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On the Koiran-Skomra's question about Hessians

Edoardo Ballico and Emanuele Ventura

Abstract. Extending a construction due to B. Segre, we give a negative answer to a question of Koiran and Skomra about Hessians, motivated by Kayal's algorithm for the equivalence problem to the Fermat polynomial. We conjecture that our counterexamples are the only ones. We also study a local version of their question.

1. Introduction

Let $g \in \mathbb{C}[x_1, \ldots, x_n]_d$ be a homogeneous polynomial of degree d and let $\mathrm{Hess}(g)$ be its Hessian matrix. The *Hessian map* is the polynomial map

$$H : \mathbb{C}[x_1, \ldots, x_n]_d \longrightarrow \mathbb{C}[x_1, \ldots, x_n]_{n(d-2)},$$

defined by $H(g) = \det(\mathrm{Hess}(g))$. This determinant is often simply called the *Hessian* of g. *Kayal's algorithm* takes as input a homogeneous polynomial $g \in \mathbb{C}[x_1, \ldots, x_n]_d$ and determines whether g is in the $\mathrm{GL}(\mathbb{C}^n)$-orbit of (or *it is equivalent to*) the Fermat polynomial $f = x_1^d + \cdots + x_n^d$; if it is so, the algorithm outputs linearly independent linear forms $\ell_i \in \mathbb{C}[x_1, \ldots, x_n]_1$ such that $g = \sum_{i=1}^n \ell_i^d$. See [**11**] and [**13**, §3.1] for a detailed account. This algorithm is based on three steps:

(1) Check that the Hessian $H(g)$ is nonzero and can be factored as $H(g) = \alpha \prod_{i=1}^n h_i^{d-2}$ where $h_i \in \mathbb{C}[x_1, \ldots, x_n]_1$ are linear forms. If it not possible, reject.
(2) Try to find complex constants $\alpha_i \in \mathbb{C}$ such that $g = \sum_{i=1}^n \alpha_i \ell_i^d$. If this is not possible, reject.
(3) Declare g to be equivalent to f and output $\ell_i = \beta_i h_i$ where $\beta_i^d = \alpha_i$.

Motivated by Kayal's algorithm, Koiran and Skomra [**13**, Question 1] asked the following question:

Question 1.1 (Koiran-Skomra). Let $n \geq 2$ and $d \geq 3$. Let $f \in \mathbb{C}[x_1, \ldots, x_n]_d$ be such that $H(f) = \alpha(x_1 \cdots x_n)^{d-2}$, for some $\alpha \neq 0$.

Does it follow that $f = \alpha_1 x_1^d + \cdots + \alpha_n x_n^d$, for some constants $\alpha_j \in \mathbb{C}$?

This question is interesting also because it is so simple to state and has a delightful invariant theory flavour. Hessians of forms have been the subject of

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many classical works in algebraic geometry and commutative algebra (for instance, Hessians are related to Lefschetz properties of artinian Gorenstein algebras); see e.g. [**17**, Chapter 7] and [**3**], along with the several references therein. Hessians are also naturally related to ranks of homogeneous polynomials [**4, 8**]. Beniamino Segre [**18**, p. 174] found a family of counterexamples in four variables of even degree to the Koiran-Skomra question; his aim was to construct polynomials whose Hessians equal to a square in the field of fractions of the corresponding hypersurfaces. We extend Segre's counterexamples to an arbitrary number of variables and composite degrees in Theorem 2.1.
Employing the algorithm in [**14**, §4.1] to decide whether the factorization in step (1) of Kayal's algorithm *exists* (and not to compute the explicit factorization), a positive answer to Question 1.1 would provide a polynomial time algorithm for the equivalence problem over $\mathbb{C}$. In their article [**13**], Koiran and Skomra did provide a polynomial time algorithm for the equivalence problem over $\mathbb{C}$ in degree $d = 3$, which was later extended by Koiran and Saha [**12**] to $d > 3$. However, their algorithm has to work in a very different way than Kayal's algorithm because of our Theorem 2.1: this shows that over any field $K \subset \mathbb{C}$, the answer is negative for infinite pairs of integers (n, d). An instance of interest in [**13**] is $d = 3$: in this case, the answer is positive if and only if $n = 2$ (Corollary 2.5).

Our monomial counterexamples are *homaloidal polynomials*, i.e. their first partial derivatives define a Cremona transformation. Theorem 2.1 also shows that these monomials are *totally Hessian* [**3**, Remark 3.5]. Therefore they give examples of totally Hessian polynomials (although reducible) of arbitrarily large degree in any number of variables $n \geq 2$.

In Theorem 2.8 and Theorem 2.9, we deal with the case of singular binary homogeneous polynomials, providing the only ones whose Hessians have the desired form. Our approach is combinatorial and we wonder whether an invariant theoretic strategy could be put in place. We conjecture that for any $n \geq 2$ the only smooth homogeneous polynomials satisfying the assumptions of Question 1.1 are the Fermat polynomials; see Conjecture 2.16.

Computing the differential of the Hessian map, we formulate a local version of Koiran-Skomra's question, which has a positive answer when $d \geq n+1$ (Proposition 3.2).

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2. Koiran-Skomra's question

2.1. Negative result. The first counterexamples to Question 1.1 were constructed by Beniamino Segre [**18**, p. 174], where he considers polynomials in four variables of degree $d = 2k$ of the form $g(x_1, x_2, x_3, x_4) = x_1^{2k} + (1-2k)x_2^{2k} - (x_3 \cdot x_4)^k$, whose Hessian matrix has determinant

$$H(g) = \left(2k^2(2k-1)^2(x_1 \cdot x_2 \cdot x_3 \cdot x_4)^{k-1}\right)^2.$$

We extend this family of counterexamples as follows.

THEOREM 2.1. *Let $K \subset \mathbb{C}$ be a field. The answer to Question 1.1 over K is negative for all $n \geq 2$ and $d \geq 3$ such that there exists $2 \leq q \leq n$ with $q \mid d$.*

PROOF. Let $d = kq$ with $k \geq 1$ (or $k \geq 2$ if $q = 2$). Then define

$$g = (x_1 \cdots x_q)^k + x_{q+1}^d + \cdots + x_n^d \in K[x_1, \ldots, x_n].$$

We show that

$$H((x_1 \cdots x_q)^k) = (1-d)(-k)^q (x_1 \cdots x_q)^{d-2} \neq 0. \tag{2.1}$$

This is enough to conclude that $H(g)$ has the desired form. We first prove that every monomial in the determinant expansion of the Hessian above is $(x_1 \cdots x_q)^{d-2}$.

To see this, fix an index $\ell \in \{1, \ldots, q\}$ and look at the total degree in x_ℓ of an arbitrary monomial in the determinant expansion. By symmetry, we may assume $x_\ell = x_1$.

Let $f_{i,j} = \left(\text{Hess}((x_1 \cdots x_q)^k)\right)_{i,j}$ be the (i,j)-entry of the Hessian matrix of $(x_1 \cdots x_q)^k$. Let m be a monomial in the expansion of $H((x_1 \cdots x_q)^k)$. This is a product of entries $f_{i,j}$, where the product ranges over the indices of all rows and columns, by definition of determinant. Then exactly one of the following cases can occur:

(I) m is a multiple of the $(1,1)$-entry (this choice gives no contribution for $k = 1$);

(II) m is a multiple of two distinct entries in the first row and in the first column.

In case (I), the total degree of m with respect to x_1 is $(k-2)+(q-1)k = qk-2 = d-2$. In case (II), the total degree of m with respect to x_1 is $2(k-1)+(q-2)k = qk-2 = d-2$. In other words, for any monomial in the expansion of $H((x_1 \cdots x_q)^k)$, the total degree of every variable is $d-2$. Let C be the $q \times q$ matrix whose (i,j)-entry is the coefficient of the monomial $f_{i,j}$. One has

$$C = \begin{bmatrix} k(k-1) & k^2 & \cdots & k^2 \\ k^2 & k(k-1) & \cdots & k^2 \\ \vdots & \vdots & \vdots & \vdots \\ k^2 & k^2 & \cdots & k(k-1) \end{bmatrix}. \tag{2.2}$$

The argument above shows that

$$H((x_1 \cdots x_q)^k) = \det(C)(x_1 \cdots x_q)^{d-2}.$$

Write $C = -kI_q + k^2 J_q$, where I_q is the $q \times q$ identity matrix and J_q is the $q \times q$ matrix whose entries are all ones. Note that $J_q = \mathbf{1}_q \cdot (\mathbf{1}_q)^T$, where $\mathbf{1}_q$ is the column vector with q ones.

One may write $C = -kI_q \cdot C'$, where $C' = I_q - \mathbf{1}_q \cdot (k\mathbf{1}_q)^T$. Note that

(2.3)

$$\begin{bmatrix} I_q & 0 \\ (k\mathbf{1}_q)^T & 1 \end{bmatrix} \cdot \begin{bmatrix} I_q - (\mathbf{1}_q) \cdot (k\mathbf{1_q})^T & -\mathbf{1}_q \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} I_q & 0 \\ -(k\mathbf{1}_q)^T & 1 \end{bmatrix} = \begin{bmatrix} I_q & -\mathbf{1}_q \\ 0 & 1 - (k\mathbf{1}_q)^T \cdot \mathbf{1}_q \end{bmatrix}.$$

Then $\det(C')$ is the determinant of the left-hand side of (2.3), thus the determinant of the right-hand side too. So $\det(C') = 1 - (k\mathbf{1}_q)^T \cdot \mathbf{1}_q = 1 - qk$ and $\det(C) = (1-d)(-k)^q \neq 0$, which shows (2.1). Finally, the polynomial g has Hessian $H(g) = \alpha(x_1 \cdots x_n)^{d-2} \neq 0$ and is not a smooth homogeneous polynomial over K. Hence g cannot be equivalent to a Fermat polynomial over K. □

Remark 2.2. A homogeneous polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is *homaloidal* when its first partial derivatives define a birational gradient map $\mathrm{grad}(f) : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ (i.e. a Cremona transformation). Equivalently, their *polar degree* is one. Let $g = f^k$ with $k \geq 2$. By the chain rule, we have

$$\mathrm{grad}(g) = kf^{k-1}\mathrm{grad}(f).$$

Hence the two rational maps $\mathrm{grad}(g)$ and $\mathrm{grad}(f)$ coincide outside the locus $\mathcal{V}(f) = \{f = 0\}$. In particular, they have the same polar degree; see also [**5**, Corollary 2] for a more general deep result. The polynomial $f = x_1 \cdots x_n$ is homaloidal, i.e. the rational map $\mathrm{grad}(f) : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ is an isomorphism onto its image outside $\mathcal{V}(f)$, and so is $g = (x_1 \cdots x_n)^k$. Theorem 2.1 also shows that these polynomials are *totally Hessian* [**3**, Remark 3.5] (although reducible) of arbitrarily large degree in any number of variables $n \geq 2$. These observations lead to an alternative and more conceptual proof of Theorem 2.1, kindly suggested to us by an anonymous referee. Since $\mathrm{grad}(f)$ and $\mathrm{grad}(g)$ coincide outside $\mathcal{V}(f)$, the ramification divisors of the birational maps $\mathrm{grad}(f)$ and $\mathrm{grad}(g)$ have the same support. Hence $H(g) = \alpha \cdot x_1^{b_1} \cdots x_n^{b_n}$, with $b_1 + \cdots + b_n = n(nk-2)$. By symmetry, all the b_i's must be equal and so $b_i = nk - 2$. Thus $H((x_1 \cdots x_n)^k) = \alpha \cdot (x_1 \cdots x_n)^{nk-2}$.

Remark 2.3. The homogeneous polynomial g in the proof of Theorem 2.1 is a sum of pairwise coprime monomials. Then [**1**, Theorem 3.2] shows that the complex Waring rank of g satisfies the equality $\mathrm{rk}_K(g) \geq \mathrm{rk}_{\mathbb{C}}(g) = (k+1)^{q-1} + (n-q) > n = \mathrm{rk}_K(f)$, where $f = x_1^d + \cdots + x_n^d$.

Remark 2.4. There exist homogeneous polynomials whose Hessians are monomials distinct from $(x_1 \cdots x_n)^{d-2}$. Three examples:

(i) for $k \geq 2$, let $f = x_2^{k-1}x_1^2 - x_3^{k+1} \in \mathbb{C}[x_1, x_2, x_3]_{k+1}$. It has singularities at $(0:1:0)$ and $(1:0:0)$, and $H(f) = -2k^2(k+1)(k-1)x_1^2x_2^{2(k-2)}x_3^{k-1}$.
(ii) $g = x_1(x_2^2 + x_1x_3 + \sum_{i=4}^n x_i^2) \in \mathbb{C}[x_1, x_2, x_3, \dots, x_n]_3$ is a singular cubic form which splits as a product between a smooth quadric and a tangent hyperplane to it. Then $H(f) = -2^n x_1^n$.
(iii) The determinant of a generic sub-Hankel matrix [**3**, 4.1.1] is a homaloidal polynomial whose Hessian is a power of a variable [**3**, Theorem 4.4]. Since these are homaloidal of degree $d \geq 3$, they are singular.

A case of interest in [**13**] is when $d = 3$.

Corollary 2.5. Let $d = 3$. Then the answer to Question 1.1 is positive if and only if $n = 2$.

Proof. If $n \geq 3 = d$, by Theorem 2.1 the answer is negative. Suppose $n = 2$ and let $f = a_3x_1^3 + a_2x_1^2x_2 + a_1x_1x_2^2 + a_0x_2^3$. So

$$H(f) = (-4a_2^2 + 12a_3a_1)x_1^2 + (-4a_2a_1 + 36a_3a_0)x_1x_2 + (-4a_1^2 + 12a_2a_0)x_2^2,$$

where $H(f) = \alpha x_1x_2 \neq 0$. If $a_2 = 0$, then $a_3 \neq 0$ and so $a_1 = 0$. But then $f = a_3x_1^3 + a_0x_2^3$. If not, by symmetry, we may assume $a_1, a_2 \neq 0$. Therefore

$a_0 = a_1^2/3a_2$ and $a_3 = a_2^2/3a_1$, which implies that the coefficient of the monomial x_1x_2 in $H(f)$ is zero. □

For $n = d = 3$, the following statement is based on a well-known classification and we record it from our perspective.

PROPOSITION 2.6. For $n = d = 3$, the only cubic homogenous polynomials satisfying the assumptions in Question 1.1 are $f = a_1x_1^3 + a_2x_2^3 + a_3x_3^3$ and $f = a_{123}x_1x_2x_3$ (with $a_1a_2a_3 \neq 0$ and $a_{123} \neq 0$ resp.).

PROOF. Let $f \in \mathbb{C}[x_1, x_2, x_3]$ be a cubic form. The Hessian map $H : \mathbb{C}[x_1, x_2, x_3]_3 \to \mathbb{C}[x_1, x_2, x_3]_3$, $H : g \mapsto H(g)$ is an $\mathrm{SL}(\mathbb{C}^3)$-coinvariant [**7**, §5.1]. Thus, it is enough to check the $\mathrm{SL}(\mathbb{C}^3)$-orbits of ternary cubic homogenous polynomials, which are well-known. Among these, we see that the only cubic homogenous polynomials whose Hessian splits into three linearly independent linear forms are the Fermat polynomials $g = \ell_1^3 + \ell_2^3 + \ell_3^3$ and the triangles $g = \ell_1\ell_2\ell_3$. Since by assumption $H(g) = \alpha x_1x_2x_3 \neq 0$, by Lemma 2.10 proved below, we see that $\ell_i = x_i$ (up to relabeling and scaling). This shows the statement. □

2.2. Singular binary homogeneous polynomials. We deal with the case of singular binary homogeneous polynomials, providing the only ones whose Hessians have the desired form.

REMARK 2.7. For $n \geq 2$, let $f \in \mathbb{C}[x_1, \dots, x_n]_d$ and let $X_f \subset \mathbb{P}^{n-1}$ be the reduced projective hypersurface defined by f. Then the reduced singular locus $\mathrm{Sing}(X_f)$ sits inside $X_{H(f)}$, where $H(f) = \det(\mathrm{Hess}(f))$. This may be seen by interpreting the Hessian as the Jacobian of the $\frac{\partial f}{\partial x_i}$'s.

THEOREM 2.8. *For $n = 2$ and $d \geq 3$ odd, there are no singular homogeneous polynomials satisfying the assumptions in Question 1.1.*

PROOF. Let $d = 2p + 1$. Let $f = f(x_1, x_2) = \sum_{i=0}^{d} a_i x_1^i x_2^{d-i} \in \mathbb{C}[x_1, x_2]_d$ be such that $H(f) = \alpha(x_1x_2)^{d-2} \neq 0$. Up to the automorphism of $\mathbb{C}[x_1, x_2]$ swapping x_1 and x_2 and by Remark 2.7, we may assume that $f = \sum_{i\geq 2}^{d} a_i x_1^i x_2^{d-i}$, i.e. $a_0 = a_1 = 0$. The coefficient of the monomial $x_1^j x_2^{2d-j-4}$ in $H(f)$ is of the form

$$\sum_{k+\ell=j+2} \gamma_{(k,\ell)} a_k a_\ell.$$

For all $2 \leq s \leq p$ and $j = 2(s-1)$, the coefficient $\gamma_{(s,s)}$ in front of a_s^2 appearing in the monomial $x_1^{2s-2}x_2^{2d-2s-2}$ in $H(f)$ is

$$\gamma_{(s,s)} = -s(d-1)(d-s) \neq 0.$$

Now, suppose that for some $1 \leq j' < p$ we have that $a_{j''} = 0$ for all $0 \leq j'' \leq j'$. We claim that $a_{j'+1} = 0$.

To show the claim, look at the coefficient of the monomial $x_1^{2j'}x_2^{2d-2j'-4}$ given by

$$\sum_{k+\ell=2j'+2} \gamma_{(k,\ell)} a_k a_\ell. \tag{2.4}$$

Since $k + \ell = 2(j' + 1)$, if $\ell \geq j' + 2$ then $k \leq j'$. By assumption, in this case, we have $a_k a_\ell = 0$ and so this summand does not contribute to (2.4). Then the only summand left in (2.4) is $\gamma_{(j'+1,j'+1)} a_{j'+1}^2$. Since $j' < p$, $2j' \leq 2(p-1) = 2p-2 =$

$d-3$. Thus the monomial $x_1^{2j'}x_2^{2d-2j'-2}$ does not appear in $H(f)$, by assumption. Therefore $\gamma_{(j'+1,j'+1)}a_{j'+1}^2 = 0$ and since $\gamma_{(j'+1,j'+1)} \neq 0$, we find $a_{j'+1} = 0$.

Since $a_0 = a_1 = 0$, then the argument above proves that $a_j = 0$ for all $j \leq p$. Now, look at the coefficient of $x^{d-2}y^{d-2}$. This is of the form

$$\sum_{k+\ell=d} \gamma_{(k,\ell)} a_k a_\ell.$$

If $\ell \geq p+1$, then $k \leq p$. So $a_k a_\ell = 0$ by what we have shown. Hence the coefficient above vanishes. This is in contradiction with the assumption on $H(f)$. □

Theorem 2.9. *For $n = 2$ and $d \geq 4$ even, the only singular binary homogeneous polynomials satisfying the assumptions in Question 1.1 are of the form $f = a_{d/2}x_1^{d/2}x_2^{d/2}$.*

Proof. Let $d = 2p$. Let $f = f(x_1, x_2) = \sum_{i=0}^d a_i x_1^i x_2^{d-i} \in \mathbb{C}[x_1, x_2]_d$ be such that $H(f) = \alpha(x_1x_2)^{d-2} \neq 0$. As in the proof of Theorem 2.8, we may assume that $f = \sum_{i\geq 2}^d a_i x_1^i x_2^{d-i}$, i.e. $a_0 = a_1 = 0$. Moreover, again looking at $\gamma_{j,j}$ for $j \leq p-1$, we find that $a_j = 0$ for all $j \leq p-1$ and $a_p = a_{d/2} \neq 0$.

Let $1 \leq t \leq p$ and consider the coefficient a_{p+t} of the monomial $x_1^{p+t}x_2^{d-p-t}$ in f. Look at the coefficient of the monomial $x_1^{2p+t-2}x_2^{2p-t-2}$ in $H(f)$. This is of the form

$$\sum_{k+\ell=2p+t} \gamma_{(k,\ell)} a_k a_\ell \tag{2.5}$$

and, since $2p+t-2 > 2p-2 = d-2$, it must vanish by the assumption on $H(f)$. Since $k+\ell = 2p+t$, if $\ell \geq p+t+1$ then $k \leq p-1$. In this case, $a_k a_\ell = 0$ by what we have shown above. So the only left summand in (2.5) is $\gamma_{(p,p+t)}a_p a_{p+t}$, where $a_p \neq 0$ but $\gamma_{(p,p+t)}a_p a_{p+t} = 0$.

If $\gamma_{(p,p+t)} \neq 0$, for any given p and every $1 \leq t \leq p$, then $a_{p+t} = 0$ for all $1 \leq t \leq p$ and we would be done.

The coefficient $\gamma_{(p,p+t)}$ is given by

$$\gamma_{(p,p+t)} = 2p(2p-1)(t^2 - p). \tag{2.6}$$

In other words, given $p \geq 2$, $\gamma_{(p,p+t)} = 0$ if and only if $t = \sqrt{p}$ (i.e. when p is a perfect square and t is its positive square root).

From now on, we shall then assume that p is a perfect square, otherwise we are done. Since $\gamma_{(p,p+t')} \neq 0$ for all $1 \leq t' \leq \sqrt{p}-1$, then the coefficient (2.5) for the monomial $x_1^{2p+t'-2}x_2^{2p-t'-2}$ is

$$\gamma_{(p,p+t')}a_p a_{p+t'} = 0.$$

Hence $a_{p+t'} = 0$ for all $1 \leq t' \leq \sqrt{p}-1$.

Claim 1: Let $1 \leq t \leq p$ be an integer such that $m\sqrt{p} < t < (m+1)\sqrt{p}$, for some integer $0 \leq m \leq \sqrt{p}-1$. Suppose that $a_{p+t'} = 0$ for all t' such that $m'\sqrt{p} < t' < (m'+1)\sqrt{p}$ for all $0 \leq m' < m$. Then $a_{p+t} = 0$.

Proof. Write $t = m\sqrt{p} + n$, where $1 \leq n < \sqrt{p}$. We look at the coefficient of the monomial $x_1^{2p+t-2}x_2^{2p-t-2} = x_1^{2p+m\sqrt{p}+n-2}x_2^{2p-m\sqrt{p}-n}$. This is of the form

$$\sum_{k+\ell=2p+m\sqrt{p}+n} \gamma_{(k,\ell)} a_k a_\ell. \tag{2.7}$$

Note that $2p + m\sqrt{p} + n - 2 > 2p - 2 = d - 2$ and so the coefficient (2.7) must vanish. If ℓ or k are of the form $p + q\sqrt{p} + q'$ with $0 \leq q \leq m-1$ and $1 \leq q' < \sqrt{p}$, then the product $a_k a_\ell = 0$, by assumption. If $\ell = p + z\sqrt{p}$ with $0 \leq z \leq m-1$, then $k = p + z'\sqrt{p} + n$ with $1 \leq z' \leq m-1$. Thus the product $a_k a_\ell = 0$, by assumption. In conclusion, the only left summand in the coefficient (2.7) is $\gamma_{p,p+m\sqrt{p}+n} a_p a_{p+m\sqrt{p}+n} = 0$. Since $a_p \neq 0$ and $\gamma_{p,p+m\sqrt{p}+n} \neq 0$ (because $m\sqrt{p} + n \neq \sqrt{p}$ for all choices of m, n defined above), we find $a_{p+m\sqrt{p}+n} = a_{p+t} = 0$. $\square$

Note that we have already proven that $a_{p+t'} = 0$ for all $1 \leq t' \leq \sqrt{p} - 1$. This is the case $m = 1$ in the assumption of **Claim 1**. Then applying the conclusion of **Claim 1** iteratively, we find that all the coefficients of f vanish, *unless* they are the following $(\sqrt{p}+1)$ coefficients: $a_p, a_{p+\sqrt{p}}, a_{p+2\sqrt{p}}, \ldots, a_{2p}$. To conclude, we have to show that, except a_p, they all must vanish.

To this aim, for $0 \leq s, r \leq \sqrt{p}$ and $(s,r) \neq (0,0)$, we look at the coefficient of $a_{p+s\sqrt{p}} a_{p+r\sqrt{p}}$ in the coefficient of the monomial $x_1^{2p+(s+r)\sqrt{p}-2} x_2^{2p-(s+r)\sqrt{p}-2}$ in $H(f)$. This has the form $\gamma_{p+s\sqrt{p},p+r\sqrt{p}} = 2p(2p-1)(ps^2 - 2srp + pr^2 + sr - p)$. We study the vanishing (and the sign) of $F(p,s,r) = ps^2 - 2srp + pr^2 + sr - p$. Write $F(p,s,r) = p(s-r)^2 + sr - p$. If $s = r$, then $s = r < \sqrt{p}$, because the exponent of x_1 must satisfy $2p + (s+r)\sqrt{p} - 2 \leq 4p - 4 = 2d - 4$. Thus, if $s = r$ then $F(p,s,s) < 0$. If $s, r \geq 1$ and $s \neq r$, then $p(s-r)^2 - p \geq 0$ and the product $sr \geq 1$. So $F(p,s,r) > 0$. If $s = 0$ (or $r = 0$), then $F(p,0,r) = pr^2 - p = 0$ if and only $r = 1$ (or $s = 1$). Otherwise, if $s = 0$, then $F(p,0,r) > 0$ for any $r \geq 2$. In conclusion, $\gamma_{p+s\sqrt{p},p+r\sqrt{p}} \neq 0$ unless $s = 0$ and $r = 1$ (or the way around). This is exactly the case of (2.6).

Claim 2: Suppose that for all $1 \leq i \leq u - 1 < \sqrt{p} - 1$ we know that $a_{p+(\sqrt{p}-i)\sqrt{p}} = a_{2p-i\sqrt{p}} = 0$. Then $a_{p+(\sqrt{p}-u)\sqrt{p}} = 0$.

Proof. The remaining coefficients are $a_{p+\sqrt{p}}, a_{p+2\sqrt{p}}, \ldots, a_{2p}$ (besides a_p) and all the others vanish.

Note that $2p - i\sqrt{p} > p + (\sqrt{p} - u)\sqrt{p}$. In other words, we are assuming that all the a_h with $h > p + (\sqrt{p} - u)\sqrt{p}$ vanish.

We look at the coefficient of the monomial $x_1^{4p-2u\sqrt{p}-2} x_2^{2u\sqrt{p}-2}$ in $H(f)$, which must vanish by assumption. This coefficient has the form

$$\sum_{k+\ell=4p-2u\sqrt{p}} \gamma_{(k,\ell)} a_k a_\ell = 0. \tag{2.8}$$

Write $k = p + h'\sqrt{p}$ and $\ell = p + h''\sqrt{p}$. Then $k + \ell = 4p - 2u\sqrt{p} = 2p + (h' + h'')\sqrt{p}$. Therefore $h' + h'' = 2(\sqrt{p} - u)$. If $h' \geq \sqrt{p} - u + 1$, then $k = p + h'\sqrt{p} > p + (\sqrt{p} - u)\sqrt{p}$ and hence $a_k = 0$, by assumption. Thus the coefficient (2.8) becomes $\gamma_{(p+(\sqrt{p}-u)\sqrt{p},p+(\sqrt{p}-u)\sqrt{p})} a^2_{p+(\sqrt{p}-u)\sqrt{p}} = 0$. Since $\gamma_{(p+(\sqrt{p}-u)\sqrt{p},p+(\sqrt{p}-u)\sqrt{p})} \neq 0$ by the part before this claim, we find $a_{p+(\sqrt{p}-u)\sqrt{p}} = 0$. $\square$

Now, look at the coefficient of the monomial $x_1^{4p-\sqrt{p}-2} x_2^{\sqrt{p}-2}$. The only coefficient appearing here is the product $\gamma_{(2p,p+(\sqrt{p}-1)\sqrt{p})} a_{2p} a_{p+(\sqrt{p}-1)\sqrt{p}}$, which must vanish. Since $\gamma_{(2p,p+(\sqrt{p}-1)\sqrt{p})} \neq 0$ by the part of the proof before **Claim 2**, either a_{2p} or $a_{p+(\sqrt{p}-1)\sqrt{p}}$ vanishes.

If $a_{2p} = 0$, applying **Claim 2**, we are done. If $a_{p+(\sqrt{p}-1)\sqrt{p}} = 0$, then applying **Claim 2**, all coefficients $a_{p+\sqrt{p}}, a_{p+2\sqrt{p}}, \ldots, a_{2p}$ are zero, except possibly a_{2p}.

However, assuming $a_{2p} \neq 0$ (with all the other coefficients being zero, except a_p) leads to an immediate contradiction with the assumption on $H(f)$. In conclusion, $f = a_{d/2} x_1^{d/2} x_2^{d/2}$ and this establishes the statement. $\square$

2.3. Stabilizers and Jacobian rings.

Lemma 2.10. *One has the following descriptions for the stabilizers:*

(i) $\mathrm{Stab}_{\mathrm{GL}(\mathbb{C}^n)}((x_1 \cdots x_n)^{d-2}) \cong (\mathbb{C}^*)^{n-1} \rtimes \mathfrak{S}_n$.

(ii) *Let* $h = (x_1 \cdots x_n)^{d-2}$ *and let* $\mathcal{V}(h)$ *be the corresponding projective hypersurface in the projective space* $\mathbb{P}^{n-1}_{\mathbb{C}}$ *equipped with the natural action of* $\mathrm{PGL}(\mathbb{C}^n)$. *Then* $\mathrm{Stab}_{\mathrm{PGL}(\mathbb{C}^n)}(\mathcal{V}(h)) \cong (\mathbb{C}^*)^n \rtimes \mathfrak{S}_n / \mathbb{C}^*$.

Proof. (i). Let $A \in \mathrm{Stab}_{\mathrm{GL}(\mathbb{C}^n)}((x_1 \cdots x_n)^{d-2})$ be an element of the stabilizer in $\mathrm{GL}(\mathbb{C}^n)$ of the indicated polynomial. Since a polynomial ring is a unique factorization domain and the x_i's are irreducible in this ring, for each i we find $A \circ x_i = \gamma_i x_{\sigma(i)}$, for some $\sigma \in \mathfrak{S}_n$ and $\gamma_i \in \mathbb{C}^*$ with $\prod_{i=1}^n \gamma_i^{d-2} = 1$. The last condition provides an isomorphism with the $(n-1)$-dimensional algebraic torus. Statement (ii) is proven similarly (here we have to further quotient out by the $\mathbb{C}^*$ given by the scalar multiples of the identity). $\square$

Lemma 2.11. *Let* $f \in \mathbb{C}[x_1, \ldots, x_n]_d$ *and* $h = H(f) \in \mathbb{C}[x_1, \ldots, x_n]_{n(d-2)}$ *be its Hessian.*

(i) *Their stabilizers in* $\mathrm{SL}(\mathbb{C}^n)$ *satisfy* $\mathrm{Stab}_{\mathrm{SL}(\mathbb{C}^n)}(f) \subset \mathrm{Stab}_{\mathrm{SL}(\mathbb{C}^n)}(h)$.

(ii) *Suppose* $h = \alpha(x_1 \cdots x_n)^{d-2} \neq 0$ *and let* $s = \dim \mathrm{Stab}_{\mathrm{SL}(\mathbb{C}^n)}(f)$. *Then every irreducible component of the fiber* $H^{-1}(h)$ *containing* f *has dimension* $\geq (n-1) - s$.

Proof. (i). Since the Hessian map is an $\mathrm{SL}(\mathbb{C}^n)$-coinvariant [**7**, §5.1], for any $A \in \mathrm{Stab}_{\mathrm{SL}(\mathbb{C}^n)}(f)$, we have $h = H(A \circ f) = A \circ h$, which proves the containment. (ii). Let $X \subset H^{-1}(h)$ be an irreducible component containing f and let G be the connected component of the identity in $\mathrm{Stab}_{\mathrm{SL}(\mathbb{C}^n)}(h)$, which has dimension $\dim(G) = n-1$. Note that the orbit $G \cdot f \subset X$ and so $\dim X \geq \dim(G \cdot f) \geq n - 1 - s$. $\square$

The two statements in Lemma 2.11 may be used as necessary conditions for the equality $H(f) = \alpha(x_1 \cdots x_n)^{d-2} \neq 0$ to be satisfied by a given $f \in \mathbb{C}[x_1, \ldots, x_n]_d$. An application of that comes next.

Proposition 2.12. Let $n \geq 2$ and $d \geq 3$. There is no $g \in \mathbb{C}[x_1, \ldots, x_n]_d$ such that $H(g) = \alpha(x_1 \cdots x_n)^{d-2} \neq 0$ with complex Waring rank $\mathrm{rk}_{\mathbb{C}}(g) = n+1$.

Proof. Up to the action of $\mathrm{GL}(\mathbb{C}^n)$, we may assume that $g = x_1^d + \cdots + x_n^d + \ell^d$, where $\ell = \alpha_1 x_1 + \cdots + \alpha_n x_n$, for some $\alpha_i \in \mathbb{C}$. If $\ell \in \langle x_{i_1}, \ldots, x_{i_s} \rangle$ for some $s \leq n-1$, then setting $h = g - \sum_{j \neq i_1, \ldots, i_s} x_j^d \in \mathbb{C}[x_{i_1}, \ldots, x_{i_s}]$, we find that $H(g) = \beta H(h) \cdot \prod_{j \neq i_1, \ldots, i_s} x_j^{d-2}$. Thus we reduce to the case where $\alpha_i \neq 0$ for all i. It is a direct computation to see that $\dim \mathrm{Stab}_{\mathrm{GL}(\mathbb{C}^n)}(g) = 0$. By the assumption $H(g) = \alpha(x_1 \cdots x_n)^{d-2} \neq 0$ and Lemma 2.11, $\dim \mathrm{Ker}(dH_g) \geq n-1$. (This is because if the dimension of the fiber at $H(g)$ satisfies $\dim H^{-1}(H(g)) \geq q$ then $\dim \mathrm{Ker}(dH_g) \geq q$, for any $q \in \mathbb{N}$.) Now, following the argument in the proof of [**2**, Lemma 7.2], we find that $\mathrm{Ker}(dH_g) = 0$, which is a contradiction. $\square$

One of the tools at our disposal is a finite-dimensional algebra attached to a smooth polynomial called the *Jacobian ring*. Although in Proposition 2.15 we will establish a sufficient condition on a smooth homogeneous polynomial with a monomial Hessian to be a Fermat polynomial, this approach seems to be weak towards answering Question 1.1 in the smooth case.

DEFINITION 2.13. Let $g \in \mathbb{C}[x_1, \ldots, x_n]_d$. The Jacobian ring of g is the quotient ring $R(g) = \mathbb{C}[x_1, \ldots, x_n]/J(g)$, where $J(g) = (\partial g/\partial x_1, \ldots, \partial g/\partial x_n)$. If g is a smooth homogeneous polynomial, then $R(g)$ is a zero-dimensional local ring and a finite-dimensional graded Gorenstein $\mathbb{C}$-algebra [**9**, Corollary 4.2]. Moreover, the highest non-zero degree summand (the *socle*) of $R(g)$ is $R(g)_{n(d-2)} \cong \mathbb{C}$ and generated by the class of its Hessian, i.e., $H(g) \notin J(g)$.

PROPOSITION 2.14. Let f be a Fermat polynomial. Suppose a smooth homogeneous polynomial $g \in \mathbb{C}[x_1, \ldots, x_n]$ of degree $d \geq 3$ in $n \geq 2$ variables is such that its Jacobian ring satisfies $R(g) = R(f)$ and its Hessian $H(g) = \alpha(x_1 \cdots x_n)^{d-2}$. Then g is a Fermat polynomial.

PROOF. Since $R(g) = R(f)$, a result of Donagi [**9**, Proposition 4.9] shows that there exists $A \in \mathrm{PGL}(\mathbb{C}^n)$ such that $A \circ \mathcal{V}(f) = \mathcal{V}(g)$, where these are the corresponding degree d smooth projective hypersurfaces. Note that A must fix $\mathcal{V}(H(f))$, as $H(g)$ is a multiple of $H(f)$. Therefore $A \in \mathrm{Stab}_{\mathrm{PGL}(\mathbb{C}^n)}(\mathcal{V}(H(f)))$. By Lemma 2.10(ii), one then finds $g = \beta_1 x_1^d + \cdots + \beta_n x_n^d$. □

PROPOSITION 2.15. Suppose a smooth homogenous polynomial $g \in S = \mathbb{C}[x_1, \ldots, x_n]$ of degree $d \geq 3$ in $n \geq 2$ variables is such that $H(g) = x_1^{b_1} \cdots x_n^{b_n}$. Suppose its Jacobian ideal $J(g)$ is a monomial ideal. Then $H(g) = (x_1 \cdots x_n)^{d-2}$ and g is a Fermat polynomial.

PROOF. The dimension of the Jacobian ring $R(g)$ as a complex vector space depends only on n and d; see [**9**, Proposition 4.3]. Hence it coincides with the one of $R(f)$, where f is a Fermat polynomial. One then has $\dim_{\mathbb{C}} R(g) = \dim_{\mathbb{C}} R(f) = (d-1)^n$. To see the last equality, let $f = \alpha_1 x_1^d + \cdots + \alpha_n x_n^d$ be a Fermat polynomial. Its Jacobian ring is the quotient $R(f) = S/(x_1^{d-1}, \ldots, x_n^{d-1})$. A monomial $\mathbb{C}$-basis for this algebra is formed by all monomial divisors of $H(f) = \alpha(x_1 \cdots x_n)^{d-2}$, whose cardinality is $(d-1)^n$. Since $R(g)$ is a finite-dimensional graded artinian Gorenstein $\mathbb{C}$-algebra, Macaulay's theorem [**10**, Lemma 2.12] gives a bijection between these Gorenstein algebras, whose socle is in degree $n(d-2)$, and homogenous polynomials of degree $n(d-2)$ up to $\mathbb{C}^*$. This implies that there exists $h \in T = \mathbb{C}[y_1, \ldots, y_n]_{n(d-2)}$ such that $J(g) = \mathrm{Ann}(h)$, where S acts by differentiation on T. Recall that $H(g) = x_1^{b_1} \cdots x_n^{b_n} \notin J(g) = \mathrm{Ann}(h)$. Since $\mathrm{Ann}(h)$ is a monomial ideal and $\dim_{\mathbb{C}} R(g)_{n(d-2)} = 1$, every monomial of degree $n(d-2)$ different from $H(g)$ is in $\mathrm{Ann}(h)$. Thus, writing a monomial expansion of h, we find that $h = y_1^{b_1} \cdots y_n^{b_n}$, up to scaling. So $\mathrm{Ann}(h) = (x_1^{b_1+1}, \ldots, x_n^{b_n+1})$ and then $(d-1)^n = \dim_{\mathbb{C}} R(g) = \dim_{\mathbb{C}} T/\mathrm{Ann}(h) = \prod_{i=1}^n (b_i + 1)$. Moreover, by definition of the Hessian, we have $\sum_{i=1}^n (b_i + 1) = (\sum_{i=1}^n b_i) + n = n(d-2) + n = n(d-1)$.

For any finite collection of nonnegative real numbers $\{b_i+1\}_{i\in[n]}$, we have the inequality between their arithmetic and geometric means (the AM-GM inequality):

$$d-1=\frac{\sum_{i=1}(b_i+1)}{n}\geq\sqrt[n]{\prod_{i=1}^{n}(b_i+1)}=d-1, \tag{2.9}$$

where the first and last equalities are consequences of the two identities on the b_i's. Equality in the AM-GM inequality (2.9) is verified if and only if $b_1+1=\cdots=b_n+1$. This is the case, and so $b_i=d-2$ for all $1\leq i\leq n$. Therefore we find $H(g)=(x_1\cdots x_n)^{d-2}$ and $J(g)=\mathrm{Ann}(h)=(x_1^{d-1},\ldots,x_n^{d-1})=J(f)$. By Proposition 2.14, g is a Fermat polynomial. □

We do not know whether after removing the assumption on $H(g)$ from Proposition 2.15 the conclusion still holds true. Note that the homogeneous polynomials in Remark 2.4 are all singular.

Conjecture 2.16. Suppose a homogeneous polynomial $g\in S=\mathbb{C}[x_1,\ldots,x_n]$ of degree $d\geq 3$ in $n\geq 2$ variables is such that its Hessian $H(g)$ is a monomial. Then g is smooth if and only if $H(g)=(x_1\cdots x_n)^{d-2}$ and g is a Fermat polynomial.

Remark 2.17. Conjecture 2.16 is true also for $n=3$ and $d=4$. This follows from a result of Kuribayashi and Komiya [**15**] stating that any smooth quartic curve with 12 hyperflexes and with monomial Hessian is projectively equivalent to a Fermat quartic.

Definition 2.18. Let $A\in\mathrm{GL}(\mathbb{C}^n)$ and let I_n be the $n\times n$ identity matrix. The linear transformation A is a *unitary reflection* if it has finite order and $\mathrm{Ker}(A-I_n)$ is a codimension-one subspace, i.e. A has finite order and fixes pointwise a hyperplane in $\mathbb{C}^n$. A finite group generated by unitary reflections is called a *unitary group generated by reflections* (*u.g.g.r.*).

Remark 2.19. A complete classification of u.g.g.r. was found by Shephard and Todd [**20**]. These include as particular cases the finite Euclidean reflection groups (called *Coxeter groups*). An interesting subfamily of u.g.g.r. is the one of *Shephard groups* arising as a symmetry group of a *regular complex polytope*, defined and classified by Shephard [**19**]. The equivalence stated in [**16**, Theorem 5.10] characterises Shephard and Coxeter groups among all the u.g.g.r. groups. Item (v) in *loc. cit.* states that the Hessian of a minimal degree invariant form under a Shephard group G is a suitable product of powers of linear functionals (each functional corresponds to a hyperplane fixed by an element in G). This very last statement and its similarity with Question 1.1 was our motivation to look at these finite groups.

Example 2.20. The group $G(d,1,n)\cong\mathbb{Z}_d^n\rtimes\mathfrak{S}_n\subset(\mathbb{C}^*)^n\rtimes\mathfrak{S}_n$ is a Shephard group. As a matrix group, it is the group of permutation matrices whose nonzero entries are d-th roots of unity.

Example 2.21. Let $n,d\geq 2$ and $n|d$. As a matrix group, the subgroup $G(d,n,n)\subset G(d,1,n)$ consists of permutation matrices of the form $\mathrm{Diag}(\theta^{a_1},\ldots,\theta^{a_n})\circ\sigma$, where $\sigma\in\mathfrak{S}_n$ acts by permutation and Diag is a diagonal matrix with the indicated entries, θ is a primitive d-th root of unity and $\sum_{i=1}^n a_i\equiv 0 \mod n$. The group $G(d,n,n)$ is a u.g.g.r. but not a Shephard group. In particular, [**16**, Theorem 5.10](v) fails.

REMARK 2.22. The groups described above are contained in the stabilizers of homogeneous polynomials that are related to Question 1.1.

(i) Let $f = x_1^d + \cdots + x_n^d$. Then $\mathrm{Stab}_{\mathrm{GL}(\mathbb{C}^n)}(f) = G(d,1,n)$.
(ii) Let $n, d \geq 2$ and $n|d$. If $f = (x_1 \cdots x_n)^{d/n}$ then $G(d,n,n) \subset \mathrm{Stab}_{\mathrm{GL}(\mathbb{C}^n)}(f)$.

PROPOSITION 2.23. Let $n \geq 2$ and $d \geq 3$ with $n|d$. Let $f \in \mathbb{C}[x_1, \ldots, x_n]_d$ be such that its Hessian $H(f) = \alpha(x_1 \cdots x_n)^{d-2} \neq 0$. Then $f = \beta(x_1 \cdots x_n)^{d/n}$ or $f = \beta(x_1^d + \cdots + x_n^d)$ if and only if $G(d,n,n) \subset \mathrm{Stab}_{\mathrm{GL}(\mathbb{C}^n)}(f)$.

PROOF. One implication is Remark 2.22. Suppose $G(d,n,n) \subset \mathrm{Stab}_{\mathrm{GL}(\mathbb{C}^n)}(f)$, i.e. f is an invariant under $G(d,n,n)$. Then $f = \alpha_1(x_1^d + \cdots + x_n^d) + \alpha_2(x_1 \cdots x_n)^{d/n}$ as the invariant ring $\mathbb{C}[x_1, \ldots, x_n]^{G(d,n,n)}$ is the polynomial ring generated by the elementary symmetric functions of degrees $1, \ldots, n-1$ evaluated at $x_1^d, \ldots, x_n^d$ and by $(x_1 \cdots x_n)^{d/n}$ [**20**, Part II, §6]. (Indeed, by the Jacobian criterion these n invariant polynomials are algebraically independent and the product of their degrees is equal to $d^n \cdot (n-1)!$, the cardinality of $G(d,n,n)$; see [**20**, Theorem 5.1].) Suppose $\alpha_1, \alpha_2 \neq 0$. In the monomial expansion of $H(f)$ we look for monomials divisible by $(x_1 \cdots x_{n-2})^{d-2+2\frac{d}{n}}$. Equivalently, we search for monomials m in the expansion of $H(f)$ such that the x_i-degree of m, for $i \in \{1, \cdots, n-2\}$, is at least $d-2+2\frac{d}{n}$. One may write $\deg_{x_i}(m) = h_1(d-2) + h_2\left(\frac{d}{n}-2\right) + h_3\left(\frac{d}{n}-1\right) + h_4\frac{d}{n} \geq d-2+2\frac{d}{n}$, where h_1, h_2, h_3 gives a choice of a monomial in the i-th row of $\mathrm{Hess}(f)$, i.e. the row where the entries are of the form $\frac{\partial^2 f}{\partial x_i \partial x_j}$, and h_4 represents a choice of a monomial outside the i-th row. Thus $h_1 + h_2 + h_3 = 1$ and $h_1 + h_2 + h_3 + h_4 \leq n$. Suppose $h_1 = 0$. If $h_2 = 0$, then $h_3 = 1$ and $h_4 \leq n-1$. So $h_3\left(\frac{d}{n}-1\right) + h_4\frac{d}{n} \leq \frac{d}{n} - 1 + (n-1)\frac{d}{n} = d - 1 < d - 2 + 2\frac{d}{n}$ and hence this case is not possible. Similarly, if $h_3 = 0$, then $h_2 = 1$ and this case is not possible. Therefore $h_1 = 1$ and $h_2 = h_3 = 0$. Hence $h_1(d-2) + h_4\frac{d}{n} = d-2+h_4\frac{d}{n} \geq d-2+2\frac{d}{n}$, which implies $h_4 \geq 2$. Since this holds for every $i \in \{1, \ldots, n-2\}$, the monomials we are looking for can only be monomials in the expansion of the product between $\gamma(x_1 \cdots x_{n-2})^{d-2}$ ($\gamma \neq 0$ because $\alpha_1 \neq 0$) and the determinant of the lower-right 2×2 minor of $\mathrm{Hess}(f)$ given by

$$\begin{vmatrix} \alpha_1 d(d-1)x_{n-1}^{d-2} + \alpha_2 \frac{d}{n}\left(\frac{d}{n}-1\right)(x_1 \ldots x_{n-2})^{\frac{d}{n}} x_{n-1}^{\frac{d}{n}-2} x_n^{\frac{d}{n}} & \alpha_2\left(\frac{d}{n}\right)^2 (x_1 \ldots x_{n-2})^{\frac{d}{n}} x_{n-1}^{\frac{d}{n}-1} x_n^{\frac{d}{n}-1} \\ \alpha_2\left(\frac{d}{n}\right)^2 (x_1 \ldots x_{n-2})^{\frac{d}{n}} x_{n-1}^{\frac{d}{n}-1} x_n^{\frac{d}{n}-1} & \alpha_1 d(d-1)x_n^{d-2} + \alpha_2 \frac{d}{n}\left(\frac{d}{n}-1\right)(x_1 \ldots x_{n-2})^{\frac{d}{n}} x_{n-1}^{\frac{d}{n}} x_n^{\frac{d}{n}-2} \end{vmatrix}. \tag{2.10}$$

From (2.10), we see that the monomial $(x_1 \cdots x_{n-2})^{d-2+2\frac{d}{n}} x_{n-1}^{2\frac{d}{n}-2} x_n^{2\frac{d}{n}-2}$ appearing in $H(f)$ has coefficient

$$\gamma\alpha_2^2 \cdot \left[\left(\frac{d}{n}\right)^2 \left(\frac{d}{n}-1\right)^2 - \left(\frac{d}{n}\right)^4\right] = \gamma\alpha_2^2 \left(\frac{d}{n}\right)^2 \left(1 - 2\frac{d}{n}\right) \neq 0.$$

This is a contradiction and hence either $\alpha_1 = 0$ or $\alpha_2 = 0$. $\square$

CONJECTURE 2.24. Let $n \geq 2$ and $d \geq 3$. Let $f \in \mathbb{C}[x_1, \ldots, x_n]_d$ be such that its Hessian is $H(f) = \alpha(x_1 \cdots x_n)^{d-2} \neq 0$. Then, up to conjugation, $\mathrm{Stab}_{\mathrm{GL}(\mathbb{C}^n)}(f)$ contains a product of groups of the form $G(d,1,n-q) \times G(d,q,q)$ (with $G(d,1,0)$ and $G(d,0,0)$ being trivial groups by convention), and f is an invariant of minimal degree for such a group.

3. Local Koiran-Skomra's question

We formulate a local version of the Koiran-Skomra's Question 1.1: whether there exist homogeneous polynomial solutions g to the equality $H(g) = \alpha(x_1 \cdots x_n)^{d-2} \neq 0$, that are close to a Fermat polynomial $f = \alpha_1 x_1^d + \cdots + \alpha_n x_n^d$. To deal with this local version, we first compute the differential of the Hessian map.

Lemma 3.1 ([**2**, Lemma 7.2]). *Let $f, g \in \mathbb{C}[x_1, \ldots, x_n]_d$ be two homogeneous polynomials of degree d. The linear function $dH_f : \mathbb{C}[x_1, \ldots, x_n]_d \to \mathbb{C}[x_1, \ldots, x_n]_{n(d-2)}$, defined as $g \mapsto \frac{d}{dt} H(f + tg)_{|t=0}$, is the differential at f of the Hessian map $H : \mathbb{C}[x_1, \ldots, x_n]_d \to \mathbb{C}[x_1, \ldots, x_n]_{n(d-2)}$. The image $dH_f(g)$ is the sum of the determinants of n matrices H^i whose i-th row is $H^i_{ij} = \frac{\partial^2 g}{\partial x_i \partial x_j}$, and whose k-th row for $k \neq i$ is $H^i_{kj} = \frac{\partial^2 f}{\partial x_k \partial x_j}$.*

Proof. Viewing $p(t) = H(f + tg) = \det(\mathrm{Hess}(f + tg))$ as a polynomial in t, the image of g under dH_f is by definition the coefficient of t in $p(t)$. By definition of determinant, every contribution to this coefficient must involve a unique row of the matrix $\mathrm{Hess}(f + tg)$: the i-th such contribution is the determinant of H^i in the statement. □

Proposition 3.2. Let $f = \sum_{i=1}^n x_i^d \in \mathbb{C}[x_1, \ldots, x_n]_d$.

(i) If $d \geq n + 1$, then $\dim(\mathrm{Ker}(dH_f)) = n - 1$.

(ii) If $d \leq n$, then $\dim(\mathrm{Ker}(dH_f)) = n - 1 + \binom{n}{d}$.

Proof. (i). Let $g \in \mathbb{C}[x_1, \ldots, x_n]_d$. Then, using the formula in Lemma 3.1, we find $dH_f(g) = (d(d-1))^{n-1} \cdot \sum_{i=1}^n \prod_{j \neq i} x_j^{d-2} \frac{\partial^2 g}{\partial x_i^2}$. Denote $g_i = \prod_{j \neq i} x_j^{d-2} \frac{\partial^2 g}{\partial x_i^2}$. For two distinct indices $0 \leq h, k \leq n$, consider the monomial expansion of g_h and g_k. Note that $\deg(g_h) = \deg(g_k) = n(d-2)$. A monomial shared by g_h and g_k must be divisible by both $\prod_{j \neq h} x_j^{d-2}$ and $\prod_{j \neq k} x_j^{d-2}$. Thus such a monomial is divisible by $\prod_{i=1}^n x_i^{d-2}$. Since any monomial of the g_i has degree $n(d-2)$, the only shared monomial by any two of the g_i's is $\prod_{i=1}^n x_i^{d-2}$.

Suppose $g \in \mathrm{Ker}(dH_f)$. Then the condition $dH_f(g) = 0$ implies that the coefficients of all monomials appearing only in a single g_i vanish. Since $d \geq n+1$, all the coefficients of g appear in a unique g_i, except the coefficients β_i of the powers x_i^d appearing in the monomial expansion of g. The fact that the only shared monomial by all the g_i's is $\prod_{i=1}^n x_i^{d-2}$ implies the linear equation $\sum_{i=1}^n \beta_i = 0$. (Note that this is the tangent space of the algebraic torus in the fiber at f.) Thus $\dim(\mathrm{Ker}(dH_f)) = n - 1$.

(ii). Keep the notation from above. Since $d \leq n$, the coefficients of monomials in g annihilated by all the quadratic differentials, i.e. the square-free monomials $x_{i_1} \cdots x_{i_d}$ for some $1 \leq i_1, \ldots, i_d \leq n$, do not appear in any of the g_i's summands. There are $\binom{n}{d}$ of those. This shows the lower bound $\dim(\mathrm{Ker}(dH_f)) \geq n - 1 + \binom{n}{d}$. The equality comes from the direct verification that any other coefficient in g appears as coefficient of a unique monomial in a unique g_i and thus must vanish. □

Remark 3.3. Proposition 3.2 implies that, when $d \geq n + 1$, the local version of Koiran-Skomra's Question 1.1 has a positive answer.

For binary homogeneous polynomials, we offer a description of the differential at all monomials. We believe that the differential is important to understand better the Hessian map. For this part we were inspired by [**2**], where the authors provide ample evidence on how useful the differential is.

PROPOSITION 3.4. Let $d \geq 2$ and let $f = x_1^k x_2^{d-k} \in \mathbb{C}[x_1, x_2]_d$, with $0 < k \leq d$.

(i) If $k = d$, then $\dim(\mathrm{Ker}(dH_f)) = 2$.
(ii) If $k = d-1$, then $\dim(\mathrm{Ker}(dH_f)) = 1$, unless $d = 2$. In the last case, $\dim(\mathrm{Ker}(dH_f)) = 2$.
(iii) If $k \leq d-2$, then $\dim(\mathrm{Ker}(dH_f))$ is the number of indices $0 \leq j \leq d$ such that $d(k-j)^2 - d(k+j) + 2kj = 0$.
(iv) Let $d \geq 3$ and $k \leq d-1$. Then $\dim(\mathrm{Ker}(dH_f)) \leq 1$, unless $d = 2k$ and k is a square. In this case $\mathrm{Ker}(dH_f)$ is the span of $x_1^{k+\sqrt{k}} x_2^{k-\sqrt{k}}$ and $x_1^{k-\sqrt{k}} x_2^{k+\sqrt{k}}$.

PROOF. Statement (i) is a direct computation using the formula in Lemma 3.1. By Lemma 3.1, for $g \in \mathbb{C}[x_1, x_2]_d$, we have

$$dH_f(g) = \begin{vmatrix} \partial^2 g/\partial x_1^2 & \partial^2 g/\partial x_1 \partial x_2 \\ (d-k)kx_1^{k-1}x_2^{d-k-1} & (d-k)(d-k-1)x_1^k x_2^{d-k-2} \end{vmatrix} + \begin{vmatrix} k(k-1)x_1^{k-2}x_2^{d-k} & (d-k)kx_1^{k-1}x_2^{d-k-1} \\ \partial^2 g/\partial x_1 \partial x_2 & \partial^2 g/\partial x_2^2 \end{vmatrix}.$$

Let $g = \sum_{i=0}^d a_i x_1^i x_2^{d-i}$. Thus

$$dH_f(g) = (d-k)(d-k-1)\left[\sum_{i=0}^d a_i i(i-1)x_1^{k+i-2}x_2^{2d-k-i-2}\right] +$$
$$-2(d-k)k\left[\sum_{i=0}^d a_i(d-i)ix_1^{k+i-2}x_2^{2d-k-i-2}\right] +$$
$$+k(k-1)\left[\sum_{i=0}^d a_i(d-i)(d-i-1)x_1^{k+i-2}x_2^{2d-k-i-2}\right].$$

The coefficient of the monomial $a_j x_1^{k+j-2}x_2^{2d-k-j-2}$ is

$$g(d,k,j) = (d-k)(d-k-1)j(j-1) - 2(d-k)k(d-j)j + k(k-1)(d-j)(d-j-1) = \tag{3.1}$$
$$= (d-1)\left[d(k-j)^2 - d(k+j) + 2kj\right].$$

(ii). Suppose $k = d-1$. Then $g(d, d-1, j) = (d-1)(d-j)(d^2 - dj - 3d + 2)$. Thus we have $g(d, d-1, d) = 0$. For $j = d-s \leq d-1$, we find that $g(d, d-1, d-s) = (d-1)(d-j)(sd - 3d + 2)$. So $g(d, d-1, d-s) = 0$ if and only if $d = -\frac{2}{s-3}$, which is only possible if $s = 1$ or $s = 2$. If $s = 1$, then $d = 1$, which is excluded from the assumptions as it is a trivial case. For $s = 2$, we have $d = 2$ and $j = 0$. In this case, in the expansion of $dH_f(g)$ the only vanishing coefficients are those of a_0 and a_2 and whence $\dim(\mathrm{Ker}(dH_f)) = 2$. Otherwise, when $k = d-1$, in the expansion of $dH_f(g)$ the only vanishing coefficient is the one of a_d and so $\dim(\mathrm{Ker}(dH_f)) = 1$.
(iii). Using the approach in (ii) above, when $k \leq d-2$ the only vanishing coefficients of the monomials $a_j x_1^{k+j-2}x_2^{2d-k-j-2}$ are for those indices j satisfying $g(d,k,j) = 0$. Then it is immediate to see that $\dim(\mathrm{Ker}(dH_f))$ is the number of indices j such that $g(d,k,j) = 0$. (Clearly, the equation is quadratic in j and hence $\dim(\mathrm{Ker}(dH_f)) \leq 2$.)
(iv). By part (iii), $\dim \mathrm{Ker}(H_f) > 1$ if and only if the following polynomial in j

$$j^2 + j\left(-2k - 1 + \frac{2p}{d}\right) + k^2 - k = 0 \tag{3.2}$$

has two integral roots $0 \leq j_1 < j_2 \leq d$. Assume the existence of such j_1, j_2. Thus

$$j_1 j_2 = k(k-1) \text{ and } j_1 + j_2 = 2k + 1 - \frac{2k}{d}. \tag{3.3}$$

Since $0 < k < d$, (3.3) gives $d = 2k$ and $j_1 + j_2 = 2k$. So $j_1 = k - \sqrt{k}$ and $j_2 = k + \sqrt{k}$. Assuming the existence of these integral solutions, $d = 2k$ where k is a square. □

In some cases, one finds a large subset of binary homogeneous polynomials whose differential is injective:

Corollary 3.5. Let $d \geq 3$ be a prime and $2 \leq k \leq d-2$. Let $W_k \subset \mathbb{C}[x_1, x_2]_d$ be the set of homogenous polynomials of the form $\sum_{i \geq k}^{d} a_i x_1^i x_2^{d-i}$ such that $a_k \neq 0$. Then, for any $f \in W_k$, $\dim(\operatorname{Ker}(dH_f)) = 0$.

Proof. For $\lambda \in \mathbb{C}^*$ and $f \in W_k$, define $f_\lambda(x_1, x_2) = f(\lambda x_1, x_2)/\lambda^k$. One has $\dim(\operatorname{Ker}(dH(f_\lambda))) = \dim(\operatorname{Ker}(dH_f))$. Note that $g = \lim_{\lambda \to 0} f_\lambda(x_1, x_2) = a_k x_1^k x_2^{d-k} \neq 0$. By semicontinuity, we have $\dim(\operatorname{Ker}(dH_f)) \leq \dim(\operatorname{Ker}(dH(g)))$. By Proposition 3.4(iii), the latter is the number of indices $0 \leq j \leq d$ satisfying the equation $d(k-j)^2 - d(k+j) + 2kj = 0$. If $j = 0$, then $k = 0$ or $k = 1$, which are excluded. For $j \neq 0$, since d is a prime dividing the nonzero integer $2kj$ and $k \leq d-2$, we must have $j = d$. For $j = d$, one finds $d(k-j)^2 - d(k+j) + 2kj = d(k-d)(k-d+1)$. So, given $2 \leq k \leq d-2$, there is no j satisfying the equation above. Thus $\dim(\operatorname{Ker}(dH_f)) = 0$ for all $f \in W_k$. □

In Theorem 2.9, the exceptional cases appear when $d = 2m^2$. These are also somewhat exceptional instances from the perspective of the differential of the Hessian map, as witnessed by Proposition 3.4(iv) and, as a consequence, by the next result.

Theorem 3.6. *Let $d \geq 3$ and let $f \in \mathbb{C}[x_1, x_2]_d$ with $H(f) \neq 0$. Then* $\dim H^{-1}(H(f)) \leq 1$ *and* $\dim(\operatorname{Ker}(dH_f)) \leq 1$, *unless $d = 2m^2$ and $f = \ell_1^{m^2} \ell_2^{m^2}$ (where ℓ_i are linearly independent linear forms). In the latter case,* $\dim(\operatorname{Ker}(dH_f)) = 2$.

Proof. First, suppose $d \neq 2m^2$. It is sufficient to prove that $\dim \operatorname{Ker}(dH_f) \leq 1$. (This is because if the dimension of the fiber at $H(f)$ satisfies $\dim H^{-1}(H(f)) \geq q$ then $\dim(\operatorname{Ker}(dH_f)) \geq q$, for any $q \in \mathbb{N}$.) For any $k \geq 1$, let W_k be the set of all $g \in \mathbb{C}[x_1, x_2]_d$ such that $g = \sum_{i=k}^{d} a_i x_1^i x_2^{d-i}$ and $a_k \neq 0$. Up to the action of $\mathrm{GL}(\mathbb{C}^2)$, we may assume $f(0,1) = 0$, i.e. we may assume $f = \sum_{i \geq 1}^{d} a_i x_1^i x_2^{d-i}$. Let $i_{\min}$ be the smallest positive integer such that $a_{i_{\min}} \neq 0$ and hence $f \in W_{i_{\min}}$. We have $i_{\min} < d$, because $H(f) \neq 0$ and so $1 \leq i_{\min} \leq d-1$. As in the proof of Corollary 3.5, for any $\lambda \in \mathbb{C}^*$, define $g_\lambda(x_1, x_2) = f(\lambda x_1, x_2)/\lambda^{i_{\min}}$. Note that $g = a_{i_{\min}} x_1^{i_{\min}} x_2^{d-i_{\min}} = \lim_{\lambda \to 0} g_\lambda$. One then has $\dim(\operatorname{Ker}(dH_{g_\lambda})) = \dim(\operatorname{Ker}(dH_f))$ for all $\lambda \in \mathbb{C}^*$. By semicontinuity, we have $\dim(\operatorname{Ker}(dH_f)) \leq \dim(\operatorname{Ker}(dH_g)) \leq 1$, where the last inequality is Proposition 3.4(iv).
Now, assume $d = 2m^2$. Since $H(f) \neq 0$ and $d \geq 3$, we may assume that f has at least two distinct zeros. Up to the action of $\mathrm{GL}(\mathbb{C}^2)$, we may assume $f(0,1) = f(1,0) = 0$, i.e. we may assume $a_0 = a_d = 0$. The first part of this proof works if $i_{\min} \neq m^2$. Thus, we may assume $i_{\min} = m^2$. Let $j_{\min}$ be the minimal positive integer such that $a_{d-j_{\min}} \neq 0$ by Proposition 3.4(iv). Since $a_d = 0$, $j_{\min} > 0$.

Define $h_\lambda(x_1, x_2) = f(x_1, \lambda x_2)/\lambda^{j_{\min}}$. Since $h = a_{d-j_{\min}} x_1^{d-j_{\min}} x_2^{j_{\min}} = \lim_{\lambda\to 0} h_\lambda$, by semicontinuity we obtain $\dim(\mathrm{Ker}(dH_f)) \leq \dim(\mathrm{Ker}(dH_h))$. If $j_{\min} < m^2$, then Proposition 3.4(iv) gives $\dim(\mathrm{Ker}(dH_f)) \leq 1$. Otherwise, $i_{\min} = j_{\min} = m^2$ and so $f = x_1^{m^2} x_2^{m^2}$. In this case, Proposition 3.4(iv) shows that $\dim(\mathrm{Ker}(H_f)) = 2$. □

REMARK 3.7. Theorem 3.6 also shows that the Hessian map at the monomial counterexamples is not necessarily a local embedding.

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