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On Weighted Extropies

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Abstract

The extropy is a measure of information introduced by Lad et al. (2015) as dual to entropy. It is a shift-independent information measure just as the entropy. We introduce here the notion of weighted extropy, a shift-dependent information measure which gives higher weights to larger values of random variables. We also study the weighted residual and past extropies as weighted versions of extropy for residual and past lifetimes. Bivariate versions extropy and weighted extropy are also described. Several examples are presented through out to illustrate all the concepts introduced here.

1 Introduction

1.1 Hazard and reversed hazard rates

Let X be a non-negative absolutely continuous random variable with probability density function (pdf) f , cumulative distribution function (cdf) F , and survival function \bar{F} . In reliability theory, hazard rate function of X (also known as the force of mortality or the failure rate), where X is the lifetime of a system or a component, has found many key applications. In the same way, the reversed hazard rate of X has also attracted much attention in the literature. In a certain sense, it is the dual function of hazard rate and it bears some interesting features useful in reliability analysis; see Block and Savits (1998) and Finkelstein (2002).

For x such that $\bar{F}(x) > 0$, the hazard rate function of X at x , $r(x)$, is defined as

$$\begin{aligned} r(x) &= \lim_{\Delta x \rightarrow 0^+} \frac{\mathbb{P}(x < X \leq x + \Delta x | X > x)}{\Delta x} \\ &= \frac{1}{\bar{F}(x)} \lim_{\Delta x \rightarrow 0^+} \frac{\mathbb{P}(x < X \leq x + \Delta x)}{\Delta x} = \frac{f(x)}{\bar{F}(x)}. \end{aligned}$$

Analogously, for x such that $F(x) > 0$, the reversed hazard rate function of X at x , $q(x)$, is defined as

$$\begin{aligned} q(x) &= \lim_{\Delta x \rightarrow 0^+} \frac{\mathbb{P}(x - \Delta x < X \leq x | X \leq x)}{\Delta x} \\ &= \frac{1}{F(x)} \lim_{\Delta x \rightarrow 0^+} \frac{\mathbb{P}(x - \Delta x < X \leq x)}{\Delta x} = \frac{f(x)}{F(x)}. \end{aligned}$$

The hazard rate $r(x)$ is interpreted as the rate of instantaneous failure occurring immediately after the time point x , given that the unit has survived up to time x . Similarly, the reversed hazard rate $q(x)$ is interpreted as the rate of instantaneous failure occurring immediately before the time point x , given that the unit has not survived time x . The hazard rate function and the reversed hazard rate function uniquely determine the distribution of X ; see Barlow and Proschan (1996) for details.

1.2 Residual and past lifetimes

In reliability theory, the residual and past lifetimes have a great importance. The residual lifetime of X at time t is defined as $X_t = (X - t | X > t)$, which is a random variable taking values in $(t, +\infty)$ with pdf $f_{X_t}(x) = \frac{f(x+t)}{\bar{F}(t)}$ and survival function $\bar{F}_{X_t}(x) = \frac{\bar{F}(x+t)}{\bar{F}(t)}$, $x > 0$. The past lifetime of X at time t is defined as ${}_tX = (X | X \leq t)$, which is a random variable that takes values in $(0, t]$ with pdf $f_{{}_tX}(x) = \frac{f(x)}{F(t)}$ and distribution function $F_{{}_tX}(x) = \frac{F(x)}{F(t)}$.

1.3 Entropy, weighted entropy and extropy

The differential entropy, or Shannon entropy, of a random variable X is a basic concept as a measure of discrimination and information and is defined as

$$H(X) = -\mathbb{E}[\log f(X)] = -\int_0^{+\infty} f(x) \log f(x) dx, \quad (1)$$

where \log is the natural logarithm; see Shannon (1948).

Lad et al. (2015) introduced the concept of extropy as dual to entropy and it facilitates the comparison of uncertainties of two random variables. If the extropy of X is less than that of another variable Y , then X is said to have more uncertainty than Y . For a non-negative absolutely continuous random variable X , the extropy is defined as

$$J(X) = -\frac{1}{2}\mathbb{E}[f(X)] = -\frac{1}{2}\int_0^{+\infty} f^2(x) dx. \quad (2)$$

As a measure of information, the Shannon entropy is position-free, i.e., a random variable X has the same Shannon entropy of $X + b$, for any $b \in \mathbb{R}$. To avoid this problem, the concept of weighted entropy has been introduced (see Di Crescenzo and Longobardi (2006)) as

$$H^w(X) = -\mathbb{E}[X \log f(X)] = -\int_0^{+\infty} x f(x) \log f(x) dx. \quad (3)$$

To measure the uncertainty about the residual lifetime of X at time t , Ebrahimi (1996) introduced the residual entropy as

$$H(X_t) = -\int_t^{+\infty} \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx, \quad (4)$$

and it is the differential entropy of the residual lifetime X_t . It is also possible to study the uncertainty about the past lifetime [Di Crescenzo and Longobardi (2002)] defined by

$$H({}_tX) = - \int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} dx, \quad (5)$$

and it is the differential entropy of the past lifetime ${}_tX$.

Analogous to residual entropy, Qiu (2017) defined the extropy for residual lifetime X_t , called the residual extropy at time t , as

$$J(X_t) = -\frac{1}{2} \int_0^{+\infty} f_{X_t}^2(x) dx = -\frac{1}{2\bar{F}^2(t)} \int_t^{+\infty} f^2(x) dx. \quad (6)$$

In an analogous manner, Krishnan et al. (2020) and Kamari and Buono (2020) studied the past extropy defined as

$$J({}_tX) = -\frac{1}{2} \int_0^{+\infty} f_{{}_tX}^2(x) dx = -\frac{1}{2\bar{F}^2(t)} \int_0^t f^2(x) dx. \quad (7)$$

Di Crescenzo and Longobardi (2006) discussed weighted versions of the residual and past entropies. The weighted residual entropy is defined as

$$H^w(X_t) = - \int_t^{+\infty} x \frac{f(x)}{\bar{F}(t)} \log \frac{f(x)}{\bar{F}(t)} dx, \quad (8)$$

while the weighted past entropy is defined as

$$H^w({}_tX) = - \int_0^t x \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)} dx. \quad (9)$$

Asadi and Zohrevand (2007) and Di Crescenzo and Longobardi (2009a) introduced the dynamic cumulative residual entropy and the dynamic cumulative entropy, respectively. Sathar and Nair (2019) introduced and studied the dynamic survival extropy $J_s(X_t)$ defined as

$$J_s(X_t) = -\frac{1}{2\bar{F}^2(t)} \int_t^{+\infty} \bar{F}^2(x) dx. \quad (10)$$

Recently, many authors have discussed different versions of entropy and extropy and their applications; see, for example, Rao et al. (2004) for cumulative residual entropy, Di Crescenzo and Longobardi (2009b, 2013) for cumulative entropy, Mirali et al. (2017) for weighted cumulative residual entropy, and Jahanshani et al. (2019) for cumulative residual extropy.

1.4 Scope of this work

The rest of this paper is organized as follows. In Section 2, we introduce and study the weighted extropy discussing some of its properties and also present some illustrative examples. In Section 3, we introduce the bivariate version of extropy and its weighted form. In Section 4, we define and study the weighted residual and past entropies, and present some characterization results and bounds for these measures. Finally, in Section 5, some concluding remarks are made.

2 Weighted extropy

Analogous to the weighted entropy, in order to efficiently model statistical data we introduce a new measure of information named weighted extropy. It is defined as

$$J^w(X) = -\frac{1}{2}\mathbb{E}[Xf(X)] = -\frac{1}{2}\int_0^{+\infty} xf^2(x)dx, \quad (11)$$

which can also be alternatively expressed as

$$J^w(X) = -\frac{1}{2}\int_0^{+\infty} f^2(x)\int_0^x dydx = -\frac{1}{2}\int_0^{+\infty} dy\int_y^{+\infty} f^2(x)dx. \quad (12)$$

We now present two examples of distributions with the same extropy, but different weighted extropy. In the first example, we note that the weighted extropy is indeed shift-dependent.

Example 1. Let X and Y be two random variables such that $X \sim U(0, b)$, $Y \sim U(a, a + b)$, where $a, b > 0$. We have $f_X(x) = \frac{1}{b}$, for $x \in (0, b)$, and $f_Y(y) = \frac{1}{b}$, for $y \in (a, a + b)$, and then

$$J(X) = -\frac{1}{2}\int_0^b \frac{1}{b^2}dx = -\frac{1}{2b},$$
$$J(Y) = -\frac{1}{2}\int_a^{a+b} \frac{1}{b^2}dy = -\frac{1}{2b},$$

i.e., X and Y have the same extropy. But, they have different weighted extropy as seen below:

$$J^w(X) = -\frac{1}{2} \int_0^b x \frac{1}{b^2} dx = -\frac{1}{2b^2} \frac{b^2}{2} = -\frac{1}{4},$$

$$J^w(Y) = -\frac{1}{2} \int_a^{a+b} y \frac{1}{b^2} dy = -\frac{1}{2b^2} \frac{(a+b)^2 - a^2}{2} = -\frac{b^2 + 2ab}{4b^2} = -\frac{b+2a}{4b},$$

and so if $a \neq 0$, i.e., X and Y are not identically distributed, then $J^w(X) \neq J^w(Y)$.

Example 2. Let X be a random variable with piecewise constant pdf

$$f(x) = \sum_{k=1}^n c_k \mathbf{1}_{[k-1, k)}(x),$$

where $c_k \geq 0$, $k = 1, \dots, n$, $\sum_{k=1}^n c_k = 1$, and $\mathbf{1}_{[k-1, k)}(x)$ is the indicator function of x in the interval $[k-1, k)$. Then, the extropy and the weighted extropy of X are

$$J(X) = -\frac{1}{2} \int_0^n \sum_{k=1}^n c_k^2 \mathbf{1}_{[k-1, k)}(x) dx = -\frac{1}{2} \sum_{k=1}^n \int_{k-1}^k c_k^2 dx = -\frac{1}{2} \sum_{k=1}^n c_k^2,$$

$$J^w(X) = -\frac{1}{2} \int_0^n x \sum_{k=1}^n c_k^2 \mathbf{1}_{[k-1, k)}(x) dx = -\frac{1}{2} \sum_{k=1}^n \int_{k-1}^k x c_k^2 dx$$

$$= -\frac{1}{2} \sum_{k=1}^n c_k^2 \frac{k^2 - (k-1)^2}{2} = -\frac{1}{4} \sum_{k=1}^n c_k^2 (2k-1).$$

Because we obtain different distributions through a permutation of c_1, \dots, c_n , we observe that they have the same extropy, but different weighted extropy (except in some special cases).

We now present an example of distributions with the same weighted extropy, but different extropy.

Example 3. Let X be a random variable such that $X \sim U(0, b)$, with $b > 0$. In Example 1 earlier, we observed that $J(X) = -\frac{1}{2b}$ and $J^w(X) = -\frac{1}{4}$, and so the extropy depends on b while the weighted extropy does not. Thus, if we consider $Y \sim U(0, a)$, with $a > 0$ and $a \neq b$, we have random variables X and Y with the same weighted extropy, but different extropy.

Let us now evaluate the weighted extropy of some random variables.

Example 4. (a) Suppose X is exponentially distributed with parameter λ . Then,

$$J^w(X) = -\frac{1}{2} \int_0^{+\infty} x \lambda^2 e^{-2\lambda x} dx = \frac{1}{2} \left[\frac{\lambda x}{2} e^{-2\lambda x} \Big|_0^{+\infty} - \int_0^{+\infty} \frac{\lambda}{2} e^{-2\lambda x} dx \right] = -\frac{1}{8}.$$

(b) Suppose X is uniformly distributed over (a, b) . Then,

$$J^w(X) = -\frac{1}{2} \int_a^b x \frac{1}{(b-a)^2} dx = -\frac{1}{2(b-a)^2} \frac{b^2 - a^2}{2} = -\frac{1}{4} \frac{b+a}{b-a}.$$

Observe that in this case the weighted extropy can be expressed as the product

$$J^w(X) = J(X)\mathbb{E}(X),$$

where $\mathbb{E}(X) = \frac{a+b}{2}$ and $J(X) = -\frac{1}{2(b-a)}$. Then, $J(X) \leq J^w(X)$ if, and only if, $\mathbb{E}(X) \leq 1$, due to the fact that extropy and weighted extropy are non-positive.

Let us now look for values of a and b such that the weighted extropy for $Exp(\lambda)$ and $U(a, b)$ are the same. We then have

$$-\frac{1}{8} = -\frac{1}{4} \frac{b+a}{b-a} \iff 2(b-a) = b+a \iff b = 3a.$$

Then, $Exp(\lambda)$ and $U(1, 3)$ have the same weighted extropy.

(c) Suppose X is gamma distributed with parameters α and β , and with pdf

$$f(x) = \begin{cases} \frac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}, & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Then, we have

$$\begin{aligned} J^w(X) &= -\frac{1}{2} \int_0^{+\infty} x \frac{x^{2\alpha-2} e^{-2x/\beta}}{\beta^{2\alpha} \Gamma^2(\alpha)} dx \\ &= -\frac{1}{2\beta^{2\alpha} \Gamma^2(\alpha)} \int_0^{+\infty} x^{2\alpha-1} e^{-2x/\beta} dx \\ &= -\frac{1}{2^{2\alpha+1}} \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)}, \end{aligned}$$

which is free of the scale parameter β .

(d) Suppose X is beta distributed with parameters α and β , and with pdf

$$f(x) = \begin{cases} \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise,} \end{cases}$$

where

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$$

is the complete beta function. Then, we have

$$\begin{aligned} J^w(X) &= -\frac{1}{2} \int_0^1 x \frac{x^{2\alpha-2}(1-x)^{2\beta-2}}{B^2(\alpha, \beta)} dx \\ &= -\frac{1}{2} \frac{B(2\alpha, 2\beta - 1)}{B^2(\alpha, \beta)} \end{aligned}$$

if $2\beta - 1 > 0$, i.e., $\beta > \frac{1}{2}$, but if $0 < \beta \leq \frac{1}{2}$ we have $J^w(X) = -\infty$.

Remark 1. Let us now focus our attention on the integrability of $xf^2(x)$ on the support of X . If the support is unbounded, i.e., $(a, +\infty)$, with $a \geq 0$, and the function is bounded, we have to investigate the behaviour at infinity. As $\int_a^{+\infty} f(x)dx = 1$, we have $\lim_{x \rightarrow +\infty} f(x) = 0$ and $f(x)$ is infinitesimal of higher order with respect to $\frac{1}{x^{1+\varepsilon}}$, for $x \rightarrow +\infty$, for some $\varepsilon > 0$. Then, $xf^2(x)$ is infinitesimal of higher order with respect to $\frac{1}{x^{1+2\varepsilon}}$, for $x \rightarrow +\infty$ and so it is integrable, i.e., the weighted extropy is finite. If the support and the density are unbounded, we also have to study the behaviour at a . Suppose $a > 0$. If $\lim_{x \rightarrow a^+} f(x) = +\infty$, by normalization condition, we know that $f(x)$ is infinity of lower order with respect to $\frac{1}{(x-a)^{1-\varepsilon}}$, for $x \rightarrow a^+$, for some $0 < \varepsilon < 1$. Hence, $xf^2(x)$ is infinity of lower order with respect to $\frac{1}{(x-a)^{2-2\varepsilon}}$, for $x \rightarrow a^+$, and so is integrable if $\varepsilon \in (\frac{1}{2}, 1)$. If $a = 0$, by normalization condition, we know that $f(x)$ is infinity of lower order with respect to $\frac{1}{x^{1-\varepsilon}}$, for $x \rightarrow 0^+$, for some $0 < \varepsilon < 1$. Hence, $xf^2(x)$ is infinity of lower order with respect to $\frac{1}{x^{1-2\varepsilon}}$, for $x \rightarrow 0^+$, and so is integrable. If the support is bounded and f is bounded, then $xf^2(x)$ is bounded and is integrable. If the support is bounded and f is unbounded, then we can refer to the previous cases. Observe that if the support is $(0, +\infty)$, the weighted extropy is always finite.

In the following proposition, we give a characterization involving the weighted extropy vanishing.

Proposition 2.1. *Let X be a non-negative and absolutely continuous random variable. Then, $J^w(X) = 0$ if, and only if, X is degenerate.*

Proof. Suppose X is degenerate at point a . Then, by the definition of degenerate random variable and (11), we get $J^w(X) = 0$. Conversely, suppose $J^w(X) = 0$, i.e.,

$$\int_0^{+\infty} x f^2(x) dx = 0. \quad (13)$$

Thus, by noting that the integrand of (13) is non-negative, we conclude that the pdf of X vanishes for almost all $x \in (0, +\infty)$ and we have a degenerate random variable. \square

In the following theorem, we study weighted extropy under monotone transformation.

Theorem 2.1. *Let $Y = \Phi(X)$, with Φ being strictly monotone, continuous and differentiable, with derivative Φ' . Then, we have*

$$J^w(Y) = \begin{cases} -\frac{1}{2} \int_0^{+\infty} \frac{\Phi(x)}{\Phi'(x)} f_X^2(x) dx, & \text{if } \Phi \text{ is strictly increasing} \\ -\frac{1}{2} \int_0^{+\infty} \frac{\Phi(x)}{|\Phi'(x)|} f_X^2(x) dx, & \text{if } \Phi \text{ is strictly decreasing.} \end{cases} \quad (14)$$

Proof. From (11), we have

$$J^w(Y) = -\frac{1}{2} \int_0^{+\infty} x \frac{f_X^2(\Phi^{-1}(x))}{(\Phi'(\Phi^{-1}(x)))^2} dx.$$

Let Φ be strictly increasing. Then, with a change of variable in the above integral, we get

$$J^w(Y) = -\frac{1}{2} \int_0^{+\infty} \frac{\Phi(x)}{\Phi'(x)} f_X^2(x) dx,$$

giving the first expression in (14). If Φ is strictly decreasing, we similarly obtain

$$J^w(Y_t) = -\frac{1}{2} \int_0^{+\infty} \frac{\Phi(x)}{|\Phi'(x)|} f_X^2(x) dx,$$

which is the second expression in (14). \square

Remark 2. If in Theorem 2.1 we take $\Phi(X) = F_X(X)$, then we get

$$J^w(Y) = -\frac{1}{2} \int_0^{+\infty} F_X(x) f_X(x) dx = -\frac{1}{4},$$

which agrees with the result for Uniform(0, 1) distribution, since it is known in this case that the probability integral transformation $Y = F_X(X)$ is Uniform(0, 1).

In the following proposition, we recall how extropy changes under linear transformations.

Proposition 2.2. *Let X be a non-negative absolutely continuous random variable, and $Y = aX + b$, with $a > 0, b \geq 0$. Then, $J(Y) = \frac{1}{a}J(X)$.*

Proof. From $Y = aX + b$, we readily have $f_Y(x) = \frac{1}{a}f_X\left(\frac{x-b}{a}\right)$, $x > b$. So,

$$J(Y) = -\frac{1}{2} \int_b^{+\infty} \frac{1}{a^2} f_X^2\left(\frac{x-b}{a}\right) dx = -\frac{1}{2} \int_0^{+\infty} \frac{1}{a} f_X^2(x) dx = \frac{1}{a} J(X).$$

□

Remark 3. In Proposition 2.2, if we choose $a = 1$, we get the known property that extropy is invariant under translations.

In the following proposition, we discuss how weighted extropy behaves under linear transformations.

Proposition 2.3. *Let X be a non-negative absolutely continuous random variable, and $Y = aX + b$, with $a > 0, b \geq 0$. Then, $J^w(Y) = J^w(X) + \frac{b}{a}J(X)$.*

Proof. From $Y = aX + b$, we have $f_Y(x) = \frac{1}{a}f_X\left(\frac{x-b}{a}\right)$, $x > b$, and so

$$\begin{aligned} J^w(Y) &= -\frac{1}{2} \int_b^{+\infty} x \frac{1}{a^2} f_X^2\left(\frac{x-b}{a}\right) dx = -\frac{1}{2} \int_0^{+\infty} (ax + b) \frac{1}{a} f_X^2(x) dx \\ &= -\frac{1}{2} \int_0^{+\infty} x f_X^2(x) dx - \frac{1}{2} \frac{b}{a} \int_0^{+\infty} f_X^2(x) dx = J^w(X) + \frac{b}{a} J(X). \end{aligned}$$

□

Remark 4. In Proposition 2.3, if we choose $b = 0$, we see that the weighted extropy is invariant for proportional random variables, as seen earlier in Example 3 for the case of uniform distribution.

In the following proposition, we give bounds for random variables with support $(0, b)$, where b is finite, or $(a, +\infty)$, $a > 0$.

Proposition 2.4. (i) Let X be a continuous random variable with support $(0, b)$, $b < +\infty$.

Then,

$$J^w(X) \geq bJ(X); \quad (15)$$

(ii) Let X be a continuous random variable with support $(a, +\infty)$, $a > 0$. Then,

$$J^w(X) \leq aJ(X). \quad (16)$$

Proof. From the definitions of extropy and weighted extropy, (15) and (16) follow in a straightforward manner. \square

In the following theorem, we obtain a lower bound for the weighted extropy of the sum of two independent random variables.

Theorem 2.2. Let X and Y be two non-negative independent random variables with densities f_X and f_Y , respectively. Then,

$$J^w(X + Y) \geq -2 \{J(X)J^w(Y) + J^w(X)J(Y)\}. \quad (17)$$

Proof. As X and Y are non-negative independent random variables, the density function of $Z = X + Y$ is given, for $z > 0$, by

$$f_Z(z) = \int_0^z f_X(x)f_Y(z-x)dx.$$

Hence, the weighted extropy of Z is given by

$$J^w(Z) = -\frac{1}{2} \int_0^{+\infty} z \left[\int_0^z f_X(x)f_Y(z-x)dx \right]^2 dz.$$

Upon using Jensen's inequality, we then have

$$\begin{aligned} J^w(Z) &\geq -\frac{1}{2} \int_0^{+\infty} z \int_0^z f_X^2(x)f_Y^2(z-x)dx dz \\ &= -\frac{1}{2} \int_0^{+\infty} f_X^2(x) \int_x^{+\infty} z f_Y^2(z-x)dz dx \\ &= -\frac{1}{2} \int_0^{+\infty} f_X^2(x) \int_0^{+\infty} (z+x)f_Y^2(z)dz dx \\ &= \int_0^{+\infty} f_X^2(x)(J^w(X) + xJ(Y))dx \\ &= -2J(X)J^w(Y) - 2J(Y)J^w(X), \end{aligned}$$

as required. □

Remark 5. In particular, if X and Y are independent and identically distributed in Theorem 2.2, we simply deduce

$$J^w(X + Y) \geq -4J(X)J^w(X).$$

3 Bivariate version of extropy and weighted extropy

It is possible to introduce bivariate version of extropy. If X and Y are non-negative absolutely continuous random variables, the bivariate version of extropy, denoted by $J(X, Y)$, is defined as

$$J(X, Y) = \frac{1}{4}\mathbb{E}[f(X, Y)] = \frac{1}{4} \int_0^{+\infty} \int_0^{+\infty} f^2(x, y) dx dy, \quad (18)$$

where $f(x, y)$ is the joint density of (X, Y) .

Remark 6. This measure can also be defined for a general k -dimensional vector. In that case, in the definition of $J(X_1, X_2, \dots, X_k)$, the multiplying factor for the integral will be $(-\frac{1}{2})^k$.

In the following proposition, we discuss how bivariate version of extropy behaves in the case of independence.

Proposition 3.1. *Let X and Y be non-negative absolutely continuous random variables. If X and Y are independent, then $J(X, Y) = J(X)J(Y)$.*

Proof. If X and Y are independent, then $f(x, y) = f_X(x)f_Y(y)$, and hence

$$\begin{aligned} J(X, Y) &= \frac{1}{4}\mathbb{E}[f(X, Y)] = \frac{1}{4}\mathbb{E}[f_X(X)f_Y(Y)] \\ &= \frac{1}{4}\mathbb{E}[f_X(X)]\mathbb{E}[f_Y(Y)] = J(X)J(Y). \end{aligned}$$

□

In a spirit similar to that of the bivariate version of extropy, we can also introduce bivariate weighted extropy as

$$J^w(X, Y) = \frac{1}{4} \mathbb{E}[XY f(X, Y)] = \frac{1}{4} \int_0^{+\infty} \int_0^{+\infty} xy f^2(x, y) dx dy, \quad (19)$$

where X and Y are non-negative random variables with joint density function $f(x, y)$.

In the following proposition, we discuss the bivariate extropy in the case of independence.

Proposition 3.2. *Let X and Y be non-negative absolutely continuous random variables. If X and Y are independent, then $J^w(X, Y) = J^w(X)J^w(Y)$.*

Proof. If X and Y are independent, then $f(x, y) = f_X(x)f_Y(y)$, and hence

$$\begin{aligned} J^w(X, Y) &= \frac{1}{4} \mathbb{E}[XY f(X, Y)] = \frac{1}{4} \mathbb{E}[XY f_X(X) f_Y(Y)] \\ &= \frac{1}{4} \mathbb{E}[X f_X(X)] \mathbb{E}[Y f_Y(Y)] = J^w(X) J^w(Y). \end{aligned}$$

□

Example 5. (a) Let (X, Y) be a bivariate beta random variable with joint density function [see Kotz, Balakrishnan and Johnson (2000) for details on this distribution]

$$f(x, y) = \frac{1}{B(\alpha, \beta, \gamma)} x^{\alpha-1} (y-x)^{\beta-1} (1-y)^{\gamma-1}, \quad 0 < x < y < 1, \quad \alpha, \beta, \gamma > 0,$$

where $B(\alpha, \beta, \gamma) = \frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma)}$ is the bivariate complete beta function. Then, we readily find the bivariate extropy as

$$\begin{aligned} J(X, Y) &= \frac{1}{4} \mathbb{E}[f(X, Y)] \\ &= \frac{1}{4B^2(\alpha, \beta, \gamma)} \iint_{0 < x < y < 1} x^{2\alpha-2} (y-x)^{2\beta-2} (1-y)^{2\gamma-2} dx dy \\ &= \frac{B(2\alpha-1, 2\beta-1, 2\gamma-1)}{4B^2(\alpha, \beta, \gamma)} \end{aligned}$$

provided $\alpha, \beta, \gamma > \frac{1}{2}$.

(b) For the above bivariate beta distribution, we readily find the bivariate weighted extropy to be

$$\begin{aligned}
J^w(X, Y) &= \frac{1}{4} \mathbb{E}[XY f(X, Y)] \\
&= \frac{1}{4B^2(\alpha, \beta, \gamma)} \iint_{0 < x < y < 1} x^{2\alpha-1} y (y-x)^{2\beta-2} (1-y)^{2\gamma-2} dx dy \\
&= \frac{1}{4B^2(\alpha, \beta, \gamma)} [B(2\alpha, 2\beta, 2\gamma-1) + B(2\alpha+1, 2\beta-1, 2\gamma-1)]
\end{aligned}$$

provided $\alpha > 0$ and $\beta, \gamma > \frac{1}{2}$.

(c) In the special case of bivariate uniform distribution (i.e., $\alpha = \beta = \gamma = 1$), the above expressions readily reduce to

$$J(X, Y) = \frac{1}{2} \quad \text{and} \quad J^w(X, Y) = \frac{1}{8}.$$

4 Weighted residual and past extropies

In this section, we introduce and study weighted residual extropy and weighted past extropy.

Definition 1. Let X be a non-negative absolutely continuous random variable. For all t in the support of f , we define

(i) the weighted residual extropy of X at time t as

$$J^w(X_t) = -\frac{1}{2\overline{F}^2(t)} \int_t^{+\infty} x f^2(x) dx; \quad (20)$$

(ii) the weighted past extropy of X at time t as

$$J^w({}_tX) = -\frac{1}{2F^2(t)} \int_0^t x f^2(x) dx. \quad (21)$$

Remark 7. The definition in Equation (20) is in conformance with other definitions of residual entropies in the literature (see, for instance, Di Crescenzo and Longobardi (2006))

and Sekeh et al. (2014)). Moreover, we can refer to (20) as the weighted residual extropy of the first type and introduce the weighted residual extropy of the second type as

$$J^{w*}(X_t) = -\frac{1}{2\bar{F}^2(t)} \int_t^{+\infty} (x-t)f^2(x)dx.$$

These measures are related by a simple relationship involving the residual extropy

$$J^{w*}(X_t) = J^w(X_t) - tJ(X_t).$$

We point out that the use of the first or the second type of weighted residual extropy is based on the way in which we want to give a weight to the observations. In fact, in the first case, we take into account the time t in the weight, while in the second case we use as weight the time elapsed between t and the value assumed by X .

Remark 8. We observe that

$$\lim_{t \rightarrow 0^+} J^w(X_t) = \lim_{t \rightarrow +\infty} J^w(tX) = J^w(X). \quad (22)$$

In the following lemma, we evaluate the derivative of the weighted residual extropy and the derivative of the weighted past extropy.

Lemma 4.1. *Let X be a non-negative absolutely continuous random variable with weighted residual extropy $J^w(X_t)$ and weighted past extropy $J^w(tX)$. Then,*

- (i) $\frac{d}{dt} J^w(X_t) = 2r(t) \left[J^w(X_t) + \frac{t r(t)}{4} \right]$, where $r(t)$ is the hazard rate function of X ;
- (ii) $\frac{d}{dt} J^w(tX) = -2q(t) \left[J^w(tX) + \frac{t q(t)}{4} \right]$, where $q(t)$ is the reversed hazard rate function of X .

Proof. (i) From the definitions of weighted residual extropy and hazard rate function, we have

$$\begin{aligned} \frac{d}{dt} J^w(X_t) &= -\frac{1}{\bar{F}^3(t)} f(t) \int_t^{+\infty} x f^2(x) dx + \frac{1}{2\bar{F}^2(t)} t f^2(t) \\ &= 2r(t) \left[J^w(X_t) + \frac{t r(t)}{4} \right]; \end{aligned}$$

(ii) From the definition of weighted past extropy and reversed hazard rate function, we have

$$\begin{aligned}\frac{d}{dt} J^w({}_tX) &= \frac{1}{F^3(t)} f(t) \int_0^t x f^2(x) dx - \frac{1}{2F^2(t)} t f^2(t) \\ &= -2q(t) \left[J^w({}_tX) + \frac{t q(t)}{4} \right].\end{aligned}$$

□

Remark 9. We may ask a natural question here whether the weighted residual extropy could be constant over the support of a non-negative absolutely continuous random variable. In this regard, if $J^w(X_t)$ is constant, then for all $t > 0$, we have

$$J^w(X_t) + \frac{t r(t)}{4} = 0$$

and then

$$\int_t^{+\infty} x f^2(x) dx = \frac{t}{2} f(t) \bar{F}(t).$$

Differentiating both sides, we get

$$2t f^2(t) = f(t) \bar{F}(t) + t f'(t) \bar{F}(t).$$

It is known that $r'(t) = \frac{f'(t)}{\bar{F}(t)} + r^2(t)$ and so dividing by $\bar{F}^2(t)$ both sides of the above equality, we obtain

$$r'(t) = -\frac{r(t)}{t} + 3r^2(t),$$

which is a Bernoulli differential equation with initial condition $r(t_0) = r_0 > 0$, $t_0 > 0$. Upon solving this differential equation, we get

$$r(t) = \frac{1}{t(C - 3 \log t)},$$

where $C = \frac{1}{t_0 r_0} + 3 \log t_0$. We know that the hazard rate function is non-negative and so this condition is satisfied if and only if $t \leq e^C/3$. Hence, the weighted residual extropy can not be constant over $(0, +\infty)$.

Proposition 4.1. *The weighted extropy, the weighted residual extropy and the weighted past extropy satisfy the following relationship:*

$$J^w(X) = F^2(t)J^w({}_tX) + \overline{F}^2(t)J^w(X_t). \quad (23)$$

Proof. From (11), (20) and (21), we have

$$\begin{aligned} J^w(X) &= -\frac{1}{2} \int_0^{+\infty} x f^2(x) dx = -\frac{1}{2} \int_0^t x f^2(x) dx - \frac{1}{2} \int_t^{+\infty} x f^2(x) dx \\ &= -\frac{1}{2} F^2(t) \int_0^t x \frac{f^2(x)}{F^2(t)} dx - \frac{1}{2} \overline{F}^2(t) \int_t^{+\infty} x \frac{f^2(x)}{\overline{F}^2(t)} dx \\ &= F^2(t)J^w({}_tX) + \overline{F}^2(t)J^w(X_t), \end{aligned}$$

as required. □

Theorem 4.1. *If X is a non-negative absolutely continuous random variable and if $J^w(X_t)$ is increasing in $t > 0$, then $J^w(X_t)$ uniquely determines the distribution of X .*

Proof. From Lemma 4.1, we have

$$\frac{d}{dt} J^w(X_t) = 2r(t) \left[J^w(X_t) + \frac{t r(t)}{4} \right].$$

Consider the function

$$g(x) = 2x \left[J^w(X_t) + \frac{tx}{4} \right] - \frac{d}{dt} J^w(X_t).$$

We know that $g(r(t)) = 0$ and $g(0) = -\frac{d}{dt} J^w(X_t) \leq 0$ because $J^w(X_t)$ is increasing in $t > 0$. Moreover, $\lim_{x \rightarrow +\infty} g(x) = +\infty$. If we obtain the derivative of $g(x)$, we observe that there is only one point at which it vanishes; in fact,

$$\frac{d}{dx} g(x) = 2J^w(X_t) + tx$$

and so

$$\frac{d}{dx} g(x) = 0 \iff x = -\frac{2}{t} J^w(X_t) (\geq 0).$$

Then, $g(x) = 0$ has a unique solution and it is $r(t)$. From Barlow and Proschan (1996), we know that the hazard rate function uniquely determines the distribution and so $J^w(X_t)$ uniquely determines the distribution as well. □

Theorem 4.2. *If X is a non-negative absolutely continuous random variable and if $J^w({}_tX)$ is decreasing in $t > 0$, then $J^w({}_tX)$ uniquely determines the distribution of X .*

Proof. From Lemma 4.1, we have

$$\frac{d}{dt}J^w({}_tX) = -2q(t) \left[J^w({}_tX) + \frac{t q(t)}{4} \right].$$

Consider the function

$$h(x) = 2x \left[J^w({}_tX) + \frac{tx}{4} \right] + \frac{d}{dt}J^w({}_tX).$$

We know that $h(q(t)) = 0$ and $h(0) = \frac{d}{dt}J^w({}_tX) \leq 0$ because $J^w({}_tX)$ is decreasing in $t > 0$. Moreover, $\lim_{x \rightarrow +\infty} h(x) = +\infty$. If we obtain the derivative of $h(x)$, we observe that there is only one point at which it vanishes; in fact,

$$\frac{d}{dx}h(x) = 2J^w({}_tX) + tx$$

and so

$$\frac{d}{dx}h(x) = 0 \iff x = -\frac{2}{t}J^w({}_tX) (\geq 0).$$

Then, $h(x) = 0$ has a unique solution and it is $q(t)$. From Barlow and Proschan (1996), we know that the reversed hazard rate function uniquely determines the distribution and so $J^w({}_tX)$ uniquely determines the distribution as well. \square

In the following two theorems, we obtain bounds for the weighted residual extropy and the weighted past extropy under the monotonicity of hazard rate and reversed hazard rate functions.

Theorem 4.3. *If the hazard rate function $r(t)$ is increasing, then*

$$J^w(X_t) \leq t r^2(t) J_s(X_t), \tag{24}$$

where $J_s(X_t)$ is the dynamic survival extropy defined in (10).

Proof. From the definition of the weighted residual extropy, we have

$$J^w(X_t) = -\frac{1}{2\bar{F}^2(t)} \int_t^{+\infty} x f^2(x) dx = -\frac{1}{2\bar{F}^2(t)} \int_t^{+\infty} x r^2(x) \bar{F}^2(x) dx.$$

As the hazard rate function is increasing by assumption, we have

$$\begin{aligned} -\frac{1}{2\bar{F}^2(t)} \int_t^{+\infty} x r^2(x) \bar{F}^2(x) dx &\leq -\frac{r^2(t)}{2\bar{F}^2(t)} \int_t^{+\infty} x \bar{F}^2(x) dx \\ &\leq -t \frac{r^2(t)}{2\bar{F}^2(t)} \int_t^{+\infty} \bar{F}^2(x) dx \\ &= t r^2(t) J_s(X_t), \end{aligned}$$

as required. □

Theorem 4.4. *If the reversed hazard rate function $q(t)$ is decreasing, then*

$$J^w({}_tX) \leq \frac{1}{4} \left(\frac{1}{2} - t q(t) \right). \quad (25)$$

Proof. From the definition of the weighted past extropy, we have

$$J^w({}_tX) = -\frac{1}{2F^2(t)} \int_0^t x f^2(x) dx = -\frac{1}{2F^2(t)} \int_0^t x q(x) F(x) f(x) dx.$$

Integration by parts gives

$$J^w({}_tX) = -\frac{1}{2F^2(t)} \left[t q(t) \frac{F^2(t)}{2} - \int_0^t (q(x) + x q'(x)) \frac{F^2(x)}{2} dx \right];$$

moreover, due to the assumption of monotonicity of the reversed hazard rate function, we get

$$J^w({}_tX) \leq -\frac{t q(t)}{4} + \frac{1}{8}$$

yielding (25). □

Example 6. For the exponential distribution with parameter $\lambda > 0$, the hazard function $r(t) = \lambda$ is constant and so satisfies the conditions of Theorem 4.3. The weighted residual extropy is given by

$$J^w(X_t) = -\frac{1}{2e^{-2\lambda t}} \int_t^{+\infty} x \lambda^2 e^{-2\lambda x} dx = -\frac{\lambda t}{4} - \frac{1}{8},$$

while the upper bound in Theorem 4.3 is

$$-t \frac{\lambda^2}{2e^{-2\lambda t}} \int_t^{+\infty} e^{-2\lambda x} dx = -\frac{\lambda t}{4},$$

and so (24) is fulfilled. Moreover, the reversed hazard rate function is decreasing and so we can also apply Theorem 4.4. Note that the bound in Theorem 4.4 is relevant when the RHS of (25) is negative. In the case of the exponential distribution of parameter λ , this condition holds for $t < \frac{1}{2\lambda}$.

In the following theorem, we discuss weighted residual extropy and weighted past extropy under monotone transformation.

Theorem 4.5. *Let $Y = \Phi(X)$, with Φ being strictly monotone, continuous and differentiable, with derivative Φ' . Then, for all $t > 0$, we have*

$$J^w(Y_t) = \begin{cases} -\frac{1}{2\bar{F}_X^2(\Phi^{-1}(t))} \int_{\Phi^{-1}(t)}^{+\infty} \frac{\Phi(x)}{\Phi'(x)} f_X^2(x) dx, & \text{if } \Phi \text{ is strictly increasing} \\ -\frac{1}{2\bar{F}_X^2(\Phi^{-1}(t))} \int_0^{\Phi^{-1}(t)} \frac{\Phi(x)}{|\Phi'(x)|} f_X^2(x) dx, & \text{if } \Phi \text{ is strictly decreasing} \end{cases} \quad (26)$$

and

$$J^w({}_tY) = \begin{cases} -\frac{1}{2\bar{F}_X^2(\Phi^{-1}(t))} \int_0^{\Phi^{-1}(t)} \frac{\Phi(x)}{\Phi'(x)} f_X^2(x) dx, & \text{if } \Phi \text{ is strictly increasing} \\ -\frac{1}{2\bar{F}_X^2(\Phi^{-1}(t))} \int_{\Phi^{-1}(t)}^{+\infty} \frac{\Phi(x)}{|\Phi'(x)|} f_X^2(x) dx, & \text{if } \Phi \text{ is strictly decreasing.} \end{cases} \quad (27)$$

Proof. From (20), we have

$$J^w(Y_t) = -\frac{1}{2\bar{F}_X^2(\Phi^{-1}(t))} \int_t^{+\infty} x \frac{f_X^2(\Phi^{-1}(x))}{(\Phi'(\Phi^{-1}(x)))^2} dx.$$

Now, let Φ be strictly increasing. Then, with a change of variable in the above integral, we get

$$J^w(Y_t) = -\frac{1}{2\bar{F}_X^2(\Phi^{-1}(t))} \int_{\Phi^{-1}(t)}^{+\infty} \frac{\Phi(x)}{\Phi'(x)} f_X^2(x) dx,$$

giving the first expression in (26). If Φ is strictly decreasing, we similarly obtain

$$J^w(Y_t) = -\frac{1}{2\bar{F}_X^2(\Phi^{-1}(t))} \int_0^{\Phi^{-1}(t)} \frac{\Phi(x)}{|\Phi'(x)|} f_X^2(x) dx,$$

giving the second expression in (26). The proof of (27) is quite similar and is therefore not presented here for brevity. \square

5 Concluding remarks

In this paper, some new measures of information have been introduced and studied. We have defined the weighted extropy as well as weighted residual and past extropies. We have presented some bounds and characterization results under monotonicity of hazard and reversed hazard functions. We have also presented bivariate versions of extropy and weighted extropy. We have given numerous examples to illustrate all the concepts introduced here.

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References

- [1] Asadi, M., Zohrevand, Y. (2007). On the dynamic cumulative residual entropy. *Journal of Statistical Planning and Inference*, **137**, 1931–1941.
- [2] Barlow, R. E., Proschan, F. J. (1996). *Mathematical Theory of Reliability*. Philadelphia: Society for Industrial and Applied Mathematics.
- [3] Block, H. W., Savits, T. H. (1998). The reversed hazard rate function. *Probability in the Engineering and Informational Sciences*, **12**, 69–90.
- [4] Di Crescenzo, A., Longobardi, M. (2002). Entropy-based measure of uncertainty in past lifetime distributions. *Journal of Applied Probability*, **39**, 434–440.
- [5] Di Crescenzo, A., Longobardi, M. (2006). On weighted residual and past entropies. *Scientiae Mathematicae Japonicae*, **64** (2), 255–266.
- [6] Di Crescenzo, A., Longobardi, M. (2009a). On cumulative entropies. *Journal of Statistical Planning and Inference*, **139**, 4072–4087.

- [7] Di Crescenzo, A., Longobardi, M. (2009b). On cumulative entropies and lifetime estimations, In: J. Mira, J.M. Ferrandez, J.R. Alvarez Sanchez, F. Paz, J. Toledo (Eds.), *Methods and Models in Artificial and Natural Computation*, IWINAC 2009, Part I, in: LNCS, vol. 5601, Springer-Verlag, Berlin, Heidelberg, pp. 132–141.
- [8] Di Crescenzo, A., Longobardi, M. (2013) Stochastic comparisons of cumulative entropies, In: H. Li, X. Li (Eds.), *Stochastic Orders in Reliability and Risk, in Honor of Professor Moshe Shaked*, Lecture Notes in Statistics, vol. 208, Springer, New York, pp. 167–182.
- [9] Ebrahimi, N. (1996). How to measure uncertainty in the residual life time distribution. *Sankhyā: Series A*, **58**, 48–56.
- [10] Finkelstein, M. S. (2002). On the reversed hazard rate. *Reliability Engineering & System Safety*, **78**, 71–75.
- [11] Jahanshani, S. M. A., Zarei, H., Khammar, A. H. (2019). On cumulative residual extropy. *Probability in the Engineering and Informational Sciences*, DOI:10.1017/S0269964819000196.
- [12] Kamari, O., Buono, F. (2020). On extropy of past lifetime distribution. *Ricerche di Matematica*, DOI: 10.1007/s11587-020-00488-7.
- [13] Kotz, S., Balakrishnan, N., Johnson, N. L. (2000). *Continuous Multivariate Distributions – Vol. 1*, Second edition. New York: John Wiley & Sons.
- [14] Krishnan, A. S., Sunoj S. M., Nair N. U. (2020). Some reliability properties of extropy for residual and past lifetime random variables. *Journal of the Korean Statistical Society*, <https://doi.org/10.1007/s42952-019-00023-x>.
- [15] Lad, F., Sanfilippo, G., Agrò, G. (2015). Extropy: complementary dual of entropy. *Statistical Science*, **30**, 40–58.
- [16] Mirali, M., Baratpour, S., Fakoor, V. (2017). On weighted cumulative residual entropy. *Communications in Statistics – Theory and Methods*, **46**, 2857–2869.
- [17] Qiu, G. (2017). The extropy of order statistics and record values. *Statistics & Probability Letters*, **120**, 52–60.
- [18] Rao, M., Chen, Y., Vemuri, B.C., Wang, F. (2004). Cumulative residual entropy: a new measure of information, *IEEE Transactions on Information Theory*, **50**, 1220–1228.

- [19] Sathar, A. E. I., Nair, D. R. (2019). On dynamic survival extropy. *Communications in Statistics – Theory and Methods*, <https://doi.org/10.1080/03610926.2019.1649426>.
- [20] Sekeh, S. Y., Mohtashami Borzadaran, G. R., Rezaei Roknabadi, A. H. (2014). Some results based on a version of the generalized dynamic entropies. *Communications in Statistics – Theory and Methods*, **43**(14), 2989–3006.
- [21] Shannon, C. E. (1948). A mathematical theory of communication. *Bell System Technical Journal*, **27**, 379–423.