Opinion Dynamics With Set-Based Confidence: Convergence Criteria and Periodic Solutions

Opinion Dynamics With Set-Based Confidence: Convergence Criteria and Periodic Solutions

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***Abstract*—This letter introduces a new multidimensional extension of the Hegselmann-Krause (HK) opinion dynamics model, where opinion proximity is not determined by a norm or metric. Instead, each agent trusts opinions within the Minkowski sum $\xi + \mathcal{O}$, where ξ is the agent's current opinion and $\mathcal{O}$ is the confidence set defining acceptable deviations. During each iteration, agents update their opinions by simultaneously averaging the trusted opinions. Unlike traditional HK systems, where $\mathcal{O}$ is a ball in some norm, our model allows the confidence set to be non-convex and even unbounded. The new model, referred to as SCOD (Set-based Confidence Opinion Dynamics), can exhibit properties absent in the conventional HK model. Some solutions may converge to non-equilibrium points in the state space, while others oscillate periodically. These "pathologies" disappear if the set $\mathcal{O}$ is symmetric and contains zero in its interior: similar to the usual HK model, the SCOD then converge in a finite number of iterations to one of the equilibrium points. The latter property is also preserved if one agent is "stubborn" and resists changing their opinion, yet still influences the others; however, two stubborn agents can lead to oscillations.**

***Index Terms*—Agents-based systems, network analysis and control, emerging control applications.**

I. Introduction

The Hegselmann-Krause (HK) model [1] can be viewed as a deterministic averaging consensus algorithm with an opinion-dependent interaction graph, illustrating the principle of *homophily* in social interactions: agents trust like-minded individuals and readily assimilate their opinions, while approaching dissimilar opinions with discretion. For historical discussions and an overview of the HK model's development over the past 20 years, refer to surveys [2], [3], [4].

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The original model from [1] addresses scalar opinions, but many opinions are better represented as vectors, capturing individuals' positions on multiple topics, like belief systems [5], [6] or experts' assessments of multifaceted problems, such as probability distributions [7] or resource allocation between multiple entities [8]. This led to the development of multidimensional HK models [9], where opinion formation involves averaging opinions within a multidimensional ball centered on the agent's opinion, ignoring those outside. The key consideration is the norm used to measure the proximity of opinions, which is usually ℓ_2, ℓ_1 (Manhattan) [10] or ℓ_∞ [11]. The HK system with the Euclidean norm allows for convenient Lyapunov functions [9], [12] and a mechanical kinetic energy analogue employed in many convergence analyses [13], [14], [15].

At the same time, there is no substantial experimental support for using the Euclidean or any specific norm to assess opinion proximity within the cognitive mechanisms underlying social homophily. Moreover, as the dimension of the opinion space grows, the "nearest-neighbor" rules in opinion assimilation are undermined by the phenomenon of distance concentration, studied in data science [16], [17], [18], where distances between all pairs of points in high-dimensional random data tend to become equal. Using the ℓ_p distance, higher values of p exacerbate this phenomenon. Even in 2, 3, and 4 dimensions, ℓ_1 norm outperforms the Euclidean norm in evaluating data similarity, but is surpassed by ℓ_p distances with $p < 1$ [17].

Objectives: The goal of this letter is to explore how much the properties of bounded confidence opinion dynamics depend on the distance-based homophily mechanism. To this end, introduce a generalized model, termed **SCOD** (Set-based Confidence Opinion Dynamics), where the confidence ball is replaced by a *set* of admissible opinion discrepancies, $\mathcal{O}$. An agent with opinion ξ trusts opinions within the Minkowski sum $\xi + \mathcal{O}$, ignoring those outside; the averaging opinion update mechanism remains the same as in the HK model.

Contributions: We explore the properties of the SCOD system by identifying its similarities and differences with the standard HK model and examining the role of the set $\mathcal{O}$:

(i) The SCOD model inherits the HK model's convergence properties when $\mathcal{O}$ is symmetric and contains zero in its interior: the group splits into clusters with equal opinions, and the dynamics terminate after a finite number of stages; *(ii)* Under the same conditions, opinions remain convergent even with one stubborn agent who never changes their opinion but influences others. However, several stubborn agents can

give rise to periodic oscillations unless their opinions coincide. *(iii)* If these conditions on $\mathcal{O}$ are violated, the SCOD model can exhibit behaviors untypical for the HK model, e.g., some solutions oscillate or converge to non-equilibrium points.

Structure of the text: The SCOD model is introduced in Section II, showing that even a small-size SCOD system with a general set $\mathcal{O}$ can behave very differently from the conventional HK model. In Section III, we formulate our main result, establishing the convergence of the SCOD in the case of *symmetric* $\mathcal{O}$. The proof of this main theorem is given in Section IV. Section V concludes this letter.

II. The Model Definition and Examples

The SCOD model introduced below naturally extends the multidimensional HK model introduced in [9].

1) Opinions: Denote the set of agents by $\mathcal{V}$ and their number by[1] $n = |\mathcal{V}|$. At period $t = 0, 1, \ldots,$ agent $i \in \mathcal{V}$ holds an opinion vector $\xi^i(t) \in \mathbb{R}^d$, whose element ξ^i_k stands for the agent's position on topic $k \in \{1, \ldots, d\}$. The system's state is naturally written as the $n \times d$ matrix [4], [5], [7]

$$\Xi(t) \triangleq \left(\xi^i_k(t)\right)^{i\in\mathcal{V}}_{k=1,\ldots,d}.$$

2) Confidence Graph: Each agent forms their opinions based on the "similar" opinions of their peers, with "similarity" relations defined by the *confidence set* $\mathcal{O} \subseteq \mathbb{R}^d$ and conveniently characterized by a *confidence graph* $\mathcal{G}(\Xi) = (\mathcal{V}, \ \mathcal{E}(\Xi))$. In this graph, the nodes represent the agents, and an arc $i \to j$ exists (agent i trusts agent j's opinion) if and only if $\xi^j - \xi^i \in \mathcal{O}$. Node $i \in \mathcal{V}$ has the set of (out-)neighbors

$$\mathcal{N}_i(\Xi) \triangleq \left\{j \in \mathcal{V} : \xi^j \in \xi^i + \mathcal{O}\right\}. \quad (1)$$

We adopt the following assumption, entailing that $i \in \mathcal{N}_i(\Xi)\, \forall i \in \mathcal{V}$ (i.e., each node has a self-loop).

Assumption 1 (Self-Confidence): $\mathbf{0} \in \mathcal{O}$.

3) The SCOD (Opinion Update Rule): The mechanism of opinion evolution is same as in the HK Model. The opinion of agent i is formed by averaging the trusted opinions,

$$\xi^i(t+1) = \frac{1}{|\mathcal{N}_i(\Xi(t))|} \sum_{j\in\mathcal{N}_i(\Xi(t))} \xi^j(t), \quad i \in \mathcal{V}. \quad (2)$$

4) Extension: Stubborn Agents: The SCOD model can be generalized to include *stubborn agents* whose opinions always remain unchanged. The SCOD with a set of stubborn individuals $\mathcal{V}_s \subset \mathcal{V}$ and set of *ordinary* agents $\mathcal{V} \setminus \mathcal{V}_s$ is the system (2), where $\mathcal{N}_i$ for ordinary agents $i \in \mathcal{V} \setminus \mathcal{V}_s$ is defined by (1), whereas $\mathcal{N}_i(\Xi) \equiv \{i\} \quad \forall i \in \mathcal{V}_s$.

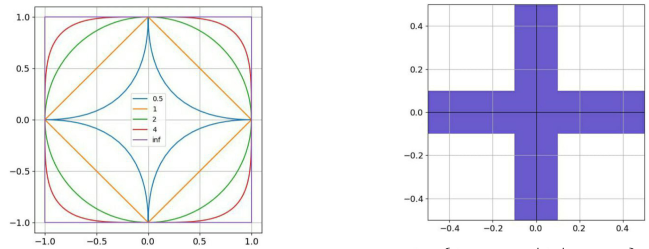
A. The SCOD vs. Previously Known Models

In the standard HK model the opinions are scalar ($d = 1$), and $\mathcal{O} = (-R, R)$ is an interval.[2] Later asymmetric intervals $\mathcal{O} = (-\ell, \ u)$ have been studied [20]. Multidimensional HK models are special cases of the SCOD, where $\mathcal{O}$ is a ball centered at $\mathbf{0}$ with respect to some norm or metrics [9], [10], [11]. Usually, $\mathcal{O}$ is the ℓ_p-ball (Fig. 1(a))

$$\mathcal{O} = \mathcal{O}_{p,R} \triangleq \left\{\xi \in \mathbb{R}^d : |\xi_1|^p + \ldots + |\xi_d|^p \leq R^p\right\}.$$

[1]Hereinafter, the cardinality of a set N is denoted by $|N|$.

[2]In some works [19], closed intervals $[-R, \ R]$ have also been considered.



(a) unit ℓ_p-balls

(b) $\{\xi : \min_{i=1,2} |\xi_i| \leq 0.1\}$.

Fig. 1. Examples of confidence sets.

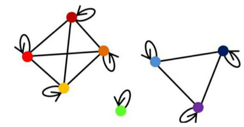


Fig. 2. A union of disconnected cliques.

Some models considered in the literature deal with unbounded confidence sets, e.g., the *averaged-based* HK model from [11] is a special case of (2) with $\mathcal{O} = \{\xi \in \mathbb{R}^d : |\xi_1 + \ldots + \xi_d| \leq R\}$ being a "stripe" between two hyperplanes.

Another interesting example is inspired by a more sophisticated dynamical model from [21]. One may suppose that an agent with opinion vector ξ can be influenced by another individual with opinion ξ' if their positions $\xi_k, \ \xi'_k$ on *some* topic $k \in \{1, \ldots, d\}$ are close: $\mathcal{O} = \{\xi : |\xi_l| \leq \varepsilon_l$ for some $k = 1, \ldots, d\}$. Fig. 1(b) demonstrates this set for the special case of $d = 2$ and $\varepsilon_1 = \varepsilon_2 = 0.1$.

B. Gallery of Untypical Behaviors

Before analyzing the general behavior of the SCOD system, we consider small-scale examples showing that with a general confidence set $\mathcal{O}$, it can behave very differently from standard HK models, where $\mathcal{O} = \{\xi : \|\xi\| \leq R\}$ is a ball. Namely, in the HK model (a) all solutions converge to equilibrium points in finite time, and (b) the agents split into clusters: those within a cluster reach consensus, while those in different clusters do not trust each other [22]. None of these properties are generally valid for the SCOD model.

1) Non-Clustered Equilibria: The SCOD model can have equilibria, which are absent in the HK model.

Definition 1: Opinion matrix (the system state) Ξ is *clustered* if for all $i, \ j \in \mathcal{V}$ either $\xi^i = \xi^j$ or $\xi^j - \xi^i \notin \mathcal{O}$.

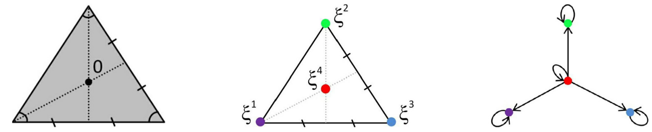
A clustered matrix Ξ is an equilibrium of the SCOD (2), and the graph $\mathcal{G}(\Xi)$ is a union of disjoint complete graphs, or *cliques* (Fig. 2). Unlike the HK model with norm-based confidence, SCOD systems admit *non-clustered* equilibria.

Example 1: Choosing $\mathcal{O}$ as an equilateral triangle centered at the origin (Fig. 3(a)) and choosing the opinions of $n = 4$ agents as shown in Fig. 3(b), one gets an equilibrium of the SCOD that is not clustered as the strongly connected components of $\mathcal{G}(\Xi)$ are not disconnected (Fig. 3(c)).

2) Periodic Solutions: We next show that small-size SCOD systems can exhibit periodic solutions.

Example 2: Consider $n = 3$ agents and the confidence set

$$\mathcal{O} = (-7, 7) \setminus \mathcal{M}, \ \mathcal{M} = \{\pm 1, \pm 3, \pm 5, -4, -2, \ 6\} \quad (3)$$



(a) set $\mathcal{O}$ (b) opinions (c) graph $\mathcal{G}(\Xi)$

Fig. 3. Non-clustered equilibrium of the SCOD.

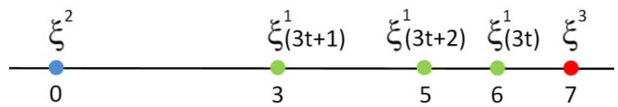
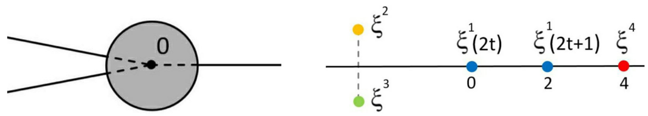


Fig. 4. Example 2. ξ^1 oscillates with period 3.



(a) set $\mathcal{O}$ (b) $n = 4$ opinions

Fig. 5. Example 3. (a) the confidence set; (b) opinions.

Then, the system (2) has a periodic solution with $\xi^2 \equiv 0$, $\xi^3 \equiv 7$ (their sets of neighbors $\mathcal{N}_2 \equiv \{2\}$, $\mathcal{N}_3 \equiv \{3\}$ are constant) and $\xi^1(t)$, $\mathcal{N}_1(t)$ switching with period 3 (Fig. 4):

$$6 \xrightarrow[\mathcal{N}_1=\{1,2\}]{} 3 \xrightarrow[\mathcal{N}_1=\{1,3\}]{} 5 \xrightarrow[\mathcal{N}_1=\{1,3\}]{} 6. \tag{4}$$

Remark 1: Notably, periodic solutions do not exist in the case where $\mathcal{O}$ is an interval, containing 0 [4], [20], [23]; in this case the dynamics terminate in time polynomially depending on n. Recent works [24], [25], focused on achieving of practical consensus under homophily and heterophily effects, also prove convergence in presence of a "deadzone" around 0, in which case $\mathcal{O} = (-\ell, -\varepsilon) \cup \{0\} \cup (\varepsilon,\ u)$.

Our next example demonstrates that, when dealing with multidimensional opinions, periodic solutions are possible even with a confidence set being *star-shaped* at $\mathbf{0}$.

Definition 2: Set $\mathcal{O}$ is *star-shaped* at point ξ^* if $[\xi^*,\ x] \triangleq \{a\xi^* + (1-a)x : a \in [0, 1]\} \subseteq \mathcal{O}$ for any $x \in \mathcal{O}$. For instance, a convex set is star-shaped at any of its points.

If $\mathcal{O}$ is star-shaped at $\mathbf{0}$, then the following natural property holds: If an agent with opinion ξ trusts another opinion ξ', they trust all "intermediate" opinions from the interval $[\xi,\ \xi']$.

Example 3: Consider a confidence set $\mathcal{O} \subset \mathbb{R}^2$ which is a union of rays $\{\xi : \xi_1 > 0, \xi_2 = 0\}$, $\{\xi : \xi_2 = \xi_1/5 < 0\}$, $\{\xi : \xi_2 = -\xi_1/5 > 0\}$ and the unit circle (Fig. 5(a)). Then, (2) has a periodic solution (see Fig. 5(b)) with $\xi^2 \equiv (-3,\ 1)$, $\xi^3 \equiv (-3, -1)$, $\xi^4 \equiv (4,\ 0)$ and $\xi^1(t)$, $\mathcal{N}_1(t)$ switching as follows:

$$\xi^1 = (0, 0) \xrightarrow[\mathcal{N}_1=\{1,4\}]{} (2, 0) \xrightarrow[\mathcal{N}_1=\{1,2,3,4\}]{} (0, 0). \tag{5}$$

Remark 2: In the latter example, unlike in Example 2, $\mathcal{O}$ is closed, but the periodic solution remains unchanged replacing $\mathcal{O}$ by its small open neighborhood.

Revisiting Examples 1-3, an important feature is noted: the confidence set is asymmetric with respect to $\mathbf{0}$. This is not coincidental: as discussed in the next section, the symmetry ($\mathcal{O} = -\mathcal{O}$) *excludes* the possibility of diverging solutions and

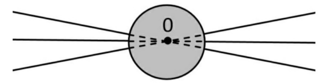


Fig. 6. Example 4: set $\mathcal{O}$.

(a) set $\mathcal{O}$ (b) opinions

Fig. 7. Example 5: (a) confidence set; (b) opinions.

non-clustered equilibria in the SCOD model without stubborn agents. However, the periodic solutions reemerge if the SCOD system with $\mathcal{O} = -\mathcal{O}$ includes stubborn agents ($\mathcal{V}_s \neq \emptyset$), as demonstrated by our next example.

Example 4: Consider a confidence set $\mathcal{O}$ which is a union of lines $\{\xi_2 = 0\}$, $\{\xi_2 = \xi_1/5\}$ and $\{\xi_2 = -\xi_1/5\}$ with the ball of unit radius (see Fig. 6). The SCOD with $n = 4$, the set of stubborn agents $\mathcal{V}_s = \{2, 3, 4\}$ and the initial opinions from Fig. 5(b) exhibits the oscillations in opinion ξ^1 as in (5). Similarly, consider $n = 3$ agents whose initial opinions are chosen as in Examples 2, but $\mathcal{O} = (-7,\ 7) \setminus \{\pm 1, \pm 3, \pm 5\}$. If agents 1, 3 are stubborn, then ξ^2 oscillates as in (4).

Remark 3: Note that in Examples 2-4, oscillations arise due to presence of static opinions, enabled by the geometry of set $\mathcal{O}$ or stubbornness of some agents. This effect, where static opinions induce oscillations, is well-known in models with randomized asynchronous interactions [26], [27]. Our examples show that the same effect occurs in the *deterministic* SCOD model with asymmetry or stubborn individuals.[3]

3) Convergent Solutions Absent in HK Models: Even if $\mathcal{O}$ is symmetric, solutions of the SCOD may converge in *infinite* time and reach *non-equilibrium* states.[4] This behavior is demonstrated by the following example.

Example 5: Let the two-dimensional confidence set be the union of two lines: $\xi_1 = 0$ and $\xi_2 = 0$ (Fig. 7(a)). The initial opinions of $n = 5$ agents are shown in Fig. 7(b): four opinions are the vertices of the square $(\pm 1, \pm 1)$, while $\xi^5 = (0,\ a)$, where $a > 1$. Evidently, ξ^5 is static, while ξ^i, $i = 1, \ldots, 4$ converge to $\mathbf{0}$. The resulting opinion profile is not an equilibrium. Removing the fifth agent, the solution converges over the *infinite time* to the null equilibrium.

III. The SCOD With a Symmetric Confidence Set

Using the theory of averaging algorithms and inequalities [29], it can be shown that for a *symmetric* confidence set

[3] Notice that the systems in Examples 2, 3 are very different from their counterparts in Example 4, although the trajectories $\Xi(t)$ for the specific initial condition are same. In the former two examples, none of agents is stubborn, although some agents remain "isolated" ($\mathcal{N}_i \equiv \{i\}$) in the sense that they do not trust to the others because of the specific geometry of set $\mathcal{O}$ and the opinion trajectory $\Xi(t)$. In the latter example, some agents are stubborn and keep constant opinions *for all possible initial conditions*.

[4] Similar behaviors are reported in continuous-time HK systems with generalized solutions [28] yet are absent in the discrete-time HK model.

$\mathcal{O} = -\mathcal{O}$ the asymptotic behaviors of the SCOD model are similar to those of conventional HK models in the absence of stubborn agents. In the HK model based on the Euclidean norm, stubborn agents do not destroy convergence [15], which, however, is not the case for the SCOD (see Example 4). Convergence can be guaranteed, however, in special situations, e.g., when only one agent is stubborn or all stubborn individuals share the same opinion.

We first introduce the three key assumptions.

Assumption 2 (Symmetric Confidence Set): $\mathcal{O} = -\mathcal{O}$.

Assumption 3 (Trust in Similar Opinions): $\mathcal{O}$ contains $\mathbf{0}$ along with a small neighborhood[5]: a radius $R > 0$ exists such that $\mathcal{O} \supseteq \{\xi : \|\xi\| < R\}$.

Assumption 2 entails that the relations of trust are reciprocal: if i trusts j, then j trusts i for each opinion matrix Ξ, in particular, graph $\mathcal{G}(\Xi)$ is undirected. Assumption 3 is a stronger form of Assumption 1, requiring the agent to trust all opinions that are sufficiently close (in the sense of usual distance) to their own. For instance, the sets in Fig. 1 and Fig. 6 satisfy Assumptions 2 and 3. The set in Fig. 5(a) satisfies Assumption 3 but violates Assumption 2, while the set in Fig. 7(a) satisfies Assumption 2 but not Assumption 3.

Assumption 4 (Homogeneous Stubborn Agents): All stubborn agents (if they exist) share the same opinion[6]:

$$\xi^i(0) = \xi^* \quad \forall i \in \mathcal{V}_s. \tag{6}$$

Main Result: Convergence and Equilibria

The following theorem examines the convergence of the SCOD trajectories $\Xi(t)$ and structures of their limits.

Theorem 1: If $\mathcal{O}$ obeys Assumptions 1, 2, and the stubborn agents obey Assumption 4, the following statements are true:

(A) $\Xi(0)$ is an equilibrium if and only if it is clustered.

(B) All opinions have finite limits $\xi^i(\infty) = \lim_{t\to\infty} \xi^i(t)$, and $\xi^i(\infty) = \xi^j(\infty)$ whenever agents i, j trust each other infinitely often: $\xi^j(t_k) - \xi^i(t_k) \in \mathcal{O}$ for a sequence $t_k \to \infty$.

If Assumption 3 also holds, then:

(C) The terminal state $\Xi(\infty)$ is a (clustered) equilibrium.

(D) If $\mathcal{V}_s = \emptyset$ (no stubborn agents), the dynamics terminate in a finite number of steps. Otherwise, every opinion $\xi^i(t)$ either converges to the stubborn agents' common opinion ξ^* from (6) or stops changing after a finite number of steps.

A. Numerical Example

The following numerical example illustrates the behavior of the SCOD with the set $\mathcal{O}$ from Fig. 1(b) for $n = 100$ agents and $\xi^* = \mathbf{0}$. The left plot in Fig. 8 demonstrates the case where $|\mathcal{V}_s| = 1$ and two clusters emerge. The right plot is for $|\mathcal{V}_s| = 50$: the group reaches consensus at $\mathbf{0}$. The opinions of regular agents are sampled uniformly from $[-1,\ 1]^2$.

One may notice that the convergence to the stubborn opinion is quite slow; the estimate of the convergence rate in the SCOD models remains a non-trivial open problem.

B. Discussion

The assumptions of Theorem 1, while formally only sufficient, are essential and cannot be readily discarded.

[5]Since all norms on $\mathbb{R}^d$ are equivalent, the norm here is unimportant.

[6]We assume that (6) holds automatically if $\mathcal{V}_s = \emptyset$.

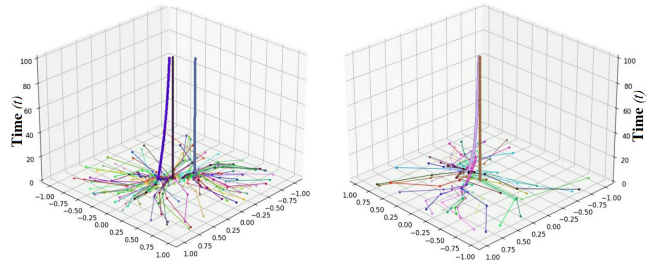


Fig. 8. The SCOD with 1 (left) and 50 (right) stubborn agents.

Assumption 1, besides making (2) well-defined ($|\mathcal{N}_i| \neq \emptyset$), also excludes trivial periodicity. For instance, if $\mathcal{O} = \mathbb{R}^d \setminus \{\mathbf{0}\}$, the trivial SCOD dynamics $\xi^1(t+1) = \xi^2(t)$, $\xi^2(t+1) = \xi^1(t)$ violates (B). Furthermore, every pair of different opinions is a clustered yet a non-equilibrium state, so (A) also fails to hold.

Discarding *Assumption 2* can result in oscillatory solutions (Examples 2 and 3). Example 1 shows that non-clustered equilibria may exist. Even in the absence of stubborn agents, both (A) and (B) may be violated without the symmetry of $\mathcal{O}$.

Assumption 4 also cannot be fully discarded, as shown by Example 4: two stubborn agents with different opinions can lead to periodic solutions, even if $\mathcal{O}$ obeys Assumptions 1-3.

As Example 5 shows, (C) and (D) may be violated without *Assumption 3*, and $\Xi(\infty)$ may fail to be non-equilibrium.

Note that HK models with stubborn agents have mostly been studied for scalar opinions [4], except in [14], [15], where confidence sets are Euclidean balls. These results, estimating the system's "kinetic energy", have not been extended to arbitrary norms. Hence, to the best of our knowledge, Theorem 1 is new and non-trivial even for the norm-based HK models.

IV. Proof of Theorem 1

We will use the following lemma on the convergence of recurrent averaging inequalities [29, Th. 5]

$$x(t+1) \leq W(t)x(t), \quad t = 0, 1 \ldots, \tag{7}$$

where $x(t)$ are m-dimensional column vectors, $W(t)$ are *row-stochastic* $m \times m$ matrices and the inequality is elementwise.

Lemma 1: Let matrices $W(t)$ be *type-symmetric*, that is, for some constant $K \geq 1$ one has $K^{-1}w_{ji}(t) \leq w_{ij}(t) \leq Kw_{ji}(t)$ for all pairs $i \neq j$ and all $t = 0, 1, \ldots$ Assume also that the diagonal entries are uniformly positive: $w_{ii}(t) \geq \delta > 0$ for all i and $t \geq 0$. Then, any solution $x(t)$ of (7) that is bounded from below enjoys the following properties:

(a) a finite limit $x(\infty) \stackrel{\Delta}{=} \lim_{t\to\infty} x(t)$ exists;

(b) $x_i(\infty) = x_j(\infty)$ for all pairs of agents i, j that interact *persistently*, that is, $\sum_{t=0}^{\infty} w_{ij}(t) = \infty$;

(c) the residuals $\Delta(t) \stackrel{\Delta}{=} W(t)x(t) - x(t+1)$ are ℓ_1-summable, that is, $\sum_{t=0}^{\infty} \Delta(t) < \infty$.

Remark 4: Lemma 1 is well-known for averaging consensus algorithms $x(t+1) = W(t)x(t)$, whose trajectories are always bounded from below and satisfy (7). Under the assumptions of Lemma 1, the consensus dynamics thus enjoys properties (a) and (b), with (c) being trivial. This statement, in a more general setting, appeared in [30, Th. 1], while its special case dates back to the seminal paper [31].

A. Case I: No Stubborn Agents

Henceforth Assumptions 1 and 2 are supposed to be valid.

We first prove Theorem 1 in the case where $\mathcal{V}_s = \emptyset$. The proof retraces one for the usual HK model [22]. For a fixed solution $\Xi(t)$, the SCOD dynamics (2) entails that

$$\xi^i(t+1) = \sum_{j\in\mathcal{V}} \bar{w}_{ij}(t)\xi^j(t), \tag{8}$$

where matrices $\bar{W}(t) = (\bar{w}_{ij}(t))$ are determined by

$$\bar{w}_{ij}(t) \triangleq \begin{cases} \frac{1}{|\mathcal{N}_i(t)|}, & \text{if } j \in \mathcal{N}_i(t) \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

and satisfy the assumptions of Lemma 1 thanks to Assumptions 1 and 2. Furthermore, $\bar{w}_{ij}(t) \in \{0\} \cup [1/n, \infty)$, and hence i, j trust each other infinitely often if and only if

$$\sum_{t=0}^{\infty} \bar{w}_{ij}(t) = \infty. \tag{10}$$

To prove (B), fix a coordinate $k \in \{1, \dots, d\}$. The vectors $x(t) \triangleq (\xi_k^1(t), \dots, \xi_k^n(t))^\top$ obey the consensus dynamics $x(t+1) = \bar{W}(t)x(t)$. Thus, the limits $x(\infty) = \lim_{t\to\infty} x(t)$ exist, and $x_i(\infty) = x_j(\infty)$ if (10) holds by virtue of Lemma 1. Applying this for all k, statement (B) follows.

To prove (A), notice that for an equilibrium $\Xi(t) \equiv \Xi$ the respective matrix $\bar{W}(t) \equiv \bar{W}$ is also constant. If two agents i, j trust each other at the state Ξ, then $\xi^i = \xi^j$ in view of (B). This implies that every equilibrium is clustered (Definition 1): agents i, j cannot trust each other unless their opinions coincide. Trivially, clustered states are equilibria.

Assume now that Assumption 3 additionally holds. We will prove that the SCOD terminate in a finite number of steps, which implies both (C) and (D). Notice first that $\mathcal{N}_i(t) \triangleq \mathcal{N}_i(\Xi(t)) = \{j : \xi^j(\infty) = \xi^i(\infty)\}$ for t being large. Indeed, if $\lim_{t\to\infty} \xi^i(t) = \lim_{t\to\infty} \xi^j(t)$, then $\|\xi^i(t) - \xi^j(t)\| < R$ for t being large, where R is the radius from Assumption 3, whence $\xi^j(t) - \xi^i(t) \in \mathcal{O}$. On the other hand, we know that if $\xi^j(\infty) \neq \xi^i(\infty)$, then $j \notin \mathcal{N}_i(t)$ starting from some step $t = t_{ij}$. Hence, in a finite number of steps the graph $\mathcal{G}(\Xi(t))$ splits into several disconnected cliques (Fig. 2) and stops changing. In view of (2), at the next step the agents in each clique reach consensus, arriving at an equilibrium. This finishes the proof.

B. Case II: Stubborn Agent are Present

The core idea of the proof is to demonstrate that the set of ordinary agents, denoted $\mathcal{V}' \triangleq \mathcal{V} \setminus \mathcal{V}_s$, can be divided into two (potentially, empty) groups. The first group, I, consists of agents influenced by the stubborn individuals, directly or indirectly; as a result, their opinions eventually converge to ξ^*. The second group, J, comprises agents whose dependence on stubborn agents and those in I ceases at some time instant t_*. After this moment, their evolution follows a reduced-order SCOD model, which was analyzed in the previous step. To formalize this approach, we introduce an auxiliary recurrent averaging inequality (7) and examine it using Lemma 1.

Without loss of generality, we assume that $\mathcal{V}' = \{1, \dots, m\}$, where agents $\mathcal{V}_s = \{m+1, \dots, n\}$. For each regular agent, denote $x_i(t) \triangleq \|\xi^i(t) - \xi^*\|$, where $\|\cdot\|$ is some norm on $\mathbb{R}^d$.

Step 1 - Recurrent Averaging Inequality: Note that vectors $x(t) = (x_1(t), \dots, x_m(t))^\top \in \mathbb{R}^m$ satisfy inequality (7), where the $m \times m$ stochastic matrices $W(t)$ are as follows

$$w_{ij}(t) \triangleq \begin{cases} \bar{w}_{ij}(t), & i, j \in \mathcal{V}',\ i \neq j, \\ \bar{w}_{ii}(t) + \sum_{\ell\in\mathcal{V}_s} \bar{w}_{i\ell}(t), & i = j \in \mathcal{V}', \end{cases} \tag{11}$$

where $\bar{w}_{ij}(t)$ are defined in (9). Indeed, using (8), the relation $\sum_{j\in\mathcal{V}} \bar{w}_{ij}(t) = 1$ and the norm's convexity, one arrives at

$$\begin{aligned} x_i(t+1) &\overset{(8)}{=} \left\| \sum_{j\in\mathcal{V}} \bar{w}_{ij}(t)\big(\xi^j(t) - \xi^*\big) \right\| \leq \\ &\leq \sum_{j\in\mathcal{V}'} \bar{w}_{ij}(t) \underbrace{\left\|\xi^j(t) - \xi^*\right\|}_{=x_j(t)} + \sum_{j\in\mathcal{V}_s} \bar{w}_{ij}(t) \underbrace{\left\|\xi^j(t) - \xi^*\right\|}_{=0} \\ &= \sum_{j\in\mathcal{V}'} \bar{w}_{ij}(t)x_j(t) \leq \sum_{j\in\mathcal{V}'} w_{ij}(t)x_j(t) \qquad \forall i \in \mathcal{V}'. \end{aligned}$$

Here we used the fact that $\xi^j(t) \equiv \xi^j(0) = \xi^*$ for each stubborn agent $j \in \mathcal{V}_s$ and inequalities $\bar{w}_{ij}(t) \leq w_{ij}(t)$, which hold due to (11). Hence, the residuals can be estimated as

$$\begin{aligned} \Delta_i\ (t) &\triangleq \sum_{j\in\mathcal{V}'} w_{ij}(t)x_j(t) - x_i(t+1) \geq \\ &\geq \sum_{j\in\mathcal{V}'} \big(w_{ij}(t) - \bar{w}_{ij}(t)\big)x_j(t) \overset{(11)}{=} x_i(t) \sum_{\ell\in\mathcal{V}_s} \bar{w}_{i\ell}(t). \end{aligned} \tag{12}$$

Matrices (11) satisfy the conditions of Lemma 1 thanks to Assumptions 1 and 2. In view of Lemma 1, the limit exists $x_i(\infty) \triangleq \lim_{t\to\infty} x_i(t)$ for each ordinary agent $i \in \mathcal{V}'$. According to Lemma 1, $x_i(\infty) = x_j(\infty)$ whenever (10) holds, that is, two agents $i, j \in \mathcal{V}'$ trust each other infinitely often.

Step 2 - The Group Splitting and Reduction to Case I: We now introduce two group of agents denoted by $I \triangleq \{i \in \mathcal{V}' : x_i(\infty) = 0\}$ (i.e., $\xi^i(t) \xrightarrow[t\to\infty]{} \xi^*$ for $i \in I$) and $J \triangleq \mathcal{V}' \setminus I$.

Two agents $i \in I$ and $j \in J$ don't trust each other ($\bar{w}_{ij}(t) = 0$) for t being large, because $x_j(\infty) \neq 0 = x_i(\infty)$ for all $i \in I$ and $j \in J$. Using statement (c) in Lemma 1 and (12), one proves that an agent $j \in J$ does not trust stubborn agents ($\bar{w}_{j\ell}(t) = 0\, \forall \ell \in \mathcal{V}_s$) for t being large. In other words, for large t we have $\mathcal{N}_j(t) \subseteq J$ for all $j \in J$. If J is non-empty, the opinions $\tilde{\Xi}(t) \triangleq (\xi^j(t))_{j\in J}$ evolve *independently* of the remaining group, following a SCOD model of size $|J|$.

Statements (B)-(D) now reduce to Case I. Since the opinions of agents in both sets I, J converge, (B) is valid. If Assumption 3 holds, $\tilde{\Xi}(t) = \tilde{\Xi}(\infty)$ for t being large, whereas the opinions of remaining agents from $I \cup \mathcal{V}_s$ converge to ξ^*. This proves (D). To prove (C), recall that $\xi^* - \xi^j(\infty) = \xi^* - \xi^j(t) \notin \mathcal{O}$ for $j \in J$ and t being large and, according to Case I, $\tilde{\Xi}(\infty)$ is clustered under Assumption 3. Hence, $\Xi(\infty)$ is also clustered, containing clusters of $\tilde{\Xi}(\infty)$ and the cluster constituted by agents from $I \cup \mathcal{V}_s$ with the final opinion ξ^*.

To prove (A), consider an equilibrium solution $\Xi(t) \equiv \Xi = \lim_{t\to\infty} \Xi(t)$ and the corresponding sets I, J. As has been shown, $\xi^i = \lim_{t\to\infty} \xi^i(t) = \xi^*$ for $i \in I \cup \mathcal{V}_s$, agents from J do not trust to agents from I, and $\tilde{\Xi} = (\xi^j)_{j\in J}$ is an equilibrium of the reduced-order SCOD model, proved to be clustered (Case I). Hence, every equilibrium of the SCOD is clustered. The inverse statement is straightforward.

V. Conclusion and Open Problems

This letter extends the multidimensional HK model by replacing the distance-based opinion rejection mechanism with a general set-based mechanism. We analyze the resulting SCOD model, highlighting its similarities and differences with the usual HK model, and show that some properties of the HK model, such as finite-time convergence and equilibrium structure, extend to a symmetric confidence set containing 0 in its interior. However, this behavior can be disrupted by *stubborn* individuals, whose presence may lead to periodic oscillations in the opinions. Examples in Section II illustrate that for *asymmetric* confidence set the SCOD model behaves differently from usual HK models, exhibiting convergence to non-equilibrium points and oscillations.

Finally, we mention several directions for future research.

Stubborn Agents and Oscillations: While Assumption 4 cannot be fully discarded, it seems to be only sufficient for SCOD convergence. A natural question arises: when do stubborn agents give rise to oscillating trajectories?

Convergence Rate: A limitation of the averaging inequalities method [29] is the absence of explicit estimates on the convergence time or rate of the solutions. Simulations with various sets $\mathcal{O}$ suggest the conjecture that, under Assumptions 1-3 and $\mathcal{V}_s = \emptyset$, the termination time depends polynomially on n.

Heterogeneity Effects and Oscillatory Solution Existence: A natural extension of the SCOD model is the *heterogeneous* SCOD, where each agent has its own confidence set $\mathcal{O}_i$. Heterogeneous SCOD can have periodic solutions even if all $\mathcal{O}_i$ are open and symmetric.[7] On the other hand, heterogeneous HK models with balls of different radii are believed to converge [22], [32], although a formal proof seems to be unavailable. This raises a natural question: under which assumptions on $\mathcal{O}_i$ does the heterogeneous SCOD model have periodic and other oscillatory solutions?

[7] The case $n = 3$ in Example 4 can be modified by replacing stubborn agents with agents having small confidence intervals $\mathcal{O}_i = (-\varepsilon,\ \varepsilon)$.

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