{L}_{\boldsymbol{X}}\alpha+\int_{-\boldsymbol{X}(U)}\boldsymbol{\alpha}=0.\end{aligned}\tag{271}$$

For $\boldsymbol{\alpha}$ $\text{Diff}(M)$-equivariant only, and $\mathcal{H}_{\text{loc}}$-basic, we have invariance of its integral: $\boldsymbol{\alpha}_U{}^{(\psi,\gamma)}=\boldsymbol{\alpha}_U$, and (271) holds substituting $(\boldsymbol{X},\boldsymbol{\lambda})\to(X,\lambda)$.

For $\boldsymbol{\alpha}$ $\text{Diff}(M)$-tensorial *and* $\mathcal{H}_{\text{loc}}$-basic, we thus have that $\boldsymbol{\alpha}_U=\mathcal{I}(\boldsymbol{\alpha},U)$ induces a well-defined object on the bundle of regions $\bar{\boldsymbol{U}}(M)=\Phi\times\boldsymbol{U}(M)/\!\sim$ (so one may define an associated equivariant $\boldsymbol{U}(M)$-valued function on Φ). It also holds that $\boldsymbol{d}\left(\boldsymbol{\alpha}_U{}^{(\boldsymbol{\psi},\boldsymbol{\gamma})}\right)=\boldsymbol{d\alpha}_U$, i.e.,

$$\begin{aligned}\boldsymbol{d}\langle\boldsymbol{\alpha}^{(\boldsymbol{\psi},\boldsymbol{\gamma})},\boldsymbol{\psi}^{-1}(U)\rangle\;&=\langle\boldsymbol{d\alpha},U\rangle=\boldsymbol{d}\langle\boldsymbol{\alpha},U\rangle\\&\to\boldsymbol{d}\int_{\boldsymbol{\psi}^{-1}(U)}\boldsymbol{\psi}^*\boldsymbol{\alpha}=\int_U\boldsymbol{d\alpha}=\boldsymbol{d}\int_U\boldsymbol{\alpha}.\end{aligned}\tag{272}$$

Besides, specializing (258), we get the identities

$$\begin{aligned}(\boldsymbol{d\alpha}_U)^{(\boldsymbol{\psi},\boldsymbol{\gamma})}&=\boldsymbol{d\alpha}_U+\langle\mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\alpha},U\rangle,\\\langle\boldsymbol{d\alpha},U\rangle^{(\boldsymbol{\psi},\boldsymbol{\gamma})}&=\langle\boldsymbol{d\alpha},U\rangle+\langle\mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\alpha},U\rangle.\end{aligned}\tag{273}$$

As we now show, all this is relevant to local gRGFT and the variational principle. When $\boldsymbol{\alpha}$ is a $\mathcal{H}_{\text{loc}}$-invariant Lagrangian 0-form on Φ and top form on $(U\subset)\,M$, equations (272)–(273) yield a *well-defined variational principle*, meaning that the space of solutions is preserved under $C^\infty\left(\Phi,\text{Diff}(M)\ltimes\mathcal{H}_{\text{loc}}\right)\simeq\boldsymbol{Diff}_v(\Phi)$ transformations.

5.1.3. Lagrangian, Action and Geometric Prescription for the Variational Principle in Field Theory

Lagrangian, Action and Anomalies: A gRGFT over a region $U\subset M$ is usually given by a choice of Lagrangian form $L:\Phi\to\Omega^n(U)$, $\phi\mapsto L(\phi)$, with $n=\dim M$.[20] A priori, it may support a non-trivial action of $\text{Diff}(M)\ltimes\mathcal{H}_{\text{loc}}$, so that

$$R^\star_{(\psi,\gamma)}L=\psi^*(L^\gamma)=\psi^*\left(L+c(\,;\gamma)\right), \tag{274}$$

where $c:\Phi\times\mathcal{H}_{\text{loc}}\to\Omega^n(U)$ is a $\mathcal{H}_{\text{loc}}$-1-cocycle (with value in an additive Abelian group) satisfying indeed $c(\phi;\gamma\gamma')=c(\phi;\gamma)+c(\phi^\gamma;\gamma')$. It is automatic, following from the definition $c(\,;\gamma)\coloneqq R^\star_\gamma L-L=L^\gamma-L$. Defining similarly the $\text{Diff}(M)$-1-cocycle $c:\Phi\times\text{Diff}(M)\to\Omega^n(U)$, $(\phi,\psi)\mapsto c(\phi;\psi)\coloneqq R^\star_\psi L-L=$

[20] Or an element of the d-cohomology class of L, as they give rise to the same field equations, as we are about to see. Only the presymplectic potential of the theory differs between members of the same class, which may have relevant consequences regarding the symplectic structures of the theories, and their quantization. We may write $L=L'+d\ell$, with ℓ the so-called "boundary Lagrangian".

$\psi^* L - L$, one finds the total $(\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}})$-1-cocycle to be

$$c\big(\phi; (\psi, \gamma)\big) := R^\star_{(\psi,\gamma)} L - L = c(\phi; \psi) + \psi^* c(\phi; \gamma). \tag{275}$$

Linearizing this 1-cocycle we get the $\big(\mathfrak{diff}(M) \oplus \mathrm{Lie}\mathcal{H}\big)$-1-cocycle $a : \Phi \times \big(\mathfrak{diff}(M) \oplus \mathrm{Lie}\mathcal{H}\big) \to \Omega^n(U)$:

$$a((X, \lambda);\ ) := \tfrac{d}{d\tau} c(\ ; (\psi_\tau, \gamma_\tau))\Big|_{\tau=0} = \tfrac{d}{d\tau}\Big(R^\star_{(\psi_\tau,\gamma_\tau)} L - L\Big)\Big|_{\tau=0} = \boldsymbol{L}_{(X,\lambda)^v} L. \tag{276}$$

In other words, a Lagrangian is a priori a twisted tensorial form, $L \in \Omega^0_{\text{tens}}\big(\Phi, \Omega^n(U)\big)$. One then finds that

$$\begin{aligned}
\boldsymbol{L}_{(X,\lambda)^v} L = \tfrac{d}{d\tau} c\big(\ ; (\psi_\tau, \gamma_\tau)\big)\Big|_{\tau=0} &= \tfrac{d}{d\tau}\Big(c(\ ; \psi_\tau) + \tfrac{d}{d\tau}\psi_\tau^* c(\ ; \gamma_\tau)\Big)\Big|_{\tau=0} \\
&= a(X;\ ) + \mathfrak{L}_X c(\ ; \gamma_{\tau=0}) + \psi^*_{\tau=0} a(\lambda;\ ) \\
&= a(X;\ ) + a(\lambda;\ ) \\
&= \mathfrak{L}_X L + \delta_\lambda L = a\big((X, \lambda);\ \big).
\end{aligned} \tag{277}$$

We used the fact that $\gamma_{\tau=0} = \mathrm{id}_{\mathcal{H}_{\mathrm{loc}}}$, $\psi_{\tau=0} = \mathrm{id}_M$, so $c(\ ; \mathrm{id}_{\mathcal{H}_{\mathrm{loc}}}) = 0$. The term $a\left((X, \lambda);\ \right)$ is what one may call the *classical* $(\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}})$-anomaly. As a special case of (199), it satisfies the identity

$$\begin{aligned}
&(X, \lambda)^v a\big((X', \lambda'); \phi\big) - \big(X', \lambda'\big)^v a((X, \lambda); \phi) \\
&\quad = a\big(\big[(X, \lambda), (X', \lambda')\big]_{\mathrm{Lie}}; \phi\big),
\end{aligned} \tag{278}$$

which is the classical analogue of the "Wess-Zumino consistency condition", and contains the combined consistency conditions for both the $\mathrm{Diff}(M)$-anomaly $a(X; \phi)$ and the $\mathcal{H}_{\mathrm{loc}}$-anomaly $a(\lambda; \phi)$. Indeed, we remind that $(X, \lambda)^v = X^v + \lambda^v$ and that the Lie bracket in $\mathfrak{diff}(M) \oplus \mathrm{Lie}\mathcal{H}_{\mathrm{loc}}$ is (180), so (278) splits as:

$$X^v a(X'; \phi) - X'^v a(X; \phi) = a\big([X, X']_{\mathfrak{diff}(M)}; \phi\big) \tag{279}$$

$$+ \qquad\qquad +$$

$$\lambda^v a(\lambda'; \phi) - \lambda'^v a(\lambda; \phi) = a\big([\lambda, \lambda']_{\mathrm{Lie}H}; \phi\big) \tag{280}$$

$$+ \qquad\qquad +$$

$$\begin{aligned}
&X^v\, a(\lambda'; \phi) + \lambda^v a(X'; \phi) = a\big(-X'(\lambda) + X(\lambda'); \phi\big). \\
&\quad - X'^v\, a(\lambda; \phi) - \lambda'^v\, a(X; \phi)
\end{aligned} \tag{281}$$

The line (279) is the $\mathrm{Diff}(M)$-anomaly consistency condition, (280) is that of the $\mathcal{H}_{\mathrm{loc}}$-anomaly, while (281) is the mutual consistency condition between the two, reflecting (again) the semi-direct structure of $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$.

Remembering that L has to be a top form on M (so $dL \equiv 0$), and requiring that the field equations be preserved under $\mathcal{H}_{\mathrm{loc}}$-transformations (i.e., imposing $\delta_\lambda L = d\beta(\lambda;\ )$), one may write

$$\begin{aligned}
\boldsymbol{L}_{(X,\lambda)^v} L = a((X, \lambda);\ ) = a(X;\ ) + a(\lambda;\ ) &= \mathfrak{L}_X L + \delta_\lambda L \\
&= d\left(\iota_X L + \beta(\lambda;\ )\right) =: d\beta((X, \lambda);\ ).
\end{aligned} \tag{282}$$

This requirement is essential for a consistent treatment via covariant phase space methods, as we shall see elsewhere.

The Lagrangian being tensorial, its $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty(\Phi, \mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}})$-transformation is thus easily found to be

$$L^{(\boldsymbol{\psi},\boldsymbol{\gamma})} := \Xi^\star L = \boldsymbol{\psi}^*(L^{\boldsymbol{\gamma}}), \tag{283}$$

while its $\mathfrak{diff}_v(\Phi) \simeq C^\infty(\Phi, \mathfrak{diff}(M) \oplus \mathrm{Lie}\mathcal{H}_{\mathrm{loc}})$-transformation is

$$\boldsymbol{L}_{(\boldsymbol{X},\boldsymbol{\lambda})^v} L = a\big((\boldsymbol{X}, \boldsymbol{\lambda}); \phi\big) = \mathfrak{L}_{\boldsymbol{X}} L + \delta_{\boldsymbol{\lambda}} L. \tag{284}$$

The anomaly for ϕ-dependent parameters satisfies, as a special case of (208), the consistency condition

$$\begin{aligned}
&(\boldsymbol{X}, \boldsymbol{\lambda})^v a\big((\boldsymbol{X}', \boldsymbol{\lambda}'); \phi\big) - \big(\boldsymbol{X}', \boldsymbol{\lambda}'\big)^v a\big((\boldsymbol{X}, \boldsymbol{\lambda}); \phi\big) \\
&\quad = a\big(\big\{(\boldsymbol{X}, \boldsymbol{\lambda}), (\boldsymbol{X}', \boldsymbol{\lambda}')\big\}; \phi\big),
\end{aligned} \tag{285}$$

which, using the bracket of $C^\infty\left(\Phi, \mathfrak{diff}(M) \oplus \mathrm{Lie}\mathcal{H}_{\mathrm{loc}}\right)$ featuring in (191), can be shown to reduce to (278).

The action functional over U is the map

$$\begin{aligned}
&S : \Omega^0\big(\Phi, \Omega^n(U)\big) \times \bar{U}(M) \to \Omega^0(\Phi), \\
&\qquad (L, U) \mapsto S = \langle L, U\rangle := \int_U L.
\end{aligned} \tag{286}$$

It is always $\mathrm{Diff}(M)$-invariant, by (260), so there can be no $\mathrm{Diff}(M)$-anomaly for the action. A priori, there may be a $\mathcal{H}_{\mathrm{loc}}$-anomaly: By (264), the action of $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$ on the action is

$$\begin{aligned}
S^{(\psi,\gamma)} &:= \tilde{R}^\star_{(\psi,\gamma)} S = \int_{\psi^{-1}(U)} R^\star_{(\psi,\gamma)} L = \int_{\psi^{-1}(U)} \psi^*(L^\gamma) \\
&= \int_U L + c(\ ; \gamma) =: S + \mathsf{c}(\ ; \gamma).
\end{aligned} \tag{287}$$

The term $\mathsf{c} : \Phi \times \mathcal{H}_{\mathrm{loc}} \to \mathbb{R}$, $(\phi, \gamma) \mapsto \mathsf{c}(\phi; \gamma)$, is a $\mathcal{H}_{\mathrm{loc}}$-1-cocycle. It is the classical counterpart of the so-called WZ term, or yet "integrated anomaly", as its linearization is

$$\mathsf{a}(\lambda; \phi) := \tfrac{d}{d\tau}\mathsf{c}(\phi; \gamma_\tau)\Big|_{\tau=0} = \int_U \tfrac{d}{d\tau} c(\phi; \gamma_\tau)\Big|_{\tau=0} =: \int_U a(\lambda; \phi), \tag{288}$$

featuring the $\mathcal{H}_{\mathrm{loc}}$-anomaly. Naturally, it satisfies the consistency condition

$$\lambda^v\, \mathsf{a}(\lambda'; \phi) - \lambda'^v\, \mathsf{a}(\lambda; \phi) = \mathsf{a}\big([\lambda, \lambda']_{\mathrm{Lie}H}; \phi\big) \tag{289}$$

analogue to the non-integrated one (280). One may write the linearization of (287), by (266) and (261):

$$\begin{aligned}
\delta_{(X,\lambda)} S &= \int_U \boldsymbol{L}_{(X,\lambda)^v} L + \int_{-X(U)} L = \int_U \delta_\lambda L + \mathfrak{L}_X L \\
&\quad + \int_{-X(U)} L = \int_U \delta_\lambda L = \int_U a(\lambda; \phi) = \mathsf{a}(\lambda; \phi),
\end{aligned} \tag{290}$$

which shows indeed that there is no classical $\mathrm{Diff}(M)$-anomaly. Observe that, since $\mathcal{H}_{\mathrm{loc}}$ does not act on $\bar{U}(M)$, by (250) it holds that: $\delta_{(X,\lambda)} S = \boldsymbol{L}_{\lambda^v} S = \iota_{\lambda^v} \boldsymbol{d} S$.

The action can thus be seen as a twisted tensorial object. The action of $C^\infty\left(\Phi, \mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\right)$ is, by (264),

$$S^{(\boldsymbol{\psi},\boldsymbol{\gamma})} := \tilde{\Xi}^\star S = \int_{\boldsymbol{\psi}^{-1}(U)} \Xi^* L = \int_U L + c(\ ;\boldsymbol{\gamma}) =: S + \mathsf{c}(\ ;\boldsymbol{\gamma}), \tag{291}$$

while the action of $C^\infty\left(\Phi, \mathfrak{diff}(M) \oplus \mathrm{Lie}\mathcal{H}_{\mathrm{loc}}\right)$ is, by (266)

$$\delta_{(X,\lambda)} S = \int_U \boldsymbol{L}_{(X,\lambda)^v} L + \int_{-\boldsymbol{X}(U)} L = \int_U \delta_\lambda L = \mathsf{a}(\lambda;\phi). \tag{292}$$

Again, one has that $\delta_{(X,\lambda)} S = \boldsymbol{L}_{\lambda^v} S = \iota_{\lambda^v} \boldsymbol{d}S$. One may thus define the twisted tensorial form $Z \in \Omega^0_{\mathrm{tens}}(\Phi, C)$,

$$\begin{aligned} &Z := \exp i\, S, \quad \text{ s.t. } \quad Z^{(\psi,\gamma)} = C(\ ;\gamma)^{-1} Z, \quad \text{ with} \\ &C(\ ;\gamma) := \exp\{-i\,\mathsf{c}(\ ;\gamma)\}, \end{aligned} \tag{293}$$

where $C : \Phi \times \mathcal{H}_{\mathrm{loc}} \to U(1)$ is a $\mathcal{H}_{\mathrm{loc}}$-1-cocycle, with phase the WZ term. As a twisted object, it must be acted upon via a twisted covariant derivative, $\bar{\boldsymbol{D}} := \boldsymbol{d}\ + \boldsymbol{\varpi}$, where, by (228), the twisted connection satisfies:

$$\begin{aligned} &\boldsymbol{\varpi}\left((X,\lambda)^v\right) = \tfrac{d}{d\tau} C(\phi;\gamma_\tau)\Big|_{\tau=0} = -i\,\mathsf{a}(\lambda;\phi) \in \mathfrak{u}(1) = i\,\mathbb{R}, \\ &\quad R^\star_{(\psi,\gamma)} \boldsymbol{\varpi} = \boldsymbol{\varpi} + C(\ ;\gamma)^{-1} \boldsymbol{d}C(\ ;\gamma) = \boldsymbol{\varpi} - i\boldsymbol{d}\mathsf{c}(\ ;\gamma). \end{aligned} \tag{294}$$

The horizontality of the curvature $\bar{\boldsymbol{\Omega}} = \boldsymbol{d\varpi} \in \Omega^2_{\mathrm{tens}}(\Phi, C)$, i.e., $\bar{\boldsymbol{\Omega}}(\lambda^v, \lambda'^v) \equiv 0$, encodes/reproduces the consistency condition (289). It is by definition the case that $\bar{\boldsymbol{D}} Z \in \Omega^1_{\mathrm{tens}}(\Phi, C)$, but then since

$$\bar{\boldsymbol{D}} Z = \boldsymbol{d} Z + \boldsymbol{\varpi} Z = (i\boldsymbol{d}S + \boldsymbol{\varpi}) Z,$$

$$\text{it follows that} \quad i\boldsymbol{d}S + \boldsymbol{\varpi} \in \Omega^1_{\mathrm{basic}}(\Phi), \tag{295}$$

because Z carries the (twisted) equivariance. The object (295), that one may write $\boldsymbol{d}S - i\boldsymbol{\varpi}$, is what one may call a “generalized WZ improved action”: Indeed, the usual WZ improved action is recovered as a special case for flat twisted connections $\boldsymbol{\varpi}_0$, which are necessarily written in terms of dressing fields, as we will see in the next section. The WS trick is then explained as coming from twisted covariant derivation.

When there is no $\mathcal{H}_{\mathrm{loc}}$-anomaly, i.e., when S is $\left(\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\right)$-invariant, it induces a well-defined section on the bundle of regions $\bar{\boldsymbol{U}}(M) = \Phi \times \boldsymbol{U}(M)/\!\sim$. It thus equivalently defines a $(\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}})$-equivariant (tensorial) $\bar{\boldsymbol{U}}(M)$-valued 0-form,

$$\begin{aligned} &\boldsymbol{U}_S(\phi) := \pi_U(\phi, U)_{|\,S=\mathrm{cst}} \equiv U, \\ &\boldsymbol{U}_S\left(\psi^*(\phi^\gamma)\right) := \pi_U\left(\psi^*(\phi^\gamma), \psi^{-1}(U)\right)_{|\,S=\mathrm{cst}} \equiv \psi^{-1}(U). \end{aligned} \tag{296}$$

Its $C^\infty\left(\Phi, \mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\right)$-transformation is then

$$\boldsymbol{U}_S^{(\boldsymbol{\psi},\boldsymbol{\gamma})} = \boldsymbol{\psi}^{-1}(\boldsymbol{U}_S), \quad \text{so} \quad \boldsymbol{L}_{(X,\lambda)^v} \boldsymbol{U}_S = -\boldsymbol{X}(\boldsymbol{U}_S). \tag{297}$$

The map $\boldsymbol{U}_S$ we may interpret as defining $\mathrm{Diff}(M)$-equivariant “ϕ-relative” regions of M. This may be understood as a first formal step towards be the concrete translation of the conceptual insight, exposed in Section 2, that theories implementing the symmetries of gRGFT (of GR in particular) – i.e., enforcing principles of epistemic democracy – have a relational definition of spacetime: that is, define spacetime regions relative to the d.o.f. of their fields content.[21] It is of course only half the full picture, which requires to see how fields d.o.f. relationally co-define each other. A more complete formal relational picture is provided in Section 5.2.

We remark that a quantum theory is also a priori a twisted tensorial object, even for an invariant action S:

$$\mathsf{Z} := \int \delta\phi\, \exp \tfrac{i}{\hbar} S, \quad \text{s.t.} \quad \mathsf{Z}^{(\psi,\gamma)} = C\left(\ ;(\psi,\gamma)\right)^{-1} \mathsf{Z}, \tag{298}$$

with $\delta\phi$ is an integration measure on Φ, and where $C : \Phi \times \left(\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\right) \to U(1)$ is a 1-cocycle, whose linearization is the the combined $\left(\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\right)$-anomaly. It is the transformation of the measure $\delta\phi$ that may generate a non-zero cocycle. Therefore, a $\mathrm{Diff}(M)$-anomaly potentially occurs only in quantum field theories. The twisted covariant derivative adapted to $\mathsf{Z} \in \Omega^0_{\mathrm{tens}}(\Phi, C)$ is $\bar{\boldsymbol{D}}\mathsf{Z} = \boldsymbol{d}\mathsf{Z} + \boldsymbol{\varpi}\mathsf{Z} \in \Omega^1_{\mathrm{tens}}(\Phi, C)$, with twisted connection satisfying

$$\begin{aligned} &\boldsymbol{\varpi}\left((X,\lambda)^v\right) = \tfrac{d}{d\tau} C\left(\phi;(\psi_\tau,\gamma_\tau)\right)\Big|_{\tau=0} = -\tfrac{i}{\hbar}\mathsf{a}((X,\lambda);\phi) \in \mathfrak{u}(1) = i\,\mathbb{R}, \\ &\quad R^\star_{(\psi,\gamma)} \boldsymbol{\varpi} = \boldsymbol{\varpi} + C\left(\ ;(X,\lambda)\right)^{-1} \boldsymbol{d}C\left(\ ;(X,\lambda)\right). \end{aligned} \tag{299}$$

The horizontality of the curvature $\bar{\boldsymbol{\Omega}} = \boldsymbol{d\varpi} \in \Omega^2_{\mathrm{tens}}(\Phi, C)$, i.e., $\bar{\boldsymbol{\Omega}}\left((X,\lambda)^v, (X',\lambda'^v)\right) \equiv 0$, encodes/reproduces the WZ consistency condition of the combined $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$ quantum anomaly, directly analogous to (278). We may observe that acting only with $\boldsymbol{d}$ on Z is a well-defined geometric operation only if $\mathsf{Z} \in \Omega^0_{\mathrm{basic}}(\Phi)$ – i.e., in the absence of quantum anomaly – as then $\boldsymbol{d}$ is indeed a covariant derivative on basic forms, so that $\boldsymbol{d}\mathsf{Z} \in \Omega^1_{\mathrm{basic}}(\Phi)$.

Variational Principle in Local Field Theory: We want to write the variational principle for a gRGFT, and examine how the objects involved, notably the field equations, transform under the action of $C^\infty\left(\Phi, \mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\right)$. We write the variational principle as

$$\begin{aligned} \boldsymbol{d}S_{|\phi} = 0 \quad \text{with} \quad \boldsymbol{d}L_{|\phi} &= \boldsymbol{E}_{|\phi} + d\boldsymbol{\theta}_{|\phi} \\ &= E(\boldsymbol{d}\phi;\phi) + d\theta(\boldsymbol{d}\phi;\phi), \end{aligned} \tag{300}$$

with $\boldsymbol{E} \in \Omega^1(\Phi, \Omega^n(U))$ the field equations 1-form, and $\boldsymbol{\theta} \in \Omega^1(\Phi, \Omega^{n-1}(U))$ the so-called “presymplectic potential” of the theory. Remember that in $\boldsymbol{d}S$, the derivative $\boldsymbol{d}$ is extended from Φ to $\Phi \times \boldsymbol{U}(M)$ by (250). The space of solutions is defined as $\mathcal{S} := \left\{\phi \in \Phi \mid \boldsymbol{E}_{|\phi} = 0\right\}$. Our goal is to assess its stability under the action of $\boldsymbol{Diff}_v(\Phi)$.

[21] It is only a first step because, as we have shown, M is not spacetime. Still, $\boldsymbol{U}_S$ formally expresses, through its $\mathrm{Diff}(M)$-equivariance, the covariant relation between field d.o.f. and physical points/regions of spacetime. We may call manifolds defined via the set of values (open sets) of objects like $\boldsymbol{U}_S$, “*manifields*”.

The equivariance and verticality properties of $\boldsymbol{d}L$ are easily found. By, (274) we have

$$R^\star_{(\psi,\gamma)}\boldsymbol{d}L = \boldsymbol{d}R^\star_{(\psi,\gamma)}L = \boldsymbol{d}\,\psi^*\big(L + c(\ ;\gamma)\big) = \psi^*\big(\boldsymbol{d}L + \boldsymbol{d}c(\ ;\gamma)\big). \tag{301}$$

The verticality property of $\boldsymbol{d}L$ is just another way to see the infinitesimal equivariance of L, given by (282), so:

$$\begin{aligned}\iota_{(X,\lambda)^v}\boldsymbol{d}L &= a\big((X,\lambda);\ \big) = a(X;\ ) + a(\lambda;\ )\\ &= d\,\beta\big((X,\lambda);\ \big) = d\left(\iota_X L + \beta(\lambda;\ )\right).\end{aligned} \tag{302}$$

This allows to compute geometrically the $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\right)$-transformation of $\boldsymbol{d}L$, using (195):

$$\begin{aligned}\boldsymbol{d}L^{(\psi,\gamma)}_{|\phi}(\mathfrak{X}_{|\phi}) &:= \Xi^\star\boldsymbol{d}L_{|\phi}(\mathfrak{X}_{|\phi}) = \boldsymbol{d}L_{|\,\Xi(\phi)}\left(\Xi_\star\mathfrak{X}_{|\phi}\right)\\
&= \boldsymbol{d}L_{|\,\Xi(\phi)}\left(R_{(\psi,\gamma)\star}\mathfrak{X}_{|\phi} + \left\{\left(\boldsymbol{\psi}_*^{-1}\boldsymbol{d\psi},\ \boldsymbol{\psi}^*\left(\boldsymbol{\gamma}^{-1}\boldsymbol{d\gamma}\right)\right)_{|\phi}(\mathfrak{X}_{|\phi})\right\}^v_{|\,\Xi(\phi)}\right)\\
&= R^\star_{(\psi,\gamma)}\boldsymbol{d}L_{|\,\Xi(\phi)}(\mathfrak{X}_{|\phi})\\
&\quad + \boldsymbol{d}L_{\Xi(\phi)}\left(\left\{\left(\boldsymbol{\psi}_*^{-1}\boldsymbol{d\psi},\ \boldsymbol{\psi}^*\left(\boldsymbol{\gamma}^{-1}\boldsymbol{d\gamma}\right)\right)_{|\phi}(\mathfrak{X}_{|\phi})\right\}^v_{|\,\Xi(\phi)}\right)\\
&= \boldsymbol{\psi}(\phi)*\big(\boldsymbol{d}L + \boldsymbol{d}c\left(\ ;\boldsymbol{\gamma}(\phi)\right)\big)_{|\phi}(\mathfrak{X}_{|\phi})\\
&\quad + \left[a\left(\boldsymbol{\psi}_*^{-1}\boldsymbol{d\psi};\ \Xi(\phi)\right) + a\left(\boldsymbol{\psi}^*(\boldsymbol{\gamma}^{-1}\boldsymbol{d\gamma});\ \Xi(\phi)\right)\right]_{|\phi}(\mathfrak{X}_{|\phi}).\end{aligned} \tag{303}$$

The second term is readily found to be

$$\begin{aligned}a\big(\boldsymbol{\psi}_*^{-1}\boldsymbol{d\psi};\ \Xi(\phi)\big) &= \mathfrak{L}_{\boldsymbol{\psi}_*^{-1}\boldsymbol{d\psi}}L\big(\boldsymbol{\psi}^*(\phi^{\boldsymbol{\gamma}})\big) = \mathfrak{L}_{\boldsymbol{\psi}_*^{-1}\boldsymbol{d\psi}}\boldsymbol{\psi}^*L(\phi^{\boldsymbol{\gamma}})\\
&= \boldsymbol{\psi}^*\mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}L(\phi^{\boldsymbol{\gamma}})\\
&= \boldsymbol{\psi}^*\big(\mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\left(L + c(\ ;\boldsymbol{\gamma})\right)\big).\end{aligned} \tag{304}$$

The last term is a bit more subtle: using the definition of the infinitesimal cocycle a and the defining property of c,

$$\begin{aligned}&a\big(\boldsymbol{\psi}^*(\boldsymbol{\gamma}^{-1}\boldsymbol{d\gamma})_{|\phi}(\mathfrak{X}_{|\phi});\ \Xi(\phi)\big)\\
&= \boldsymbol{\psi}^*\,a\big(\boldsymbol{\gamma}^{-1}\boldsymbol{d\gamma}_{|\phi}(\mathfrak{X}_{|\phi});\ \phi^{\boldsymbol{\gamma}}\big)\\
&= \boldsymbol{\psi}^*\,\tfrac{d}{d\tau}\,c\big(\phi^{\boldsymbol{\gamma}};\ \exp\left\{\tau\left(\boldsymbol{\gamma}^{-1}\boldsymbol{d\gamma}\right)_{|\phi}(\mathfrak{X}_{|\phi})\right\}\big)\Big|_{\tau=0}\\
&= \boldsymbol{\psi}^*\,\tfrac{d}{d\tau}\,c\big(\phi^{\boldsymbol{\gamma}};\ \boldsymbol{\gamma}(\phi)^{-1}\exp\left\{\tau\left(\boldsymbol{d\gamma\gamma}^{-1}\right)_{|\phi}(\mathfrak{X}_{|\phi})\right\}\boldsymbol{\gamma}(\phi)\big)\Big|_{\tau=0}\\
&= \boldsymbol{\psi}^*\,\tfrac{d}{d\tau}\,c\big(\phi;\ \exp\left\{\tau\left(\boldsymbol{d\gamma\gamma}^{-1}\right)_{|\phi}(\mathfrak{X}_{|\phi})\right\}\boldsymbol{\gamma}(\phi)\big) - c\big(\phi;\boldsymbol{\gamma}(\phi)\big)\Big|_{\tau=0}\\
&= \boldsymbol{\psi}^*\,\boldsymbol{d}c(\phi;\ )_{|\boldsymbol{\gamma}(\phi)}\circ\boldsymbol{d\gamma}_{|\phi}(\mathfrak{X}_{|\phi})\\
&= \boldsymbol{\psi}^*\,\boldsymbol{d}c(\phi;\boldsymbol{\gamma})_{|\phi}(\mathfrak{X}_{|\phi}).\end{aligned} \tag{305}$$

Gathering all results, we get

$$\begin{aligned}\boldsymbol{d}L^{(\psi,\gamma)}_{|\phi}(\mathfrak{X}_{|\phi}) &= \boldsymbol{\psi}(\phi)^*\left(\boldsymbol{d}L_{|\phi} + \boldsymbol{d}c\left(\ ;\boldsymbol{\gamma}(\phi)\right)_{|\phi} + \boldsymbol{d}c(\phi;\boldsymbol{\gamma})_{|\phi}\right.\\
&\quad\left. + \mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\left(L + c(\ ;\boldsymbol{\gamma})\right)_{|\phi}\right)(\mathfrak{X}_{|\phi})\\
&= \boldsymbol{\psi}(\phi)^*\left(\boldsymbol{d}L_{|\phi} + \boldsymbol{d}c(\ ;\boldsymbol{\gamma})_{|\phi}\right.\\
&\quad\left. + \mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\left(L + c(\ ;\boldsymbol{\gamma})\right)_{|\phi}\right)(\mathfrak{X}_{|\phi}).\end{aligned} \tag{306}$$

Stating the result as valid $\forall\phi \in \Phi$ and $\forall\mathfrak{X} \in \Gamma(T\Phi)$, this is

$$\boldsymbol{d}L^{(\psi,\gamma)} = \boldsymbol{\psi}^*\big(\boldsymbol{d}L + \boldsymbol{d}c(\ ;\boldsymbol{\gamma}) + \mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\left(L + c(\ ;\boldsymbol{\gamma})\right)\big). \tag{307}$$

This can be cross-checked by using $[\Xi^\star, \boldsymbol{d}] = 0$ and computing $\boldsymbol{d}\,\Xi^\star L = \boldsymbol{d}(L^{(\psi,\gamma)})$. This generalizes Equation (233) in ref. [8].

We observe that for the $\left(\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\right)$-transformed Lagrangian $L^{(\psi,\gamma)} = R^\star_{(\psi,\gamma)}L$ to have the same field equations as L, it must be the case that

$$\boldsymbol{d}c(\phi;\ \gamma) = d\boldsymbol{b}(\phi;\ \gamma), \tag{308}$$

where $\boldsymbol{b}(\phi;\gamma) = b\,(\boldsymbol{d}\phi;\phi,\gamma)$ is linear in $\boldsymbol{d}\phi$. This is a separate hypothesis, that must be stressed as such. It is realized for example in the case of 3D Chern–Simons theory, where $c(A;\gamma) = \mathrm{Tr}\left(d(\gamma d\gamma^{-1}A) - \frac{1}{3}(\gamma^{-1}d\gamma)^3\right)$ so that $\boldsymbol{b}(A;\gamma) = \mathrm{Tr}\left(\gamma d\gamma^{-1}\boldsymbol{d}A\right)$. Under hypothesis (308), the result (307) gives

$$\begin{aligned}\boldsymbol{d}L^{(\psi,\gamma)} &= \boldsymbol{\psi}^*\big(\boldsymbol{d}L + d\boldsymbol{b}(\ ;\boldsymbol{\gamma}) + \mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\left(L + c(\ ;\boldsymbol{\gamma})\right)\big)\\
&= \boldsymbol{\psi}^*\big(\boldsymbol{d}L + d\left[\boldsymbol{b}(\ ;\boldsymbol{\gamma}) + \iota_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}L + \iota_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}c(\ ;\boldsymbol{\gamma})\right]\big),\end{aligned} \tag{309}$$

where we use the fact that L and $c(\ ;\gamma)$ are top forms on $U \subset M$. Hence, $\boldsymbol{d}L$ transforms under $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\right)$ by a boundary term.

By (264), we find the $C^\infty\left(\Phi, \mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\right)$-transformation of $\boldsymbol{d}S$ to be

$$\boldsymbol{d}S^{(\psi,\gamma)} = \int_{\boldsymbol{\psi}^{-1}(U)}\boldsymbol{d}L^{(\psi,\gamma)} = \int_U \boldsymbol{d}L + \boldsymbol{d}c(\ ;\boldsymbol{\gamma}) + \mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\left(L + c(\ ;\boldsymbol{\gamma})\right). \tag{310}$$

Under the hypotesis (308), this gives

$$\begin{aligned}\boldsymbol{d}S^{(\psi,\gamma)} &= \int_U \boldsymbol{d}L + d\left[\boldsymbol{b}(\ ;\boldsymbol{\gamma}) + \iota_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}L + \iota_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}c(\ ;\boldsymbol{\gamma})\right]\\
&= \boldsymbol{d}S + \int_{\partial U}\boldsymbol{b}(\ ;\boldsymbol{\gamma}) + \iota_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}L + \iota_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}c(\ ;\boldsymbol{\gamma}).\end{aligned} \tag{311}$$

We see that if L is $\mathcal{H}_{\mathrm{loc}}$-invariant, so that $c = 0 = \boldsymbol{b}$, this result is indeed a special case of (273). The fact that $\boldsymbol{d}L$, and $\boldsymbol{d}S$, transform with a boundary term hint at the fact that the variational principle remains well-defined under field-dependent transformations, and that the space of solutions is preserved.

The fact of the latter matter can be assessed by directly computing the transformation of the field equations $\boldsymbol{E}$. For this, we need both the equivariance and verticality properties of $\boldsymbol{E}$. By (300)–(301), we get

$$R^\star_{(\psi,\gamma)}\boldsymbol{d}L = R^\star_{(\psi,\gamma)}\boldsymbol{E} + R^\star_{(\psi,\gamma)}d\boldsymbol{\theta} = \psi^*\left(\boldsymbol{E} + d\boldsymbol{\theta} + \boldsymbol{d}c(\ ;\gamma)\right). \tag{312}$$

As things stand, we would not know how to split the $\mathcal{H}_{\mathrm{loc}}$-equivariance of $\boldsymbol{d}L$ as contributions coming from that of $\boldsymbol{E}$ and $d\boldsymbol{\theta}$. There are only two options to consider. The first would be that $\boldsymbol{E}$ contributes non-trivially, alongside $\boldsymbol{\theta}$: In such cases, the action of $\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$ does not preserve the space of solutions $\mathcal{S}$ (a $\mathcal{H}_{\text{loc}}$-transformation moves us "off-shell"). As this assumption explicitly contravenes the foundational principles of gRGFT as reminded in Section 2, we reject it.[22] The only option compatible with gRGFT is that $\boldsymbol{E}$ is $\mathcal{H}_{\text{loc}}$-invariant. This then implies that (308) holds, a condition thus revealed to be essential to gRGFT. We therefore have the equivariance

$$R^{\star}_{(\psi,\gamma)}\boldsymbol{E} = \psi^{*}\boldsymbol{E} \quad \text{and} \quad R^{\star}_{(\psi,\gamma)}\boldsymbol{\theta} = \psi^{*}\big(\boldsymbol{\theta} + \boldsymbol{b}(\ \gamma)\big). \tag{313}$$

We need now only find the verticality property of $\boldsymbol{E}$. First, from (282) and (300) we get:

$$\begin{aligned} \iota_{(X,\lambda)^v}\boldsymbol{E} &= d\big(\beta\big((X,\lambda);\phi\big) - \iota_{(X,\lambda)^v}\boldsymbol{\theta}\big) \\ \iota_{X^v}\boldsymbol{E} \ + \ \iota_{\lambda^v}\boldsymbol{E} &= d\big(\iota_X L - \iota_{X^v}\boldsymbol{\theta}\big) \ + \ d\big(\beta(\lambda;\phi) - \iota_{\lambda^v}\boldsymbol{\theta}\big). \end{aligned} \tag{314}$$

We may use these expressions to compute $\boldsymbol{E}^{(\psi,\gamma)}$, but the result would not be conceptually enlightening. It turns out to be possible to write each of the two contributions on the r.h.s. above in terms of the functional expression of $\boldsymbol{E}$ itself. To see this requires of us to have a closer look at how (300) is obtained.

Since we are considering general-relativistic gauge field theories, we remind that our space of fields is made of $\phi = \{A, b\}$, where $A = A_{\text{YM}} + A_{\text{Cartan}}$ with A_{YM} the local representative of a Ehresmann connection (i.e., a Yang-Mills gauge potential) and A_{Cartan} the local representative of a Cartan connection (i.e., a gravitational gauge potential), and $b = \{\varphi, F, \dots\}$ are $\mathcal{H}_{\text{loc}}$-tensorial fields, with φ a matter field and $F = F_{\text{YM}} + F_{\text{Cartan}}$ the Yang-Mills and gravitational field strengths. We have $DA = F$ and $Db = db + \rho_*(A)b$. Since $DF = 0$, the Bianchi identity, and $DDb = \rho_*(F)b$, one shows that $D^{2p}b = \rho_*(F^p)b$ and $D^{2p+1}b = \rho_*(F^p)Db$. Thus, $\{\phi, D\phi\}$ is an algebraically closed set of variables under the action of D; we may write this $D\{\phi\} \subset \{\phi\}$. One easily shows that $\mathcal{L}_X A = D(\iota_X A) + \iota_X F$ and $\mathcal{L}_X b = [\iota_X, D]b - \rho_*(\iota_X A)b$, so we may write the generic formula

$$\iota_{X^v}\boldsymbol{d}\phi = \mathcal{L}_X\phi = \iota_X D\phi + D(\iota_X\phi) - \rho_*(\iota_X A)\phi. \tag{315}$$

The Lagrangian L in gRGFT must be (or in most relevant cases, is) built from an invariant polynomial P on $\text{Lie}H$ and representation V of H – in the latter case P is usually simply a H-invariant bilinear form on V: i.e., it is s.t. $P\big(\rho(h)v_1, \dots, \rho(h)v_i, \dots\big) = P\big(v_1, \dots, v_i, \dots\big)$ for $h \in H$ and v_i $\text{Lie}H$- and/or V-valued variables ($\rho = \text{Ad}$ in the former case). More symbolically, $P \circ \rho(H) = P$.[23] By linearizing, this implies $\Sigma_i P(\dots, \rho_*(\text{Lie}H)v_i, \dots) = 0$, and the identity $\Sigma_i (-)^{p(k_1 + \dots + k_{i-1})} P\big(\dots, \rho_*(\eta)v_i, \dots\big) = 0$, for η a $\text{Lie}H$-valued p-form and the variables v_i are now k_i-forms.

Now, we may assume the Lagrangian to have the form $L(\phi) = \tilde{L}(\phi; D\phi)$ – meaning it is defined on $J^1\Phi$, the 1st jet bundle of field space. Then we have,

$$\boldsymbol{d}L_{|\phi} = \tilde{L}_0(\boldsymbol{d}\phi; \{\phi\}) + \tilde{L}_1(\boldsymbol{d}D\phi; \{\phi\}), \tag{316}$$

where $\{\phi\}$ means the collection of remaining ϕ and $D\phi$ in the respective functional expressions $\tilde{L}_{0/1}$, which are linear in their first argument. All functional expressions L, $\tilde{L}$ and $\tilde{L}_{0/1}$ are built from the H-invariant polynomial P. Using that $\boldsymbol{d}D\phi = D(\boldsymbol{d}\phi) + \rho_*(\boldsymbol{d}A)\phi$ and integrating by parts, we obtain the formal expressions of the field equation and presymplectic potential:

$$\begin{aligned} \boldsymbol{d}L_{|\phi} &= \tilde{L}_0\big(\boldsymbol{d}\phi; \{\phi\}\big) + \tilde{L}_1\big(D(\boldsymbol{d}\phi) + \rho_*(\boldsymbol{d}A)\phi; \{\phi\}\big), \\ &= \tilde{L}_0\big(\boldsymbol{d}\phi; \{\phi\}\big) + d\tilde{L}_1\big(\boldsymbol{d}\phi; \{\phi\}\big) - (-)^{|\phi|}\tilde{L}_1\big(\boldsymbol{d}\phi; d\{\phi\}\big) \\ &\quad + \tilde{L}_1\big(\rho_*(A)\boldsymbol{d}\phi; \{\phi\}\big) + \tilde{L}_1\big(\rho_*(\boldsymbol{d}A)\phi; \{\phi\}\big) \\ &= \tilde{L}_0\big(\boldsymbol{d}\phi; \{\phi\}\big) + d\tilde{L}_1\big(\boldsymbol{d}\phi; \{\phi\}\big) - (-)^{|\phi|}\tilde{L}_1\big(\boldsymbol{d}\phi; d\{\phi\}\big) \\ &\quad - (-)^{|\phi|}\tilde{L}_1\big(\boldsymbol{d}\phi; \rho_*(A)\{\phi\}\big) + \tilde{L}_1\big(\rho_*(\boldsymbol{d}A)\phi; \{\phi\}\big) \\ &= \tilde{L}_0\big(\boldsymbol{d}\phi; \{\phi\}\big) - (-)^{|\phi|}\tilde{L}_1\big(\boldsymbol{d}\phi; D\{\phi\}\big) \\ &\quad + \tilde{L}_1\big(\rho_*(\boldsymbol{d}A)\phi; \{\phi\}\big) + d\tilde{L}_1\big(\boldsymbol{d}\phi; \{\phi\}\big) \\ &=: E(\boldsymbol{d}\phi; \phi) + d\theta(\boldsymbol{d}\phi; \phi) \\ &= \boldsymbol{E}_{|\phi} + d\boldsymbol{\theta}_{|\phi}, \end{aligned} \tag{317}$$

where we denote the form degree $|\boldsymbol{d}\phi| = |\phi|$, and we used the identity $\tilde{L}_1\big(\rho_*(A)\boldsymbol{d}\phi; \{\phi\}\big) + (-)^{|\phi|}\tilde{L}_1\big(\boldsymbol{d}\phi; \rho_*(A)\{\phi\}\big) = 0$ stemming from the fact that $\tilde{L}_1$ is built from P.

By a similar formal computation, and using (315), we get the expression of the evaluation of the Lagrangian n-form on a vector field $X \in \Gamma(TM) \simeq \mathfrak{diff}(M)$, acting as a derivation (like $\boldsymbol{d}$):

$$\begin{aligned} \iota_X L(\phi) &= \iota_X \tilde{L}(\phi; D\phi) \\ &= \tilde{L}_0\big(\iota_X\phi; \{\phi\}\big) + \tilde{L}_1\big(\iota_X D\phi; \{\phi\}\big), \\ &= \tilde{L}_0\big(\iota_X\phi; \{\phi\}\big) + \tilde{L}_1\big(\iota_{X^v}\boldsymbol{d}\phi; \{\phi\}\big) \\ &\quad - \tilde{L}_1\big(D(\iota_X\phi); \{\phi\}\big) + \tilde{L}_1\big(\rho_*(\iota_X A)\phi; \{\phi\}\big), \\ &= \tilde{L}_0\big(\iota_X\phi; \{\phi\}\big) + \tilde{L}_1(\iota_{X^v}\boldsymbol{d}\phi; \{\phi\}) - d\tilde{L}_1(\iota_X\phi; \{\phi\}) \\ &\quad + (-)^{|\iota_X\phi|}\tilde{L}_1(\iota_X\phi; D\{\phi\}) + \tilde{L}_1\big(\rho_*(\iota_X A)\phi; \{\phi\}\big) \\ &= \tilde{L}_0\big(\iota_X\phi; \{\phi\}\big) - (-)^{|\phi|}\tilde{L}_1(\iota_X\phi; D\{\phi\}) + \tilde{L}_1\big(\rho_*(\iota_X A)\phi; \{\phi\}\big) \\ &\quad + \tilde{L}_1(\iota_{X^v}\boldsymbol{d}\phi; \{\phi\}) - d\tilde{L}_1(\iota_X\phi; \{\phi\}) \\ &= E(\iota_X\phi; \phi) + \iota_{X^v}\boldsymbol{\theta} - d\theta(\iota_X\phi; \phi). \end{aligned} \tag{318}$$

From this we obtain $\iota_X L - \iota_{X^v}\boldsymbol{\theta} = E(\iota_X\phi; \phi) - d\theta(\iota_X\phi; \phi)$, so that we get the first contribution in (314)

$$\iota_{X^v}\boldsymbol{E} = dE(\iota_X\phi; \phi). \tag{319}$$

There remains only to find $\iota_{\lambda^v}\boldsymbol{E}$. The most straightforward way to do so is to attempt to compute it formally from the expressiond derived in (317). Using the relevant part in (210), i.e., $\iota_{\lambda^v}\boldsymbol{d}\phi = \delta_\lambda\phi$,

[22] We remark that the extreme case where only $\boldsymbol{E}$ contributes, and $\boldsymbol{\theta}$ is invariant, is realized e.g., in the case of Massive Yang-Mills theory. See appendix E in ref. [65].

[23] This means that a Lagrangian $L = P(b)$ would be $\mathcal{H}_{\text{loc}}$-invariant: a prototypical example is YM theory, $L_{\text{YM}}(\phi) = P(F, * F) = \text{Tr}(F * F)$. Such is not always the case, e.g., 3D Chern–simons theory is $L_{\text{CS}}(\phi) = P(A, F) = \text{Tr}(AF - 1/3A^3)$. Remark that in both cases $P = \text{Tr}$.

considering further that $\delta_\lambda \phi = (\delta_\lambda A, \delta_\lambda b) = (D\lambda, -\rho_*(\lambda) b)$, and integrating by part as above, one may be easily convinced that

$$\begin{aligned} \iota_{\lambda^v} \boldsymbol{E} &= d\big(\tilde{L}_0\big(\lambda;\{\phi\}\big) - (-)^{|\phi|}\tilde{L}_1(\lambda; D\{\phi\}) + \tilde{L}_1\big(\rho_*(\lambda)\phi;\{\phi\}\big)\big) \\ &\quad + \tilde{L}_2\big(\lambda;\{\phi\}\big) \\ &= dE(\lambda;\phi) + \tilde{L}_2\big(\lambda;\{\phi\}\big), \end{aligned} \tag{320}$$

where $\tilde{L}_2$ is a functional expression (depending on $\tilde{L}_0$ and $\tilde{L}_1$) linear in its first argument, an *underived* λ. But since from more general considerations we have that $\iota_{\lambda^v}\boldsymbol{E} = d\left(\beta(\lambda;\phi) - \iota_{\lambda^v}\boldsymbol{\theta}\right)$, i.e., $\iota_{\lambda^v}\boldsymbol{E}$ is d-exact, it must be that $\tilde{L}_2 \equiv 0$. So we get the result

$$\iota_{\lambda^v}\boldsymbol{E} = dE(\lambda;\phi). \tag{321}$$

Finally, we thus get to write the verticality property of $\boldsymbol{E}$:

$$\iota_{(X,\lambda)^v}\boldsymbol{E} = dE\big(\iota_X\phi;\phi\big) + dE(\lambda;\phi) =: dE\big((\iota_X\phi,\lambda);\phi\big). \tag{322}$$

If one fails to be convinced by the above argumentation, one may take (322), especially (321), as an hypothesis or working assumption that needs to be checked in specific theories whenever one wishes to apply/use the results presented hereafter.

The $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\right)$-transformation of $\boldsymbol{E}$ is then easily found to be, using (195):

$$\begin{aligned} \boldsymbol{E}^{(\boldsymbol{\psi},\boldsymbol{\gamma})}_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi}) &:= \Xi^\star \boldsymbol{E}_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi}) = \boldsymbol{E}_{|\Xi(\phi)}\big(\Xi_\star \boldsymbol{\mathfrak{X}}_{|\phi}\big) \\ &= \boldsymbol{E}_{|\Xi(\phi)}\left(R_{(\boldsymbol{\psi},\boldsymbol{\gamma})\star}\left[\boldsymbol{\mathfrak{X}}_{|\phi} + \left\{\left(\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}, \boldsymbol{d\gamma\gamma}^{-1} - \boldsymbol{\gamma}\mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\gamma}^{-1}\right)_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi})\right\}^v_{|\phi}\right]\right) \\ &= R^\star_{(\boldsymbol{\psi},\boldsymbol{\gamma})}\boldsymbol{E}_{|\Xi(\phi)}\left(\boldsymbol{\mathfrak{X}}_{|\phi} + \left\{\left(\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}, \boldsymbol{d\gamma\gamma}^{-1} - \boldsymbol{\gamma}\mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\gamma}^{-1}\right)_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi})\right\}^v_{|\phi}\right) \\ &= \boldsymbol{\psi}^*\boldsymbol{E}_{|\phi}\left(\boldsymbol{\mathfrak{X}}_{|\phi} + \left\{\left(\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}, \boldsymbol{d\gamma\gamma}^{-1} - \boldsymbol{\gamma}\mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\gamma}^{-1}\right)_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi})\right\}^v_{|\phi}\right) \\ &= \boldsymbol{\psi}^*\boldsymbol{E}_{|\phi}\big(\boldsymbol{\mathfrak{X}}_{|\phi}\big) + \boldsymbol{\psi}^* dE\Big(\iota_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi})}\phi;\phi\Big) + \boldsymbol{\psi}^* dE\Big(\big(\boldsymbol{d\gamma\gamma}^{-1} - \boldsymbol{\gamma}\mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\gamma}^{-1}\big)_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi});\phi\Big). \end{aligned} \tag{323}$$

Hence, written as valid at all points $\phi \in \Phi$ and $\forall \boldsymbol{\mathfrak{X}} \in \Gamma(T\Phi)$, this is

$$\begin{aligned} \boldsymbol{E}^{(\boldsymbol{\psi},\boldsymbol{\gamma})} = \boldsymbol{\psi}^*\big(&\boldsymbol{E} + dE\big(\iota_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\phi;\phi\big) \\ &+ dE\left(\boldsymbol{d\gamma}\,\boldsymbol{\gamma}^{-1} - \boldsymbol{\gamma}\,\mathfrak{L}_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\gamma}^{-1};\phi\right)\big). \end{aligned} \tag{324}$$

This is one of the most important equations of this paper. It shows that the space of solutions $\mathcal{S}$ of a general-relativistic gauge theory is stable under field-dependent transformations: $\mathcal{S}$ is fibered by the action of $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\right)$, and hence a principal bundle in its own right. Which means that gRGFT admits a much bigger set of symmetries than initially assumed ($\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$) and encoded in its foundational principles. To the best of our knowledge, this fact was first noticed in the case of GR by refs. [21, 66], and later by ref. [22], where metric-dependent diffeomorphisms were considered.[24] The result (324) specializes to

$$\begin{aligned} \boldsymbol{E}^{\boldsymbol{\psi}} &= \boldsymbol{\psi}^*\big(\boldsymbol{E} + dE\big(\iota_{\boldsymbol{d\psi}\circ\boldsymbol{\psi}^{-1}}\phi;\phi\big)\big), \\ \boldsymbol{E}^{\boldsymbol{\gamma}} &= \boldsymbol{E} + dE\big(\boldsymbol{d\gamma}\,\boldsymbol{\gamma}^{-1};\phi\big), \end{aligned} \tag{325}$$

which reproduce and/or generalize respectively e.g., Equation (233) in ref. [8], and Equation (103) in ref. [6] and Equation (74) in ref. [42].

By the same logic, from (313) and using (318) and (321) to find the verticality property if $\boldsymbol{\theta}$, one may obtain its $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\right)$-transformation – showing explicitly that $\boldsymbol{d}L^{(\boldsymbol{\psi},\boldsymbol{\gamma})} = \boldsymbol{E}^{(\boldsymbol{\psi},\boldsymbol{\gamma})} + \boldsymbol{d\theta}^{(\boldsymbol{\psi},\boldsymbol{\gamma})}$. This result, as well as the transformation of the symplectic 2-form $\boldsymbol{\Theta} := \boldsymbol{d\theta}$, are relevant to the covariant phase space analysis of gRGFT. A study we postpone to a forthcoming part of this series, which will recover and expand on previous works such as refs. [6, 8, 42, 67, 68]. For now, our focus is on the relational reformulation of local gRGFT via dressing.

5.2. Relational Formulation via Dressing

The DFM for local theory parallels the treatment of the global case detailed in Section 4. As the local case has many interesting specific features, we again provide as much details as needed to get the clearest conceptual picture.

5.2.1. Building Basic Forms on Local Field Space

We remind that $Q \to N$ is the reference bundle with base N featuring in the global DFM. We define the space of $(\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}})$-dressing fields as

[24] And where instances of the FN bracket for metric-dependent vector fields was computed heuristically. See Equations (3.1)–(3.2) in ref. [21] and Equation (2.1) in ref. [22].

$$\begin{aligned}\mathcal{D}r[N;M,H] \coloneqq \big\{&(\upsilon,\mathrm{u}) \text{ with } \upsilon: N\to M \text{ and } \mathrm{u}: M\to H \mid \\ &(\upsilon,\mathrm{u})^{(\psi,\gamma)} \coloneqq \big(\psi^{-1}\circ\upsilon,\ \psi^*(\mathrm{u}^\gamma)\big) \\ &\coloneqq \big(\psi^{-1}\circ\upsilon,\ \psi^*(\gamma^{-1}\mathrm{u})\big) = (\psi,\gamma)^{-1}\cdot(\upsilon,\mathrm{u})\big\},\end{aligned} \tag{326}$$

where $(\psi,\gamma)^{-1} = (\psi^{-1}, \psi^*\gamma^{-1})$, and the group product law (179) (/(A12)) is used (and extended) to write the last equality in the defining $(\mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}})$-transformation of the dressing field (υ,u). The linearization of which is

$$\delta_{(X,\lambda)}(\upsilon,\mathrm{u}) = \big(-X\circ\upsilon,\ \mathfrak{L}_X\mathrm{u} - \lambda\mathrm{u}\big). \tag{327}$$

We will usually refer to (υ,u) as *a* dressing field, despite it being a pair with υ a $\mathrm{Diff}(M)$-dressing field as defined in ref. [8] and u a $\mathcal{H}_{\mathrm{loc}}$-dressing field as originally defined in refs. [4–6].

We define the space of dressed fields $\Phi^{(\upsilon,\mathrm{u})}$ by

$$\begin{aligned} |^u : \Phi &\to \Phi^{(\upsilon,\mathrm{u})}, \\ \phi &\mapsto \phi^{(\upsilon,\mathrm{u})} \coloneqq \upsilon^*(\phi^{\mathrm{u}}).\end{aligned} \tag{328}$$

We refer to $\phi^{(\upsilon,\mathrm{u})}$ as a dressed field, or the dressing of ϕ. The notation ϕ^{u} means the same functional expression as the $\mathcal{H}_{\mathrm{loc}}$-transformation of ϕ, i.e., ϕ^γ, but with $\gamma\in\mathcal{H}_{\mathrm{loc}}$ replaced by u. For example, the $\mathcal{H}_{\mathrm{loc}}$-dressing of a gauge potential A is $A^{\mathrm{u}} \coloneqq \mathrm{u}^{-1}A\mathrm{u} + \mathrm{u}^{-1}d\mathrm{u}$. The $\mathcal{H}_{\mathrm{loc}}$-dressing of a tensorial field b is $b^{\mathrm{u}} \coloneqq \rho(\mathrm{u}^{-1})\,b$. In particular the $\mathcal{H}_{\mathrm{loc}}$-dressing of a field strength F is $F^{\mathrm{u}} \coloneqq \mathrm{u}^{-1}F\mathrm{u}$, while that of a matter fields φ is $\varphi^{\mathrm{u}} \coloneqq \rho(\mathrm{u}^{-1})\,\varphi$. As is clear from these examples, we have that $(\phi^{\mathrm{u}})^\gamma = (\phi^\gamma)^{\mathrm{u}^\gamma} = (\phi^\gamma)^{\gamma^{-1}\mathrm{u}} = \phi^{\mathrm{u}}$. By construction, dressed fields are $\mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}$-invariant:

$$\begin{aligned}\big(\phi^{(\upsilon,\mathrm{u})}\big)^{(\psi,\gamma)} &= \big(\phi^{(\psi,\gamma)}\big)^{(\upsilon,\mathrm{u})^{(\psi,\gamma)}} = \big(\phi^{(\psi,\gamma)}\big)^{(\psi,\gamma)^{-1}\cdot(\upsilon,\mathrm{u})} \\ &= \phi^{(\psi,\gamma)\cdot(\psi,\gamma)^{-1}\cdot(\upsilon,\mathrm{u})} = \phi^{(\upsilon,\mathrm{u})}.\end{aligned} \tag{329}$$

Or, more explicitly,

$$\begin{aligned}\big(\phi^{(\upsilon,\mathrm{u})}\big)^{(\psi,\gamma)} &= \big(\phi^{(\psi,\gamma)}\big)^{(\upsilon,\mathrm{u})^{(\psi,\gamma)}} = \big(\phi^{(\psi,\gamma)}\big)^{\left(\psi^{-1}\circ\upsilon,\ \psi^*(\gamma^{-1}\mathrm{u})\right)} \\ &= (\psi\circ\upsilon)^*\Big([\phi^{(\psi,\gamma)}]^{\psi^*(\gamma^{-1}\mathrm{u})}\Big) = (\psi\circ\upsilon)^*\Big(\psi^*[\phi^\gamma]^{\psi^*(\gamma^{-1}\mathrm{u})}\Big) \\ &= \upsilon^*\circ\psi^{-1}\Big(\psi^*[(\phi^\gamma)^{\gamma^{-1}\mathrm{u}}]\Big) \\ &= \upsilon^*(\phi^{\mathrm{u}}) = \phi^{(\upsilon,\mathrm{u})}.\end{aligned} \tag{330}$$

Observe that we have automatic compatibility conditions for υ and u, analogous to those displayed in Section 4.2, ensuring the stepwise reduction (155): Indeed, from (326) we have on the one hand that $\mathrm{u}^\psi = \psi^*\mathrm{u}$ which ensures that the $\mathcal{H}_{\mathrm{loc}}$-invariant dressed field ϕ^{u} still has a well-defined $\mathrm{Diff}(M)$-transformation, that one can dress for via υ. On the other hand, we have $\upsilon^\gamma = \upsilon$, which secures the fact that the $\mathcal{H}_{\mathrm{loc}}$-invariance achieved via u is not spoiled after further dressing via υ. So, whenever dressed fields are defined $\phi^{(\upsilon,\mathrm{u})} \in \Phi^{(\upsilon,\mathrm{u})} \sim [\phi] \in \mathcal{M}$.

We now define *field-dependent* $(\mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}})$-dressing fields as

$$\begin{aligned}(\boldsymbol{\upsilon},\mathbf{u}) : \Phi &\to \mathcal{D}r[N;M,H], \\ \phi &\mapsto (\boldsymbol{\upsilon}(\phi),\mathbf{u}(\phi)), \quad \text{s.t.}\quad R^\star_{(\psi,\gamma)}(\boldsymbol{\upsilon},\mathbf{u}) = (\psi,\gamma)^{-1}\cdot(\boldsymbol{\upsilon},\mathbf{u}) \\ &\qquad = \big(\psi^{-1}\circ\boldsymbol{\upsilon},\psi^*(\gamma^{-1}\mathbf{u})\big), \\ &\qquad \text{i.e.,}\quad \big(\boldsymbol{\upsilon}\big(\psi^*(\phi^\gamma)\big),\mathbf{u}\big(\psi^*(\phi^\gamma)\big)\big) \\ &\qquad = (\psi,\gamma)^{-1}\cdot\big(\boldsymbol{\upsilon}(\phi),\mathbf{u}(\phi)\big).\end{aligned} \tag{331}$$

The linear version of this defining equivariance is

$$\boldsymbol{L}_{(X,\lambda)^v}(\boldsymbol{\upsilon},\mathbf{u}) = \big(-X\circ\boldsymbol{\upsilon},\ \mathfrak{L}_X\mathbf{u} - \lambda\mathbf{u}\big). \tag{332}$$

Such a dressing field can be understood as an equivariant 0-form, i.e., tensorial, with values in $\mathcal{D}r[N;M,H]$ seen as a representation space for (a right action of) $\mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}$. Therefore, its $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\big(\Phi, \mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}\big)$ and $\boldsymbol{\mathfrak{diff}}_v(\Phi) \simeq C^\infty\big(\Phi, \mathfrak{diff}(M)\oplus \mathrm{Lie}\mathcal{H}_{\mathrm{loc}}\big)$ transformations are

$$\begin{aligned}&(\boldsymbol{\upsilon},\mathbf{u})^{(\boldsymbol{\psi},\boldsymbol{\gamma})} = (\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\cdot(\boldsymbol{\upsilon},\mathbf{u}), \quad \text{and} \\ &\boldsymbol{L}_{(\boldsymbol{X},\boldsymbol{\lambda})^v}(\boldsymbol{\upsilon},\mathbf{u}) = \big(-\boldsymbol{X}\circ\boldsymbol{\upsilon},\ \mathfrak{L}_{\boldsymbol{X}}\mathbf{u} - \boldsymbol{\lambda}\mathbf{u}\big).\end{aligned} \tag{333}$$

A ϕ-dependent dressing field induces a map

$$\begin{aligned}F_{(\boldsymbol{\upsilon},\mathbf{u})} &: \Phi \to \mathcal{M}, \\ \phi &\mapsto F_{(\boldsymbol{\upsilon},\mathbf{u})}(\phi) \coloneqq \boldsymbol{\upsilon}(\phi)^*\big(\phi^{\mathbf{u}(\phi)}\big) = \phi^{(\boldsymbol{\upsilon},\mathbf{u})} \sim [\phi], \quad \text{s.t.} \\ &\qquad F_{(\boldsymbol{\upsilon},\mathbf{u})}\circ R_{(\psi,\gamma)} = F_{(\boldsymbol{\upsilon},\mathbf{u})}.\end{aligned} \tag{334}$$

It tells us that a $\mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}$-orbit, a point $[\phi]\in\mathcal{M}$ is represented by a dressed field $\phi^{(\boldsymbol{\upsilon},\mathbf{u})}$, so the image of $F_{(\boldsymbol{\upsilon},\mathbf{u})}$ is a *relational coordinatization* of $\mathcal{M}$: Indeed, the expression of the $\mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}$-invariant variable $\phi^{(\boldsymbol{\upsilon},\mathbf{u})} = \boldsymbol{\upsilon}(\phi)^*\big(\phi^{\mathbf{u}(\phi)}\big)$ is an explicit field-dependent coordinatization of the physical d.o.f. with respect to each other.

Let us unpack this further. The field $\phi^{\mathbf{u}(\phi)}$ can be understood to coordinatize the internal d.o.f. of ϕ w.r.t each other, exhibiting how the true *physical internal* d.o.f. encoded in ϕ are co-defined – Remind that ϕ is a priori a collection of fields. Thus, $\phi^{\mathbf{u}(\phi)}$ is a $\mathcal{H}_{\mathrm{loc}}$-invariant assignation of internal d.o.f. over points of M. It is the formal implementation, locally on M, of the core conceptual gauge theoretic insight stemming from the internal hole argument and the internal point-coincidence argument as detailed in Section 2.

The field $\boldsymbol{\upsilon}(\phi)^*\big(\phi^{\mathbf{u}(\phi)}\big)$ can be understood to coordinatize the external d.o.f. of $\phi^{\mathbf{u}(\phi)}$ w.r.t each other. It exhibits that the true *physical spatio-temporal* d.o.f. encoded in $\phi^{\mathbf{u}(\phi)}$ are co-defined. It is the formal implementation of the core general-relativistic insight stemming from articulating of the hole argument and the point-coincidence argument... Or at least half of it: The fact that $\boldsymbol{\upsilon}(\phi)^*\big(\phi^{\mathbf{u}(\phi)}\big)$ is a field on N further stresses that M was never the physical spacetime. We will indicate shortly, when considering dressed integrals below, how the DFM makes manifest how the physical spacetime is defined via the field content.

In short, the dressed field $\phi^{(\boldsymbol{v},\mathbf{u})}$ represents the physical spatio-temporal and internal d.o.f. in a manifestly $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$-invariant and *manifestly relational* way.

It is clear that the map (334) realizes the bundle projection, $F_{(\boldsymbol{v},\mathbf{u})} \sim \pi$, and thus allows to build basic forms on the local field space Φ via $\Omega^\bullet_{\text{basic}}(\Phi) = \mathrm{Im}\,\pi^\star \simeq \mathrm{Im}\,F^\star_{(\boldsymbol{v},\mathbf{u})}$. Our aim will be in general to build the basic counterpart of a form $\boldsymbol{\alpha} = \alpha\,(\wedge^\bullet \boldsymbol{d}\phi; \phi) \in \Omega^\bullet(\Phi)$. For this we must first consider its formal analogue on the base space $\mathcal{M}$ of Φ, $\bar{\boldsymbol{\alpha}} = \alpha\,(\wedge^\bullet \boldsymbol{d}[\phi]; [\phi]) \in \Omega^\bullet(\mathcal{M})$, and then define the *dressing* of $\boldsymbol{\alpha}$ as

$$\boldsymbol{\alpha}^{(\boldsymbol{v},\mathbf{u})} := F^\star_{(\boldsymbol{v},\mathbf{u})}\bar{\boldsymbol{\alpha}} = \alpha\Big(\wedge^\bullet F^\star_{(\boldsymbol{v},\mathbf{u})}\boldsymbol{d}[\phi];\ F_{(\boldsymbol{v},\mathbf{u})}(\phi)\Big) \quad \in \Omega^\bullet_{\text{basic}}(\Phi). \tag{335}$$

Since it is basic by construction, it is automatically invariant under $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \mathrm{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\right)$.

To obtain a concrete expression of $\boldsymbol{\alpha}^{(\boldsymbol{v},\mathbf{u})}$, one may use the usual rule of thumb of the DFM: replace in the expression for the transformation $\boldsymbol{\alpha}^{(\boldsymbol{\psi},\boldsymbol{\gamma})}$ the parameters $(\boldsymbol{\psi},\boldsymbol{\gamma})$ by the dressing field $(\boldsymbol{v},\mathbf{u})$. But for the sake of completness, let us see how we may confirm the validity of this rule.

From (335), it is clear that all we need is $F^\star_{(\boldsymbol{v},\mathbf{u})}\boldsymbol{d}[\phi]$ expressed in terms of $\boldsymbol{d}\phi$ and $(\boldsymbol{v},\mathbf{u})$. The obvious way to try and find it is to write $F^\star_{(\boldsymbol{v},\mathbf{u})}\boldsymbol{d}[\phi]_{|F_{(\boldsymbol{v},\mathbf{u})}(\phi)}(\boldsymbol{\mathfrak{X}}_{|\phi}) = \boldsymbol{d}[\phi]_{|F_{(\boldsymbol{v},\mathbf{u})}(\phi)}(F_{(\boldsymbol{v},\mathbf{u})\star}\boldsymbol{\mathfrak{X}}_{|\phi})$, for a generic $\boldsymbol{\mathfrak{X}} \in \Gamma(T\Phi)$ with flow $\varphi_\tau : \Phi \to \Phi$, and s.t. $\boldsymbol{\mathfrak{X}}_{|\phi} = \frac{d}{d\tau}\varphi_\tau(\phi)|_{\tau=0} = \boldsymbol{\mathfrak{X}}(\phi)\frac{\delta}{\delta\phi}$. So, we need the pushforward $F_{(\boldsymbol{v},\mathbf{u})\star} : T_\phi\Phi \to T_{F_{(\boldsymbol{v},\mathbf{u})}(\phi)}\mathcal{M}$, $\boldsymbol{\mathfrak{X}}_{|\phi} \mapsto F_{(\boldsymbol{v},\mathbf{u})\star}\boldsymbol{\mathfrak{X}}_{|\phi}$. Using the expression of the map (334) we have

$$\begin{aligned}
F_{(\boldsymbol{v},\mathbf{u})\star}\boldsymbol{\mathfrak{X}}_{|\phi} &:= F_{(\boldsymbol{v},\mathbf{u})\star}\tfrac{d}{d\tau}\varphi_\tau(\phi)\Big|_{\tau=0} = \tfrac{d}{d\tau}F_{(\boldsymbol{v},\mathbf{u})}\left(\varphi_\tau(\phi)\right)\Big|_{\tau=0} \\
&= \tfrac{d}{d\tau}\boldsymbol{v}\big(\varphi_\tau(\phi)\big)^*\left(\phi^{\mathbf{u}(\varphi_\tau(\phi))}\right)\Big|_{\tau=0} \\
&= \tfrac{d}{d\tau}\boldsymbol{v}\big(\varphi_\tau(\phi)\big)^*\left(\phi^{\mathbf{u}(\phi)}\right)\Big|_{\tau=0} + \tfrac{d}{d\tau}\boldsymbol{v}(\phi)^*\left(\varphi_\tau(\phi)^{\mathbf{u}(\phi)}\right)\Big|_{\tau=0} \\
&\quad + \tfrac{d}{d\tau}\boldsymbol{v}(\phi)^*\left(\phi^{\mathbf{u}(\varphi_\tau(\phi))}\right)\Big|_{\tau=0}.
\end{aligned}$$

The result may be written as $\tilde{\boldsymbol{\mathfrak{X}}}([\phi])\frac{\delta}{\delta[\phi]}$ as a derivation of $C^\infty(\mathcal{M})$. Let us work out each term in turn. Inserting $\boldsymbol{v}(\phi)^{-1} \circ \boldsymbol{v}(\phi) = \mathrm{id}_M$, the 1st term is $\frac{d}{d\tau}\boldsymbol{v}(\phi)^*\boldsymbol{v}(\phi)^{-1*}\boldsymbol{v}(\varphi_\tau(\phi))^*(\phi^{\mathbf{u}(\phi)})|_{\tau=0} = \boldsymbol{v}(\phi)^*\frac{d}{d\tau}(\boldsymbol{v}(\varphi_\tau(\phi)) \circ \boldsymbol{v}(\phi)^{-1})^*(\phi^{\mathbf{u}(\phi)})|_{\tau=0}$. The term $\boldsymbol{v}(\varphi_\tau(\phi)) \circ \boldsymbol{v}(\phi)^{-1}$ is a curve in M, so $\frac{d}{d\tau}\boldsymbol{v}(\varphi_\tau(\phi)) \circ \boldsymbol{v}(\phi)^{-1}|_{\tau=0} = \boldsymbol{d}\boldsymbol{v} \circ \boldsymbol{v}^{-1}{}_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi}) \in \Gamma(TM)$. Thus,

$$\begin{aligned}
&\tfrac{d}{d\tau}\boldsymbol{v}\big(\varphi_\tau(\phi)\big)^*\left(\phi^{\mathbf{u}(\phi)}\right)\Big|_{\tau=0} \\
&\quad = \boldsymbol{v}(\phi)^*\tfrac{d}{d\tau}\left(\boldsymbol{v}\big(\varphi_\tau(\phi)\big) \circ \boldsymbol{v}(\phi)^{-1}\right)^*\left(\phi^{\mathbf{u}(\phi)}\right)\Big|_{\tau=0} \\
&\quad = \boldsymbol{v}(\phi)^*\left(\mathfrak{L}_{\boldsymbol{d}\boldsymbol{v}\circ\boldsymbol{v}^{-1}{}_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi})}\phi^{\mathbf{u}(\phi)}\right) \\
&\quad = \boldsymbol{v}(\phi)^*\left[\left(\mathfrak{L}_{\boldsymbol{d}\boldsymbol{v}\circ\boldsymbol{v}^{-1}{}_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi})}\phi + \delta_{\left(\mathfrak{L}_{\{\boldsymbol{d}\boldsymbol{v}\circ\boldsymbol{v}^{-1}{}_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi})\}}\mathbf{u}(\phi)\right)\mathbf{u}(\phi)^{-1}}\phi\right)^{\mathbf{u}(\phi)}\right],
\end{aligned} \tag{336}$$

where we used a direct analogue of the identity (215) in the last step. Inserting $\mathbf{u}(\phi)\mathbf{u}(\phi)^{-1} = \mathrm{id}_{\mathcal{H}_{\text{loc}}}$ in the 3rd term we get $\boldsymbol{v}(\phi)^*\frac{d}{d\tau}(\phi^{\mathbf{u}(\phi)\mathbf{u}(\phi)^{-1}\mathbf{u}(\varphi_\tau(\phi))})|_{\tau=0} = \boldsymbol{v}(\phi)^*\frac{d}{d\tau}(\phi^{\mathbf{u}(\phi)})^{\mathbf{u}(\phi)^{-1}\mathbf{u}(\varphi_\tau(\phi))}|_{\tau=0}$. The term $\mathbf{u}(\phi)^{-1}\mathbf{u}(\varphi_\tau(\phi))$ is a curve in $\mathcal{H}_{\text{loc}}$ through the identity, so $\frac{d}{d\tau}\mathbf{u}(\phi)^{-1}\mathbf{u}(\varphi_\tau(\phi))|_{\tau=0} = \mathbf{u}^{-1}\boldsymbol{d}\mathbf{u}_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi}) \in \mathrm{Lie}\mathcal{H}_{\text{loc}}$. Thus,

$$\begin{aligned}
\tfrac{d}{d\tau}\boldsymbol{v}(\phi)^*\left(\phi^{\mathbf{u}\left(\varphi_\tau(\phi)\right)}\right)\Big|_{\tau=0} &= \boldsymbol{v}(\phi)^*\delta_{\{\mathbf{u}^{-1}\boldsymbol{d}\mathbf{u}_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi})\}}\left(\phi^{\mathbf{u}(\phi)}\right) \\
&= \boldsymbol{v}(\phi)^*\left[\left(\delta_{\{\boldsymbol{d}\mathbf{u}\mathbf{u}^{-1}{}_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi})\}}\phi\right)^{\mathbf{u}(\phi)}\right],
\end{aligned} \tag{337}$$

where we used an exact analogue of (214) in the last step. Finally, the 2nd term is $\frac{d}{d\tau}\boldsymbol{v}(\phi)^*(\varphi_\tau(\phi)^{\mathbf{u}(\phi)})|_{\tau=0} =: \frac{d}{d\tau}F_{(\boldsymbol{v}(\phi),\mathbf{u}(\phi))}(\varphi_\tau(\phi))|_{\tau=0} = F_{(\boldsymbol{v}(\phi),\mathbf{u}(\phi))\star}\boldsymbol{\mathfrak{X}}_{|\phi}$. As a vector on $\mathcal{M}$, we may find its expression as a derivation by applying it to $\mathbf{g} \in C^\infty(\mathcal{M})$:

$$\begin{aligned}
\left[F_{(\boldsymbol{v}(\phi),\mathbf{u}(\phi))\star}\boldsymbol{\mathfrak{X}}\,(\mathbf{g})\right]([\phi]) &= \tfrac{d}{d\tau}\,\mathbf{g}\Big(F_{(\boldsymbol{v}(\phi),\mathbf{u}(\phi))}\left(\varphi_\tau(\phi)\right)\Big)\Big|_{\tau=0} \\
&= \underbrace{\tfrac{d}{d\tau}F_{(\boldsymbol{v}(\phi),\mathbf{u}(\phi))}\left(\varphi_\tau(\phi)\right)\Big|_{\tau=0}}_{\left[\boldsymbol{\mathfrak{X}}(F_{(\boldsymbol{v}(\phi),\mathbf{u}(\phi))})\right](\phi)} \\
&\quad \times \left(\tfrac{\delta}{\delta[\phi]}\mathbf{g}\right)\left(F_{(\boldsymbol{v}(\phi),\mathbf{u}(\phi))}(\phi)\right) \\
&= \underbrace{\left(\tfrac{\delta}{\delta\phi}F_{(\boldsymbol{v}(\phi),\mathbf{u}(\phi))}\right)(\phi)}_{\frac{\delta}{\delta\phi}\,\boldsymbol{v}(\phi)^*\left(\phi^{\mathbf{u}(\phi)}\right) = \boldsymbol{v}(\phi)^*\left((\ \ )^{\mathbf{u}(\phi)}\right)} \\
&\quad \times \boldsymbol{\mathfrak{X}}(\phi)\left(\tfrac{\delta}{\delta[\phi]}\mathbf{g}\right)\left(\underbrace{\boldsymbol{v}(\phi)^*\left(\phi^{\mathbf{u}(\phi)}\right)}_{\sim[\phi]}\right) \\
&= \left[\boldsymbol{v}(\phi)^*\left(\boldsymbol{\mathfrak{X}}(\phi)^{\mathbf{u}(\phi)}\right)\tfrac{\delta}{\delta[\phi]}(\mathbf{g})\right]([\phi]).
\end{aligned}$$

Gathering all three terms we have finally

$$\begin{aligned}
F_{(\boldsymbol{v}(\phi),\mathbf{u}(\phi))\star}\boldsymbol{\mathfrak{X}}_{|\phi} = \boldsymbol{v}(\phi)^*\Bigg[&\Big(\boldsymbol{d}\phi_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi}) + \mathfrak{L}_{\boldsymbol{d}\boldsymbol{v}\circ\boldsymbol{v}^{-1}{}_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi})}\phi \\
&+ \delta_{\left(\mathfrak{L}_{\{\boldsymbol{d}\boldsymbol{v}\circ\boldsymbol{v}^{-1}{}_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi})\}}\mathbf{u}(\phi)\right)\mathbf{u}(\phi)^{-1}}\phi + \delta_{\{\boldsymbol{d}\mathbf{u}\mathbf{u}^{-1}{}_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi})\}}\phi\Big)^{\mathbf{u}(\phi)}\Bigg] \\
&\times \frac{\delta}{\delta[\phi]}_{|F_{(\boldsymbol{v},\mathbf{u})}(\phi)}
\end{aligned}$$

from which we find

$$\begin{aligned}
\boldsymbol{d}\phi^{(\boldsymbol{v},\mathbf{u})}{}_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi}) &:= F^\star_{(\boldsymbol{v},\mathbf{u})}\boldsymbol{d}[\phi]_{|F_{(\boldsymbol{v},\mathbf{u})}(\phi)}(\boldsymbol{\mathfrak{X}}_{|\phi}) = \boldsymbol{d}[\phi]_{|F_{(\boldsymbol{v},\mathbf{u})}(\phi)}\left(F_{(\boldsymbol{v},\mathbf{u})\star}\boldsymbol{\mathfrak{X}}_{|\phi}\right) \\
&= \boldsymbol{v}(\phi)^*\Bigg[\Big(\boldsymbol{d}\phi_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi}) + \mathfrak{L}_{\boldsymbol{d}\boldsymbol{v}\circ\boldsymbol{v}^{-1}{}_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi})}\phi \\
&\quad + \delta_{\left(\mathfrak{L}_{\{\boldsymbol{d}\boldsymbol{v}\circ\boldsymbol{v}^{-1}{}_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi})\}}\mathbf{u}(\phi)\right)\mathbf{u}(\phi)^{-1}}\phi + \delta_{\{\boldsymbol{d}\mathbf{u}\mathbf{u}^{-1}{}_{|\phi}(\boldsymbol{\mathfrak{X}}_{|\phi})\}}\phi\Big)^{\mathbf{u}(\phi)}\Bigg].
\end{aligned} \tag{338}$$

Which defines the dressing of $\boldsymbol{d}\phi$, the basis 1-form of basic forms on Φ:

$$\boldsymbol{d}\phi^{(\boldsymbol{\upsilon},\mathbf{u})} = \boldsymbol{\upsilon}^*\left[\left(\boldsymbol{d}\phi + \mathfrak{L}_{\boldsymbol{d}\boldsymbol{\upsilon}\circ\boldsymbol{\upsilon}^{-1}}\phi + \delta_{\left(\mathfrak{L}_{\{\boldsymbol{d}\boldsymbol{\upsilon}\circ\boldsymbol{\upsilon}^{-1}\}}\mathbf{u}\right)\mathbf{u}^{-1}}\phi + \delta_{\boldsymbol{d}\mathbf{u}\mathbf{u}^{-1}}\phi\right)^{\mathbf{u}}\right] \in \Omega^1_{\text{basic}}(\Phi). \tag{339}$$

This specializes to

$$\begin{aligned} \boldsymbol{d}\phi^{\boldsymbol{\upsilon}} &= \boldsymbol{\upsilon}^*\left(\boldsymbol{d}\phi + \mathfrak{L}_{\boldsymbol{d}\boldsymbol{\upsilon}\circ\boldsymbol{\upsilon}^{-1}}\phi\right),\\ \boldsymbol{d}\phi^{\mathbf{u}} &= \left(\boldsymbol{d}\phi + \delta_{\boldsymbol{d}\mathbf{u}\mathbf{u}^{-1}}\phi\right)^{\mathbf{u}}, \end{aligned} \tag{340}$$

which are respectively the Diff(M)-dressing and $\mathcal{H}_{\text{loc}}$-dressing of $\boldsymbol{d}\phi$, showing that (339) generalizes Equation (149) in ref. [8], as well as Equations (38)–(38) in ref. [6] and Equation (50) in ref. [42]. Such formulas feature in the sub-literature of covariant phase space dealing with "edge modes".[35]

Inserting (339) into (335), this yields the dressing of $\boldsymbol{\alpha}$:

$$\boldsymbol{\alpha}^{(\boldsymbol{\upsilon},\mathbf{u})} = \alpha\left(\wedge^{\bullet}\boldsymbol{d}\phi^{(\boldsymbol{\upsilon},\mathbf{u})};\ \phi^{(\boldsymbol{\upsilon},\mathbf{u})}\right) \ \in\ \Omega^{\bullet}_{\text{basic}}(\Phi). \tag{341}$$

On account of the formal similarity between $\Xi(\phi) = \boldsymbol{\psi}^*(\phi^{\gamma})$ and $F_{(\boldsymbol{\upsilon},\mathbf{u})}(\phi) = \boldsymbol{\upsilon}^*(\phi^{\mathbf{u}})$, and between $\boldsymbol{d}\phi^{(\boldsymbol{\psi},\boldsymbol{\gamma})}$ (212) and $\boldsymbol{d}\phi^{(\boldsymbol{\upsilon},\mathbf{u})}$ (339), resulting into the close formal expressions of $\boldsymbol{\alpha}^{(\boldsymbol{\psi},\boldsymbol{\gamma})}$ and $\boldsymbol{\alpha}^{(\boldsymbol{\upsilon},\mathbf{u})}$ (335)–(341), we confirm the rule of thumb to obtain the dressing of any form $\boldsymbol{\alpha}$ on Φ: First compute the $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\right)$ transformation $\boldsymbol{\alpha}^{(\boldsymbol{\psi},\boldsymbol{\gamma})}$, then substitute $(\boldsymbol{\psi},\boldsymbol{\gamma}) \rightarrow (\boldsymbol{\upsilon},\mathbf{u})$ in the resulting expression to obtain $\boldsymbol{\alpha}^{(\boldsymbol{\upsilon},\mathbf{u})}$. This rule will be used systematically in what follows.

5.2.2. Dressing and Flat Connections

As we already observed in Section 4.1.1, dressing fields induce flat connections, and flat connections are expressible in terms of dressing fields.

- A ϕ-dependent dressing field induces the *flat Ehresmann connection*

$$\begin{aligned} \boldsymbol{\omega}_0 &:= -\boldsymbol{d}(\boldsymbol{\upsilon},\mathbf{u})\cdot(\boldsymbol{\upsilon},\mathbf{u})^{-1}\\ &\in \Omega^1_{\text{eq}}\left(\Phi, \mathfrak{diff}(M) \oplus \text{Lie}\mathcal{H}_{\text{loc}}\right),\\ \left(\boldsymbol{\omega}_{0\,|\,\text{Diff}},\ \boldsymbol{\omega}_{0\,|\,\mathcal{H}_{\text{loc}}}\right) &= -\left(\boldsymbol{d}\boldsymbol{\upsilon}\circ\boldsymbol{\upsilon}^{-1},\ \boldsymbol{d}\mathbf{u}\,\mathbf{u}^{-1} - \mathbf{u}\mathfrak{L}_{\boldsymbol{d}\boldsymbol{\upsilon}\circ\boldsymbol{\upsilon}^{-1}}\mathbf{u}^{-1}\right), \end{aligned} \tag{342}$$

using (B19)–(B21) in the second equality. Its Ad-equivariance is easily proven,

$$\begin{aligned} R^{\star}_{(\psi,\gamma)}\boldsymbol{\omega}_0 &= -\boldsymbol{d}\left(R^{\star}_{(\psi,\gamma)}(\boldsymbol{\upsilon},\mathbf{u})\right)\cdot\left(R^{\star}_{(\psi,\gamma)}(\boldsymbol{\upsilon},\mathbf{u})\right)^{-1}\\ &= -(\psi,\gamma)^{-1}\cdot\boldsymbol{d}(\boldsymbol{\upsilon},\mathbf{u})\cdot(\boldsymbol{\upsilon},\mathbf{u})^{-1}\cdot(\psi,\gamma)\\ &= \text{Ad}_{(\psi,\gamma)^{-1}}\boldsymbol{\omega}_0. \end{aligned} \tag{343}$$

The verticality is also easily found, using (332) and (B21),

$$\begin{aligned} \iota_{(X,\lambda)^v}\boldsymbol{\omega}_0 &= -\iota_{(X,\lambda)^v}\boldsymbol{d}(\boldsymbol{\upsilon},\mathbf{u})\cdot(\boldsymbol{\upsilon},\mathbf{u})^{-1}\\ &= -\left(-X\circ\boldsymbol{\upsilon},\ \mathfrak{L}_X\mathbf{u} - \lambda\mathbf{u}\right)\cdot(\boldsymbol{\upsilon},\mathbf{u})^{-1}\\ &= -\left(-X\circ\boldsymbol{\upsilon}\circ\upsilon^{-1}, (\mathfrak{L}_X\mathbf{u} - \lambda\mathbf{u})\mathbf{u}^{-1} - \mathbf{u}\mathfrak{L}_{\{-X\circ\boldsymbol{\upsilon}\circ\upsilon^{-1}\}}\mathbf{u}^{-1}\right)\\ &= (X,\lambda). \end{aligned} \tag{344}$$

Or, done another way,

$$\begin{aligned} \iota_{(X,\lambda)^v}\boldsymbol{\omega}_0 &= -\iota_{(X,\lambda)^v}\left(\boldsymbol{d}\boldsymbol{\upsilon}\circ\boldsymbol{\upsilon}^{-1}, \boldsymbol{d}\mathbf{u}\mathbf{u}^{-1} - \mathbf{u}\mathfrak{L}_{\boldsymbol{d}\boldsymbol{\upsilon}\circ\boldsymbol{\upsilon}^{-1}}\mathbf{u}^{-1}\right)\\ &= -\left(-X\circ\boldsymbol{\upsilon}\circ\upsilon^{-1}, (\mathfrak{L}_X\mathbf{u}\mathbf{u}^{-1} - \lambda\mathbf{u}\mathbf{u}^{-1}) - \mathbf{u}\mathfrak{L}_{\{-X\circ\boldsymbol{\upsilon}\circ\upsilon^{-1}\}}\mathbf{u}^{-1}\right)\\ &= (X,\lambda). \end{aligned} \tag{345}$$

This shows that $\boldsymbol{\omega}_0$ satisfies the defining properties of a Ehresmann connection (219). Its flatness is manifest from $\boldsymbol{\omega}_0 := -\boldsymbol{d}(\boldsymbol{\upsilon},\mathbf{u})\cdot(\boldsymbol{\upsilon},\mathbf{u})^{-1}$, but less trivial to check from the explicit form $\boldsymbol{\omega}_0 = -\left(\boldsymbol{d}\boldsymbol{\upsilon}\circ\boldsymbol{\upsilon}^{-1}, \boldsymbol{d}\mathbf{u}\mathbf{u}^{-1} - \mathbf{u}\mathfrak{L}_{\boldsymbol{d}\boldsymbol{\upsilon}\circ\boldsymbol{\upsilon}^{-1}}\mathbf{u}^{-1}\right)$ and the bracket (180).

$$\begin{aligned} &\boldsymbol{\Omega}_0 = \boldsymbol{d}\boldsymbol{\omega}_0 + \tfrac{1}{2}[\boldsymbol{\omega}_0,\boldsymbol{\omega}_0]_{\text{Lie}} = 0,\\ \hookrightarrow\ &\left(\boldsymbol{\Omega}_{0\,|\,\text{Diff}}, \boldsymbol{\Omega}_{0\,|\,\mathcal{H}_{\text{loc}}}\right) = \Big(\boldsymbol{d}\boldsymbol{\omega}_{0\,|\,\text{Diff}} + \tfrac{1}{2}\left[\boldsymbol{\omega}_{0\,|\,\text{Diff}}, \boldsymbol{\omega}_{0\,|\,\text{Diff}}\right]_{\mathfrak{diff}(M)},\\ &\qquad \times\ \boldsymbol{d}\boldsymbol{\omega}_{0\,|\,\mathcal{H}_{\text{loc}}} + \tfrac{1}{2}\left[\boldsymbol{\omega}_{0\,|\,\mathcal{H}_{\text{loc}}}, \boldsymbol{\omega}_{0\,|\,\mathcal{H}_{\text{loc}}}\right]_{\text{Lie}}\\ &\qquad - \omega_{0\,|\,\text{Diff}}\left(\omega_{0\,|\,\mathcal{H}_{\text{loc}}}\right)\Big) = (0,0). \end{aligned} \tag{346}$$

The non-trivial part to check is $\boldsymbol{\Omega}_{0\,|\,\mathcal{H}_{\text{loc}}} = 0$. Let us show it explicitly, for the benefit of the reader:

$$\begin{aligned} \boldsymbol{\Omega}_{0\,|\,\mathcal{H}_{\text{loc}}} &= \boldsymbol{d}\left(-\boldsymbol{d}\mathbf{u}\mathbf{u}^{-1} + \mathbf{u}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1}\right)\\ &\quad + \tfrac{1}{2}\left[-\boldsymbol{d}\mathbf{u}\mathbf{u}^{-1} + \mathbf{u}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1}, -\boldsymbol{d}\mathbf{u}\mathbf{u}^{-1} + \mathbf{u}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1}\right]\\ &\quad - (-\boldsymbol{d}\upsilon\circ\upsilon)\left(-\boldsymbol{d}\mathbf{u}\mathbf{u}^{-1} + \mathbf{u}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1}\right)\\ &= \cancel{\boldsymbol{d}(-\boldsymbol{d}\mathbf{u}\mathbf{u}^{-1})} + \boldsymbol{d}\left(\mathbf{u}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1}\right)\\ &\quad + \tfrac{1}{2}\cancel{\left[\boldsymbol{d}\mathbf{u}\mathbf{u}^{-1}, \boldsymbol{d}\mathbf{u}\mathbf{u}^{-1}\right]} - \left[\mathbf{u}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1}, \boldsymbol{d}\mathbf{u}\mathbf{u}^{-1}\right]\\ &\quad + \tfrac{1}{2}\left[\mathbf{u}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1}, \mathbf{u}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1}\right]\\ &\quad - \mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}(\boldsymbol{d}\mathbf{u}\mathbf{u}^{-1}) + \mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\left(\mathbf{u}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1}\right)\\ &= \cancel{\boldsymbol{d}\mathbf{u}\left(\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1}\right)} + \mathbf{u}\boldsymbol{d}\left(\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1}\right)\\ &\quad - \mathbf{u}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1}\cdot\boldsymbol{d}\mathbf{u}\mathbf{u}^{-1} - \cancel{\boldsymbol{d}\mathbf{u}\mathbf{u}^{-1}\mathbf{u}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1}}\\ &\quad + \cancel{\mathbf{u}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1}\cdot\mathbf{u}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1}}\\ &\quad - \mathbf{u}\mathbf{u}^{-1}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\left(\boldsymbol{d}\mathbf{u}\mathbf{u}^{-1}\right) + \cancel{\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1}}\\ &\quad + \mathbf{u}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1}\\ &= \mathbf{u}\,\mathfrak{L}_{\boldsymbol{d}(\boldsymbol{d}\upsilon\circ\upsilon^{-1})}\mathbf{u}^{-1} - \cancel{\mathbf{u}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\boldsymbol{d}\mathbf{u}^{-1}}\\ &\quad - \cancel{\mathbf{u}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\left(\mathbf{u}^{-1}\boldsymbol{d}\mathbf{u}\mathbf{u}^{-1}\right)} + \mathbf{u}\,\tfrac{1}{2}\left[\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}, \mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\right]_{\Gamma(TM)}\mathbf{u}^{-1}\\ &= \mathbf{u}\,\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1} + \mathbf{u}\,\tfrac{1}{2}\mathfrak{L}_{\left[\boldsymbol{d}\upsilon\circ\upsilon^{-1}, \boldsymbol{d}\upsilon\circ\upsilon^{-1}\right]_{\Gamma(TM)}}\mathbf{u}^{-1}\\ &= -\mathbf{u}\,\mathfrak{L}_{-\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1} - \mathbf{u}\,\tfrac{1}{2}\mathfrak{L}_{\left[\boldsymbol{d}\upsilon\circ\upsilon^{-1}, \boldsymbol{d}\upsilon\circ\upsilon^{-1}\right]_{\mathfrak{diff}(M)}}\mathbf{u}^{-1}\\ &= -\mathbf{u}\,\tfrac{1}{2}\mathfrak{L}_{\{\Omega_{0\,|\,\text{Diff}}\}}\mathbf{u}^{-1} = 0. \end{aligned}$$

As usual, ω_0 induces a covariant derivative $D_0 = d\ + \rho_*(\omega_0)$ on $\Omega^\bullet_{\text{tens}}(\Phi, \rho)$. In this case is satisfies $D_0^2 = 0$.

- Consider a G-valued 1-cocycle C for the action of $\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$ on Φ, satisfying by definition

$$C\big(\phi; (\psi, \gamma)\cdot(\psi', \gamma')\big) = C\big(\phi; (\psi, \gamma)\big)\cdot C\big(\phi^{(\psi,\gamma)}; (\psi', \gamma')\big), \tag{347}$$

where the product "$\cdot$" on the r.h.s. is in the group G. From this we have that the inverse is

$$C\big(\phi; (\psi, \gamma)\big)^{-1} = C\big(\phi^{(\psi,\gamma)}; (\psi, \gamma)^{-1}\big). \tag{348}$$

Its functional form allows to define a *twisted dressing field* defined by $C\big(\ ; (\boldsymbol{v}, \mathbf{u})\big)$ whose equivariance is indeed,

$$\begin{aligned}\Big[R^\star_{(\psi,\gamma)} C\big(\ ; (\boldsymbol{v}, \mathbf{u})\big)\Big](\phi) &= C\big(\phi^{(\psi,\gamma)}; (\psi, \gamma)^{-1}(\boldsymbol{v}, \mathbf{u})\big)\\ &= C\big(\phi^{(\psi,\gamma)}; (\psi, \gamma)^{-1}\big)\cdot C\big(\phi; (\boldsymbol{v}, \mathbf{u})\big)\\ &= C\big(\phi; (\psi, \gamma)\big)^{-1}\cdot C\big(\phi; (\boldsymbol{v}, \mathbf{u})\big),\end{aligned}$$

$$\text{so}\quad R^\star_{(\psi,\gamma)} C\big(\ ; (\boldsymbol{v}, \mathbf{u})\big) = C\big(\ ; (\psi, \gamma)\big)^{-1}\cdot C\big(\ ; (\boldsymbol{v}, \mathbf{u})\big). \tag{349}$$

The linear version of which is

$$\boldsymbol{L}_{(X,\lambda)^v} C\big(\ ; (\boldsymbol{v}, \mathbf{u})\big) = -a\big((X, \lambda);\ \big)\cdot C\big(\ ; (\boldsymbol{v}, \mathbf{u})\big). \tag{350}$$

Such a twisted dressing field is a tensorial 0-form, so its $\text{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\right)$ and $\mathfrak{diff}_v(\Phi) \simeq C^\infty\left(\Phi, \mathfrak{diff}(M) \oplus \text{Lie}\mathcal{H}_{\text{loc}}\right)$ transformations are, as a special case of (207)

$$\begin{aligned}&C\big(\ ; (\boldsymbol{v}, \mathbf{u})\big)^{(\boldsymbol{\psi},\boldsymbol{\gamma})} = C(\boldsymbol{\psi}, \boldsymbol{\gamma})^{-1}\cdot C\big(\ ; (\boldsymbol{v}, \mathbf{u})\big) \quad \text{and}\\ &\boldsymbol{L}_{(X,\lambda)^v} C\big(\ ; (\boldsymbol{v}, \mathbf{u})\big) = -a(\boldsymbol{X}, \boldsymbol{\lambda})\cdot C\big(\ ; (\boldsymbol{v}, \mathbf{u})\big),\end{aligned} \tag{351}$$

and we remind the notation $[C(\boldsymbol{\psi}, \boldsymbol{\gamma})](\phi) \coloneqq C\left(\phi; \left(\boldsymbol{\psi}(\phi), \boldsymbol{\gamma}(\phi)\right)\right)$ and $[a(\boldsymbol{X}, \boldsymbol{\lambda})](\phi) \coloneqq a\left((\boldsymbol{X}(\phi), \boldsymbol{\lambda}(\phi)); \phi\right)$, so that we may likewise write the twisted dressing field as $C(\boldsymbol{v}, \mathbf{u})$.

The latter induces the *flat twisted connection*

$$\boldsymbol{\varpi}_0 \coloneqq -\boldsymbol{d}C(\boldsymbol{v}, \mathbf{u})\cdot C(\boldsymbol{v}, \mathbf{u})^{-1} \quad \in \Omega^1_{\text{eq}}(\Phi, C), \tag{352}$$

whose equivariance is easily found to be

$$\begin{aligned}R^\star_{(\psi,\gamma)}\boldsymbol{\varpi}_0 &= -\boldsymbol{d}\Big(R^\star_{(\psi,\gamma)} C(\boldsymbol{v}, \mathbf{u})\Big)\cdot\Big(R^\star_{(\psi,\gamma)} C(\boldsymbol{v}, \mathbf{u})\Big)^{-1}\\ &= -C\big(\ ; (\boldsymbol{\psi}, \boldsymbol{\gamma})\big)^{-1}\Big[\boldsymbol{d}C(\boldsymbol{v}, \mathbf{u})\cdot C(\boldsymbol{v}, \mathbf{u})^{-1}\Big]\\ &\quad\times C\big(\ ; (\boldsymbol{\psi}, \boldsymbol{\gamma})\big) - \boldsymbol{d}C\big(\ ; (\boldsymbol{\psi}, \boldsymbol{\gamma})\big)^{-1}\cdot C\big(\ ; (\boldsymbol{\psi}, \boldsymbol{\gamma})\big)\\ &= \text{Ad}_{C(\ ;(\psi,\gamma))^{-1}}\boldsymbol{\varpi}_0 + C\big(\ ; (\boldsymbol{\psi}, \boldsymbol{\gamma})\big)^{-1}\boldsymbol{d}C\big(\ ; (\boldsymbol{\psi}, \boldsymbol{\gamma})\big).\end{aligned} \tag{353}$$

Its verticality property is derived from (350),

$$\iota_{(X,\lambda)^v}\boldsymbol{\varpi}_0 = -\iota_{(X,\lambda)^v}\boldsymbol{d}C(\boldsymbol{v}, \mathbf{u})\cdot C(\boldsymbol{v}, \mathbf{u})v^{-1} = a\big((X, \lambda);\ \big) \quad \in \mathfrak{g}. \tag{354}$$

These are indeed the defining properties (228) of a twisted connection. Its flatness, $\bar{\boldsymbol{\Omega}}_0 = \boldsymbol{d}\boldsymbol{\varpi}_0 + \frac{1}{2}[\boldsymbol{\varpi}_0, \boldsymbol{\varpi}_0]_{\mathfrak{g}} = 0$, is obvious. As usual, $\boldsymbol{\varpi}_0$ induces a twisted covariant derivative $\bar{\boldsymbol{D}}_0 \coloneqq \boldsymbol{d}\ + \boldsymbol{\varpi}_0$ on $\Omega^\bullet_{\text{tens}}(\Phi, C)$, s.t. $\bar{\boldsymbol{D}}_0^2 = 0$.

In case $G = U(1)$, we have $C(\boldsymbol{v}, \mathbf{u}) = \exp\{-i\,\mathsf{c}(\boldsymbol{v}, \mathbf{u})\}$, with the *Abelian twisted dressing field*

$$\begin{aligned}&\mathsf{c}(\boldsymbol{v}, \mathbf{u}) \coloneqq \mathsf{c}\big(\ ; (\boldsymbol{v}, \mathbf{u})\big),\\ \text{s.t.}\quad &R^\star_{(\psi,\gamma)}\mathsf{c}(\boldsymbol{v}, \mathbf{u}) = \mathsf{c}(\boldsymbol{v}, \mathbf{u}) - \mathsf{c}\big(\ ; (\psi, \gamma)\big).\end{aligned} \tag{355}$$

The associated twisted connection is $\boldsymbol{\varpi}_0 \coloneqq -\boldsymbol{d}C(\boldsymbol{v}, \mathbf{u})\cdot C(\boldsymbol{v}, \mathbf{u})^{-1} = i\boldsymbol{d}\mathsf{c}(\boldsymbol{v}, \mathbf{u})$, i.e., it is $\boldsymbol{d}$-exact as one would expect, so that its flatness is manifest: $\bar{\boldsymbol{\Omega}}_0 = \boldsymbol{d}\boldsymbol{\varpi}_0 = 0$. For a twisted tensorial form $\boldsymbol{\alpha} = \exp\{i\boldsymbol{\theta}\} \in \Omega^\bullet_{\text{tens}}(\Phi, C)$, the twisted covariant derivative gives

$$\begin{aligned}&\bar{\boldsymbol{D}}_0\alpha = \boldsymbol{d}\boldsymbol{\alpha} + \boldsymbol{\varpi}_0\boldsymbol{\alpha} = i\boldsymbol{d}\big(\boldsymbol{\theta} + \mathsf{c}(\boldsymbol{v}, \mathbf{u})\big)\,\boldsymbol{\alpha} \quad \in \Omega^{\bullet+1}_{\text{tens}}(\Phi, C),\\ &\text{where}\quad \boldsymbol{\theta} + \mathsf{c}(\boldsymbol{v}, \mathbf{u}) \quad \in \Omega^\bullet_{\text{basic}}(\Phi).\end{aligned} \tag{356}$$

The invariant object $\boldsymbol{\theta} + \mathsf{c}(\boldsymbol{v}, \mathbf{u})$ is also seen to be the phase of the dressing of $\boldsymbol{\alpha}$:

$$\boldsymbol{\alpha}^{(\boldsymbol{v},\mathbf{u})} = C(\boldsymbol{v}, \mathbf{u})^{-1}\boldsymbol{\alpha} = \exp i\{\boldsymbol{\theta} + \mathsf{c}(\boldsymbol{v}, \mathbf{u})\} =: \exp i\boldsymbol{\theta}^{(\boldsymbol{v},\mathbf{u})} \in \Omega^\bullet_{\text{basic}}(\Phi). \tag{357}$$

This is the geometry underlying the notion of action/Lagrangian "improved" via WZ counterterms implementing anomaly cancelation – see e.g., section 12.3 in ref. [69], Chap. 15 in ref. [64], or the end of Chap.4 in ref. [63].

5.2.3. Residual Transformations, and Composition of Dressing Operations

We here details the topic of possible residual transformations within the local DFM, counterpart of Section 4.3.

Residual Transformations of the First Kind: These are expected when one has a dressing field by which to reduce only a subgroup of $\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$. For residual transformations to be well-defined as a group $\mathcal{J}$, the equivariance group of the dressing field, i.e., the group under which it has its defining transformation – which may of may not be identical the group in which it takes value, called its target group – must be a *normal subgroup* $\mathcal{K}$. Then, the residual transformation group is $\mathcal{J} = \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}})/\mathcal{K}$. The residual $\mathcal{J}$-transformations of the dressed fields $\phi^{(\boldsymbol{v},\mathbf{u})}$ is then determined by the $\mathcal{J}$-transformation of ϕ, which is known immediately as a special case of their $(\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}})$-transformation, and of the $\mathcal{K}$-dressing field $(\boldsymbol{v}, \mathbf{u})$, which should naturally arise from its ϕ-dependence. From this will follow the residual $\mathcal{J}$-equivariance and verticality property of any $\mathcal{K}$-dressed/basic quantity $\boldsymbol{\alpha}^{(\boldsymbol{v},\mathbf{u})}$ – which is $\boldsymbol{\mathcal{K}} = C^\infty(\Phi, \mathcal{K})$-invariant – and thus its transformation under $\boldsymbol{\mathcal{J}} \coloneqq C^\infty\left(\Phi, \mathcal{J}\right) \subset \boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\right)$.

In favorable circumstances, the $\mathcal{J}$-transformation of the dressing field $(\boldsymbol{v}, \mathbf{u})$ is s.t. $\phi^{(\boldsymbol{v},\mathbf{u})}$ have *standard* transformations under $\mathcal{J}$ – i.e., identical to the $\mathcal{J}$-transformations of the bare fields ϕ. Thus, the residual $\mathcal{J}$-equivariance and verticality property of a

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dressed quantity $\boldsymbol{\alpha}^{(\upsilon,\mathrm{u})}$ will be the same as that of $\boldsymbol{\alpha}$, and therefore will share the same $\boldsymbol{\mathcal{J}} := C^\infty(\Phi, \mathcal{J})$-transformation. This leaves open the possibility that a second $\mathcal{J}$-dressing field may be found and used to produce completely invariant dressed variables, provided that this second dressing field is also $\mathcal{K}$-invariant. These are the compatibility conditions analogous to (150)–(153) discussed in the global case, Section 4.2.

As observed below (330), this occurs naturally when considering the $\mathcal{H}_{\text{loc}}$-dressing $\mathbf{u}$, in which case we have $\mathcal{K} = \mathcal{H}_{\text{loc}} \triangleleft (\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}})$ and $\mathcal{J} = \text{Diff}(M)$. As shown there, the dressed field ϕ^{u} is $\mathcal{H}_{\text{loc}}$-invariant. Since furthermore, by (331), $R^\star_\psi \mathbf{u} = \psi^*\mathbf{u}$, the residual $\mathcal{J} = \text{Diff}(M)$-transformation of ϕ^{u} is the standard form $R_\psi \phi^{\mathrm{u}} = \psi^*(\phi^{\mathrm{u}})$, as one would expect. Therefore any dressed form $\boldsymbol{\alpha}^{\mathrm{u}} = \alpha\,(\curlywedge \boldsymbol{d}\phi^{\mathrm{u}}; \phi^{\mathrm{u}})$, for which one may use (340) or a special case of (341), will have the same $\mathcal{J} = \text{Diff}(M)$-equivariance and verticality property as $\boldsymbol{\alpha}$, and thus the same $\boldsymbol{\mathcal{J}} = C^\infty(\Phi, \text{Diff}(M))$-transformation.

This allows for the possibility of using a second $\mathcal{J} = \text{Diff}(M)$-dressing field $\boldsymbol{\upsilon}$, which again by (331) satisfies the compatiblity condition $R^\star_\gamma \boldsymbol{\upsilon} = \upsilon$, i.e., it is $\mathcal{K} = \mathcal{H}_{\text{loc}}$-invariant. Hence the possibility to write the completely invariant dressed field $(\phi^{\mathrm{u}})^{\upsilon} = \phi^{(\upsilon,\mathrm{u})}$, and to build basic objects $(\boldsymbol{\alpha}^{\mathrm{u}})^{\upsilon} = \boldsymbol{\alpha}^{(\upsilon,\mathrm{u})}$.

Another such case presents itself when reducing only part of the gauge group $\mathcal{H}_{\text{loc}}$ via a $\mathcal{K}$-dressing field $\mathbf{u}$, with $\mathcal{K} \triangleleft \mathcal{H}_{\text{loc}}$ so that there is a residual gauge group $\mathcal{H}^{\text{res}}_{\text{loc}} := \mathcal{H}_{\text{loc}}/\mathcal{K}$, and the residual transformation group is $\mathcal{J} = \text{Diff}(M) \ltimes \mathcal{H}^{\text{res}}_{\text{loc}}$. We may then have an analogue of Prop.1 in Section 4.3.1.

Proposition 3. *Given a $\mathcal{K}$-dressing field $\mathbf{u}$, if its $\mathcal{J}$-residual equivariance is*

$$R^\star_{(\psi,\eta)} \mathbf{u} = \psi^*(\mathbf{u}^\eta) = \psi^*\left(\eta^{-1}\mathbf{u}\eta\right) \quad \textit{for} \ (\psi,\eta) \in \mathcal{J} = \text{Diff}(M) \ltimes \mathcal{H}^{res}_{loc}, \tag{358}$$

then the dressed fields ϕ^{u} are standard $\mathcal{H}^{res}_{loc}$-gauge fields, i.e., have $\mathcal{H}^{res}_{loc}$-transformations identical to their bare counterparts ϕ: $R_{(\psi,\eta)}\phi^{\mathrm{u}} = \psi^*\left((\phi^{\mathrm{u}})^\eta\right)$. Any dressed form $\boldsymbol{\alpha}^{\mathrm{u}}$ is $\mathcal{K}$-basic but has identical $\mathcal{J}$-equivariance and verticality, hence the same $\boldsymbol{\mathcal{J}} = C^\infty\left(\Phi, \text{Diff}(M) \ltimes \mathcal{H}^{res}_{loc}\right)$-transformation, as its bare counterpart $\boldsymbol{\alpha}$. For example, for $\boldsymbol{d}\phi^{\mathrm{u}}$ we have

$$\begin{aligned}
(\boldsymbol{d}\phi^{\mathrm{u}})^{\boldsymbol{\gamma}} &= \boldsymbol{d}\phi^{\mathrm{u}} \quad \textit{for} \ \boldsymbol{\gamma} \in \boldsymbol{\mathcal{K}}, \\
(\boldsymbol{d}\phi^{\mathrm{u}})^{(\boldsymbol{\psi},\boldsymbol{\eta})} &= \boldsymbol{\psi}^*\left[\left(\boldsymbol{d}\phi^{\mathrm{u}} + \mathfrak{L}_{\boldsymbol{d}\boldsymbol{\psi}\circ\boldsymbol{\psi}^{-1}}\phi^{\mathrm{u}} + \delta_{(\mathfrak{L}_{\boldsymbol{d}\boldsymbol{\psi}\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\eta})\boldsymbol{\eta}^{-1}}\phi^{\mathrm{u}} + \delta_{\boldsymbol{d}\boldsymbol{\eta}\boldsymbol{\eta}^{-1}}\phi^{\mathrm{u}}\right)^{\boldsymbol{\eta}}\right], \\
&\qquad \textit{for} \ (\boldsymbol{\psi},\boldsymbol{\eta}) \in \boldsymbol{\mathcal{J}},
\end{aligned} \tag{359}$$

the last line being entirely analogous to $\boldsymbol{d}\phi^{(\boldsymbol{\psi},\boldsymbol{\gamma})}$ (212).

If one happens to identify a $\mathcal{H}^{\text{res}}_{\text{loc}}$-dressing field $\mathbf{u}'$ satisfying the compatibility condition

$$R^\star_{(\psi,\gamma)}\mathbf{u}' = \psi^*\mathbf{u}' \quad \textbf{i.e.,} \quad R^\star_\gamma \mathbf{u}' = \mathbf{u}' \quad \text{for} \ \gamma \in \mathcal{K}, \tag{360}$$

then one may build the $\mathcal{H}_{\text{loc}}$-invariant dressed fields $(\phi^{\mathrm{u}})^{\mathrm{u}'} = \phi^{\mathrm{u}\,\mathrm{u}'}$, where the composite $\mathbf{u}\,\mathbf{u}'$ is a $\mathcal{H}_{\text{loc}}$-dressing field, since (358) and (360) together ensure that $R^\star_\gamma \mathbf{u}\,\mathbf{u}' = \gamma^{-1}\mathbf{u}\,\mathbf{u}'$ and $R^\star_\eta \mathbf{u}\,\mathbf{u}' = \eta^{-1}\mathbf{u}\,\mathbf{u}'$. We can thus produce the $\mathcal{H}_{\text{loc}}$-basic form $(\boldsymbol{\alpha}^{\mathrm{u}})^{\mathrm{u}'} = \boldsymbol{\alpha}^{\mathrm{u}\,\mathrm{u}'}$.

By (360) still, it is manifest that $\phi^{\mathrm{u}\,\mathrm{u}'}$ have standard $\text{Diff}(M)$-residual transformation: $R_\psi\, \phi^{\mathrm{u}\,\mathrm{u}'} = \psi^*(\phi^{\mathrm{u}\,\mathrm{u}'})$. So any dressed form $\boldsymbol{\alpha}^{\mathrm{u}\,\mathrm{u}'}$ shares the same $\text{Diff}(M)$-equivariance and verticality as its bare counterpart $\boldsymbol{\alpha}$, and thus the same $C^\infty\left(\Phi, \text{Diff}(M)\right)$-transformation. For example, for $\boldsymbol{d}\phi^{\mathrm{u}\,\mathrm{u}'}$,

$$\begin{aligned}
(\boldsymbol{d}\phi^{\mathrm{u}\,\mathrm{u}'})^{\boldsymbol{\gamma}} &= \boldsymbol{d}\phi^{\mathrm{u}} \quad \text{for} \ \boldsymbol{\gamma} \in \boldsymbol{\mathcal{H}}_{\text{loc}}, \\
(\boldsymbol{d}\phi^{\mathrm{u}\mathrm{u}'})^{\boldsymbol{\psi}} &= \boldsymbol{\psi}^*\left(\boldsymbol{d}\phi^{\mathrm{u}\mathrm{u}'} + \mathfrak{L}_{\boldsymbol{d}\boldsymbol{\psi}\circ\boldsymbol{\psi}^{-1}}\phi^{\mathrm{u}\mathrm{u}'}\right) \quad \text{for} \ \boldsymbol{\psi} \in C^\infty(\Phi, \text{Diff}(M)).
\end{aligned} \tag{361}$$

The residual $\text{Diff}(M)$-transformation can be reduced if one finds a $\text{Diff}(M)$-dressing field $\boldsymbol{\upsilon}$ satisfying the compatibility condition

$$R^\star_{(\psi,\gamma)}\boldsymbol{\upsilon} = \psi^{-1}\circ\boldsymbol{\upsilon} \quad \text{i.e.,} \quad R^\star_\gamma \boldsymbol{\upsilon} = \boldsymbol{\upsilon} \ \text{for} \ \gamma \in \mathcal{H}_{\text{loc}}. \tag{362}$$

Then the final dressed fields $(\phi^{\mathrm{u}\,\mathrm{u}'})^{\upsilon} = \phi^{(\upsilon,\mathrm{u}\,\mathrm{u}')}$ are fully $\left(\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\right)$-invariant, and one may produce finally basic forms $(\boldsymbol{\alpha}^{\mathrm{u}\,\mathrm{u}'})^{\upsilon} = \boldsymbol{\alpha}^{(\upsilon,\mathrm{u}\,\mathrm{u}')} \in \Omega^\bullet_{\text{basic}}(\Phi)$.[25]

We will provide an example illustrating this in Section 5.2.7 below, when treating Einsein–Maxwell theory with charged matter modeled by a complex scalar field.

Residual Transformations of the Second Kind: This distinct type represents a parametrization of the ambiguity, or arbitrariness, in the choice of dressing field. It amounts to the statement that two dressing fields (υ', u') and (υ, u) in $\mathcal{D}r[N; M, H]$ are a priori related by an element $(\varphi, \zeta) \in \text{Diff}(N) \times \mathcal{G}_{\text{loc}}$, with $\mathcal{G}_{\text{loc}} := \left\{\zeta : U \subset M \to \mathrm{G} \mid \zeta^\gamma = \zeta\right\}$ and G is the target Lie group of the dressing field u (which may be either s.t. $\mathrm{G} \subset H$ or s.t. $\mathrm{G} \supset H$, and was initially put as $\mathrm{G} = H$ in (326) for simplicity of exposition). That is, we have

$$(\upsilon', \mathrm{u}') = (\upsilon \circ \varphi,\ \mathrm{u}\zeta\,). \tag{363}$$

Defining $\bar{\zeta} := \upsilon^*\zeta : N \to \mathrm{G}$, and $\bar{\mathcal{G}}_{\text{loc}} := \upsilon^*\mathcal{G}_{\text{loc}}$, the above can be rewritten as

$$(\upsilon', \mathrm{u}') = \left(\upsilon \circ \varphi,\ \mathrm{u}\,(\upsilon^{-1})^*\bar{\zeta}\,\right) = (\upsilon, \mathrm{u}) \cdot (\varphi, \zeta), \tag{364}$$

extending the semi-direct product rule (179). Furthermore, we write

$$\begin{aligned}
(\upsilon'', \mathrm{u}'') &= \left(\upsilon' \circ \varphi',\ \mathrm{u}'\zeta'\,\right) \\
&= \left(\upsilon' \circ \varphi',\ \mathrm{u}'(\upsilon'^{-1})^*\bar{\zeta}'\,\right) \\
&= \left(\upsilon \circ \varphi \circ \varphi', \mathrm{u}\,(\upsilon^{-1})^*\bar{\zeta}\,[(\upsilon\circ\varphi)^{-1}]^*\bar{\zeta}'\right) \\
&= \left(\upsilon \circ \varphi \circ \varphi', \mathrm{u}\,(\upsilon^{-1})^*\left(\bar{\zeta}\,(\varphi^{-1})^*\bar{\zeta}'\right)\right) \\
&= (\upsilon, \mathrm{u}) \cdot \left(\varphi \circ \varphi', \bar{\zeta}\,(\varphi^{-1})^*\bar{\zeta}'\right) \\
&= (\upsilon, \mathrm{u}) \cdot \left((\varphi, \bar{\zeta}) \cdot (\varphi', \bar{\zeta}')\right),
\end{aligned} \tag{365}$$

[25] Conceivably one may have reduced for only a (normal) subgroup of $\text{Diff}(M)$, e.g., compactly supported diffemorphisms, but this seems of limited interest. See ref. [8] for a brief discussion.

where in the last step we define the semi-direct product in $\mathrm{Diff}(N) \ltimes \bar{\mathcal{G}}_{\text{loc}}$. The last two equations show that there is a right action of $\mathrm{Diff}(N) \ltimes \bar{\mathcal{G}}_{\text{loc}}$ on the space of dressing fields $Dr[N; M, H]$:

$$Dr[N; M, H] \times \left(\mathrm{Diff}(N) \ltimes \bar{\mathcal{G}}_{\text{loc}}\right) \to Dr[N; M, H],$$
$$((\upsilon, \mathrm{u}), (\varphi, \bar{\zeta})) \mapsto R_{(\varphi,\bar{\zeta})}(\upsilon, \mathrm{u}) := (\upsilon, \mathrm{u}) \cdot (\varphi, \bar{\zeta}). \tag{366}$$

This can be understood as the local counterpart of Section 4.3.2, where $\mathrm{Diff}(N) \ltimes \bar{\mathcal{G}}_{\text{loc}}$ is the local version of $\mathrm{Aut}(Q)$. This action formally extends, a priori, to ϕ-dependent dressing fields $(\boldsymbol{\upsilon}, \mathbf{u})$.

Then, since ϕ are local representatives of fields on P, they do not support the action of $\mathrm{Diff}(N) \ltimes \bar{\mathcal{G}}_{\text{loc}}$, which we may denote $\phi^{(\varphi,\zeta)} = \phi$. This, combined to (366), implies that there is a right action of $\mathrm{Diff}(N) \ltimes \bar{\mathcal{G}}_{\text{loc}}$ on the space of dressed fields $\Phi^{(\upsilon,\mathrm{u})}$

$$\Phi^{(\upsilon,\mathrm{u})} \times \left(\mathrm{Diff}(N) \ltimes \bar{\mathcal{G}}_{\text{loc}}\right) \to \Phi^{(\upsilon,\mathrm{u})},$$
$$\left(\phi^{(\upsilon,\mathrm{u})},\ (\varphi, \bar{\zeta})\right) \mapsto R_{(\varphi,\zeta)}\, \phi^{(\upsilon,\mathrm{u})} := \varphi^*\left(\left[\phi^{(\upsilon,\mathrm{u})}\right]^{\bar{\zeta}}\right). \tag{367}$$

It is all but analogous to the action (177) of $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$ on Φ. Therefore, the space of dressed fields $\Phi^{(\upsilon,\mathrm{u})}$ is, a priori, a $\left(\mathrm{Diff}(N) \ltimes \bar{\mathcal{G}}_{\text{loc}}\right)$-principal bundle. It is then possible to elaborate on all the stuctures – tangent bundle, vertical bundle, spaces of forms, etc. – that exist on $\Phi^{(\upsilon,\mathrm{u})}$ as we did for Φ. In particular, dressed forms $\boldsymbol{\alpha}^{(\upsilon,\mathrm{u})}$ (341) are clearly naturally interpretable as forms on $\Phi^{(\upsilon,\mathrm{u})}$. We might then write down their vertical transformations under $\boldsymbol{Diff}\left(\Phi^{(\upsilon,\mathrm{u})}\right) \simeq C^\infty\left(\Phi^{(\upsilon,\mathrm{u})}, \mathrm{Diff}(N) \ltimes \bar{\mathcal{G}}_{\text{loc}}\right)$. It is easy to see that they would be formally identical to the $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \mathrm{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\right)$-transformation of their bare counterpart $\boldsymbol{\alpha}$ on Φ. By this rule of thumb, we immediately write for example, as an exact analogue of (212),

$$\left(\boldsymbol{d}\phi^{(\upsilon,\mathrm{u})}\right)^{(\boldsymbol{\varphi},\bar{\boldsymbol{\zeta}})} = \boldsymbol{\varphi}^*\Bigg[\left(\boldsymbol{d}\phi^{(\upsilon,\mathrm{u})} + \mathfrak{L}_{\boldsymbol{d\varphi}\circ\boldsymbol{\varphi}^{-1}}\phi^{(\upsilon,\mathrm{u})}\right.$$
$$\left. + \delta_{(\mathfrak{L}_{\boldsymbol{d\varphi}\circ\boldsymbol{\varphi}^{-1}}\bar{\boldsymbol{\zeta}})\bar{\boldsymbol{\zeta}}^{-1}}\phi^{(\upsilon,\mathrm{u})} + \delta_{\boldsymbol{d}\bar{\boldsymbol{\zeta}}\bar{\boldsymbol{\zeta}}^{-1}}\phi^{(\upsilon,\mathrm{u})}\right)^{\bar{\boldsymbol{\zeta}}}\Bigg]. \tag{368}$$

We may observe that (366)–(368) reproduce and generalize formulas found in the edge mode literature, and pertaining to what is called there "surface symmetries" or "corner symmetries". Compare (366) with e.g., Equation (2.43) and Equation (3.44) in ref. [35] or Equation (3.68) and Equation (4.64) in ref. [70]. The name stems from the fact that in this literature, dressing fields are introduced as "edge modes", i.e., as d.o.f. living on a codimension 2 submanifold of M (a corner), which is then also the support of a transformation that they alone support, and is none other than our (366).

As we have stated in introducing the topic, the group $\mathrm{Diff}(N) \ltimes \bar{\mathcal{G}}_{\text{loc}}$ parametrizes the a priori ambiguity in the choice of dressing field. It is essentially isomorphic to $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$, and thus encodes neither more nor less information. Which is expected from the global fact that the dressing is a bundle morphism $\mathbf{u} : Q \to P$, so one indeed would expect the natural morphism $\mathrm{Aut}(Q) \simeq \mathrm{Aut}(P)$, from which descends the morphism $\mathrm{Diff}(N) \ltimes \bar{\mathcal{G}}_{\text{loc}} \simeq \mathrm{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$.

This "residual" group is naturally there, and hardly avoidable, when one introduces an ad hoc dressing field in a theory. By which we mean that it is introduced as d.o.f. independent from the original set of fields ϕ and for the main purpose of either implementing a $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$ symmetry in a theory which initially does not enjoy it, or, relatedly, to "restore" a $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$ symmetry broken by a fixed background structure – in effect the latter moves amount to concealing the background structure. In such cases, ad hoc dressing fields generalize Stueckelberg fields.

The only hope for the residual group to be reduced, perhaps even to a discrete group, is with ϕ-dependent dressing field, i.e., those built from the original d.o.f. of the theory, when the relational interpretation of the formalism is the most natural and compelling. We will comment further on this in Section 5.2.5.

Before, we need to complete the technical and conceptual picture of the local DFM by discussing dressed integrals and how it instantiate a notion of integration on *physical* spacetime.

5.2.4. Dressed Integrals and Integration on Spacetime

We can now elaborate on the local version of Section 4.4. In Section 5.1.2, we observed that for $U \in \boldsymbol{U}(M)$ and $\boldsymbol{\alpha} \in \Omega^\bullet\left(\Phi, \Omega^{\text{top}}(U)\right)$, integrals $\boldsymbol{\alpha}_U = \langle \boldsymbol{\alpha}, U\rangle = \int_U \boldsymbol{\alpha}$ are objects on $\Phi \times \boldsymbol{U}(M)$ with values in $\boldsymbol{\Omega}^\bullet(\Phi)$. Furthermore, integrals with tensorial integrand are invariant under the action of $C^\infty\left(\Phi, \mathrm{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\right)$ as defined by (264). Their projection along $\bar{\pi} : \Phi \times \boldsymbol{U}(M) \to \bar{\boldsymbol{U}}(M)$, $(\phi, U) \mapsto [\phi, U] = [\psi^*\phi, \psi^{-1}(U)]$ is well-defined, so that they can be said "basic" w.r.t. $\bar{\pi}$. Such integrals descend to the associated bundle of regions $\bar{\boldsymbol{U}}(M) := \Phi \times \boldsymbol{U}(M)/\sim$, which is the quotient of the product space by the action (237) of $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$.

In Section 5.2.1, dressed forms on Φ were defined as basic on Φ, i.e., in $\mathrm{Im}\,\pi^\star$ with the projection realized via a dressing field, $F_{(\upsilon,\mathbf{u})} \sim \pi$. As argued there, we can then rely on the formal similarity of the actions of $F_{\boldsymbol{u}}$ and $\Xi \in \boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \mathrm{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\right)$ to read the dressing $\boldsymbol{\alpha}^{(\boldsymbol{\psi},\boldsymbol{u})}$ of a form $\boldsymbol{\alpha}$ from its vertical transformation $\boldsymbol{\alpha}^{(\boldsymbol{\psi},\boldsymbol{\gamma})}$. Likewise, we define *dressed integrals* as being basic on $\Phi \times \boldsymbol{U}(M)$, i.e., in $\mathrm{Im}\,\bar{\pi}^\star$, with the projection realized as:

$$\bar{F}_{(\boldsymbol{\upsilon},\mathbf{u})} : \Phi \times \boldsymbol{U}(M) \to \bar{\boldsymbol{U}}(M) \simeq \Phi^{\boldsymbol{u}} \times \boldsymbol{U}(N),$$
$$(\phi, U) \mapsto \bar{F}_{(\boldsymbol{\upsilon},\mathbf{u})}(\phi, U) := \left(F_{(\boldsymbol{\upsilon},\mathbf{u})}(\phi), \boldsymbol{\upsilon}^{-1}(U)\right) \tag{369}$$
$$= \left(\boldsymbol{\upsilon}^*(\phi^{\mathrm{u}}), \boldsymbol{\upsilon}^{-1}(U)\right).$$

We remind that, by definition, $U \in \boldsymbol{U}(M)$ is acted upon trivially by $\mathcal{H}_{\text{loc}}$.

Let us highlight that (369) features the notion of a *dressed region* $U^{\boldsymbol{\upsilon}} := \boldsymbol{\upsilon}^{-1}(U)$, which is a map

$$U^{\boldsymbol{\upsilon}} : \Phi \times \boldsymbol{U}(M) \to \boldsymbol{U}(N), \tag{370}$$

s.t. for $(\psi, \gamma) \in \mathrm{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$,

$$(U^{\boldsymbol{\upsilon}})^{(\psi,\gamma)} := \bar{R}^\star_{(\psi,\gamma)}\, U^{\boldsymbol{\upsilon}} = (R^\star_{(\psi,\gamma)}\boldsymbol{\upsilon})^{-1} \circ \psi^{-1}(U)$$
$$= (\psi^{-1} \circ \boldsymbol{\upsilon})^{-1} \circ \psi^{-1}(U) = \boldsymbol{\upsilon}^{-1}(U) =: U^{\boldsymbol{\upsilon}}. \tag{371}$$

From which follows that under the action of $(\boldsymbol{\psi}, \boldsymbol{\gamma}) \in C^\infty\big(\Phi, \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\big)$ we have,

$$(U^{\upsilon})^{(\psi,\gamma)} := \bar{\Xi}^\star U^{\upsilon} = U^{\upsilon}. \tag{372}$$

In other words, U^{υ} is a ϕ-dependent $\big(\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\big)$- *and* $C^\infty\big(\Phi, \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\big)$-invariant region of the physical *relationally defined* spacetime, which we may denote $M^{(\upsilon,\mathrm{u})}$. As reminded in Section 2, the hole argument and the point-coincidence argument establish that physical spacetime is defined *relationally*, in a $\text{Diff}(M)$-invariant way, by its gauge field content. This fact is usually tacitly encoded by the $\text{Diff}(M)$-covariance of general-relativistic gauge field theories, but made manifest via the DFM: U^{υ} are manifesly $\text{Diff}(M)$-invariant and manifestly ϕ-relationally defined regions, representing faithfully regions of the physical spacetime, on which relationally defined and $\big(\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\big)$-invariant fields $\phi^{(\upsilon,\mathrm{u})} := \boldsymbol{\upsilon}^*(\phi^{\mathrm{u}})$ live, and can be integrated over. This is the second part of the picture whose first part, unpacked below (334), is that ϕ^{u} implements the internal point-coincidence argument, showing how the internal d.o.f. of the fields ϕ co-define each other in a $\mathcal{H}_{\text{loc}}$-invariant way. These invariant internal d.o.f. in turn coordinatize the "points" of the internal structure (the fiber) of the enriched physical spacetime.

With (369) we thus get the complete formal implementation of the relational core of gRGFT arising from the generalized hole and point-coincidence arguments. But observe that we further get a relational definition of spacetime invariant under the much bigger *field-dependent* transformation group $C^\infty\big(\Phi, \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\big)$, which is indeed the largest symmetry group of gRGFT, as observed at the end of Section 5.1.3.

Observe that an immediate consequence of the above is that the physical *relational boundary* ∂U^{υ} of a physical spacetime region U^{υ} is necessarily $\big(\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\big)$-invariant. This invalidate the claims, often repeated in the GFT and gravity literature, that boundaries "break" either $\text{Diff}(M)$-invariance or $\mathcal{H}_{\text{loc}}$-invariance, or both. These claims are indeed logically equivalent to the hole argument and its generalization, as they commit the conceptual mistake of thinking of points of M as physical entities defined independently of the field content. Such "boundary problems" overlook the relational resolution offered by the (generalized) point-coincidence argument.[26] We shall elaborate further on this precise point in a forthcoming paper.[71]

We have all we need to defined integration *on the physical spacetime*. On $\Omega^\bullet\big(\Phi, \Omega^{\text{top}}(U)\big) \times \boldsymbol{U}(M)$ we define:

$$\begin{aligned} \tilde{F}_{(\upsilon,\mathrm{u})} : \Omega^\bullet\big(\Phi, \Omega^{\text{top}}(U)\big) \times \boldsymbol{U}(M) &\to \Omega^\bullet_{\text{basic}}\big(\Phi, \Omega^{\text{top}}(N)\big) \times \boldsymbol{U}(N), \\ (\boldsymbol{\alpha}, U) &\mapsto \tilde{F}^\star_{(\upsilon,\mathrm{u})}(\boldsymbol{\alpha}, U) \\ &:= \big(\boldsymbol{\alpha}^{(\upsilon,\mathrm{u})}, \boldsymbol{\upsilon}^{-1}(U)\big), \end{aligned} \tag{373}$$

with indeed $\Omega^\bullet_{\text{basic}}\big(\Phi, \Omega^{\text{top}}(N)\big) \simeq \Omega^\bullet\big(\Phi^{(\upsilon,\mathrm{u})}, \Omega^{\text{top}}(N)\big)$, as dressed fields $\phi^{(\upsilon,\mathrm{u})}$ live on $\text{Im}(\boldsymbol{\upsilon}^{-1}) \subset N$. Then, in formal analogy with (264), the dressing of an integral $\boldsymbol{\alpha}_U := \langle \boldsymbol{\omega}, U\rangle = \int_U \boldsymbol{\alpha}$ is

$$\begin{aligned} \boldsymbol{\alpha}_U{}^{(\upsilon,\mathrm{u})} &:= \langle\ ,\ \rangle \circ \tilde{F}_{(\upsilon,\mathrm{u})}(\boldsymbol{\alpha}, U) = \langle \boldsymbol{\alpha}^{(\upsilon,\mathrm{u})}, \boldsymbol{\upsilon}^{-1}(U)\rangle = \int_{\boldsymbol{\upsilon}^{-1}(U)} \boldsymbol{\alpha}^{(\upsilon,\mathrm{u})} \\ &= \int_{U^{\upsilon}} \boldsymbol{\alpha}^{(\upsilon,\mathrm{u})} =: \big(\boldsymbol{\alpha}^{(\upsilon,\mathrm{u})}\big)_{U^{\upsilon}}. \end{aligned} \tag{374}$$

Such dressed integrals define integration on physical spacetime, insofar as U^{υ} is interpretable as an invariantly and relationally defined region of spacetime.[27] Clearly, when $\boldsymbol{d}$ acts on such an integral, it sees the dressed region U^{υ} due to its ϕ-dependence. Thefore we have, in analogy with (265),

$$\boldsymbol{d}\Big(\big(\boldsymbol{\alpha}^{(\upsilon,\mathrm{u})}\big)_{U^{\upsilon}}\Big) = \big(\boldsymbol{d}\boldsymbol{\alpha}_U\big)^{(\upsilon,\mathrm{u})} - \langle \mathfrak{L}_{\upsilon_*^{-1}d\upsilon}\, \boldsymbol{\alpha}^{(\upsilon,\mathrm{u})}, U^{\upsilon}\rangle. \tag{375}$$

That is, $[\tilde{F}^\star_{(\upsilon,\mathrm{u})}, \boldsymbol{d}] \neq 0$ and the commutator is a boundary term, $\langle \iota_{\psi_*^{-1}d\psi}\, \boldsymbol{\alpha}^{(\upsilon,\mathrm{u})}, \partial\,(U^{\upsilon})\,\rangle$, by (262) and the fact that $\boldsymbol{\alpha}^{(\upsilon,\mathrm{u})}$ is a top form on U^{υ}. Observe that is means basicity is lost, the boundary term being manifestly non-horizontal. This impacts our discussion of the *relational variational principle* in the next Section 5.2.6.

For $\boldsymbol{\alpha} \in \Omega^\bullet_{\text{tens}}\big(\Phi, \Omega^{\text{top}}(U)\big)$, i.e., $\boldsymbol{\alpha}^{(\psi,\gamma)} = \boldsymbol{\psi}^*(\boldsymbol{\alpha}^{\gamma})$ by (267), a dressed integral is

$$\begin{aligned} \boldsymbol{\alpha}_U{}^{(\upsilon,\mathrm{u})} &= \big(\boldsymbol{\alpha}^{(\upsilon,\mathrm{u})}\big)_{U^{\upsilon}} = \langle \boldsymbol{\alpha}^{(\upsilon,\mathrm{u})}, U^{\upsilon}\rangle = \langle \boldsymbol{\upsilon}^*(\boldsymbol{\alpha}^{\mathrm{u}}), \boldsymbol{\upsilon}^{-1}(U)\rangle \\ &= \langle \boldsymbol{\alpha}^{\mathrm{u}}, U\rangle = \int_U \boldsymbol{\alpha}^{\mathrm{u}} =: (\boldsymbol{\alpha}^{\mathrm{u}})_U, \end{aligned} \tag{376}$$

by the invariance property (260) of the integration pairing. Meaning that the integral of the $\mathcal{H}_{\text{loc}}$-invariant dressed object $\boldsymbol{\alpha}^{\mathrm{u}}$ over U is numerically identical to the integral of the physical quantity $\boldsymbol{\alpha}^{(\upsilon,\mathrm{u})}$ over the spacetime region U^{υ}. Then we have

$$d\big(\boldsymbol{\alpha}_U{}^{(\upsilon,\mathrm{u})}\big) = \boldsymbol{d}\langle \boldsymbol{\psi}^*(\boldsymbol{\alpha}^{\mathrm{u}}), \boldsymbol{\upsilon}^{-1}(U)\rangle = \boldsymbol{d}\langle \boldsymbol{\alpha}^{\mathrm{u}}, U\rangle = \langle \boldsymbol{d}(\boldsymbol{\alpha}^{\mathrm{u}}), U\rangle. \tag{377}$$

And the relation (375) specializes to

$$\begin{aligned} \boldsymbol{d}\Big(\big(\boldsymbol{\alpha}^{(\upsilon,\mathrm{u})}\big)_{U^{\upsilon}}\Big) &= \big(\boldsymbol{d}\boldsymbol{\alpha}_U\big)^{(\upsilon,\mathrm{u})} - \langle \mathfrak{L}_{\upsilon_*^{-1}d\upsilon}\, \boldsymbol{\upsilon}^*(\boldsymbol{\alpha}^{\mathrm{u}}), U^{\upsilon}\rangle \\ &= \big(\boldsymbol{d}\boldsymbol{\alpha}_U\big)^{(\upsilon,\mathrm{u})} - \langle \boldsymbol{\upsilon}^*\mathfrak{L}_{d\upsilon\circ\upsilon^{-1}}(\boldsymbol{\alpha}^{\mathrm{u}}), \boldsymbol{\upsilon}^{-1}(U)\rangle \\ &= \big(\boldsymbol{d}\boldsymbol{\alpha}_U\big)^{(\upsilon,\mathrm{u})} - \langle \mathfrak{L}_{d\upsilon\circ\upsilon^{-1}}(\boldsymbol{\alpha}^{\mathrm{u}}), U\rangle. \end{aligned} \tag{378}$$

The last two relation imply the following lemma:

$$\boldsymbol{d}(\boldsymbol{\upsilon}^*\boldsymbol{\alpha}) = \boldsymbol{\upsilon}^*\big(\boldsymbol{d}\boldsymbol{\alpha} + \mathfrak{L}_{d\upsilon\circ\upsilon^{-1}}\boldsymbol{\alpha}\big). \tag{379}$$

[26] The notion of "edge modes" d.o.f. – i.e., ad hoc dressing fields – sometimes introduced, and interpreted as the Goldstone bosons associated to these $\text{Diff}(M)$ or $\mathcal{H}_{\text{loc}}$ symmetry breaking, is then a solution to an arguably non-existing problem. Or rather to a problem artificially introduced in gRGFT, i.e., treating a boundary as a background structure, and then trying to conceal that background structure. See Section 5.2.5 next.

[27] We remark that, if it exists, the residual transformation of the 2nd kind $C^\infty\big(\Phi^{(\upsilon,\mathrm{u})}, \text{Diff}(N) \ltimes \bar{\mathcal{G}}_{\text{loc}}\big)$ may act on dressed integrals, analogously to (264), as

$$\big((\boldsymbol{\alpha}_U)^{(\upsilon,\mathrm{u})}\big)^{(\varphi,\bar{\xi})} = \int_{\varphi^{-1}(U^{\upsilon})} \big(\boldsymbol{\alpha}^{(\upsilon,\mathrm{u})}\big)^{(\varphi,\bar{\xi})}.$$

But again, this situation may likely arise only for ad hoc dressing fields, when the relational interpretation of the DFM is unavailable, or implausible. See next section.

In case $\boldsymbol{\alpha}$ is $\mathrm{Diff}(M)$-equivariant *and* $\mathcal{H}_{\mathrm{loc}}$-invariant, it is s.t. $\boldsymbol{\alpha}^{\mathbf{u}} = \boldsymbol{\alpha}$ by the DFM rule of thumb, so

$$\boldsymbol{\alpha}_U{}^{(\boldsymbol{v},\mathbf{u})} = \left(\boldsymbol{\alpha}^{(\boldsymbol{v},\mathbf{u})}\right)_{U^{\boldsymbol{v}}} = \langle \boldsymbol{\alpha}^{(\boldsymbol{v},\mathbf{u})}, U^{\boldsymbol{v}} \rangle = \langle \boldsymbol{v}^*\boldsymbol{\alpha}, \boldsymbol{v}^{-1}(U) \rangle = \langle \boldsymbol{\alpha}, U \rangle = \boldsymbol{\alpha}_U. \tag{380}$$

Which means that the integral of the bare unphysical quantity $\boldsymbol{\alpha}$ over U is numerically identical to the integral of the physical quantity $\boldsymbol{\alpha}^{(\boldsymbol{v},\mathbf{u})}$ over the true spacetime region $U^{\boldsymbol{v}}$. It is interesting to notice that this is an instance where a quantity computed in the "bare" formalism gives the correct results for the corresponding physical quantity. In the latter case we have, furthermore, $\boldsymbol{d}\boldsymbol{\alpha}_U = \boldsymbol{d}\left(\boldsymbol{\alpha}_U{}^{(\boldsymbol{v},\mathbf{u})}\right)$, and using (258)–(273) we have that:

$$\begin{aligned} \left(\boldsymbol{d}\boldsymbol{\alpha}_U\right)^{(\boldsymbol{v},\mathbf{u})} &= \boldsymbol{d}\boldsymbol{\alpha}_U + \langle \mathfrak{L}_{\boldsymbol{d}\boldsymbol{v}\circ\boldsymbol{v}^{-1}}\boldsymbol{\alpha}, U \rangle, \\ \langle \boldsymbol{d}\boldsymbol{\alpha}, U \rangle^{(\boldsymbol{v},\mathbf{u})} &= \langle \boldsymbol{d}\boldsymbol{\alpha}, U \rangle + \langle \mathfrak{L}_{\boldsymbol{d}\boldsymbol{v}\circ\boldsymbol{v}^{-1}}\boldsymbol{\alpha}, U \rangle. \end{aligned} \tag{381}$$

This has immediate consequences for the variational principle in field theory, to which we turn after taking stock of a few noteworthy conceptual points.

5.2.5. Discussion

As we have established, the DFM is a systematic and formal way to obtain objects, built from fields ϕ and their variations $\boldsymbol{d}\phi$, that are invariant under $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$: they are *basic* objects on field space Φ and on $\Phi \times \boldsymbol{U}(M)$, so that they descend to, or represent, objects on the moduli space $\mathcal{M}$ and on the bundle of regions $\bar{\boldsymbol{U}}(M) = \Phi \times \boldsymbol{U}(M)/\sim$. Together the latter contain the relevant physical quantities: physical field d.o.f. for $\mathcal{M}$, integral quantities over physical spacetime for $\bar{\boldsymbol{U}}(M)$. Dressed fields and dressed integrals can be understood as coordinatizations for them.

Here we elaborate on the constraints in the existence of dressing fields, on the distinction to be made between ad hoc dressings and genuine ones, and on the issue of locality of observables within the DFM, as well as how "physical integrals" can be understood as local observables.

Existence and Globality of Dressings: The DFM is a *conditional* proposition: *If* one can find, or build, a dressing field, *then* it gives the algorithm to produce these invariants (and analyse possible residual symmetries). Let us discuss some of what may hide behind the conditional "if".

It should be stressed that the existence of global dressing fields (υ, u) is not guaranteed, with at least two senses of the word "global" to be distinguished. Firstly, it is clear that ϕ-independent $\mathrm{Diff}(M)$-dressing fields of the type $\upsilon \in \mathcal{D}r[\mathbb{R}^n, M]$, i.e., $\upsilon : \mathbb{R}^n \to M$, are nothing but *global* coordinate charts for M, which may not exist depending on the topology of M. The same is true in the ϕ-dependent case $\boldsymbol{v} : \mathbb{R}^n \to M$, as topological obstructions may imply that no field ϕ is globally defined, or non-vanishing, everywhere on M and fit to serve as a global coordinatization. Likewise, a ϕ-independent $\mathcal{H}_{\mathrm{loc}}$-dressing field $\mathrm{u} \in \mathcal{D}r[H, H]$, i.e., $\mathrm{u} : M \to H$, is the local representative of a global dressing $u : P \to H$, the latter providing a global trivialization of P. So, if P is non-trivial, $\mathcal{H}_{\mathrm{loc}}$-dressing u fields exist only locally over $U \subset M$. This holds the same for a ϕ-dependent $\mathbf{u}$. In such cases, the patching of multiple dressing fields necessary to globally cover M is achieved via what we called above residual symmetries of the 2nd kind. Again, only in the ϕ-dependent case may the group $\mathrm{Diff}(N) \ltimes \bar{\mathcal{G}}_{\mathrm{loc}}$ controlling the patching relations be discrete, counting the finite number or ways to built the dressings from ϕ.

Secondly, as noted in Section 5.2.2, dressing fields induce flat Ehresmann connections. If they are defined globally *on field space* Φ, so are their associated flat connections, which implies Φ is globally trivial as a bundle. If Φ is non-trivial, dressing fields would only be available locally, over open subsets $\Phi_{|\mathcal{U}}$ with $\mathcal{U} \subset \mathcal{M}$ (being compatible with the local triviality of Φ). In that respect, the situation is not unlike the Gribov-Singer obstruction to global gauge-fixings, i.e., global sections $\boldsymbol{\sigma} : \mathcal{M} \to \Phi$.[60,61,72,73]

Relational View vs. ad hoc Dressing Fields: As we have argued extensively, ϕ-dependent dressing fields $\boldsymbol{u} = \boldsymbol{u}(\phi)$, whose d.o.f. are extracted from the original field space Φ, allow a formal implementation of a *relational description* of the physics of gRGFT: The invariant dressed fields $\phi^{(\boldsymbol{v}(\phi),\mathbf{u}(\phi))}$ are readily understood to represent physical d.o.f. relationally because the d.o.f. of the fields ϕ are *coordinatized relative to each other*. As a result, only $\left(\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\right)$-invariant, physical relational d.o.f. (both spatiotemporal and internal) are manifest. Relatedly, as observed several times, only by *constructively* producing a ϕ-dependent dressing field is there any chance for the residual symmetry of the 2nd kind $\mathrm{Diff}(N) \ltimes \bar{\mathcal{G}}_{\mathrm{loc}}$ to be reduced to a small, discrete, or trivial subgroup. In concrete situations, it may be reduced to a discrete choice: that of reference coordinatizing field among the collection ϕ.

So, explicit in the DFM is the proposition that the dressing field is to be found among, or built from, the existing set of fields ϕ of a theory. This is the situation of most fundamental physical significance for the analysis of gRGFT: it is when the relational interpretation of the formalism is the most compelling and transparent. But then one must be careful when considering cases where a dressing fields is introduced in a theory *independently* from the original set of d.o.f. ϕ: i.e., when extending the field space to $\Phi' = \Phi + (\upsilon, \mathrm{u})$ and admitting $\boldsymbol{d}(\upsilon, \mathrm{u}) \neq 0$. Naturally, in that extended context, one may see the newly introduced dressing field as trivially field-dependent, by writing

$$\begin{aligned} &(\boldsymbol{v}, \mathbf{u}) : \Phi' \to \mathcal{D}r[N; M, H], \\ &\left\{\phi, (\upsilon, \mathrm{u})\right\} \mapsto \boldsymbol{u}\left(\phi, (\upsilon, \mathrm{u})\right) := (\upsilon, \mathrm{u}), \end{aligned} \tag{382}$$

and $\boldsymbol{d}$ being understood as the exterior derivative on Φ'. This amounts to considering a different theory, or model, where not only the kinematics and dynamics are altered, but also where the *physical signature* of the symmetries can be radically different.

There is a priori no issue if the initial theory one starts with already has a native $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$-symmetry, and one simply considers a variant with another dynamical field that happens to be a dressing field. Identifying the dressing field in the variant model, with Φ', may be considered a case of "constructive" building of dressing, as just mentioned. Such is the case e.g., in variants on the theme of "scalar coordinatizations" of GR, as we will illustrate in Section 5.2.7. Both the initial and extended theories, with respective field spaces Φ and Φ', enjoy $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$ symmetry, which in both cases encodes what we may call their "general relationality", characteristic of the gRGFT framework: i.e., the fact, illustrated in Figure 2, that the physical invariant field d.o.f. and spacetime points are relationally and dynamically defined, so that there is no non-dynamical non-relational background structures.

The situation is quite different when one starts with a theory *without* $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$ symmetries, i.e., having some background (non-dynamical, non-relational) structures "breaking" $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$, and one introduces by *fiat* a dressing field (v, u) whose sole purpose is to "implement" or "restore" $\big(\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\big)$-invariance. The latter we call ad hoc dressing fields. We may signal two typical instances covered by this ad hoc case of the DFM, appearing quite frequently in the literature.

The first is the starting premise of the literature on "edge modes" as introduced, and elaborated on, in refs. [28, 33, 35, 36, 56, 68, 70, 74–76]: One starts from a theory where $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$ is seemingly broken by the boundary ∂U of a $(n-1)$-dimensional region $U \subset M$, and then introduces extra d.o.f. living on ∂U, the so-called edge modes, whose transformation properties are so that they compensate for the symmetry breaking at ∂U, and thus restore $\big(\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\big)$-invariance there. A significant drawback of this whole approach is that, since such edge modes are exactly ad hoc dressing fields, introduced by *fiat*, there is a priori no way to restrict the ambiguity in their choice represented by the residual group $\mathrm{Diff}(N) \ltimes \bar{\mathcal{G}}_{\mathrm{loc}}$. In the edge mode literature, the latter group is (or was, initially) seen as supported, like edge modes themselves, on ∂U and is thus call "surface symmetry" or "corner symmetry" (∂U being a "corner", a codimension 2 submanifold in M). It is manifest that, contrary to what is claimed in the edge mode literature, the residual group encodes neither more nor less information than the original (physically meaningful) group $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$.

It should be further stressed that the starting premise of the edge mode approach, that boundaries "break" $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$, the so-called "boundary problem", is faulty. As already previewed in Section 5.2.4, above (373), it has indeed the same logical structure as the hole argument: It considers boundaries ∂U "drawn" on M, and their constitutive points, as physical entities defined independently of the field content. As such it artificially introduces a background structure, a *non-relational boundary*, and then introduces an ad hoc dressing field (edge modes) whose sole purpose is to conceal it. It is clear from Section 2 that such "boundary problems" are a non-starter once the point-coincidence argument is brought to bear: As argued there, points of M are not spacetime points, nor its regions are spacetime regions. Rather, spacetime points and regions are defined *relationally* by the fields ϕ, as is formally implemented by dressed regions $U^{v(\phi)}$ (370)–(372). So a physical spacetime boundary $\partial U^{v(\phi)}$ is relational and necessarily $\big(\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\big)$-invariant.

A second situation is when one starts from a theory which does not enjoy a $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$ symmetry as it has non-relational/non-dynamical background structures featuring in its field equations – e.g., a background metric or geometry (preserved by a subgroup of $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$ which is the genuine symmetry of the theory). But then a $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$ symmetry is enforced via the introduction of an ad hoc dressing field: those generalize Stueckelberg fields,[77–79] and the move is usually seen as "restoring" or "implementing $\big(\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}\big)$-invariance". Such is typically the case in the literature on massive gravity, bi-gravity, and sometimes string theory – see e.g., refs. [80–82]. It should be clear that such artificially enforced $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$ symmetries do not have the same physical status as native ones, as they are grafted onto a theory essentially to hide its background non-dynamical structures.

This is what motivates their denomination as "*artificial* symmetries" in the philosophy of physics literature.[83–86] Indeed, enforcing a symmetry via ad hoc DFM (generalizing the Stueckelberg trick) is an instance of "Kretschmannization" of a theory, i.e., rewriting it so as to display a strictly formal symmetry devoid of physical signature or content. The name stems from the notorious "Kretschmann objection" against GC as a fundamental feature of GR, according to which any theory can be made $\mathrm{Diff}(M)$-invariant (or generally covariant). A closer analysis resulted in the realization that one may distinguish between *substantive* $\mathrm{Diff}(M)$/GC as a distinctive native feature of a theory, whose physical signature is that all physical structures and d.o.f. are dynamically and relationally defined – so that no background structures are needed – from *artificial* $\mathrm{Diff}(M)$/GC which is implemented by hand (often via the introduction of extra objects/fields, e.g., à la Stueckelberg) and thus hides background structures.

A comparable distinction has been developed for internal gauge symmetries, in response to a "generalized Kretschmann objection" to the GP[7,41,85] – according to which any field theory can be rewritten so as to display a gauge symmetry. In that case, a consensual proposition regarding the physical signature of *substantive gauge symmetries* is the trade-off between gauge invariance and locality of a theory, which is absent for *artificial gauge symmetries*. See also ref. [1] for a possible counterpoint, and a broader discussion on the physical meaning of local symmetries.

Theories obtained through Kretschmannization are only superficially and formally alike gRGFT, but betray their key physical insight, i.e., the *complete* (or general) relationality of physical structures implying the absence of any background structure/field, which is what a "substantive" $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$ transformation encodes. So, if one can still hold on to a "partially relational" view of the DFM when analyzing such theories, they in fact do not enjoy the signature "general relationality" of genuine general-relativistic gauge field theories.

From the above follows that the DFM as developed here is the common geometric framework underpinning a number of notions appearing in recent years in the literature on gravity and gauge theories. For example, as mentioned, the ad hoc DFM underlies massive gravity and bi-gravity theories,[80] as well as the notion, closely related to edge modes, of "embedding fields"[28,54–56,87] – see also ref. [51]. The DFM also grounds the notion of "gravitational dressings" as proposed in refs. [44–48], as well as that of "dynamical reference frames" as proposed in refs. [52, 53] – which also has an explicit relational angle.

Locality of Observables, and Relational Integrals as Local Observables: Let us offer some comments on the issue of locality vs. non-locality of observables in the DFM. We should start by defining our terms. Here, by "local" we shall mean "field theoretically local", which may be understood to consists in at least three desiderata: (1) Relativistic causality: Physical processes (carrying energy/information) propagate within the light-cone structure of spacetime. (2) Field locality: Physical properties/magnitudes are (smoothly) assigned to arbitrarily small regions of spacetime, or, in the idealized limit, to spacetime points, and thus interact pointwisely in regions where they do not vanish. Relatedly one may consider (3) Separability: The physical properties/magnitudes (fields configurations) of regions of spacetime are recovered from physical properties/magnitudes of its constitutive subregions. Relativistic causality (1) remains non-negotiable, so failure of non-locality means allowing infringement of (2) and/or (3). Both have been suggested. It is famously the case that accounts of the Aharonov-Bohm effect, which has often been argued to require a form of non-locality meant to imply a "non-separability" that is argued to be a systemic feature of gauge theory – see e.g., refs. [88, 89].

The DFM accommodates both the possibilities that observables be local or non-local. If $\big(\boldsymbol{\upsilon}(\phi), \mathbf{u}(\phi)\big)$ is a local functional of the bare fields ϕ, then the invariant dressed fields $\phi^{(\upsilon,\mathbf{u})}$ are local relational variables and so is any dressed form $\boldsymbol{\alpha}^{(\upsilon,\mathbf{u})} = \alpha\left(\wedge^{\bullet}\boldsymbol{d}\phi^{(\upsilon,\mathbf{u})}; \phi^{(\upsilon,\mathbf{u})}\right)$ built from them: these are potential local observables. If $\big(\boldsymbol{\upsilon}(\phi), \mathbf{u}(\phi)\big)$ is a non-local functional of ϕ, involving e.g., integrals or inversion of differential operators, then $\phi^{(\upsilon,\mathbf{u})}$ are invariant non-local relational variables, idem for $\boldsymbol{\alpha}^{(\upsilon,\mathbf{u})}$. The framework developed here is therefore a priori neutral regarding the issue of locality of the observables. Which of the above cases actually obtains depends essentially on the model under consideration, i.e., on the precise field content available. See e.g., refs. [7, 41] for discussions of this point in the context of internal gauge theories.

As an important example, consider the natural class of invariants, and potential observables, given by integrals $\langle \boldsymbol{\alpha}; U\rangle = \int_U \boldsymbol{\alpha}$ of $\mathcal{H}_{\text{loc}}$-invariant local functionals $\boldsymbol{\alpha}$ of the fields ϕ – by which we mean that the value of $\boldsymbol{\alpha}$ at $x \in U \subset M$ depends only on the r-jet of ϕ at x for finite r. Such integrals may be considered non-local in the sense that a priori they invariantly assign real numbers not to points, but rather to whole regions $U \subset M$. This appears to be somewhat in tension with desideratum (2). Some of these integral quantities, representing key physical quantities, have indeed been called "quasi-local": e.g., the quasi-local mass/energy of isolated gravitational systems.[90] However, a relational understanding of such integral quantities – as only *indirectly* representing integration on spacetime – allows to see that they do actually satisfy (2). They indeed arise from the physical properties of invariant relationally defined physical field d.o.f. $\phi^{(\upsilon,\mathbf{u})}$ over relationally defined *physical spacetime points* and subregions U^{υ}. This is shown in a manifest way by the notion of dressed integrals (374), which by (380) coincide with "bare" integrals for $\text{Diff}(M)$-tensorial and $\mathcal{H}_{\text{loc}}$-basic integrand $\boldsymbol{\alpha}$.

5.2.6. Relational Formulation of General-Relativistic Gauge Field Theory

We have developed all that is needed to obtain a relational reformulation of gRGFT. Let us consider a Lagrangian $L \in \Omega^0_{\text{tens}}\left(\Phi, \Omega^{\text{top}}(U)\right)$ as supporting a non-trivial action of $\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$ (274). Its $C^\infty\left(\Phi, \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\right) \simeq \boldsymbol{Diff}_v(\Phi)$-transformation is given by (283), so by the DFM rule of thumb its dressing is immediately found to be

$$L^{(\upsilon,\mathbf{u})} = \boldsymbol{\upsilon}^*(L^{\mathbf{u}}) = \boldsymbol{\upsilon}^*\big(L + c(\ ;\mathbf{u})\big) \quad \in \Omega^0_{\text{basic}}\big(\Phi, \Omega^{\text{top}}(U)\big), \tag{383}$$

where the 1-cocycle is extended to obtain the Abelian twisted dressing field $c(\ ;\mathbf{u})$ as in (355), Section 5.2.2. The dressed Lagrangian, explicitly $L^{(\upsilon,\mathbf{u})}(\phi) = L\left(\phi^{(\upsilon,\mathbf{u})}\right) = L\left(\boldsymbol{\upsilon}^*(\phi^{\mathbf{u}})\right)$, is a manifestly relational and $\big(\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\big)$-invariant reformulation of the bare theory L.

Being basic on Φ, a dressed Lagrangian (383) represents (or coordinatizes) a physical quantity on the moduli space $\mathcal{M}$. It should not be confused with a *gauge-fixed* Lagrangian: A gauge-fixing is the datum of a (local) section $\boldsymbol{\sigma} : \mathcal{V} \subset \mathcal{M} \to \Phi_{|\mathcal{V}}$, a gauge-fixed Lagrangian is then $\boldsymbol{\sigma}^\star L = L \circ \boldsymbol{\sigma} \in \Omega^0\left(\mathcal{V}, \Omega^{\text{top}}(U)\right)$. Observe furthermore that under transformation by $(\psi, \gamma) \in \text{Diff} \ltimes \mathcal{H}_{\text{loc}}$, a new gauge-fixing section is $\boldsymbol{\sigma}' := R_{(\psi,\gamma)} \circ \boldsymbol{\sigma}$, and the gauge-fixed Lagrangian transforms to $\boldsymbol{\sigma}'^\star L = L \circ \boldsymbol{\sigma}' = L \circ R_{(\psi,\gamma)} \circ \boldsymbol{\sigma} = \left(R^\star_{(\psi,\gamma)} L\right) \circ \boldsymbol{\sigma} = \big(\psi^*\left(L + c(\ ;\gamma)\right)\big) \circ \sigma \neq \boldsymbol{\sigma}^\star L$. In other words, a gauge-fixed Lagrangian is not invariant under the action of $\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$. These two elements should help make clear how dressed and gauge-fixed Lagrangians differ: the former is basic on Φ, the latter lives on $\mathcal{V} \subset \mathcal{M}$ and is not invariant under $\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$. This indeed reflects the fundamental distinction between dressing, which realizes the projection $\pi : \Phi \to \mathcal{M}$, and gauge-fixing, which goes the exact opposite way $\boldsymbol{\sigma} : \mathcal{M} \to \Phi$.

Let us also keep in mind that dressings have a natural relational interpretation that gauge-fixings do not. See ref. [91] for a comprehensive exposition of the distinction between dressing and gauge fixing, which further stresses how what is often considered a case of the latter is actually a case of the former (e.g., the Lorenz "gauge" in classical electrodynamics). Many standard "gauge fixings" turn out to actually be instances of dressings. As shown in ref. [43], the observation extends to supersymmetric field theory and supergravity, with the transverse and divergenceless "gauges" for the Rarita–Schwinger field and the gavitino.

The action $S = \langle L, U\rangle = \int_U L$ transforms under $C^\infty\left(\Phi, \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\right)$ by (291), from which we read its dressing

$$S^{(\upsilon,\mathbf{u})} := \int_{U^\upsilon} L^{(\upsilon,\mathbf{u})} = \int_U L + c(\ ;\mathbf{u}) =: S + \mathsf{c}(\ ;\mathbf{u}), \tag{384}$$

which indeed is a special case of (374)–(376). The dressed action $S^{(\upsilon,\mathbf{u})}$ is basic on $\Phi \times \boldsymbol{U}(M)$ and thus represents a physical quantity defined on the bundle of regions $\bar{\boldsymbol{U}}(M)$. It is clear that $S^{(\upsilon,\mathbf{u})}$ is not a gauge-fixed action, as we see *by definition* integration over the physical region of spacetime U^υ, and there is certainly nothing like a "gauge-fixed region of spacetime" in the standard bare formalism.

Remark that for an $\mathcal{H}_{\text{loc}}$-invariant Lagrangian we have $L^{\mathbf{u}} = L$, and the above specializes to $S^{(\upsilon,\mathbf{u})} = S$, so that the computation of the action in the bare formalism gives the correct physical result. This is a first hint as to why, in theories strictly respecting the principle of gRGFT – in that case, the GP, so that $L^\gamma = L$ – the bare formalism can give sensible results even without solving first the issue of extracting the physical d.o.f. of the theory.

We may observe that, writing $Z = \exp i S$ as in (293), the dressed action (384) is also obtained from the *twisted* covariant derivative induced by the flat twisted $\mathcal{H}_{\text{loc}}$-connection $\boldsymbol{\varpi}_0 = i\boldsymbol{d}\mathsf{c}(\ ;\mathbf{u})$, writing

$$\bar{\boldsymbol{D}}_0 Z = \boldsymbol{d}Z + \boldsymbol{\varpi}_0 Z = i\boldsymbol{d}\big(S + \mathsf{c}(\ ;\mathbf{u})\big)\, Z, \tag{385}$$

which is a special case of (356)–(357) – itself a special case of the non-flat case (295).

We further highlights that the dressed Lagrangian and action (383)–(384) have by definition a built-in (classical) $\mathcal{H}_{\text{loc}}$-anomaly cancelation mechanism: the twisted dressing field $\mathsf{c}(\ ;\mathbf{u})$ plays the role of a WZ term whose linear transformation cancels that of the Lagrangian – as described e.g., in section 12.3 of ref. [69], Chap.15 in ref. [64], or the end of Chap.4 in ref. [63]. Indeed, the defining property of a WZ term is that expected from the prepotential of a flat twisted connection: $\boldsymbol{L}_{\lambda^v}\mathsf{c}(\ ;\mathbf{u}) = \iota_{\lambda^v}\boldsymbol{d}\mathsf{c}(\ ;\mathbf{u}) = -\iota_{\lambda^v} i\,\boldsymbol{\varpi}_0 := \mathsf{a}(\lambda;\phi)$, for $\lambda \in \text{Lie}\mathcal{H}_{\text{loc}}$ and $\mathsf{a}(\lambda;\phi) = \int_U a(\lambda;\phi)$ the integraded $\mathcal{H}_{\text{loc}}$-anomaly.

We now consider the behavior under dressing of the variational principle: $\boldsymbol{d}L = \boldsymbol{E} + d\boldsymbol{\theta}$ and $\boldsymbol{d}S = 0 \xrightarrow{b.c.} \boldsymbol{E} = 0$, with boundary conditions (b.c.) imposed at ∂U. We want to establish a *relational variational principle*. The most immediate thing to propose would be to vary the dressed action, $\boldsymbol{d}S^{(\upsilon,\mathbf{u})}$. However, by (378), this is

$$\begin{aligned} \boldsymbol{d}S^{(\upsilon,\mathbf{u})} &= (\boldsymbol{d}S)^{(\upsilon,\mathbf{u})} - \langle \mathfrak{L}_{\upsilon_*^{-1}\boldsymbol{d}\upsilon} L^{(\upsilon,\mathbf{u})}),\, U^{\upsilon}\rangle, \\ &= (\boldsymbol{d}S)^{(\upsilon,\mathbf{u})} - \langle \iota_{\upsilon_*^{-1}\boldsymbol{d}\upsilon} L^{(\upsilon,\mathbf{u})},\, \partial U^{\upsilon}\rangle, \end{aligned} \tag{386}$$

which is not basic, since as a rule $[\tilde{F}^{\star}_{(\upsilon,\mathbf{u})}, \boldsymbol{d}] \neq 0$. So, this first guess does not qualify as a good candidate for relational variational principle. However, the quantity $(\boldsymbol{d}S)^{(\upsilon,\mathbf{u})}$ is basic by definition, and differ from $\boldsymbol{d}S^{(\upsilon,\mathbf{u})}$ by a boundary term. Hence, both will give the same field equations, which furthermore will be basic. Therefore, the basic object $(\boldsymbol{d}S)^{(\upsilon,\mathbf{u})}$, representing a well-defined quantity on the bundle of regions $\bar{\boldsymbol{U}}(M) := \Phi \times \boldsymbol{U}(M)/\sim$, is the natural quantity implementing a relational variational principle.[28]

It is easy to write it down: The 1-form $\boldsymbol{d}L$ transforms under $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\right)$ by (307), from which we read its dressing

$$(\boldsymbol{d}L)^{(\upsilon,\mathbf{u})} = \boldsymbol{\upsilon}^*\big(\boldsymbol{d}L + \boldsymbol{d}c(\ ;\mathbf{u}) + \mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\big(L + c(\ ;\mathbf{u})\big)\big). \tag{387}$$

Here we have indeed, $(\boldsymbol{d}L)^{(\upsilon,\mathbf{u})} = \boldsymbol{d}L^{(\upsilon,\mathbf{u})}$ since $[F^{\star}_{(\upsilon,\mathbf{u})}, \boldsymbol{d}] = 0$ on $\Phi \to \mathcal{M}$. Correspondingly,

$$(\boldsymbol{d}S)^{(\upsilon,\mathbf{u})} = \int_{U^{\upsilon}} L^{(\upsilon,\mathbf{u})} = \int_U \boldsymbol{d}L + \boldsymbol{d}c(\ ;\mathbf{u}) + \mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\big(L + c(\ ;\mathbf{u})\big). \tag{388}$$

For a Lagrangian satisfying the principles of gRGFT, i.e., satisfying $\boldsymbol{d}c(\phi;\gamma) = db(\phi;\gamma)$ under hypothesis (308), this specializes to

$$\begin{aligned} (\boldsymbol{d}S)^{(\upsilon,\mathbf{u})} &= \int_U \boldsymbol{d}L + d\big[b(\ ;\mathbf{u}) + \iota_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}} L + \iota_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\, c(\ ;\mathbf{u})\big] \\ &= \boldsymbol{d}S + \int_{\partial U} b(\ ;\mathbf{u}) + \iota_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}} L + \iota_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\, c(\ ;\mathbf{u}). \end{aligned} \tag{389}$$

The fact that $\boldsymbol{d}S$ and its dressing $(\boldsymbol{d}S)^{(\upsilon,\mathbf{u})}$ differ by boundary terms implies that the bare variational principle and relational variational principle are "equivalent", in that the space $S^{\mathbf{u}}$ of relational solutions is isomorphic to the space S of solutions of the bare theory. This we may show explicitly.

The relational (dressed) field equations are directly found via $(\boldsymbol{d}L)^{(\upsilon,\mathbf{u})} = \boldsymbol{E}^{(\upsilon,\mathbf{u})} + d\boldsymbol{\theta}^{(\upsilon,\mathbf{u})}$, and read immediately from the $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\right)$-transformation (324) of the bare field equations $\boldsymbol{E}$,

$$\boldsymbol{E}^{(\upsilon,\mathbf{u})} = \boldsymbol{\upsilon}^*\big(\boldsymbol{E} + dE\big(\iota_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\phi;\phi\big) + dE\big(\boldsymbol{d}\mathbf{u}\mathbf{u}^{-1} - \mathbf{u}\mathfrak{L}_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\mathbf{u}^{-1};\phi\big)\big). \tag{390}$$

This reproduces and generalizes results of refs. [6, 8, 42], as it specializes to general-relativistic and gauge field theories respectively as

$$\begin{aligned} \boldsymbol{E}^{\upsilon} &= \boldsymbol{\upsilon}^*\big(\boldsymbol{E} + dE\big(\iota_{\boldsymbol{d}\upsilon\circ\upsilon^{-1}}\phi;\phi\big)\big), \\ \boldsymbol{E}^{\mathbf{u}} &= \boldsymbol{E} + dE\big(\boldsymbol{d}\mathbf{u}\,\mathbf{u}^{-1};\phi\big). \end{aligned} \tag{391}$$

Equation (390) is our most important equation. It implies that the field equations $\boldsymbol{E}^{(\upsilon,\mathbf{u})} = E\left(\boldsymbol{d}\phi^{(\upsilon,\mathbf{u})};\phi^{(\upsilon,\mathbf{u})}\right) = 0$ satisfied by the physical relational variables $\phi^{(\upsilon,\mathbf{u})}$ are functionally identical as those $\boldsymbol{E} = E(\boldsymbol{d}\phi;\phi) = 0$ satisfied by the bare variables ϕ. It is of foundational importance as it explains why it has been possible to successfully apply gRGFT *before* solving the issue of its fundamental physical d.o.f. and observables. Indeed, what we actually confront to observations are the field equations $\boldsymbol{E}^{(\upsilon,\mathbf{u})} = 0$ of the dressed/relational theory $L^{(\upsilon,\mathbf{u})} = L(\phi^{(\upsilon,\mathbf{u})})$ – and *not* a "gauge-fixed" version of it, as is often claimed – but due to the fundamental principles underlying gRGFT, the latter are formally equivalent to their bare version $\boldsymbol{E} = 0$ using bare fields ϕ – which are only "partial observables" in the terminology of refs. [58, 93]. Both are even equal or in strict covariance relation when the contributions of the variation $(\boldsymbol{d}\upsilon, \boldsymbol{d}\mathbf{u})$ of the "coordinatizing" dressing field can be neglected – or when boundary terms are neglected.

A final crucial observation to be made about the dressed field equations (390) is that they describe a manifestly $\left(\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}\right)$-invariant and fully relational dynamics, where physical d.o.f. evolves w.r.t. each other, *without* a predetermined time variable. Which means not only that there are no issues with determinism, the generalized point-coincidence argument being automatically implemented, but also that there is no so-called "problem of time". The latter, also called the "frozen formalism problem",[94,95] may be formulated as the worry that, since proper time evolution on M is a special case of diffeomorphism transformation, "time evolution is pure gauge in general-relativistic physics" so that physical observables have no dynamics. The issue arises only when overlooking that M is not spacetime, and forgetting the relational core of general-relativistic physics. This "problem of time" is dissolved in the relational formulation of gRGFT, or rather never arises in the first place. We will further elaborate on the importance of this fact in a forthcoming part of this series dealing with relational quantization[96] – closely followed by a separate paper where we present a relational formulation of non-relativistic Quantum Mechanics.[97]

[28] The fact that the most natural quantity $\boldsymbol{d}S^{(\upsilon,\mathbf{u})}$ fails is a hint at the fact that a genuine relational variational principle for gRGFT should involve a variational operator on $\Phi \times \boldsymbol{U}(M)$ extending $\boldsymbol{d}$ on Φ – which only varies fields. We shall develop this viewpoint in a forthcoming paper.[92]

In the following last section, we present two applications of the above relational formalism: The case of scalar coordinatization of GR, and the case of Einstein gravity coupled to EM and a $\mathbb{C}$-scalar field.

5.2.7. Relational Einstein Equations

General Relativity with Scalar Matter: We consider GR – with cosmological constant Λ – coupled to matter described phenomenologically as a set of scalar fields. We separate the discussion of the kinematics from that of the dynamics so as to highlight that the DFM/relational reformulation takes place at the kinematical level first.

Kinematics: The field space is $\Phi = \{\phi\} = \{g, \varphi\}$, where g is a metric field on M and $\varphi : U \subset M \to N = \mathbb{R}^n$. It means, we consider the connection to be Levi-Civita, $\Gamma = \Gamma(g)$, i.e., there is no torsion $T = 0$. The action of $\text{Diff}(M) \ltimes \mathcal{H}_{\text{loc}}$ is

$$R_{(\psi,\gamma)}\,\phi = R_{(\psi,\gamma)}\left(g, \varphi\right) = \left(\psi^* g, \psi^* \varphi\right) = \left(\psi^* g, \varphi \circ \psi\right). \tag{392}$$

Naturally in this model there is no gauge group $\mathcal{H}_{\text{loc}}$, so the structure group of Φ is actually just $\text{Diff}(M)$.

The first task at hand is to identify a $\text{Diff}(M)$-dressing field. Considering the *section* of φ, $\bar{\varphi} : N = \mathbb{R}^n \to U \subset M$ s.t. $\varphi \circ \bar{\varphi} = \text{id}_N$ – i.e., the "right inverse", so $\bar{\varphi} = \varphi^{-1}$ – we may define the dressing field

$$\begin{aligned} &\boldsymbol{\upsilon} = \boldsymbol{\upsilon}(\varphi) := \bar{\varphi} : N \to M, \\ \text{s.t.}\quad &R^\star_{(\psi,\gamma)}\boldsymbol{\upsilon}(\varphi) = \boldsymbol{\upsilon}\big(R_{(\psi,\gamma)}\varphi\big) = \boldsymbol{\upsilon}(\varphi \circ \psi) := \psi^{-1} \circ \bar{\varphi} =: \psi^{-1} \circ \boldsymbol{\upsilon}(\varphi). \end{aligned} \tag{393}$$

Therefore, under $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty\left(\Phi, \text{Diff}(M)\right)$ the dressing field transforms as $\boldsymbol{\upsilon}^{(\boldsymbol{\psi},\gamma)} = \boldsymbol{\psi}^{-1} \circ \boldsymbol{\upsilon}$, as is expected. It allows to define dressed regions of spacetime:

$$U^{\upsilon} := \boldsymbol{\upsilon}^{-1}(U), \quad \text{s.t.} \quad (U^{\upsilon})^{\boldsymbol{\psi}} = U^{\upsilon}. \tag{394}$$

This gives a $\text{Diff}(M)$- *and* $C^\infty\left(\Phi, \text{Diff}(M)\right)$-invariant relational definition of physical spacetime M^{υ} via matter φ as a reference physical system, as is expected from the point-coincidence argument. The dressed field space is then

$$\Phi^{\upsilon} = \{\phi^{\upsilon}\} = \{g^{\upsilon}, \varphi^{\upsilon}\} = \{\boldsymbol{\upsilon}^* g, \text{id}_N\}. \tag{395}$$

The $C^\infty\left(\Phi, \text{Diff}(M)\right)$-invariant dressed metric g^{υ} encodes the geometric properties of spacetime M^{υ}. It can be understood as the bare metric g "written in" ("dressed by") the coordinate system supplied by the matter distribution φ: writing in abstract index notation we have

$$(g^{\upsilon})_{ab} = \frac{\partial x^\mu}{\partial \boldsymbol{\upsilon}^a}\frac{\partial x^\nu}{\partial \boldsymbol{\upsilon}^b}\, g_{\mu\nu}. \tag{396}$$

Clearly, $\varphi^{\upsilon} = \text{id}_N$ simply expresses the fact of coordinatizing matter distribution w.r.t. itself, i.e., that it is (invariantly) at rest in its own reference frame.

Dynamics: The (bare) Lagrangian of the theory L is $\text{Diff}(M)$-equivariant and (trivially) $\mathcal{H}_{\text{loc}}$-invariant,

$$\begin{aligned} L(g, \varphi) &= L_{\text{EH}+\Lambda}(g) + L_{\text{Matter}}(g, \varphi) \\ &= \tfrac{1}{2\kappa}\,\text{vol}_g\left(\mathsf{R}(g) - 2\Lambda\right) + L_{\text{Matter}}(g, \varphi), \end{aligned} \tag{397}$$

where $\kappa = \frac{8\pi G}{c^4}$ is the gravitational coupling constant, $\text{vol}_g = \sqrt{|g|}\,d^n x$ is the volume form induced by g, and $\mathsf{R}(g)$ is the Ricci scalar: in abstract index notation $\mathsf{R}(g) := g^{\mu\nu}R_{\mu\nu}$, or $\mathsf{R}(g) := g^{-1}\cdot \text{Ricc}$, with Ricc the Ricci tensor. The field equations associated to the variation w.r.t. g are

$$\begin{aligned} &\boldsymbol{E} = E(\boldsymbol{d}g; \{g, \varphi\}) = \boldsymbol{d}g \cdot \left(G(g) + \Lambda g - \kappa T(g, \varphi)\right)\text{vol}_g, \\ \text{s.t.}\quad &\boldsymbol{E} = 0 \quad\Rightarrow\quad G(g) + \Lambda g = \kappa T(g, \varphi), \end{aligned} \tag{398}$$

where $G(g) := \text{Ricc} - \frac{1}{2}\mathsf{R}g$ is the Einstein tensor, and the energy-momentum tensor associated to the matter distribution φ is defined by $\boldsymbol{d}g \cdot T(g, \varphi)\,\text{vol}_g := -\frac{1}{2}\boldsymbol{d}L_{\text{Matter}}$. Since φ is not a fundamental field, i.e., non-variational, the equation of motion of one of its constituting particles with 4-velocity v is given by the geodesic equation $\nabla^g v = dv + \Gamma(g)v = 0$. Clearly $R^\star_{(\psi,\gamma)}\boldsymbol{E} = \psi^*\boldsymbol{E}$, so satisfies hypothesis (313) as one would expect in gRGFT. By application of (324)–(325), the $\boldsymbol{Diff}_v(\Phi) \simeq C^\infty(\Phi, \text{Diff}(M))$-transformation of $\boldsymbol{E}$ is

$$\begin{aligned} &\boldsymbol{E}^{\boldsymbol{\psi}} = \boldsymbol{\psi}^*\left(\boldsymbol{E} + dE(\iota_{d\boldsymbol{\psi}\circ\boldsymbol{\psi}^{-1}}g; \{g, \varphi\})\right), \\ &\Rightarrow\ \boldsymbol{d}g^{\boldsymbol{\psi}} \cdot \left(G^{\boldsymbol{\psi}} + \Lambda g^{\boldsymbol{\psi}} - \kappa T^{\boldsymbol{\psi}}\right)\text{vol}_{g^{\boldsymbol{\psi}}} \\ &\quad = \boldsymbol{\psi}^*\Big[\boldsymbol{d}g \cdot (G(g) + \Lambda g -, \kappa T(g, \varphi))\,\text{vol}_g \\ &\qquad + d\big(\iota_{d\boldsymbol{\psi}\circ\boldsymbol{\psi}^{-1}}g \cdot (G(g) + \Lambda g - \kappa T(g, \varphi))\,\text{vol}_g\big)\Big]. \end{aligned} \tag{399}$$

Which implies that if $G + \Lambda g = \kappa T$ then $G^{\boldsymbol{\psi}} + \Lambda g^{\boldsymbol{\psi}} = \kappa T^{\boldsymbol{\psi}}$, for any *$\phi$-dependent diffeomorphism* $\boldsymbol{\psi}$. This shows that the covariance group of GR is much bigger than $\text{Diff}(M)$ as required by hypothesis, and as usually understood: it includes ϕ-dependent diffeomorphisms $C^\infty\left(\Phi, \text{Diff}(M)\right)$ – as already observed, and first noticed by refs. [21, 66].

We can now write down the relational reformulation of GR via DFM. The dressed Lagrangian is

$$\begin{aligned} L^{\upsilon}(g, \varphi) := \boldsymbol{\upsilon}^* L(g, \varphi) = L(g^{\upsilon}, \varphi^{\upsilon}) &= L_{\text{EH}+\Lambda}(g^{\upsilon}) + L_{\text{Matter}}(g^{\upsilon}, \varphi^{\upsilon}) \\ &= \tfrac{1}{2\kappa}\,\text{vol}_{g^{\upsilon}}\left(\mathsf{R}(g^{\upsilon}) - 2\Lambda\right) \\ &\quad + L_{\text{Matter}}(g^{\upsilon}, \varphi^{\upsilon}), \end{aligned} \tag{400}$$

with corresponding *dressed Einstein equations*

$$\begin{aligned} \boldsymbol{E}^{\upsilon} &= E(\boldsymbol{d}g^{\upsilon}; \{g^{\upsilon}, \varphi^{\upsilon}\}) \\ &= \boldsymbol{d}g^{\upsilon} \cdot \left(G(g^{\upsilon}) + \Lambda g^{\upsilon} - \kappa T(g^{\upsilon}, \varphi^{\upsilon})\right)\text{vol}_g, \\ \text{s.t.}\quad \boldsymbol{E}^{\upsilon} = 0 \quad&\Rightarrow\quad G^{\upsilon} + \Lambda g^{\upsilon} = \kappa T^{\upsilon}. \end{aligned} \tag{401}$$

These are the strictly $\mathrm{Diff}(M)$- *and* $C^\infty\big(\Phi, \mathrm{Diff}(M)\big)$-invariant relational Einstein field equations, which have a well-posed Cauchy problem. We stress that $\boldsymbol{E}^{\boldsymbol{v}}$ are *not* a gauge-fixed version of the bare Einstein equations $\boldsymbol{E}$, as argued at the end of Section 5.2.6. Their explicit form in terms of the bare field equations $\boldsymbol{E}$ is read from (399) and applying the DFM rule of thumb:

$$\begin{aligned}
&\boldsymbol{E}^{\boldsymbol{v}} = \boldsymbol{v}^*\big(\boldsymbol{E} + dE(\iota_{d\boldsymbol{v}\circ\boldsymbol{v}^{-1}}g; \{g,\varphi\})\big),\\
\Rightarrow\ & \boldsymbol{d}g^{\boldsymbol{v}} \cdot \big(G^{\boldsymbol{v}} + \Lambda g^{\boldsymbol{v}} - \kappa T^{\boldsymbol{v}}\big)\,\mathrm{vol}_{g^{\boldsymbol{v}}} = \boldsymbol{v}^*\Big[\boldsymbol{d}g\cdot(G(g) + \Lambda g\\
&\qquad - \kappa T(g,\varphi))\,\mathrm{vol}_g\\
&\qquad + d\big(\iota_{d\boldsymbol{v}\circ\boldsymbol{v}^{-1}}g\cdot(G(g)\\
&\qquad + \Lambda g - \kappa T(g,\varphi))\,\mathrm{vol}_g\big)\Big].
\end{aligned} \tag{402}$$

It shows the equivalence between the bare Einstein equations $G + \Lambda g = \kappa T$ and the relational Einstein equations $G^{\boldsymbol{v}} + \Lambda g^{\boldsymbol{v}} = \kappa T^{\boldsymbol{v}}$. We also see that if the variation of the coordinatizing (dressing) field $\boldsymbol{dv}$ is neglected, the two are in simple covariance relation: $\boldsymbol{E}^{\boldsymbol{v}} = \boldsymbol{v}^*\boldsymbol{E}$. This, as we observed on more than one occasion, explains why confrontation of GR with observations has been so successful early on, even though the problem of identifying its fundamental physical d.o.f. and observables has remained a longstanding open issue. It is thus not a gauge-fixed version of GR that passes observational tests, as is sometimes claimed, but its relational dressed version (400)–(401).

This model can be considered to apply to the historical version of GR where gravity is coupled to matter which is described heuristically, or effectively, as a fluid (gas, particles, dust, etc.). The scalar fields φ^a could then represents the 4-velocity field of matter. This encompasses and extends various "scalar coordinatizations" of GR, such as refs. [98–100]. A variant of it may be applied to pure gravity: then the dressing field is metric dependent, $\boldsymbol{v} = \boldsymbol{v}(g)$, and supplies a g-dependent $\mathrm{Diff}(M)$-invariant definition of space time regions $U^{\boldsymbol{v}}$. Or, seen otherwise, a g-dependent coordinate system. This encompasses e.g., the classic works[21,66,101] where $\boldsymbol{v}$ is built from n $(= 4)$ independent (non-vanishing) scalar invariants, like the Kretschmann–Komar invariants – compare e.g., (396) to Equation (2.2).[101]

Finally, let us highlight the fact that the relational formulation suggests a natural heuristic argument for the necessity of quantization of gravity. An often repeated such heuristics relies on the dynamics: In view of the bare Einstein equations (398), $G(g) + \Lambda g = \kappa T(g,\varphi)$, the left-hand side contains only metric d.o.f., while the right-hand side contains also the d.o.f. of matter. Now, as the argument goes, if the matter d.o.f. are quantized, so that the energy-momentum becomes an operator $\hat{T}$ with vacuum expected value $\langle\hat{T}\rangle = T$, one expects that the left-hand side should also contain quantized metric d.o.f. $\hat{g}$ s.t. $\langle\hat{g}\rangle = g$, and so that a "quantum Einstein equation" $(\hat{G} + \Lambda\hat{g})\,|\Psi\rangle = \kappa\hat{T}\,|\Psi\rangle$ may hold.

In view of the relational Einstein equation (401), $G^{\boldsymbol{v}} + \Lambda g^{\boldsymbol{v}} = \kappa T^{\boldsymbol{v}}$, this argument is flawed. Indeed, the matter d.o.f. involved in the dressing field $\boldsymbol{v}$ are seen to actually contribute, like the metric d.o.f., to both sides of the field equations. Actually, on the basis of the relational formulation, a stronger, more compelling argument can be made, that does not rely on the classical on-shell dynamics, but rather holds *at the kinematical level*. Indeed, the very definition of the physical relational metric (395)–(396) involves the matter d.o.f., hence if the latter are quantized one expects to naturally obtain a notion of quantized gravitational d.o.f., so that $g^{\boldsymbol{v}} \mapsto \widehat{g^{\boldsymbol{v}}}$ and $\langle\widehat{g^{\boldsymbol{v}}}\rangle = g^{\boldsymbol{v}}$. Furthermore, by the same argument, one may obtain a notion of quantized (relationally defined) spacetime regions $U^{\boldsymbol{v}} \mapsto U^{\hat{\boldsymbol{v}}}$, that *does not* imply a quantization of the underlying manifold M – which is unobservable. This kinematical argument can be summarized via the following sketch:

Relational d.o.f. in GR + Quantum reference system $\Downarrow$ Kinematical quantum gravity

We will revisit and expand on these observations in a forthcoming paper.[96] We now turn to the second example of gravity coupled to EM and a charged complex scalar matter field.

Einstein-$\mathbb{C}$-Maxwell Model: We shall now consider GR plus cosmological constant coupled to EM and a charged complex scalar matter field ϕ.

Kinematics: The field space is $\Phi = \{\phi\} = \{g, A, \phi\}$, where g is a metric field, A is a $\mathfrak{u}(1)$-valued EM potential, and ϕ is a $\mathbb{C}$-valued scalar field. It means here $\mathcal{H}_{\text{loc}} = \mathcal{U}(1)$. The EM field strength is $F = dA$, and the minimal coupling between the EM potential and the matter field is $D\phi = d\phi + A\phi$. The action of $\mathrm{Diff}(M) \ltimes \mathcal{U}(1)$ is

$$\begin{aligned}
R_{(\psi,\gamma)}\phi &= R_{(\psi,\gamma)}\,(g, A, \phi) = \big(\psi^*g,\ \psi^*(A^\gamma),\ \psi^*(\phi^\gamma)\big)\\
&= \big(\psi^*g,\ \psi^*(A + \gamma^{-1}d\gamma), \psi^*(\gamma^{-1}\phi)\big).
\end{aligned} \tag{403}$$

In this model, it is possible to dress in steps, for $\mathcal{U}(1)$ first, and then for $\mathrm{Diff}(M)$, as indicated in Section 5.2.3.

The first step in thus to identify a $\mathcal{U}(1)$-dressing field. It is easily done: Using a polar decomposition of the matter field $\phi = \rho\exp i\theta$, where $\rho = |\phi|^{1/2}$ is the modulus and θ is the phase, it is clear by (403) that we may define

$$\begin{aligned}
&\mathbf{u} = \mathbf{u}(\phi) \coloneqq \exp i\theta,\\
\text{s.t.}\quad & R^\star_{(\psi,\gamma)}\mathbf{u}(\phi) = \mathbf{u}\big(R_{(\psi,\gamma)}\phi\big) \coloneqq \psi^*\big(\gamma^{-1}\exp i\theta\big) \eqqcolon \psi^*\big(\gamma^{-1}\mathbf{u}(\phi)\big).
\end{aligned} \tag{404}$$

By the dressing property $R^\star_\gamma\mathbf{u} = \gamma^{-1}\mathbf{u}$, we define the $\mathcal{U}(1)$-invariant fields

$$\Phi^{\mathbf{u}} = \{\phi^{\mathbf{u}}\} = \{g^{\mathbf{u}},\ A^{\mathbf{u}},\ \phi^{\mathbf{u}}\} \coloneqq \{g,\ A + \mathbf{u}^{-1}d\mathbf{u},\ \rho\}, \tag{405}$$

with $\phi^u = \mathbf{u}^{-1}\phi = \rho$ by definition. The variable $A^{\mathbf{u}}$ implements the idea that the gauge invariant internal d.o.f. of the EM potential are coordinatized w.r.t. that of the scalar matter field (i.e., its phase θ), in accordance with the insight gained from the internal point-coincidence argument from Section 2. Meanwhile $\phi^{\mathbf{u}} = \rho$ signifies the self-coordinatization of the internal d.o.f. of matter, i.e., it has constant 0 phase w.r.t. itself: this is the internal version of "being at rest" in its own reference frame, analogue to the case of the real scalar field in the previous example.

The dressed EM field strength is $F^{\mathbf{u}} \coloneqq dA^{\mathbf{u}} = \mathbf{u}^{-1}F\mathbf{u} = F$, and happens to coincide with the bare EM field strength, which as is well-known is the only (local) gauge-invariant field in the bare formulation of classical EM. Also, observe that the dressed covariant derivative is $(D\phi)^{\mathbf{u}} = \mathbf{u}^{-1}(D\phi) = D^{\mathbf{u}}\phi^{u} = D^{\mathbf{u}}\rho = d\rho + A^{\mathbf{u}}\rho$, which represents the minimal coupling between the invariant d.o.f. of the EM potential and the matter field. It shows that in the relational formulation it is possible to write the coupling of invariantly defined fields – which is partially in tension with some comments around the main thesis proposed in ref. [93]. As a side comment, we remark that the variables $\{A^{\mathbf{u}}, \rho\}$ allow in particular to model the Aharanov–Bohm effect in an invariant *and local* way, i.e., as arising from the pointwise (field-local) interaction between physical fields, described by $D^{\mathbf{u}}\rho = d\rho + A^{\mathbf{u}}\rho$. See e.g., refs. [102] or [7] Chap.5, and also ref. [1].

By the compatibility condition $R^{\star}_{\psi}\mathbf{u} = \psi^{*}\mathbf{u}$, the $\mathcal{U}(1)$-invariant ("internally relational") variables have well-defined $\mathrm{Diff}(M)$-transformations: the action of $\mathrm{Diff}(M)$ on $\Phi^{\mathbf{u}}$ being as expected

$$R_{\psi}\phi^{\mathbf{u}} = R_{\psi}\left(g,\, A^{\mathbf{u}},\, \rho\right) = \left(\psi^{*}g,\, \psi^{*}(A^{\mathbf{u}}),\, \psi^{*}\rho\right). \tag{406}$$

This makes it natural to look for a $\mathrm{Diff}(M)$-dressing field. The invariant EM potential is $A^{\mathbf{u}} = A^{\mathbf{u}}{}_{\mu}\, dx^{\mu}$, and its components form a scalar field $\mathsf{A}^{\mathbf{u}} \coloneqq A^{\mathbf{u}}{}_{\mu} : M \to N = \mathbb{R}^{n}$. We consider its section $\overline{\mathsf{A}^{\mathbf{u}}} : N = \mathbb{R}^{n} \to M$, which is s.t. $\mathsf{A}^{\mathbf{u}} \circ \overline{\mathsf{A}^{\mathbf{u}}} = \mathrm{id}_{N}$ (a right inverse). We then define the $\mathrm{Diff}(M)$-dressing field

$$\begin{aligned} &\boldsymbol{\upsilon} = \boldsymbol{\upsilon}(A^{\mathbf{u}}) = \boldsymbol{\upsilon}(A,\phi) \coloneqq \overline{\mathsf{A}^{\mathbf{u}}} : N \to M, \\ \text{s.t.}\quad & R^{\star}_{(\psi,\gamma)}\boldsymbol{\upsilon}(A,\phi) = R^{\star}_{(\psi,\gamma)}\boldsymbol{\upsilon}(A^{\mathbf{u}}) = \boldsymbol{\upsilon}\big(R_{(\psi,\gamma)}A^{\mathbf{u}}\big) = \boldsymbol{\upsilon}\big(\psi^{*}(A^{\mathbf{u}})\big) \\ &\coloneqq \psi^{-1}\circ \overline{\mathsf{A}^{\mathbf{u}}} \eqqcolon \psi^{-1}\circ\boldsymbol{\upsilon}(A,\phi). \end{aligned} \tag{407}$$

Therefore, under $\boldsymbol{Diff}_{v}(\Phi) \simeq C^{\infty}\left(\Phi, \mathrm{Diff}(M)\right)$ this dressing field transforms as $\boldsymbol{\upsilon}^{(\boldsymbol{\psi},\boldsymbol{\gamma})} = \boldsymbol{\psi}^{-1}\circ\boldsymbol{\upsilon}$, as is expected. It allows to define dressed regions of spacetime:

$$U^{\upsilon} \coloneqq \boldsymbol{\upsilon}^{-1}(U), \quad \text{s.t.} \quad (U^{\upsilon})^{(\boldsymbol{\psi},\boldsymbol{\gamma})} = U^{\upsilon}. \tag{408}$$

As expected from the point-coincidence argument, this gives a $\big(\mathrm{Diff}(M)\ltimes\mathcal{U}(1)\big)$- *and* $C^{\infty}\left(\Phi, \mathrm{Diff}(M)\ltimes\mathcal{U}(1)\right)$-invariant relational definition of spacetime, with the electromagnetic sector field content $\{A,\phi\}$, or more precisely the invariant physical electromagnetic d.o.f. $A^{\mathbf{u}}$, being the reference physical system.

The final dressed field space is then

$$\Phi^{(\upsilon,\mathbf{u})} = \{\phi^{(\upsilon,\mathbf{u})}\} = \{g^{\upsilon},\, A^{(\upsilon,\mathbf{u})},\, \rho^{\upsilon}\} = \{\boldsymbol{\upsilon}^{*}g,\, \boldsymbol{\upsilon}^{*}(A^{\mathbf{u}}),\, \boldsymbol{\upsilon}^{*}\rho\}. \tag{409}$$

These are living on the *physical* spacetime, whose regions are defined by (408). For example, the scalar matter field is s.t. $\rho^{\upsilon} = \rho\circ\boldsymbol{\upsilon} : U^{\upsilon} \to \mathbb{R}$. This again illustrates the point-coincidence argument, and the co-definition of the physical field d.o.f., as the values of ρ^{υ} are given *at* values of $A^{(\upsilon,\mathbf{u})}$ (via υ), this coincidence of values defining, "tagging" or identifying, a physical spacetime point. The minimal coupling of both fields is given by the dressed covariant derivative $D^{(\upsilon,\mathbf{u})}\rho^{\upsilon} = d\rho^{\upsilon} + A^{(\upsilon,\mathbf{u})}\rho^{\upsilon}$, which is then actually an *invariant* derivative. Likewise, the $C^{\infty}\left(\Phi, \mathrm{Diff}(M)\ltimes\mathcal{U}(1)\right)$-invariant dressed field g^{υ} describes the metric properties of the physical, relationally defined, spacetime $M^{(\upsilon,\mathbf{u})}$. It can be seen as the bare metric g "written" in the coordinate system supplied by the (invariant) electromagnetic field content $A^{\mathbf{u}} \sim (A,\phi)$: in abstract index notation

$$(g^{\upsilon})_{ab} = \frac{\partial x^{\mu}}{\partial \boldsymbol{\upsilon}^{a}}\frac{\partial x^{\nu}}{\partial \boldsymbol{\upsilon}^{b}}\, g_{\mu\nu}, \qquad \text{with} \qquad \boldsymbol{\upsilon} = \boldsymbol{\upsilon}(A^{\mathbf{u}}) = \boldsymbol{\upsilon}(A,\phi). \tag{410}$$

From the invariant metric g^{υ} one derives the invariant (physical) Levi–Civita connection field $\Gamma^{\upsilon} = \Gamma(g^{\upsilon})$, thus the invariant covariant derivative $\nabla^{g^{\upsilon}}$, which defines the physical parallel transport and geodesic on spacetime $M^{(\upsilon,\mathbf{u})}$. The invariant Riemann tensor of spacetime is $\mathrm{Riem}^{\upsilon} = \mathrm{Riem}(g^{\upsilon})$, fom which one derive the invariant Ricci tensor Ricc^{υ} and Ricci scalar $\mathsf{R}(g^{\upsilon}) = (g^{\upsilon})^{-1}\cdot\mathrm{Ricc}^{\upsilon}$. The physical Einstein tensor is thus $G^{\upsilon} = G(g^{\upsilon}) = \mathrm{Ricc}^{\upsilon} - \tfrac{1}{2}\mathsf{R}^{\upsilon}g^{\upsilon}$.

This achieves our analysis of the relational kinematics of the theory. We now consider its dynamics.

Dynamics: The (bare) Lagrangian of the theory L is

$$\begin{aligned} L(g,A,\phi) &= L_{\mathrm{EH}+\Lambda}(g) + L_{\mathrm{EM}}(g,A) + L_{\mathrm{KG}}(g,A,\phi) \\ &= \tfrac{1}{2\kappa}\,\mathrm{vol}_{g}\left(\mathsf{R}(g) - 2\Lambda\right) + \tfrac{1}{2}F *_{g} F \\ &\quad + \tfrac{1}{2}\big(\langle D\phi, *_{g} D\phi\rangle + m^{2}\langle\phi, *_{g}\phi\rangle\big), \end{aligned} \tag{411}$$

where $\langle\ ,\ \rangle : \mathbb{C}\times\mathbb{C}\to\mathbb{C}$, $(v,w)\mapsto\langle v,w\rangle \coloneqq v^{\dagger}w$, with $v^{\dagger}$ the conjugate of v, and $*_{g}$: $\Omega^{p}(U)\to\Omega^{n-p}(U)$ is the Hodge dual operator. The field equation 1-form is

$$\begin{aligned} \boldsymbol{E} = E(\boldsymbol{d}\phi;\phi) &= E(\boldsymbol{d}g;\phi) + E(\boldsymbol{d}A;\phi) + E(\boldsymbol{d}\phi;\phi) \\ &= \boldsymbol{d}g\cdot\big(G(g) + \Lambda g - \kappa T(g,A,\phi)\big)\,\mathrm{vol}_{g} \\ &\quad + \boldsymbol{d}A\big(d *_{g} F - J(A,\phi)\big) + \langle\boldsymbol{d}\phi, D *_{g} D\phi + m^{2}\phi\rangle, \end{aligned} \tag{412}$$

where T and J are, respectively, the energy-momentum tensor of the EM field and charge matter field, and the EM current. Explicitly,

$$\begin{aligned} T(g,A,\phi) &= T_{\mu\nu}(g,A) + T_{\mu\nu}(g,\phi) = \Big(F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta} - \tfrac{1}{4}g_{\mu\nu}F\cdot F\Big)\,\mathrm{vol}_{g} \\ &\quad + \Big(\partial_{\mu}\phi\,\partial_{\nu}\phi - \tfrac{1}{2}g_{\mu\nu}\,\partial\phi\cdot\partial\phi\Big)\,\mathrm{vol}_{g}, \\ J(A,\phi) &\coloneqq \tfrac{1}{2}\big(\langle *_{g} D\phi, \phi\rangle - \langle\phi, *_{g} D\phi\rangle\big). \end{aligned} \tag{413}$$

Then, $\boldsymbol{E} = 0$ implies

$$G(g) + \Lambda g = \kappa T(g,A,\phi), \quad d *_{g} F = J(A,\phi), \quad \big(D *_{g} D + m^{2}\big)\phi = 0. \tag{414}$$

These are respectively the Einstein equation, the Maxwell equation, and the Klein–Gordon equation.

It is easy to check that $R^{\star}_{(\psi,\gamma)}\boldsymbol{E} = \psi^{*}(\boldsymbol{E}^{\gamma}) = \psi^{*}\boldsymbol{E}$, satisfying the general hypothesis (313) of gRGFT. One may also verify that, applying (324)–(325), the $\boldsymbol{Diff}_{v}(\Phi) \simeq C^{\infty}\left(\Phi, \mathrm{Diff}(M)\ltimes\mathcal{U}(1)\right)$-transformation $\boldsymbol{E}^{(\boldsymbol{\psi},\boldsymbol{\gamma})}$ are s.t. $\boldsymbol{E} = 0 \Rightarrow \boldsymbol{E}^{(\boldsymbol{\psi},\boldsymbol{\gamma})} = 0$. Which means that the covariance group of the theory is much bigger than $\mathrm{Diff}(M)\ltimes\mathcal{U}(1)$, and encompass ϕ-dependent transformations:

its full covariance group is $C^\infty\left(\Phi, \mathrm{Diff}(M)\ltimes\mathcal{U}(1)\right)$. An observation that generalizes the one made in refs. [21, 66].

We now write down the relational reformulation of the theory via DFM. The dressed Lagrangian reads

$$\begin{aligned}L^{(\boldsymbol{v},\mathbf{u})}(g,A,\phi) &:= \boldsymbol{v}^* L^{\mathbf{u}}(g,A,\phi) = L(g^{\boldsymbol{v}}, A^{(\boldsymbol{v},\mathbf{u})}, \rho^{\boldsymbol{v}})\\ &= L_{\mathrm{EH}+\Lambda}(g^{\boldsymbol{v}}) + L_{\mathrm{EM}}(g^{\boldsymbol{v}}, A^{(\boldsymbol{v},\mathbf{u})}) + L_{\mathrm{KG}}(g^{\boldsymbol{v}}, A^{(\boldsymbol{v},\mathbf{u})}, \rho^{\boldsymbol{v}})\\ &= \tfrac{1}{2\kappa}\,\mathrm{vol}_{g^{\boldsymbol{v}}}\left(\mathsf{R}(g^{\boldsymbol{v}}) - 2\Lambda\right) + \tfrac{1}{2} F^{(\boldsymbol{v},\mathbf{u})} *_{g^{\boldsymbol{v}}} F^{(\boldsymbol{v},\mathbf{u})}\\ &\quad + \tfrac{1}{2}\left(\langle D^{(\boldsymbol{v},\mathbf{u})}\rho^{\boldsymbol{v}}, *_{g^{\boldsymbol{v}}} D^{(\boldsymbol{v},\mathbf{u})}\rho^{\boldsymbol{v}}\rangle + m^2\langle \rho^{\boldsymbol{v}}, *_{g^{\boldsymbol{v}}}\rho^{\boldsymbol{v}}\rangle\right),\end{aligned} \tag{415}$$

with corresponding dressed field equation 1-form

$$\begin{aligned}\boldsymbol{E}^{(\boldsymbol{v},\mathbf{u})} &= E(\boldsymbol{d}g^{\boldsymbol{v}};\phi^{(\boldsymbol{v},\mathbf{u})}) + E(\boldsymbol{d}A^{(\boldsymbol{v},\mathbf{u})};\phi^{(\boldsymbol{v},\mathbf{u})}) + E(\boldsymbol{d}\rho^{\boldsymbol{v}};\phi^{(\boldsymbol{v},\mathbf{u})})\\ &= \boldsymbol{d}g^{\boldsymbol{v}}\cdot\left(G(g^{\boldsymbol{v}}) + \Lambda g^{\boldsymbol{v}} - \kappa T(g^{\boldsymbol{v}}, A^{(\boldsymbol{v},\mathbf{u})}, \rho^{\boldsymbol{v}})\right)\mathrm{vol}_{g^{\boldsymbol{v}}}\\ &\quad + \boldsymbol{d}A^{(\boldsymbol{v},\mathbf{u})}\left(d *_{g^{\boldsymbol{v}}} F^{(\boldsymbol{v},\mathbf{u})} - J(A^{(\boldsymbol{v},\mathbf{u})}, \rho^{\boldsymbol{v}})\right)\\ &\quad + \langle \boldsymbol{d}\rho^{\boldsymbol{v}}, D^{(\boldsymbol{v},\mathbf{u})} *_{g^{\boldsymbol{v}}} D^{(\boldsymbol{v},\mathbf{u})}\rho^{\boldsymbol{v}} + m^2\rho^{\boldsymbol{v}}\rangle,\end{aligned} \tag{416}$$

so that $\boldsymbol{E}^{(\boldsymbol{v},\mathbf{u})} = 0$ implies

$$\begin{aligned}&G^{\boldsymbol{v}} + \Lambda g^{\boldsymbol{v}} = \kappa T^{(\boldsymbol{v},\mathbf{u})}, \quad d *_{g^{\boldsymbol{v}}} F^{(\boldsymbol{v},\mathbf{u})} = J(A^{(\boldsymbol{v},\mathbf{u})}, \rho^{\boldsymbol{v}}),\\ &\left(D^{(\boldsymbol{v},\mathbf{u})} *_{g^{\boldsymbol{v}}} D^{(\boldsymbol{v},\mathbf{u})} + m^2\right)\rho^{\boldsymbol{v}} = 0.\end{aligned} \tag{417}$$

The latter are the strictly $\left(\mathrm{Diff}(M)\ltimes\mathcal{U}(1)\right)$- *and* $C^\infty\left(\Phi, \mathrm{Diff}(M)\ltimes\mathcal{U}(1)\right)$-invariant relational Einstein, Maxwell, Klein–Gordon field equations, which all have a well-posed initial value (Cauchy) problem. One may check, using (390)–(391) to express $\boldsymbol{E}^{(\boldsymbol{v},\mathbf{u})}$ in terms of the bare $\boldsymbol{E}$, that $\boldsymbol{E} = 0 \Rightarrow \boldsymbol{E}^{(\boldsymbol{v},\mathbf{u})} = 0$. This is the reason why the bare formalism gives sensible physical results, as it is actually the relational version (415)–(417) of the theory which is confronted to experimental tests, not a gauge-fixed version of it as is often said – we indeed insist that $L^{(\boldsymbol{v},\mathbf{u})}$ and $\boldsymbol{E}^{(\boldsymbol{v},\mathbf{u})}$ are *not* gauge-fixed versions of the bare L and $\boldsymbol{E}$.

This model points towards an *enriched* physical spacetime whose *points internal structure* is that of a compact, 1-dimensional manifold, and whose geometric structure may be represented by a $U(1)$-principal bundle. See ref. [1].

Finally, let us observe that this relational formulation provides a straighforward heuristic argument for the naturality of quantization of both the EM and the gravitational field.

Such an argument, for the electromagnetic field, is a dynamical one and relies on the bare Maxwell equation $d *_g F = J(A, \phi)$: if the matter d.o.f. – both internal and external – on the right-hand side are quantized, one would expect that the EM d.o.f. on the left-hand side are also quantized. But looking at the $\mathcal{U}(1)$-invariant, *internally relational* Maxwell equation $d *_g F^{\mathbf{u}} = J(A^{\mathbf{u}}, \rho)$, we see that actually the matter d.o.f. contribute on both sides, via $\mathbf{u} = \mathbf{u}(\phi)$. The relational formulation actually allows to make the argument at the kinematical level: In view of the $\mathcal{U}(1)$-invariant relational variable $A^{\mathbf{u}} = A + \mathbf{u}^{-1}d\mathbf{u}$, with $\mathbf{u} = \mathbf{u}(\phi)$, if ϕ is quantized, it seems unavoidable that the invariant EM potential $A^{\mathbf{u}}$, and the EM field strength $F^{\mathbf{u}}$, should be too. This argument can be thus summarized as follows:

$$\boxed{\begin{array}{c}\text{Relational d.o.f. in scalar EM + Quantum (internal) ref. system}\\ \Downarrow\\ \text{Kinematical quantum EM field}\end{array}}$$

Similarly, and as seen in the previous example, looking at the bare Einstein equations $G(g) + \Lambda g = \kappa T(g, A, \phi)$, it may seem that quantization of the d.o.f. of ϕ on the right-hand side would suggest the quantization of the the d.o.f. of g, while leaving A a priori unaffected. One would need to appeal to the bare Maxwell equation to hint at quantization of A. But inspection of the relational Einstein equations $G(g^{\boldsymbol{v}}) + \Lambda g^{\boldsymbol{v}} = \kappa T(g^{\boldsymbol{v}}, A^{(\boldsymbol{v},\mathbf{u})}, \rho^{\boldsymbol{v}})$ shows that the matter d.o.f *and* the EM d.o.f. contribute to both sides via $\boldsymbol{v} = \upsilon(A^{\mathbf{u}}) = \boldsymbol{v}(A, \phi)$. Also, just by looking at the right-hand side, it already appears that quantization of ϕ would kinematically imply that of $A^{(\boldsymbol{v},\mathbf{u})}$, without using the relational Maxwell equation $d *_{g^{\boldsymbol{v}}} F^{\boldsymbol{v}} = J(A^{(\boldsymbol{v},\mathbf{u})}, \rho^{\boldsymbol{v}})$.

This indicates that the relational formulation via DFM makes for a solely kinematical argument: Simply by looking at the relational variables $A^{(\boldsymbol{v},\mathbf{u})} = \boldsymbol{v}^*(A^{\mathbf{u}})$ and $g^{\boldsymbol{v}} = \boldsymbol{v}^* g$, with $\mathbf{u} = \mathbf{u}(\phi)$ and $\boldsymbol{v} = \boldsymbol{v}(A^{\mathbf{u}}) = \boldsymbol{v}(A, \phi)$, it appears that quantization of the d.o.f. of ϕ suggests that of the physical EM field $A^{(\boldsymbol{v},\mathbf{u})}$, as well as that of the physical gravitational field $g^{\boldsymbol{v}}$. We summarize this by

$$\boxed{\begin{array}{c}\text{Relational d.o.f. in GR-EM + Quantum reference system}\\ \Downarrow\\ \text{Kinematical quantum EM}\\ \Downarrow\\ \text{Kinematical quantum gravity}\end{array}}$$

The topic of relational quantization, expanding on these observations and going beyond these heuristic arguments, will be addressed in ref. [96].

6. Conclusion

In this paper we have developed a relational formulation for gRGFT based on the DFM. To do so, we exploited the bundle geometry of field space Φ, highlighting in particular the structures encoding the physical d.o.f., i.e., the moduli space $\mathcal{M}$ and what we called the associated bundle of regions $\bar{\boldsymbol{U}}(M) := \Phi \times \boldsymbol{U}(M)/\sim$. The latter in particular encodes the *physical* spacetime regions, as understood via the point-coincidence argument. A key result stemming from understanding the geometry of the field space of gRGFT is (324),

$$\boxed{\boldsymbol{E}^{(\boldsymbol{\psi},\boldsymbol{\gamma})} = \boldsymbol{\psi}^*\Big(\boldsymbol{E} + dE\big(\iota_{d\boldsymbol{\psi}\circ\boldsymbol{\psi}^{-1}}\phi;\phi\big) + dE\big(\boldsymbol{d\gamma\gamma}^{-1} - \boldsymbol{\gamma}\mathfrak{L}_{d\boldsymbol{\psi}\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\gamma}^{-1};\phi\big)\Big)}$$

which shows that, while the covariance group of gRGFT is usually understood – or defined – to be $\mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}$, it actually is the much bigger group $C^\infty\left(\Phi, \mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}\right)$ of *field-dependent transformations*. This generalizes the observation by refs. [21, 66].

The DFM is then, in a nutshell, a way to realize *basic* objects on Φ and $\Phi\times\boldsymbol{U}(M)$, which can be naturally interpreted as relational coordinatizations on $\mathcal{M}$ and $\bar{\boldsymbol{U}}(M)$. The relational formulation via DFM therefore technically implements the core insight of gRGFT stemming from the generalized hole and point-coincidence arguments, as described in Section 2 – see also ref. [1]. An essential result of this paper is (390), giving the *relational field equations* of gRGFT,

$$\boxed{\boldsymbol{E}^{(\boldsymbol{v},\mathbf{u})} = \boldsymbol{v}^*\big(\boldsymbol{E} + d\boldsymbol{E}\big(\iota_{d\boldsymbol{v}\circ\boldsymbol{v}^{-1}}\phi;\phi\big) + d\boldsymbol{E}\big(\boldsymbol{d}\mathbf{u}\mathbf{u}^{-1} - \mathbf{u}\mathfrak{L}_{d\boldsymbol{v}\circ\boldsymbol{v}^{-1}}\mathbf{u}^{-1};\phi\big)\big)}$$

which are strictly $C^\infty\left(\Phi, \mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}\right)$-invariant, and thus have a well-posed Cauchy problem. The relational equations $\boldsymbol{E}^{(\boldsymbol{v},\mathbf{u})} = 0$ are the ones tacitly used in practical experimental situations. The fact that, as the result shows, they are "equivalent" to the bare field equations $\boldsymbol{E} = 0$, explains why gRGFT was successfully compared to experiments much before the problem of identifying the fundamental physical d.o.f. and observables was solved.

Another fundamental result of the relational reformulation via DFM is the definition of $C^\infty\left(\Phi, \mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}\right)$-invariant *dressed regions* $U^{\boldsymbol{v}(\phi)}$, which formally implement the point-coincidence argument and provides a definition of spacetime regions by its field content ϕ. Hence one obtains a relational definition of spacetime $M^{(\boldsymbol{v},\mathbf{u})}$. From this, we observed, follows that the so-called "boundary problem", i.e., the often repeated claim that "boundaries break $\mathrm{Diff}(M)$ and/or $\mathcal{H}_{\mathrm{loc}}$ symmetry", is erroneous: a physical relational boundary $\partial U^{\boldsymbol{v}} \subset M^{(\boldsymbol{v},\mathbf{u})}$ is of necessity $\left(\mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}\right)$-invariant, and even invariant under the full covariance group $C^\infty\left(\Phi, \mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}\right)$ of gRGFT. Such "boundary problems" dissolve in the relational understanding of physics. We shall investigate further the consequences of this in a forthcoming paper.[71]

We considered a natural relational reformulation of GR, featuring the *relational Einstein equations*

$$\boxed{G^{\boldsymbol{v}} + \Lambda g^{\boldsymbol{v}} = \kappa T^{(\boldsymbol{v},\mathbf{u})}}$$

which encompass all manners of "scalar coordinatization" such as refs. [98–100] and [21, 66, 101]. This formulation of GR has as many natural applications: black holes physics, gravitational waves physics, and cosmological perturbation theory,[29] etc.; We may also assess the degree to which the relational formulation influences dark matter models. In the above relational Einstein equations, it should be stressed that the matter d.o.f., like those of the (invariant) metric $g^{\boldsymbol{v}}$, appear on both sides, allowing for a new take on the heuristics motivating the necessity of quantum gravity.

This brings us to the next phase of our program dealing with relational path integral quantization, or *relational Quantum Field Theory* (rQFT), which will be explored in ref. [96]. As we have shown in Section 5.2, the relational reformulation of a theory with bare action S failing to be $\mathcal{H}_{\mathrm{loc}}$-invariant implements an automatic mechanism of cancelation of *classical* $\mathcal{H}_{\mathrm{loc}}$-anomaly, where a *twisted dressing field* plays the role of WZ term. We expect that the same will hold in the QFT context. We also highlighted that anomalies, either classical or quantum, are to be understood via the *twisted connections* on field space Φ. It is all but certain that the geometry of twisted connections will play a key role in relational quantization.

Furthermore, as a separate item of our program, in ref. [105] we will also investigate the *relational covariant phase space* formalism of gRGFT. This may be the starting point of a (formal) relational geometric quantization, where again we expect twisted geometry to play a non-trivial role.

Appendix A: Semi-Direct Product Structure of Aut(P)

As is well-known, the group of vertical automorphisms $\mathrm{Aut}_v(P)$ of a principal bundle P is a normal subgroup of its automorphisms group $\mathrm{Aut}(P)$: $\mathrm{Aut}_v(P) \triangleleft \mathrm{Aut}(P)$. Their quotient is thus a group, isomorphic to the group of diffeomorphisms of the orbit space $P/H = M$, $\mathrm{Aut}(P)/\,\mathrm{Aut}_v(P) =: \overline{\mathrm{Diff}} \simeq \mathrm{Diff}(P/H) = \mathrm{Diff}(M)$. One shows that $\mathrm{Aut}(P)$ has a natural structure of *inner* semi-direct product:

$$\begin{aligned} \mathrm{Aut}(P) &= \overline{\mathrm{Diff}} \ltimes \mathrm{Aut}_v(P), \\ \psi &= (\bar\psi, \eta), \end{aligned} \tag{A1}$$

with product $\quad \psi' \circ \psi = \left(\bar\psi' \circ \bar\psi,\ \eta' \circ \mathrm{Conj}(\bar\psi')\,\eta\right),$

with the group morphism $\mathrm{Conj}: \overline{\mathrm{Diff}} \to \mathrm{Aut}\left(\mathrm{Aut}_v(P)\right)$, $\bar\psi \mapsto \mathrm{Conj}(\bar\psi)$, and $\mathrm{Conj}(\bar\psi)\,\eta := \bar\psi \circ \eta \circ \bar\psi^{-1}$. Indeed, writing an automorphsism as $\psi = \eta \circ \bar\psi$, as read from the following graph

$$\begin{tikzcd} P \arrow[r, "\bar\psi"'] \arrow[d] \arrow[rr, bend left, "\psi"] & P \arrow[r, "\eta"'] \arrow[d] & P \arrow[d] \\ M \arrow[r, "\psi"'] & M \arrow[r, equal] & M \end{tikzcd} \tag{A2}$$

we get the composition

$$\begin{aligned} \psi' \circ \psi &= \left(\eta' \circ \bar\psi'\right) \circ \left(\eta \circ \bar\psi\right) \\ &= \left(\eta' \circ \bar\psi'\right) \circ \left(\eta \circ \bar\psi'^{-1} \circ \bar\psi' \circ \bar\psi\right) \\ &= \left(\eta' \circ \left(\bar\psi' \circ \eta \circ \bar\psi'^{-1}\right)\right) \circ \left(\bar\psi' \circ \bar\psi\right). \end{aligned} \tag{A3}$$

We have then that a form $\phi \in \Omega^\bullet(P)$ transform under $\mathrm{Aut}(P)$ as $\psi^*\phi = \bar\psi^*(\eta^*\phi)$, where $\eta^*\phi$ defines the $\mathrm{Aut}_v(P) \simeq \mathcal{H}$-gauge transformation of ϕ, while $\bar\psi^*\phi$ defines its $\overline{\mathrm{Diff}}$-transformation.

The alternative decomposition of an automorphism, given by the following graph:

$$\begin{tikzcd} P \arrow[r, "\eta"'] \arrow[d] \arrow[rr, bend left, "\psi"] & P \arrow[r, "\bar\psi"'] \arrow[d] & P \arrow[d] \\ M \arrow[r, equal] & M \arrow[r, "\psi"'] & M \end{tikzcd} \tag{A4}$$

[29] Where we may make contact with e.g., refs. [103, 104].

gives rise to the semi-direct product rule

$$\psi' \circ \psi = \left(\bar{\uppsi}' \circ \bar{\uppsi},\, \mathrm{Conj}(\bar{\uppsi}^{-1})\, \eta' \circ \eta\right), \tag{A5}$$

from the composition

$$\begin{aligned} \psi' \circ \psi &= \left(\bar{\uppsi}' \circ \eta'\right) \circ \left(\bar{\uppsi} \circ \eta\right) \\ &= \left(\bar{\uppsi}' \circ \bar{\uppsi} \circ \bar{\uppsi}^{-1} \circ \eta'\right) \circ \left(\bar{\uppsi} \circ \eta\right) \\ &= \left(\bar{\uppsi}' \circ \bar{\uppsi}\right) \circ \left(\left(\bar{\uppsi}^{-1} \circ \eta' \circ \bar{\uppsi}\right) \circ \eta\right). \end{aligned} \tag{A6}$$

This decomposition may be seen as a special case of the canonically split of bundle morphisms arising from the pullback bundle construction: Given a bundle $Q \to N$ and a diffeomorphism $\uppsi : N \to M$, one may define the pullback bundle $\uppsi^* P \to N$. Then, there is a bundle morphism $\bar{\uppsi} : \uppsi^* P \to P$ covering $\uppsi$, i.e., s.t. $\bar{\pi}(\bar{\uppsi}) = \uppsi$. By the (categorical) universality property of the pullback, for a bundle morphism $\psi : Q \to P$ covering $\uppsi$, there is a unique bundle morphism $\eta : Q \to \uppsi^* P$ covering id_N. This is synthesized in the following graph:

$$\begin{tikzcd} Q \arrow[r, "\eta"'] \arrow[rr, bend left, "\psi"] \arrow[d] & \uppsi^* P \arrow[r, "\bar{\uppsi}"'] \arrow[d] & P \arrow[d] \\ N \arrow[r, equal] & N \arrow[r, "\uppsi"'] & M \end{tikzcd} \tag{A7}$$

On the other hand, the natural semi-direct structure (A1)–(A2) may be understood to follow as a special case of the split of a bundle morphism given by the graph

$$\begin{tikzcd} Q \arrow[r, "\bar{\uppsi}"'] \arrow[rr, bend left, "\psi"] \arrow[d] & P \arrow[r, "\eta"'] \arrow[d] & P \arrow[d] \\ N \arrow[r, "\uppsi"'] & M \arrow[r, equal] & M \end{tikzcd} \tag{A8}$$

We may fit the two options into a single graph:

$$\begin{tikzcd} Q \arrow[r, "\eta"] \arrow[rrd, "\bar{\uppsi}"'] \arrow[rrrd, dashed, "\psi"] \arrow[dd] & \uppsi^* P \arrow[rrd, "\bar{\uppsi}"] \arrow[ldd, dotted] & & \\ & & P \arrow[r, "\eta"'] \arrow[dd] & P \arrow[ddl] \\ N \arrow[rrd, "\uppsi"] & & & \\ & & M & \end{tikzcd} \tag{A9}$$

The decomposition (A8) is the one we refer to in Section 4.2 to split an $\mathrm{Aut}(P)$-dressing into an $\mathrm{Aut}_v(P)$-dressing and a $\overline{\mathrm{Diff}}$-dressing, as it is the one respecting the necessary compatibility conditions (152)–(153).

A.1. Semi-Direct Structure of the Local Symmetry Group

From (A3) follows a semi-direct product on $\overline{\mathrm{Diff}} \ltimes \mathcal{H} \simeq \overline{\mathrm{Diff}} \ltimes \mathrm{Aut}_v(P) = \mathrm{Aut}(P)$. Indeed, one finds that the $\mathrm{Aut}_v(P)$ part of $\psi' \circ \psi$ is

$$\begin{aligned} \left[\eta' \circ \left(\bar{\uppsi}' \circ \eta \circ \bar{\uppsi}'^{-1}\right)\right](p) &= \eta' \circ \bar{\uppsi}'\left(\bar{\uppsi}'^{-1}(p)\, \gamma\left(\bar{\uppsi}'^{-1}(p)\right)\right) \\ &= \eta'\left(\bar{\uppsi}'\left(\bar{\uppsi}'^{-1}(p)\right) \gamma\left(\bar{\uppsi}'^{-1}(p)\right)\right) \\ &= \eta'(p)\, \gamma\left(\bar{\uppsi}'^{-1}(p)\right) \\ &= p\, \gamma'(p)\, \gamma\left(\bar{\uppsi}'^{-1}(p)\right) \\ &= p \left(\gamma' \cdot \left(\bar{\uppsi}'^{-1*} \gamma\right)\right)(p). \end{aligned} \tag{A10}$$

One has thus the product structure

$$\left(\bar{\uppsi}', \gamma'\right) \cdot \left(\bar{\uppsi}, \gamma\right) = \left(\bar{\uppsi}' \circ \bar{\uppsi},\, \gamma' \cdot \left(\bar{\uppsi}'^{-1*} \gamma\right)\right). \tag{A11}$$

This directly induces the local version on $(U \subset)\, M$, i.e., the semi-direct product of $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$:

$$\left(\uppsi', \gamma'\right) \cdot \left(\uppsi, \gamma\right) = \left(\uppsi' \circ \uppsi,\, \gamma' \cdot \left(\uppsi'^{-1*} \gamma\right)\right), \tag{A12}$$

with $\mathcal{H}_{\mathrm{loc}}$ defined in (176). This is directly relevant to (local) field theory as discussed in Section 5.1. The inverse element is $(\uppsi, \gamma)^{-1} = \left(\uppsi^{-1}, \uppsi^* \gamma^{-1}\right)$.

From this one may derive the adjoint action of $\mathrm{Diff}(M) \ltimes \mathcal{H}_{\mathrm{loc}}$ on its Lie algebra $\mathfrak{diff}(M) \oplus \mathrm{Lie}\mathcal{H}_{\mathrm{loc}}$: For an element $(X, \lambda) := \left(\frac{d}{d\tau} \uppsi_\tau\big|_{\tau=0}, \frac{d}{ds} \gamma_s\big|_{s=0}\right)$ of the Lie algebra, we get

$$\begin{aligned} &\mathrm{Ad}_{(\uppsi,\gamma)}(X, \lambda) \\ &\quad := \tfrac{d^2}{d\tau ds} \mathrm{Conj}(\uppsi, \gamma)(X, \lambda)\Big|_{\tau=0, s=0} \\ &\quad = \tfrac{d^2}{d\tau ds} (\uppsi, \gamma) \circ (\uppsi_\tau, \gamma_s) \circ \left(\uppsi^{-1}, \uppsi^* \gamma^{-1}\right)\Big|_{\tau=0, s=0} \\ &\quad = \tfrac{d^2}{d\tau ds} (\uppsi, \gamma) \circ \left(\uppsi_\tau \circ \uppsi^{-1}, \gamma_s \cdot \uppsi_\tau^{-1*} \uppsi^* \gamma^{-1}\right)\Big|_{\tau=0, s=0} \\ &\quad = \tfrac{d^2}{d\tau ds} \left(\uppsi \circ \uppsi_\tau \circ \uppsi^{-1}, \gamma \cdot \uppsi^{-1*} \gamma_s \cdot \uppsi^{-1*} \uppsi_\tau^{-1*} \uppsi^* \gamma^{-1}\right)\Big|_{\tau=0, s=0} \\ &\quad = \tfrac{d^2}{d\tau ds} \left(\uppsi \circ \uppsi_\tau \circ \uppsi^{-1}, \gamma \cdot \uppsi^{-1*} \gamma_s \cdot \left(\uppsi \circ \uppsi_\tau \circ \uppsi^{-1}\right)^{-1*} \gamma^{-1}\right)\Big|_{\tau=0, s=0}, \end{aligned}$$

$$\begin{aligned} &\mathrm{Ad}_{(\uppsi,\gamma)}(X, \lambda) \\ &\quad = \left(\uppsi_* X \circ \uppsi^{-1}, \mathrm{Ad}_\gamma\left(\uppsi^{-1*} \lambda\right) - \gamma \cdot \mathfrak{L}_{\uppsi_* X \circ \uppsi^{-1}} \gamma^{-1}\right). \end{aligned} \tag{A13}$$

The identity (13) is used in the last step. Similarly, one gets

$$\begin{aligned} &\mathrm{Ad}_{(\uppsi,\gamma)^{-1}}(X, \lambda) \\ &\quad := \tfrac{d^2}{d\tau ds}\, (\uppsi, \gamma)^{-1} \circ (\uppsi_\tau, \gamma_s) \circ (\uppsi, \gamma)\, \Big|_{\tau=0,\, s=0} \\ &\quad = \tfrac{d^2}{d\tau ds} \left(\uppsi^{-1} \circ \uppsi_\tau \circ \uppsi,\ \uppsi^* \gamma^{-1} \cdot \uppsi^* \gamma_s \cdot \left(\uppsi^{-1} \circ \uppsi_\tau\right)^{-1*} \gamma\right)\Big|_{\tau=0,\, s=0} \\ &\quad = \tfrac{d^2}{d\tau ds} \left(\uppsi^{-1} \circ \uppsi_\tau \circ \uppsi,\ \uppsi^*\left(\gamma^{-1} \cdot \gamma_s \cdot \uppsi_\tau^{-1*} \gamma\right)\right)\Big|_{\tau=0,\, s=0} \end{aligned}$$

$$= \begin{cases} \left(\psi_*^{-1}X\circ\psi,\, \psi^*\left(\mathrm{Ad}_{\gamma^{-1}}\lambda - \gamma^{-1}\mathfrak{L}_X\gamma\right)\right), \\ \left(\psi_*^{-1}X\circ\psi,\, \mathrm{Ad}_{\psi^*\gamma^{-1}}(\psi^*\lambda) - \psi^*\gamma^{-1}\cdot \mathfrak{L}_{\psi_*^{-1}X\circ\psi}\psi^*\gamma\right), \end{cases} \tag{A14}$$

using the fact that $\psi_\tau^{-1*} = \psi_{-\tau}^*$ and, in the last line, (14).

Appendix B: Lie Algebra (Anti-)Isomorphisms and Pushforward by a Vertical Diffeomorphism of (Local) Field Space

B.1. Lie Algebra Morphisms

We prove here that the verticality map $|^v : \mathfrak{aut}(P) \to \Gamma(V\Phi)$, $X \mapsto X^v$, is a morphism of Lie algebra. We write the flow through $\phi \in \Phi$ of $X^v \in \Gamma(V\Phi)$ as $\tilde{\psi}_\tau(\phi) := R_{\psi_\tau}\phi := \psi_\tau^*\phi$, with $X = \frac{d}{d\tau}\psi_\tau\Big|_{\tau=0} \in \Gamma(TP)$, so that

$$X^v_{|\phi} = \tfrac{d}{d\tau}\tilde{\psi}_\tau(\phi)\Big|_{\tau=0} \quad \Big(=X(\phi)^v\tfrac{\delta}{\delta\phi},\ \text{written as a derivation of } C^\infty(\Phi)\Big). \tag{B1}$$

One can thus write the bracket of two vertical vector fields:

$$\begin{aligned} [X^v, Y^v]_{|\phi} = \boldsymbol{L}_{X^v}Y^v{}_{|\phi} &:= \tfrac{d}{d\tau}(\tilde{\psi}_\tau^{-1})_\star Y^v_{|\tilde{\psi}_\tau(\phi)}\Big|_{\tau=0} \\ &:= \tfrac{d}{d\tau}\tfrac{d}{ds}\left(\tilde{\psi}_\tau^{-1}\circ\tilde{\eta}_s\circ\tilde{\psi}_\tau\right)(\phi)\Big|_{s=0}\Big|_{\tau=0} \\ &:= \tfrac{d}{d\tau}\tfrac{d}{ds}R_{\psi_\tau^{-1}}\circ R_{\eta_s}\circ R_{\psi_\tau}\phi\Big|_{s=0}\Big|_{\tau=0} \\ &:= \tfrac{d}{d\tau}\tfrac{d}{ds}R_{(\psi_\tau\circ\eta_s\circ\psi_\tau^{-1})}\phi\Big|_{s=0}\Big|_{\tau=0} \\ &:= \tfrac{d}{d\tau}\tfrac{d}{ds}\underbrace{(\psi_\tau\circ\eta_s\circ\psi_\tau^{-1})}_{\text{flow of } -[X,Y]}{}^*\phi\Big|_{s=0}\Big|_{\tau=0} \\ &=: \mathfrak{L}_{-[X,Y]}\phi =: (-[X,Y]_{\Gamma(TP)})^v_{|\phi} = ([X,Y]_{\mathfrak{aut}(P)})^v_{|\phi}. \end{aligned} \tag{B2}$$

In the 1st-2nd and 5th–6th lines the definition of $\tilde{\psi}_\tau$ and ψ_τ-relatedness are used, while in the 1st and 6th lines, we have used the definitions of the Lie derivatives of a vector field on Φ and of a field on P.

We now prove that the map $|^v : \mathfrak{aut}(P) \to \Gamma_{\text{inv}}(V\Phi)$, $\boldsymbol{X} \mapsto \boldsymbol{X}^v$, is a Lie algebra *anti*-morphism. The flow through $\phi \in \Phi$ of $\boldsymbol{X}^v \in \Gamma_{\text{inv}}(V\Phi)$ as $\tilde{\boldsymbol{\psi}}_\tau(\phi) := (R_{\boldsymbol{\psi}_\tau})(\phi) = R_{\boldsymbol{\psi}_\tau(\phi)}\phi := (\boldsymbol{\psi}_\tau(\phi))^*\phi$, with $\boldsymbol{X} = \frac{d}{d\tau}\boldsymbol{\psi}_\tau|_{\tau=0} \in \Gamma(TP)$. So,

$$\begin{aligned} [\boldsymbol{X}^v, \boldsymbol{Y}^v]_{|\phi} = \boldsymbol{L}_{X^v}\boldsymbol{Y}^v{}_{|\phi} &:= \tfrac{d}{d\tau}(\tilde{\boldsymbol{\psi}}_\tau^{-1})_\star \boldsymbol{Y}^v_{|\tilde{\boldsymbol{\psi}}_\tau(\phi)}\Big|_{\tau=0} \\ &= \tfrac{d}{d\tau}\tfrac{d}{ds}\left(\tilde{\boldsymbol{\psi}}_\tau^{-1}\circ\tilde{\boldsymbol{\eta}}_s\circ\tilde{\boldsymbol{\psi}}_\tau\right)(\phi)\Big|_{s=0}\Big|_{\tau=0}. \end{aligned} \tag{B3}$$

The equivariance property of the elements of the gauge group has been used. We have then

$$\begin{aligned} \tilde{\boldsymbol{\eta}}_s\circ\tilde{\boldsymbol{\psi}}_\tau(\phi) &= R_{\boldsymbol{\eta}_s(\tilde{\boldsymbol{\psi}}_\tau(\phi))}\,\tilde{\boldsymbol{\psi}}_\tau(\phi) = R_{\boldsymbol{\eta}_s\left(R_{\boldsymbol{\psi}_\tau(\phi)}\phi\right)}\,R_{\boldsymbol{\psi}_\tau(\phi)}\phi \\ &= R_{\boldsymbol{\psi}_\tau(\phi)^{-1}\circ\boldsymbol{\eta}_s(\phi)\circ\boldsymbol{\psi}_\tau(\phi)}\,R_{\boldsymbol{\psi}_\tau(\phi)}\phi = R_{\boldsymbol{\eta}_s(\phi)\circ\boldsymbol{\psi}_\tau(\phi)}\,\phi \\ &= \left(R_{\boldsymbol{\eta}_s\circ\boldsymbol{\psi}_\tau}\right)\phi. \end{aligned} \tag{B4}$$

So,

$$\begin{aligned} \left(\tilde{\boldsymbol{\psi}}_\tau^{-1}\circ\tilde{\boldsymbol{\eta}}_s\circ\tilde{\boldsymbol{\psi}}_\tau\right)(\phi) &= \tilde{\boldsymbol{\psi}}_\tau^{-1}\left(\tilde{\boldsymbol{\eta}}_s\circ\tilde{\boldsymbol{\psi}}_\tau(\phi)\right) \\ &= R_{\boldsymbol{\psi}_\tau^{-1}\left(R_{\boldsymbol{\eta}_s(\phi)\circ\boldsymbol{\psi}_\tau(\phi)}\phi\right)}\,R_{\boldsymbol{\eta}_s(\phi)\circ\boldsymbol{\psi}_\tau(\phi)}\phi \\ &= R_{[\boldsymbol{\eta}_s(\phi)\circ\boldsymbol{\psi}_\tau(\phi)]^{-1}\circ\boldsymbol{\psi}_\tau(\phi)^{-1}\circ[\boldsymbol{\eta}_s(\phi)\circ\boldsymbol{\psi}_\tau(\phi)]}\,R_{\boldsymbol{\eta}_s(\phi)\circ\boldsymbol{\psi}_\tau(\phi)}\phi \\ &= R_{\boldsymbol{\psi}_\tau(\phi)^{-1}\circ\boldsymbol{\eta}_s(\phi)\circ\boldsymbol{\psi}_\tau(\phi)}\phi. \end{aligned} \tag{B5}$$

Finally,

$$\begin{aligned} [\boldsymbol{X}^v, \boldsymbol{Y}^v]_{|\phi} &= \tfrac{d}{d\tau}\tfrac{d}{ds}R_{\boldsymbol{\psi}_\tau(\phi)^{-1}\circ\boldsymbol{\eta}_s(\phi)\circ\boldsymbol{\psi}_\tau(\phi)}\,\phi\Big|_{s=0}\Big|_{\tau=0} \\ &= \tfrac{d}{d\tau}\tfrac{d}{ds}\underbrace{(\boldsymbol{\psi}_\tau(\phi)^{-1}\circ\boldsymbol{\eta}_s(\phi)\circ\boldsymbol{\psi}_\tau(\phi))}_{\text{flow of } [\boldsymbol{X},\boldsymbol{Y}]}{}^*\phi\Big|_{s=0}\Big|_{\tau=0} \\ &=: \mathfrak{L}_{[\boldsymbol{X},\boldsymbol{Y}]}\phi =: ([\boldsymbol{X},\boldsymbol{Y}]_{\Gamma(TP)})^v_{|\phi} = (-[\boldsymbol{X},\boldsymbol{Y}]_{\mathfrak{aut}(P)})^v_{|\phi}, \end{aligned} \tag{B6}$$

which ends the proof of the assertion. Observe that the above computations hold the same for both $\mathfrak{aut}_v(P)$[6,42] and $\mathfrak{diff}(M)$.[8]

B.2. Pushforward by a Vertical Diffeomorphism of Field Space

We consider a generic vector field $\mathfrak{X} \in \Gamma(T\Phi)$ with flow φ_τ and a vertical diffeomorphism $\Xi \in \boldsymbol{Diff}_v(\phi)$ to which corresponds $\boldsymbol{\psi} \in C^\infty\left(\Phi, \mathrm{Aut}(P)\right)$. The pushforward of $\mathfrak{X}_{|\phi} \in T_\phi\Phi$ by $\boldsymbol{\psi}$ is

$$\begin{aligned} \Xi_\star\mathfrak{X}_{|\phi} &= \tfrac{d}{d\tau}\,\Xi\left(\varphi_\tau(\phi)\right)\Big|_{\tau=0} = \tfrac{d}{d\tau}\,R_{\boldsymbol{\psi}\left(\varphi_\tau(\phi)\right)}\varphi_\tau(\phi)|_{\tau=0} \\ &= \tfrac{d}{d\tau}\,R_{\boldsymbol{\psi}\left(\varphi_\tau(\phi)\right)}\phi\Big|_{\tau=0} + \tfrac{d}{d\tau}\,R_{\boldsymbol{\psi}(\phi)}\varphi_\tau(\phi)|_{\tau=0} \\ &= \tfrac{d}{d\tau}\,R_{\boldsymbol{\psi}\left(\varphi_\tau(\phi)\right)}\phi\Big|_{\tau=0} + R_{\boldsymbol{\psi}(\phi)\star}\mathfrak{X}_{|\phi}. \end{aligned} \tag{B7}$$

In the last equality the definition of the pushforward by the right action of $\boldsymbol{\psi}(\phi) \in \mathrm{Aut}(P)$ is used. The remaining term is manifestly a vertical vector field. The question is to find the element of $\mathfrak{aut}(P)$ that generates it, knowing that it must be anchored at the point $\Xi(\phi) = R_{\boldsymbol{\psi}(\phi)}\phi = \boldsymbol{\psi}(\phi)^*\phi$ of field space:

$$\begin{aligned} \tfrac{d}{d\tau}\,R_{\boldsymbol{\psi}\left(\varphi_\tau(\phi)\right)}\phi\Big|_{\tau=0} &= \tfrac{d}{d\tau}\,R_{\boldsymbol{\psi}(\phi)\circ\boldsymbol{\psi}(\phi)^{-1}\circ\boldsymbol{\psi}\left(\varphi_\tau(\phi)\right)}\phi\Big|_{\tau=0} \\ &= \tfrac{d}{d\tau}\,R_{\boldsymbol{\psi}(\phi)^{-1}\circ\boldsymbol{\psi}\left(\varphi_\tau(\phi)\right)}\,R_{\boldsymbol{\psi}(\phi)}\phi\Big|_{\tau=0}. \end{aligned}$$

So, $\boldsymbol{\psi}(\phi)^{-1}\circ\boldsymbol{\psi}\left(\varphi_\tau(\phi)\right)$ is the flow of the vector field on $\mathrm{Aut}(P)$ we are looking for. Now, on the one hand we have

$$\tfrac{d}{d\tau}\,\boldsymbol{\psi}\left(\varphi_\tau(\phi)\right)\Big|_{\tau=0} = \boldsymbol{d\psi}_{|\phi}\left(\mathfrak{X}_{|\phi}\right) = \boldsymbol{\psi}_\star\mathfrak{X}_{|\phi} \in T_{\boldsymbol{\psi}(\phi)}\mathrm{Aut}(P),$$

since $\psi : \Phi \to \mathrm{Aut}(P)$. On the other hand, the Maurer-Cartan form – given by the left translation – on $\mathrm{Aut}(P)$ is

$$\begin{aligned} L_{\psi^{-1}\star} : T_\psi \,\mathrm{Aut}(P) &\to T_{\mathrm{id}_P}\mathrm{Aut}(P) = \mathfrak{aut}(P) \simeq \Gamma(TP), \\ \mathcal{X}_{|\psi} &\mapsto L_{\psi^{-1}\star}\mathcal{X}_{|\psi} := (\psi^{-1})_* \mathcal{X}_{|\psi}. \end{aligned} \tag{B8}$$

So we have

$$\begin{aligned} L_{\psi(\phi)^{-1}\star} &: T_{\psi(\phi)}\mathrm{Aut}(P) \to \mathfrak{aut}(P) \simeq \Gamma(TP), \\ \big[d\psi_{|\phi}(\mathfrak{X}_{|\phi})\big]_{|\psi(\phi)} &\mapsto L_{\psi(\phi)^{-1}\star}\big[d\psi_{|\phi}(\mathfrak{X}_{|\phi})\big]_{|\psi(\phi)} \\ &= (\psi(\phi)^{-1})_*\big[d\psi_{|\phi}(\mathfrak{X}_{|\phi})\big]_{|\psi(\phi)} \\ &= L_{\psi(\phi)^{-1}\star}\tfrac{d}{d\tau}\,\psi\big(\varphi_\tau(\phi)\big)\Big|_{\tau=0} \\ &= \tfrac{d}{d\tau}\,\psi(\phi)^{-1}\circ\psi\big(\varphi_\tau(\phi)\big)\Big|_{\tau=0}. \end{aligned} \tag{B9}$$

Thus, $(\psi(\phi)^{-1})_*\,\big[d\psi_{|\phi}\,(\mathfrak{X}_{|\phi})\big]$ is the generating vector field of P we were searching for. We then get

$$\begin{aligned} \tfrac{d}{d\tau}\,R_{\psi(\varphi_\tau(\phi))}\phi\Big|_{\tau=0} &= \tfrac{d}{d\tau}\,R_{\psi(\phi)^{-1}\circ\psi(\varphi_\tau(\phi))}\,R_{\psi(\phi)}\phi\Big|_{\tau=0} \\ &= \big\{(\psi(\phi)^{-1})_*\big[d\psi_{|\phi}(\mathfrak{X}_{|\phi})\big]\big\}^v_{|R_{\psi(\phi)}\phi}. \end{aligned} \tag{B10}$$

Hence, finally we get

$$\begin{aligned} \Xi_\star\mathfrak{X}_{|\phi} &= R_{\psi(\phi)\star}\mathfrak{X}_{|\phi} + \big\{\psi(\phi)^{-1}_* d\psi_{|\phi}(\mathfrak{X}_{|\phi})\big\}^v_{|\Xi(\phi)} \\ &= R_{\psi(\phi)\star}\Big(\mathfrak{X}_{|\phi} + \big\{d\psi_{|\phi}(\mathfrak{X}_{|\phi})\circ\psi(\phi)^{-1}\big\}^v_{|\phi}\Big), \end{aligned} \tag{B11}$$

where in the second line the property (20) of the fundamental vertical vector fields under pushforward by the right action of the structure group has been used. This result is essential to compute geometrically the vertical and gauge transformations of forms on Φ.

B.3. Pushforward by a Vertical Diffeomorphism of Local Field Space

We here derive the local version of the above result, important for the standard formulation of field theory. Despite some repetition, we deem it pedagogically useful. The local field space Φ is now that of local representatives of fields on $(U \subset)\ M$, and its structure group is $\mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}$. A vertical diffeomorphism $\Xi \in \boldsymbol{Diff}_v(\Phi)$ is generated by $(\psi,\gamma)\in C^\infty\big(\Phi, \mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}\big)$. As above, we consider the pushforward along Ξ of a generic vector field $\mathfrak{X}\in\Gamma(T\Phi)$ with flow φ_τ:

$$\begin{aligned} \Xi_\star\mathfrak{X}_{|\phi} &= \tfrac{d}{d\tau}\,\Xi\big(\varphi_\tau(\phi)\big)\Big|_{\tau=0} = \tfrac{d}{d\tau}\,R_{(\psi,\gamma)\circ(\varphi_\tau(\phi))}\,\varphi_\tau(\phi)\Big|_{\tau=0} \\ &= \tfrac{d}{d\tau}\,R_{(\psi,\gamma)\circ(\varphi_\tau(\phi))}\phi\Big|_{\tau=0} + R_{(\psi(\phi),\gamma(\phi))\star}\mathfrak{X}_{|\phi}. \end{aligned} \tag{B12}$$

The first term is clearly a vertical vector field, which must be anchored at the point $\Xi(\phi) = R_{(\psi(\phi),\gamma(\phi))}\phi$. One needs only to find its $\mathfrak{diff}(M)\oplus\mathrm{Lie}\mathcal{H}_{\mathrm{loc}}$-generating element.

For notational convenience, let us write $(\psi,\gamma)_\tau := (\psi,\gamma)\circ\big(\varphi_\tau(\phi)\big)$, and $\big(\psi(\phi),\gamma(\phi)\big) = (\psi,\gamma)$. We have

$$\tfrac{d}{d\tau}\,R_{(\psi,\gamma)_\tau}\phi\,\Big|_{\tau=0} = \tfrac{d}{d\tau}\,R_{(\psi,\gamma)^{-1}\cdot(\psi,\gamma)_\tau}\,R_{(\psi,\gamma)}\phi\,\Big|_{\tau=0},$$

where $(\psi,\gamma)^{-1}\cdot(\psi,\gamma)_\tau$ is the flow of the vector field on $\mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}$ we are seeking. This is a curve through the identity $(\mathrm{id}_M, \mathrm{id}_{\mathcal{H}_{\mathrm{loc}}})$ in the group. Given that the left action, by an inverse element, on $\mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}$ is

$$\begin{aligned} L_{(\psi,\gamma)^{-1}}(\psi',\gamma') &:= (\psi,\gamma)^{-1}\cdot(\psi',\gamma') = (\psi^{-1},\psi^*\gamma^{-1})\cdot(\psi',\gamma') \\ &= \big(\psi^{-1}\circ\psi', \psi^*(\gamma^{-1}\gamma')\big), \end{aligned} \tag{B13}$$

the Maurer–Cartan form is

$$\begin{aligned} L_{(\psi,\gamma)^{-1}\star} : T_{(\psi,\gamma)}\big(\mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}\big) &\to T_{(\mathrm{id}_M,\mathrm{id}_{\mathcal{H}_{\mathrm{loc}}})}\big(\mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}\big) \\ &= \mathfrak{diff}(M)\oplus\mathrm{Lie}\mathcal{H}_{\mathrm{loc}} \\ T_\psi\mathrm{Diff}(M)\oplus T_\gamma\mathcal{H}_{\mathrm{loc}} &\to T_{\mathrm{id}_M}\mathrm{Diff}(M)\oplus T_{\mathrm{id}_{\mathcal{H}_{\mathrm{loc}}}}\mathcal{H}_{\mathrm{loc}} \\ (\mathcal{X}_{|\psi},\mathcal{Y}_{|\gamma}) &\mapsto L_{(\psi,\gamma)^{-1}\star}(\mathcal{X}_{|\psi},\mathcal{Y}_{|\gamma}) \\ &:= (\psi,\gamma)^{-1}\cdot(\mathcal{X}_{|\psi},\mathcal{Y}_{|\gamma}) \\ &= \big(\psi^{-1}_*\mathcal{X}_{|\psi}, \psi^*(\gamma^{-1}\mathcal{Y}_{|\gamma})\big). \end{aligned} \tag{B14}$$

Now, writing $(\psi,\gamma)_\tau = (\psi_\tau,\gamma_\tau)$, we have that

$$\begin{aligned} &\tfrac{d}{d\tau}(\psi,\gamma)^{-1}\cdot(\psi,\gamma)_\tau\Big|_{\tau=0} \\ &= \tfrac{d}{d\tau}\,L_{(\psi,\gamma)^{-1}}(\psi,\gamma)_\tau\,\Big|_{\tau=0} = L_{(\psi,\gamma)^{-1}\star}\tfrac{d}{d\tau}\,(\psi_\tau,\gamma_\tau)\,\Big|_{\tau=0} \\ &= L_{(\psi,\gamma)^{-1}\star}\Big(d\psi_{|\phi}(\mathfrak{X}_{|\phi}), d\gamma_{|\phi}(\mathfrak{X}_{|\phi})\Big) \\ &= (\psi,\gamma)^{-1}\cdot\Big(d\psi_{|\phi}(\mathfrak{X}_{|\phi}), d\gamma_{|\phi}(\mathfrak{X}_{|\phi})\Big) \\ &= \Big(\psi^{-1}_*\big[d\psi_{|\phi}(\mathfrak{X}_{|\phi})\big], \psi^*\gamma^{-1}\big[d\gamma_{|\phi}(\mathfrak{X}_{|\phi})\big]\Big) \in \mathfrak{diff}\oplus\mathrm{Lie}\mathcal{H}_{\mathrm{loc}}, \end{aligned} \tag{B15}$$

with $d\psi_{|\phi}(\mathfrak{X}_{|\phi}) \in T_\psi\,\mathrm{Diff}(M)$ and $d\gamma_{|\phi}(\mathfrak{X}_{|\phi}) \in T_\gamma\mathcal{H}_{\mathrm{loc}}$. So finally,

$$\begin{aligned} \Xi_\star\mathfrak{X}_{|\phi} &= R_{(\psi,\gamma)\star}\mathfrak{X}_{|\phi} + \Big\{\big[(\psi,\gamma)^{-1}\cdot(d\psi,d\gamma)\big]_{|\phi}(\mathfrak{X}_{|\phi})\Big\}^v_{|\Xi(\phi)} \\ &= R_{(\psi,\gamma)\star}\mathfrak{X}_{|\phi} + \Big\{\big(\psi^{-1}_* d\psi, \psi^*(\gamma^{-1}d\gamma)\big)_{|\phi}(\mathfrak{X}_{|\phi})\Big\}^v_{|\Xi(\phi)}. \end{aligned} \tag{B16}$$

There is an alternative form of the above result, which is most useful in concrete computations. It relies on the pushforward of fundamental vector fields (183): we have indeed that

$$\begin{aligned} &\big\{(\psi,\gamma)^{-1}\cdot(d\psi,d\gamma)_{|\phi}(\mathfrak{X}_{|\phi})\big\}^v_{|\Xi(\phi)} \\ &= R_{(\psi,\gamma)\star}\Big\{\mathrm{Ad}_{(\psi,\gamma)}\Big(\big[(\psi,\gamma)^{-1}\cdot(d\psi,d\gamma)\big]_{|\phi}(\mathfrak{X}_{|\phi})\Big)\Big\}^v_{|\phi}. \end{aligned} \tag{B17}$$

We compute the $\mathfrak{diff}(M) \oplus \mathrm{Lie}\mathcal{H}_{\mathrm{loc}}$-valued 1-form on the right to be

$$\begin{aligned}
&\mathrm{Ad}_{(\boldsymbol{\psi},\boldsymbol{\gamma})}\left((\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\cdot(d\boldsymbol{\psi},d\boldsymbol{\gamma})\right)\\
&\quad=\mathrm{Ad}_{(\boldsymbol{\psi},\boldsymbol{\gamma})}\left(\boldsymbol{\psi}_*^{-1}d\boldsymbol{\psi},\ \boldsymbol{\psi}^*(\boldsymbol{\gamma}^{-1}d\boldsymbol{\gamma})\right)\\
&\quad=\Big(\boldsymbol{\psi}_*\left(\boldsymbol{\psi}_*^{-1}d\boldsymbol{\psi}\right)\circ\boldsymbol{\psi}^{-1},\mathrm{Ad}_{\boldsymbol{\gamma}}\left(\boldsymbol{\psi}^{-1*}\left[\boldsymbol{\psi}^*(\boldsymbol{\gamma}^{-1}d\boldsymbol{\gamma})\right]\right)\\
&\qquad-\boldsymbol{\gamma}\,\mathfrak{L}_{\boldsymbol{\psi}_*(\boldsymbol{\psi}_*^{-1}d\boldsymbol{\psi})\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\gamma}^{-1}\Big)\\
&\quad=\left(d\boldsymbol{\psi}\circ\boldsymbol{\psi}^{-1},\ d\boldsymbol{\gamma}\,\boldsymbol{\gamma}^{-1}-\boldsymbol{\gamma}\,\mathfrak{L}_{d\boldsymbol{\psi}\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\gamma}^{-1}\right). \qquad (B18)
\end{aligned}$$

This result we denote by

$$(d\boldsymbol{\psi},d\boldsymbol{\gamma})\cdot(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\equiv\left(d\boldsymbol{\psi}\circ\boldsymbol{\psi}^{-1},d\boldsymbol{\gamma}\,\boldsymbol{\gamma}^{-1}-\boldsymbol{\gamma}\,\mathfrak{L}_{d\boldsymbol{\psi}\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\gamma}^{-1}\right). \qquad (B19)$$

This is justified by the fact that it can be understood via the linearization of the right action

$$\begin{aligned}
R_{(\psi,\gamma)^{-1}}(\psi',\gamma')&=(\psi',\gamma')\cdot(\psi,\gamma)^{-1}=(\psi',\gamma')\cdot(\psi^{-1},\psi^*\gamma^{-1})\\
&=\left(\psi'\circ\psi^{-1},\gamma'\,\psi'^{-1*}(\psi^*\gamma^{-1})\right). \qquad (B20)
\end{aligned}$$

So, one has that

$$\begin{aligned}
R_{(\psi,\gamma)^{-1}\star}:T_{(\psi,\gamma)}\big(\mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}\big)&\to T_{(\mathrm{id}_M,\mathrm{id}_{\mathcal{H}_{\mathrm{loc}}})}\big(\mathrm{Diff}(M)\ltimes\mathcal{H}_{\mathrm{loc}}\big)\\
&=:\mathfrak{diff}(M)\oplus\mathrm{Lie}\mathcal{H}_{\mathrm{loc}},\\
\left(\mathcal{X}_{|\psi},\mathcal{Y}_{|\gamma}\right)&\mapsto R_{(\psi,\gamma)^{-1}\star}\left(\mathcal{X}_{|\psi},\mathcal{Y}_{|\gamma}\right)\\
&=\left(\mathcal{X}_{|\psi},\ \mathcal{Y}_{|\gamma}\right)\cdot(\psi,\gamma)^{-1}\\
&:=\Big(\mathcal{X}_{|\psi}\circ\psi^{-1},\ \mathcal{Y}_{|\gamma}\gamma^{-1}\\
&\qquad-\gamma\,\mathfrak{L}_{\mathcal{X}_{|\psi}\circ\psi^{-1}}\gamma^{-1}\Big). \qquad (B21)
\end{aligned}$$

Thus, we get the final alternative result

$$\begin{aligned}
\Xi_\star\mathfrak{X}_{|\phi}&=R_{(\boldsymbol{\psi},\boldsymbol{\gamma})\star}\left(\mathfrak{X}_{|\phi}+\left\{(d\boldsymbol{\psi},d\boldsymbol{\gamma})_{|\phi}(\mathfrak{X}_{|\phi})\cdot(\boldsymbol{\psi},\boldsymbol{\gamma})^{-1}\right\}^v_{|\phi}\right)\\
&=R_{(\boldsymbol{\psi},\boldsymbol{\gamma})\star}\Big(\mathfrak{X}_{|\phi}+\Big\{\big(d\boldsymbol{\psi}\circ\boldsymbol{\psi}^{-1},d\boldsymbol{\gamma}\boldsymbol{\gamma}^{-1}\\
&\qquad-\boldsymbol{\gamma}\mathfrak{L}_{d\boldsymbol{\psi}\circ\boldsymbol{\psi}^{-1}}\boldsymbol{\gamma}^{-1}\big)_{|\phi}(\mathfrak{X}_{|\phi})\Big\}^v_{|\phi}\Big). \qquad (B22)
\end{aligned}$$

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Conflict of Interest

The authors declare no conflict of interest.

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[1] J. François, L. Ravera, On the Meaning of Local Symmetries: Epistemic-Ontological Dialectic, arXiv:2404.17449 [physics.hist-ph], **2024**.
[2] P. A. M. Dirac, *Can. J. Phys.* **1955**, *33*, 650.
[3] P. A. M. Dirac, *The principles of Quantum Mechanics*, 4th ed., Oxford University Press, Oxford **1958**.
[4] C. Fournel, J. François, S. Lazzarini, T. Masson, *Int. J. Geom. Methods Mod. Phys.* **2014**, *11*, 1450016.
[5] J. François, *Reduction of Gauge Symmetries: A New Geometrical Approach*, Thesis, Aix-Marseille Université, Marseille **2014**.
[6] J. François, *J. High Energy Phys.* **2021**, *2021*, 225.
[7] P. Berghofer, J. François, S. Friederich, H. Gomes, G. Hetzroni, A. Maas, R. Sondenheimer, *Gauge Symmetries, Symmetry Breaking, and Gauge-Invariant Approaches*, Elements in the Foundations of Contemporary Physics, Cambridge University Press, Cambridge **2023**.
[8] J. François André, *J. Phys. A* **2024**, *57*, 305203.
[9] M. Zajac, *J. Geom. Mech.* **2023**, *15*, 128.
[10] J. Norton, *Rep. Prog. Phys.* **1993**, *56*, 791.
[11] J. Stachel, *Einstein and the History of General Relativity*, (Eds: D. Howard, J. Stachel), Birkhäuser, Boston **1989**, pp. 1–63.
[12] J. Stachel, *Living Rev. Relativ.* **2014**, *17*, 1.
[13] M. Giovanelli, *Eur. J. Philos. Sci.* **2021**, *11*, 45.
[14] J. François, *J. High Energy Phys.* **2024**, *8*, 040.
[15] A. Frölicher, A. Kriegl, *Linear Spaces and Differentiation Theory*, Pure and Applied Mathematics, John Wiley & Sons Ltd, New York **1988**.
[16] A. Kriegl, P. P. Michor, *The Convenient Setting of Global Analysis*, Mathematical Surveys and Monographs, Vol. *53*, American Mathematical Society, Providence, R.I. **1997**.
[17] K. C. H. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids*, London mathematical society lecture notes Series, Vol. *213*, Cambridge University Press, Cambridge **2005**.
[18] A. Schmeding, C. Wockel, *Ann. Glob. Anal. Geom.* **2015**, *48*, 87.
[19] J. Pierard de Maujouy, *Field Theory and Generalised Cartan geometry*, PhD thesis, Université Paris Cité, Institut Mathématiques de Jussieu-Paris Rive Gauche, Paris **2022**.
[20] M. Crainic, R. L. Fernandes, *Geom. Topol. Monogr.* **2011**, *17*, 1.
[21] P. G. Bergmann, A. Komar, *Int. J. Theor. Phys.* **1972**, *5*, 15.
[22] D. C. Salisbury, K. Sundermeyer, *Phys. Rev. D* **1983**, *27*, 740.
[23] G. Barnich, C. Troessaert, *Phys. Rev. Lett.* **2010**, *105*, 111103.
[24] L. Freidel, R. Oliveri, D. Pranzetti, S. Speziale, *J. High Energy Phys.* **2021**, *2021*, 83.
[25] L. Freidel, R. Oliveri, D. Pranzetti, S. Speziale, *J. High Energy Phys.* **2021**, *2021*, 170.
[26] H. Gomes, F. Hopfmüller, A. Riello, *Nucl. Phys. B* **2019**, *941*, 249.

[27] V. Chandrasekaran, É. Flanagan, I. Shehzad, A. J. Speranza, *Int. J. Mod. Phys. A* **2022**, *37*, 2250105.
[28] A. J. Speranza, *J. High Energy Phys.* **2022**, *2022*, 29.
[29] G. Barnich, *AIP Conf. Proc.* **2010**, *1307*, 7.
[30] I. Kolar, P. Michor, J. Slovak, *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin **1993**.
[31] J. François, *Adv. Theor. Math. Phys.* **2021**, *25*, 1389.
[32] F. Hopfmüller, L. Freidel, *Phys. Rev. D* **2018**, *97*, 124029.
[33] V. Chandrasekaran, A. Speranza, *J. High Energy Phys.* **2021**, *2021*, 137.
[34] G. Odak, A. Rignon-Bret, S. Speziale, *Phys. Rev. D* **2023**, *107*, 084028.
[35] W. Donnelly, L. Freidel, *J. High Energy Phys.* **2016**, *2016*, 102.
[36] A. Speranza, *J. High Energy Phys.* **2018**, *2018*, 21.
[37] S. Kobayashi, *Transformation Groups in Differential Geometry*, Springer, Berlin **1972**.
[38] J. Attard, J. François, S. Lazzarini, T. Masson, in *Foundations of Mathematics and Physics one Century After Hilbert: New Perspectives*, Springer, Cham **2018**.
[39] J. Attard, J. François, *Class. Quantum Grav.* **2017**, *34*.
[40] J. Attard, J. François, *Adv. Theor. Math. Phys.* **2018**, *22*, 1831.
[41] J. François, *Philos. Sci.* **2019**, *86*, 472.
[42] J. François, N. Parrini, N. Boulanger, *J. High Energy Phys.* **2021**, *2021*, 186.
[43] J. François, L. Ravera, *J. High Energ. Phys.* **2024**, 41.
[44] W. Donnelly, S. B. Giddings, *Phys. Rev. D* **2016**, *93*, 024030.
[45] W. Donnelly, S. B. Giddings, *Phys. Rev. D* **2016**, *94*, 104038.
[46] S. B. Giddings, A. Kinsella, *J. High Energy Phys.* **2018**, *2018*, 74.
[47] S. B. Giddings, *Phys. Rev. D* **2019**, *100*, 126001.
[48] S. B. Giddings, S. Weinberg, *Phys. Rev. D* **2020**, *102*, 026010.
[49] V. Chandrasekaran, R. Longo, G. Penington, E. Witten, *J. High Energy Phys.* **2023**, *2023*, 82.
[50] K. Jensen, J. Sorce, A. J. Speranza, *J. High Energy Phys.* **2023**, *2023*, 20.
[51] V. Kabel, W. Wieland, *Phys. Rev. D* **2022**, *106*, 064053.
[52] S. Carrozza, S. Eccles, P. Hoehn, *SciPost Phys.* **2024**, *17*, 048.
[53] C. Goeller, P. A. Hoehn, J. Kirklin, Diffeomorphism-invariant observables and dynamical frames in gravity: reconciling bulk locality with general covariance, **2022**.
[54] A. J. Speranza, Geometrical tools for embedding fields, submanifolds, and foliations, **2019**.
[55] L. Ciambelli, R. G. Leigh, P. C. Pai, *Phys. Rev. Lett.* **2022**, *128*, 171302.
[56] L. Ciambelli, *PoS Modave* **2023**, *2022*, 002.
[57] J. Tamborino, *SIGMA* **2012**, *8*.
[58] C. Rovelli, *Phys. Rev. D* **2002**, *65*, 124013.
[59] J. Attard, J. François, S. Lazzarini, *Phys. Rev. D* **2016**, *93*, 085032.
[60] I. M. Singer, *Commun. Math. Phys.* **1978**, *60*, 7.
[61] I. M. Singer, *Phys. Scr.* **1981**, *24*, 817.
[62] S. Lazzarini, T. Masson, *J. Geom. Phys* **2012**, *62*, 387.
[63] R. A. Bertlmann, *Anomalies In Quantum Field Theory*, International Series of Monographs on Physics, Vol. *91*, Oxford University Press, Oxford **1996**.
[64] L. Bonora, *Fermions and Anomalies in Quantum Field Theories*, Theoretical and Mathematical Physics, 1st ed., Springer, Cham **2023**.
[65] T. Frankel, *The Geometry of Physics: An Introduction*, 3rd ed., Cambridge University Press, Cambridge **2011**.
[66] P. G. Bergmann, *Phys. Rev.* **1961**, *124*, 274.
[67] K. Prabhu, *Class. Quantum Gravity.* **2017**, *34*, 035011.
[68] M. Geiller, *J. High Energy Phys.* **2018**, *2018*, 29.
[69] M. Göckeler, T. Schücker, *Differential Geometry, Gauge Theory and Gravity*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, Cambridge **1987**.
[70] M. Geiller, *Nucl. Phys. B* **2017**, *924*, 312.
[71] J. François, L. Ravera, The boundary problem as the hole argument in disguise, *In preparation*, **2024**.
[72] V. N. Gribov, *Nucl. Phys. B* **1978**, *139*, 1.
[73] J. Fuchs, M. G. Schmidt, C. Schweigert, *Nucl. Phys. B* **1994**, *426*, 107.
[74] L. Freidel, M. Geiller, D. Pranzetti, *J. High Energy Phys.* **2020**, *2020*, 26.
[75] L. Freidel, M. Geiller, D. Pranzetti, *J. High Energy Phys.* **2020**, *2020*, 27.
[76] L. Freidel, M. Geiller, D. Pranzetti, *J. High Energy Phys.* **2021**, *2021*, 100.
[77] R. Ruegg, M. Ruiz-Altaba, *Int. J. Mod. Phys A* **2004**, *19*, 3265.
[78] E. C. G. Stueckelberg, *Helv. Phys. Acta* **1938**, *11*, 225.
[79] E. C. G. Stueckelberg, *Helv. Phys. Acta* **1938**, *11*, 299.
[80] C. de Rham, *Living Rev. Relativ.* **2014**, *17*.
[81] M. B. Green, C. B. Thorn, *Nucl. Phys. B* **1991**, *367*, 462.
[82] W. Siegel, *Phys. Rev. D* **1994**, *49*, 4144.
[83] J. B. Pitts, *Stud. Hist. Philos. Mod. Phys.* **2005**, *37*, 347.
[84] J. B. Pitts, *General Covariance, Artificial Gauge Freedom and Empirical Equivalence*, PhD thesis, Graduate School of the University of Notre Dame, Notre Dame, IN **2008**.
[85] J. B. Pitts, Empirical equivalence, artificial gauge freedom and a generalized Kretschmann objection, *arXiv:0911.5400*, **2009**.
[86] J. B. Pitts, *Stud. Hist. Philos. Mod. Phys.* **2012**, *43*, 1.
[87] L. Freidel, A canonical bracket for open gravitational system, arXiv:2111.14747 [hep-th], 11 **2021**.
[88] J. Nguyen, N. J. Teh, L. Wells, *Br. J. Philos. Sci.* **2020**, *71*, 665.
[89] J. Dougherty, *Philos. Sci.* **2017**, *84*, 1189.
[90] M. T. Wang, S. T. Yau, *Phys. Rev. Lett.* **2009**, *102*, 021101.
[91] P. Berghofer, J. François, Dressing vs. Fixing: On How to Extract and Interpret Gauge-Invariant Content **2024**, 4.
[92] J. François, L. Ravera, Geometric relational variational principle for general-relativistic gauge field theory, **2024**.
[93] C. Rovelli, *Found. Phys.* **2014**, *44*, 91.
[94] C. J. Isham, *Integrable Systems, Quantum Groups, and Quantum Field Theories*, Vol. *409*, Springer, Dordrecht **1992**, pp. 157–287.
[95] E. Anderson, *Ann. Phys.* **2012**, *524*, 757.
[96] J. François, L. Ravera, Geometric relational framework for general-relativistic gauge field theories. Part II: Quantization, In preparation, **2024**.
[97] J. François, L. Ravera, Bundle geometric relational formulation of non-relativistic Quantum Mechanics, In preparation, **2024**.
[98] C. Rovelli, *Classical Quant. Grav.* **1991**, *8*, 297.
[99] J. D. Brown, K. V. Kuchar, *Phys. Rev. D* **1995**, *51*, 5600.
[100] C. Rovelli, *Phys. Rev. D* **2002**, *65*, 044017.
[101] A. Komar, *Phys. Rev.* **1958**, *111*, 1182.
[102] D. Wallace, Deflating the Aharonov-Bohm Effect, https://arxiv.org/abs/1407.5073, **2014**.
[103] K. Giesel, S. Hofmann, T. Thiemann, O. Winkler, *Classical Quant. Grav.* **2010**, *27*, 055005.
[104] K. Giesel, A. Herzog, P. Singh, *Classical Quant. Grav.* **2018**, *35*, 155012.
[105] J. François, L. Ravera, Geometric relational framework for general-relativistic gauge field theories. Part III: Covariant phase space & Black hole physics, **2024**.