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On extropy of past lifetime distribution

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Abstract

Recently Qiu et al. (2017) have introduced residual extropy as measure of uncertainty in residual lifetime distributions analogues to residual entropy (1996). Also, they obtained some properties and applications of that. In this paper, we study the extropy to measure the uncertainty in a past lifetime distribution. This measure of uncertainty is called past extropy. Also it is showed a characterization result about the past extropy of largest order statistics.

Keywords: Reversed residual lifetime, Past extropy, Characterization, Order statistics.

AMS Subject Classification: 94A17, 62B10, 62G30

1 Introduction

The concept of Shannon entropy as a seminal measure of uncertainty for a random variable was proposed by Shannon (1948). Shannon entropy $H(f)$ for a non-negative and absolutely continuous random variable X is defined as follows:

$$H(f) = -\mathbb{E}[\log f(x)] = -\int_0^{+\infty} f(x) \log f(x) dx, \quad (1)$$

where F and f are cumulative distribution function (CDF) and probability density function (pdf), respectively. There are huge literatures devoted to the applications, generalizations and properties of Shannon entropy (see, e.g. Cover and Thomas, 2006).

Recently, a new measure of uncertainty was proposed by Lad et al. (2015) called extyopy as a complement dual of Shannon entropy (1948). For a non-negative random variable X the extropy is defined as below:

$$J(X) = -\frac{1}{2} \int_0^{+\infty} f^2(x) dx. \quad (2)$$

It's obvious that $J(X) \leq 0$.

One of the statistical applications of extropy is to score the forecasting distributions using the total log scoring rule.

Study of duration is an interest subject in many fields of science such as reliability, survival analysis, and forensic science. In these areas, the additional life time given that the component or system or a living organism has survived up to time t is termed the residual life function of the component. If X is the life of a component, then $X_t = (X - t | X > t)$ is called the residual life function. If a component is known to have survived to age t then extropy is no longer useful to measure the uncertainty of remaining lifetime of the component.

Therefore, Ebrahimi (1996) defined the entropy for residual lifetime $X_t = (X - t | X > t)$ as a dynamic form of uncertainty called the residual entropy at time t and defined as

$$H(X; t) = - \int_t^{+\infty} \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx,$$

where $\bar{F}(t) = \mathbb{P}(X > t) = 1 - F(t)$ is the survival (reliability) function of X .

Analogous to residual entropy, Qiu et al. (2017) defined the extropy for residual lifetime X_t called the residual extropy at time t and defined as

$$J(X_t) = -\frac{1}{2} \int_0^{+\infty} f_{X_t}^2(x) dx = -\frac{1}{2\bar{F}^2(t)} \int_t^{+\infty} f^2(x) dx. \quad (3)$$

In many situations, uncertainty can relate to the past. Suppose the random variable X is the lifetime of a component, system or a living organism, having an absolutely continuous distribution function $F_X(t)$ and the density function $f_X(t)$. For $t > 0$, let the random variable ${}_tX = (t - X | X < t)$ be the time elapsed after failure till time t , given that the component has already failed at time t . We denote the random variable ${}_tX$, the reversed residual life (past lifetime). For instance, at time t , one has under gone a medical test to check for a certain disease. Suppose that the test result is positive. If X is the age when the patient was infected, then it is known that $X < t$. Now the question is, how much time has elapsed since the patient has been infected by this disease? Based on this idea, Di Crescenzo and Longobardi (2002) introduced the entropy of the reversed residual lifetime ${}_tX$ as a dynamic measure of uncertainty called past entropy as follows:

$$H(X; [t]) = - \int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} dx.$$

This measure is dual of residual entropy introduced by Ebrahimi (1996).

In this paper, we study the extropy for ${}_tX$ as dual of residual extropy that is called past extropy and it is defined as below (see also Krishnan et al. (2020)):

$$J({}_tX) = -\frac{1}{2} \int_0^{+\infty} f_{tX}^2(x) dx = -\frac{1}{2F^2(t)} \int_0^t f^2(x) dx, \quad (4)$$

where $f_{tX}(x) = \frac{f(t-x)}{F(t)}$, for $x \in (0, t)$. It can be seen that for $t \geq 0$, $J({}_tX)$ possesses all the properties of $J(X)$.

Remark 1. It's clear that $J({}_{+\infty}X) = J(X)$.

Past extropy has applications in the context of information theory, reliability and survival analysis, insurance, forensic science and other related fields because in that a lifetime distribution truncated above is of utmost importance.

The paper is organized as follows: in section 2, an approach to measure of uncertainty in the past lifetime distribution is proposed. Then it is studied a characterization result with the reversed failure rate. Following a characterization result is given based on past extropy of the largest order statistics in section 3.

2 Past extropy and some characterizations

Analogous to residual extropy (Qiu et al. (2017)), the extropy for ${}_tX$ is called past extropy and for a non-negative random variable X is as below:

$$J({}_tX) = -\frac{1}{2} \int_0^{+\infty} f_{tX}^2(x) dx = -\frac{1}{2F^2(t)} \int_0^t f^2(x) dx, \quad (5)$$

where $f_{tX}(x) = \frac{f(t-x)}{F(t)}$, for $x \in (0, t)$ is the density function of ${}_tX$. It's clear that $J({}_tX) \leq 0$ while the residual entropy of a continuous distribution may take any value on $[-\infty, +\infty]$. Also, $J({}_{+\infty}X) = J(X)$.

Example 1. a) If $X \sim Exp(\lambda)$, then $J({}_tX) = -\frac{\lambda}{4} \frac{1+e^{-\lambda t}}{1-e^{-\lambda t}}$ for $t > 0$. This shows that the past extropy of exponential distribution is an increasing function of t .

b) If $X \sim U(0, b)$, then $J({}_tX) = -\frac{1}{2t}$.

c) If X has power distribution with parameter $\alpha > 0$, i.e. $f(x) = \alpha x^{\alpha-1}$, $0 < x < 1$, then $J({}_tX) = \frac{-\alpha^2}{2(2\alpha-1)t}$.

d) If X has Pareto distribution with parameters $\theta > 0, x_0 > 0$, i.e. $f(x) = \frac{\theta}{x_0} \frac{x_0^{\theta+1}}{x^{\theta+1}}$, $x > x_0$, then $J({}_tX) = \frac{\theta^2}{2(2\theta+1)(t^\theta - x_0^\theta)^2} \left[\frac{x_0^{2\theta}}{t} - \frac{t^{2\theta}}{x_0} \right]$.

There is a functional relation between past extropy and residual extropy as follows:

$$J(X) = F^2(t)J({}_tX) + \bar{F}^2(t)J(X_t), \forall t > 0.$$

In fact

$$\begin{aligned} F^2(t)J({}_tX) + \bar{F}^2(t)J(X_t) &= -\frac{1}{2} \int_t^{+\infty} f^2(x)dx - \frac{1}{2} \int_0^t f^2(x)dx \\ &= -\frac{1}{2} \int_0^{+\infty} f^2(x)dx = J(X). \end{aligned}$$

From (4) we can rewrite the following expression for the past extropy:

$$J({}_tX) = \frac{-\tau^2(t)}{2f^2(t)} \int_0^t f^2(x)dx,$$

where $\tau(t) = \frac{f(t)}{F(t)}$ is the reversed failure rate.

Definition 1. A random variable is said to be increasing (decreasing) in past extropy if $J({}_tX)$ is an increasing (decreasing) function of t .

Theorem 2.1. $J({}_tX)$ is increasing (decreasing) if and only if $J({}_tX) \leq (\geq) \frac{-1}{4}\tau(t)$.

Proof. From (5) we get

$$\frac{d}{dt}J({}_tX) = -2\tau(t)J({}_tX) - \frac{1}{2}\tau^2(t).$$

Then $J({}_tX)$ is increasing if and only if

$$2\tau(t)J({}_tX) + \frac{1}{2}\tau^2(t) \leq 0,$$

but $\tau(t) \geq 0$ so this is equivalent to

$$J({}_tX) \leq -\frac{1}{4}\tau(t).$$

□

Theorem 2.2. The past extropy $J({}_tX)$ of X is uniquely determined by $\tau(t)$.

Proof. From (5) we get

$$\frac{d}{dt}J({}_tX) = -2\tau(t)J({}_tX) - \frac{1}{2}\tau^2(t).$$

So we have a linear differential equation of order one and it can be solved in the following way

$$J({}_tX) = e^{-2 \int_{t_0}^t \tau(s)ds} \left[J({}_{t_0}X) - \int_{t_0}^t \frac{1}{2} \tau^2(s) e^{2 \int_{t_0}^s \tau(y)dy} ds \right],$$

where we can use the boundary condition $J({}_{+\infty}X) = J(X)$, so we get

$$J({}_tX) = e^{2 \int_t^{+\infty} \tau(s)ds} \left[J(X) + \int_t^{+\infty} \frac{1}{2} \tau^2(s) e^{-2 \int_s^{+\infty} \tau(y)dy} ds \right]. \quad (6)$$

□

Example 2. Let $X \sim Exp(\lambda)$, with reversed failure rate $\tau(t) = \frac{\lambda e^{-\lambda t}}{1 - e^{-\lambda t}}$. It follows from (6) that

$$\begin{aligned} J({}_tX) &= e^{2 \int_t^{+\infty} \frac{\lambda e^{-\lambda s}}{1 - e^{-\lambda s}} ds} \left[J(X) + \int_t^{+\infty} \frac{1}{2} \frac{\lambda^2 e^{-2\lambda s}}{(1 - e^{-\lambda s})^2} e^{-2 \int_s^{+\infty} \frac{\lambda e^{-\lambda y}}{1 - e^{-\lambda y}} dy} ds \right] \\ &= (1 - e^{-\lambda t})^{-2} \left[J(X) + \frac{1}{2} \int_t^{+\infty} \lambda^2 e^{-2\lambda s} ds \right] \\ &= \frac{\lambda}{4} \frac{e^{-2\lambda t} - 1}{(1 - e^{-\lambda t})^2} = -\frac{\lambda}{4} \frac{1 + e^{-\lambda t}}{1 - e^{-\lambda t}}, \end{aligned}$$

so we find again the same result of example 1.

Using the following definition (see Shaked and Shanthikumar, 2007), we show that $J({}_tX)$ is increasing in t .

Definition 2. Let X and Y be two non-negative variables with reliability functions \bar{F}, \bar{G} and pdfs f, g respectively. X is smaller than Y

- a) in the likelihood ratio order, denoted by $X \leq_{lr} Y$, if $\frac{f(x)}{g(x)}$ is decreasing in $x \geq 0$;
- b) in the usual stochastic order, denoted by $X \leq_{st} Y$ if $\bar{F}(x) \leq \bar{G}(x)$ for $x \geq 0$.

Remark 2. It is well known that if $X \leq_{lr} Y$ then $X \leq_{st} Y$ and $X \leq_{st} Y$ if and only if $\mathbb{E}(\varphi(Y)) \leq (\geq) \mathbb{E}(\varphi(X))$ for any decreasing (increasing) function φ .

Theorem 2.3. Let X be a random variable with CDF F and pdf f . If $f(F^{-1}(x))$ is decreasing in $x \geq 0$, then $J({}_tX)$ is increasing in $t \geq 0$.

Proof. Let U_t be a random variable with uniform distribution on $(0, F(t))$ with pdf $g_t(x) = \frac{1}{F(t)}$ for $x \in (0, F(t))$, then based on (4) we have

$$\begin{aligned} J({}_tX) &= -\frac{1}{2F^2(t)} \int_0^{F(t)} f(F^{-1}(u)) du = -\frac{1}{2F(t)} \int_0^{F(t)} g_t(u) f(F^{-1}(u)) du \\ &= -\frac{1}{2F(t)} \mathbb{E}[f(F^{-1}(U_t))]. \end{aligned}$$

Let $0 \leq t_1 \leq t_2$. If $0 < x \leq F(t_1)$, then $\frac{g_{t_1}(x)}{g_{t_2}(x)} = \frac{F(t_2)}{F(t_1)}$ is a non-negative constant. If $F(t_1) < x \leq F(t_2)$, then $\frac{g_{t_1}(x)}{g_{t_2}(x)} = 0$. Therefore $\frac{g_{t_1}(x)}{g_{t_2}(x)}$ is decreasing in $x \in (0, F(t_2))$, which implies $U_{t_1} \leq_{lr} U_{t_2}$. Hence $U_{t_1} \leq_{st} U_{t_2}$ and so

$$0 \leq \mathbb{E}[f(F^{-1}(U_{t_2}))] \leq \mathbb{E}[f(F^{-1}(U_{t_1}))]$$

using the assumption that $f(F^{-1}(U_t))$ is a decreasing function. Since $0 \leq \frac{1}{F(t_2)} \leq \frac{1}{F(t_1)}$ then

$$J({}_{t_1}X) = -\frac{1}{2F(t_1)} \mathbb{E}[f(F^{-1}(U_{t_1}))] \leq -\frac{1}{2F(t_2)} \mathbb{E}[f(F^{-1}(U_{t_2}))] = J({}_{t_2}X).$$

□

Remark 3. Let X be a random variable with CDF $F(x) = x^2$, for $x \in (0, 1)$. Then $f(F^{-1}(x)) = 2\sqrt{x}$ is increasing in $x \in (0, 1)$. However $J(tX) = -\frac{2}{3t}$ is increasing in $t \in (0, 1)$. So the condition in theorem 2.3 that $f(F^{-1}(x))$ is decreasing in x is sufficient but not necessary.

3 Past extropy of order statistics

Let X_1, X_2, \dots, X_n be a random sample with distribution function F , the order statistics of the sample are defined by the arrangement $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ from the minimum to the maximum by $X_{(1)}, X_{(2)}, \dots, X_{(n)}$. Qiu and Jia (2017) defined the residual extropy of the i -th order statistics and showed that the residual extropy of order statistics can determine the underlying distribution uniquely. Let X_1, X_2, \dots, X_n be continuous and i.i.d. random variables with CDF F indicate the lifetimes of n components of a parallel system. Also $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ be the ordered lifetimes of the components. Then $X_{n:n}$ represents the lifetime of parallel system with CDF $F_{X_{n:n}}(x) = (F(x))^n$, $x > 0$. The CDF of $[t - X_{n:n} | X_{n:n} < t]$ is $1 - \left(\frac{F(t-x)}{F(t)}\right)^n$ where $[t - X_{n:n} | X_{n:n} < t]$ is called reversed residual lifetime of the system. Now past extropy for reversed residual lifetime of parallel system with distribution function $F_{X_{n:n}}(x)$ is as follows:

$$J(tX_{n:n}) = -\frac{n^2}{2(F(t))^{2n}} \int_0^t f^2(x)[F(x)]^{2n-2} dx.$$

Theorem 3.1. *If X has an increasing pdf f on $[0, T]$, with $T > t$, then $J(tX_{n:n})$ is decreasing in $n \geq 1$.*

Proof. The pdf of $(X_{n:n} | X_{n:n} \leq t)$ can be expressed as

$$g_{n:n}^t(x) = \frac{nf(x)F^{n-1}(x)}{F^n(t)}, \quad x \leq t.$$

We note that

$$\frac{g_{2n-1:2n-1}^t(x)}{g_{2n+1:2n+1}^t(x)} = \frac{2n-1}{2n+1} \frac{F^2(t)}{F^2(x)}$$

is decreasing in $x \in [0, t]$ and so $(X_{2n-1:2n-1} | X_{2n-1:2n-1} \leq t) \leq_{lr} (X_{2n+1:2n+1} | X_{2n+1:2n+1} \leq t)$ which implies $(X_{2n-1:2n-1} | X_{2n-1:2n-1} \leq t) \leq_{st} (X_{2n+1:2n+1} | X_{2n+1:2n+1} \leq t)$. If f is increasing on $[0, T]$ we have

$$\mathbb{E}[f(X_{2n-1:2n-1}) | X_{2n-1:2n-1} \leq t] \leq \mathbb{E}[f(X_{2n+1:2n+1}) | X_{2n+1:2n+1} \leq t].$$

From the definition of the past extropy it follows that

$$\begin{aligned} J(tX_{n:n}) &= -\frac{n^2}{2F^{2n}(t)} \int_0^t f^2(x)F^{2n-2}(x)dx \\ &= \frac{-n^2}{2(2n-1)F(t)} \int_0^t \frac{(2n-1)F^{2n-2}(x)f(x)}{F^{2n-1}(t)} f(x)dx \\ &= \frac{-n^2}{2(2n-1)F(t)} \mathbb{E}[f(X_{2n-1:2n-1}) | X_{2n-1:2n-1} \leq t]. \end{aligned}$$

Then it follows that

$$\begin{aligned} \frac{J(tX_{n:n})}{J(tX_{n+1:n+1})} &= \frac{n^2}{(n+1)^2} \frac{2n-1}{2n+1} \frac{\mathbb{E}[f(X_{2n-1:2n-1}) | X_{2n-1:2n-1} \leq t]}{\mathbb{E}[f(X_{2n+1:2n+1}) | X_{2n+1:2n+1} \leq t]} \\ &\leq \frac{\mathbb{E}[f(X_{2n-1:2n-1}) | X_{2n-1:2n-1} \leq t]}{\mathbb{E}[f(X_{2n+1:2n+1}) | X_{2n+1:2n+1} \leq t]} \leq 1. \end{aligned}$$

Since the past extropy of a random variable is non-negative we have $J(tX_{n:n}) \geq J(tX_{n+1:n+1})$ and the proof is completed. \square

Example 3. Let X be a random variable distributed as a Weibull with two parameters, $X \sim W2(\alpha, \lambda)$, i.e. $f(x) = \lambda \alpha x^{\alpha-1} \exp(-\lambda x^\alpha)$. It can be showed that for $\alpha > 1$ this pdf has a maximum point $T = (\frac{\alpha-1}{\lambda \alpha})^{\frac{1}{\alpha}}$. Let us consider the case in which X has a Weibull distribution with parameters $\alpha = 2$ and $\lambda = 1$, $X \sim W2(2, 1)$ and so $T = \frac{\sqrt{2}}{2}$. The hypothesis of the theorem 3.1 are satisfied for $t = 0.5 < T = \frac{\sqrt{2}}{2}$. Figure 1 shows that $J(0.5X_{n:n})$ is decreasing in $n \in \{1, 2, \dots, 10\}$. Moreover the result of the theorem 3.1 does not hold for the smallest order statistic as shown in figure 2.

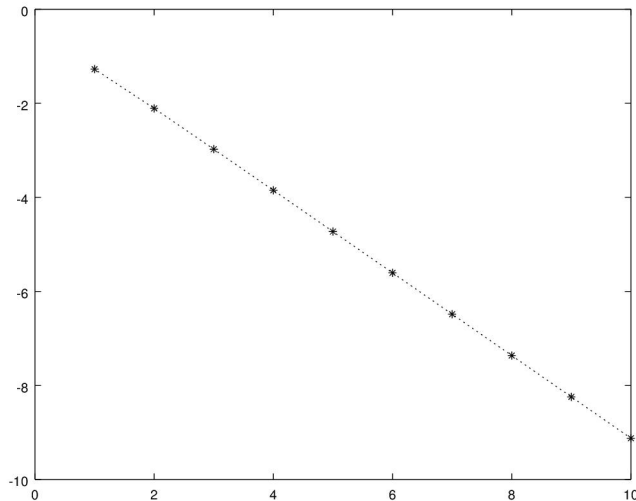


Figure 1: $J(0.5X_{n:n})$ of a $W2(2, 1)$ for $n = 1, 2, \dots, 10$

In the case in which X has an increasing pdf on $[0, T]$ with $T > t$ we give a lower bound for $J(tX)$.

Theorem 3.2. *If X has an increasing pdf f on $[0, T]$, with $T > t$, then $J(tX) \geq -\frac{\tau(t)}{2}$.*

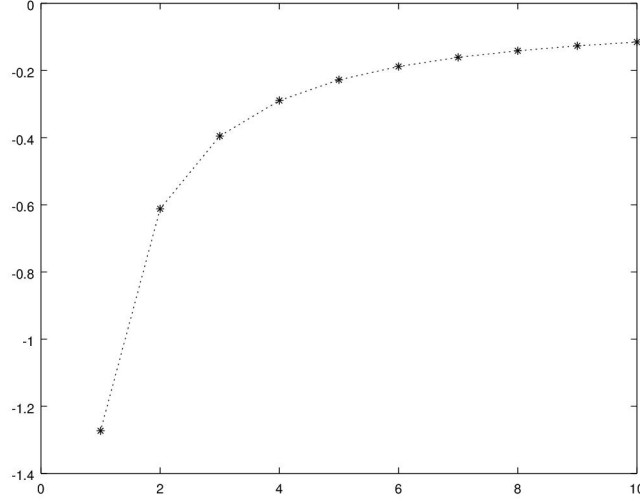


Figure 2: $J(0.5X_{1:n})$ of a $W2(2, 1)$ for $n = 1, 2, \dots, 10$

Proof. From the definition we get

$$\begin{aligned}
 J({}_tX) &= -\frac{1}{2F^2(t)} \int_0^t f^2(x)dx \\
 &= \frac{-f(t)}{2F(t)} + \frac{1}{2F^2(t)} \int_0^t F(x)f'(x)dx \\
 &\geq -\frac{\tau(t)}{2}.
 \end{aligned}$$

□

Example 4. Let $X \sim W2(2, 1)$, as in example 3, so we know that its pdf is increasing in $[0, T]$ with $T = \frac{\sqrt{2}}{2}$. The hypothesis of the theorem 3.2 are satisfied for $t < T = \frac{\sqrt{2}}{2}$. Figure 3 shows that the function $-\frac{\tau(t)}{2}$ (in red) is a lower bound for the past extropy (in black). We remark that the theorem gives information only for $t \in [0, T]$, in fact for larger values of t the function $-\frac{\tau(t)}{2}$ could not be a lower bound anymore, as showed in figure 3.

Qiu (2016), Qiu and Jia (2017) showed that extropy of the i -th order statistics and residual extropy of the i -th order statistics can characterize the underlying distribution uniquely. In the following theorem, whose proof requires next lemma, we show that the past extropy of the largest order statistic can uniquely characterize the underlying distribution.

Lemma 3.1. *Let X and Y be non-negative random variables such that $J(X_{n:n}) = J(Y_{n:n})$, $\forall n \geq 1$. Then $X \stackrel{d}{=} Y$.*

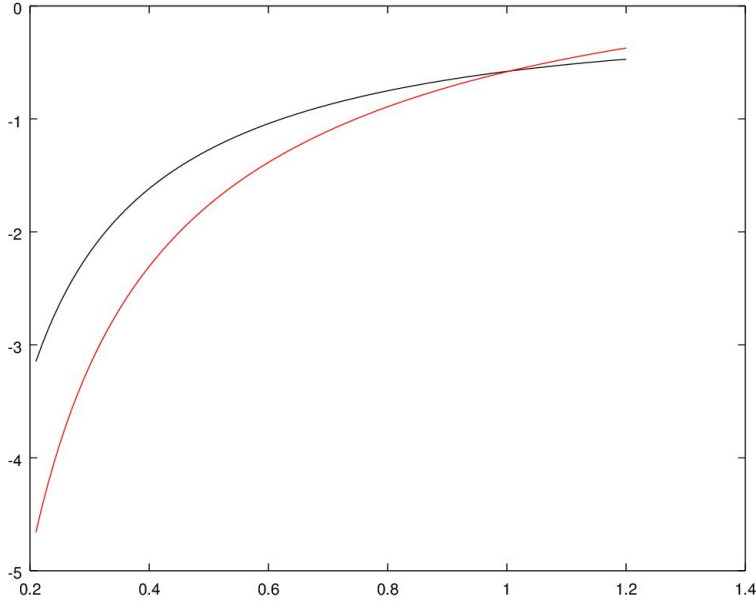


Figure 3: $J(tX)$ (in black) and $-\frac{\tau(t)}{2}$ (in red) of a $W2(2,1)$

Proof. From the definition of the extropy, $J(X_{n:n}) = J(Y_{n:n})$ holds if and only if

$$\int_0^{+\infty} F_X^{2n-2}(x) f_X^2(x) dx = \int_0^{+\infty} F_Y^{2n-2}(x) f_Y^2(x) dx$$

i.e. if and only if

$$\int_0^{+\infty} F_X^{2n-2}(x) \tau_X(x) dF_X^2(x) = \int_0^{+\infty} F_Y^{2n-2}(x) \tau_Y(x) dF_Y^2(x).$$

Putting $u = F_X^2(x)$ in the left side of the above equation and $u = F_Y^2(x)$ in the right side we have

$$\int_0^1 u^{n-1} \tau_X(F_X^{-1}(\sqrt{u})) du = \int_0^1 u^{n-1} \tau_Y(F_Y^{-1}(\sqrt{u})) du.$$

that is equivalent to

$$\int_0^1 u^{n-1} [\tau_X(F_X^{-1}(\sqrt{u})) - \tau_Y(F_Y^{-1}(\sqrt{u}))] du = 0 \quad \forall n \geq 1.$$

Then from Lemma 3.1 of Qui (2017) we get $\tau_X(F_X^{-1}(\sqrt{u})) = \tau_Y(F_Y^{-1}(\sqrt{u}))$ for all $u \in (0, 1)$. By taking $\sqrt{u} = v$ we have $\tau_X(F_X^{-1}(v)) = \tau_Y(F_Y^{-1}(v))$ and so $f_X(F_X^{-1}(v)) = f_Y(F_Y^{-1}(v))$ for all $v \in (0, 1)$. This is equivalent to $(F_X^{-1})'(v) = (F_Y^{-1})'(v)$ i.e. $F_X^{-1}(v) = F_Y^{-1}(v) + C$, for all $v \in (0, 1)$ with C constant. But for $v = 0$ we have $F_X^{-1}(0) = F_Y^{-1}(0) = 0$ and so $C = 0$. \square

Theorem 3.3. *Let X and Y be two non-negative random variables with cumulative distribution functions $F(x)$ and $G(x)$, respectively. Then F and G belong to the same family of distributions if and only if for $t \geq 0$, $n \geq 1$,*

$$J({}_tX_{n:n}) = J({}_tY_{n:n}).$$

Proof. It suffices to prove the sufficiency. $J({}_tX_{n:n})$ is the past extropy for $X_{n:n}$ but it is also the extropy for the variable ${}_tX_{n:n}$. So through lemma 3.1 we get ${}_tX \stackrel{d}{=} {}_tY$. Then $\frac{F(t-x)}{F(t)} = \frac{G(t-x)}{G(t)}$, for $x \in (0, t)$. If exists t' such that $F(t') \neq G(t')$ then in $(0, t')$ $F(x) = \alpha G(x)$ with $\alpha \neq 1$. But for all $t > t'$, exists $x \in (0, t)$ such that $t - x = t'$ and so $F(t) \neq G(t)$ and as in the precedent step we have $F(x) = \alpha G(x)$ for $x \in (0, t)$. Letting t to $+\infty$ we have a contradiction because F and G are both distribution function and their limit is 1. \square

4 Conclusion

In this paper we studied a measure of uncertainty, the past extropy. It is the extropy of the inactivity time. It is important in the moment in which with an observation we find our system down and we want to investigate about how much time has elapsed after its fail. Moreover we studied some connections with the largest order statistic.

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References

- [1] Cover, T. M. and Thomas, J. A., 2006. Elements of Information Theory, (2nd edn). Wiley, New York.
- [2] Di Crescenzo, A., Longobardi, M., 2002. Entropy-based measure of uncertainty in past lifetime distributions, Journal of Applied Probability, 39, 434–440.
- [3] Ebrahimi, N., 1996. How to measure uncertainty in the residual life time distribution. Sankhya: The Indian Journal of Statistics, Series A, 58, 48–56.
- [4] Krishnan, A. S., Sunoj S. M., Nair N. U., 2020. Some reliability properties of extropy for residual and past lifetime random variables. Journal of the Korean Statistical Society. <https://doi.org/10.1007/s42952-019-00023-x>.

- [5] Lad, F., Sanfilippo, G., Agrò, G., 2015. Extropy: complementary dual of entropy. *Statistical Science* 30, 40–58.
- [6] Qiu, G., 2017. The extropy of order statistics and record values. *Statistics & Probability Letters*, 120, 52–60.
- [7] Qiu, G., Jia, K., 2018. The residual extropy of order statistics. *Statistics & Probability Letters*, 133, 15–22.
- [8] Shaked, M., Shanthikumar, J. G., 2007. *Stochastic orders*. Springer Science & Business Media.
- [9] Shannon, C. E., 1948. A mathematical theory of communication. *Bell System Technical Journal*, 27, 379–423.