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Francesco Buono[1*†] and Jorge Navarro[2†]

[1*]Università di Napoli Federico II, Via Cintia, Napoli, I-80126, Napoli, Italy.

[2]Facultad de Matemáticas, Universidad de Murcia, Campus de Espinardo, Murcia, 30100, Murcia, Spain.

*Corresponding author(s). E-mail(s): francesco.buono3@unina.it; Contributing authors: jorgenav@um.es;

†These authors contributed equally to this work.

Abstract

In this paper, some properties of the order dependent time-homogeneous load-sharing model are obtained, including an algorithmic procedure to simulate samples from this model. Then, the problem of how to get predictions of the future failure times is analysed in a sample from censored data (early failures). Punctual predictions based on the median, the mean and the convolutions of exponential distributions are proposed and prediction bands are obtained. Some illustrative examples show how to apply the theoretical results. An application in the study of lifetimes of coherent systems is proposed as well.

Keywords: Hazard rate function, Load-sharing model, Quantile regression, Simulation, Predictions.

1 Introduction

The hazard rate function is an efficient tool to describe and characterize distributions. It is widely used and applied in several fields of probability and statistics, especially in reliability theory and survival analysis, in order to study aging properties and make comparisons among distributions. This function is commonly considered for univariate distributions, but the interest towards more concrete and realistic models brought to ponder the possibility of defining a corresponding multivariate version.

An appropriate formulation of a multivariate version of hazard rate should result an efficient tool to describe stochastic dependence among failure-times, under a scheme of longitudinal observations. This requirement led to the formulation of the concept of multivariate hazard rates. The first steps in the study of such a concept were made by Cox [2] for bivariate distributions and since then this problem has been extensively studied and generalized by many other researchers, see e.g. [10, 15, 16] and the references therein.

When the description of stochastic dependence among failure-times is based on the multivariate conditional hazard rates, the interesting class of time-homogeneous load sharing models emerges in a natural way. These models form a subclass of the load-sharing models and their interest is based on some shared properties with the exponential distributions. A wide and still developing literature has been devoted to this topic. As classical references one can refer to [3, 12, 14]. In the time-homogeneous model the load shared by the components in a system (or the systemic risk in an economic situation) does not depend on time and it only changes when a component fails (or e.g. a bank goes bankrupt). In several practical situations this is a reasonable assumption since when a component fails, the other components have to cover the load assigned to the failed component. This assumption allows us to estimate the parameters (constant hazard rates) of the models in practice. Load-sharing models have been extensively studied in [18, 19] in the case of parallel systems with two components. There the components start operating by sharing the total load L with load proportions α and $1-\alpha$; then, when one component fails, the other one continue working with the total load L.

In this paper, some properties of and time-homogeneous load-sharing models are considered for a new generalized version of this model recently proposed in [5, 6] and called the *order dependent* version. Then, two objectives are pursued. Firstly, a method for simulate samples from these models is discussed and an algorithmic procedure to do so is proposed. Secondly, a study on the predictions of future failure values under these models is performed. The statistical analysis about predictions is carried out by assuming different levels of knowledge about the sample. Predictions intervals are obtained as well by using convolutions of exponential distributions. Finally, it is explored the problem of predicting the lifetime of a coherent system whose components are distributed according to an order dependent time-homogeneous load-sharing model.

The rest of the paper is organized as follows. In Section 2, the formal definition of multivariate conditional hazard rate functions and the order dependent time-homogeneous load-sharing model are given and some properties of this model are explored. The study about predictions of future values is performed in Section 3 where different ways of obtaining predictions are presented. In Section 4, a procedure has been proposed to generate a sample from an order dependent time-homogeneous load-sharing model and an example is provided.

In Section 5, several examples about predictions are discussed and an application to the analysis systems is provided. Finally, Section 6 contains the conclusions and summarizes the results of the paper.

2 The models

Let $X_1, \ldots, X_n$ be non-negative random variables with an absolutely continuous joint distribution. For a fixed index $j \in [n] = \{1, \ldots, n\}$ and $i_1, \ldots, i_k \in [n]$ with $j \notin I = \{i_1, \ldots, i_k\}$, and an ordered sequence $0 \le t_1 \le \cdots \le t_k$, the multivariate conditional hazard rate (MCHR) function $\lambda_j(t|i_1, \ldots, i_k;\ t_1, \ldots, t_k)$ is defined as follows [16]:

$$\begin{aligned} &\lambda_j(t|i_1, \ldots, i_k;\ t_1, \ldots, t_k) \\ &= \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \mathbb{P}\left(X_j \le t + \Delta t \,\middle|\, X_{i_1} = t_1, \ldots, X_{i_k} = t_k, \min_{h \notin I} X_h > t \right). \end{aligned} \tag{1}$$

Furthermore, we use the following notation for the MCHR functions with no failures

$$\lambda_j(t|\emptyset) = \lim_{\Delta t \to 0^+} \frac{1}{\Delta t} \mathbb{P}\left(X_j \le t + \Delta t \,|X_{1:n} > t\,\right), \tag{2}$$

where $X_{1:n} = \min(X_1, \ldots, X_n)$.

From Equation (1), it readily follows that the function $\lambda_j(t|i_1, \ldots, i_k;\ t_1, \ldots, t_k)$ describes the hazard rate of X_j at time t given an observed history, from 0 to t, in which the failure of components $X_{i_1}, \ldots, X_{i_k}$ have been observed at times $t_1, \ldots, t_k$, respectively. Moreover, the function $\lambda_j(t|\emptyset)$ in Equation (2) describes the hazard rate of X_j at time t, conditional on the observation $(X_{1:n} > t)$ (i.e., no failures in $[0, t]$) and is sometimes called in the literature risk-specific, or initial, failure rate (see [16]).

If the random variables $X_1, \ldots, X_n$ are independent, then the multivariate conditional hazard rate functions degenerate into the classical marginal hazard rate functions, in the sense that, for $j \notin \{i_1, \ldots, i_k\}$, $\lambda_j(t|i_1, \ldots, i_k;\ t_1, \ldots, t_k) = r_j(t)$ for all $t > 0$ regardless of $i_1, \ldots, i_k$ and failure times $t_1, \ldots, t_k$, where $r_j(\cdot)$ is the hazard rate function of X_j. Furthermore, if the random variables are exchangeable, i.e., $(X_1, \ldots, X_n) =_{ST} (X_{\pi(1)}, \ldots, X_{\pi(n)})$ for any permutation π of $[n]$, where $=_{ST}$ denotes the equality in law, then the multivariate conditional hazard rate functions do not depend on j and $i_1, \ldots, i_k$ but only on k and the failure times $t_1, \ldots, t_k$. Then, in the exchangeable case, the quantities defined by Equations (1) and (2) respectively become

$$\lambda_j(t|i_1, \ldots, i_k;\ t_1, \ldots, t_k) = \lambda^{(k)}(t|t_1, \ldots, t_k), \qquad \lambda_j(t|\emptyset) = \lambda^{(0)}(t),$$

for $k \in \{1, 2, \ldots, n-1\}$ and $0 \le t_1 \le t_2 \le \cdots \le t_k \le t$.

The joint probability density function f of $(X_1, \ldots, X_n)$ can be determined and computed in terms of the multivariate conditional hazard rate functions. The result can be stated as follows.

Proposition 1 *The joint probability density function f of $(X_1, \ldots, X_n)$ can be obtained for $0 \le t_1 \le t_2 \le \cdots \le t_n$ as*

$$\begin{aligned} f(t_1, \ldots, t_n) &= \lambda_1(t_1|\emptyset) \exp\left[-\int_0^{t_1} \left(\sum_{j=1}^n \lambda_j(u|\emptyset)\right) du\right] \\ &\cdot \lambda_2(t_2|1;\ t_1) \exp\left[-\int_{t_1}^{t_2} \left(\sum_{j=2}^n \lambda_j(u|1;\ t_1)\right) du\right] \\ &\cdots \lambda_{k+1}(t_k|1, \ldots, k;\ t_1, \ldots, t_k) \exp\left[-\int_{t_k}^{t_{k+1}} \left(\sum_{j=k+1}^n \lambda_j(u|1, \ldots, k;\ t_1, \ldots, t_k)\right) du\right] \\ &\cdots \lambda_n(t_n|1, \ldots, n-1;\ t_1, \ldots, t_{n-1}) \exp\left[-\int_{t_{n-1}}^{t_n} \lambda_n(u|1, \ldots, n-1;\ t_1, \ldots, t_{n-1}) du\right]. \end{aligned} \tag{3}$$

Similar expressions hold when $t_1, \ldots, t_n$ are such that $0 \le t_{\pi(1)} \le \cdots \le t_{\pi(n)}$ for some permutation π of the set $[n]$.

For details on the proof of this proposition, one may refer to [15].

The multivariate conditional hazard rate functions are, in particular, a useful tool to study the minimum among dependent random variables as showed in [4]. In that paper, the authors proved that, for any vector of dependent random variables, the probabilities of the events related to the behavior of the minimum are equal to the probabilities of the same events for a vector of independent random variables.

The multivariate conditional hazard rate functions are efficient tools to describe the joint distribution of lifetimes subject to load-sharing situations, see [11, 12, 14] for some applications. If the MCHR functions do not depend on the failure times of the components, $t_1, \ldots, t_k$, then we have a load-sharing model. In this case, the current hazard of a working component only depends on the calendar time t and on the set of working components. Moreover, if in addition the MCHR functions do not depend on the calendar time t, then, they are constant functions and we talk about time-homogeneous load-sharing models. In particular, they can be seen as a natural generalization of the joint distribution of a vector of independent and exponentially distributed random variables.

For a review on general properties of load-sharing (LS) models and time-homogeneous load-sharing (THLS) models see [12, 14, 17]. The formal definitions of LS and THLS models can be stated as follows.

Definition 1 Let $(X_1, \ldots, X_n)$ be a random vector with absolutely continuous joint distribution. It is distributed according to a load-sharing model (LS) if, for any $i_1, \ldots, i_k \in [n]$ and $j \notin I = \{i_1, \ldots, i_k\}$, there exist functions $\mu_j(t|I)$ such that, for all $0 \le t_1 \le \cdots \le t_k \le t$,

$$\lambda_j(t|i_1, \ldots, i_k;\ t_1, \ldots, t_k) = \mu_j(t|I).$$

Furthermore, a load-sharing model is time-homogeneous (THLS) when there exist non-negative numbers $\mu_j(I)$ and $\mu_j(\emptyset)$ such that, for any $t > 0$ and any $j \notin I$,

$$\begin{aligned} \mu_j(t|I) &= \mu_j(I), \\ \lambda_j(t|\emptyset) &= \mu_j(\emptyset). \end{aligned}$$

In this paper, we will consider a more general class of models introduced in [6] which contains LS and THLS models as particular cases. This is the formal definition.

Definition 2 Let $(X_1, \ldots, X_n)$ be a random vector with absolutely continuous joint distribution. It is distributed according to an order dependent load-sharing (ODLS) model if, for any $i_1, \ldots, i_k \in [n]$ and $j \notin \{i_1, \ldots, i_k\}$, there exist functions $\mu_j(t|i_1, \ldots, i_k)$ such that, for all $0 \le t_1 \le \cdots \le t_k \le t$,

$$\lambda_j(t|i_1, \ldots, i_k;\ t_1, \ldots, t_k) = \mu_j(t|i_1, \ldots, i_k).$$

Furthermore, an order dependent load-sharing model is time-homogeneous (shortly written as ODTHLS) when there exist non-negative numbers $\mu_j(i_1, \ldots, i_k)$ and $\mu_j(\emptyset)$ such that, for any $t > 0$ and any $j \notin I$,

$$\begin{aligned} \mu_j(t|i_1, \ldots, i_k) &= \mu_j(i_1, \ldots, i_k), \\ \lambda_j(t|\emptyset) &= \mu_j(\emptyset). \end{aligned}$$

In the rest of the paper, by requiring that the vector $(X_1, \ldots, X_n)$ follows an ODTHLS model, we will implicitly assume that the parameters of the model are given as $\mu_j(\emptyset)$, with $j \in [n]$, and $\mu_j(i_1, \ldots, i_k)$, with $I = \{i_1, \ldots, i_k\} \subset [n]$ and $j \notin I$.

Remark 1 If for any non-empty set $I \subset [n]$ and any $j \notin I$, the function $\mu_j(t|i_1, \ldots, i_k)$ is invariant under permutations of $i_1, \ldots, i_k$, then the ODLS model reduces to a LS model. In the same way, if for any non-empty set $I \subset [n]$ and any $j \notin I$ the number $\mu_j(i_1, \ldots, i_k)$ is invariant under permutations of $i_1, \ldots, i_k$, then the ODTHLS model reduces to a THLS model. Note that this model includes a kind of weak exchangeability property since the multivariate conditional hazard rate functions just depend on the set of broken components $I = \{i_1, \ldots, i_k\}$ instead of the vector of ordered failures $(i_1, \ldots, i_k)$ used in the ODLS model.

Under the assumption of a ODTHLS model, the expression of the joint probability density function given in Equation (3) simplifies considerably as can be seen in the following proposition.

Proposition 2 *The joint probability density function f of $(X_1, \ldots, X_n)$ under the ODTHLS model can be obtained for $0 \le t_1 \le t_2 \le \cdots \le t_n$ as*

$$\begin{aligned} f(t_1, \ldots, t_n) &= \mu_1(\emptyset) \exp\left[-t_1 \sum_{j=1}^{n} \mu_j(\emptyset)\right] \cdot \mu_2(1) \exp\left[-(t_2 - t_1) \sum_{j=2}^{n} \mu_j(1)\right] \cdot \\ &\quad \ldots \cdot \mu_{k+1}(1, \ldots, k) \exp\left[-(t_{k+1} - t_k) \sum_{j=k+1}^{n} \mu_j(1, \ldots, k)\right] \cdot \ldots \\ &\quad \cdot \mu_n(1, \ldots, n-1) \exp\left[-(t_n - t_{n-1}) \mu_n(1, \ldots, n-1)\right], \end{aligned}$$

for $0 \le t_1 \le t_2 \le \cdots \le t_n$. Similar expressions hold when $t_1, \ldots, t_n$ are such that $t_{\pi(1)} \le \cdots \le t_{\pi(n)}$ for some permutation π of the set $[n]$.

Dealing with an ODTHLS model, the following quantities are of great interest

$$M(i_1, \ldots, i_k) = \sum_{h \notin \{i_1, \ldots, i_k\}} \mu_h(i_1, \ldots, i_k); \tag{4}$$

$$\rho_j(i_1, \ldots, i_k) = \frac{\mu_j(i_1, \ldots, i_k)}{M(i_1, \ldots, i_k)}. \tag{5}$$

They are very useful in the study of the order statistics of $(X_1, \ldots, X_n)$ as stated in the following proposition extracted from [17] and adapted here to the more general ODTHLS model (see also Section 3 of [5]).

Proposition 3 *Let $(X_1, \ldots, X_n)$ be distributed according to an ODTHLS model and let π be a fixed permutation of $[n]$. Then, for $r = 1, 2, \ldots, n-1$*

$$\begin{aligned} &\mathbb{P}(X_{1:n} = X_{\pi(1)}, \ldots, X_{r:n} = X_{\pi(r)}) = \\ &\qquad \rho_{\pi(1)}(\emptyset) \rho_{\pi(2)}(\pi(1)) \rho_{\pi(3)}(\pi(1), \pi(2)) \cdot \ldots \cdot \rho_{\pi(r)}(\pi(1), \ldots, \pi(r-1)) \end{aligned}$$

and

$$\begin{aligned} &\mathbb{P}(X_{1:n} = X_{\pi(1)}, \ldots, X_{n:n} = X_{\pi(n)}) = \\ &\qquad \rho_{\pi(1)}(\emptyset) \rho_{\pi(2)}(\pi(1)) \rho_{\pi(3)}(\pi(1), \pi(2)) \cdot \ldots \cdot \rho_{\pi(n-1)}(\pi(1), \ldots, \pi(n-2)). \end{aligned} \tag{6}$$

In order to state the following result, let us denote by $\Lambda^{(r)}$ a vector $(\lambda_1, \ldots, \lambda_r) \in \mathbb{R}_+^r$ and by $\overline{G}_{\Lambda^{(r)}}(t)$ the survival function of the random variable $S_r = \sum_{s=1}^{r} \Gamma_s$, where $\Gamma_1, \ldots, \Gamma_r$ are independent random variables with exponential distributions of parameters (hazard rates) $\lambda_1, \ldots, \lambda_r$, respectively. Moreover, for a permutation π of $[n]$ and $r \in [n]$, we place

$$\Lambda^{(r)}(\pi) = (M(\emptyset), M(\pi(1)), \ldots, M(\pi(1), \ldots, \pi(r-1))).$$

Then, we have the following proposition. It is the adapted version to the ODTHLS model of a result in [17] for the THLS model.

Proposition 4 *Let $(X_1, \ldots, X_n)$ be distributed according to an ODTHLS model. Then, for any $t > 0$ and $j \in [n]$,*

$$\mathbb{P}(X_{1:n} > t | X_{1:n} = X_j) = \exp(-tM(\emptyset)),$$

and for any permutation π of $[n]$ and $k \in \{2, \ldots, n\}$,

$$\mathbb{P}(X_{k:n} > t | X_{1:n} = X_{\pi(1)}, \ldots, X_{k-1:n} = X_{\pi(k-1)}, X_{k:n} = X_{\pi(k)}) = \overline{G}_{\Lambda^{(k)}(\pi)}(t). \tag{7}$$

To prove Proposition 4, an important property of interarrival times of ODTHLS models follows. In fact, in [17] it is observed that conditioning on the event $(X_{1:n} = X_{\pi(1)}, \ldots, X_{k:n} = X_{\pi(k)})$, the interarrival times $X_{1:n}, X_{2:n} - X_{1:n}, \ldots, X_{k:n} - X_{k-1:n}$ can be seen as independent random variables, exponentially distributed with parameters $M(\emptyset), M(\pi(1)), \ldots, M(\pi(1), \ldots, \pi(k-1))$, respectively. Two simple consequences of this fact, which will reveal to be key tools for our predictions and simulations, are presented in the following two remarks.

Remark 2 From Proposition 4, the independence between the events $(X_{1:n} > t)$ and $(X_{1:n} = X_j)$ readily follows. In fact

$$\begin{aligned}
\mathbb{P}(X_{1:n} > t) &= \sum_{j=1}^{n} \mathbb{P}(X_{1:n} = X_j)\mathbb{P}(X_{1:n} > t | X_{1:n} = X_j) \\
&= \exp(-tM(\emptyset)) \sum_{j=1}^{n} \mathbb{P}(X_{1:n} = X_j) \\
&= \exp(-tM(\emptyset)).
\end{aligned}$$

Remark 3 We note that $M(\emptyset), M(\pi(1)), \ldots, M(\pi(1), \ldots, \pi(k-1))$ do not depend on $\pi(k)$ and then the interarrival times (or spacings) $X_{1:n}, X_{2:n} - X_{1:n}, \ldots, X_{k:n} - X_{k-1:n}$, can be seen as independent random variables, exponentially distributed with parameters $M(\emptyset), M(\pi(1)), \ldots, M(\pi(1), \ldots, \pi(k-1))$, respectively, given $(X_{1:n} = X_{\pi(1)}, \ldots, X_{k-1:n} = X_{\pi(k-1)})$. Hence, under this conditioning event, from (7), the distribution of $X_{k:n}$ is a convolution of k independent exponential distributions.

3 Predictions

In this section, we consider the problem of predicting future failure times in the ODTHLS model. More precisely, we analyze different scenarios given by different levels of knowledge. We start by giving the prediction of $X_{k+1:n}$ given the observed history

$$\mathcal{H}_k = \{X_{1:n} = X_{\pi(1)} = t_1, X_{2:n} = X_{\pi(2)} = t_2, \ldots, X_{k:n} = X_{\pi(k)} = t_k\} \tag{8}$$

for $k < n$, where π is a permutation of $[n]$.

Proposition 5 *Let $(X_1,\ldots,X_n)$ follow an ODTHLS model. Given the history $\mathcal{H}_k$ in (8) for $k<n$, the median prediction of the next failure time $X_{k+1:n}$ is given by*

$$\widehat{X}_{k+1:n}=\mathfrak{m}(t_k)=t_k+\frac{\log 2}{M(\pi(1),\ldots,\pi(k))}, \tag{9}$$

where t_k is the value taken by the k-th order statistic, and the mean prediction of $X_{k+1:n}$ is given by

$$\tilde{X}_{k+1:n}=t_k+\frac{1}{M(\pi(1),\ldots,\pi(k))}.$$

Moreover, a prediction band of size $\gamma=\beta-\alpha$, with $\alpha,\beta,\gamma\in(0,1)$, is given by $[t_k+q_\alpha,t_k+q_\beta]$, where q_α and q_β are the respective quantiles of the exponential distribution with parameter $M(\pi(1),\ldots,\pi(k))$.

Proof We start by recalling that for the ODTHLS models the dependence on the values taken by the variables is lost under conditioning with respect to the past history, so that the relevant information contained in $\mathcal{H}_k$ is the same as the one contained in $(X_{1:n}=X_{\pi(1)},X_{2:n}=X_{\pi(2)},\ldots,X_{k:n}=X_{\pi(k)})$. As observed above under this model, given $(X_{1:n}=X_{\pi(1)},X_{2:n}=X_{\pi(2)},\ldots,X_{k:n}=X_{\pi(k)})$, the interarrival times $X_{1:n},X_{2:n}-X_{1:n},\ldots,X_{k:n}-X_{k-1:n}$, $X_{k+1:n}-X_{k:n}$ can be seen as independent random variables, exponentially distributed with parameters $M(\emptyset)$, $M(\pi(1)),\ldots,$ $M(\pi(1),\ldots,\pi(k-1)),M(\pi(1),\ldots,\pi(k))$, respectively. Hence, conditioning on the given history, the interarrival time $Z_{k+1}=X_{k+1:n}-X_{k:n}$ is exponential with parameter $M(\pi(1),\ldots,\pi(k))$ and its value can be estimated by its median. If we denote by $m_{M(\pi(1),\ldots,\pi(k))}=\frac{\log 2}{M(\pi(1),\ldots,\pi(k))}$ the median of an exponential distribution with parameter $M(\pi(1),\ldots,\pi(k))$, then the median prediction of $X_{k+1:n}$ is given by

$$\widehat{X}_{k+1:n}=\mathfrak{m}(t_k)=t_k+m_{M(\pi(1),\ldots,\pi(k))}=t_k+\frac{\log 2}{M(\pi(1),\ldots,\pi(k))},$$

which proves (9).

An alternative way of predicting $X_{k+1:n}$ is based on the mean of the exponential distribution with parameter $M(\pi(1),\ldots,\pi(k))$ and in this framework the prediction is

$$\tilde{X}_{k+1:n}=t_k+\mathbb{E}(X_{k+1:n}-X_{k:n}|\mathcal{H}_k)=t_k+\frac{1}{M(\pi(1),\ldots,\pi(k))}.$$

If we want to get a confidence $\gamma=\beta-\alpha$, where $\alpha,\beta,\gamma\in(0,1)$ and q_α and q_β are the respective quantiles of the exponential distribution with parameter $M(\pi(1),\ldots,\pi(k))$, then we use that

$$\mathbb{P}\left(t_k+q_\alpha\leq X_{k+1:n}\leq t_k+q_\beta|X_{1:n}=X_{\pi(1)},\ldots,X_{k:n}=X_{\pi(k)}=t_k\right)=\gamma,$$

where we have omitted $t_1,\ldots,t_{k-1}$ in the conditioning event since they are not relevant for our purposes. This concludes the proof. □

For example, in the above proposition, the centered 90% prediction band is obtained with $\beta=0.95$ and $\alpha=0.05$ as

$$\begin{aligned}C_{90}&=[t_k+q_{0.05},t_k+q_{0.95}]\\&=\left[t_k-\frac{\log(0.95)}{M(\pi(1),\ldots,\pi(k))},t_k-\frac{\log(0.05)}{M(\pi(1),\ldots,\pi(k))}\right].\end{aligned}$$

Here, we prefer to use the predictions given by the median $\mathfrak{m}(t_k)$, instead of the ones based on the mean, since they are obtained by using quantiles as well as the prediction bands.

We now turn to consider another scenario concerning a different observed history. We suppose to know how the realizations of the variables are ordered up to a certain index, for instance k, without knowing the values taken. Hence, we have information just about $\pi(1), \ldots, \pi(k)$ but not on $t_1, \ldots, t_k$ and our purpose is to predict $X_{k+1:n}$.

Proposition 6 *Let $(X_1, \ldots, X_n)$ follow an ODTHLS model. Let us suppose to know the history $X_{1:n} = X_{\pi(1)}, X_{2:n} = X_{\pi(2)}, \ldots, X_{k:n} = X_{\pi(k)}$, for $k < n$. The median and the mean prediction of the next failure time $X_{k+1:n}$ are respectively given by*

$$\widehat{X}_{k+1:n} = m_{M(\emptyset)} + m_{M(\pi(1))} + \cdots + m_{M(\pi(1),\ldots,\pi(k))}, \tag{10}$$

$$\tilde{X}_{k+1:n} = \frac{1}{M(\emptyset)} + \frac{1}{M(\pi(1))} + \cdots + \frac{1}{M(\pi(1),\ldots,\pi(k))}. \tag{11}$$

Furthermore, the prediction for $X_{k+1:n}$ can be obtained by the median of the convolution of $k+1$ independent exponential distributions with parameters $M(\emptyset)$, $M(\pi(1)), \ldots, M(\pi(1), \ldots, \pi(k))$.

Proof Two reasonable ways of predicting $X_{k+1:n}$ are given by estimating each inter-arrival time through the median or the mean and then provide the estimate of $X_{k+1:n}$ as

$$\widehat{X}_{k+1:n} = m_{M(\emptyset)} + m_{M(\pi(1))} + \cdots + m_{M(\pi(1),\ldots,\pi(k))},$$
$$\tilde{X}_{k+1:n} = \frac{1}{M(\emptyset)} + \frac{1}{M(\pi(1))} + \cdots + \frac{1}{M(\pi(1),\ldots,\pi(k))},$$

where the first is the prediction based on the median and the second on the mean.

Moreover, by observing that

$$X_{k+1:n} = X_{1:n} + (X_{2:n} - X_{1:n}) + \cdots + (X_{k+1:n} - X_{k:n}),$$

another option to predict $X_{k+1:n}$ is to obtain the median of the convolution given above, i.e., the convolution of $k+1$ independent exponential distributions with parameters $M(\emptyset)$, $M(\pi(1)), \ldots, M(\pi(1), \ldots, \pi(k))$. Note that the mean of that convolution coincides with (11). □

Now, let us suppose $k < n - 1$ and that our purpose is to predict $X_{k+2:n}$. By the assumptions of the model, the value of $X_{k+2:n}$ will depend on which path will be traversed to move from $X_{k:n}$ to $X_{k+2:}$, i.e., on which of the $n-k$ available alternatives will be assumed for $X_{k+1:n}$. Thus we obtain the following result.

Proposition 7 *Let $(X_1,\ldots,X_n)$ follow an ODTHLS model. Given the history $\mathcal{H}_k$ in (8) for $k<n-1$, the prediction of $X_{k+2:n}$ is given by*

$$\widehat{X}_{k+2:n} = \widehat{X}_{k+1:n} + \sum_{j\notin\{\pi(1),\ldots,\pi(k)\}} \rho_j(\pi(1),\ldots,\pi(k))\frac{\log 2}{M(\pi(1),\ldots,\pi(k),j)},$$

where $\widehat{X}_{k+1:n}$ is the median prediction of $X_{k+1:n}$ obtained as in Proposition 5.

Proof Let $j\notin\{\pi(1),\ldots,\pi(k)\}$. We have

$$\mathbb{P}(X_{k+1:n}=X_j|X_{1:n}=X_{\pi(1)},\ldots,X_{k:n}=X_{\pi(k)})=\rho_j(\pi(1),\ldots,\pi(k)), \tag{12}$$

where the dependence on $t_1,t_2,\ldots,t_k$ in the conditioning event is lost due to the properties of the model. Proceeding as above, we can predict the value of $X_{k+1:n}$, namely $\widehat{X}_{k+1:n}$. Then, by using this value and the median regression, we can predict the value of $X_{k+2:n}$ in $n-k$ different ways depending on which variable is the $(k+1)$-th order statistic. Hence, for each $j\notin\{\pi(1),\ldots,\pi(k)\}$, we obtain a prediction of $X_{k+2:n}$, namely $\widehat{X}^{(j)}_{k+2:n}$. Finally, based on (12), the final prediction of $X_{k+2:n}$ is obtained by the weighted mean of all $n-k$ predictions as

$$\widehat{X}_{k+2:n} = \sum_{j\notin\{\pi(1),\ldots,\pi(k)\}} \rho_j(\pi(1),\ldots,\pi(k))\widehat{X}^{(j)}_{k+2:n},$$

or, equivalently, since the values $\rho_j(\pi(1),\ldots,\pi(k))$ sum to one,

$$\begin{aligned}\widehat{X}_{k+2:n} &= \widehat{X}_{k+1:n} + \sum_{j\notin\{\pi(1),\ldots,\pi(k)\}} \rho_j(\pi(1),\ldots,\pi(k))m_{M(\pi(1),\ldots,\pi(k),j)}\\ &= \widehat{X}_{k+1:n} + \sum_{j\notin\{\pi(1),\ldots,\pi(k)\}} \rho_j(\pi(1),\ldots,\pi(k))\frac{\log 2}{M(\pi(1),\ldots,\pi(k),j)}.\end{aligned}$$

□

Again, we remark that the above prediction can be done also in terms of the mean instead of the median. In order to set a different prediction and to obtain the related prediction bands, we state the following result.

Proposition 8 *Let $(X_1,\ldots,X_n)$ follow an ODTHLS model. Let π be a fixed permutation of $[n]$ and $k<n-1$. Then,*

$$\mathbb{P}(X_{k+2:n}-X_{k:n}>t|\mathcal{H}_k) = \sum_{j\neq\pi(1),\ldots,\pi(k)} \rho_j(\pi(1),\ldots,\pi(k))\overline{G}_{\Upsilon_j^{(k)}(\pi)}(t), \tag{13}$$

where $\mathcal{H}_k$ is the history in (8), $\overline{G}_{\Upsilon_j^{(k)}(\pi)}(t)$ is the survival function of the random variable Y_1+Y_2, where Y_1 and Y_2 are independent random variables with exponential distributions of parameters (hazard rates) $M(\pi(1),\ldots,\pi(k))$ and $M(\pi(1),\ldots,\pi(k),j)$, respectively.

Proof The result follows by the law of total probability and by Proposition 4 observing that $X_{k+2:n}-X_{k:n}$ can be seen as the sum of two independent interarrival times, $X_{k+2:n}-X_{k:n}=(X_{k+2:n}-X_{k+1:n})+(X_{k+1:n}-X_{k:n})$ with exponential distributions. □

Conditioning on the observed history, the interarrival time $X_{k+2:n} - X_{k:n}$ is a mixture of $n-k$ distributions which are sums of two independent exponential distributions, not necessarily with the same parameters. We refer for example to (4.8) and (4.9) in [9] for the analytical expressions of the survival functions of such distributions, see also [13] p. 299. In particular, it is necessary to distinguish between the case in which the exponential distributions have the same parameter or not. If Y_1 and Y_2 are independent and exponentially distributed with parameters λ and μ, $\lambda \neq \mu$, then the survival function of $Y = Y_1 + Y_2$ is

$$\bar{F}_Y(t) = \frac{\mu}{\mu - \lambda} e^{-\lambda t} - \frac{\lambda}{\mu - \lambda} e^{-\mu t}, \tag{14}$$

for $t \geq 0$. In the case $\lambda = \mu$, the survival function of Y is given, for $t \geq 0$, as

$$\bar{F}_Y(t) = (1 + \lambda t) e^{-\lambda t}. \tag{15}$$

The median of such distributions can also be a good prediction for $X_{k+2:n}$. Numerical methods should be used to get that median from (13), (14) and (15). Then, if we want to get a confidence $\gamma = \beta - \alpha$, where $\alpha, \beta, \gamma \in (0,1)$ and q_α and q_β are the respective quantiles of the distribution given in Proposition 8, we use that

$$\mathbb{P}\left(t_k + q_\alpha \leq X_{k+2:n} \leq t_k + q_\beta | \mathcal{H}_k\right) = \gamma.$$

Remark 4 By proceeding in this way, it is possible to estimate each $X_{s:n}$ for $s > k$. As seen above, with the increase of s there will be more terms in the convolutions. In particular, by supposing to know the history $\mathcal{H}_k$ in (8), the estimation of $X_{s:n}$ will be based on the sum of $\frac{(n-k)!}{(n-s+1)!}$ terms. Moreover, it is also possible to construct prediction bands by giving a result similar to Proposition 8. In this case, we will need distributions constructed as the sum of $s-k$ independent exponential distributions. Such distributions have been studied in [1, 7].

Remark 5 The prediction techniques described in this section can be applied also to the residual lifetimes of an ODTHLS model. Suppose to know the observed history $\mathcal{H}_k$ in (8) for $k < n$, and that the remaining variables $X_{j_1}, \ldots, X_{j_{n-k}}$ are greater than a time t ($t > t_k$). Then, the joint distribution of the residual lifetimes $X_{j_1} - t, \ldots, X_{j_{n-k}} - t$ is still an ODTHLS model and its parameters are given in terms of the ones of $(X_1, \ldots, X_n)$ (see [4]). For instance, we simply have $\tilde{\mu}_{j_l}(\emptyset) = \mu_{j_l}(\pi(1), \ldots, \pi(k))$, for $l \in \{1, \ldots, n-k\}$ and the minimum of the residual lifetimes, $X_{\pi(k+1)} - t$, can be estimated by the median of the exponential distribution with parameter $M(\pi(1), \ldots, \pi(k))$ giving $t + \frac{\log 2}{M(\pi(1), \ldots, \pi(k))}$ as prediction for $X_{\pi(k+1)}$.

4 Simulated samples from the ODTHLS model

Let $(X_1, \dots, X_n)$ be a random vector satisfying the ODTHLS model with parameters $\mu_j(\emptyset)$ and $\mu_j(i_1, \dots, i_k)$, $I = \{i_1, \dots, i_k\} \subset [n]$ and $j \notin I$. There are $n!$ ways to order the variables $X_1, \dots, X_n$ and, for each one, the probability that such an order corresponds to the sequence given by the order statistics is given in Proposition 3. These probabilities depend on the parameters of the model, hence, once they are fixed, it is possible to choose one of the permutations by a random generation. For instance, it is possible to simulate the random choice between all the permutations by generating a uniform number in $(0, 1)$.

Suppose that the permutation π is randomly selected in the set of all the permutations of $[n]$ according to the probabilities given in (6) and the fixed parameters of the model. Hence, we have $X_{1:n} = X_{\pi(1)}, \dots, X_{n:n} = X_{\pi(n)}$. By Proposition 4 and Remark 2, the minimum is distributed as an exponential random variable with parameter $M(\emptyset)$, and it is not affected by which is the random variable in which it is assumed. Then, it is possible to simulate the minimum by a random generator of an exponential random variable with parameter (hazard rate) $M(\emptyset)$.

Let k be a natural number between 2 and n and suppose we have already simulated $X_{1:n}, \dots, X_{k-1:n}$. Now, by using that conditioning on the event $(X_{1:n} = X_{\pi(1)}, \dots, X_{k-1:n} = X_{\pi(k-1)})$, the interarrival times $X_{1:n}, X_{2:n} - X_{1:n}, \dots, X_{k:n} - X_{k-1:n}$ can be seen as independent random variables, exponentially distributed with parameters $M(\emptyset), M(\pi(1)), \dots, M(\pi(1), \dots, \pi(k-1))$, respectively, the interarrival time $X_{k:n} - X_{k-1:n}$ can be simulated by generating an exponential number with parameter $M(\pi(1), \dots, \pi(k-1))$. Then, the simulation of $X_{k:n}$ is obtained by summing this exponential number with the simulation of $X_{k-1:n}$ obtained in the previous step.

We want to emphasize that, once the permutation is fixed, the interarrival times can be generated all at the same time. If we denote by Z_j, $j \in [n]$, the j-th interarrival time, i.e., $Z_1 = X_{1:n}$, $Z_2 = X_{2:n} - X_{1:n}, \dots, Z_n = X_{n:n} - X_{n-1:n}$ obtained for the permutation π, then the simulation of the k-th order statistic is given as $X_{k:n} = \sum_{j=1}^{k} Z_j$. Finally, by using the permutation π, the simulated values for $X_1, \dots, X_n$ are obtained as $X_{\pi(1)} = X_{1:n}, \dots, X_{\pi(n)} = X_{n:n}$.

The algorithm procedure can be summarized as follows.

Step 1. Choose π according to the probabilities given in Proposition 3.
Step 2. Simulate n independent exponential distributions $Z_1, \dots, Z_n$ with respective parameters $M(\emptyset), M(\pi(1)), \dots, M(\pi(1), \dots, \pi(n-1))$.
Step 3. Compute $X_{k:n} = Z_1 + \dots + Z_k$, for $k = 1, \dots, n$.
Step 4. Compute $X_{\pi(k)} = X_{k:n}$, for $k = 1, \dots, n$.

Note that this algorithm can also be applied to simulate samples from THLS models since they are particular models of ODTHLS ones.

Let us see an example.

Example 1 Let (X_1, X_2, X_3) be distributed according to an ODTHLS model with parameters defined as follows

$$\begin{aligned}
&\mu_1(\emptyset) = 1, \quad \mu_1(2) = 2, \quad \mu_1(3) = 1, \quad \mu_1(2,3) = \mu_1(3,2) = 3,\\
&\mu_2(\emptyset) = 2, \quad \mu_2(1) = 1, \quad \mu_2(3) = 3, \quad \mu_2(1,3) = \mu_2(3,1) = 2,\\
&\mu_3(\emptyset) = 2, \quad \mu_3(1) = 2, \quad \mu_3(2) = 1, \quad \mu_3(1,2) = \mu_3(2,1) = 2.
\end{aligned}$$

We note that it is also a THLS model since $\mu_i(j,k) = \mu_i(k,j)$ for all distinct i, j and k. Hence, from (4) and (5) we have

$$\begin{aligned}
&M(\emptyset) = 5, \quad M(1) = 3, \quad M(2) = 3, \quad M(3) = 4,\\
&M(1,2) = M(2,1) = 2, \quad M(1,3) = M(3,1) = 2, \quad M(2,3) = M(3,2) = 3,
\end{aligned}$$

from which

$$\begin{aligned}
&\rho_1(\emptyset) = \frac{1}{5}, \quad \rho_2(\emptyset) = \frac{2}{5}, \quad \rho_3(\emptyset) = \frac{2}{5},\\
&\rho_2(1) = \frac{1}{3}, \quad \rho_3(1) = \frac{2}{3},\\
&\rho_1(2) = \frac{2}{3}, \quad \rho_3(2) = \frac{1}{3},\\
&\rho_1(3) = \frac{1}{4}, \quad \rho_2(3) = \frac{3}{4},
\end{aligned}$$

and, naturally,

$$\rho_1(2,3) = \rho_1(3,2) = \rho_2(1,3) = \rho_2(3,1) = \rho_3(1,2) = \rho_3(2,1) = 1.$$

For $n = 3$ there are six possible permutations and from Proposition 3 the corresponding probabilities are given as follows

$$\begin{aligned}
\mathbb{P}(X_{1:3} = X_1, X_{2:3} = X_2, X_{3:3} = X_3) &= \frac{1}{15},\\
\mathbb{P}(X_{1:3} = X_1, X_{2:3} = X_3, X_{3:3} = X_2) &= \frac{2}{15},\\
\mathbb{P}(X_{1:3} = X_2, X_{2:3} = X_1, X_{3:3} = X_3) &= \frac{4}{15},\\
\mathbb{P}(X_{1:3} = X_2, X_{2:3} = X_3, X_{3:3} = X_1) &= \frac{2}{15},\\
\mathbb{P}(X_{1:3} = X_3, X_{2:3} = X_1, X_{3:3} = X_2) &= \frac{1}{10},\\
\mathbb{P}(X_{1:3} = X_3, X_{2:3} = X_2, X_{3:3} = X_1) &= \frac{3}{10}.
\end{aligned}$$

By generating a uniform number in $(0,1)$ and accordingly to the probabilities given above, the permutation $(2,1,3)$ is chosen. Hence, three exponential numbers are generated with parameters $M(\emptyset) = 5$, $M(2) = 3$, and $M(2,1) = 2$, respectively. In this way, the simulated interarrival times obtained are $0.17166,\ 0.14498,\ 0.25606$ and then the simulated values of the order statistics of our model are, respectively, $0.17166,\ 0.31663 = 0.17166 + 0.14498,\ 0.57270 = 0.31663 + 0.25606$. Furthermore, since we have fixed the permutation $(2,1,3)$, the values $0.17166, 0.31663$ and 0.57270 represent a simulation of $X_2,\ X_1$ and X_3, respectively.

5 Examples

In this section, we will give some examples to apply the results given in Section 3. In order to do this, we will use the simulation procedure studied in Section 4 to obtain samples. Finally, in the last part of the section, we will consider the problem of predicting the lifetime of a coherent system whose components are distributed according to an ODTHLS model.

5.1 Predictions for ODTHLS models

In this section, we consider the ODTHLS model given in Example 1 and suppose that the realization of the sample is the one that we have simulated there, i.e., $X_1 = 0.31663$, $X_2 = 0.17166$ and $X_3 = 0.57270$. Suppose we just know $X_{1:3} = X_2 = 0.17166$ and that our purpose is to predict $X_{2:3}$ and $X_{3:3}$. Proceeding as described in Proposition 5, the mean and the median predictions of $X_{2:3} = 0.31663$ are given by

$$\tilde{X}_{2:3} = X_{1:3} + \frac{1}{M(2)} = 0.50499$$

and

$$\widehat{X}_{2:3} = \mathfrak{m}(X_{1:3}) = X_{1:3} + \frac{\log 2}{M(2)} = 0.40270,$$

respectively. Furthermore, we can obtain the centered 90% and 50% prediction bands that are given as

$$C_{90} = \left[X_{1:3} - \frac{\log(0.95)}{M(2)}, X_{1:3} - \frac{\log(0.05)}{M(2)}\right] = [0.18875, 1.17023]$$

and $C_{50} = [0.26755, 0.63375]$. In this case, the true value of $X_{2:3}$ belongs to both regions. Once $X_{2:3}$ has been predicted, proceeding as described in Proposition 7, also $X_{3:3}$ can be predicted. In this case the prediction of $X_{3:3} = 0.57270$ is given by

$$\begin{aligned}\widehat{X}_{3:3} &= \widehat{X}_{2:3} + \rho_1(2)\frac{\log 2}{M(2,1)} + \rho_3(2)\frac{\log 2}{M(2,3)} \\ &= 0.40270 + \frac{2}{3} \cdot \frac{\log 2}{2} + \frac{1}{3} \cdot \frac{\log 2}{3} = 0.71077.\end{aligned}$$

From Proposition 8 we can get a different prediction for $X_{3:3}$. We have

$$\begin{aligned}\bar{G}_{3|1}(t) &= \mathbb{P}(X_{3:3} - X_{1:3} > t | X_{1:3} = X_2 = 0.17166) \\ &= \rho_1(2)\overline{G}_{Y_{1,1}+Y_{1,2}}(t) + \rho_3(2)\overline{G}_{Y_{2,1}+Y_{2,2}}(t),\end{aligned}$$

where $Y_{1,1}$ and $Y_{1,2}$ are independent and exponentially distributed with parameters $M(2) = 3$ and $M(2,1) = 2$, respectively, and $Y_{2,1}$ and $Y_{2,2}$ are

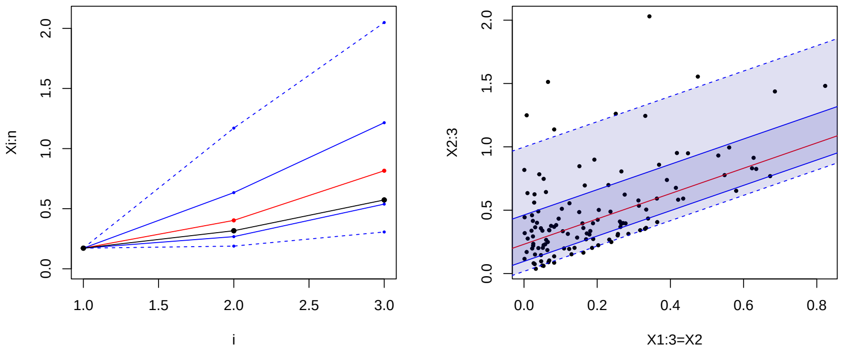

Fig. 1 Predictions (red) for $X_{s:3}$ from $X_{1:3}$ for $s = 2, 3$ jointly with the exact values (black points) for a simulated sample from an ODTHLS model (see Section 5.1). The blue lines represent the limits for the 50% (continuous lines) and the 90% (dashed lines) prediction intervals (left). Scatterplots of a simulated sample from $(X_{1:3}, X_{2:3})$, for the case $X_{1:3} = X_2$, for the ODTHLS model in Section 5.1 jointly with the theoretical median regression curves (red) and 50% (dark grey) and 90% (light grey) prediction bands (right).

independent and exponentially distributed with parameters $M(2) = 3$ and $M(2, 3) = 3$, respectively. Hence, by referring to the analytical expressions given in (14) and (15), we obtain

$$\bar{G}_{3|1}(t) = \rho_1(2)\frac{M(2)e^{-M(2,1)t} - M(2,1)e^{-M(2)t}}{M(2) - M(2,1)} + \rho_3(2)(1 + M(2)t)e^{-M(2)t},$$

where the second term is related to the sum of two independent exponential distributions with the same parameter $M(2) = M(2, 3) = 3$. Hence, by resolving $\bar{G}_{3|1}(t) = 0.5$ we obtain a prediction for the difference $X_{3:3} - X_{1:3}$ that is 0.64409, from which

$$\widehat{X}_{3:3} = t_1 + 0.64409 = 0.81575.$$

By resolving $\bar{G}_{3|1}(t) = \alpha$, for $\alpha = 0.05, 0.25, 0.75, 0.95$, we obtain the 90% and 50% centered prediction bands as $C_{90} = [0.30639, 2.04858]$ and $C_{50} = [0.53811, 1.21520]$. We observe that $X_{3:3} = 0.57270$ belongs to both regions. In Figure 1, left, we plot these predictions (red) for $X_{2:3}, X_{3:3}$ from $X_{1:3}$ jointly with the exact values (black points) and the prediction bands.

Analysis by using more samples

To see better what happens with these predictions we simulate $N = 300$ predictions of this kind, that is, 300 samples of size 3. Let us consider the case in which we predict $X_{2:3}$ from $X_{1:3}$. In order to give the results in a more readable way, we group them in three classes based on which is the

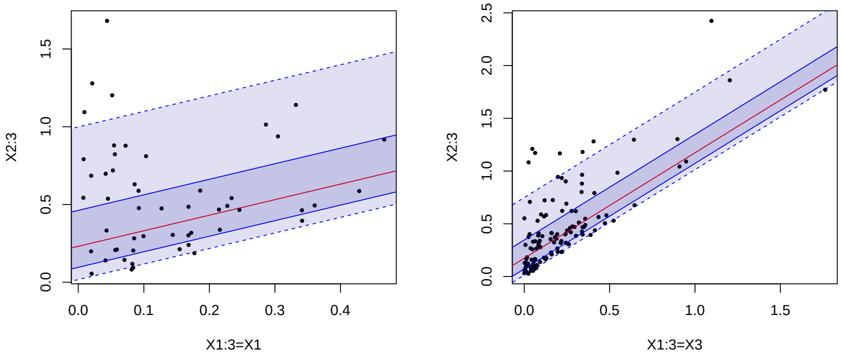

Fig. 2 Scatterplots of a simulated sample from $(X_{1:3}, X_{2:3})$ for the ODTHLS model in Section 5.1 jointly with the theoretical median regression curves (red) and 50% (dark grey) and 90% (light grey) prediction bands for the case $X_{1:3} = X_1$ (left) and $X_{1:3} = X_3$ (right).

component corresponding to the minimum order statistic. The data are plotted in Figures 1 and 2. There we can see that the prediction bands represent very well the dispersion of the majority of data (except some extreme values). In these samples, the minimum is assumed in X_1, X_2 and X_3 for 52, 122 and 126 times, respectively. These values are consistent with the expected values given by $\rho_1(\emptyset) \cdot 300 = 60$, $\rho_2(\emptyset) \cdot 300 = 120$ and $\rho_3(\emptyset) \cdot 300 = 120$. If the minimum is assumed in X_1, C_{50} contains 24 data and C_{90} contains 45 while 4 data are above the upper limit and 3 are below the bottom limit. If the minimum is assumed in X_2, C_{50} contains 69 data and C_{90} contains 109 while 6 data are above the upper limit and 7 are below the bottom limit. If the minimum is assumed in X_3, C_{50} contains 58 data and C_{90} contains 112 while 7 data are above the upper limit and 7 are below the bottom limit. Note that the prediction bands depend on which component fails first.

Predictions without knowing the times

Now, suppose we know the minimum is assumed by X_2 and we have no information about its value. Then, as described in (10) and (11), predictions for the first and the second order statistics based on the median (left) and the mean (right) are given by

$$\widehat{X}_{1:3} = \frac{\log 2}{M(\emptyset)} = 0.13863, \qquad \tilde{X}_{1:3} = \frac{1}{M(\emptyset)} = 0.2,$$

$$\widehat{X}_{2:3} = \frac{\log 2}{M(\emptyset)} + \frac{\log 2}{M(2)} = 0.36968, \qquad \tilde{X}_{2:3} = \frac{1}{M(\emptyset)} + \frac{1}{M(2)} = 0.53333.$$

Moreover, as described in Proposition 6, the prediction of $X_{2:3}$ can be obtained also by the median of the convolution $X_{1:3} + (X_{2:3} - X_{1:3})$. In fact, given that $X_{1:3} = X_2$, these interarrival times are independent and exponential with parameters $M(\emptyset) = 5$ and $M(2) = 3$ and the survival function of their convolution is obtained by (14). The median of such a distribution can be numerically computed and gives another prediction for $X_{2:3}$ as 0.44139. Of course, if we use the mean of the convolution we get 0.53333 as above.

Furthermore, if we know that the first and the second order statistics are assumed in X_2 and X_1, respectively, the maximum $X_{3:3}$ can be predicted by the median and the mean, respectively, as

$$\widehat{X}_{3:3} = \frac{\log 2}{M(\emptyset)} + \frac{\log 2}{M(2)} + \frac{\log 2}{M(2,1)} = 0.71625,$$

and

$$\tilde{X}_{3:3} = \frac{1}{M(\emptyset)} + \frac{1}{M(2)} + \frac{1}{M(2,1)} = 1.03333.$$

In addition, we can obtain the prediction of $X_{3:3}$ based on the convolution $Y = X_{1:3} + (X_{2:3} - X_{1:3}) + (X_{3:3} - X_{2:3})$, given that $X_{1:3} = X_2, X_{2:3} = X_1$. The interarrival times are independent and have exponential distributions with parameters $M(\emptyset) = 5$, $M(2) = 3$ and $M(2,1) = 2$. The survival function of this convolution can be obtained by specializing the result of [1] to the case of three exponential distributions with different parameters and it is expressed, for $t \geq 0$, as

$$\begin{aligned} \bar{G}_Y(t) = & \frac{M(2)M(2,1)}{(M(2)-M(\emptyset))(M(2,1)-M(\emptyset))} e^{-M(\emptyset)t} \\ & + \frac{M(\emptyset)M(2,1)}{(M(\emptyset)-M(2))(M(2,1)-M(2))} e^{-M(2)t} \\ & + \frac{M(\emptyset)M(2)}{(M(\emptyset)-M(2,1))(M(2)-M(2,1))} e^{-M(2,1)t}. \end{aligned} \tag{16}$$

The median of such a distribution can be numerically computed and gives another prediction for $X_{3:3}$ as $X^*_{3:3} = 0.90225$. Note that $\bar{G}_Y$ can also be used to get the prediction intervals for $X_{3:3}$. We have $C_{90} = [0.26708, 2.24684]$ and $C_{50} = [0.57337, 1.35021]$. The exact value 0.57270 belongs to C_{90} but does not belong to C_{50}.

Predictions based only on the knowledge of the group of failed variables

Now, suppose we have even less information and we just know that the first and the second order statistics are assumed by X_1 and X_2 but we have not the possibility to establish which one is $X_{1:3}$ or $X_{2:3}$. There are two possible scenarios corresponding to the permutations $(1,2,3)$ and $(2,1,3)$. Conditioning on the information $X_{3:3} = X_3$, it follows

$$\begin{aligned}\mathbb{P}(X_{1:3} = X_1, X_{2:3} = X_2 | X_{3:3} = X_3) &= \frac{1}{5},\\ \mathbb{P}(X_{1:3} = X_2, X_{2:3} = X_1 | X_{3:3} = X_3) &= \frac{4}{5}.\end{aligned}$$

Thus, the predictions for the first, second and third order statistics are obtained as

$$\begin{aligned}\widehat{X}_{1:3} &= \frac{\log 2}{M(\emptyset)} = 0.13863,\\ \widehat{X}_{2:3} &= \frac{\log 2}{M(\emptyset)} + \frac{1}{5}\cdot\frac{\log 2}{M(1)} + \frac{4}{5}\cdot\frac{\log 2}{M(2)} = 0.36968,\\ \widehat{X}_{3:3} &= \frac{\log 2}{M(\emptyset)} + \frac{1}{5}\left(\frac{\log 2}{M(1)} + \frac{\log 2}{M(1,2)}\right) + \frac{4}{5}\left(\frac{\log 2}{M(2)} + \frac{\log 2}{M(2,1)}\right) = 0.71625.\end{aligned}$$

In this particular case, we have obtained the same predictions of the case in which we know that $X_{1:3} = X_2$ and $X_{2:3} = X_1$, but this is only due to the assumptions $M(1) = M(2)$ and $M(1,2) = M(2,1)$ and the same holds for the predictions based on the mean or on the convolutions. Now, if we consider the same model except for $\mu_3(1,2) = 3$, and then $M(1,2) = 3 \neq M(2,1) = 2$, the median prediction of $X_{3:3}$ knowing that $X_{1:3} = X_2$ and $X_{2:3} = X_1$ is still 0.71625, but without knowing which one between X_1 and X_2 is the minimum and which one the second order statistic, the prediction of $X_{3:3}$ becomes $\widehat{X}_{3:3} = 0.69315$. Under these assumptions, the mean prediction of $X_{3:3}$ is $\tilde{X}_{3:3} = 1$.

Finally, we obtain a prediction based on convolutions by giving a weight of 0.2 and 0.8, respectively, to the medians of the convolutions of independent and exponential distributions with parameters $M(\emptyset) = 5, M(1) = 3, M(1,2) = 3$ and $M(\emptyset) = 5, M(2) = 3, M(2,1) = 2$. The survival function of the latter is equal to the one given in (16), whereas the former has a different expression since two of the three parameters coincide. In particular, the convolution of three independent exponential distributions of parameters $M(\emptyset), M(1), M(1)(= M(1,2))$ has the following survival function, which can be derived from [1], for $t \geq 0$,

$$\begin{aligned}\bar{G}(t) = &\frac{M(1)^2}{(M(1) - M(\emptyset))^2} e^{-M(\emptyset)t} - \frac{M(\emptyset)M(1)}{(M(\emptyset) - M(1))^2} e^{-M(1)t}\\ &+ \frac{M(\emptyset)M(1)}{M(\emptyset) - M(1)} t e^{-M(1)t} + \frac{M(\emptyset)}{M(\emptyset) - M(1)} e^{-M(1)t},\end{aligned} \tag{17}$$

and its median is 0.76649. Hence, the prediction for $X_{3:3}$ based on the convolutions is given as

$$X^*_{3:3} = \frac{1}{5}\cdot 0.76649 + \frac{4}{5}\cdot 0.90225 = 0.87510.$$

Moreover, a different prediction for $X_{3:3}$ can be obtained by using the median of the mixture of the survival functions given in (16) and (17) with 0.8 and 0.2 as weights, respectively. The prediction obtained numerically in this way is $X^*_{3:3} = 0.87229$ and its advantage is that we can give the prediction regions. The centered 90% and 50% prediction bands are $[0.25848, 2.17710]$ and $[0.55452, 1.30560]$, respectively. Observe that the exact value $X_{3:3} = 0.57270$ belongs to both the regions.

5.2 The case of unknown parameters

In this section, we analyze the problem of the predictions dealing with an ODTHLS model for which we do not know the values of the parameters. Hence, in the first part of the analysis, we obtain estimation for the parameters used in the predictions.

The estimation of the parameters

Let (X_1, X_2, X_3) be distributed according to an ODTHLS model with parameters defined as follows

$$\begin{array}{lllll} \mu_1(\emptyset) = 1, & \mu_1(2) = 2, & \mu_1(3) = 1, & \mu_1(2,3) = 3, & \mu_1(3,2) = 1, \\ \mu_2(\emptyset) = 2, & \mu_2(1) = 1, & \mu_2(3) = 3, & \mu_2(1,3) = 2, & \mu_2(3,1) = 1, \\ \mu_3(\emptyset) = 2, & \mu_3(1) = 2, & \mu_3(2) = 1, & \mu_3(1,2) = 2, & \mu_3(2,1) = 1. \end{array}$$

Hence, we have

$$\begin{array}{lllll} M(\emptyset) = 5, & M(1) = 3, & M(2) = 3, & M(3) = 4, & M(1,2) = 2, \\ M(2,1) = 1, & M(1,3) = 2, & M(3,1) = 1, & M(2,3) = 3, & M(3,2) = 1. \end{array}$$

Suppose we do not know the parameters of the model and we have historical data related to $N = 300$ samples. For those samples we know how X_1, X_2 and X_3 are ordered and their values. Then, we can estimate the parameters of the model through the values of interarrival times. Since the minimum is distributed as an exponential distribution with parameter $M(\emptyset)$, it can be estimated as

$$\widehat{M}(\emptyset) = \frac{N}{\sum_{i=1}^{N} X_{1:3}^{(i)}} = 5.19128,$$

where $X_{1:3}^{(i)}$ is the minimum in the i-th sample.

In order to estimate the other parameters, we need to group the data with the corresponding permutation. Let $\pi_1 = (1,2,3)$, $\pi_2 = (1,3,2)$, $\pi_3 = (2,1,3)$, $\pi_4 = (2,3,1)$, $\pi_5 = (3,1,2)$ and $\pi_6 = (3,2,1)$ and define $\mathcal{P}_j$ as the set of the observed samples ordered according to π_j, $j = 1, 2, \ldots, 6$.

Then, by recalling $Z_2 = X_{2:3} - X_{1:3}$, the estimations of $M(1)$, $M(2)$ and $M(3)$ are obtained as

$$\widehat{M}(1) = \frac{|\mathcal{P}_1 \cup \mathcal{P}_2|}{\sum_{i \in \mathcal{P}_1 \cup \mathcal{P}_2} Z_2^{(i)}} = 2.48951,$$
$$\widehat{M}(2) = \frac{|\mathcal{P}_3 \cup \mathcal{P}_4|}{\sum_{i \in \mathcal{P}_3 \cup \mathcal{P}_4} Z_2^{(i)}} = 3.34077,$$
$$\widehat{M}(3) = \frac{|\mathcal{P}_5 \cup \mathcal{P}_6|}{\sum_{i \in \mathcal{P}_5 \cup \mathcal{P}_6} Z_2^{(i)}} = 4.10161.$$

Finally, about the parameters $M(h,k)$, $h,k = 1,2,3$, $h \neq k$, by using $Z_3 = X_{3:3} - X_{2:3}$, we have

$$\widehat{M}(1,2) = \frac{|\mathcal{P}_1|}{\sum_{i \in \mathcal{P}_1} Z_3^{(i)}} = 2.67262, \qquad \widehat{M}(1,3) = \frac{|\mathcal{P}_2|}{\sum_{i \in \mathcal{P}_2} Z_3^{(i)}} = 2.11041,$$
$$\widehat{M}(2,1) = \frac{|\mathcal{P}_3|}{\sum_{i \in \mathcal{P}_3} Z_3^{(i)}} = 0.96048, \qquad \widehat{M}(2,3) = \frac{|\mathcal{P}_4|}{\sum_{i \in \mathcal{P}_4} Z_3^{(i)}} = 3.89834,$$
$$\widehat{M}(3,1) = \frac{|\mathcal{P}_5|}{\sum_{i \in \mathcal{P}_5} Z_3^{(i)}} = 0.91519, \qquad \widehat{M}(3,2) = \frac{|\mathcal{P}_6|}{\sum_{i \in \mathcal{P}_6} Z_3^{(i)}} = 0.82732.$$

Predictions based on the estimated parameters

Assume fully knowledge about the first and second order statistics, i.e., the value and the corresponding component, and we want to predict the maximum order statistic. Then, we can predict the interarrival time by using the quantile regression with the estimated parameters $\widehat{M}(h,k)$. We repeat this procedure for the 300 samples and the results are presented in Figure 3 where they are grouped by the different permutations. In order to compare with the predictions based on the fully knowledge of the model, in the figures we also plot the theoretical median regression lines (green), whereas the theoretical prediction bands are omitted for the readability of the plots. Moreover, since the parameters have been estimated, here the theoretical coverage percentage of the prediction bands is not exactly 50% or 90% and we refer to them as $\widehat{C}_{50}^{\pi_j}$ and $\widehat{C}_{90}^{\pi_j}$, $j \in \{1, \ldots, 6\}$. The number of elements in these regions are reported in Table 1.

5.3 Reliability of systems

In this section, we consider a coherent system whose components are distributed according to an ODTHLS model and, by using the observed hystory, we obtain predictions for the lifetime of the system.

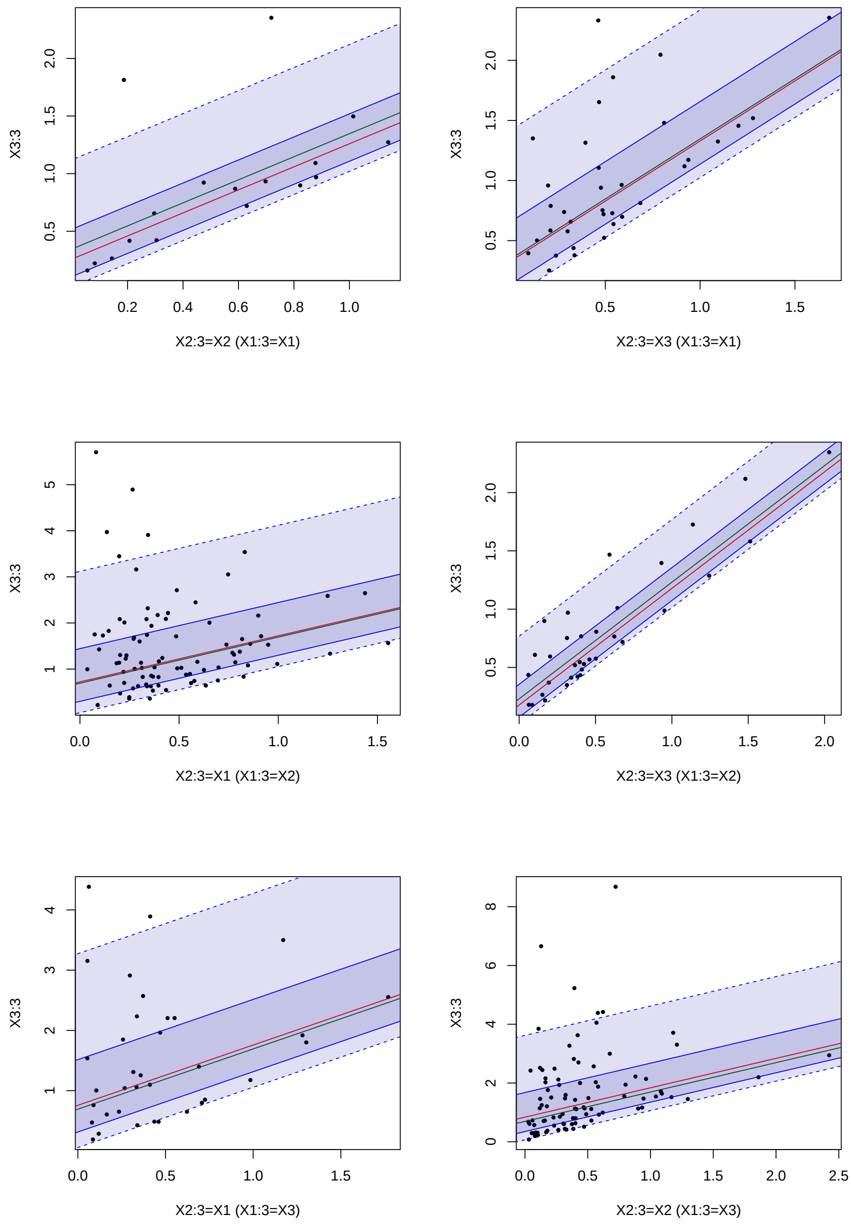

Fig. 3 Scatterplots of a simulated sample from $(X_{2:3}, X_{3:3})$ for the ODTHLS model in Section 5.2 jointly with the median regression curves (red) and 50% (dark grey) and 90% (light grey) prediction bands obtained by estimating the parameters and with the theoretical median regression curve (green).

Real coverage in π_j	$\widehat{C}_{90}^{\pi_j}$	$\widehat{C}_{50}^{\pi_j}$
$j=1$	88.23%	64.70%
$j=2$	97.14%	51.43%
$j=3$	89.77%	54.54%
$j=4$	97.06%	32.35%
$j=5$	85.71%	45.71%
$j=6$	90.11%	47.25%
Weighted average	91.00%	49.00%

Table 1 Percentage of exact data in $\widehat{C}_{90}^{\pi_j}$ and $\widehat{C}_{50}^{\pi_j}$, $j \in \{1, \ldots, 6\}$, and weighted average in Section 5.2.

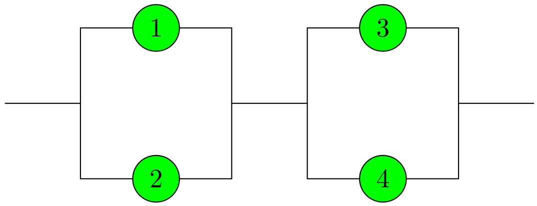

Fig. 4 The structure of the system in Example 2.

Example 2 Let us consider a coherent system formed by four components X_1, X_2, X_3, X_4 and whose lifetime T is described as

$$T = \min\{\max\{X_1, X_2\}, \max\{X_3, X_4\}\},$$

and whose structure is displayed in Figure 4. Let us suppose that (X_1, X_2, X_3, X_4) is distributed according to an ODTHLS model and that $X_{1:4} = X_1 = t_1$. Our purpose is to predict the lifetime of the system. The parameters of the model are (we give just the ones interesting for our purposes)

$$\begin{aligned}
&\mu_1(\emptyset) = 4, \quad \mu_2(\emptyset) = 1, \quad \mu_3(\emptyset) = 1, \quad \mu_4(\emptyset) = 2,\\
&\mu_2(1) = 1, \quad \mu_2(1,3) = 2, \quad \mu_2(1,4) = 2, \quad \mu_2(1,3,4) = 2, \quad \mu_2(1,4,3) = 3,\\
&\mu_3(1) = 3, \quad \mu_3(1,2) = 3, \quad \mu_3(1,4) = 3, \quad \mu_3(1,2,4) = 1, \quad \mu_3(1,4,2) = 2,\\
&\mu_4(1) = 2, \quad \mu_4(1,2) = 3, \quad \mu_4(1,3) = 1, \quad \mu_4(1,2,3) = 3, \quad \mu_4(1,3,2) = 2.
\end{aligned}$$

By knowing the first failure and the structure of the system, we deduce that T will be equal to the second order statistic if it is assumed by X_2 whereas it will be the third order statistic if the second failure is assumed by X_3 or X_4. Hence, we obtain a prediction for the lifetime of the system by using the predictions of the second and third order statistics appropriately weighted. More precisely, from Proposition 3, the weight for the prediction of the second order statistic will be

$$\mathbb{P}(X_{2:4} = X_2 | X_{1:4} = X_1) = \rho_2(1) = \frac{\mu_2(1)}{\mu_2(1) + \mu_3(1) + \mu_4(1)} = \frac{1}{6}.$$

About the third order statistic, we have to consider two different predictions, one for the case $X_{2:4} = X_3$ and one for $X_{2:4} = X_4$. The corresponding weights are given by

$$\mathbb{P}(X_{2:4} = X_3 | X_{1:4} = X_1) = \rho_3(1) = \frac{1}{2},$$
$$\mathbb{P}(X_{2:4} = X_4 | X_{1:4} = X_1) = \rho_4(1) = \frac{1}{3}.$$

Hence, the prediction for the lifetime of the system is obtained as

$$\widehat{T} = \rho_2(1) \cdot \widehat{X}_{2:4} + \rho_3(1) \cdot \widehat{X}^{(3)}_{3:4} + \rho_4(1) \cdot \widehat{X}^{(4)}_{3:4},$$

where $\widehat{X}^{(j)}_{3:4}$, $j = 3, 4$, denotes the prediction of the third order statistic given $(X_{1:4} = X_1 = t_1, X_{2:4} = X_j)$.

Consider the following (simulated) realization of the sample

$$X_{1:4} = X_1 = 0.10728,\ X_{2:4} = X_3 = 0.17977,$$
$$X_{3:4} = X_2 = 0.35048,\ X_{4:4} = X_4 = 0.99044.$$

Then, we have $T = X_{3:4} = 0.35048$. Suppose we know only $X_{1:4} = X_1 = 0.10728$, hence, by proceeding as described above we obtain

$$\begin{aligned}
\widehat{X}_{2:4} &= t_1 + \frac{\log 2}{M(1)} = 0.22281, && M(1) = \mu_2(1) + \mu_3(1) + \mu_4(1) = 6,\\
\widehat{X}^{(3)}_{3:4} &= \widehat{X}_{2:4} + \frac{\log 2}{M(1,3)} = 0.45386, && M(1,3) = \mu_2(1,3) + \mu_4(1,3) = 3,\\
\widehat{X}^{(4)}_{3:4} &= \widehat{X}_{2:4} + \frac{\log 2}{M(1,4)} = 0.36144, && M(1,4) = \mu_2(1,4) + \mu_3(1,4) = 5,
\end{aligned}$$

from which it follows that the prediction for $T = 0.35048$ is

$$\widehat{T} = \frac{1}{6} \cdot 0.22281 + \frac{1}{2} \cdot 0.45386 + \frac{1}{3} \cdot 0.36144 = 0.38454.$$

If the system does not fail at $X_{2:4}$, i.e. the second order statistic is assumed by X_3 or X_4, and we just know that $t_2 = X_{2:4} = 0.17977$, then the prediction for the lifetime of the system will be

$$\widehat{T} = t_2 + \frac{3}{5} \cdot \frac{\log 2}{M(1,3)} + \frac{2}{5} \cdot \frac{\log 2}{M(1,4)} = 0.37385,$$

or, by using the median of the mixture of two exponential distributions with parameters $M(1,3)$ and $M(1,4)$ and weights 0.6 and 0.4, respectively, $\widehat{T} = 0.36645$. If we also know that $X_{2:4} = X_3$, then the prediction will be

$$\widehat{T} = t_2 + \frac{\log 2}{M(1,3)} = 0.41082.$$

In both cases we can obtain prediction bands for T. In the first case we have a mixture of two exponential distributions and in the second an exponential distribution with parameter 3. The prediction bands in the first case for the point estimation $\widehat{T} = 0.36645$ are $C_{90} = [0.19329, 1.04527]$ and $C_{50} = [0.25621, 0.56174]$, and in the second case, for $\widehat{T} = 0.41082$, we have $C_{90} = [0.19687, 1.17835]$ and $C_{50} = [0.27566, 0.64187]$.

6 Conclusions

In this paper, some properties of the order dependent time-homogeneous load-sharing (ODTHLS) model are discussed. The non-order dependent model is recovered as a particular case and hence all the results are valid and applicable for it too. The problem of the predictions of future failure times in these models is addressed. Then a procedure is described to generate samples which follow the ODTHLS model. Different scenarios based on the level of information are considered and several examples are proposed. These procedures can also be applied to predict system failures. Prediction bands for these future values are provided as well. For predictions without assuming the ODTHLS model see [8] and the references therein.

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Statements and Declarations

Competing interest. The authors declare no conflict of interest.

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