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# Fractionalization of Forchheimer’s correction to Darcy’s law in porous media in large deformations

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## Abstract

This work presents a theoretical and numerical study of the flow of the interstitial fluid saturating a porous medium, principally aimed at modeling a bio-mimetic material and assumed to experience a dynamic regime different from the Darcian one, as is typically hypothesized in biomechanical scenarios. The main aspect of our research is the conjecture according to which, for a particular mechanical state of the porous medium, the fluid exhibits two types of deviation from Darcy’s law. One is due to the inertial forces characterizing the pore scale dynamics of the fluid. This aspect can be resolved by turning to the *Forchheimer correction* to Darcy’s law, which introduces non-linearities in the relationship between the fluid filtration velocity and the dissipative forces describing the interactions between the fluid and the solid matrix. The second source of discrepancies from classical Darcy’s law emerges, for example, when pore scale disturbances to the flow, such as obstructions of the fluid path or clogging of the pores, result in a time delay between drag force and filtration velocity. Recently, models have been proposed in which such delay is described through constitutive laws featuring fractional operators. Whereas, to the best of our knowledge, the above-mentioned behaviors have been studied separately or in small deformations, we present a model of fluid flow in a deformable porous medium undergoing large deformations in which the fluid motion obeys a fractionalized Forchheimer’s correction. After reviewing Forchheimer’s formulation, we present a fractionalization of the Darcy-Forchheimer law, and we explain the numerical procedure adopted to solve the highly non-linear boundary value problem resulting from the presence of the two considered deviations from the Darcian regime. We complete our study by highlighting the way in which the fractional order of the model tunes the magnitude of the pore pressure and fluid filtration velocity.

**Keywords:** Flow in deformable porous media; Darcy’s law; Forchheimer’s correction; Media with memory; Integro-differential constitutive equations; Fractional Calculus; Fractional integrals and derivatives.

## 1 Introduction

According to a rather consolidated modeling picture in the biomechanical literature[1], a biological tissue classified as *soft* and *hydrated* is regarded, at least, as a biphasic medium[2], constituted by a sufficiently

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40 compliant solid porous matrix and a fluid that participates in a variety of biophysical, biochemical, and  
41 mechanical processes, all essential for sustaining the tissue itself[2, 3, 4, 1, 5].

42 The characterization of the mechanical properties of the solid matrix of soft tissues, be they hydrated  
43 or not, has been the subject of several studies with increasing level of complexity: whereas the first, pio-  
44 neering models looked at the essence of phenomenology, and, for their purposes, considered tissues (see,  
45 e.g.,[2] for articular cartilage) homogeneous and isotropic, more recent works studied the consequences  
46 of inhomogeneity and anisotropy, especially in connection with the presence of reinforcing collagen  
47 fibers[6, 7, 8, 9, 10, 11], often assumed to be statistically oriented[12, 13, 14, 15, 16, 17, 18].

48 Collagen fibers represent a very important chapter in the mechanical and hydraulic analysis of  
49 biological tissues. Indeed, besides exerting a structural action that contributes to the overall mechanical  
50 response of a given tissue, they influence considerably also the tendency of the tissue to enhance, or to  
51 inhibit, the circulation of fluid in its interior. At the macroscale, this property is referred to as *permeability*.  
52 For example, in the case of articular cartilage, Maroudas and Bullough[6] have hypothesized that the  
53 tissue's permeability depends on the distribution and orientation of the collagen fibers. Subsequent studies  
54 in this direction, conceived to examine Maroudas and Bullough's hypothesis[6] have been conducted, e.g.,  
55 in[19, 20], and set themselves in a line of research dedicated to the theoretical and numerical modeling  
56 of the biomechanics of fiber-reinforced, anisotropic tissues[8, 21, 22, 23, 24, 25, 26, 17, 27, 18].

57 To the authors' knowledge, since Holmes and Mow's permeability model[2] for articular cartilage,  
58 the explicit coupling between this transport property and the tissue's deformation has been a leading  
59 topic in many other publications on the subject (see, e.g.,[21, 22, 24, 20]). In all these works, emphasis is  
60 put on the importance of understanding how the mechanics of the tissue combines with its permeability  
61 in order to provide acceptable descriptions of the fluid's behavior, especially in terms of its mechanical  
62 state. This is motivated by the fact that being able to predict, for example, the fluid pressure allows to  
63 estimate possible remarkable aspects of a tissue, like its global health[28, 7, 29].

64 Rather typical approaches having the purpose of studying the mechanics of soft and hydrated tissues,  
65 like articular cartilage, and, above all, of giving prominence to the coupling discussed above, are based  
66 on several formulations of poro-elasticity, in terms either of Biot's or of biphasic theory [1, 22, 19, 24,  
67 20, 17, 30, 31]. A common feature of the majority of these approaches is that they rely on the hypothesis  
68 that the flow of the fluid obeys Darcy's law (see, e.g.,[32, 33, 34, 1, 22]), thereby presuming, in the most  
69 classical formulation, a linear relationship between the fluid filtration velocity and the pressure gradient  
70 realized in the tissue. More precisely, the filtration velocity is obtained by multiplying the medium  
71 permeability (which, in general, is a second-order tensor field) by the opposite of the pressure gradient.  
72 The resulting flow model has the advantage of being computationally cheap, because of the linearity of  
73 this relationship with respect to the pressure gradient, and it is sufficient to capture the coupling between  
74 flow and deformation through a suitable definition of the permeability tensor (see, e.g., [2]). In fact,  
75 this coupling is also capable of considering nonlinear deformations. In spite of this capability, however,  
76 in the literature there have also been attempts to elaborate flow models that account for non-Darcian  
77 behaviors of the fluid, like, for instance, those predicted by Forchheimer or Brinkman's corrections to  
78 Darcy's law (see, e.g.,[35, 36, 30]).

79 In the context of articular cartilage, in[36, 30], the authors have hypothesized that, under certain  
80 loading conditions, as could be the case in compression tests in which the load is applied with a relatively  
81 high velocity, the mechanical behavior of the fluid is better approximated by the Darcy-Forchheimer  
82 model of the flow. In fact, adopting Forchheimer's correction means accounting for non-linear terms in  
83 the constitutive relationships between the filtration velocity and the drag force that may generally result  
84 in slower flows and higher pressures than those predicted by Darcy's law. This, in turn, calls for the  
85 introduction of additional parameters to describe the flow, the identification of which may depend on the  
86 structure of the porous medium[37], the model of permeability[22, 19], and the experimental procedure  
87 employed to estimate the numerical values of the quantities at hand. In addition, it has been shown  
88 in[30] that resorting to the Darcy-Forchheimer law may be used to switch from a model of permeability  
89 to another one by attributing the resulting variations in the behavior of the fluid to the correction of the  
90 flow rather than to different assumptions on the permeability.

91 Another phenomenon that is not accounted for in the “classical” formulation of the Darcy or Darcy-  
92 Forchheimer models is the anomalous “diffusion” of the fluid flow (see, e.g.,[38]). In particular, Darcy’s  
93 law has proved to be non appropriate for fluid flow in high porosity media due to the influence of inertial,  
94 thermal, and convective terms, and because of solid-fluid boundary effects that are not contemplated in  
95 Darcy’s model[39].

96 Recently, a body of work has gone into collecting experimental evidence of anomalous “diffusion”  
97 (another type of non-Darcian behavior) for different classes of porous media, from tissues, such meniscal  
98 tissue[40], to rocks and porous building materials[41]. The predominant matter is the explicit time-  
99 dependence of the permeability, which results in an explicitly time-dependent flow rate.

100 In [42], clogging of the pores and explicit time-dependence of the permeability of hydro-geological  
101 porous media are described by means of an integro-differential operator that keeps track of the time history  
102 of permeability. This study offers a very important point of departure for the introduction of Fractional  
103 Calculus in modeling flow in porous media, especially for describing deviations from Darcian transport,  
104 as is the case for subdiffusion or superdiffusion processes, both observable experimentally[38, 43].

105 Confined compression tests in meniscal tissues have shown that anomalous transport phenomena are  
106 well captured by fractional poroelastic models (e.g., of Biot-type) in which the pore pressure diffusion  
107 equation results from a modified version of Darcy’s law involving fractional derivatives[40, 44, 45]. The  
108 diffusion is then anomalous and the order of the fractional derivatives involved in the models rules the  
109 fluid flow. Fittings of experimental data proved to be better than adopting classical Biot or biphasic  
110 models, and the fractional poroelastic model has been —for the first time— validated [44]. The values  
111 of the fractional order, obtained by fitting the experimental data across various samples, vary in the range  
112 [0.51, 0.73][44], and are considerably higher than the ones found by Bulle et al.[40]. Hence, the memory  
113 effects related to the fluid flow in the meniscus seem to be, in general, not negligible. In addition, since  
114 the model in[44] is transversely isotropic for both the fractional and the standard case, the fractional  
115 derivative captures the deviation from classical Darcy’s law triggered by the presence of a hierarchical  
116 network of collagen channels in the meniscus, which is not inherently related to the anisotropy of the  
117 medium. By our understanding, this result shows that biological tissues can exhibit a behavior well  
118 captured by the methods of Fractional Calculus, and related to the tortuosity of the pores. Moreover, the  
119 model developed in[44] accounts only for small deformations. In this sense, one of the aims of our work  
120 is to provide the mathematical infrastructure to deal with temporal fractional derivatives within a large  
121 deformation setting, while complying with the principle of objectivity.

122 Other studies[46, 47] highlight the role of poroelasticity in the anomalous “diffusion” processes  
123 that can be observed on meniscus samples. In the literature, some investigations have been done to  
124 capture the relationship between the memory effects of the flow of interstitial fluid, which are due to the  
125 interactions between the fluid and the pore network, and the behavior of the solid phase. In particular,  
126 in[31] a fractional Darcy’s law was studied in the setting of small elastic strains, while in[48] classical  
127 Darcy’s law was coupled with a solid phase experiencing “*material hereditariness*”[49, 50, 51], i.e.,  
128 dependence of the stress on the past history of strain, which was described by means of a fractional-order  
129 “*hereditariness*” model[52, 53, 54, 55].

130 With respect to the review of literature done above, the novelty of our work resides in the search for  
131 memory effects associated with a *fractional Darcy-Forchheimer* flow model developed in the framework  
132 of *finite deformations*. After presenting the constitutive theory on which our study relies, we simulate  
133 an unconfined compression test, performed over a cylindrical specimen of a hypothetical tissue that  
134 has “borrowed” some properties from articular cartilage[7, 9, 29, 56, 20], but is assumed here to be  
135 homogeneous and isotropic. Note also that we speak of a “hypothetical tissue” because, for the time  
136 being, we do not have experimental values for the parameters defining the fractional operators adopted  
137 in the sequel. We take the elastic coefficients of articular cartilage because of the studies available in  
138 the literature that address explicitly memory effects in this tissue and employ Fractional Calculus (see  
139 e.g.[57, 58], although the framework established therein is very different from ours). In addition, we  
140 select the unconfined compression test since this is a rather standard experimental set-up and is able to  
141 provide information in quite a simple manner about the relationship between specimen deformation and

142 fluid flow.

143 As remarked above, the assumption of homogeneity and isotropy is not realistic, but we need it now  
144 in order to focus on the novelties of our model without having to deal with other sources of computational  
145 complications. However, a generalization to an inhomogeneous and anisotropic medium, with statistical  
146 orientation of reinforcing collagen fibers, is not too demanding from the modeling point of view, since the  
147 literature in the field is quite rich[8, 21, 22, 23, 24, 25, 26, 17, 27, 18], although it necessarily increases  
148 the computational burden.

149 Before proceeding, we clarify that, at the moment, we *are not* aiming at reproducing any experiment  
150 conducted on a real medium, be it a bio-mimetic scaffold or a true biological tissue. Rather, we are  
151 presenting a study that is meant to indicate, through numerical simulations, new research directions  
152 in the field of Fractional Calculus applied to Biomechanics. In this sense, the numerical simulations  
153 presented in the sequel may provide guidance in devising experimental procedures that aim at quantifying  
154 the presence of possible memory effects in the flow of the interstitial fluid of the medium under study.  
155 The model and the associated simulations, in fact, should act like a magnifying glass on the internal  
156 mechanics of the medium and of the memory effects (non-local in time) taking place in it. We believe  
157 that such information could aid in designing experiments.

158 Our principal results are: (i) the formulation of a fractional constitutive equation that expresses the  
159 dissipative drag force stemming from the fluid-solid interactions as a functional of the fluid filtration  
160 velocity; and (ii) the numerical procedure developed to solve this equation together with the momentum  
161 and mass balance laws characterizing the fractional Darcy-Forchheimer model in finite deformations.  
162 The main outcomes of our simulations predict the influence of the fractionalization of Forchheimer’s  
163 correction on pore pressure and on the magnitude of the fluid filtration velocity.

## 164 2 Kinematics of biphasic mixtures

165 In this section, we briefly present the kinematics of solid-fluid mixtures in the framework delineated in  
166 [59, 60]. The solid and the fluid phase are represented by two smooth material manifolds  $\mathcal{M}_s$  and  $\mathcal{M}_f$ ,  
167 and the embedding of the solid phase in the three dimensional Euclidean space  $\mathcal{S}$  is called *reference*  
168 *placement* of the solid phase  $\mathcal{B} \subset \mathcal{S}$ .

169 The class of media taken as target in this work refers mainly to bio-mimetic scaffolds, to be employed,  
170 for example, in the replacement of injured tissues, as could be the case of articular cartilage, which  
171 comprises cells, extracellular matrix, and collagen fibers. Although such target media may feature  
172 complicated internal structures a simplified approach is followed in the sequel. This is done because, for  
173 a given target tissue, the focus of our work is not a detailed description of the tissue’s structure.

174 In this work, we adhere to the description of the solid phase put forward in[30]. For each instant of  
175 time  $t$  of the time window  $\mathcal{I} \subset ]0, +\infty[$  in which we keep track of the evolution of the system, the motion  
176  $\chi(\cdot, t) : \mathcal{B} \rightarrow \mathcal{S}$  of the solid phase maps the reference placement  $\mathcal{B}$  into the *current placement*  $\chi(\mathcal{B}, t)$ .  
177 This description can be extended also to the case in which the solid phase features several constituents  
178 (e.g., extracellular matrix, cells, and collagen fibers) by hypothesizing, as is often done in this context,  
179 that the “points” of  $\mathcal{M}_s$  share the same motion. Furthermore, for each  $t \in \mathcal{I}$ , the motion of the fluid  
180 is described by means of a one-parameter family of embeddings  $\mathfrak{f}(\cdot, t) : \mathcal{M}_f \rightarrow \mathcal{S}$  that attaches fluid  
181 particles  $\mathfrak{X}_f \in \mathcal{M}_f$  to a point in the Euclidean space  $\mathcal{S}$ . The portion of  $\mathcal{S}$  in which the solid and the fluid  
182 phases coexist is denoted by  $\mathcal{B}_t := \chi(\mathcal{B}, t) \cap \mathfrak{f}(\mathcal{M}_f, t)$ [59, 60] and constitutes the solid-fluid mixture.  
183 Furthermore, for each time  $t \in \mathcal{I}$ , we assume, with a slight abuse of notation, that the inverse mapping  
184 in space  $[\chi(\cdot, t)]^{-1} : \mathcal{B}_t \rightarrow \mathcal{B}$  is surjective with respect to the reference placement of the solid phase,  
185 so that for each point of the mixture there is a corresponding point in the reference placement of the solid  
186 phase (since  $\chi(\cdot, t)$  is injective but nor surjective, the inverse  $[\chi(\cdot, t)]^{-1}$  has to be understood, with an  
187 abuse of notation, as the inverse of the auxiliary map  $\hat{\chi}(\cdot, t) : \mathcal{B} \rightarrow \mathcal{B}_t$ , with  $\hat{\chi}(X, t) \equiv \chi(X, t)$ ).

188 In summary, the hydrated media considered hereafter, even with multiple solid constituents, are seen  
189 at the macroscale as mixtures with a solid component and a fluid one. In particular, following [61, 20],  
190 and under the hypothesis that the heterogeneities at the medium’s fine scale do not affect the behavior

191 of the medium at the macroscale [62], we introduce an admissible representative element [62] and the  
 192 fraction of relative volume which is occupied by the solid or by the fluid phase. These quantities are  
 193 called, respectively, the solid volumetric fraction and the fluid volumetric fraction, and are denoted by  
 194  $\phi_\alpha : \mathcal{B}_t \rightarrow ]0, 1[$ , with  $\alpha = s, f$ .

195 For each point  $x \in \mathcal{B}_t$  in the current placement and each point  $X \in \mathcal{B}$  in the reference placement,  
 196 we introduce the tangent spaces  $T_x\mathcal{S}$  and  $T_X\mathcal{B}$ , and the dual spaces  $T_x^*\mathcal{S}$  and  $T_X^*\mathcal{B}$ , respectively, as well  
 197 as the tangent bundles  $T\mathcal{S} := \bigsqcup_{x \in \mathcal{B}_t} T_x\mathcal{S}$  and  $T\mathcal{B} := \bigsqcup_{X \in \mathcal{B}} T_X\mathcal{B}$ . Similarly, we define the cotangent  
 198 bundles  $T^*\mathcal{S} := \bigsqcup_{x \in \mathcal{B}_t} T_x^*\mathcal{S}$  and  $T^*\mathcal{B} := \bigsqcup_{X \in \mathcal{B}} T_X^*\mathcal{B}$  (see e.g. [63] for further details).

199 The velocity of a solid particle passing at the time  $t$  through the spatial point  $x = \chi(X, t)$  is  
 200 denoted by  $\mathbf{v}_s(x, t) = \dot{\chi}(X, t) \in T_x\mathcal{S}$ , with the superimposed dot meaning partial differentiation with  
 201 respect to time, while the velocity of a fluid particle passing through the same spatial point  $x \in \mathcal{B}_t$  is  
 202 obtained as  $\mathbf{v}_f(x, t) = \dot{\mathbf{f}}(\mathbf{x}_f, t) \in T_x\mathcal{S}$ . The above defined velocities  $\mathbf{v}_s$  and  $\mathbf{v}_f$  are also known as *spatial*  
 203 velocities, while the relative motion of the fluid with respect to the solid phase is described by the relative  
 204 velocity as  $\mathbf{w}_{fs}(x, t) := \mathbf{v}_f(x, t) - \mathbf{v}_s(x, t)$ . For the fluid phase, we also introduce the *filtration velocity*  
 205  $\mathbf{q}(x, t) := \phi_f(x, t)\mathbf{w}_{fs}(x, t) \in T_x\mathcal{S}$ , which represents the specific mass flux vector of fluid passing through  
 206  $x \in \mathcal{B}_t$  at time  $t$  (i.e., the mass flux vector normalized by the fluid true mass density  $\rho_f$ )[33].

207 Finally, we introduce the tangent map of the motion of the solid phase,  $\mathbf{F}(X, t) = T\chi(X, t) \equiv$   
 208  $D\chi(X, t)$ , where  $D\chi$  is the Jacobian tensor associated with  $\chi$ , known as the *deformation gradient*  
 209 *tensor*  $\mathbf{F}(X, t) : T_X\mathcal{B} \rightarrow T_{\chi(X, t)}\mathcal{S}$ , which transforms vectors of  $T_X\mathcal{B}$  into vectors of  $T_x\mathcal{S}$ , with  
 210  $x = \chi(X, t)$ . In order for a motion to be admissible, the determinant of  $\mathbf{F}$  is required to satisfy the  
 211 condition  $J(X, t) := \det\mathbf{F}(X, t) > 0$ , for all  $X \in \mathcal{B}$  and  $t \in \mathcal{I}$ , so that  $\mathbf{F}$  is non-singular. Similarly, we  
 212 define the inverse, the transpose, and the inverse transpose tensors of  $\mathbf{F}$ , that is,  $\mathbf{F}^{-1}(x, t) : T_x\mathcal{S} \rightarrow T_X\mathcal{B}$ ,  
 213  $\mathbf{F}^T(x, t) : T_x^*\mathcal{S} \rightarrow T_X^*\mathcal{B}$ , and  $\mathbf{F}^{-T}(X, t) : T_X^*\mathcal{B} \rightarrow T_x^*\mathcal{S}$ , respectively, with  $X = [\chi(\cdot, t)]^{-1}(x)$ .  
 214 As usual, the Cauchy-Green deformation tensor  $\mathbf{C}(X, t) : T_X\mathcal{B} \rightarrow T_X^*\mathcal{B}$  is defined as  $\mathbf{C}(X, t) =$   
 215  $\mathbf{F}^T(x, t)\boldsymbol{\eta}(x)\mathbf{F}(X, t)$ , with  $x = \chi(X, t)$ , and  $\boldsymbol{\eta}(x) : T_x\mathcal{S} \rightarrow T_x^*\mathcal{S}$  being the spatial metric tensor attached  
 216 at the spatial point  $x \in \mathcal{S}$ [63]. When there is no room for confusion, also the less rigorous notations  
 217  $\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F} \equiv \mathbf{F}^T \boldsymbol{\eta} \mathbf{F}$  will be employed, in which the dot “ $\cdot$ ” is an abbreviation for the spatial metric tensor  
 218 field  $\boldsymbol{\eta}$ .

219 *Remark 1* (Composition of maps and notation).

220 To pass from the current placement to the reference placement, we compose any field  $\zeta : \mathcal{B}_t \times \mathcal{I} \rightarrow \mathbb{S}$ ,  
 221 valued in  $\mathbb{S} \equiv \mathbb{R}$  or in higher-order vector or tensor spaces, with  $\chi$  and the time map  $\mathfrak{T} : \mathcal{B} \times \mathcal{I} \rightarrow \mathcal{B}_t$ ,  
 222 so that it holds that  $\zeta \circ (\chi, \mathfrak{T}) : \mathcal{B} \times \mathcal{I} \rightarrow \mathbb{S}$  (the formalism is adapted from [64]).

223 In the sequel, in order to lower the notational complexity, we will omit the explicit composition  
 224 with  $(\chi, \mathfrak{T})$  and drop the dependency on  $X$  and  $t$  whenever the reported expressions are already well-  
 225 established in the literature. For example, the material volumetric fraction of the  $\alpha$ -th phase, which is  
 226 defined as  $\Phi_\alpha(X, t) := J(X, t)\phi_\alpha(\chi(X, t), t)$ , will be simply indicated as  $\Phi_\alpha := J\phi_\alpha$ .

### 227 3 Fundamental balance equations

228 In this section, we recall the fundamental balance equations for the modeling problem at hand, i.e., the  
 229 balance of mass and the balance of linear momentum for the solid and for the fluid phase.

230 Our target material, assumed to represent, e.g., a bio-mimetic tissue, is viewed as a solid-fluid mixture,  
 231 in which the solid phase comprises all the solid constituents of the tissue (in the present framework, these  
 232 could be identified with the extracellular matrix and the structural components of the cells), while the  
 233 fluid phase accounts for the interstitial fluid that flows through the pores.

234 As is often the case in the biomechanical modeling of soft hydrated tissues[1, 22, 20, 36, 30], both  
 235 the solid and the fluid phase are regarded as incompressible (more specifically, we will assume that  
 236 their true mass densities are constant), and their presence in the mixture under study is measured by  
 237 their *volumetric fractions*, denoted by  $\phi_s$  and  $\phi_f$ , respectively. Through these quantities, we define the  
 238 *apparent mass densities*  $\rho_s\phi_s$  and  $\rho_f\phi_f$ , with  $\rho_s$  and  $\rho_f$  being the constant mass densities of the solid and  
 239 the fluid. Since the mixture considered in our work is saturated, the condition  $\phi_s + \phi_f = 1$  applies.

240 By composing the mass balances for the fluid and solid phase in the current placement (see Remark  
 241 1 and Appendix A) with the pair of maps  $(\chi, \mathfrak{L}) : \mathcal{B} \times \mathcal{I} \rightarrow \mathcal{S} \times \mathcal{I}$ , the mass balance laws for the solid  
 242 phase and for the mixture as a whole can be written with respect to the reference placement as

$$\dot{\Phi}_s = 0, \quad \text{in } \mathcal{B}, \quad (1a)$$

$$J + \text{Div} \mathbf{Q} = 0, \quad \text{in } \mathcal{B}, \quad (1b)$$

243 where, with a slight abuse of notation,  $\Phi_s := J\phi_s$  and  $\mathbf{Q} := J\mathbf{F}^{-1}\mathbf{q}$  are the solid phase *material volumetric*  
 244 *fraction* and the *material filtration velocity* representing the pull-backs of  $\phi_s$  and  $\mathbf{q}$ , respectively [5, 20, 30].  
 245 It follows from Equation (1a) that  $\Phi_s$  is constant in time. Hence, we set  $\Phi_s(X, t) \equiv \Phi_{sR}(X)$  (see Appendix  
 246 1).

247 We emphasize that, in spite of the terminology ‘‘filtration velocity’’,  $\mathbf{q}$  is *not* a true velocity. Rather,  
 248 it is a *specific mass flux vector*, i.e., a mass flux vector defined by the multiplication of the velocity of  
 249 the fluid relative to the solid, i.e.,  $\mathbf{w}_{fs}$ , by the volumetric fraction of the fluid  $\phi_f$ . As remarked in [65],  
 250 this is an important clarification, since it predicts how  $\mathbf{q}$  transforms. Indeed, since  $\mathbf{q}$  is a flux vector, it  
 251 has to be identified with a *pseudo-vector* and, as such, its material counterpart, obtained by computing  
 252 its backward Piola transformation, reads  $\mathbf{Q} = J\mathbf{F}^{-1}\mathbf{q}$  [66, 20, 65, 30].

253 In the present framework, the balance of linear momentum (see Appendix A) can be reformulated in  
 254 the reference placement as

$$\text{Div}(\mathbf{T}_s + \mathbf{T}_f) + [\Phi_s \varrho_s + (J - \Phi_s) \varrho_f] \mathbf{g} = \mathbf{0}, \quad (2a)$$

$$\text{Div} \mathbf{T}_f + \mathbf{F}^{-T} \mathbf{\Pi}_f + (J - \Phi_s) \varrho_f \mathbf{g} = \mathbf{0}, \quad (2b)$$

255 where  $\mathbf{T}_\alpha := J\boldsymbol{\sigma}_\alpha \mathbf{F}^{-T}$ , with  $\alpha \in \{s, f\}$ , is the first Piola-Kirchhoff stress tensor associated with the  $\alpha$ th  
 256 phase,  $\mathbf{\Pi}_f = J\mathbf{F}^T \boldsymbol{\pi}_f \in T_X^* \mathcal{B}$  is the pull-back of  $\boldsymbol{\pi}_f$  to the reference placement, with  $\boldsymbol{\pi}_f$  being the force  
 257 density due to the exchanges of linear momentum between the fluid and the solid phase.

## 258 4 General constitutive relations

259 In view of the computational burden that will be introduced for describing the flow, for the purposes  
 260 of our present study we assume that the solid phase is isotropic, homogeneous, and characterized by a  
 261 Neo-Hookean hyperelastic strain energy density function  $\Psi_s(\mathbf{C})$  [67], which, written per unit volume of  
 262 the reference placement, takes on the form

$$\Psi_s(\mathbf{C}) = \frac{1}{2} \Phi_s \mu_s [I_1 - 3] - \frac{1}{2} \Phi_s \mu_s \log I_3 + \frac{1}{8} \Phi_s \lambda_s [\log I_3]^2, \quad (3)$$

263 where  $\lambda_s$  and  $\mu_s$  are Lamé’s parameters, and  $I_1 = \text{tr} \mathbf{C}$ , and  $I_3 = \det \mathbf{C} = J^2$  are two of the three  
 264 principal invariants of the Cauchy-Green tensor  $\mathbf{C}$  (we recall that the second principal invariant is  
 265  $I_2 = \frac{1}{2} \{[\text{tr} \mathbf{C}]^2 - \text{tr} \mathbf{C}^2\}$ ).

266 Before going further, it is important to remark that, for biological tissues, there exist strain en-  
 267 ergy densities that are more appropriate than the Neo-Hookean one. A rather typical example is the  
 268 Holmes&Mow [2] strain energy density function, which has been extensively used and generalized in  
 269 many works addressing the mechanics of articular cartilage in the biphasic context, especially when  
 270 fibers are included in order to make the model at least transversely isotropic [8, 23, 13, 16, 19, 24, 20,  
 271 25, 26, 30, 17, 27, 18].

272 By viewing  $I_1$  and  $I_3$  as functions of  $\mathbf{C}$ , and  $\mathbf{C}$  as a function of  $\mathbf{F}$ , we can rewrite  $\Psi_s(\mathbf{C})$  as  
 273  $\Psi_s(\mathbf{C}) \equiv W_s(\mathbf{F})$ , and, thus, we determine  $\mathbf{T}_{sc}$  as

$$\mathbf{T}_{sc} = \frac{\partial W_s}{\partial \mathbf{F}}(\mathbf{F}) = \boldsymbol{\eta} \mathbf{F} \left[ 2 \frac{\partial \Psi_s}{\partial \mathbf{C}}(\mathbf{C}) \right]. \quad (4)$$

274 In the sequel,  $\mathbf{T}_{sc}$  will be referred to as the *constitutive part of the first Piola-Kirchhoff stress tensor*  
 275 associated with the solid phase. Its explicit expression will be supplied below, when discussing some

276 numerical aspects of the problem at hand. Here, we simply notice that, since  $\mathbf{T}_{\text{sc}}$  is defined constitutively,  
 277  $\mathbf{T}_s$  is fully defined in terms of  $\mathbf{T}_{\text{sc}}$  and of the pore pressure  $P(X, t) = p(\chi(X, t), t)$  (i.e., the pore pressure  
 278 expressed as a function of the points of  $\mathcal{B}$  and of time), and since  $\mathbf{T}_f$  depends only on  $P$  (as is customary  
 279 in the Darcy-Forchheimer model, the fluid stress tensor does not feature any dissipative part)), then all  
 280 the stresses featuring in the balance laws of interest are completely expressed in terms of the unknowns  
 281  $\chi$  (through the deformation gradient tensor) and  $P$ .

282 In fact, the structure of the inner part of stress, i.e.,  $\boldsymbol{\sigma}_I = \boldsymbol{\sigma}_s + \boldsymbol{\sigma}_f$  (see Appendix B), yields the  
 283 inner first and second Piola-Kirchhoff stress tensors  $\mathbf{T}_I = -JPF^{-T} + \mathbf{T}_{\text{sc}}$ , and  $\mathbf{S}_I = -JPC^{-1} + \mathbf{S}_{\text{sc}}$ ,  
 284 where  $\mathbf{S}_{\text{sc}}$  is the constitutive part of the solid phase second Piola-Kirchhoff stress tensor and is defined  
 285 as  $\mathbf{S}_{\text{sc}} = JF^{-1}\boldsymbol{\eta}^{-1}\boldsymbol{\sigma}_{\text{sc}}F^{-T}$ . Moreover, since the solid phase is assumed to be hyperelastic,  $\mathbf{S}_I$  can be  
 286 determined by differentiating an *augmented* strain energy density  $\Psi_s^a$ , obtained through the addition of  
 287 the pressure term  $-[J - 1]P$  to  $\Psi_s(\mathbf{C})$ , i.e.,

$$\Psi_s^a(\mathbf{C}, P) := \Psi_s(\mathbf{C}) - [J - 1]P = \frac{1}{2}\Phi_s\mu_s[\text{tr}\mathbf{C} - 3] - \Phi_s\mu_s \log J + \frac{1}{2}\Phi_s\lambda_s [\log J]^2 - [J - 1]P, \quad (5a)$$

$$\mathbf{S}_I = 2\frac{\partial\Psi_s^a}{\partial\mathbf{C}}(\mathbf{C}, P) = -JPC^{-1} + \mathbf{S}_{\text{sc}} = -JPC^{-1} + 2\frac{\partial\Psi_s}{\partial\mathbf{C}}(\mathbf{C}). \quad (5b)$$

288 For future use, we also introduce the strain energy density  $W_s^a(\mathbf{F}, P) \equiv \Psi_s^a(\mathbf{C}, P)$ .

289 There remains to determine  $\boldsymbol{\pi}_{\text{fd}}$  and, to do so, we proceed with the study of the dissipation  
 290 inequality[32, 68, 34, 22, 5, 30].

## 291 5 Constitutive representation of the dissipative forces

292 We are interested here in evaluating the influence that the fluid phase non-Darcian dynamics may have  
 293 on the medium's overall mechanical behavior. In particular, to account for loading conditions that do  
 294 not fully justify the hypothesis of negligibility of inertial forces, we consider Forchheimer's correction to  
 295 Darcy's law[62, 69, 32, 70, 35, 33]. Moreover, to account for dissipative flow features that, in the literature  
 296 (see e.g.[71, 72, 73, 74, 38]), have conducted to flow laws non-local in time, we propose a fractionalization  
 297 of Forchheimer's correction. To this end, we suggest a relation between the fluid phase filtration velocity  
 298 and pressure gradient that is highly non-linear, and is expressed through integro-differential operators of  
 299 fractional type describing a possible non-locality in time in the constitutive representation of the drag  
 300 forces as functionals of the fluid filtration velocity.

301 Under the hypotheses done so far, by assuming that the sole source of energetic loss is due to the  
 302 momentum exchanged between the fluid and the solid phase, and adhering to the frameworks developed  
 303 in[32, 34], and, subsequently, in[5], it can be proven that the local form of the residual dissipation per  
 304 unit of volume of  $\mathcal{B}_t$  is given by

$$\mathfrak{D}^{(a)} = -\boldsymbol{\pi}_{\text{fd}}\mathbf{w}_{\text{fs}} = -\boldsymbol{\pi}_{\text{fd}}\phi_f^{-1}\mathbf{q} \geq 0. \quad (6)$$

305 We recall that, throughout this work, all the force densities, and, thus, also  $\boldsymbol{\pi}_{\text{fd}}$ , are identified as  
 306 pseudo co-vectors, while all the velocities are defined as vectors. Hence,  $\boldsymbol{\pi}_{\text{fd}}(x, t)$  has to be regarded  
 307 as a linear map that, whenever evaluated over a vector  $\mathbf{u} \in T_x\mathcal{B}_t$ , returns a pseudo-scalar value. In  
 308 particular, the juxtapositions  $\boldsymbol{\pi}_{\text{fd}}\mathbf{w}_{\text{fs}}$  and  $\boldsymbol{\pi}_{\text{fd}}\mathbf{q}$  in Equation (6) are to be understood, in index notation, as  
 309  $\boldsymbol{\pi}_{\text{fd}}\mathbf{w}_{\text{fs}} = [\boldsymbol{\pi}_{\text{fd}}]_a[\mathbf{w}_{\text{fs}}]^a$  and  $\boldsymbol{\pi}_{\text{fd}}\mathbf{q} = [\boldsymbol{\pi}_{\text{fd}}]_a\mathbf{q}^a$ , where Einstein's convention of summation over repeated  
 310 indices applies, unless stated otherwise.

311 Expressions similar to Equation (6) can be found in several publications (see e.g. [32, 34, 1, 5, 30])  
 312 and, thus, its full derivation will not be reported here. However, we recall that the superscript "(a)" in  
 313  $\mathfrak{D}^{(a)}$  stands for "augmented", since, to obtain Equation (6), the constraint of incompressibility, imposed  
 314 to each phase of the mixture, and reflected by the mass balance law (1b), is appended to the local form of  
 315 the dissipation inequality, multiplied by the pore pressure  $p$ . This latter field, thus, acquires the meaning  
 316 of the Lagrange multiplier[34, 5] associated with the given constraint. For the advantages related with  
 317 this procedure, the reader is referred to[75, 34].

318 We hypothesize that the dissipative force density  $\pi_{fd}$  can be expressed constitutively, up to the sign,  
 319 as the result of some suitably defined operator  $O_q$ , applied to  $q$ , and in which the subscript “ $q$ ” indicates  
 320 that, in general, the operator may depend on  $q$  itself. Hence, we impose a relationship of the kind

$$\pi_{fd} \equiv -O_q q. \quad (7)$$

321 Such relationship is nonlinear in general, and, for consistency with Equation (6), it imposes that  $O_q$   
 322 complies with the dissipation inequality, so that the condition  $\mathfrak{D}^{(a)} = \phi_f^{-1}[O_q q]q \geq 0$  must be respected  
 323 at all times and at all points of the region of space occupied by the mixture. Furthermore, by substituting  
 324 the relationship (7) into the balance law of linear momentum in the spatial placement (see Appendix 2),  
 325 we find the following operator equation in the unknown  $q$ :

$$-O_q q = \phi_f [\text{grad } p - \varrho_f g]. \quad (8)$$

326 Among the various possible definitions of  $O_q$ , each of which depends on the biphasic medium that  
 327 has to be modeled, we require  $O_q$  to be such that it vanishes identically for the null filtration velocity  
 328  $q_0 \equiv \mathbf{0}$ , i.e.,

$$O_q q = O_{q_0} q_0 \equiv \mathbf{0}. \quad (9)$$

329 In addition, we require that the null vector field  $q_0 \equiv \mathbf{0}$  is the unique solution to the equation  $O_q q = \mathbf{0}$ . By  
 330 doing so, when the pressure field solves  $\text{grad } p - \varrho_f g = \mathbf{0}$ , so that also the left-hand side of Equation (8)  
 331 has to vanish, the solution is  $q = q_0$ . This requirement is important in view of the fact that a “modified”  
 332 Caputo derivative will feature in the definition of the operator  $O_q q$ , thereby implying that a field  $q$  with  
 333 non-vanishing initial value  $q(x, t_{in}) \neq \mathbf{0}$  is, in general, a nontrivial solution of the equation  $O_q q = \mathbf{0}$  (see  
 334 Equation (15)). Hence, to maintain the uniqueness of the solution  $q_0 \equiv \mathbf{0}$ , we will always assume that  $q$   
 335 has null initial value.

336 The considerations on  $O_q q$  done above imply that also the right-hand side of Equation (8) must vanish  
 337 for  $q = q_0$ , thereby recovering Stevin’s law of the statics of fluids, i.e.,  $\text{grad } p - \varrho_f g = \mathbf{0}$ . Moreover,  
 338 several other fluid behaviors are ruled out, like those characterized by non-null values of  $\pi_{fd}$  for  $q = q_0$ .  
 339 In the latter case, indeed, by denoting by  $\pi_{fd}^{\text{st}}$  the value of  $\pi_{fd}$  in static conditions, the statics of the fluid  
 340 under consideration is governed by the force balance  $\pi_{fd}^{\text{st}} - \phi_f \text{grad } p + \phi_f \varrho_f g = \mathbf{0}$ , which determines  $\pi_{fd}^{\text{st}}$   
 341 as  $\pi_{fd}^{\text{st}} = \phi_f [\text{grad } p - \varrho_f g]^1$  without constitutive prescriptions.

342 For the sake of clarity, before describing the operator  $O_q$  in detail for the case that characterizes the  
 343 main novelty of this work, we briefly discuss the (classical) definitions of  $O_q$  that return Darcy’s law and  
 344 Forchheimer’s correction to Darcy’s law. In doing this, since gravity is not expected to play a relevant  
 345 role for the problems that will be investigated in the sequel, we shall drop the buoyancy term  $\varrho_f g$  from  
 346 here on.

## 347 5.1 Darcy’s law

348 Although Darcy’s law is well-known, we find it useful to briefly review its origin and the range of its  
 349 applicability in order to give context to the need for Forchheimer’s correction and for its fractionalization.

350 Darcy’s law is widely employed in the mechanics of porous media of environmental, industrial, and  
 351 biological interest (see e.g. [69, 33, 34, 1, 22, 76], to mention just a few) to describe, at the *macroscale*,  
 352 the flow of a fluid through the pores of a given porous medium. Here, by “macroscale”, it is meant the  
 353 scale at which the porous medium and the fluid are viewed as a mixture.

354 Darcy’s regime is satisfactory when the following two main hypotheses are met:

- 355 (i) The stress tensor of the fluid is well approximated by its so-called equilibrium part, so that any  
 356 contribution due to the fluid viscosity is negligible and one can write the fluid’s Cauchy stress  
 357 tensor as  $\sigma_f = -\phi_f p \mathbf{I}^T$ .

<sup>1</sup>Note that this equation is different from Equation (103b) (see Appendix B) in that it applies in static conditions, whereas Equation (103b) holds true in dynamic regime, but in the limit of negligible inertial forces.

358 (ii) Inertial forces are negligible both at the macroscale and at the microscale. At the macroscale, this  
 359 assumption implies that no inertial effects are accounted for in the fluid’s macroscopic momentum  
 360 balance law. For what concerns the microscale, instead, the assumption of negligible inertial  
 361 effects has two meanings. On the one hand, it requires that such effects are one or more orders  
 362 of magnitude smaller than those of the other forces contributing to the flow, and, on the other  
 363 hand, that the linear momentum exchanged between the fluid and the solid at their interface does  
 364 not depend appreciably on the dynamic part of the overall mechanical stress (see e.g. [70]). In  
 365 particular, this latter statement is reflected by the fact that, at the macroscale, and in the cases in  
 366 which  $\boldsymbol{\pi}_{\text{fd}}$  can be expressed constitutively, one can prescribe  $\boldsymbol{\pi}_{\text{fd}}$  to be a linear function of  $\boldsymbol{q}$  (see  
 367 e.g. [69, 33, 34, 1]), i.e.,

$$\boldsymbol{\pi}_{\text{fd}} = \mathcal{G}^{\pi_{\text{fd}}}(\boldsymbol{q}, \dots) := \mathcal{G}^r(\dots)\boldsymbol{q} = -\boldsymbol{r}\boldsymbol{q}, \quad (10)$$

368 where  $\mathcal{G}^{\pi_{\text{fd}}}(\boldsymbol{q}, \dots)$  is the constitutive law expressing  $\boldsymbol{\pi}_{\text{fd}}$ ,  $\boldsymbol{r}$  is a second-order tensor field, referred  
 369 to as *resistivity tensor*, and  $\mathcal{G}^r(\dots)$  is its constitutive representation (here, the ellipsis means that  
 370 the considered constitutive functions depend, in general, on variables that are left unspecified at  
 371 the moment). In passing, we recall that there exist generalizations to Darcy’s law that involve  
 372 threshold phenomena, according to which, for example, relationships similar to Equation (10) can  
 373 be written only when the norm of  $\boldsymbol{\pi}_{\text{fd}}$  exceeds a certain value (see e.g. [33]). However, these  
 374 circumstances are out of the scopes of our present work.

375 According to Equation (10), in the case of Darcy’s law the identification  $\boldsymbol{O}_q\boldsymbol{q} \equiv \boldsymbol{r}\boldsymbol{q}$  applies, so that  
 376 the operator  $\boldsymbol{O}_q$  is represented by  $\boldsymbol{r}$  and is, thus, independent of  $\boldsymbol{q}$ . Furthermore, by substituting Equation  
 377 (10) into the residual dissipation inequality (6), one obtains

$$\begin{aligned} \mathfrak{D}^{(a)} &= -\boldsymbol{\pi}_{\text{fd}}\phi_f^{-1}\boldsymbol{q} = [\boldsymbol{r}\boldsymbol{q}]\phi_f^{-1}\boldsymbol{q} = \phi_f^{-1}[\text{sym}(\boldsymbol{r})\boldsymbol{q}]\boldsymbol{q} \\ &= \phi_f^{-1}\text{tr}\{\text{sym}(\boldsymbol{r})[\boldsymbol{q} \otimes \boldsymbol{q}]\} = \phi_f^{-1}[\text{sym}(\boldsymbol{r})]_{ab}[\boldsymbol{q}]^a[\boldsymbol{q}]^b \geq 0, \end{aligned} \quad (11)$$

378 which requires *per se* the symmetric part of the resistivity tensor,  $\text{sym}(\boldsymbol{r})$ , to be positive semi-definite.  
 379 Typically, however, one aims at obtaining an expression for  $\boldsymbol{q}$  in closed form by substituting Equation  
 380 (10) into the spatial balance law of linear momentum for the fluid phase, and solving for  $\boldsymbol{q}$ . Hence, to do  
 381 this, one assumes that  $\text{sym}(\boldsymbol{r})$  is positive definite (which is also consistent with the condition  $\boldsymbol{O}_{q_0}\boldsymbol{q}_0 \equiv \mathbf{0}$ )  
 382 and, often, it is also hypothesized from the outset that the resistivity tensor  $\boldsymbol{r}$  is symmetric, so that the  
 383 identity  $\boldsymbol{r} \equiv \text{sym}(\boldsymbol{r})$  is stated. Under these hypotheses, indeed, one achieves Darcy’s law in the “popular”  
 384 form

$$\boldsymbol{q} = -\frac{\boldsymbol{k}}{\mu}\text{grad } p \equiv \boldsymbol{q}_{\text{D}}, \quad \boldsymbol{r} := \phi_f\mu\boldsymbol{k}^{-1}, \quad (12)$$

385 where  $\boldsymbol{k}$  is a second-order tensor field referred to as *permeability tensor*,  $\mu$  is the fluid’s viscosity, and  
 386  $\boldsymbol{q}_{\text{D}}$  stands for “Darcy’s velocity”.

387 With respect to the reference placement of the medium, Equation (12) transforms as

$$\boldsymbol{Q} = -\frac{\boldsymbol{K}}{\mu}\text{Grad } P \equiv \boldsymbol{Q}_{\text{D}}, \quad (13)$$

388 where  $P$  is the pore pressure written as a function of the points  $X$  of the reference placement and of time,  
 389 i.e.,  $P = p \circ (\chi, \mathfrak{T})$ , while  $\boldsymbol{K}$  is referred to as *material permeability tensor* and is related to  $\boldsymbol{k}$  through  
 390 the backward Piola transformation  $\boldsymbol{K} = \boldsymbol{J}\boldsymbol{F}^{-1}\boldsymbol{k}\boldsymbol{F}^{-\text{T}}$ . Hence, the Darcian material filtration velocity  $\boldsymbol{Q}_{\text{D}}$   
 391 can be expressed in terms of the pore pressure and deformation gradient tensor.

392 Finally, having neglected the buoyancy terms in Equations (2a) and (2b), the equations to be solved  
 393 in the case of validity of Darcy’s regime can be summarized as

$$\text{Div}(-\boldsymbol{J}\boldsymbol{P}\boldsymbol{F}^{-\text{T}} + \boldsymbol{T}_{\text{sc}}) = \mathbf{0}, \quad (14a)$$

$$j = \text{Div} \left[ \frac{\mathbf{K}}{\mu} \text{Grad} P \right], \quad (14b)$$

394 where  $T_{\text{sc}} = \eta \mathbf{F} S_{\text{sc}}$ , with  $S_{\text{sc}}$  being deducible from Equation (5b), is determined constitutively as shown in  
 395 Equation (4), while the permeability tensor  $\mathbf{K}$  is specified in Equation (36) below. Moreover, the material  
 396 volumetric fractions  $\Phi_s$  and  $\Phi_f$ , which feature in the definitions of  $T_{\text{sc}}$  and  $\mathbf{K}$ , are  $\Phi_s = J \phi_s = \Phi_{\text{sR}}$  and  
 397  $\Phi_f = J \phi_f$ , and  $\Phi_{\text{sR}}$  is regarded as known. In the system of Equations (14a) and (14b), the unknowns are  
 398 pressure  $P$  and the motion  $\chi$ . The latter is accounted for by  $\mathbf{F}$  and  $J = \det \mathbf{F}$ , and  $\Phi_f$  is expressed as  
 399  $\Phi_f = J - \Phi_s$  by virtue of the backward Piola transformation of the saturation condition.

400 Before proceeding, the following two remarks are in order:

401 *Remark 2.* In the literature on the mechanics of fluid-saturated porous media, models have been proposed  
 402 that, although assuming the Darcian regime for the fluid, account for memory in the description of the  
 403 flow (see, e.g., [38]). These models have the purpose of describing, for instance, non-mechanical, micro-  
 404 structural interactions of the fluid with the pores, such as chemical processes [38], that contribute to  
 405 hinder the flow. Often, this phenomenology is studied by adopting integro-differential operators in time  
 406 that introduce a delay in the purely Darcian linear relationship between pressure gradient and filtration  
 407 velocity. For example, Deseri and Zingales [38] used operators based on the fractional Caputo derivative,  
 408 so that, under some simplifying hypotheses, classical Darcy's law can be replaced by

$$\mathbf{q}(t) + \frac{\alpha t_c^\alpha}{\Gamma(1-\alpha)} \int_{t_{\text{in}}}^t \frac{\dot{\mathbf{q}}(\tau)}{(t-\tau)^\alpha} d\tau = \mathbf{q}_D(t). \quad (15)$$

409 Equation (15) is a generalized Cattaneo's model conceived for porous media that either do not undergo  
 410 deformation or are in the regime of infinitesimal deformations [77, 38, 78, 31]. Indeed, when the  
 411 deformations have to be regarded as finite, relationships of the type provided in Equation (15) are not  
 412 objective because of the presence of the time derivative of  $\mathbf{q}$  featuring inside the integral. To avoid this  
 413 problem, we take advantage of the property of  $\mathbf{q}$  of being a pseudo-vector and, consequently, we have  
 414 recourse to the most natural way to describe its time evolution, i.e., to its Truesdell rate (see [79] for a  
 415 recent discussion on this topic).

416 *Remark 3* (Caputo fractional derivative, frame indifference and Truesdell rate).

417 Caputo's fractional derivative [80, 42] in time cannot be employed directly to obtain constitutive relations  
 418 that satisfy the principle of frame indifference, since the time derivative is not an objective rate. In  
 419 particular, by introducing a proper orthogonal tensor  $\mathbf{R}$  ( $\det \mathbf{R} = 1$ ), such that a given vector field  $\mathbf{v}$   
 420 transforms as  $\tilde{\mathbf{v}} = \mathbf{R}\mathbf{v}$ , the Caputo derivative yields

$${}^C D_t^\alpha \tilde{\mathbf{v}} = {}^C D_t^\alpha (\mathbf{R}\mathbf{v}) = \frac{1}{\Gamma(1-\alpha)} \int_{t_{\text{in}}}^t \frac{1}{(t-\tau)^\alpha} \dot{\mathbf{R}}\mathbf{v} d\tau = \frac{1}{\Gamma(1-\alpha)} \int_{t_{\text{in}}}^t \frac{1}{(t-\tau)^\alpha} (\dot{\mathbf{R}}\mathbf{v} + \mathbf{R}\dot{\mathbf{v}}) d\tau, \quad (16)$$

421 where  $\dot{\mathbf{R}}\mathbf{v}$  is the term that spoils objectivity.

422 Conversely, since in our work we need to make sure that the fractional derivative employed in  
 423 the fractionalization of constitutive relations respects objectivity, we replace the time derivative in the  
 424 integral with an appropriate objective rate. In particular, an objective rate suitable for our purposes is  
 425 the Truesdell rate, because it is conceived for fluxes and, in fact, we have to compute the derivative of  $\mathbf{q}$ ,  
 426 which is a flux. Such a modification of the Caputo fractional derivative will satisfy a relation of the type

$$\bar{C} D_t^\alpha \tilde{\mathbf{q}} = \frac{1}{\Gamma(1-\alpha)} \int_{t_{\text{in}}}^t \frac{1}{(t-\tau)^\alpha} \mathcal{T}_s \tilde{\mathbf{q}} d\tau = \frac{1}{\Gamma(1-\alpha)} \int_{t_{\text{in}}}^t \frac{1}{(t-\tau)^\alpha} \mathbf{R} \mathcal{T}_s \mathbf{q} d\tau, \quad (17)$$

427 in which  $\mathcal{T}_s$  is the Truesdell rate and the bar over "C" in the left-hand side indicates a "modified" Caputo  
 428 derivative. We recall that, in the present context, the Truesdell rate of  $\mathbf{q}$  can be computed as<sup>2</sup>

$$\mathcal{T}_s \mathbf{q}(\chi(X, \tau), \tau) \equiv \frac{1}{J(X, \tau)} \mathbf{F}(X, \tau) \text{D}_s \left[ \frac{1}{\det \mathbf{F}^{-1}} \mathbf{F}^{-1} \mathbf{q} \right](\chi(X, \tau), \tau), \quad (18)$$

<sup>2</sup>In fact, the right-hand side of Equation (18) is *not* the definition of the Truesdell rate of  $\mathbf{q}$ , but just a simple way for computing it. A more rigorous way of writing it can be found e.g. in [79].

429 where  $D_s$  is the substantial derivative with respect to the solid phase motion. In the spatial description,  
 430 Equation (18) produces the result

$$\begin{aligned}\mathcal{T}_s \mathbf{q} &= [\operatorname{div} \mathbf{v}_s] \mathbf{q} - [\operatorname{grad} \mathbf{v}_s] \mathbf{q} + D_s \mathbf{q} \\ &= [\operatorname{div} \mathbf{v}_s] \mathbf{q} - [\operatorname{grad} \mathbf{v}_s] \mathbf{q} + [\operatorname{grad} \mathbf{q}] \mathbf{v}_s + \partial_t \mathbf{q}.\end{aligned}\quad (19)$$

431 However, since we are interested in the material description of the flow, we express  $\mathcal{T}_s \mathbf{q}$  as

$$\mathcal{T}_s \mathbf{q}(\chi(X, \tau), \tau) \equiv \frac{1}{J(X, \tau)} \mathbf{F}(X, \tau) \dot{\mathbf{Q}}(X, \tau), \quad (20)$$

432 where we recall the definition of *material filtration velocity*  $\mathbf{Q}(X, t) = [(\det \mathbf{F}^{-1})^{-1} \mathbf{F}^{-1} \mathbf{q}](\chi(X, t), t)$ .

## 433 5.2 Forchheimer's correction

434 Following [70], Forchheimer's correction to Darcy's law becomes necessary when the hypothesis (ii) of  
 435 the section 5.1 is not satisfied. Indeed, as remarked in [70], the correction accounts for the inertial effects  
 436 that characterize the pore scale dynamics of the fluid, and for those that take part to the momentum  
 437 exchange between the fluid and the solid phase. In fact, it can be shown that (see e.g.[81]), at the  
 438 macroscale, the consideration of the inertial effects mentioned above can be expressed in terms of a  
 439 non-linear relationship between  $\pi_{\text{fd}}$  and  $\mathbf{q}$  of the type (see e.g. [70, 68, 82, 36, 30])

$$\pi_{\text{fd}} = \mathcal{G}^{\pi_{\text{fd}}}(\mathbf{q}, \dots) = \mathcal{G}^{\mathbf{r}_F}(\mathbf{q}, \dots) \mathbf{q} = -\mathbf{r}_F(\|\mathbf{q}\|) \mathbf{q}, \quad (21)$$

440 where  $\mathbf{r}_F(\|\mathbf{q}\|)$  can be thought of as a  $\mathbf{q}$ -dependent resistivity tensor. Note that, here and in the  
 441 following, the subscript "F" stands for "Forchheimer", and is introduced in order to highlight that  
 442 the current description differs from the Darcian one. In addition, as suggested by the identification  
 443  $\mathcal{G}^{\mathbf{r}_F}(\mathbf{q}, \dots) \equiv -\mathbf{r}_F(\|\mathbf{q}\|)$ , the resistivity tensor depends, in general, aside from  $\|\mathbf{q}\|$ , also on other  
 444 parameters characterizing the flow, although we do not report them here explicitly for the sake of a  
 445 lighter notation. We recall and clarify the procedure undertaken in [30, 36, 82] for the characterization  
 446 of Forchheimer's correction in the spatial setting and we extend it to the reference placement through the  
 447 appropriate backward Piola transformations. In fact, although the derivations are similar, the aim in [30]  
 448 was to rephrase Forchheimer's correction as a scalar coefficient that multiplies the permeability tensor,  
 449 whilst the aim of this section is to provide a clear material description for Forchheimer's coefficient.

450 As reported in [70, 68, 82, 36, 30], the resistivity tensor  $\mathbf{r}_F(\|\mathbf{q}\|)$  can be defined as

$$\mathbf{r}_F(\|\mathbf{q}\|) = \mathbf{r} + \|\mathbf{q}\| \mathbf{a} \mathbf{r} = \phi_f \mu [\mathbf{k}^{-1} + \|\mathbf{q}\| \mathbf{a} \mathbf{k}^{-1}], \quad (22)$$

451 where  $\mathbf{a}$ , in general, is a second-order tensor field denominated *Forchheimer's coefficient*, having physical  
 452 dimensions of the inverse of a characteristic velocity, and that is to be assigned constitutively (see Equation  
 453 (32) below).

454 By comparing Equation (22) with the general definition (7), we obtain the identification

$$\mathcal{O}_q \equiv \phi_f \mu [\mathbf{k}^{-1} + \|\mathbf{q}\| \mathbf{a} \mathbf{k}^{-1}] = \phi_f \mu \mathbf{k}^{-1} [\mathbf{I} + \|\mathbf{q}\| \mathbf{k} \mathbf{a} \mathbf{k}^{-1}]. \quad (23)$$

455 Moreover, by substituting Equation (23) into the constitutive representation (21) of  $\pi_{\text{fd}}$ , using the  
 456 resulting expression into the force balance (8), neglecting gravity, and invoking the definition (12) of  
 457 Darcy's velocity  $\mathbf{q}_D$ , we find that  $\mathbf{q}$  must satisfy the algebraic equation

$$\phi_f \mu \mathbf{k}^{-1} [\mathbf{I} + \|\mathbf{q}\| \mathbf{k} \mathbf{a} \mathbf{k}^{-1}] \mathbf{q} = \phi_f \mu \mathbf{k}^{-1} \mathbf{q}_D, \quad (24)$$

458 which can be put in the equivalent form (see [30], in which a slightly different notation is employed)

$$[\mathbf{I} + \|\mathbf{q}\| \mathbf{k} \mathbf{a} \mathbf{k}^{-1}] \mathbf{q} = \mathbf{q}_D. \quad (25)$$

459 The backward Piola transformation of Equation (25) produces [30]

$$[I + \|Q\|_C K \mathcal{A} K^{-1}] Q = Q_D, \quad (26)$$

460 where  $I$  is the material identity tensor,  $\|Q\|_C := J^{-1} \sqrt{[C : (Q \otimes Q)]}$  is the  $C$ -norm of  $Q$ , i.e., the  
461 norm of  $Q$  computed with respect to the deformed metric tensor induced by the right Cauchy-Green  
462 deformation tensor  $C$ , while

$$\mathcal{A}(X, t) := F^T(\chi(X, t), t) \mathbf{a}(\chi(X, t), t) F^{-T}(X, t) \quad (27)$$

463 is the backward Piola transform of Forchheimer's coefficient. Note that the norm  $\|Q\|_C$  arises because  
464 of the identity  $\|q(x, t)\| = \|Q(X, t)\|_C$  (an equivalent formulation of equation (26) can be obtained  
465 by redefining  $\|Q\|_C$  as  $J\|q\|$ , so as to emphasize that  $\|q\|$  transforms as a pseudo-scalar, and  $\mathcal{A}$  as  
466  $J^{-1} F^T \mathbf{a} F^{-T}$ ).

467 Finally, we notice that a rather suggestive reformulation of Equation (26) reads

$$R_F(\|Q\|_C) Q = \Phi_f \mu K^{-1} Q_D, \quad (28)$$

468 where we have introduced the material resistivity tensor

$$R_F(\|Q\|_C) := \Phi_f \mu K^{-1} [I + \|Q\|_C K \mathcal{A} K^{-1}], \quad (29)$$

469 related to  $r_F(\|q\|)$  through  $R_F(\|Q\|_C) = F^T [r_F(\|q\|)] F$ . The tensor function  $R_F$  depends also on the  
470 deformation gradient tensor  $F$  through  $\Phi_f$  and  $K$ , although we prefer not to emphasize this dependence  
471 here, both for notational convenience and for highlighting the fact that, since  $R_F$  is the backward Piola  
472 transformation of  $r_F$ , it depends on the  $C$ -norm of the material filtration velocity  $Q$ .

473 Although there exists some interest for the impact of Forchheimer's correction in porous media of  
474 biological relevance (see e.g. [35, 36, 30]), to the best of our knowledge the majority of the studies  
475 devoted to the identification of Forchheimer coefficient  $\mathbf{a}$  come from hydrogeology[33] and petroleum  
476 engineering[83, 84]. In fact,  $\mathbf{a}$  is often expressed through (semi-)empirical laws. For instance, Wang et  
477 al. [85] provided an expression for  $\mathbf{a}$  that, in our formalism, reads

$$\mathbf{a} := \varrho_f \eta \mu^{-1} \mathbf{k} \boldsymbol{\beta}, \quad (30)$$

478 where the tensor field  $\boldsymbol{\beta}$  is said to be *non-Darcy coefficient* [85]. As done in [30], we take an empirical  
479 formula from Thauvin and Mohanty [37] and we adapt it to our purposes, thereby expressing  $\boldsymbol{\beta}$  as

$$\boldsymbol{\beta} = c_0 \phi_f^{c_1} [\eta \mathbf{k}]^{c_2}, \quad (31)$$

480 in which  $c_0$ ,  $c_1$ , and  $c_2$  are empirical (real) constants, with  $c_0$  having to be non-negative. We remark  
481 that there are at least two ways to assign  $c_0$ , either in a theoretical way, on the basis of the geometrical  
482 features of the pores (see e.g. [86]), or by calculating it through the fitting of the experimental data (see  
483 e.g. [87]). A third approach, which interprets  $c_0$  as a parameter prescribed to switch from a permeability  
484 model to another[30], is discussed at the end of the section.

485 By substituting Equation (31) into Equation (30), and exploiting the positivity of all the eigenvalues  
486 of  $\mathbf{k}$ , we obtain

$$\mathbf{a} = c_0 \varrho_f \phi_f^{c_1} \mu^{-1} [\eta \mathbf{k}]^{1+c_2}. \quad (32)$$

487 A strong simplification is achieved when the porous medium under consideration is assumed to be  
488 isotropic and, in particular, "*unconditionally isotropic*"[22]. In this case, indeed, the spatial perme-  
489 ability tensor  $\mathbf{k}$  reduces to  $\mathbf{k} = k_{\text{iso}} \boldsymbol{\eta}^{-1}$ , where  $k_{\text{iso}}$  is referred to as *scalar permeability*; the material  
490 permeability tensor becomes  $\mathbf{K} = \kappa_{\text{iso}} C^{-1}$ , with  $\kappa_{\text{iso}} := J k_{\text{iso}}$ ; the Forchheimer coefficient  $\mathbf{a}$  reduces to

491  $\mathbf{a} = c_0 \varrho_f \phi_f^{c_1} \mu^{-1} k_{\text{iso}}^{1+c_2} \mathbf{t}^T$ , and the material Forchheimer coefficient  $\mathcal{A}$  can be written as  $\mathcal{A} = \mathcal{A}_{\text{iso}} \mathbf{I}^T$ ,  
 492 whereby it is fully represented by the scalar quantity

$$\mathcal{A}_{\text{iso}} = c_0 \varrho_f \left[ \frac{\Phi_f}{J} \right]^{c_1} \frac{1}{\mu} \left[ \frac{\kappa_{\text{iso}}}{J} \right]^{1+c_2}, \quad \text{with } \Phi_f > 0, k_{\text{iso}} > 0, \text{ and } \mu > 0. \quad (33)$$

493 By substituting this result into Equation (26), and following a procedure similar to the one described in  
 494 [82, 36, 30], we can express  $\mathbf{Q}$  as a function of  $\mathbf{Q}_D$ , i.e.,

$$\mathbf{Q} = \mathfrak{F} \mathbf{Q}_D = -\frac{\mathfrak{F} \mathbf{K}}{\mu} \text{Grad} P, \quad \mathfrak{F} := \frac{2}{1 + \sqrt{1 + 4 \mathcal{A}_{\text{iso}} \|\mathbf{Q}_D\|_{\mathcal{C}}}}, \quad (34)$$

495 where  $\mathfrak{F}$  is referred to as *material friction factor* [82, 36, 30] (note that, in [82, 36, 30], Equation (34) is  
 496 obtained in the spatial description and, thus, in the models presented therein the adjective “material” is  
 497 not present).

498 A relevant consequence of Equation (34) is that, for an “unconditionally isotropic” [22] porous  
 499 medium,  $\mathbf{Q}$  can be understood as a reformulation of Darcy’s law, in which the permeability is multiplica-  
 500 tively rescaled by means of  $\mathfrak{F}$ , which, in turn, depends again on the  $\mathcal{C}$ -norm of material Darcy’s velocity  
 501  $\|\mathbf{Q}_D\|_{\mathcal{C}}$  as well as on  $\kappa_{\text{iso}}$ , porosity, and the other flow parameters accounted for in the model. Therefore,  
 502 under the hypothesis of “unconditionally isotropic” medium, the governing equations read as

$$\text{Div}(-J P \mathbf{F}^{-T} + \mathbf{T}_{\text{sc}}) = \mathbf{0}, \quad (35a)$$

$$\mathbf{j} = \text{Div} \left[ \frac{\mathfrak{F} \mathbf{K}}{\mu} \text{Grad} P \right]. \quad (35b)$$

503 In the sequel, we adopt an expression of  $\kappa_{\text{iso}}$  taken from Holmes&Mow [2], given by

$$\kappa_{\text{iso}} = J k_{\text{ref}} \left[ \frac{J - \Phi_s}{1 - \Phi_s} \right]^{m_0} \exp \left( \frac{m_1}{2} [J^2 - 1] \right), \quad (36)$$

504 where  $k_{\text{ref}}$  is a reference permeability, while  $m_0$  and  $m_1$  are non-negative material parameters.

### 505 5.3 Fractional Forchheimer’s correction

506 From the point of view of mathematical modeling, this section is the heart of the present work since we  
 507 propose here a fractionalization of the constitutive law (21), which we provide in the form

$$\boldsymbol{\pi}_{\text{fd}}(t) := -\mathbf{r}_F(\|\mathbf{q}(t)\|) \mathbf{q}(t) - \frac{\alpha t_c^\alpha}{\Gamma(1-\alpha)} \int_{t_{\text{in}}}^t \frac{\mathbf{r}_F(\|\mathbf{q}(\tau)\|)}{(t-\tau)^\alpha} \mathcal{T}_s \mathbf{q}(\tau) d\tau, \quad (37)$$

508 where  $\mathbf{r}_F(\|\mathbf{q}(t)\|)$  is defined in Equation (22),  $t_c$  is a characteristic time scale of the flow,  $\alpha \in ]0, 1[$   
 509 another characteristic parameter of the flow, and  $\mathcal{T}_s \mathbf{q}(\tau)$  denotes the *Truesdell rate* of  $\mathbf{q}$ , computed with  
 510 respect to the velocity of the solid phase, and evaluated at time  $\tau \in [t_{\text{in}}, t[$ . Note that, with the exception  
 511 of  $\alpha$ ,  $t_c$ , and the independent variables  $t$  and  $\tau$ , all the quantities featuring in Equation (37) have to be  
 512 understood as functions of spatial points and time, although we report explicitly the sole dependence on  
 513 time for the sake of a lighter notation. Equation (37), which, to the best of our knowledge, is novel and  
 514 constitutes the starting point of the fractionalization of Forchheimer’s correction, has been inspired by  
 515 the works [38, 31], in which similar models have been proposed to fractionalize Darcy’s law.

516 By substituting Equation (37) into the force balance  $-\phi_f \text{grad} p + \boldsymbol{\pi}_{\text{fd}} = \mathbf{0}$ , we obtain

$$\mathbf{r}_F(\|\mathbf{q}(t)\|) \mathbf{q}(t) + \frac{\alpha t_c^\alpha}{\Gamma(1-\alpha)} \int_{t_{\text{in}}}^t \frac{\mathbf{r}_F(\|\mathbf{q}(\tau)\|)}{(t-\tau)^\alpha} \mathcal{T}_s \mathbf{q}(\tau) d\tau = \phi_f(t) \mu \mathbf{k}^{-1}(t) \mathbf{Q}_D(t), \quad (38)$$

517 which, by virtue of Equation (20), can be recast in the form

$$\mathbf{R}_F(\|\mathbf{Q}(t)\|_{\mathcal{C}(t)}) \mathbf{Q}(t) + \frac{\alpha t_c^\alpha}{\Gamma(1-\alpha)} \int_{t_{\text{in}}}^t \frac{J(t)}{J(\tau)} \frac{\mathbf{F}^T(t) \mathbf{F}^{-T}(\tau) \mathbf{R}_F(\|\mathbf{Q}(\tau)\|_{\mathcal{C}(\tau)})}{(t-\tau)^\alpha} \dot{\mathbf{Q}}(\tau) d\tau$$

$$= \Phi_f(t) \mu \mathbf{K}^{-1}(t) \mathbf{Q}_D(t). \quad (39)$$

518 With reference to Remark 2, we notice that, due to the natural choice of employing the Truesdell rate,  
 519 the backward Piola transformation of Equation (38) to the medium's reference placement yields Equation  
 520 (39), which features the time derivative of the material filtration velocity  $\mathbf{Q}$ . In this case, because of the  
 521 presence of the deformation, Cattaneo equation is not directly recovered under the sole assumptions that  
 522  $\mathbf{a}$  is null and that  $\phi_f$ ,  $\mu$ , and  $\mathbf{k}$  are constant in time.

523 In the case of “*unconditionally isotropic*” [22] porous medium, the resistivity tensor given in Equation  
 524 (29) reads

$$\mathbf{R}_F(\|\mathbf{Q}\|_C) = \frac{\Phi_f \mu}{\kappa_{\text{iso}}} [1 + \mathcal{A}_{\text{iso}} \|\mathbf{Q}\|_C] \mathbf{C} = \mathcal{R}_F(\mathbf{F}, \mathbf{Q}) \mathbf{C}, \quad (40)$$

525 where  $\mathcal{A}_{\text{iso}}$  is defined in Equation (33), and  $\mathcal{R}_F(\mathbf{F}, \mathbf{Q})$  is a *scalar resistivity coefficient* defined by

$$\mathcal{R}_F(\mathbf{F}, \mathbf{Q}) := \frac{\Phi_f \mu}{\kappa_{\text{iso}}} [1 + \mathcal{A}_{\text{iso}} \|\mathbf{Q}\|_C]. \quad (41)$$

526 Therefore, after some algebraic passages, Equation (39) becomes

$$\begin{aligned} \mathcal{R}_F(\mathbf{F}(t), \mathbf{Q}(t)) \mathbf{Q}(t) + \frac{\alpha t_c^\alpha}{\Gamma(1-\alpha)} \int_{t_{\text{in}}}^t \frac{J(t)}{J(\tau)} \frac{\mathcal{R}_F(\mathbf{F}(\tau), \mathbf{Q}(\tau))}{(t-\tau)^\alpha} \mathbf{F}^{-1}(t) \mathbf{F}(\tau) \dot{\mathbf{Q}}(\tau) d\tau \\ = \mathcal{R}_D(\mathbf{F}(t)) \mathbf{Q}_D(t), \end{aligned} \quad (42)$$

527 with  $\mathcal{R}_D(\mathbf{F}) := \Phi_f \mu / \kappa_{\text{iso}}$ , and  $\mathcal{R}_D$  depending on  $\mathbf{F}$  being through  $\Phi_f$  and  $\kappa_{\text{iso}}$ .

528 In conclusion, for the fractional version of Forchheimer's correction analyzed in this section, the  
 529 model equations to be solved are given by

$$\text{Div}(-JPF^{-T} + \mathbf{T}_{\text{sc}}) = \mathbf{0}, \quad (43a)$$

$$\dot{J} + \text{Div} \mathbf{Q} = 0, \quad (43b)$$

$$\begin{aligned} \mathcal{R}_F(\mathbf{F}(t), \mathbf{Q}(t)) \frac{1}{J(t)} \mathbf{F}(t) \mathbf{Q}(t) + \frac{\alpha t_c^\alpha}{\Gamma(1-\alpha)} \int_{t_{\text{in}}}^t \frac{\mathcal{R}_F(\mathbf{F}(\tau), \mathbf{Q}(\tau))}{(t-\tau)^\alpha} \frac{1}{J(\tau)} \mathbf{F}(\tau) \dot{\mathbf{Q}}(\tau) d\tau \\ = \mathcal{R}_D(\mathbf{F}(t)) \frac{1}{J(t)} \mathbf{F}(t) \mathbf{Q}_D(t). \end{aligned} \quad (43c)$$

530 Equations (43a)-(43c) are equivalent to a set of seven scalar equations in the seven unknowns represented  
 531 by the three components of the solid phase motion  $\chi$ , pore pressure  $P$  (which features both in the  
 532 momentum balance law (43a) and in Darcy's velocity  $\mathbf{Q}_D$ , as specified in Equation (13)), and the three  
 533 components of the filtration velocity  $\mathbf{Q}$ . Thus, to close the model, it suffices to assign the solid phase  
 534 volumetric fraction in the reference placement, i.e.,  $\Phi_s$ , which is independent of time in the present study,  
 535 and, as it was already done in the previous sections, to prescribe constitutively the first Piola-Kirchhoff  
 536 stress tensor of the solid phase, i.e.,  $\mathbf{T}_{\text{sc}}$ , the scalar permeability  $\kappa_{\text{iso}}$ , and the coefficient  $c_0$ .

## 537 6 Numerical implementation of the model equations

538 In this section, we introduce the most fundamental aspects of the determination of the numerical solution  
 539 of the fractional Darcy-Forchheimer model (43a)-(43c). We split our study into two parts: first, we  
 540 concentrate on the discretization in time of Equation (43c) and, subsequently, we present the main  
 541 introductory steps to the finite element implementation of the whole system (43a)-(43c).

## 542 6.1 Time discretization of the fractional Darcy-Forchheimer model

543 The starting point for the numerical implementation of Equation (43c) is the renaming  $\alpha = 1 - \beta$ , so that

$$\frac{1}{\Gamma(1 - \alpha)} \int_{t_{\text{in}}}^t \frac{1}{(t - \tau)^\alpha} \mathbf{h}(\tau) d\tau \underset{1 - \alpha = \beta}{=} \frac{1}{\Gamma(\beta)} \int_{t_{\text{in}}}^t \frac{1}{(t - \tau)^{1 - \beta}} \mathbf{h}(\tau) d\tau, \quad (44)$$

544 where  $\mathbf{h}$  is a scalar- or tensor-valued function for which the considered integrals exist. By direct inspection  
545 of Equation (43c), the function  $\mathbf{h}$  is identified with the expression

$$\mathbf{h}(\tau) \equiv \mathcal{R}_{\mathbb{F}}(\mathbf{F}(\tau), \mathbf{Q}(\tau)) \frac{1}{J(\tau)} \mathbf{F}(\tau) \dot{\mathbf{Q}}(\tau), \quad (45)$$

546 with  $\mathcal{R}_{\mathbb{F}}(\mathbf{F}, \mathbf{Q})$  given in Equation (41).

547 The next step is the representation of the fractional operator featuring in Equation (43c) in the  
548 form suggested by Podlubny[80] for the numerical approximation of the Grünwald-Letnikov fractional  
549 derivative, which, for our purposes, we slightly modify as follows:

$$\begin{aligned} \frac{1}{\Gamma(\beta)} \int_{t_{\text{in}}}^t (t - \tau)^{\beta - 1} \mathbf{h}(\tau) d\tau &= \lim_{N \rightarrow \infty} \left( \frac{t - t_{\text{in}}}{N} \right)^\beta \sum_{n=0}^N \begin{bmatrix} \beta \\ n \end{bmatrix} \mathbf{h} \left( t - n \frac{t - t_{\text{in}}}{N} \right) \\ &\approx \left( \frac{t - t_{\text{in}}}{N_0} \right)^\beta \sum_{n=0}^{N_0} \begin{bmatrix} \beta \\ n \end{bmatrix} \mathbf{h} \left( t - n \frac{t - t_{\text{in}}}{N_0} \right), \end{aligned} \quad (46)$$

550 for  $\beta \in ]0, 1[$ , where  $\mathbf{h}$  is assumed to be continuous over the interval  $[t_{\text{in}}, t]$ ,  $N \in \mathbb{N}$  is the number of  
551 sub-intervals partitioning  $[t_{\text{in}}, t]$ ,  $N_0 \in \mathbb{N}$  is a sufficiently large value of  $N$  above which the value of the  
552 sum in the limit does not change appreciably within a given tolerance, and the symbol

$$\begin{bmatrix} \beta \\ 0 \end{bmatrix} = 1 \quad \text{and} \quad \begin{bmatrix} \beta \\ n \end{bmatrix} = \frac{\prod_{i=1}^n (\beta + i - 1)}{n!}, \quad \text{for } n \geq 1, \quad (47)$$

553 generalizes the binomial factor to the case in which  $\beta$  is not a natural number[80].

554 We notice that, in our simulations, the value of  $N_0$  that truncates the series representing the Grünwald-  
555 Letnikov fractional derivative of  $\mathbf{h}$ , as specified in Equation (46), will be taken as a function of the  
556 instant of time at which the corresponding sum is evaluated. To proceed, we introduce two time  
557 grids: one discretizes the time interval  $[t_{\text{in}}, t_{\text{fin}}]$  over which the system is observed, and is given by  
558  $\mathcal{T} := \{t_0, \dots, t_m, \dots, t_M\} \subseteq [t_{\text{in}}, t_{\text{fin}}]$ , so that  $t_0 \equiv t_{\text{in}}$ ,  $t_M \equiv t_{\text{fin}}$ , and  $m = 0, \dots, M$ , with  $M \in \mathbb{N}$ ,  
559  $M > 1$ , being the number of sub-intervals by which  $[t_0, t_M]$  is partitioned, and  $\Delta t := t_m - t_{m-1}$  being the  
560 constant time step; the other grid is used to handle the fractional integral (46) numerically, and discretizes  
561 each interval  $[t_0, t_m]$ ,  $m = 1, \dots, M$ , in  $\hat{N}_0(t_m) > m$  sub-intervals of uniform amplitude

$$s = \frac{t_m - t_0}{\hat{N}_0(t_m)}. \quad (48)$$

562 With reference to the second grid, the nodes are enumerated as  $t_{m,n} = t_m - ns$ , with  $n = 0, \dots, \hat{N}_0(t_m)$ ,  
563 so that  $t_{m,0} = t_m$ , and  $t_{m,\hat{N}_0(t_m)} = t_m - \hat{N}_0(t_m)s = t_0$ . Moreover, we subdivide each interval  $[t_{m-1}, t_m]$   
564 uniformly, we denote by  $n_\sigma \in \mathbb{N}$ ,  $n_\sigma > 1$ , the number of sub-intervals partitioning  $[t_{m-1}, t_m]$ , and we  
565 have  $n_\sigma s = \Delta t$ , and  $t_m - ns \in [t_{m-1}, t_m]$ , for  $n = 0, \dots, n_\sigma$ . Specifically, we obtain  $(t_m - t_{m-1})/n_\sigma =$   
566  $\Delta t/n_\sigma = s$  and  $\hat{N}_0(t_m) = n_\sigma m$ , for all  $m = 1, \dots, M$ .

567 According to the discretization outlined above, Equation (46), evaluated at  $t = t_m$  and rewritten for  
568  $\beta = 1 - \alpha$ , becomes:

$$\frac{1}{\Gamma(1 - \alpha)} \int_{t_0}^{t_m} \frac{1}{(t_m - \tau)^\alpha} \frac{\mathcal{R}_{\mathbb{F}}(\mathbf{F}(\tau), \mathbf{Q}(\tau))}{J(\tau)} \mathbf{F}(\tau) \dot{\mathbf{Q}}(\tau) d\tau$$

$$\begin{aligned}
&\approx \left( \frac{t_m - t_0}{\hat{N}_0(t_m)} \right)^{1-\alpha} \sum_{n=0}^{\hat{N}_0(t_m)} \begin{bmatrix} 1-\alpha \\ n \end{bmatrix} \mathbf{h} \left( t_m - n \frac{t_m - t_0}{\hat{N}_0(t_m)} \right) \\
&= s^{1-\alpha} \mathbf{h}(t_m) + s^{1-\alpha} \sum_{n=1}^{n_\sigma-1} \begin{bmatrix} 1-\alpha \\ n \end{bmatrix} \mathbf{h}(t_m - ns) + s^{1-\alpha} \sum_{n=n_\sigma}^{\hat{N}_0(t_m)} \begin{bmatrix} 1-\alpha \\ n \end{bmatrix} \mathbf{h}(t_m - ns). \quad (49)
\end{aligned}$$

569 The crucial point of the procedure described by Equation (49) is that  $\mathbf{h}(t_m)$  and  $\mathbf{h}(t_m - ns)$  feature  
570 the time derivatives  $\dot{\mathbf{Q}}(t_m)$  and  $\dot{\mathbf{Q}}(t_m - ns)$ . However, the numerical scheme that we use for this  
571 work provides an explicit expression for such derivatives only at the nodes of the *coarser* time grid  
572  $\mathcal{T} = \{t_0, \dots, t_m, \dots, t_M\}$ , i.e., only for  $\dot{\mathbf{Q}}(t_m)$ , which we approximate as

$$\dot{\mathbf{Q}}(t_m) \approx \dot{\mathbf{Q}}_{\text{app}}(t_m) = \frac{\mathbf{Q}(t_m) - \mathbf{Q}(t_{m-1})}{\Delta t}. \quad (50)$$

573 Hence,  $\mathbf{h}(t_m)$  reads

$$\mathbf{h}(t_m) = \frac{\mathcal{R}_F(\mathbf{F}(t_m), \mathbf{Q}(t_m))}{J(t_m)} \mathbf{F}(t_m) \frac{\mathbf{Q}(t_m) - \mathbf{Q}(t_{m-1})}{\Delta t}, \quad (51)$$

574 where all the quantities evaluated at  $t_m$  are unknown, and  $\mathbf{Q}(t_m)$  is *the* unknown for which the discretized  
575 version of Equation (43c) has to be solved. On the other hand, the quantity  $\mathbf{h}(t_m - ns)$  as a whole, for  
576  $m = 2, \dots, M$ , is computed through the interpolation

$$\mathbf{h}(t_m - ns) = \begin{cases} \left(1 - \frac{ns}{\Delta t}\right) \mathbf{h}(t_m) + \frac{ns}{\Delta t} \mathbf{h}(t_{m-1}), & \text{if } n = 0, \dots, n_\sigma - 1, \\ \mathbf{h}(t_m - p\Delta t), & \text{if } n = pn_\sigma, p = 1, \dots, m, \\ \left(1 - \frac{ls}{\Delta t}\right) \mathbf{h}(t_{m-p}) + \frac{ls}{\Delta t} \mathbf{h}(t_{m-p-1}), & \text{if } n = pn_\sigma + l, p = 1, \dots, m-1, l = 1, \dots, n_\sigma-1, \end{cases} \quad (52)$$

577 where  $\mathbf{h}(t_{m-1}), \mathbf{h}(t_{m-2}), \dots$  are obtained by applying Equation (51) recursively. Note that, for  $m = 1$ ,  
578 only the first two rows of Equation (52) apply, with  $p = 1$ .

579 Following a convergence study (data not shown) in which we solved the discretized poroelastic system  
580 (see Equations (87a) and (87b) below), we set  $n_\sigma = 5$  for all  $m = 1, \dots, M$ .

581 By enforcing the scheme outlined so far, the second addend on the left-hand side of Equation (43c)  
582 is approximated as

$$\begin{aligned}
&\frac{\alpha t_c^\alpha}{\Gamma(1-\alpha)} \int_{t_0}^{t_m} \frac{1}{(t_m - \tau)^\alpha} \frac{\mathcal{R}_F(\mathbf{F}(\tau), \mathbf{Q}(\tau))}{J(\tau)} \mathbf{F}(\tau) \dot{\mathbf{Q}}(\tau) d\tau \\
&\approx \alpha t_c^\alpha s^{1-\alpha} \mathbf{h}(t_m) + \alpha t_c^\alpha s^{1-\alpha} \sum_{n=1}^{n_\sigma-1} \begin{bmatrix} 1-\alpha \\ n \end{bmatrix} \mathbf{h}(t_m - ns) + \alpha t_c^\alpha s^{1-\alpha} \sum_{n=n_\sigma}^{\hat{N}_0(t_m)} \begin{bmatrix} 1-\alpha \\ n \end{bmatrix} \mathbf{h}(t_m - ns) \\
&= \alpha t_c^\alpha s^{1-\alpha} \mathbf{h}(t_m) + \alpha t_c^\alpha s^{1-\alpha} \sum_{n=1}^{n_\sigma-1} \begin{bmatrix} 1-\alpha \\ n \end{bmatrix} \left\{ \left(1 - \frac{ns}{\Delta t}\right) \mathbf{h}(t_m) + \frac{ns}{\Delta t} \mathbf{h}(t_{m-1}) \right\} \\
&\quad + \alpha t_c^\alpha s^{1-\alpha} \sum_{n=n_\sigma}^{\hat{N}_0(t_m)} \begin{bmatrix} 1-\alpha \\ n \end{bmatrix} \mathbf{h}(t_m - ns) \\
&= \alpha t_c^\alpha s^{1-\alpha} \zeta_\alpha \mathbf{h}(t_m) + \alpha t_c^\alpha \mathcal{F}_\alpha(t_m), \quad (53)
\end{aligned}$$

583 where we define

$$\zeta_\alpha := 1 + \sum_{n=1}^{n_\sigma-1} \begin{bmatrix} 1-\alpha \\ n \end{bmatrix} \left(1 - \frac{ns}{\Delta t}\right) = \sum_{n=0}^{n_\sigma-1} \begin{bmatrix} 1-\alpha \\ n \end{bmatrix} \left(1 - \frac{ns}{\Delta t}\right), \quad (54a)$$

$$\mathcal{F}_\alpha(t_m) := s^{1-\alpha} \left( \sum_{n=1}^{n_\sigma-1} \begin{bmatrix} 1-\alpha \\ n \end{bmatrix} \frac{ns}{\Delta t} \right) \frac{\mathcal{R}_F(\mathbf{F}(t_{m-1}), \mathbf{Q}(t_{m-1}))}{J(t_{m-1})} \mathbf{F}(t_{m-1}) \dot{\mathbf{Q}}_{\text{app}}(t_{m-1})$$

$$+ s^{1-\alpha} \sum_{n=n_\sigma}^{\hat{N}_0(t_m)} \begin{bmatrix} 1-\alpha \\ n \end{bmatrix} \mathbf{h}(t_m - ns). \quad (54b)$$

584 In conclusion, by collecting the results obtained so far, the time discretized form of Equation (43c), or  
585 equivalently, of Equation (42), reads:

$$\begin{aligned} & \mathcal{R}_F(\mathbf{F}(t_m), \mathbf{Q}(t_m))\mathbf{Q}(t_m) + \alpha t_c^\alpha s^{1-\alpha} \zeta_\alpha \mathcal{R}_F(\mathbf{F}(t_m), \mathbf{Q}(t_m)) \dot{\mathbf{Q}}_{\text{app}}(t_m) + \alpha t_c^\alpha J(t_m) \mathbf{F}^{-1}(t_m) \mathcal{F}_\alpha(t_m) \\ & = \mathcal{R}_D(\mathbf{F}(t_m))\mathbf{Q}_D(t_m). \end{aligned} \quad (55)$$

586 Moreover, to single out the unknown to be determined through the solution of Equation (55), i.e.,  $\mathbf{Q}(t_m)$ ,  
587 and in view of the linearization procedure that will be employed for the finite element simulations  
588 performed in the sequel, we take into account the expression of  $\dot{\mathbf{Q}}_{\text{app}}$  in Equation (50), we highlight the  
589 dependence of  $\mathbf{Q}_D$  on  $\mathbf{F}$  and  $\text{Grad}P$  by writing  $\mathbf{Q}_D = \mathcal{G}^{\mathbf{Q}_D}(\mathbf{F}, \text{Grad}P)$ , and we recast Equation (55) as

$$\begin{aligned} \mathcal{Z}(\mathbf{F}_m, \text{Grad}P_m, \mathbf{Q}_m) := & \left(1 + \frac{\alpha t_c^\alpha s^{1-\alpha} \zeta_\alpha}{\Delta t}\right) \mathcal{R}_F(\mathbf{F}_m, \mathbf{Q}_m)\mathbf{Q}_m - \frac{\alpha t_c^\alpha s^{1-\alpha} \zeta_\alpha}{\Delta t} \mathcal{R}_F(\mathbf{F}_m, \mathbf{Q}_m)\mathbf{Q}_{m-1} \\ & + \alpha t_c^\alpha J_m \mathbf{F}_m^{-1} \mathcal{F}_\alpha(t_m) - \mathcal{R}_D(\mathbf{F}_m) \mathcal{G}^{\mathbf{Q}_D}(\mathbf{F}_m, \text{Grad}P_m) = \mathbf{0}, \end{aligned} \quad (56)$$

590 where, for any generic physical quantity  $\Psi$ , the notation  $\Psi(t_m) \equiv \Psi_m$  has been employed to express that  
591 it is evaluated at time  $t_m$  (the dependence on  $X$  is omitted, but understood).

592 We remark that, by performing some lengthy algebraic manipulations, Equation (56) can be solved  
593 analytically for  $\mathbf{Q}_m$  by turning it into a polynomial equation of grade four in  $\mathbf{Q}_m$ . This property descends  
594 from the dependence of  $\mathcal{R}_F(\mathbf{F}_m, \mathbf{Q}_m)$  on  $\mathbf{Q}_m$  being through the  $C_m$ -norm  $\|\mathbf{Q}_m\|_{C_m}$ . However, although  
595 the four roots of an equation of this type can be computed analytically, it is very difficult to ascertain, for  
596 generic values of  $\mathbf{F}_m$  and  $\text{Grad}P_m$ , which solutions are physically admissible. Moreover, if more than one  
597 physically admissible solutions exist, the problem of non-uniqueness of the solution arises, and, even in  
598 the case in which the solution were unique, its analytical expression would be too complicated to study  
599 it in conjunction with the other two equations of the model. For all these reasons, we prefer to proceed  
600 with the search for a unique numeric solution, to be found through a Newton-Raphson method around a  
601 “good” initial guess. These considerations lead us to the adoption of the following procedure.

## 602 6.2 Linearization of the fractional Darcy-Forchheimer model

603 The discretized, fractional Darcy-Forchheimer equation (56) should be studied in conjunction with the  
604 discretized version of the balance laws (43a) and (43b), put in weak form in view of their finite element  
605 implementation. In this respect, we notice that we have conducted the numerical simulations of our  
606 work in ABAQUS<sup>®</sup>, partially writing our own code for solving Equations (43a), (43b) and (56), but  
607 not for the whole implementation. Thus, although we do not have complete control over the numerical  
608 procedures employed by the commercial software, some properties of Equations (43a), (43b), and (56)  
609 can be discussed, even without entering the details of their numerical analysis.

610 As anticipated above, we neglect gravity, and to render the weak forms of Equations (43a) and (43b)  
611 as simple as possible, we consider the case in which their associated boundary terms are identically zero.  
612 To comply with these conditions, we partition the boundary of  $\mathcal{B}$ , for the motion  $\chi$ , into the disjoint  
613 union of a traction-free part and Dirichlet part, and, for the pressure  $P$ , into the disjoint union of a  
614 flux-free part and, again, of a Dirichlet part. Then, within this setting, we take the procedure adopted  
615 in[66] for the purely Darcian, hyperelastic, and isotropic case, and extended in[88] for poroplasticity,  
616 and in[17] for anisotropic, fiber-reinforced porous media. In the sequel, we show the most fundamental  
617 steps of its generalization to our model, which, although being isotropic and homogeneous, takes into  
618 account Forchheimer’s correction to Darcy’s law and the interactions between the fluid and the solid  
619 phase arising because of such correction.

620 To begin with, we consider a three-field formulation of the problem at hand, which involves Equation  
621 (56) and the time-discrete, weak forms of Equations (43a) and (43b). This leads to the system

$$A(\chi_m, P_m, \mathbf{Q}_m; \mathbf{V}_v) := \int_{\mathcal{B}} \left\{ -J_m P_m \mathbf{F}_m^{-T} + \mathcal{G}^{T_{sc}}(\mathbf{F}_m) \right\} : \text{Grad} \mathbf{V}_v - \int_{\partial_N^x \mathcal{B}} (\mathbf{T}_{1m} \mathbf{N}) \mathbf{V}_v = 0, \quad (57a)$$

$$B(\chi_m, P_m, \mathbf{Q}_m; P_v) := - \int_{\mathcal{B}} \frac{J_m - J_{m-1}}{\Delta t} P_v + \int_{\mathcal{B}} \mathbf{Q}_m \text{Grad} P_v - \int_{\partial_N^p \mathcal{B}} (\mathbf{Q}_m \mathbf{N}) P_v = 0, \quad (57b)$$

$$\begin{aligned} \mathcal{Z}(\mathbf{F}_m, \text{Grad} P_m, \mathbf{Q}_m) := & \left( 1 + \frac{\alpha t_c^\alpha s^{1-\alpha} \zeta_\alpha}{\Delta t} \right) \mathcal{R}_F(\mathbf{F}_m, \mathbf{Q}_m) \mathbf{Q}_m - \frac{\alpha t_c^\alpha s^{1-\alpha} \zeta_\alpha}{\Delta t} \mathcal{R}_F(\mathbf{F}_m, \mathbf{Q}_m) \mathbf{Q}_{m-1} \\ & + \alpha t_c^\alpha J_m \mathbf{F}_m^{-1} \mathcal{F}_\alpha(t_m) - \mathcal{R}_D(\mathbf{F}_m) \mathcal{G}^{Q_D}(\mathbf{F}_m, \text{Grad} P_m) = \mathbf{0}, \end{aligned} \quad (57c)$$

622 in the unknowns  $\chi_m$ ,  $P_m$ , and  $\mathbf{Q}_m$ . To obtain Equations (57a) and (57b), we have introduced: the test  
623 functions  $\mathbf{V}_v$  and  $P_v$ , identifiable with an arbitrary virtual velocity and an arbitrary virtual pressure,  
624 respectively; the constitutive representation  $\mathcal{G}^{T_{sc}}(\mathbf{F}_m) \equiv \mathbf{T}_{sc}(t_m)$ , at time  $t_m$ , of the first Piola-Kirchhoff  
625 stress tensor of the solid phase; the portions  $\partial_N^x \mathcal{B}$  and  $\partial_N^p \mathcal{B}$  of the boundary of  $\mathcal{B}$ , i.e.,  $\partial \mathcal{B}$ , on  
626 which Neumann boundary conditions on the solid phase motion and on the pressure field are enforced,  
627 respectively; and the field of co-normals  $\mathbf{N}$  associated with the boundary of  $\mathcal{B}$ . We remark that, although  
628 we have reported the boundary terms in Equations (57a) and (57b), these are identically null in our  
629 setting.

630 The system (57a)-(57c) is highly non-linear in the motion  $\chi_m$  and in the filtration velocity  $\mathbf{Q}_m$ , and it  
631 will be solved by employing a linearization procedure. One possible way is to perform, for each time  $t_m$ ,  
632 a Newton-Raphson method in a neighborhood of an initially *guessed* triple  $(\chi_m^0, P_m^0, \mathbf{Q}_m^0)$ , with unknown  
633 increments  $(\delta \chi_m^1, \delta P_m^1, \delta \mathbf{Q}_m^1)$ , and then, by iteration, to construct the sequence of triples

$$(\chi_m^k = \chi_m^{k-1} + \delta \chi_m^k, P_m^k = P_m^{k-1} + \delta P_m^k, \mathbf{Q}_m^k = \mathbf{Q}_m^{k-1} + \delta \mathbf{Q}_m^k), \quad \text{for } k \geq 1. \quad (58)$$

634 At each time  $t_m$  and iteration  $k \geq 1$ , such a method requires the determination of the three increments  
635  $\delta \chi_m^k$ ,  $\delta P_m^k$ , and  $\delta \mathbf{Q}_m^k$ , for each of which it is necessary to provide a suitable spatial interpolation. However,  
636 rather than proceeding this way, we find it more convenient to follow a different path, as explained below.

637 **Two-field-approach by means of Dini's implicit function Theorem.** In this paragraph we present a  
638 numerical procedure that shares similarities with the one elaborated in[89]. We notice that, for  $m \geq 1$ ,  
639 there exists a non-empty open set  $\Omega_m$  of triples

$$(\mathbf{F}_m, \text{Grad} P_m, \mathbf{Q}_m) \in [T\mathcal{B}_t \otimes T^*\mathcal{B}] \times T^*\mathcal{B} \times T\mathcal{B}, \quad \text{with } \mathbf{Q}_m \neq \mathbf{0}, \quad (59)$$

640 such that the function  $\mathcal{Z}$  defined by the right-hand side of Equation (57c) is of class  $C^1(\Omega_m; T\mathcal{B})$ . Then,  
641 we assume that there exists a non-empty subset of  $\Omega_m$ , hereafter denoted by  $\Sigma_m \subset \Omega_m$ , that consists  
642 of all the triples  $(\mathbf{F}_m, \text{Grad} P_m, \mathbf{Q}_m) \in \Omega_m$  that satisfy Equation (57c) as an identity, i.e., that constitute  
643 the intersection between  $\Omega_m$  and the set of all the solutions of  $\mathcal{Z}(\mathbf{F}_m, \text{Grad} P_m, \mathbf{Q}_m) = \mathbf{0}$ , and for which  
644 the partial derivative of  $\mathcal{Z}$  with respect to  $\mathbf{Q}_m$  is a non-singular second-order tensor. Hence, by setting  
645  $(\#) := (\mathbf{F}_m, \text{Grad} P_m, \mathbf{Q}_m)$ , it holds by hypothesis that  $\det[\partial_{\mathbf{Q}_m} \mathcal{Z}(\#)] \neq 0$  for all  $(\mathbf{F}_m, \text{Grad} P_m, \mathbf{Q}_m) \in \Sigma_m$ ,  
646 and  $\partial_{\mathbf{Q}_m} \mathcal{Z}(\#)$  is given by

$$\begin{aligned} \partial_{\mathbf{Q}_m} \mathcal{Z}(\#) &= \mathcal{R}_F(\mathbf{F}_m, \mathbf{Q}_m) \left[ 1 + \frac{\alpha t_c^\alpha s^{1-\alpha} \zeta_\alpha}{\Delta t} \right] \mathbf{I} + \left\{ \mathbf{Q}_m + \frac{\alpha t_c^\alpha s^{1-\alpha} \zeta_\alpha}{\Delta t} [\mathbf{Q}_m - \mathbf{Q}_{m-1}] \right\} \otimes \frac{\partial \mathcal{R}_F}{\partial \mathbf{Q}_m}(\mathbf{F}_m, \mathbf{Q}_m) \\ &= \mathcal{R}_F(\mathbf{F}_m, \mathbf{Q}_m) \left[ 1 + \frac{\alpha t_c^\alpha s \zeta_\alpha}{s^\alpha \Delta t} \right] \mathbf{I} + \frac{\Phi_{fm} \mu}{\kappa_{isom}} \mathcal{A}_{isom} \left\{ \mathbf{Q}_m + \frac{\alpha t_c^\alpha s \zeta_\alpha}{s^\alpha \Delta t} [\mathbf{Q}_m - \mathbf{Q}_{m-1}] \right\} \otimes \frac{J_m^{-2} \mathbf{C}_m \mathbf{Q}_m}{\|\mathbf{Q}_m\| \mathbf{C}_m}, \end{aligned} \quad (60)$$

647 where  $\Phi_{fm} \equiv J_m - \Phi_{SR}$  is the pull-back of the fluid phase volumetric fraction evaluated at time  $t_m$ , while  
648  $\kappa_{isom}$  and  $\mathcal{A}_{isom}$  denote  $\kappa_{iso}$  and  $\mathcal{A}_{iso}$  at time  $t_m$ .

649 In fact, all the properties of  $\mathcal{Z}$  and of  $\partial_{\mathcal{Q}_m}\mathcal{Z}$  enunciated so far constitute the hypotheses of Dini's  
650 Implicit Function Theorem for vector-valued functions of multiple arguments. Therefore, by selecting  
651 one triple  $(\#) \equiv (\bar{F}_m, \text{Grad}\bar{P}_m, \bar{Q}_m) \in \Sigma_m$  (for which, thus,  $\mathcal{Z}(\#) = \mathbf{0}$ , and  $\det[\partial_{\mathcal{Q}_m}\mathcal{Z}(\#)] \neq 0$ ), there  
652 exists a neighborhood  $\mathcal{V}(\bar{F}_m, \text{Grad}\bar{P}_m, \bar{Q}_m) \subset \Omega_m$  of such triple such that, for the elements of the  
653 intersection  $\mathcal{V}(\bar{F}_m, \text{Grad}\bar{P}_m, \bar{Q}_m) \cap \Sigma_m \neq \emptyset$  it is possible to express  $\mathcal{Q}_m$  as a function of  $F_m$  and  $\text{Grad}P_m$   
654 for some neighborhood  $\mathcal{U}(\bar{F}_m, \text{Grad}\bar{P}_m) \subset [T\mathcal{B}_t \otimes T^*\mathcal{B}] \times T^*\mathcal{B}$  of the pair  $(\bar{F}_m, \text{Grad}\bar{P}_m)$ . By denoting  
655 this vector-valued function by

$$\mathcal{G}^{\mathcal{Q}_m} : \mathcal{U}(\bar{F}_m, \text{Grad}\bar{P}_m) \rightarrow T\mathcal{B}, \quad (F_m, \text{Grad}P_m) \mapsto \mathcal{G}^{\mathcal{Q}_m}(F_m, \text{Grad}P_m) = \mathcal{Q}_m, \quad (61)$$

656 Equation (57c) is identically satisfied by replacing  $\mathcal{Q}_m$  with  $\mathcal{G}^{\mathcal{Q}_m}(F_m, \text{Grad}P_m)$ , thereby obtaining

$$\hat{\mathcal{Z}}(F_m, \text{Grad}P_m) \equiv \mathcal{Z}(F_m, \text{Grad}P_m, \mathcal{G}^{\mathcal{Q}_m}(F_m, \text{Grad}P_m)) = \mathbf{0}, \quad (62)$$

657 for all  $(F_m, \text{Grad}P_m) \in \mathcal{U}(\bar{F}_m, \text{Grad}\bar{P}_m)$ . Hence, the just defined function  $\hat{\mathcal{Z}} : \mathcal{U}(\bar{F}_m, \text{Grad}\bar{P}_m) \rightarrow T\mathcal{B}$   
658 is constant in the neighborhood  $\mathcal{U}(\bar{F}_m, \text{Grad}\bar{P}_m)$  and, since it is also of class  $C^1$  therein, it has vanishing  
659 differential. In fact, upon setting  $Y_m := \text{Grad}P_m$  and  $(\natural) := (F_m, \text{Grad}P_m) \equiv (F_m, Y_m) \in \mathcal{U}(\bar{F}_m, \text{Grad}\bar{P}_m)$ ,  
660 from the condition of annihilation of the differential of  $\hat{\mathcal{Z}}$  along any pair of admissible increments  
661  $(\delta F_m, \delta Y_m)$ , we find:

$$\begin{aligned} d\hat{\mathcal{Z}}(\natural)(\delta F_m, \delta Y_m) &= [\partial_{F_m}\hat{\mathcal{Z}}(\natural)] : \delta F_m + [\partial_{Y_m}\hat{\mathcal{Z}}(\natural)]\delta Y_m \\ &= [\partial_{F_m}\mathcal{Z}(\#)] : \delta F_m + [\partial_{Y_m}\mathcal{Z}(\#)]\delta Y_m \\ &\quad + [\partial_{\mathcal{Q}_m}\mathcal{Z}(\#)] [\partial_{F_m}\mathcal{G}^{\mathcal{Q}_m}(\natural)] : \delta F_m + [\partial_{\mathcal{Q}_m}\mathcal{Z}(\#)] [\partial_{Y_m}\mathcal{G}^{\mathcal{Q}_m}(\natural)]\delta Y_m \\ &= \{ \partial_{F_m}\mathcal{Z}(\#) + [\partial_{\mathcal{Q}_m}\mathcal{Z}(\#)] [\partial_{F_m}\mathcal{G}^{\mathcal{Q}_m}(\natural)] \} : \delta F_m \\ &\quad + \{ \partial_{Y_m}\mathcal{Z}(\#) + [\partial_{\mathcal{Q}_m}\mathcal{Z}(\#)] [\partial_{Y_m}\mathcal{G}^{\mathcal{Q}_m}(\natural)] \} \delta Y_m = 0. \end{aligned} \quad (63)$$

662 Accordingly, the coefficients of  $\delta F_m$  and  $\delta Y_m$  must vanish independently from one another, i.e.,

$$\partial_{F_m}\mathcal{Z}(\#) + [\partial_{\mathcal{Q}_m}\mathcal{Z}(\#)] [\partial_{F_m}\mathcal{G}^{\mathcal{Q}_m}(\natural)] = \mathbf{0} \quad \Rightarrow \quad \partial_{F_m}\mathcal{G}^{\mathcal{Q}_m}(\natural) = -[\partial_{\mathcal{Q}_m}\mathcal{Z}(\#)]^{-1} \partial_{F_m}\mathcal{Z}(\#), \quad (64a)$$

$$\partial_{Y_m}\mathcal{Z}(\#) + [\partial_{\mathcal{Q}_m}\mathcal{Z}(\#)] [\partial_{Y_m}\mathcal{G}^{\mathcal{Q}_m}(\natural)] = \mathbf{0} \quad \Rightarrow \quad \partial_{Y_m}\mathcal{G}^{\mathcal{Q}_m}(\natural) = -[\partial_{\mathcal{Q}_m}\mathcal{Z}(\#)]^{-1} \partial_{Y_m}\mathcal{Z}(\#), \quad (64b)$$

663 where  $\mathbf{0}$  is the null element in the space of third-order tensors.

664 The result reported in Equation (61) permits to rephrase the system (57a)-(57c) as a system consisting  
665 of its first two equations only, i.e. [66, 88, 17],

$$\hat{A}(\chi_m, P_m; V_v) := \int_{\mathcal{B}} \{ -J_m P_m F_m^{-T} + \mathcal{G}^{T_{sc}}(F_m) \} : \text{Grad}V_v = 0, \quad (65a)$$

$$\hat{B}(\chi_m, P_m; P_v) := - \int_{\mathcal{B}} \frac{J_m - J_{m-1}}{\Delta t} P_v + \int_{\mathcal{B}} \mathcal{G}^{\mathcal{Q}_m}(F_m, \text{Grad}P_m) \text{Grad}P_v = 0, \quad (65b)$$

666 where the functionals  $\hat{A}$  and  $\hat{B}$  are highly non-linear both in  $\chi_m$  and in  $P_m$ .

667 *Remark 4.* It is important to remark that the function  $\mathcal{G}^{\mathcal{Q}_m}$ , although it exists, is not determined explicitly,  
668 since its determination would constitute a very demanding task. However, it is not necessary to find it in  
669 closed form. This is because we are going to solve Equations (65a) and (65b) through a Newton-Raphson  
670 linearization procedure, which, to determine the unknown increments of  $\chi_m$  and  $P_m$  at each iteration,  
671 only requires the knowledge of the partial derivatives of  $\mathcal{G}^{\mathcal{Q}_m}$  at the values of  $F_m$  and  $\text{Grad}P_m$  obtained  
672 at the preceding iteration. In this respect, we emphasize that, since an expression of  $\mathcal{G}^{\mathcal{Q}_m}$  as a function  
673 of  $F_m$  and  $\text{Grad}P_m$  is not available, the writing  $\hat{B}(\chi_m, P_m; P_v)$  has to be regarded as merely formal. More  
674 specifically, it has to be understood as  $\hat{B}(\chi_m, P_m; P_v) \equiv B(\chi_m, P_m, \mathcal{Q}_m; P_v)$ , in which  $\chi_m$  and  $P_m$  are the  
675 solutions to Equations (65a) and (65b), obtained by means of the procedure just mentioned, while  $\mathcal{Q}_m$   
676 will be determined separately through an additional Newton-Raphson method applied to Equation (57c),  
677 once  $F_m$  and  $\text{Grad}P_m$  are known (see also [89]).

678 **Newton-Raphson method applied to Equations (65a) and (65b).** To sketch the linearization procedure  
679 adopted to solve Equations (65a) and (65b), we set  $k \geq 1$ , with  $k \in \mathbb{N}$ , and we introduce both for  
680  $\chi_m$  and for  $P_m$  the values inherited from the  $(k-1)$ th iteration, i.e.,  $\chi_m^{k-1}$  and  $P_m^{k-1}$ , which are regarded  
681 as known, and the *unknown* increments  $\delta\chi_m^k$  and  $\delta P_m^k$ . Hence, we write[66, 88, 17]

$$\chi_m^k := \chi_m^{k-1} + \delta\chi_m^k \quad \Rightarrow \quad \delta\mathbf{F}_m^k = \text{Grad}\delta\chi_m^k, \quad (66a)$$

$$P_m^k := P_m^{k-1} + \delta P_m^k \quad \Rightarrow \quad \delta\text{Grad}P_m^k = \text{Grad}\delta P_m^k. \quad (66b)$$

682 Then, to shorten the notation, we define  $\mathbf{u}_m^{k-1} := (\chi_m^{k-1}, P_m^{k-1})$  and the *approximated* functionals

$$\hat{A}_{\text{app}}(\delta\chi_m^k, \delta P_m^k; \mathbf{V}_v) := \hat{A}(\mathbf{u}_m^{k-1}; \mathbf{V}_v) + \mathcal{D}_\chi \hat{A}(\mathbf{u}_m^{k-1}; \mathbf{V}_v)[\delta\chi_m^k] + \mathcal{D}_P \hat{A}(\mathbf{u}_m^{k-1}; \mathbf{V}_v)[\delta P_m^k], \quad (67a)$$

$$\hat{B}_{\text{app}}(\delta\chi_m^k, \delta P_m^k; P_v) := \hat{B}(\mathbf{u}_m^{k-1}; P_v) + \mathcal{D}_\chi \hat{B}(\mathbf{u}_m^{k-1}; P_v)[\delta\chi_m^k] + \mathcal{D}_P \hat{B}(\mathbf{u}_m^{k-1}; P_v)[\delta P_m^k], \quad (67b)$$

683 where for a generic functional  $\hat{L} \in \{\hat{A}, \hat{B}\}$  and a generic virtual field  $\psi_v \in \{\mathbf{V}_v, P_v\}$ ,  $\mathcal{D}_\chi \hat{L}(\mathbf{u}_m^{k-1}; \psi_v)[\delta\chi_m^k]$   
684 and  $\mathcal{D}_P \hat{L}(\mathbf{u}_m^{k-1}; \psi_v)[\delta P_m^k]$  denote the Gâteaux derivatives of  $\hat{L}$  with respect to the motion and pressure,  
685 evaluated at  $(\mathbf{u}_m^{k-1}; \psi_v)$ , and computed along the increments  $\delta\chi_m^k$  and  $\delta P_m^k$ , respectively.

686 Upon enforcing the conditions  $\hat{A}_{\text{app}}(\delta\chi_m^k, \delta P_m^k; \mathbf{V}_v) = 0$  and  $\hat{B}_{\text{app}}(\delta\chi_m^k, \delta P_m^k; P_v) = 0$ , the equations  
687 determining the increments  $\delta\chi_m^k$  and  $\delta P_m^k$  at each time  $t_m$  and  $k$ th iteration of Newton's method, for  
688  $k \geq 1$ , are given by[66, 88, 90]

$$\mathcal{D}_\chi \hat{A}(\mathbf{u}_m^{k-1}; \mathbf{V}_v)[\delta\chi_m^k] + \mathcal{D}_P \hat{A}(\mathbf{u}_m^{k-1}; \mathbf{V}_v)[\delta P_m^k] = -\hat{A}(\mathbf{u}_m^{k-1}; \mathbf{V}_v), \quad (68a)$$

$$\mathcal{D}_\chi \hat{B}(\mathbf{u}_m^{k-1}; P_v)[\delta\chi_m^k] + \mathcal{D}_P \hat{B}(\mathbf{u}_m^{k-1}; P_v)[\delta P_m^k] = -\hat{B}(\mathbf{u}_m^{k-1}; P_v). \quad (68b)$$

689 As is standard in linearization methods, the iterations stop for some positive integer  $k_* \geq 1$  such that, for  
690 all  $k \geq k_*$ , the absolute values  $|\hat{A}(\chi_m^k, P_m^k; \mathbf{V}_v)|$  and  $|\hat{B}(\chi_m^k, P_m^k; P_v)|$  are smaller than a given tolerance.

691 Finally, there remains to determine the explicit expressions of the Gâteaux derivatives reported in  
692 Equations (68a) and (68b). In fact, the Gâteaux derivatives featuring in Equation (68a) are rather standard,  
693 and especially the one evaluated along  $\delta\chi_m^k$  can be found in textbooks (see e.g.[91, 67]). However, in  
694 order to make our work self-contained, we show all the terms of Equations (68a) and (68b). To begin  
695 with, we notice that, due to the hypothesis of incompressibility of the solid and fluid phase, the stress  
696 tensor featuring in Equation (65a), which we write at time  $t_m$  and  $k$ th iteration as

$$\mathbf{T}_{Im}^k := -J_m^k P_m^k [\mathbf{F}_m^k]^{-T} + \mathcal{G}^{\text{Tsc}}(\mathbf{F}_m^k), \quad (69)$$

697 can be obtained by employing the augmented energy density  $W_s^a(\mathbf{F}_m, P_m) \equiv \Psi_s^a(\mathbf{C}_m, P_m)$ , with  $\Psi_s^a$  given  
698 in Equation (5a). Hence, upon writing

$$W_s^a(\mathbf{F}_m, P_m) = \frac{1}{2} \Phi_{\text{SR}} \mu_s [\text{tr} \mathbf{C}_m - 3] - \Phi_{\text{SR}} \mu_s \log J_m + \frac{1}{2} \Phi_{\text{SR}} \lambda_s [\log J_m]^2 - [J_m - 1] P_m, \quad (70)$$

699 where we have highlighted the dependence on  $\mathbf{F}_m$  (through  $\mathbf{C}_m$  and  $J_m$  on the right-hand side) and  $P_m$ , it  
700 holds that

$$\begin{aligned} \mathbf{T}_{Im}^k &\equiv \mathcal{G}^{\text{T}_1}(\mathbf{F}_m^k, P_m^k) \\ &= \frac{\partial W_s^a}{\partial \mathbf{F}_m}(\mathbf{F}_m^k, P_m^k) = \underbrace{\Phi_{\text{SR}} \mu_s \eta \mathbf{F}_m^k \mathbf{G}^{-1} - \Phi_{\text{SR}} \mu_s [\mathbf{F}_m^k]^{-T} + \Phi_{\text{SR}} \lambda_s [\log J_m^k] [\mathbf{F}_m^k]^{-T} - J_m^k P_m^k [\mathbf{F}_m^k]^{-T}}_{\equiv \mathcal{G}^{\text{Tsc}}(\mathbf{F}_m^k)}, \end{aligned} \quad (71)$$

701 where  $\mathbf{G}$  is the material metric tensor. Accordingly, the Gâteaux derivatives  $\mathcal{D}_\chi \hat{A}(\mathbf{u}_m^{k-1}; \mathbf{V}_v)[\delta\chi_m^k]$  and  
702  $\mathcal{D}_P \hat{A}(\mathbf{u}_m^{k-1}; \mathbf{V}_v)[\delta P_m^k]$  are given by

$$\mathcal{D}_\chi \hat{A}(\mathbf{u}_m^{k-1}; \mathbf{V}_v)[\delta\chi_m^k] = \int_{\mathcal{B}} \left[ \frac{\partial^2 W_s^a}{\partial \mathbf{F}_m^2}(\mathbf{F}_m^{k-1}, P_m^{k-1}) : \text{Grad} \delta\chi_m^k \right] : \text{Grad} \mathbf{V}_v =: [\mathbf{C}_{\chi\chi}]_m^{k-1}(\delta\chi_m^k, \mathbf{V}_v), \quad (72a)$$

$$\mathcal{D}_P \hat{A}(\mathbf{u}_m^{k-1}; \mathbf{V}_v)[\delta P_m^k] = \int_{\mathcal{B}} \left[ \frac{\partial^2 W_s^a}{\partial \mathbf{F}_m \partial P_m}(\mathbf{F}_m^{k-1}, P_m^{k-1}) \delta P_m^k \right] : \text{Grad} \mathbf{V}_v =: [C_{\chi P}]_m^{k-1}(\delta P_m^k, \mathbf{V}_v), \quad (72b)$$

703 where the notation  $[C_{\chi\chi}]_m^{k-1}(\delta\chi_m^k, \mathbf{V}_v)$  and  $[C_{\chi P}]_m^{k-1}(\delta P_m^k, \mathbf{V}_v)$  is meant to highlight the influence of the  
704 motion on itself and the one of the pore pressure on the motion, respectively.

705 We recognize that the second derivative of  $W_s^a$  with respect to  $\mathbf{F}_m$ , hereafter denoted by  $\mathbb{A}_{Im}^{k-1}$ , is the  
706 (augmented) *algorithmic first elasticity tensor*[91] of the mixture as a whole, while the mixed derivative  
707 of  $W_s^a$  with respect to  $\mathbf{F}_m$  and  $P_m$  is representative of the presence of the pore pressure, intended as a  
708 Lagrange multiplier of the present theory, in the expression of the mixture's inner stress tensor. In explicit  
709 form, these derivatives read

$$\frac{\partial^2 W_s^a}{\partial \mathbf{F}_m^2}(\mathbf{F}_m^{k-1}, P_m^{k-1}) \equiv \mathbb{A}_{Im}^{k-1} = \boldsymbol{\eta} \otimes \mathbf{S}_{Im}^{k-1} + [\boldsymbol{\eta} \mathbf{F}_m^{k-1}] \mathbb{C}_{Im}^{k-1} : [(\boldsymbol{\eta} \mathbf{F}_m^{k-1})^T \otimes \mathbf{I}^T], \quad (73a)$$

$$\frac{\partial^2 W_s^a}{\partial \mathbf{F}_m \partial P_m}(\mathbf{F}_m^{k-1}, P_m^{k-1}) = -J_m^{k-1} [\mathbf{F}_m^{k-1}]^{-T}. \quad (73b)$$

710 where  $\mathbf{S}_{Im}^{k-1} = [\mathbf{F}_m^{k-1}]^{-1} \boldsymbol{\eta}^{-1} \mathbf{T}_{Im}^{k-1}$  is the inner part of the mixture's second Piola-Kirchhoff stress tensor,  
711 and  $\mathbb{C}_{Im}^{k-1}$  is the elasticity tensor associated with it (i.e.,  $\mathbb{C}_{Im}^{k-1}$  consists of the sum of the true elasticity tensor  
712 of the solid phase and of the pressure contribution stemming from the hypothesis of incompressibility)

$$\mathbb{C}_{Im}^{k-1} = 4 \frac{\partial^2 \Psi_s^a}{\partial \mathcal{C}_m^2}(\mathbf{C}_m^{k-1}, P_m^{k-1}). \quad (74a)$$

713 Note that, in writing the last term of Equation (73a), the minor symmetry of  $\mathbb{C}_{Im}^{k-1}$  in its last pair of  
714 indices has been used. More explicitly, for the considered  $W_s^a$ , the first elasticity tensor is given by

$$\begin{aligned} \mathbb{A}_{Im}^{k-1} = & \Phi_{sR} \mu_s \boldsymbol{\eta} \otimes \mathbf{G}^{-1} + (\Phi_{sR} \mu_s - \Phi_{sR} \lambda_s \log J_m^{k-1} + J_m^{k-1} P_m^{k-1}) [\mathbf{F}_m^{k-1}]^{-T} \otimes [\mathbf{F}_m^{k-1}]^{-1} \\ & + (\Phi_{sR} \lambda_s - J_m^{k-1} P_m^{k-1}) [\mathbf{F}_m^{k-1}]^{-T} \otimes [\mathbf{F}_m^{k-1}]^{-T}. \end{aligned} \quad (75)$$

715 *Remark 5.* In order to comply with the user interface of the ‘‘UMAT’’ subroutine in ABAQUS<sup>®</sup>, the  
716 Gâteaux derivative  $\mathcal{D}_\chi \hat{A}(\mathbf{u}_m^{k-1}; \mathbf{V}_v)[\delta\chi_m^k]$  in Equation (72a) is rephrased in such a way that its integrand  
717 is calculated with respect to the symmetrized increment of the deformation rate, defined as  $\delta \mathbf{d}_m^k :=$   
718  $\text{sym}(\boldsymbol{\eta}(\text{Grad} \delta\chi_m^k)[\mathbf{F}_m^{k-1}]^{-1})$ , to the updated symmetrized ‘‘spatial’’ gradient of the Eulerian counterpart  
719 of  $\mathbf{V}_v$ , which we write as  $\mathbf{d}_{vm}^k := \text{sym}(\boldsymbol{\eta}(\text{Grad} \mathbf{V}_v)[\mathbf{F}_m^{k-1}]^{-1})$ , to the incremental deformation gradient  
720 tensor  $\delta \mathbf{l}_m^k := (\text{Grad} \delta\chi_m^k)[\mathbf{F}_m^{k-1}]^{-1}$ , and to the gradient of the Eulerian counterpart of  $\mathbf{V}_v$ , which is  
721  $\mathbf{l}_{vm}^k := (\text{Grad} \mathbf{V}_v)[\mathbf{F}_m^{k-1}]^{-1}$ . To this end, we define the push-forward of the elasticity tensor  $\mathbb{C}_{Im}^{k-1}$   
722 featuring in Equation (73a), i.e.,

$$[\mathbb{C}_{Im}^{k-1}]^{spqr} := \frac{1}{J_m^{k-1}} [\mathbb{C}_{Im}^{k-1}]^{SPQR} [\mathbf{F}_m^{k-1}]_S^s [\mathbf{F}_m^{k-1}]_P^p [\mathbf{F}_m^{k-1}]_Q^q [\mathbf{F}_m^{k-1}]_R^r, \quad (76)$$

723 and we write the second Piola-Kirchhoff stress tensor as  $\mathbf{S}_{Im}^{k-1} = J_m^{k-1} [\mathbf{F}_m^{k-1}]^{-1} \boldsymbol{\sigma}_{Im}^{k-1} [\mathbf{F}_m^{k-1}]^{-T}$ . Here,  
724 and in the sequel, the Cauchy stress tensor is defined with contravariant components, unless stated  
725 otherwise. Then, as required by the ABAQUS<sup>®</sup> ‘‘UMAT’’ subroutine, we provide the *algorithmic*  
726 *elasticity tensor*[91]

$$\hat{\mathbf{a}}_m^{k-1} := \mathbb{C}_{Im}^{k-1} + \frac{1}{2} (\boldsymbol{\eta}^{-1} \otimes \boldsymbol{\sigma}_{Im}^{k-1} + \boldsymbol{\eta}^{-1} \bar{\otimes} \boldsymbol{\sigma}_{Im}^{k-1} + \boldsymbol{\sigma}_{Im}^{k-1} \otimes \boldsymbol{\eta}^{-1} + \boldsymbol{\sigma}_{Im}^{k-1} \bar{\otimes} \boldsymbol{\eta}^{-1}), \quad (77)$$

727 whereas Equation (72a) can be reformulated by expressing  $\mathbb{A}_{Im}^{k-1}$  in terms of the quantities  $J_m^{k-1} \mathbb{C}_{Im}^{k-1}$  and  
728  $J_m^{k-1} \boldsymbol{\sigma}_{Im}^{k-1}$  (see section 4.6.1 of the Theory Manual of ABAQUS<sup>®</sup>[92]) as

$$[C_{\chi\chi}]_m^{k-1}(\delta\chi_m^k, \mathbf{V}_v) = \int_{\mathcal{B}} \mathbf{d}_{vm}^k : [J_m^{k-1} \mathbb{C}_{Im}^{k-1}] : \delta \mathbf{d}_m^k + \int_{\mathcal{B}} (\boldsymbol{\eta} \mathbf{l}_{vm}^k) : [\boldsymbol{\eta}^{-1} \otimes J_m^{k-1} \boldsymbol{\sigma}_{Im}^{k-1}] : (\boldsymbol{\eta} \delta \mathbf{l}_m^k)$$

$$=: [\hat{C}dd]_m^{k-1}(\delta \mathbf{d}_m^k, \mathbf{d}_{vm}^k) + [\hat{C}ll]_m^{k-1}(\eta \delta \mathbf{l}_m^k, \eta \mathbf{l}_{vm}^k). \quad (78)$$

729 Analogously, we can rewrite  $[C_{\chi P}]_m^{k-1}(\delta \chi_m^k, \mathbf{V}_v)$  in the equivalent form

$$[C_{\chi P}]_m^{k-1}(\delta P_m^k, \mathbf{V}_v) = - \int_{\mathcal{B}} \delta P_m^k [J_m^{k-1} \boldsymbol{\eta}^{-1}] : \mathbf{d}_{vm}^k =: [\hat{C}dP]_m^{k-1}(\delta P_m^k, \mathbf{d}_{vm}^k). \quad (79)$$

730 We compute now the Gâteaux derivatives  $\mathcal{D}_{\chi} \hat{B}(\mathbf{u}_m^{k-1}; P_v)[\delta \chi_m^k]$  and  $\mathcal{D}_P \hat{B}(\mathbf{u}_m^{k-1}; P_v)[\delta P_m^k]$ , which  
731 constitute the part of the numerical procedure at hand containing the novelty of this work. To perform  
732 these calculations, we employ Equation (55), i.e., the time-discrete form of the fractional relationship  
733 between the (material) filtration velocity and the pressure gradient. This leads to

$$\begin{aligned} \mathcal{D}_{\chi} \hat{B}(\mathbf{u}_m^{k-1}; P_v)[\delta \chi_m^k] &= - \int_{\mathcal{B}} \frac{1}{\Delta t} J_m^{k-1} \{[\mathbf{F}_m^{k-1}]^{-T} : [\text{Grad } \delta \chi_m^k]\} P_v \\ &\quad + \int_{\mathcal{B}} \left[ \frac{\partial \mathcal{G}^{\mathcal{Q}_m}}{\partial \mathbf{F}_m}(\mathbf{F}_m^{k-1}, \text{Grad } P_m^{k-1}) : \text{Grad } \delta \chi_m^k \right] \text{Grad } P_v, \end{aligned} \quad (80a)$$

$$\mathcal{D}_P \hat{B}(\mathbf{u}_m^{k-1}; P_v)[\delta P_m^k] = \int_{\mathcal{B}} \left[ \frac{\partial \mathcal{G}^{\mathcal{Q}_m}}{\partial \text{Grad } P_m}(\mathbf{F}_m^{k-1}, \text{Grad } P_m^{k-1}) \text{Grad } \delta P_m^k \right] \text{Grad } P_v. \quad (80b)$$

734 We remark that, although an explicit expression of the function  $\mathcal{G}^{\mathcal{Q}_m}$  is not available, and since it is only  
735 necessary to know the partial derivatives  $\frac{\partial \mathcal{G}^{\mathcal{Q}_m}}{\partial \mathbf{F}_m}(\mathbf{F}_m^{k-1}, \text{Grad } P_m^{k-1})$  and  $\frac{\partial \mathcal{G}^{\mathcal{Q}_m}}{\partial \text{Grad } P_m}(\mathbf{F}_m^{k-1}, \text{Grad } P_m^{k-1})$ ,  
736 which are both evaluated at the  $(k-1)$ th Newton iteration, and are, thus, known, Dini's implicit function  
737 theorem permits to determine these derivatives exactly through Equations (64a) and (64b). Therefore,  
738 Equations (80a) and (80b) become

$$\begin{aligned} \mathcal{D}_{\chi} \hat{B}(\mathbf{u}_m^{k-1}; P_v)[\delta \chi_m^k] &\equiv [C_{P\chi}]_m^{k-1}(\delta \chi_m^k, P_v) \\ &= - \int_{\mathcal{B}} \frac{1}{\Delta t} J_m^{k-1} \{[\mathbf{F}_m^{k-1}]^{-T} : [\text{Grad } \delta \chi_m^k]\} P_v \\ &\quad - \int_{\mathcal{B}} \left\{ \left[ \frac{\partial \mathcal{Z}}{\partial \mathcal{Q}_m}(\#_m^{k-1}) \right]^{-1} \left[ \frac{\partial \mathcal{Z}}{\partial \mathbf{F}_m}(\#_m^{k-1}) \right] : \text{Grad } \delta \chi_m^k \right\} \text{Grad } P_v, \end{aligned} \quad (81a)$$

$$\begin{aligned} \mathcal{D}_P \hat{B}(\mathbf{u}_m^{k-1}; P_v)[\delta P_m^k] &\equiv [C_{PP}]_m^{k-1}(\delta P_m^k, P_v) \\ &= - \int_{\mathcal{B}} \left\{ \left[ \frac{\partial \mathcal{Z}}{\partial \mathcal{Q}_m}(\#_m^{k-1}) \right]^{-1} \left[ \frac{\partial \mathcal{Z}}{\partial \text{Grad } P_m}(\#_m^{k-1}) \right] \text{Grad } \delta P_m^k \right\} \text{Grad } P_v, \end{aligned} \quad (81b)$$

739 where  $\frac{\partial \mathcal{Z}}{\partial \mathcal{Q}_m} \mathcal{Z}$  has been determined in Equation (60), while the derivatives of  $\mathcal{Z}$  with respect to  $\mathbf{F}_m$  and  
740  $\text{Grad } P_m$  are given by

$$\begin{aligned} \frac{\partial \mathcal{Z}}{\partial \mathbf{F}_m}(\#_m^{k-1}) &= \left( 1 + \frac{\alpha t_c^\alpha s^{1-\alpha} \zeta \alpha}{\Delta t} \right) \left[ \mathcal{Q}_m^{k-1} \otimes \frac{\partial \mathcal{R}_F}{\partial \mathbf{F}_m}(\mathbf{F}_m^{k-1}, \mathcal{Q}_m^{k-1}) \right] - \frac{\alpha t_c^\alpha s^{1-\alpha} \zeta \alpha}{\Delta t} \mathcal{Q}_m^{k-1} \otimes \frac{\partial \mathcal{R}_F}{\partial \mathbf{F}_m}(\mathbf{F}_m^{k-1}, \mathcal{Q}_m^{k-1}) \\ &\quad + \alpha t_c^\alpha J_m^{k-1} [\mathbf{F}_m^{k-1}]^{-1} \mathcal{F}_\alpha(t_m) \otimes [\mathbf{F}_m^{k-1}]^{-T} - \alpha t_c^\alpha J_m^{k-1} [\mathbf{F}_m^{k-1}]^{-1} \otimes \{[\mathbf{F}_m^{k-1}]^{-1} \mathcal{F}_\alpha(t_m)\} \\ &\quad - \mathcal{G}^{\mathcal{Q}_D}(\mathbf{F}_m^{k-1}, \text{Grad } P_m^{k-1}) \otimes \frac{\partial \mathcal{R}_D}{\partial \mathbf{F}_m}(\mathbf{F}_m^{k-1}) - \mathcal{R}_D(\mathbf{F}_m^{k-1}) \frac{\partial \mathcal{G}^{\mathcal{Q}_D}}{\partial \mathbf{F}_m}(\mathbf{F}_m^{k-1}, \text{Grad } P_m^{k-1}), \end{aligned} \quad (82a)$$

$$\frac{\partial \mathcal{Z}}{\partial \text{Grad } P_m}(\#_m^{k-1}) = -\mathcal{R}_D(\mathbf{F}_m^{k-1}) \frac{\partial \mathcal{G}^{\mathcal{Q}_D}}{\partial \text{Grad } P_m}(\mathbf{F}_m^{k-1}, \text{Grad } P_m^{k-1}) = (J_m^{k-1} - \Phi_{sR}) [\mathbf{C}_m^{k-1}]^{-1}, \quad (82b)$$

741 and, again, the notation  $[C_{P\chi}]_m^{k-1}(\delta \chi_m^k, P_v)$  and  $[C_{PP}]_m^{k-1}(\delta P_m^k, P_v)$  puts in evidence the influence of the  
742 pore pressure on the motion and the self-influence of the pore pressure. For completeness, we supply  
743 also the expressions of the derivatives of  $\mathcal{R}_F$ ,  $\mathcal{R}_D$ , and  $\mathcal{G}^{\mathcal{Q}_D}$  with respect to  $\mathbf{F}_m$ . To this end, we write  $\kappa_{\text{iso}}$   
744 and  $\mathcal{A}_{\text{iso}}$  as functions of  $J_m$ , i.e., we set  $\kappa_{\text{iso}} \equiv \hat{\kappa}_{\text{iso}}(J_m)$  and  $\mathcal{A}_{\text{iso}} \equiv \hat{\mathcal{A}}_{\text{iso}}(J_m)$ , and we express  $\|\mathcal{Q}_m\|_{C_m}$   
745 as a function of  $\mathbf{F}_m$ , i.e.,  $\|\mathcal{Q}_m\|_{C_m} \equiv \hat{\mathcal{Q}}(\mathbf{F}_m)$ . Then, we obtain:

$$\frac{\partial \mathcal{R}_D}{\partial \mathbf{F}_m}(\mathbf{F}_m^{k-1}) = \mathcal{R}_D(\mathbf{F}_m^{k-1}) \left[ \frac{J_m^{k-1}}{J_m^{k-1} - \Phi_{sR}} - \frac{J_m^{k-1}}{\hat{\kappa}_{\text{iso}}(J_m^{k-1})} \frac{\partial \hat{\kappa}_{\text{iso}}}{\partial J_m}(J_m^{k-1}) \right] [\mathbf{F}_m^{k-1}]^{-T}, \quad (83a)$$

$$\frac{\partial \hat{\mathcal{Q}}}{\partial \mathbf{F}_m}(\mathbf{F}_m^{k-1}) = -\|\mathbf{Q}_m^{k-1}\|_{\mathbf{C}_m^{k-1}}[\mathbf{F}_m^{k-1}]^{-\text{T}} + \frac{1}{J_m^{k-1}\|\mathbf{Q}_m^{k-1}\|_{\mathbf{C}_m^{k-1}}} \frac{\eta \mathbf{F}_m^{k-1} \mathbf{Q}_m^{k-1}}{J_m^{k-1}} \otimes \mathbf{Q}_m^{k-1}, \quad (83b)$$

$$\begin{aligned} \frac{\partial \mathcal{R}_F}{\partial \mathbf{F}_m}(\mathbf{F}_m^{k-1}, \mathbf{Q}_m^{k-1}) &= \frac{\partial \mathcal{R}_D}{\partial \mathbf{F}_m}(\mathbf{F}_m^{k-1})[1 + \hat{\mathcal{A}}_{\text{iso}}(J_m^{k-1})\|\mathbf{Q}_m^{k-1}\|_{\mathbf{C}_m^{k-1}}] \\ &\quad + \mathcal{R}_D(\mathbf{F}_m^{k-1}) \left\{ \frac{\partial \hat{\mathcal{A}}_{\text{iso}}}{\partial J_m}(J_m^{k-1}) J_m^{k-1} \|\mathbf{Q}_m^{k-1}\|_{\mathbf{C}_m^{k-1}} [\mathbf{F}_m^{k-1}]^{-\text{T}} \right. \\ &\quad \left. + \hat{\mathcal{A}}_{\text{iso}}(J_m^{k-1}) \frac{\partial \hat{\mathcal{Q}}}{\partial \mathbf{F}_m}(\mathbf{F}_m^{k-1}) \right\}, \end{aligned} \quad (83c)$$

$$\begin{aligned} \frac{\partial \mathcal{G}^{\mathcal{Q}_D}}{\partial \mathbf{F}_m}(\mathbf{F}_m^{k-1}, \text{Grad } P_m^{k-1}) &= \frac{J_m^{k-1}}{\hat{\kappa}_{\text{iso}}(J_m^{k-1})} \frac{\partial \hat{\kappa}_{\text{iso}}}{\partial J_m}(J_m^{k-1}) \mathbf{Q}_{Dm}^{k-1} \otimes [\mathbf{F}_m^{k-1}]^{-\text{T}} \\ &\quad - [\mathbf{F}_m^{k-1}]^{-1} \otimes \mathbf{Q}_{Dm}^{k-1} - [\mathbf{C}_m^{k-1}]^{-1} \otimes \eta \mathbf{F}_m^{k-1} \mathbf{Q}_{Dm}^{k-1}. \end{aligned} \quad (83d)$$

746 Finally, we notice that the definitions supplied in Equations (72a), (72b), (81a) and (81b) allow to  
747 rewrite Equations (68a) and (68b) in the more suggestive form

$$[\mathcal{C}_{\chi\chi}]_m^{k-1}(\delta\chi_m^k, \mathbf{V}_v) + [\mathcal{C}_{\chi P}]_m^{k-1}(\delta P_m^k, \mathbf{V}_v) = -\hat{A}(\mathbf{u}_m^{k-1}; \mathbf{V}_v), \quad (84a)$$

$$[\mathcal{C}_{P\chi}]_m^{k-1}(\delta\chi_m^k, P_v) + [\mathcal{C}_{PP}]_m^{k-1}(\delta P_m^k, P_v) = -\hat{B}(\mathbf{u}_m^{k-1}; P_v), \quad (84b)$$

748 with  $[\mathcal{C}_{\chi P}]_m^{k-1}(\cdot, \cdot)$  and  $[\mathcal{C}_{P\chi}]_m^{k-1}(\cdot, \cdot)$  being related through the identity[66, 17]

$$\begin{aligned} [\mathcal{C}_{P\chi}]_m^{k-1}(\delta\chi_m^k, P_v) &= \frac{1}{\Delta t} [\mathcal{C}_{\chi P}]_m^{k-1}(P_v, \delta\chi_m^k) \\ &\quad - \int_{\mathcal{B}} \left\{ \left[ \frac{\partial \mathcal{Z}}{\partial \mathbf{Q}_m}(\#_m^{k-1}) \right]^{-1} \left[ \frac{\partial \mathcal{Z}}{\partial \mathbf{F}_m}(\#_m^{k-1}) \right] : \text{Grad} \delta\chi_m^k \right\} \text{Grad} P_v. \end{aligned} \quad (85)$$

749 Equations (84a) and (84b) are a ‘‘prelude’’ to their associated algebraic form, which is achieved by  
750 introducing the finite element discretization of the problem at hand and the interpolation functions for  
751 the unknown increments  $\delta\chi_m^k$  and  $\delta P_m^k$  as well as for the virtual fields  $\mathbf{V}_v$  and  $P_v$ . In fact, each summand  
752 on the left-hand side of Equations (84a) and (84b) gives rise to a specific block of the matrix of the  
753 coefficients of the system of algebraic equations associated with Equations (68a) and (68b).

754 It is important to emphasize that, while Equation (84a) is essentially the same as the one studied  
755 in[66, 88, 17], the main differences between these previous studies and our work are condensed in  
756 Equation (84b). The first difference is given by the second term of the functional  $[\mathcal{C}_{P\chi}]_m^{k-1}(\cdot, \cdot)$ ,  
757 which collects all the modifications to the Darcian model that are associated both with Forchheimer’s  
758 correction and with its fractionalization (it can be proven, in this respect, that Darcy’s model is retrieved  
759 by setting  $\alpha = 0$  and  $\mathcal{A}_{\text{iso}} = 0$  identically). This term, in fact, describes a coupling between pressure  
760 and deformation that, because of the Jacobian  $\partial \mathcal{Z} / \partial \mathbf{Q}_m$  and of the derivative  $\partial \mathcal{Z} / \partial \mathbf{F}_m$ , is much more  
761 intricate than the Darcian one, and, in addition, it takes into account the non-locality in time of the  
762 model under investigation through  $\mathcal{F}_\alpha(t_m)$ . The second difference with the Darcian model addressed  
763 in[66, 88, 17] is related to the definition of the functional  $[\mathcal{C}_{PP}]_m^{k-1}(\cdot, \cdot)$ , which, again, keeps track  
764 of the non-locality in time and of all the interactions between the flow and the deformation through the  
765 inverse of the Jacobian  $\partial \mathcal{Z} / \partial \mathbf{Q}_m$  (cf. Equation (60)).

766 In spite of the differences just discussed, for the purpose of implementation in ABAQUS<sup>®</sup>, and, in partic-  
767 ular, due to the limitation of the ‘‘UMAT’’ and ‘‘UMATHT’’ subroutines present in the adopted software,  
768 in the numerical tests performed in this work, we neglect the second integral defining  $[\mathcal{C}_{P\chi}]_m^{k-1}(\delta\chi_m^k, P_v)$   
769 on the far right-hand side of Equations (81a) and (85). Hence, for the forthcoming simulations, we  
770 substitute the term  $[\mathcal{C}_{P\chi}]_m^{k-1}(\delta\chi_m^k, P_v)$  in Equation (84b) with its approximated counterpart

$$\begin{aligned} [\mathcal{C}_{P\chi}^{\text{app}}]_m^{k-1}(\delta\chi_m^k, P_v) &:= - \int_{\mathcal{B}} \frac{1}{\Delta t} J_m^{k-1} \{ [\mathbf{F}_m^{k-1}]^{-\text{T}} : [\text{Grad} \delta\chi_m^k] \} P_v \\ &= - \int_{\mathcal{B}} \frac{1}{\Delta t} J_m^{k-1} \text{tr}[\boldsymbol{\eta}^{-1} \delta \mathbf{d}_m^k] P_v \end{aligned}$$

$$=: [\hat{C}_{Pd}^{\text{app}}]_m^{k-1}(\delta \mathbf{d}_m^k, P_v), \quad (86)$$

771 and we solve the approximated system

$$[C_{\chi\chi}]_m^{k-1}(\delta\chi_m^k, \mathbf{V}_v) + [C_{\chi P}]_m^{k-1}(\delta P_m^k, \mathbf{V}_v) = -\hat{A}(\mathbf{u}_m^{k-1}; \mathbf{V}_v), \quad (87a)$$

$$[C_{P\chi}^{\text{app}}]_m^{k-1}(\delta\chi_m^k, P_v) + [C_{PP}]_m^{k-1}(\delta P_m^k, P_v) = -\hat{B}(\mathbf{u}_m^{k-1}; P_v). \quad (87b)$$

772 Analogously to Equation (86), also the term  $[C_{PP}]_m^{k-1}(\delta P_m^k, P_v)$  can be recast in the equivalent form

$$\begin{aligned} [C_{PP}]_m^{k-1}(\delta P_m^k, P_v) &= - \int_{\mathcal{B}} \{(\text{Grad} \delta P_m^k)[\mathbf{F}_m^{k-1}]^{-1}\} [J_m^{k-1} \mathfrak{B}_m^{k-1}] \{(\text{Grad} P_v)[\mathbf{F}_m^{k-1}]^{-1}\} \\ &=: [\hat{C}_{PP}]_m^{k-1}(\delta p_m^k, p_v), \end{aligned} \quad (88)$$

773 where  $p_m^k(\chi(X, t), t) := P_m^k(X, t)$ ,  $p_v(\chi(X, t), t) := P_v(X, t)$  is the spatial counterpart of the virtual  
774 pressure field, and we have set

$$\mathfrak{B}_m^{k-1} := \frac{1}{J_m^{k-1}} \mathbf{F}_m^{k-1} \left[ \frac{\partial \mathcal{Z}}{\partial \mathcal{Q}_m}(\#_m^{k-1}) \right]^{-1} \left[ \frac{\partial \mathcal{Z}}{\partial \text{Grad} P_m}(\#_m^{k-1}) \right] [\mathbf{F}_m^{k-1}]^T. \quad (89)$$

775 Clearly, this way of proceeding has the drawback that not all the interactions introduced by our model  
776 are equally considered in the algorithm employed. However, the algorithm makes it still possible to  
777 account for those deviations from Darcy's regime that the fractional version of Forchheimer's correction  
778 studied in our work unfolds in the term  $[C_{PP}]_m^{k-1}(\delta P_m^k, P_v)$  through  $\mathfrak{B}_m^{k-1}$  and in the residue  $\hat{B}(\mathbf{u}_m^{k-1}; P_v)$ .

779 Finally, by solving Equations (87a) and (87b) for  $\delta\chi_m^k$  and  $\delta P_m^k$ , reconstructing the motion and fluid  
780 pressure at the  $k$ th iteration as  $\chi_m^k = \chi_m^{k-1} + \delta\chi_m^k$  and  $P_m^k = P_m^{k-1} + \delta P_m^k$ , and computing the functionals  
781  $\hat{A}(\mathbf{u}_m^k, \mathbf{V}_v)$  and  $\hat{B}(\mathbf{u}_m^k, P_v)$ , the pair  $(\chi_m, P_m)$  that solves Equations (65a) and (65b) is found, as anticipated  
782 above, when, for some  $k_* \in \mathbb{N}$ , the absolute values  $|\hat{A}(\mathbf{u}_m^k, \mathbf{V}_v)|$  and  $|\hat{B}(\mathbf{u}_m^k, P_v)| \equiv |B(\chi_m^k, P_m^k, \mathcal{Q}_m^k; P_v)|$   
783 remain smaller than a given threshold for all  $k > k_*$ .

784 There is, however, a last step of the algorithm employed here that has to be commented. Indeed, to  
785 solve Equations (87a) and (87b), it is necessary to know the residue

$$\hat{B}(\mathbf{u}_m^{k-1}; P_v) \equiv \hat{B}(\chi_m^{k-1}, P_m^{k-1}; P_v) = B(\chi_m^{k-1}, P_m^{k-1}, \mathcal{Q}_m^{k-1}; P_v), \quad k \geq 1. \quad (90)$$

786 Yet, this quantity is unknown for all  $k \geq 2$ , because  $\mathcal{Q}_m^{k-1}$  has still to be determined. On the other hand,  
787  $\mathcal{Q}_m^{k-1}$  is known only for  $k = 1$ , since  $\mathcal{Q}_m^0$  is either guessed or computed by solving Equation (57c) through  
788 another Newton-Raphson procedure (see next paragraph). Hence, since  $\chi_m^0$  and  $P_m^0$  are supplied by the  
789 initial guess, also the residue  $B(\chi_m^0, P_m^0, \mathcal{Q}_m^0; P_v)$  is entirely defined. In conclusion, the filtration velocity  
790  $\mathcal{Q}_m^{k-1}$  must be computed at each  $k \geq 2$ . This is done by applying, again, the Newton-Raphson method  
791 shown in the next paragraph, and, with this procedure, also  $\mathcal{Q}_m^k$  is obtained. Therefore, the filtration  
792 velocity  $\mathcal{Q}_m$  at time  $t_m$  can be approximated with the value of  $\mathcal{Q}_m^k$  for  $k > k_*$ , with  $k_* \in \mathbb{N}$  being such  
793 that  $|\mathcal{Z}(\mathbf{F}_m^k, \text{Grad} P_m^k, \mathcal{Q}_m^k)|$  is smaller than a given threshold for all  $k > k_*$ .

794 **Determination of  $\mathcal{Q}_m$ .** The separate determination of  $\mathcal{Q}_m$  is necessary for computing the residues  
795  $\hat{B}(\chi_m^{k-1}, P_m^{k-1}; P_v) \equiv B(\chi_m^{k-1}, P_m^{k-1}, \mathcal{Q}_m^{k-1}; P_v)$ , for all  $k \geq 2$ . For  $k = 1$ , instead, the residue  $B(\chi_m^0, P_m^0, \mathcal{Q}_m^0; P_v)$   
796 is entirely defined by the initial triple  $(\chi_m^0, P_m^0, \mathcal{Q}_m^0)$ . In this work, to simplify the computational bur-  
797 den, we have opted to prescribe  $\mathcal{Q}_m^0$  arbitrarily through an "educated guess", since this does not affect  
798 considerably the convergence to the value of  $\mathcal{Q}_m$  that solves approximately Equation (57c).

799 To compute the residue  $\hat{B}(\chi_m^{k-1}, P_m^{k-1}; P_v) \equiv B(\chi_m^{k-1}, P_m^{k-1}, \mathcal{Q}_m^{k-1}; P_v)$ , for all  $k \geq 2$ , we determine  
800  $\mathcal{Q}_m^{k-1}$  as follows. First, for each  $k \geq 2$ , we write  $\mathcal{Q}_m^{k-1}$  as  $\mathcal{Q}_m^{k-1, l} := \mathcal{Q}_m^{k-1, l-1} + \delta \mathcal{Q}_m^{k-1, l}$ . Here,  $l \geq 1$ ,  
801  $l \in \mathbb{N}$ , is the counter of the Newton-Raphson procedure "nested"[89] in the  $k$ th iteration of the outer  
802 procedure, employed to calculate  $\chi_m^k$  and  $P_m^k$ , while  $\delta \mathcal{Q}_m^{k-1, l}$  is the increment of the filtration velocity  
803 at the  $l$ th iteration nested in the  $(k - 1)$ th iteration of the outer scheme. We notice that, for  $l = 1$ , the

804 quantity  $\mathbf{Q}_m^{k-1,0}$  is a guessed value of the filtration velocity that can be taken equal to  $\mathbf{Q}_m^{k-2}$ . Then, we  
 805 approximate the function  $\mathcal{Z}$  with its Taylor polynomial of the first grade in  $\delta\mathbf{Q}_m^{k-1,l}$ , thereby writing

$$\begin{aligned} & \mathcal{Z}_{\text{app}}(\mathbf{F}_m^{k-1}, \text{Grad } P_m^{k-1}, \mathbf{Q}_m^{k-1,l-1} + \delta\mathbf{Q}_m^{k-1,l}) \\ & := \mathcal{Z}(\mathbf{F}_m^{k-1}, \text{Grad } P_m^{k-1}, \mathbf{Q}_m^{k-1,l-1}) + \left[ \frac{\partial \mathcal{Z}}{\partial \mathbf{Q}_m}(\mathbf{F}_m^{k-1}, \text{Grad } P_m^{k-1}, \mathbf{Q}_m^{k-1,l-1}) \right] \delta\mathbf{Q}_m^{k-1,l}, \quad l \geq 1. \end{aligned} \quad (91)$$

806 Next, by setting  $\mathcal{Z}_{\text{app}}(\mathbf{F}_m^{k-1}, \text{Grad } P_m^{k-1}, \mathbf{Q}_m^{k-1,l-1} + \delta\mathbf{Q}_m^{k-1,l}) = \mathbf{0}$  for  $l \geq 1$ ,  $\delta\mathbf{Q}_m^{k-1,l}$  is obtained as

$$\delta\mathbf{Q}_m^{k-1,l} = - \left[ \frac{\partial \mathcal{Z}}{\partial \mathbf{Q}_m}(\mathbf{F}_m^{k-1}, \text{Grad } P_m^{k-1}, \mathbf{Q}_m^{k-1,l-1}) \right]^{-1} \mathcal{Z}(\mathbf{F}_m^{k-1}, \text{Grad } P_m^{k-1}, \mathbf{Q}_m^{k-1,l-1}), \quad l \geq 1, \quad (92)$$

807 and  $\mathbf{Q}_m^{k-1,l}$  can be reconstructed according to its definition. As usual, the iterations stop when, for some  
 808  $l_*(k) \in \mathbb{N}$ , the absolute value  $|\mathcal{Z}(\mathbf{F}_m^{k-1}, \text{Grad } P_m^{k-1}, \mathbf{Q}_m^{k-1,l})|$  remains smaller than a given tolerance for  
 809 all  $l > l_*(k)$ . Accordingly,  $\mathbf{Q}_m^{k-1}$  is formally identified with the limit  $\mathbf{Q}_m^{k-1} := \lim_{l \rightarrow +\infty} \mathbf{Q}_m^{k-1,l}$ . This  
 810 permits to calculate the residue  $\hat{B}(\chi_m^{k-1}, P_m^{k-1}; P_v) \equiv B(\chi_m^{k-1}, P_m^{k-1}, \mathbf{Q}_m^{k-1}; P_v)$  as

$$B(\chi_m^{k-1}, P_m^{k-1}, \mathbf{Q}_m^{k-1}; P_v) = - \int_{\mathcal{B}} \frac{J_m^{k-1} - J_{m-1}}{\Delta t} P_v + \int_{\mathcal{B}} \mathbf{Q}_m^{k-1} \text{Grad } P_v. \quad (93)$$

811 To conclude this paragraph, we notice that  $\mathbf{Q}_m^k$  is calculated with the same scheme employed for  $\mathbf{Q}_m^{k-1}$ ,  
 812 after determining the pair  $(\mathbf{F}_m^k, \text{Grad } P_m^k)$  by solving Equations (87a) and (87b), so that the quantity  
 813  $|\mathcal{Z}(\mathbf{F}_m^k, \text{Grad } P_m^k, \mathbf{Q}_m^k)|$  remains smaller than a given threshold. We also remark that, at a given time  
 814  $t_m$ , the stopping criterion for the aforementioned scheme is the convergence within a certain tolerance  
 815 of  $(\chi_m^k, P_m^k)$ , which is assured for  $k > k_*$ . To summarize, one more nested Newton-Raphson procedure  
 816 is required to calculate  $\mathbf{Q}_m$ . In fact, after determining the approximated solution  $(\chi_m^k, P_m^k) \equiv (\chi_m, P_m)$   
 817 of Equations (65a) and (65b), the value  $\mathbf{Q}_m^k = \lim_{l \rightarrow +\infty} \mathbf{Q}_m^{k,l}$  is formally found by calling for the nested  
 818 Newton-Raphson method (see, e.g., [89]), and  $\mathbf{Q}_m$  is found as  $\mathbf{Q}_m^k \equiv \mathbf{Q}_m$ , for  $k > k_*$ .

## 819 7 Summary of the model and benchmark tests

820 In this section, we describe the initial and boundary value problem (IBVP) employed for our numerical  
 821 experiments, which will be conducted in ABAQUS<sup>®</sup> by following the numerical procedure explained in  
 822 section 6.

823 Our simulations refer to the mathematical model conceived in the previous sections, which aims at  
 824 describing a class of media characterized, on the one hand, by non-negligible pore scale inertial effects  
 825 of the fluid and, on the other hand, by a complex microstructure of the pore network that gives rise to  
 826 flow laws modeled as non-local in time[38, 43, 31]. In particular, the class of media we are referring  
 827 to include bio-mimetic scaffolds that have elastic parameters comparable to those of a biological tissue,  
 828 and that are known to give rise to flow regimes compatible with Forchheimer's correction[87]. For  
 829 the purpose of studying this kind of media, we concentrate on simulating Equations (43a)-(43c), so  
 830 that it is possible to highlight how the overall behavior of the system under evaluation is influenced  
 831 by the fractional constitutive law of  $\mathbf{Q}$  specified in Equation (42). In this respect, we notice that the  
 832 standard Darcy-Forchheimer model, represented by Equations (35a) and (35b), can be recovered from  
 833 the fractional one by setting  $\alpha = 0$ , while standard Darcy's model can be obtained by setting  $c_0 = 0$  and  
 834  $\alpha = 0$  in Equation (32).

835 In the following simulations, we replicate the setup of an experimentally relevant uni-axial compress-  
 836 sion test in which, before the application of the load, a cylindrical sample of the hypothetical material  
 837 under study is put in a compression chamber, situated in the inner part of the experimental apparatus.  
 838 Inside the chamber, the sample is positioned between two impermeable plates, made of steel or, more  
 839 generally, of a material that does not allow for adhesion bonds with the sample itself. Moreover, in the  
 840 inner chamber, warm water circulates and maintains the sample in isothermal conditions. Then, the

841 experiment is conducted in control of displacement: the movement of the upper plate is controlled, and  
 842 exerts a prescribed compression on the sample. When the maximum prescribed displacement is reached,  
 843 the load is kept constant in order to study the relaxation of the material.

844 We perform the simulation of the just described unconfined compression test by solving Equations  
 845 (43a)-(43c) for a cylindrical sample over the time interval  $[t_{\text{in}}, t_{\text{fin}}] \equiv [0, t_{\text{fin}}]$ . The specimen has initial  
 846 radius  $R = 30$  mm and initial height  $H = 20$  mm, as shown in Fig 1. Since we do not simulate the  
 847 plates, boundary conditions are applied directly on the specimen's boundary, which coincides with the  
 848 boundary of its reference placement,  $\partial\mathcal{B}$ , and can be partitioned as  $\partial\mathcal{B} = \Gamma_{\text{U}} \cup \Gamma_{\text{L}} \cup \Gamma_{\text{B}}$ , with  $\Gamma_{\text{U}}$ ,  
 849  $\Gamma_{\text{L}}$ , and  $\Gamma_{\text{B}}$  being the specimen's upper, lateral, and bottom surface, respectively. We recall that, since  
 850 the constitutive framework has been set, the system (43a)-(43c) is equivalent to seven scalar equations  
 851 in the seven unknowns given by the three components of the motion  $\chi$ , pore pressure  $P$ , and the three  
 852 components of the material filtration velocity  $\mathbf{Q}$ .

853 To assign the boundary conditions, we introduce a reference frame, associated with  $\mathcal{B}$ , and having  
 854 origin at the center  $X_{\text{O}}$  of  $\Gamma_{\text{B}}$ , and axes directed along the unit vectors of the triad  $\mathcal{E}_{\text{O}} := \{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\} \subset$   
 855  $T_{X_{\text{O}}}\mathcal{B}$ , in which  $\mathbf{E}_3$  identifies the axial direction of the specimen, while  $\mathbf{E}_1$  and  $\mathbf{E}_2$  span the transversal  
 856 plane. We also introduce the co-normals  $N_{\text{U}}$ ,  $N_{\text{L}}$ , and  $N_{\text{B}}$  to  $\Gamma_{\text{U}}$ ,  $\Gamma_{\text{L}}$ , and  $\Gamma_{\text{B}}$ , and we notice that  $N_{\text{U}}$  and  
 857  $N_{\text{B}}$  are parallel and anti-parallel to the co-vector  $\mathbf{E}^3$  of the co-vector basis dual to  $\mathcal{E}_{\text{O}}$ . Hence, for every  
 858 time  $t \in [0, t_{\text{fin}}]$ , the experimental setup illustrated above is represented by the boundary conditions

$$\chi^3(X, t) = \chi_{\text{U}}^3(X_1, X_2, t), \quad [\mathbf{T}_{\text{I}}N_{\text{U}}]\mathbf{E}_1 = 0, \quad [\mathbf{T}_{\text{I}}N_{\text{U}}]\mathbf{E}_2 = 0, \quad \text{on } \Gamma_{\text{U}}, \quad (94a)$$

$$\mathbf{Q}N_{\text{U}} = 0, \quad \text{on } \Gamma_{\text{U}}, \quad (94b)$$

$$\mathbf{T}_{\text{I}}N_{\text{L}} = \mathbf{0}, \quad \text{on } \Gamma_{\text{L}}, \quad (94c)$$

$$P = 0, \quad \text{on } \Gamma_{\text{L}}, \quad (94d)$$

$$\chi(X, t) = (X^1, X^2, 0), \quad \text{on } \Gamma_{\text{B}}, \quad (94e)$$

$$\mathbf{Q}N_{\text{B}} = 0, \quad \text{on } \Gamma_{\text{B}}, \quad (94f)$$

859 where  $\chi_{\text{U}}^3(X_1, X_2, t)$  is the time-dependent loading function, defined by[20, 30, 93]

$$\chi_{\text{U}}^3(X_1, X_2, t) \equiv \chi^3(X^1, X^2, H, t) := \begin{cases} H - u_{\text{T}} \frac{t}{t_{\text{ramp}}}, & t \in ]0, t_{\text{ramp}}], \\ H - u_{\text{T}}, & t \in ]t_{\text{ramp}}, t_{\text{fin}}]. \end{cases} \quad (95)$$

860 These prescriptions represent the fact that a prescribed axial compression is applied onto the upper  
 861 surface of the specimen, while its bottom surface is clamped. The absolute value of the applied axial  
 862 displacement  $|\chi^3(X, t) - \chi_{\text{U}}^3(X_1, X_2, t)|$  increases in time until it reaches the maximum  $u_{\text{T}} = 7$  mm  
 863 at  $t = t_{\text{ramp}} = 2$  s and, afterwards, it is kept constant until the final time of the simulated experiment  
 864  $t = t_{\text{fin}} = 10$  s.

865 The second and third conditions in Equation (94a) indicate that no tangential tractions are applied  
 866 on  $\Gamma_{\text{U}}$ . In addition, Equations (94c) and (94d) mean that the lateral surface of the specimen  $\Gamma_{\text{L}}$  is  
 867 traction-free and that the pore pressure is atmospheric. Finally, Equations (94b) and (94f) show that the  
 868 upper and lower surfaces are both insulated, so that no fluid flow may occur through them. The fluid,  
 869 however, is free to escape through the lateral surfaces of the specimen during compression.

870 A schematic representation of the cylindrical specimen and of the boundary conditions discussed  
 871 above is shown in Fig 1.

872 The prescribed initial conditions for the IBVP are

$$\chi(X, 0) = \chi_{\text{in}}(X), \quad \text{in } \mathcal{B}, \quad (96a)$$

$$P(X, 0) = 0, \quad \text{in } \mathcal{B}, \quad (96b)$$

$$\mathbf{Q}(X, 0) = \mathbf{0}, \quad \text{in } \mathcal{B}, \quad (96c)$$

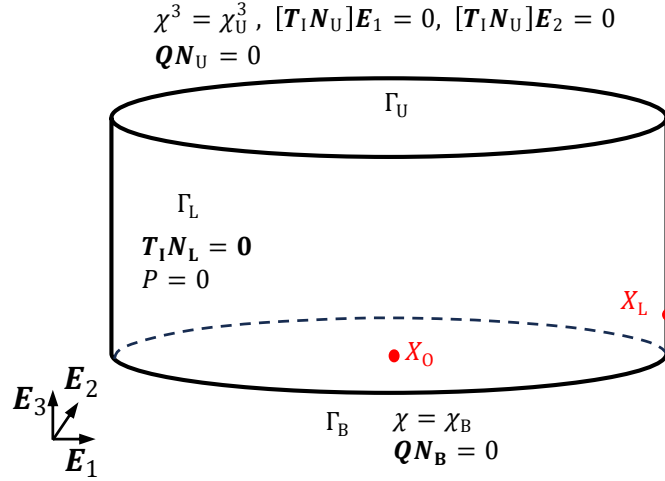


Figure 1: Geometry and boundary conditions for unconfined compression test

873 where, again, with a slight abuse of notation, we set  $\chi_{\text{in}}(X) = (X^1, X^2, X^3)$  for all the inner points of  $\mathcal{B}$ .  
874 We remark that, at the initial time  $t = t_{\text{in}} = 0$  s, Equations (43a)-(43c) are identically satisfied, whereas,  
875 for  $t \in ]0, t_{\text{fin}}]$ , it is necessary to have  $Q(X, t) \neq \mathbf{0}$  in order to meet the hypotheses of Dini's Theorem,  
876 as explained in subsection 6.2. Hence, for coding purposes, to avoid the explicit separation of the case  
877  $t = 0$  s from the case  $t \in ]0, t_{\text{fin}}]$ , the initial condition for the filtration velocity  $Q$  is taken near the machine  
878 precision.

879 Under the initial and boundary conditions (94a)-(94e) and (96a)-(96c), we study the evolution of the  
880 system for different values of the fractional order  $\alpha$ , and of the characteristic time  $t_c$  in order to simulate  
881 the evolution of the flux, of the deformation, and of the stress field over time. In particular, we perform  
882 two sets of simulations: for the first one, we assign the characteristic time  $t_c = 3$  s and we let  $\alpha$  vary  
883 as  $\alpha \in \{0.0, 0.2, 0.4, 0.6, 0.8, 0.99\}$ , and, for  $\alpha = 0.0$ , we recover the non-fractional Darcy-Forchheimer  
884 model; for the second set, we take  $\alpha = 0.4$ , and we assign the characteristic time as  $t_c \in \{1 \text{ s}, 3 \text{ s}, 50 \text{ s}\}$ .  
885 With these test cases, we aim to observe the effects of the two parameters related to the fractional model,  
886  $\alpha$  and  $t_c$ , on the behavior of the biphasic medium as a whole. The values of the material parameters  
887 adopted in the model are reported in Table 1.

888 The model is solved in ABAQUS<sup>®</sup> by having recourse to the subroutine "UMAT" for implementing  
889 Equation (43a), to the subroutine "UMATHHT" for implementing (43b), and by selecting the option  
890 "Fully coupled thermal-stress analysis" in order to solve simultaneously for the deformation and the  
891 pore pressure. The latter option is selected to insert the terms  $C_{P\chi}^{\text{app}}$  and  $C_{\chi P}$ , which introduce the coupling  
892 between the deformation and the pore pressure in the linearization of the fractional Forchheimer model.  
893 We remark that the "UMATHHT" subroutine, although originally meant for energy conservation, is used  
894 for implementing the mass conservation equation (43b) by using the similarity between these equations  
895 [45] (see Appendix C for detailed information).

896 For the simulations, a quarter of cylinder is simulated and C3D8T elements are used, which are 3D  
897 brick elements with three displacements and one "temperature" degree of freedom, which, by analogy,  
898 as shown in Appendix C, is employed to simulate the one of pore pressure. Each element has eight  
899 integration points. The model has 47372 elements and 43850 nodes. A backward time integration  
900 scheme is adopted, with constant time increment of  $\Delta t = 1$  s.

## 901 8 Results and discussion

902 In this section, we present and discuss the numerical simulations of the compression tests described  
903 in the previous section. Emphasis will be placed on commenting the memory effects introduced by  
904 fractional Forchheimer's correction (43c). Our aim is to contextualize the effects introduced by the

Parameter	Symbol	Numerical value	Unit of measure	Reference
Initial radius	$R$	30	mm	-
Initial height	$H$	20	mm	-
Referential solidity	$\Phi_s$	0.2	-	[36]
Reference permeability	$k_{\text{ref}}$	$4 \cdot 10^{-4}$	$\text{mm}^2$	[87]
Material parameter	$m_0$	0.0848	-	[94]
Material parameter	$m_1$	4.6380	-	[94]
First Lamé's constant	$\lambda_s$	$5.55 \cdot 10^5$	Pa	[20]
Second Lamé's constant	$\mu_s$	$2.22 \cdot 10^5$	Pa	[20]
Density fluid phase	$\varrho_f$	$1 \cdot 10^3$	$\text{kg}/\text{m}^3$	-
Fluid viscosity	$\mu$	$0.89 \cdot 10^{-9}$	$\text{MPa} \cdot \text{s}$	-
Forchheimer's parameter	$c_0$	0.363	-	[87]
Forchheimer's parameter	$c_1$	-5.5	-	[30]
Forchheimer's parameter	$c_2$	-0.5	-	[30]
Characteristic time	$t_c$	3	s	-
Prescribed displacement	$u_T$	7	mm	-
Loading time	$T_{\text{ramp}}$	2	s	-

Table 1: Values of the material parameters used for the numerical simulations.

905 fractional law through the comparison of the numerical simulations performed in the fractional case with  
906 those done under the assumption either of Darcy's law or of non-fractional Darcy-Forchheimer's law.  
907 We will focus on the description of the filtration velocity and on its coupling with the deformation of  
908 the solid phase through the visualization of the system's evolution. In this respect, we recall that the  
909 filtration velocity is, by definition, the product of the velocity of the fluid relative to the solid with the fluid  
910 phase volumetric fraction, which, because of the hypothesis of saturation, coincides with the porosity.  
911 Therefore, for a specimen under compression, the filtration velocity of the fluid is not a mere consequence  
912 of its kinematics relative to the solid, since there exists also a direct feedback of the deformation on the  
913 fluid volumetric fraction. The latter, indeed, decreases under compression until the compaction limit,  
914 which, in turn, places a lower bound on the volumetric deformation itself. As noticed, e.g., in[19], the  
915 natural condition  $\Phi_f(X, t) = J(X, t) - \Phi_s(X) \geq 0$  yields the "unilateral constraint"  $J(X, t) \geq \Phi_s(X)$  at  
916 all points  $X \in \mathcal{B}$  and at all times.

917 We remark that the simulated specimen consists of a hypothetical tissue, which, as anticipated above,  
918 borrows some of its mechanical and structural properties from articular cartilage, since it features a  
919 complex microstructure, and exhibits memory effects[57, 58], while possessing a high permeability that  
920 is commonly associated with Forchheimer's correction in biological scaffolds[87].

## 921 8.1 Flow through the lateral surface of the specimen

922 The maximum of the magnitude of the filtration velocity is attained on a locus of points that, due to the  
923 axial symmetry of the problem under study, coincides with the circle defined by the lower edge, i.e.,  
924  $\mathcal{E}_{\text{BL}} := \overline{\Gamma_B} \cap \overline{\Gamma_L}$ , where the superimposed bar denotes the topological closure of the set to which it is  
925 applied (note that, if reinforcing fibers were included in the model, this behavior would be different in  
926 general; e.g., in articular cartilage the fibers being parallel to the sample's symmetry axis at the bottom  
927 would contrast the transversal filtration velocity in this region). However, since the conditions on the  
928 motion imposed on the Dirichlet nodes of the mesh lying on  $\Gamma_B$  have led to small numerical artifacts in  
929 the computation of the filtration velocity, we study the evolution of the magnitude of this quantity in a  
930 relatively small, stripe-shaped subset of  $\Gamma_L$ , containing  $\mathcal{E}_{\text{BL}}$ . In particular, in this subset, we select the  
931 point of coordinates  $X_L = (30, 0, 2.8) \in \Gamma_L$  (length units are given in millimetres), and we observe the  
932 evolution of the magnitude of the filtration velocity at this point, i.e., of  $\|\mathbf{q}(X_L, t)\|$ , for different values  
933 of  $\alpha$  and  $t_c$ .

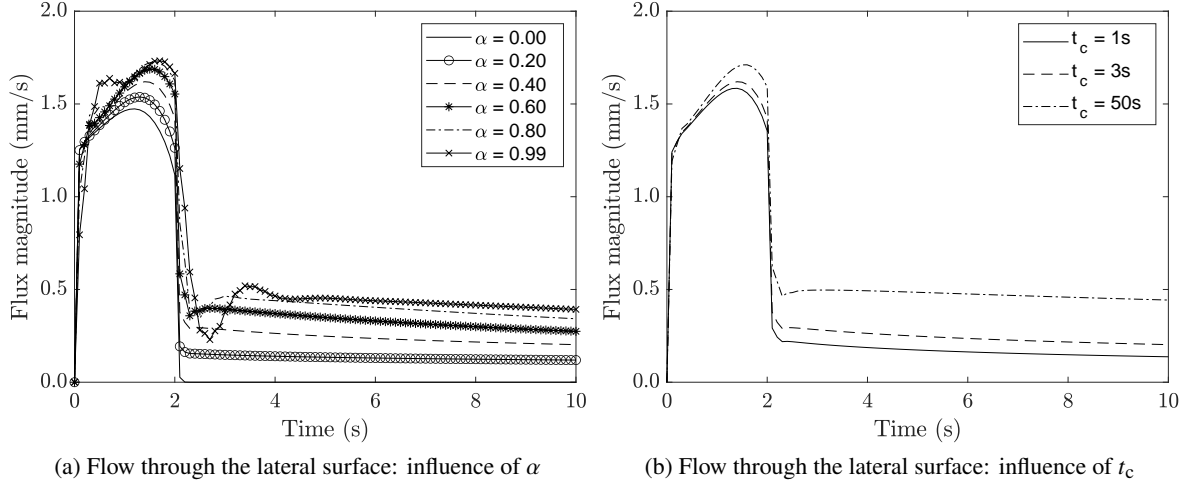


Figure 2: Time evolution of the Euclidean norm  $\|\mathbf{q}(X_L, t)\|$  of the filtration velocity (“flux magnitude” in the figures), evaluated at the node corresponding to the point  $X_L = (30, 0, 2.8) \in \Gamma_L$ , for  $\alpha = 0$  (i.e., standard Darcy-Forchheimer case) and for varying  $\alpha \in \{0.2, 0.4, 0.6, 0.8, 0.99\}$  with  $t_c = 3$  s (left panel), and for  $t_c \in \{1 \text{ s}, 3 \text{ s}, 50 \text{ s}\}$  with  $\alpha = 0.4$  (right panel)

934 By computing  $\|\mathbf{q}(X_L, t)\|$  for various values of  $\alpha$  (see Figure 2), we notice that the behavior of  
 935 the filtration velocity depends noticeably on the fractional order  $\alpha$ , whereas the value of  $t_c$  scales the  
 936 trend imposed by  $\alpha$ . In fact, in the Darcy model and in the Darcy-Forchheimer model, the maximum  
 937 of  $\|\mathbf{q}(X_L, t)\|$  is registered at relatively close time instants  $t^* \in [0, t_{\text{ramp}}]$  (see Figure 3). Yet, for the  
 938 fractional Forchheimer model, the maximum of  $\|\mathbf{q}(X_L, t)\|$  is observed at times larger than  $t^*$ . Moreover,  
 939 by setting  $t_{\text{max}}(\alpha) := \operatorname{argmax}_{t \in [0, t_{\text{fin}}]} \{\|\mathbf{q}_\alpha(X_L, t)\|\}$ , where  $\mathbf{q}_\alpha$  indicates the filtration velocity computed  
 940 for a given fractional order  $\alpha$ , we notice that  $t_{\text{max}}(\alpha)$  increases with  $\alpha$ . As a consequence of this behavior,  
 941 we also observe a widening of the time interval over which  $\|\mathbf{q}(X_L, t)\|$  grows monotonically in time.  
 942 This result constitutes a delay in the attainment of  $q_{\text{max}} := \max_{t \in [0, t_{\text{fin}}]} \{\|\mathbf{q}_\alpha(X_L, t)\|\}$ , and is an expected  
 943 feature of the model. Its physical interpretation could be related to the complexity of the microstructure,  
 944 which manifests itself, for instance, through the tortuosity of the pore network, or to some inertial effects  
 945 of the fluid taking place at the pore scale. Looking at Figure 3, we observe that, for the considered  
 946 problem, Forchheimer’s correction does not affect significantly the dynamics of the interstitial fluid,  
 947 since  $\|\mathbf{q}(X_L, t)\|$  in Darcy-Forchheimer’s case is only slightly different from the one computed with the  
 948 equivalent Darcy model (i.e., same setting and same parameters, but  $\alpha = 0$  and  $c_0 = 0$ ).

949 Moreover, it can be observed that there is a trend reversal of the magnitude of the outflow when  
 950 passing from the very first instants of loading (for  $t$  close to 0 s), in which fractional Forchheimer’s  
 951 fluid velocity is lower than the Darcian one, to the subsequent times, where, on average, the magnitude  
 952 of the outflow increases with the fractional order. Such behavior can be correlated with the evolution  
 953 of the pressure field, since it does not increase monotonically during the loading phase, as shown in  
 954 Figure 3(b). The presence of such a “breaking point” suggests a transition from a Darcian-like flow to an  
 955 alternative flow regime, whose existence depends on whether or not memory related effects are present  
 956 in the model.

957 Finally, it can be observed that the increase in the fractional order entails that the flux magnitude  
 958 relaxes more slowly towards the stationary state, as it is seen in Figures 2 and 3. In addition, the outflow  
 959 registered while the load is being maintained, i.e., for  $t$  greater than  $T_{\text{ramp}}$ , depends both on  $\alpha$  and on  $t_c$ .  
 960 For higher values of the fractional order, e.g.  $\alpha = 0.80, 0.99$ , fluctuations in the flux are visible after  
 961 the loading phase, and are quickly reabsorbed. We remark that the presence of oscillatory behavior with  
 962 exponential decay is typical, for example, of the Mittag-Leffler functions, which can be found as the  
 963 analytic solution of various integral equations and fractional models[95].

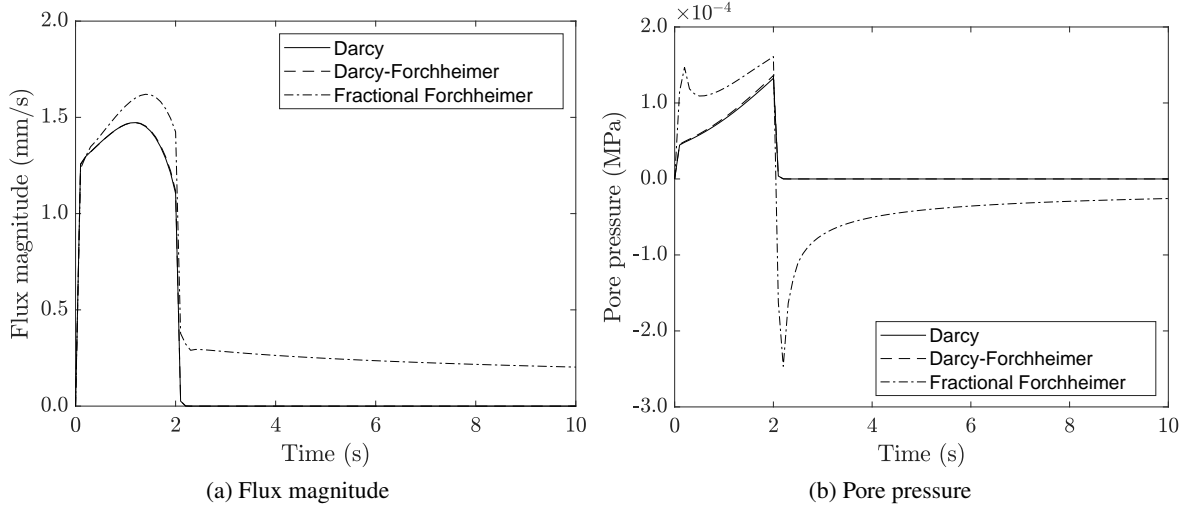


Figure 3: Comparison between the Darcy, the Darcy-Forchheimer and the fractional Forchheimer models of the time evolution of the Euclidean norm of the filtration velocity  $\|\mathbf{q}(X_L, t)\|$  (“Flux magnitude” in Figure 3(a)), evaluated at the node corresponding to the point  $X_L = (30, 0, 2.8) \in \Gamma_L$ , and of the pore pressure  $P(X_O, t)$  (“Pore pressure” in Figure 3(b)), evaluated at the node corresponding to the point  $X_O = (0, 0, 0) \in \Gamma_B$ , located at center of the bottom surface  $\Gamma_B$ . For the simulation of the fractional Forchheimer model we selected  $\alpha = 0.4$  and  $t_c = 3$  s.

## 964 8.2 Fractional effects in the central region

965 Next, we move on to analyze the dynamics of the interstitial fluid in the central region of the specimen.  
 966 Initially, in the center  $X_O$  of the bottom surface  $\Gamma_B$ , the fractional Forchheimer correction induces values  
 967 of the pore pressure that are even higher than those attained with the non-fractional Forchheimer model,  
 968 which, in turn, predicts values already higher than in Darcy’s model, as shown in Figure 3(b). However,  
 969 for  $\alpha \in [0.40, 0.99]$ , the pressure evolution is not monotonic, and for  $\alpha = 0.80, 0.99$ , the fluctuations  
 970 of the pressure field induced by the fractional flow model exceed the values in the Darcy-Forchheimer  
 971 case. Hence, upon this choice of permeability and Forchheimer’s parameters, the pressure field evolves  
 972 and decays mainly because of the memory effects.

973 Figure 4 displays the magnitude, predicted by the fractional Darcy-Forchheimer model, of the fluid  
 974 radial filtration velocity evaluated at  $X_O \in \Gamma_B$ . This magnitude coincides with that of the total filtration  
 975 velocity since  $\Gamma_B$  is in contact with the lower plate, which is impermeable. We notice that, in general,  
 976 the filtration velocity of the fluid is comparable or slightly greater than the one obtained with the non-  
 977 fractional Darcy model, i.e., for  $c_0 = 0$  and  $\alpha = 0$  (see Figure 3). However, while the loading is being  
 978 maintained, and in response to the value of  $\alpha$ , there exist cases in which the fluid filtration velocity is  
 979 higher than the one computed with the non-fractional Darcy-Forchheimer model (see Figure 4a). In  
 980 particular, the major oscillations in the pressure values translate into small swings in the tangential  
 981 filtration velocity that quickly decays while the compression is held. Depending on the tissue under  
 982 investigation, this result could be interpreted, for example, either as an accumulation of fluid in some  
 983 regions of the pore network, which, because of tortuosity or other inhibitors of the hydraulic conductivity,  
 984 may act as slowly emptying “buffers”, or as the manifestation at the tissue scale of inertial or viscous  
 985 effects and fluid-solid interactions at the pore scale. It is also interesting to note that, in  $X_O \in \Gamma_B$ ,  
 986 pore pressure does not increase monotonically for  $\alpha \in [0.40, 0.99]$  (see Figures 3(b) and 4c), while for  
 987  $\alpha = 0.2$  the behavior resembles the standard Darcy one, as the fractional effects do not induce swings in  
 988 the pressure fields in the specimen.

989 Under the steady state loading, ( $t > t_{\text{ramp}}$ ), as time goes by, the history effect decreases, and the flux  
 990 comes closer to the non-fractional model (see Figure 4a). Finally, only a very marginal impact of  $\alpha$  on  
 991 normal stress is observed (see Figure 4d), apart for the case  $\alpha = 0.99$ , in which, after the compression

992 phase ends, the negative pressure exerted on the pores affects the registered stress.

993 If some chemical substances, like salts or drugs, were considered in our models, and if one were  
 994 interested in studying the situation in which such substances, dissolved in the fluid, are for some reason  
 995 concentrated in the lower region of the specimen, then the radial filtration velocity of the fluid would be  
 996 responsible for their transport towards the outer region of the specimen itself.

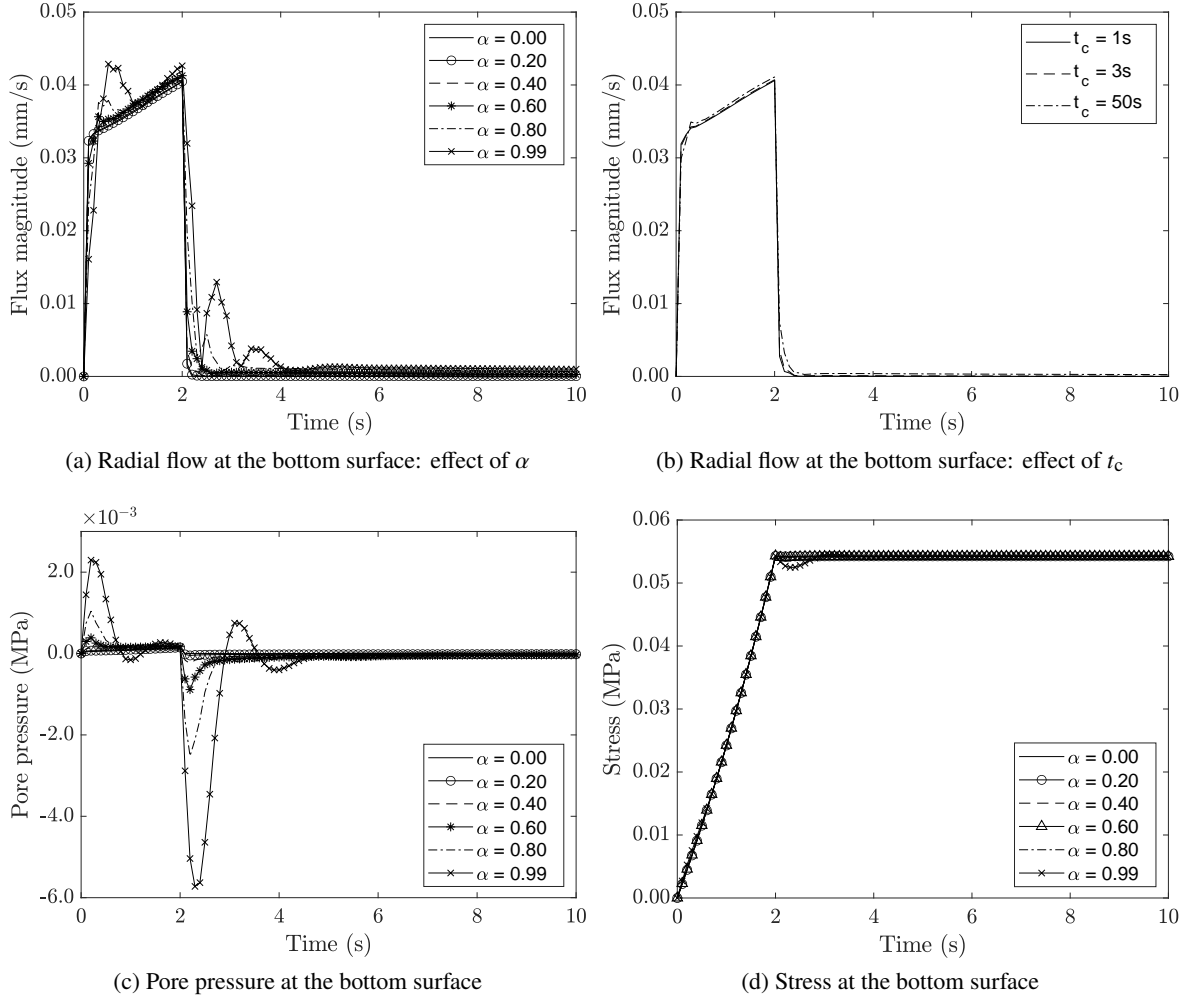


Figure 4: Time evolution of the Euclidean norm of the filtration velocity  $\|\mathbf{q}(X_O, t)\|$  (“flux magnitude” in Figure 4(a) and 4(b)), pore pressure  $P(X_O, t)$  (“pore pressure” in Figure 4(c)), and absolute value of the axial component of Cauchy stress,  $|\sigma_3^3(X_O, t)|$  (“Stress” in Figure 4(d)), evaluated at the node  $X_O = (0, 0, 0) \in \Gamma_B$ , located at center of the bottom surface  $\Gamma_B$  (corresponding to the origin of the given reference frame) for  $\alpha = 0$  (non-fractional Darcy-Forchheimer case) and for varying  $\alpha \in \{0.20, 0.40, 0.60, 0.80, 0.99\}$ , with characteristic time  $t_c = 3$  s (Figures 4(a), 4(c) and 4(d)), and for  $t_c \in \{1$  s, 3 s, 50 s}, with  $\alpha = 0.4$  (Figure 4(b)).

### 997 8.3 A “zoom in” on the system’s behavior for different material parameters

998 We now consider the case reported in[30], in which Forchheimer’s correction is considered to extend  
 999 the range of applicability of a given description of the flow by allowing to pass from a permeability  
 1000 formulation to a different one. With respect to Table 1, in order to make a comparison with the  
 1001 results observed in [30], this amounts to consider the following parameters:  $R = 1.5$  mm,  $H = 1$  mm,  
 1002  $k_{\text{ref}} = 1.88 \cdot 10^{-11}$  mm<sup>2</sup>,  $c_0 = 1.44 \cdot 10^9$  and a loading ramp with the same shape as the one considered  
 1003 before, but with  $u_T = 0.2$  mm and  $T_{\text{ramp}} = 20$  s. In this case, the simulations of the quarter of cylinder are

1004 carried out with the aid of the software ABAQUS<sup>®</sup> by employing C3D8T elements, with 23800 nodes  
 1005 and 26535 elements.

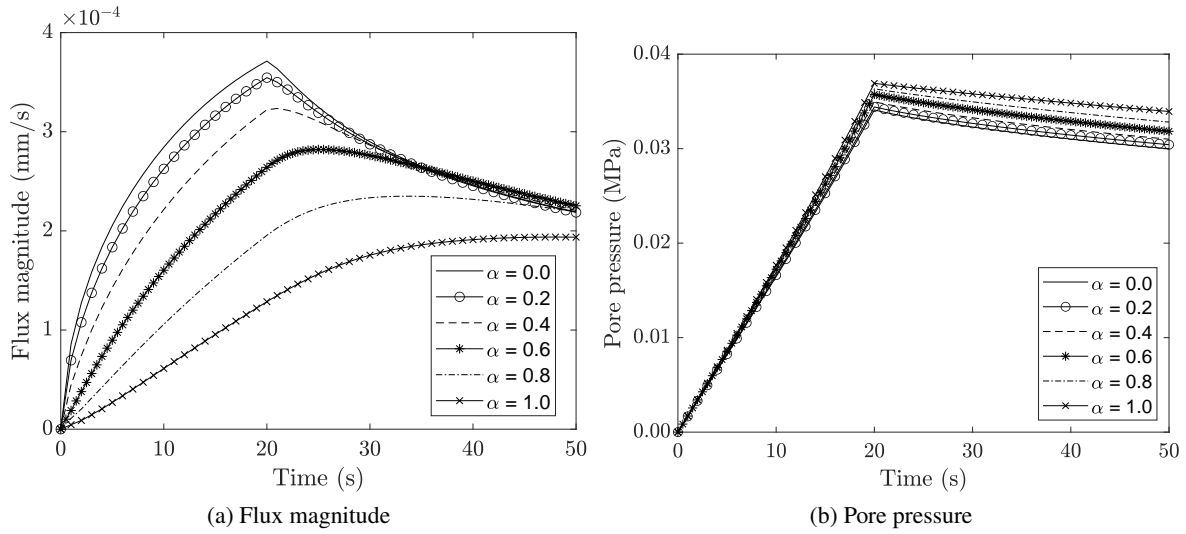


Figure 5: Time evolution of the Euclidean norm of the filtration velocity  $\|\mathbf{q}(X_L, t)\|$  (“Flux magnitude” in Figure 5(a)), evaluated at the node corresponding to the point  $X_L = (1.5, 0, 0.14) \in \Gamma_L$ , and of the pore pressure  $P(X_O, t)$  (“Pore pressure” in Figure 5(b)), evaluated at the node corresponding to the point  $X_O = (0, 0, 0) \in \Gamma_B$ , located at center of the bottom surface  $\Gamma_B$ , for different values of  $\alpha \in \{0, 0.2, 0.4, 0.6, 0.8, 1.0\}$ . For the simulation of the fractional Forchheimer model we selected  $t_c = 50$  s.

1006 In [30], the nonlinear effects associated with standard Forchheimer’s model have been interpreted as  
 1007 a correction to the “true” permeability, while the physics of the process described by the model presented  
 1008 in our work is different, and shows that those conclusions can be limiting. Indeed, the analogy with  
 1009 the correction of the permeability is evident only as long as we limit ourselves to a specific time frame  
 1010 in which the recent history of the filtration velocity is monotonically increasing or decreasing. Indeed,  
 1011 if we study  $\|\mathbf{q}(X_L, t)\|$  for  $t \in ]0, T_{\text{ramp}}]$ , Figure 5(a) shows that the flow’s history in the time integral  
 1012 affects the determination of the flux itself in a predictable way. During the loading ramp, the outflow  
 1013 grows because of the increasing compression, and the time derivative of the filtration velocity inside the  
 1014 integral of Equation (43c) is positive, and it does exert an antagonistic action with respect to equivalent  
 1015 Darcy’s velocity. In this case, the filtration velocity  $\|\mathbf{q}(X_L, t)\|$  is lower than the one in the corresponding  
 1016 standard Darcy-Forchheimer model, i.e., under the same boundary and initial conditions. However, for  
 1017  $t$  much greater than  $t_{\text{max}}$ , the outflow decreases in time, and, thus, the time derivative of the filtration  
 1018 velocity becomes negative, thereby producing results that could be associated with a higher permeability.  
 1019 Hence, depending on the history of the fluid flow, the tissue could be more or less permeable.

1020 This way of framing the study of the filtration velocity makes sense if the equivalent Darcian velocity  
 1021 is still comparable with the filtration velocity in the Darcian case, i.e., if the pressure field evolves in a  
 1022 similar way to the one in Darcy’s case. However, this is not necessarily true, as reported in Figure 3(b).  
 1023 In fact, in this case, the analogy with the purely Darcian case holds only in the instants subsequent to the  
 1024 start of the compression, during which the registered values of the outflow in the fractional Forchheimer  
 1025 case are lower than the non-fractional Darcy-Forchheimer ones.

1026 We have already observed in Figure 2(a) that, in fractional Forchheimer’s model, the maximum of  
 1027  $\|\mathbf{q}(X_L, t)\|$  is reached at a later time than in the standard Darcy-Forchheimer one. In order to quantify this  
 1028 time delay, we report in Figure 6 the time of the simulation in which the maximum outflow is computed as  
 1029 a function of the fractional parameter. Although the trend is clearly influenced by the material parameters  
 1030 considered for the benchmark, it is increasing in both cases.

1031 Finally, a visual comparison between the non-fractional Forchheimer correction (which corresponds

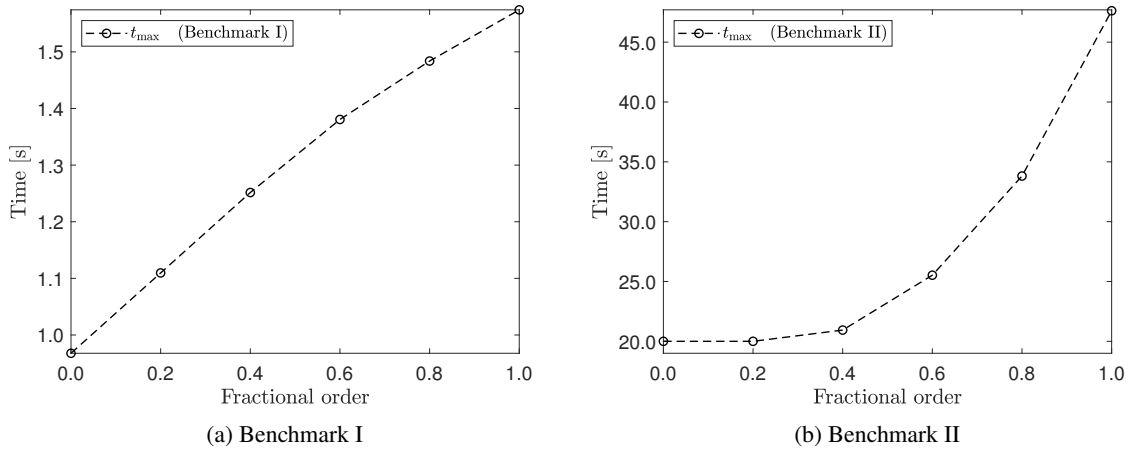


Figure 6: Time at which the maximum value of the Euclidean norm of the filtration velocity  $\|\mathbf{q}(X_L, t)\|$  is reached, evaluated at the node corresponding to the point  $X_L = (1.5, 0, 0.14) \in \Gamma_L$ , by varying  $\alpha \in ]0, 1[$ .

1032 to the case  $\alpha = 0$ ) and the fractional Forchheimer correction can be drawn by looking at Figures 7 and 8.  
 1033 At the time  $t = t_{\text{ramp}}$ , we plot the spatial distributions of pore pressure, magnitude of the filtration velocity,  
 1034 magnitude of the displacement field, and von Mises stress both for the non-fractional Darcy-Forchheimer  
 1035 model and for the fractional Forchheimer model with  $\alpha = 0.4$  and  $t_c = 50$  s, at the time  $t = 20$  s. The  
 1036 plots for the pore pressure and for the magnitude of the filtration velocity confirm that the region of  
 1037 interest for understanding the behavior of the fluid are the lower central region, where the overpressure  
 1038 area is located, and the lower lateral surface. The effect of the fractional order  $\alpha$  on the coupling of the  
 1039 fluid with the solid phase is weak, since the spatial distribution of the total deformation and von Mises  
 1040 stress are barely affected (Figure 8) by  $\alpha$ , and the evolution of the normal stress on the center of the  
 1041 bottom surface is similar (see Figure 4d).

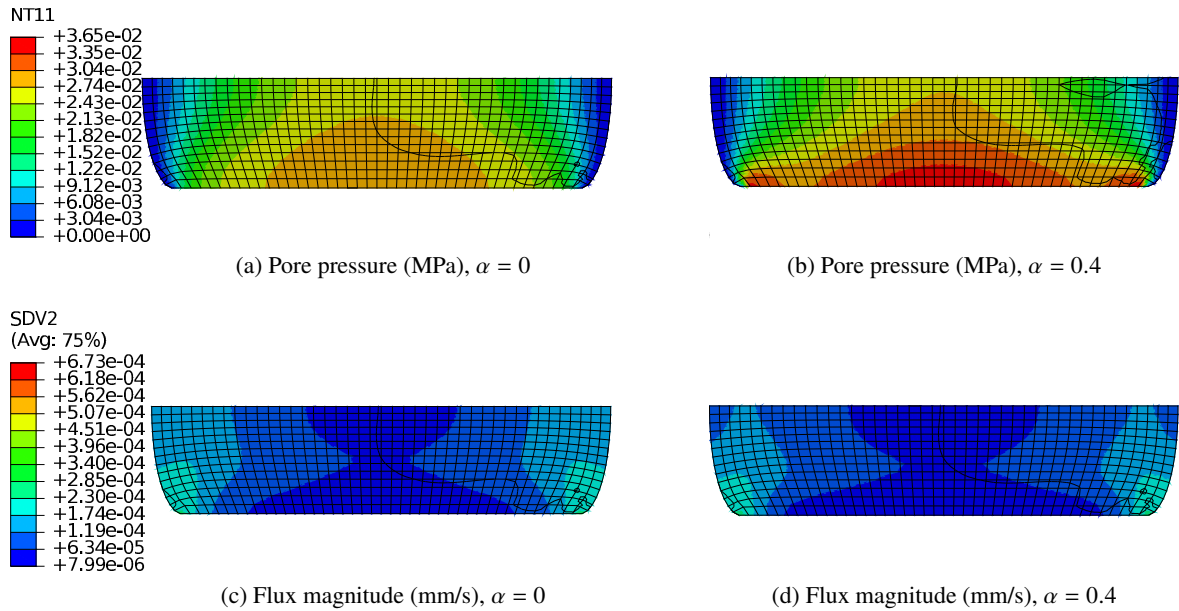


Figure 7: Comparison between the standard Darcy-Forchheimer model and the fractional Forchheimer model, with the choice of parameters  $\alpha = 0.4$  and  $t_c = 50$  s, at time  $t = 20$  s, of the spatial distributions of the pore pressure (Figures 7a and 7b) and flux magnitude (Figures 7c and 7d). The black solid lines in the plots represent different layers of elements of the finite element discretization.

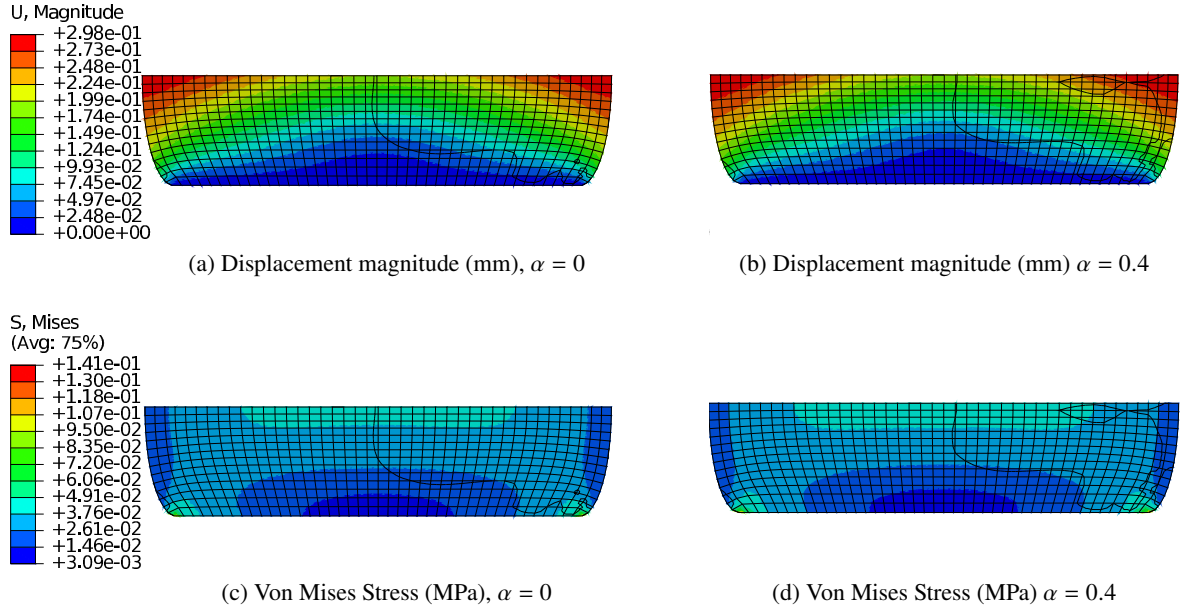


Figure 8: Comparison between the standard Darcy-Forchheimer model and the fractional Forchheimer model, with the choice of parameters  $\alpha = 0.4$  and  $t_c = 50$  s, at time  $t = 20$  s, of the spatial distributions of the displacement magnitude (Figures 8a and 8b) and of the von Mises stress (Figures 8c and 8d). The black solid lines in the plots represent different layers of elements of the finite element discretization.

## 9 Conclusions

In this work, we have described a hypothetical biological tissue, viewed as a saturated and hydrated porous medium, by formulating a mechanical model having the fractionalization of Forchheimer’s correction to Darcy’s law in finite deformations as target. This amounts to considering the concomitant effect of two deviations from the “classical” Darcian regime, and has been done with the purpose of studying a scenario that may originate in a tissue with a complex microstructure, like articular cartilage, when memory effects have to be combined with flow velocities that do not justify Darcy’s approximation. The main motivation for undertaking this study is the generalization of a class of flow models already existing in the literature, and aiming at describing Darcy’s law with memory, to the case in which the interactions of the fluid with the solid matrix require to include inertial effects.

With the recent development of numerical methods coupled with image analysis (CFD-IA,[96]), the image-based simulations from high-resolution x-ray tomography and multiphoton microscopy of native meniscal tissue [97, 98] can reveal the fluid flow at the pore scale. Ongoing work on FSI (fluid-structure interaction) - AI, which couples FEM and meshless fluid flow solvers (such as SPH), will give rise to running simulation of deforming the solid and fluid phases of native tissue architecture, retaining the complexity of the pores’ morphology. These simulations will provide the data to verify the model proposed here and elsewhere [44] as well as contribute to one of the main questions when dealing with fractional models, i.e., what the relation between the fractional parameters and the architecture of the tissue is. In other words, can we give a physical meaning to the fractional parameters?

To assess what our model predicts for a very typical benchmark problem, we have solved an initial and boundary value problem that simulates the uni-axial compression of a cylindrical specimen of the hypothetical tissue under investigation, and, to this end, we have devised a numerical procedure capable of framing fractional and highly nonlinear flow laws within the context of finite deformation poro-elasticity, and we implemented it in ABAQUS®.

In spite of the fact that, by applying the fractional operator only to the filtration velocity, we have particularized the constitutive picture presented in[38], our research encompasses two essential

1068 generalizations. The first one pertains to the definition of the fractional operator applied to the filtration  
1069 velocity, and describes the non-linearity of the flow model related to the passage from the Darcian to  
1070 the Forchheimer regime. Indeed, in Equation (38) we define a generalized Caputo derivative in which  
1071 the kernel of the integral operator features the resistivity tensor  $\mathbf{r}_F(\|\mathbf{q}(\tau)\|)$  applied to the Truesdell  
1072 derivative of  $\mathbf{q}$  at time  $\tau$ , i.e.,  $\mathcal{T}_s\mathbf{q}(\tau)$ . This yields a modified Cattaneo’s model for the filtration velocity  $\mathbf{q}$   
1073 that weighs the evolution of  $\mathbf{q}$  by means of a resistivity coefficient that depends on  $\mathbf{q}$  itself in a non-linear  
1074 way.

1075 The second generalization is inherent to the coupling between flow and deformation and consists in  
1076 the introduction of the Truesdell rate of the flow. Indeed, since our approach is entirely formulated for  
1077 finite deformations, it requires to employ the *correct* objective derivative for the kinematic parameter  
1078 chosen to describe the filtration motion of the fluid through the deforming solid matrix. In this respect,  
1079 since we have chosen the filtration velocity  $\mathbf{q}$ , which is a pseudo-vector, we have reformulated Caputo’s  
1080 classical fractional derivative of  $\mathbf{q}$  in such a way that the time derivative of  $\mathbf{q}$ , featuring under the integral  
1081 operator in the classical definition, is replaced by its Truesdell derivative,  $\mathcal{T}_s\mathbf{q}$ . Although the use of the  
1082 objective rates is well established in Continuum Mechanics, its employment in the present context makes  
1083 it clear how the deformation affects such reformulation. Indeed, looking at Equation (42), the pull-back  
1084 of the “modified” Caputo derivative, i.e., with  $\mathcal{T}_s\mathbf{q}$  in lieu of  $\dot{\mathbf{q}}$ , transforms it into a Caputo-type fractional  
1085 derivative for  $\mathbf{Q}$ , i.e., expressed in terms of  $\dot{\mathbf{Q}}(\tau)$ , at the price of introducing  $J(t)\mathbf{F}^{-1}(t)$  and  $J^{-1}(\tau)\mathbf{F}(\tau)$   
1086 in the kernel of the corresponding integral operator: the latter defines the push-forward of  $\dot{\mathbf{Q}}(\tau)$  to the  
1087 placement of the medium at time  $\tau$ , whereas the former defines the pull-back, to the reference placement,  
1088 of the integral in Equation (42), which captures the whole history of the medium from  $t_{\text{in}}$  to  $t$ .

1089 We point out that the fractional order  $\alpha$ , by analogy with Cattaneo’s model[99], can be interpreted  
1090 as a measure of how much the history of the process influences the filtration velocity  $\mathbf{q}$ . Depending  
1091 on the history, such effect can be more or less antagonizing, and, in the latter case, it can lead to an  
1092 outflow greater than the one obtainable in the standard Darcy-Forchheimer model under the same loading  
1093 conditions. We have also observed that the introduction of the fractional law can lead to a significant  
1094 alteration in the values of pressure in the central region with respect to the Darcy and Darcy-Forchheimer  
1095 models, depending on the choice of parameters, and we did observe coupling effects that could alter  
1096 significantly the stress state of the solid phase only for  $\alpha = 0.99$ . To this end, we remark that different  
1097 couplings could be studied by considering a different fractional law [38], or by introducing remodeling  
1098 effects, either structural or due to growth (a fractional model of which has been recently presented  
1099 in[100]) or due to the spatial reorientation of fibers.

1100 A different kind of nonlinear coupling that we would be interested to study in the future is the  
1101 combined effect of a fractional Forchheimer’s law for the flow and a fractional viscoelastic behavior of  
1102 the solid phase. This approach would aim at a better characterization of the mechanical behaviour of  
1103 biological tissues for which fractional models have been successful in describing the solid phase, but no  
1104 fractional law has been proposed to describe the interstitial fluid.

## 1105 **Conflict of Interests**

1106 The Authors declare that they have no conflict of interests.

## 1107 **Authors’ contributions**

1108 All authors have equally contributed to this work. This work is part of a joint research project conducted in  
1109 equal measure by the authors Sachin Gunda and Alessandro Giammarini, and constitutes an intersection  
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## 10 Appendix A

We define the *apparent mass densities*  $\varrho_s \phi_s$  and  $\varrho_f \phi_f$ , with  $\varrho_s$  and  $\varrho_f$  being the constant true mass densities of the solid and the fluid. Hence, we write the balance of mass for each phase in the mixture's current placement  $\mathcal{B}_t$  as [1, 22, 36, 30]

$$\partial_t(\varrho_s \phi_s) + \operatorname{div}(\varrho_s \phi_s \mathbf{v}_s) = 0 \quad \Rightarrow \quad \partial_t \phi_s + \operatorname{div}(\phi_s \mathbf{v}_s) = 0, \quad \text{in } \mathcal{B}_t, \quad (97a)$$

$$\partial_t(\varrho_f \phi_f) + \operatorname{div}(\varrho_f \phi_f \mathbf{v}_f) = 0 \quad \Rightarrow \quad \partial_t \phi_f + \operatorname{div}(\phi_f \mathbf{v}_f) = 0, \quad \text{in } \mathcal{B}_t. \quad (97b)$$

The absence of terms on the right-hand side of Equations (97a) and (97b) means that, at the considered timescale, we see neither growth processes nor mass exchange between the constituents.

Since the mixture considered in our work is saturated, the condition  $\phi_s + \phi_f = 1$  applies. Hence, the balance of mass for the solid phase and for the mixture as a whole, obtained by adding together Equations (97a) and (97b), can be rephrased as

$$D_s \phi_s + \phi_s \operatorname{div} \mathbf{v}_s = 0, \quad \text{in } \mathcal{B}_t, \quad (98a)$$

$$\operatorname{div} \mathbf{v}_s + \operatorname{div} \mathbf{q} = 0, \quad \text{in } \mathcal{B}_t, \quad (98b)$$

where the substantial derivative with respect to the motion of the solid phase has been introduced, i.e.,  $D_s \zeta := \partial_t \zeta + (\operatorname{grad} \zeta) \mathbf{v}_s$ , for any differentiable field  $\zeta : \mathcal{B}_t \times \mathcal{I} \rightarrow \mathbb{S}$  valued in  $\mathbb{S} \equiv \mathbb{R}$  or in higher-order vector or tensor spaces [5]. Hence, Equations (1a) and (1b) are obtained by considering the pull-back of Equations (98a) and (98b) in the reference placement.

Next, we introduce the balance of linear momentum in the current placement. Since, in the present framework, macroscopic inertial forces are assumed to be negligible from the outset, we write [19, 36, 30]

$$\operatorname{div} \boldsymbol{\sigma}_s + \boldsymbol{\pi}_s + \varrho_s \phi_s \mathbf{g} = \mathbf{0}, \quad \Rightarrow \quad \operatorname{div}(\boldsymbol{\sigma}_s + \boldsymbol{\sigma}_f) + (\varrho_s \phi_s + \varrho_f \phi_f) \mathbf{g} = \mathbf{0}, \quad \text{in } \mathcal{B}_t, \quad (99a)$$

$$\operatorname{div} \boldsymbol{\sigma}_f + \boldsymbol{\pi}_f + \varrho_f \phi_f \mathbf{g} = \mathbf{0}, \quad \Rightarrow \quad \operatorname{div} \boldsymbol{\sigma}_f + \boldsymbol{\pi}_f + \varrho_f \phi_f \mathbf{g} = \mathbf{0}, \quad \text{in } \mathcal{B}_t, \quad (99b)$$

where  $\boldsymbol{\sigma}_s$  and  $\boldsymbol{\sigma}_f$  are the Cauchy stress tensors of the solid and of the fluid phase,  $\boldsymbol{\pi}_s$  and  $\boldsymbol{\pi}_f$  are the force densities due to the exchanges of linear momentum between the phases, and  $\mathbf{g}$  is the gravity acceleration co-vector. Note that, in the equations of the first column, each balance law is associated with a single phase, i.e., either with the solid or with the fluid phase. In the second column, instead, the second equation is identical to its homologous of the first column, while the first equation expresses the balance of linear momentum for the mixture as a whole. Indeed, it is obtained by adding together the balance laws associated with each single phase and by using the hypothesis of the mixture being closed with respect to linear momentum, i.e.,  $\boldsymbol{\pi}_s + \boldsymbol{\pi}_f = \mathbf{0}$ .

Since, according to Equation (1a),  $\Phi_s$  is constant in the time interval over which the system is observed, and it is determined univocally by the initial condition  $\Phi_{sR}$ , we set  $\Phi_s(X, t) = \Phi_{sR}(X)$ , and we eliminate it from the set of unknowns featuring in the balance equations. This result, indeed, permits to write the volumetric fractions of the solid and of the fluid phase as  $\phi_s(\chi(X, t), t) = \Phi_{sR}(X)/J(X, t)$  and  $\phi_f(\chi(X, t), t) = 1 - \Phi_{sR}(X)/J(X, t)$ . Therefore, Equations (98b), (99a), and (99b) feature 7 scalar equations in 21 unknowns (assuming  $\boldsymbol{\sigma}_s$  and  $\boldsymbol{\sigma}_f$  to be symmetric): 3 for the components of the motion  $\chi$ ; 3 for the components of the filtration velocity  $\mathbf{q}$ ; 6 for the components of  $\boldsymbol{\sigma}_s$ ; 6 for the components of  $\boldsymbol{\sigma}_f$ ; and 3 for the components of  $\boldsymbol{\pi}_f$ . To these unknowns, however, a Lagrange multiplier accompanying the incompressibility constraint has to be added, so that the full number of unknowns raises to 22. Consequently, to close the model, we need to supply the Cauchy stress tensors  $\boldsymbol{\sigma}_s$  and  $\boldsymbol{\sigma}_f$  as well as the force density  $\boldsymbol{\pi}_f$  constitutively, thereby introducing the missing 15 scalar equations. This way, the *remaining unknowns* to be determined are:

$$\chi, \quad \mathbf{q}, \quad p, \quad (100)$$

where  $p$  is the pore pressure and represents the Lagrangian multiplier of the present theory.

## 1437 11 Appendix B

1438 It can be proved that, if the solid phase is hyperelastic and the macroscopic stress response of the fluid  
1439 phase is not appreciably affected by the fluid viscosity, the Cauchy stress tensors are given by[32, 34, 1,  
1440 22, 19, 20, 30]

$$\sigma_s = -\phi_s p \mathbf{I}^T + \sigma_{sc}, \quad \text{in } \mathcal{B}_t, \quad (101a)$$

$$\sigma_f = -\phi_f p \mathbf{I}^T, \quad \text{in } \mathcal{B}_t, \quad (101b)$$

1441 where  $p$  is pore pressure,  $\sigma_{sc}$  is the constitutive part of  $\sigma_s$ , and  $\mathbf{I}(x, t)$  is the identity tensor associated  
1442 with  $T_x \mathcal{S}$ . Note that, in this work, the Cauchy stress tensors are taken as linear maps from  $T_x^* \mathcal{S}$  into itself,  
1443 i.e.,  $\sigma_\alpha(x, t) : T_x^* \mathcal{S} \rightarrow T_x^* \mathcal{S}$ , for all  $x \in \mathcal{B}_t$ , and, thus, the transpose of the identity tensor  $\mathbf{I}$  is needed for  
1444 consistency, since it applies that  $\mathbf{I}^T(x, t) : T_x^* \mathcal{S} \rightarrow T_x^* \mathcal{S}$ , with  $x \in \mathcal{B}_t$ , and  $\mathbf{I}^T(x, t) \boldsymbol{\beta}(x, t) = \boldsymbol{\beta}(x, t)$ , for  
1445 every co-vector  $\boldsymbol{\beta}(x, t) \in T_x^* \mathcal{S}$ .

1446 For a given hyperelastic strain energy density function  $W_s(\mathbf{F})$ , the constitutive part of the Cauchy  
1447 stress tensor associated with the solid phase is identified by

$$\sigma_{sc}(\chi(X, t), t) = \frac{1}{J(X, t)} \left[ \frac{\partial W_s}{\partial \mathbf{F}}(\mathbf{F}(X, t)) \right] \mathbf{F}^T(\chi(X, t), t). \quad (102)$$

1448 Moreover, the balance laws (99a) and (99b) can be recast in the form

$$\text{div}(-p \mathbf{I}^T + \sigma_{sc}) + (\rho_s \phi_s + \rho_f \phi_f) \mathbf{g} = \mathbf{0}, \quad \text{in } \mathcal{B}_t, \quad (103a)$$

$$-\phi_f \text{grad} p + \boldsymbol{\pi}_{fd} + \rho_f \phi_f \mathbf{g} = \mathbf{0}, \quad \text{in } \mathcal{B}_t, \quad (103b)$$

1449 with  $\sigma_{sc}$  being given in Equation (102), and  $\boldsymbol{\pi}_{fd} := \boldsymbol{\pi}_f - p \text{grad} \phi_f$  being referred to as the *dissipative*  
1450 *part* of  $\boldsymbol{\pi}_f$  [32, 34, 1, 5].

1451 The stress tensor featuring in Equation (103a), i.e.,

$$\boldsymbol{\sigma}_I := -p \mathbf{I}^T + \sigma_{sc}, \quad (104)$$

1452 represents the *inner part*[32] of the overall Cauchy stress tensor of the solid-fluid mixture under in-  
1453 vestigation, that is, the stress tensor of the mixture exclusive of the dynamic contributions, which are  
1454 negligible in the considered regime[32, 19].

## 1455 12 Appendix C

1456 Equations (57a)-(57c) are solved within ABAQUS<sup>®</sup> by using some formal analogies among thermoelas-  
1457 ticity, poroelasticity and mass diffusion, and by having recourse to the user subroutines ‘‘UMAT’’ and  
1458 ‘‘UMATHHT’’ in the same fashion as [101, 45, 44, 102]. ABAQUS<sup>™</sup> ‘‘UMATHHT’’ solves the energy bal-  
1459 ance equation (see Equation (107)). This is similar to the weak form of the mass conservation Equation  
1460 (57b) that can be written as follows. Integration is taken over the reference placement  $\mathcal{B}$ , here assumed  
1461 to coincide with the medium’s initial placement, i.e.,

$$-\int_{\mathcal{B}} \frac{J_m - J_{m-1}}{\Delta t} P_v + \int_{\mathcal{B}} \boldsymbol{Q}_m \text{Grad} P_v - \int_{\partial_N^P \mathcal{B}} (\boldsymbol{Q}_m \mathbf{N}) P_v = 0. \quad (105)$$

1462 By converting the integrals in Equation (105) to the current placement  $\mathcal{B}_t$ , Equation (105) becomes

$$-\int_{\mathcal{B}_t} \frac{1}{\Delta t} \left[ \left( \frac{J_m - J_{m-1}}{J_m} \right) \right] p_v + \int_{\mathcal{B}_t} \mathbf{q}_m \text{grad} p_v - \int_{\partial_N^P \mathcal{B}_t} (\mathbf{q}_m \mathbf{n}) p_v = 0, \quad (106)$$

1463 where, with a slight abuse of notation, the ratio  $(J_m - J_{m-1})/J_m$  is understood here as a function of  
 1464 spatial points and time. The weak form of the energy balance equation given in ABAQUS<sup>®</sup> reference  
 1465 manual[103] reads

$$\underbrace{\frac{1}{\Delta t} \int_V \delta\theta \rho (U_{t+\Delta t} - U_t) dV}_{=0} = \int_V \delta g \cdot \underbrace{\mathbf{f}}_{\mathbf{q}_m} dV + \int_S \delta\theta \underbrace{q}_{-\mathbf{q}_m \mathbf{n}} dS + \int_V \delta\theta \underbrace{r}_{-\frac{J_m - J_{m-1}}{J_m \Delta t}} dV, \quad (107)$$

1466 where  $\delta\theta$  is a virtual variation of temperature, and, thus, plays the role of  $p_v$ , while  $\delta g$  stands for  
 1467 the spatial gradient of  $\delta\theta$ , and corresponds to our grad  $p_v$ . Further equivalences among the variables  
 1468 featuring in Equations (106) and (107) are made for making “UMATHHT” suitable for solving Equations  
 1469 (43b) and (43c). The correspondences are as follows: the temperature  $\theta$  of “UMATHHT” plays the role of  
 1470 the pore pressure  $p$ ; the rate of heat generation is identified with the rate of volumetric deformation, so  
 1471 that  $r$  corresponds to  $-((J_m - J_{m-1})/(J_m \Delta t))$ ; the heat flux  $\mathbf{f}$  corresponds to the filtration velocity  $\mathbf{q}_m$ ;  
 1472 the density  $\rho$  introduced in “UMATHHT” is set equal to zero.

1473 The pseudo-code for the implementation of our equations in ABAQUS<sup>®</sup> is provided in Algorithm  
 1474 1. Within “UMATHHT”, the filtration velocity is solved from Equation (43c) by using the methodology  
 1475 explained in subsection 6.2. Variations of flux with respect to the gradient of pore pressure are calculated  
 1476 according to Equation (89). The information of the gradient required for calculating the filtration velocity  
 1477 through Equation (55) is passed to “UMATHHT” from “UMAT” by storing it among the global variables.  
 1478 The terms that are calculated in “UMATHHT”, required as output to ABAQUS<sup>®</sup>, are given as

$$FLUX \text{ (see (92))} : \mathbf{q}_m^{k-1} = \frac{1}{J_m^{k-1}} \mathbf{F}_m^{k-1} \mathbf{Q}_m^{k-1}, \quad (108a)$$

$$DFDG \text{ (see (89))} : \mathfrak{g}_m^{k-1} := \frac{1}{J_m^{k-1}} \mathbf{F}_m^{k-1} \left[ \frac{\partial \mathbf{Z}}{\partial \mathbf{Q}_m} (\#_m^{k-1}) \right]^{-1} \left[ \frac{\partial \mathbf{Z}}{\partial \text{Grad} P_m} (\#_m^{k-1}) \right] [\mathbf{F}_m^{k-1}]^T. \quad (108b)$$

1479 The subroutine “UMAT” is used to solve the balance of linear momentum, and to define the coupling  
 1480 terms. The Neo-Hookean potential energy density is stated in Equation (70), and the *consistent Jacobian*  
 1481 *matrix* given in Equation (77) can be written for the problem solved in Section 7 as follows (see (77)):

*DDSDDE* :

$$\mathfrak{a}_m^{k-1} = \frac{\Phi_{sR} \mu_s}{2J_m^{k-1}} (\boldsymbol{\eta}^{-1} \otimes \mathbf{B}_m^{k-1} + \boldsymbol{\eta}^{-1} \otimes \mathbf{B}_m^{k-1} + \mathbf{B}_m^{k-1} \otimes \boldsymbol{\eta}^{-1} + \mathbf{B}_m^{k-1} \otimes \boldsymbol{\eta}^{-1}) + \left( \frac{\Phi_{sR} \lambda_s}{J_m^{k-1}} - P \right) \boldsymbol{\eta}^{-1} \otimes \boldsymbol{\eta}^{-1}. \quad (109)$$

1482 Here,  $\mathbf{B}$  is the left Cauchy-Green deformation tensor defined as  $\mathbf{B} := \mathbf{F} \cdot \mathbf{F}^T$ . Other terms that are  
 1483 calculated in “UMAT”, required as output to ABAQUS<sup>®</sup>, are given as

$$STRESS : \boldsymbol{\sigma}_m^{k-1} = \frac{1}{J_m^{k-1}} \boldsymbol{\eta}^{-1} \mathbf{T}_{Im}^{k-1} [\mathbf{F}_m^{k-1}]^T \quad (\text{see (71)}), \quad (110a)$$

$$DDSDDT : -\boldsymbol{\eta}^{-1} \quad (\text{see (79)}), \quad (110b)$$

$$RPL := -\frac{J_m^{k-1} - J_{m-1}}{J_m^{k-1} \Delta t}, \quad (110c)$$

$$DRPLDE := -\frac{1}{\Delta t} \boldsymbol{\eta}^{-1} \quad (\text{see (86)}), \quad (110d)$$

$$DRPLDT = 0. \quad (110e)$$

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**Algorithm 1** Pseudo Code of “UMAT” and “UMATHHT” for ABAQUS®

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1: **Common Module**

2: Define global variables to store the deformation gradient in “UMAT” to be used by “UMATHHT”, and to store the history terms for the calculation of fractional integral.

3: **UMATHHT:**

4: *Inputs:* Pore pressure, Increment of pore pressure, Current gradient of pore pressure and other terms.

5: Calculate permeability  $\kappa_{\text{iso}}$  from (36)

6: Calculate Forchheimer’s coefficient  $\mathcal{A}_{\text{iso}}$  from (33)

7: Compute  $\mathcal{R}_F$  from (41)

8: Calculate  $\mathcal{F}_\alpha$  from (54b)

9: Compute filtration velocity  $\mathcal{Q}$  using the Newton-Raphson method as described in section 6.2

10: Compute flux rate  $\dot{\mathcal{Q}}_{\text{app}}$  using (50)

11: Compute the contribution from the current time step to the History variable  $\mathcal{F}_\alpha(t_m)$  using (54b) and store it in global variables.

12: Compute flux (108a), Variation of flux with respect to pore pressure gradient using (108b).

13: *Output:* Flux at the end of the increment (FLUX), Variation of the flux vector with respect to the spatial gradients of pore pressure (DFDG).

14: **UMAT:**

15: *Input:* Deformation gradient at the increment’s start and end, Stress, Pore pressure at the start of the increment, increment of pore pressure and other terms.

16: Compute Stress (110a), Consistent Jacobian matrix (109), Variation of stress with pore pressure(110b), rate of volumetric deformation (110c) and its variation with strain increment (110d) and temperature increment (110e)

17: Store deformation gradient in the global variable.

18: *Output:* Stress at the end of the increment(STRESS), Consistent Jacobian matrix (DDSDDE), Volumetric heat generation per unit time (RPL), Variation of stress with respect to pore pressure(DDSDDT), Variation of RPL with Pore pressure(DRPLDT), Variation of RPL with strain increments(DRPLDE).

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