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DOCTORAL THESIS

On semiseparability, semifunctors and conditions up to retracts

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Abstract

In this thesis, we present the notion of *semiseparable* functor, introduced and investigated in [4], and its main properties that extend known results for separable and naturally full functors. We have defined a functor to be *semiseparable* by requiring a (von Neumann) regularity condition on its associated natural transformation on the hom-set components. A separable functor is faithful and a naturally full functor is full: semiseparability allows to reverse these implications and to treat separable and naturally full functors in a unified way. A suitable idempotent natural transformation and a canonical factorization can be attached to any semiseparable functor. We provide characterizations of semiseparibility for functors that are part of an adjunction, focusing mainly on functors attached to morphisms of rings and coalgebras, comodule categories over corings, and bimodules.

We deal with semiseparable functors in the context of Eilenberg-Moore categories and idempotent complete categories, as investigated in [5]. We present the notions of *coreflections up to retracts* (*reflections up to retracts*, respectively), i.e. functors whose idempotent completion admits a fully faithful left (resp., right) adjoint, and *bireflections up to retracts*, a stronger notion involving both a left and right adjoint. We discuss semiseparability with respect to these functors. One of the main results we have proved in this setting is that a right (resp., left) adjoint functor is semiseparable if, and only if, the associated (co)monad is (co)separable and the (co)comparison functor is a bireflection up to retracts, recovering known results in the separable case. We then consider the context of pre-triangulated categories, providing conditions for the Eilenberg-Moore category of (co)modules and for the Kleisli category of free modules to inherit the pre-triangulation from the base category by means of semiseparability.

Another aim of this thesis is to show how several properties of functors, such as faithfulness, (natural) fullness, (semi)separability, can be formulated for a semifunctor, as studied in [21]. We present the notion of *semifullness*, *semifull faithfulness* and *natural semifullness* for semifunctors. We characterize these properties for semifunctors that are part of a semiadjunction, in terms of "semisplitting" conditions for the unit and counit, and we give examples of semifunctors studied with respect to these notions.

The results presented in this thesis have been investigated mainly in [4], [5], [21]. We provide some other original results and we start exploring some variations of semiseparability, that we plan to develop in future works.

Keywords. Separability; semiseparable functor; (co)reflection; Eilenberg-Moore categories; idempotent completion; semifull semifunctor; coring; (co)module categories.

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Contents

Introduction

The notion of separability recurs in several topics in Algebra, Number Theory and Algebraic Geometry, for instance in classical Galois theory, ramification theory, Azumaya algebras, and Hochschild cohomology. The categorical notion of separable functor has been introduced by C. Năstăsescu, M. Van den Bergh and F. Van Oystaeyen in [72], where applications in the framework of group-graded rings have been considered. This notion allows to reinterpret categorically the theory of separable ring extensions, and of separable algebras. The motivating example which led to the definition of separable functor is provided by the fact that the restriction of scalars functor, attached to a ring morphism $R \to S$, results to be separable if, and only if, the ring extension S/R is separable in the classical sense [72], i.e., the multiplication $m_S : S \otimes_R S \to S$, $s \otimes_R s' \mapsto ss'$, splits as an S-bimodule map. Among their main properties, separable functors reflect split epimorphisms and split monomorphisms, satisfying a functorial version of "Maschke Theorem" (see [72, Proposition 1.2]), and this is one of the reasons for the relevant interest on this notion in the framework of module categories. Another central result is the so-called "Rafael Theorem" [78, Theorem 1.2], which provides a characterization of separability for functors that are part of an adjunction in terms of splitting properties of the (co)unit. Separable functors have then been extensively studied, for instance in the context of coalgebras [29], graded homomorphims of rings [30], comodule categories over corings [23], [42], Doi-Hopf modules [27], bimodules [78], expanding the original study of the separability for the induction and restriction of scalars functors associated to a ring homomorphism. Several results and applications of separable functors are illustrated e.g. in [28]. Interestingly, separable functors play a significant role in the context of Eilenberg-Moore categories. In [22] the notion of (co)separable (co)algebra was extended to the one of (co)separable (co)monad on a category $\mathcal{C}$ (that is, a (co)algebra in the monoidal category of endofunctors on $\mathcal{C}$). The (co)separability of a (co)monad is in fact equivalent to the separability of the forgetful functor attached to the Eilenberg-Moore category of (co)modules over the given (co)monad [17, 2.9]. Moreover, the separability of any right (resp., left) adjoint functor entails the separability of the (co)monad associated to the adjunction [31, Lemma 3.1], and under an idempotent completeness condition, a separable right (resp., left) adjoint functor is always monadic (resp., comonadic), see [64, Proposition 3.16], [31, Corollary 3.6].

Several variations of the notion of separable functor have then been investigated in the literature, among all we mention separable functors of the second kind [26], naturally full functors [7], relative separable functors [3], and heavily separable functors [11]. In [78] the original definition of separable functor has been restated as follows: a functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *separable* if, and only if, the natural transformation

$$\mathcal{F} : \mathrm{Hom}_{\mathcal{C}}(-,-) \to \mathrm{Hom}_{\mathcal{D}}(F-,F-), \quad [f : X \to Y] \mapsto \mathcal{F}_{X,Y}(f) = Ff, \tag{0.1}$$

has a left inverse, i.e., there is a natural transformation $\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F-,F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$

such that $\mathcal{P}_{X,Y} \circ \mathcal{F}_{X,Y} = \mathrm{Id}_{\mathrm{Hom}_{\mathcal{C}}(X,Y)}$, for all X, Y in $\mathcal{C}$. Since an injective map between sets has a left inverse, then a functor is faithful if every component $\mathcal{F}_{X,Y}$ has a left inverse, and it is separable if this left inverse can be chosen to be natural in X and Y. Thus, as observed in [28], separable functors could also be named "naturally faithful" functors. *Naturally full* functors, introduced in [7], arise as a dual version of separable functors, by requiring $\mathcal{F}$ to have a right inverse. Clearly, a functor is fully faithful if, and only if, it is both separable and naturally full.

By definition any separable functor is faithful, while any naturally full functor is full. A natural question one can ask is when a faithful functor is separable and when a full functor is naturally full. In [4] we have introduced the notion of *semiseparable* functor that allows to reverse the latter implications and to treat separability and natural fullness in a unified way. In this thesis, we present the results concerning semiseparable functors and their applications, investigated mainly in [4] and [5]. In particular, in [5] we have defined a functor to be a *coreflection up to retracts* (resp., *reflection up to retracts*), if its idempotent completion admits a fully faithful left (resp., right) adjoint. We have studied semiseparability with respect to these functors, showing that a semiseparable (co)reflection up to retracts gives actually rise to a stronger notion, that we have called *bireflection up to retracts*, involving both a left and right adjoint. One of the main result we have proved in this setting is an extension of the aforementioned properties holding in the separable case [31]: explicitly, a right (left) adjoint functor is semiseparable if, and only if, the associated (co)monad is (co)separable and the (co)comparison functor is a bireflection up to retracts. As a consequence, given an adjunction, the semiseparability of the right adjoint provides an equivalence after idempotent completion between the Kleisli category of free modules over the associated monad and the Eilenberg-Moore category of modules over that monad. Moreover, semiseparability has found an application in the context of pre-triangulated categories. Indeed, we have obtained a semi-analogue of [12, Theorem 4.1] for separable functors, providing conditions for the Eilenberg-Moore category of modules to inherit the pre-triangulation from the base category. Dealing with idempotent complete categories, i.e. categories where all idempotents split, we came across the notion of *semifunctor*, defined as a functor that does not necessarily preserve identities. This has led us to wonder how fullness, full faithfulness, (semi)separability and natural fullness, can be formulated for semifunctors. Since a full semifunctor reveals to be actually a functor, we have investigated in [21] a notion of semifullness (and then, semifull faithfulness and natural semifullness) for semifunctors. Another aim of this thesis is to present these semifunctorial properties and their characterizations for semifunctors that are part of a semiadjunction, and in view of them, we are able to revise some results shown in [5]. Finally, as in the separable case, we exhibit some new variations of the notion of semiseparable functor, planning to continue investigating them in future works.

We now describe the main results shown in this thesis.

Semiseparable functors

We have defined a functor $F : \mathcal{C} \to \mathcal{D}$ to be *semiseparable* [4] by requiring the natural transformation $\mathcal{F}$ in (0.1) to be a regular natural transformation, i.e., by requiring $\mathcal{F}$ to admit a natural transformation $\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ such that

$$\mathcal{F} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{F}.$$

This condition is an analogue of the one defining a von Neumann regular element of a unital ring. Separable and naturally full functors are instances of semiseparable functors and, if we add to a semiseparable functor either the assumption of faithfulness or fullness, we retrieve the notions of separable or naturally full functor, respectively (Proposition 2.5). The first difference we note, with respect to the separable and naturally full cases, is that semiseparable functors are not closed under composition (see Example 3.4). Nevertheless, the closeness is available in some cases (Lemma 2.6): for instance, given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, if either F is semiseparable and G is separable, or F is naturally full and G is semiseparable, then the composite $G \circ F : \mathcal{C} \to \mathcal{E}$ is semiseparable.

To any semiseparable functor $F : \mathcal{C} \to \mathcal{D}$ it is possible to attach, in a unique way, a suitable idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ (see Proposition 2.11), which is trivial only in case F is separable (Corollary 2.12). The existence of the *associated idempotent* allows to describe separable functors in terms of reflectivity conditions. As we have mentioned, separable functors satisfy a functorial version of "Maschke Theorem", and any separable functor is Maschke, dual Maschke or conservative, i.e., they reflect split-monomorphisms, split-epimorphisms, isomorphisms, respectively. If we assume the semiseparability condition on the functor, then through its associated idempotent e we show that the reverse implications hold (Corollary 2.18). There are further situations in which the notion of semiseparable functor collapses into the one of separable functor. For example, this is the case when we consider a functor between categories with (co)equalizers that reflects (co)equalizers, or when there exists a suitable type of generator within the source category of a functor, e.g. when the category is constant generated (Proposition 2.22), as it happens for the category Set of sets, for the category Top of topological spaces, etc.

To the idempotent e associated with a semiseparable functor one can attach a suitable quotient category $\mathcal{C}_e$ of $\mathcal{C}$, the so-called "coidentifier" [39]. Via this category we have obtained a canonical factorization for any semiseparable functor. Namely, any semiseparable functor $F : \mathcal{C} \to \mathcal{D}$ factors as the naturally full quotient functor $H : \mathcal{C} \to \mathcal{C}_e$ followed by a unique separable functor $F_e : \mathcal{C}_e \to \mathcal{D}$ (Theorem 2.33). As a consequence, a functor is semiseparable if, and only if, it factors as a naturally full functor followed by a separable one (Corollary 2.35).

Next, we look at functors which have an adjoint. We prove a "Rafael-type Theorem" (Theorem 2.36) for semiseparable functors: explicilty, a functor which has a right (resp., left) adjoint is semiseparable if, and only if, the (co)unit of the adjunction is regular as a natural transformation. It is well-known that in an adjoint triple $F \dashv G \dashv H$, the functor F is fully faithful if, and only if, so is H. As a consequence of the Rafael-type Theorem, in Proposition 2.41 we show that a similar behavior holds for semiseparability, as well. We recall that a functor is *Frobenius* if there is a functor which is both its left and right adjoint. Then, the Rafael-type Theorem can be further specified for semiseparable Frobenius functors as in Theorem 2.42. As already announced, semiseparability applies in the context of Eilenberg-Moore categories. In fact, in Lemma 2.43 we show that the semiseparability of the right (left) adjoint is sufficient to obtain the (co)separability of the associated (co)monad, and our main result here is Theorem 2.47 stating that, given an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$, the right adjoint G is semiseparable if, and only if, the monad GF is separable and the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ is naturally full, where $\mathcal{C}_{GF}$ is the Eilenberg-Moore category of modules over GF. We recover stronger characterizations for separable, naturally full and fully faithful functors.

Functors admitting a fully faithful right (resp., left) adjoint are known as *(co)reflections*, cf. [16]. In particular, functors admitting a fully faithful left and right adjoint equal and satisfying a coherence condition relating the unit and counit of the two adjunctions, are called *bireflections*, cf. [39]. We show that the semiseparability condition on a (co)reflection enforces the functor to be a bireflection (Proposition 2.63) and in Corollary 2.64 we actually prove that a (co)reflection is semiseparable if, and only if, it is naturally full if, and only if, it is Frobenius if, and only if, it is a bireflection. However, there exist (co)reflections which are not semiseparable (see e.g. Example 2.74). Moreover, we observe that the monad associated to a semiseparable reflection is a Frobenius monad.

In Proposition 2.69 we find that, given a category $\mathcal{C}$ and an idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$, the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$ is a bireflection if, and only if, e splits (e.g. if $\mathcal{C}$ is idempotent complete). As a consequence, in Corollary 2.71 we show that a factorization of a semiseparable functor as a bireflection followed by a separable functor is available if, and only if, the associated idempotent natural transformation e splits, and that such a factorization amounts to the canonical one given by the coidentifier category.

It is then natural to test the notion of semiseparability on functors traditionally connected with the study of separability. First, we look at the adjunction which motivated the notion of separable functor, i.e., the restriction of scalars functor $\varphi_* : {}_S\mathcal{M} \to {}_R\mathcal{M}$, the extension of scalars functor $\varphi^* = S \otimes_R (-) : {}_R\mathcal{M} \to {}_S\mathcal{M}$ and the coinduction functor $\varphi^! = {}_R\mathrm{Hom}(S, -) : {}_R\mathcal{M} \to {}_S\mathcal{M}$ associated to a ring morphism $\varphi : R \to S$, which form an adjoint triple $\varphi^* \dashv \varphi_* \dashv \varphi^!$ between module categories. The semiseparability of φ_* reduces into its separability as it is faithful, while the semiseparability of φ^* is equivalent to the one of $\varphi^!$, and it can be characterized either in terms of the regularity of φ as a morphism of R-bimodules (Proposition 3.1), or in terms of the existence of a suitable central idempotent element in R (Proposition 3.8). Since φ^* preserves free modules, it induces what we call the *free induction functor* $S \otimes_R (-) : {}_R\mathcal{M}_f \to {}_S\mathcal{M}_f$ between the categories of free left modules (which are not idempotent complete) and, assuming that $S \neq 0$ is free as a left R-module, it results to be semiseparable if, and only if, it is separable (Proposition 3.11).

As a new study of this thesis (not considered in [4], [5]), inspired by [18] we look at categories of (left) firm modules ${}_R\overline{\mathcal{M}}$ over a possibly non-unital ring R. The notion of firm module goes back to D. Quillen [77] and it allows to develop a module theory over non-unital rings. In Proposition 3.17 we show an extension of Proposition 3.1 to the case of functors between these categories, where R is a firm ring and S is an arbitrary ring. Moreover, we show that a result similar to [78, Proposition 2.2] holds with respect to the (semi)separability of a right adjoint functor, whose source category is a full subcategory of ${}_R\overline{\mathcal{M}}$ containing the firm ring R, that can be described by the regularity of the R-component of the counit as a morphism of R-bimodules (Proposition 3.14).

Then, we study the semiseparability of the corestriction of coscalars functor $\psi_* : \mathcal{M}^C \to \mathcal{M}^D$, and the coinduction functor $\psi^* := (-)\square_D C : \mathcal{M}^D \to \mathcal{M}^C$, attached to a coalgebra map $\psi : C \to D$ (Proposition 3.21), and in Proposition 3.29, as another new example discussed in this thesis, we characterize the semiseparability of the coinduction functor attached to a coring morphism, generalizing the one for coalgebras. Turning our attention to the induction functor $(-) \otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}^{\mathcal{C}}$, attached to an R-coring $\mathcal{C}$, in Theorem 3.24 we show that the latter is semiseparable (this is equivalent to say that $\mathcal{C}$ is a *semicosplit* coring) if, and only if, the coring counit $\varepsilon_{\mathcal{C}} : \mathcal{C} \to R$ is a regular morphism of R-bimodules. Next, given an (R, S)-bimodule M, we consider the coinduction functor $\sigma_* := \mathrm{Hom}_S(M, -) : \mathcal{M}_S \to \mathcal{M}_R$ together with its left adjoint $\sigma^* := (-) \otimes_R M : \mathcal{M}_R \to \mathcal{M}_S$. We give a semiseparable version of *M-separability over R* for the ring S [83]. In Theorem 3.31 we show that the semiseparability of σ_* is equivalent both to the *M-semiseparability of S over R*, and to a regularity of the evaluation map plus a mild condition, which is redundant when M is projective as a right S-module. In Corollary 3.34 we prove that S is M-separable over R if, and only if, S is M-semiseparable over R and M is a generator in $\mathcal{M}_S$. In Example 3.37 we see an instance where S is M-semiseparable but not M-separable over R. If we add the assumption that M is finitely generated and projective as a right S-module, further characterizations of the semiseparability of σ_* and σ^* can be provided in Proposition 3.39 and Proposition 3.40, respectively. In Proposition 3.6, Corollary 3.27, and Proposition 3.38, we explicitly exhibit the factorization of functors φ^*, $(-) \otimes_R \mathcal{C}$, and σ_* as a bireflection followed by a separable functor since their source categories are idempotent complete.

Before discussing semiseparability with respect to what we have called *conditions up to retracts*, let us review some results on semifunctors.

The role of semifunctors

The notion of *semifunctor* between categories, defined as a functor that does not need to preserve identities, appeared in [38] under the name of *weak functor*, and was investigated by S. Hayashi in [46], in order to develop a categorical semantics for non-extensional typed lambda calculus. Then, proper notions of *semiadjunction* and *seminatural transformation* [52] have been considered. Explicitly, a natural transformation $\alpha : F \to F'$ between semifunctors $F, F' : \mathcal{C} \to \mathcal{D}$ is defined as a natural transformation between functors; if in addition $\alpha_X \circ F\mathrm{Id}_X = \alpha_X$ holds true for every object X in $\mathcal{C}$, then α is a seminatural transformation. Moreover, if there exists a natural transformation $\beta : F' \to F$ such that $\beta \circ F'\mathrm{Id} = \beta$, $\alpha \circ \beta = F'\mathrm{Id}$ and $\beta \circ \alpha = F\mathrm{Id}$, then α is said to be a *natural semi-isomorphism*.

Given a semifunctor $F : \mathcal{C} \to \mathcal{D}$ and its associated natural transformation $\mathcal{F}$ given as in (0.1), then (as in the functor case) F can be defined to be a *faithful* (resp., *full*, *fully faithful*) semifunctor by requiring that $\mathcal{F}_{X,Y}$ is injective (resp., surjective, bijective) for every pair of objects $X, Y \in \mathcal{C}$. Noting that a full semifunctor is actually a functor, and motivated by the behavior of an endosemifunctor on Set (Example 4.57), in [21] we have introduced a weaker notion of fullness for semifunctors that we have called *semifullness*: a semifunctor $F : \mathcal{C} \to \mathcal{D}$ is *semifull* if for any morphism $f : FX \to FY$ in $\mathcal{D}$ there exists a morphism $g : X \to Y$ in $\mathcal{C}$ such that

$$F(g) = F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X.$$

We have then defined a semifunctor to be *semifully faithful* if it is faithful and semifull. In order to show that the semifull and semifully faithful conditions can be derived from requirements on the natural transformation $\mathcal{F}$ associated with a semifunctor, we look at particular "semisplitting" properties for seminatural transformations, i.e. at *natural semisplit-mono* and *natural semisplit-epi* seminatural transformations, investigating the corresponding semisplitting properties for morphisms whose source or target is the image of a semifunctor, cf. Section 4.2. In particular, in Corollary 4.67, we obtain that, given a semiadjunction $F \dashv_{\mathrm{s}} G : \mathcal{D} \to \mathcal{C}$ with unit η and counit ϵ, then F (resp., G) is semifully faithful if, and only if, η (resp., ϵ) is a natural semi-isomorphism.

It is clear that the natural fullness condition (which implies fullness) on a semifunctor reduces to the naturally full functor case. Thus, we have investigated a notion of natural semifullness for semifunctors. We have called a semifunctor $F : \mathcal{C} \to \mathcal{D}$ *naturally semifull* if there is a natural transformation $\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ such that

$$(\mathcal{F}_{X,Y} \circ \mathcal{P}_{X,Y})(f) = F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X,$$

for every morphism $f : FX \to FY$ in $\mathcal{D}$. A naturally semifull semifunctor is obviously semifull. In Proposition 4.79 we show that a semifunctor is semifully faithful if, and only if, it is separable and naturally semifull. We obtain a Rafael-type Theorem for naturally semifull semifunctors which are part of a semiadjunction in terms of semisplitting properties for the unit and the counit (Theorem 4.74). Explicitly, given a semiadjunction $F \dashv_{\mathrm{s}} G : \mathcal{D} \to \mathcal{C}$ with unit η and counit ϵ, we prove that F is naturally semifull if, and only if, η is a natural semisplit-epi, and that G is naturally semifull if, and only if, ϵ is a natural semisplit-mono.

For what concerns (semi)separability, we have defined a semifunctor $F : \mathcal{C} \to \mathcal{D}$ to be *(semi)separable* by requiring the same conditions on the associated natural transformation $\mathcal{F}$, as in the functorial case. The first difference with separable functors is in the Maschke Theorem. In Theorem 4.78 we show that, if F is a separable semifunctor and $Ff : FC \to FC'$ is an F_C-semisplit-mono (resp., $F_{C'}$-semisplit-epi), then f is a split-mono (resp., split-epi). Rafael-type Theorems for (semi)separable semifunctors that are part of a semiadjunction hold analogously to the ones for functors, see Theorem 4.80 and Theorem 4.91. As shown in [52, Theorem 6.9], any semiadjunction gives rise to a semicomonad and to a semimonad. We extend the definition of (co)separable (co)monad to the case of a semi(co)monad, obtaining a result similar to Lemma 2.43 for semiseparable semifunctors, that is, given a semiadjunction $F \dashv_{\mathrm{s}} G$, the semiseparability of G (resp., F) implies the (co)separability of the associated semi(co)monad (see Lemma 4.98).

Searching for examples, the first semifunctor we consider is the so-called *forgetful semifunctor* (Example 4.7.1). Given a category $\mathcal{C}$ and its idempotent completion $\mathcal{C}^\natural$, the forgetful semifunctor $\upsilon_{\mathcal{C}} : \mathcal{C}^\natural \to \mathcal{C}$ results to be semifully faithful. There exist semifunctors which are neither faithful, nor semifull in general, e.g. the *semi-product semifunctor* (Example 4.7.2) and the *constant semifunctor* (Example 4.7.3). To any idempotent seminatural transformation $e = (e_X)_{X\in\mathcal{C}} : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ on a category $\mathcal{C}$ it is possible to attach a canonical semifunctor E^e that is self-semiadjoint (Proposition 4.34). In Example 4.90 we show that it reveals to be naturally semifull, while it is separable if, and only if, $E^e = \mathrm{Id}_{\mathcal{C}}$. Given a morphism of rings $\varphi : R \to S$, we consider the extension and the restriction of scalars functors. If $e : \mathrm{Id}_{{}_R\mathcal{M}} \to \mathrm{Id}_{{}_R\mathcal{M}}$ is an idempotent seminatural transformation, then we obtain the semiadjunction $\varphi^*_e \dashv_{\mathrm{s}} \varphi^e_* : {}_S\mathcal{M} \to {}_R\mathcal{M}$, where $\varphi^*_e := \varphi^* \circ E^e$ and $\varphi^e_* := E^e \circ \varphi_*$. In Proposition 4.100 we give conditions under which φ^e_* and φ^*_e are naturally semifull.

Since a monoid can be seen as a category with a single object and arrows the elements of the monoid, any semigroup homomorphism between monoids defines a semifunctor. In Example 4.7.5 we exhibit a semifunctor between monoids that is separable, but not semifull in general, hence not even naturally semifull. Similarly, in Example 4.7.6, we see an example of a semifunctor between unital rings (viewed as categories with a single object) which is naturally semifull but not separable in general.

The notions considered so far for a semifunctor $F : \mathcal{C} \to \mathcal{D}$ are related to the corresponding functorial notions for its completion $F^\natural : \mathcal{C}^\natural \to \mathcal{D}^\natural$. The idempotent completion construction (see Subsection 4.1.1) provides a canonical way to turn semifunctors into functors. In Proposition 4.92 and Corollary 4.93 through the idempotent completion we show that a semifunctor F is semifull (resp., naturally semifull, semifully faithful, faithful, (semi)separable) if, and only if, its completion $F^\natural$ is a full (resp., naturally full, fully faithful, faithful, (semi)separable) functor.

Conditions up to retracts

Coming back to functors, though several properties of a functor transfer to its idempotent completion and viceversa, there are other properties that do not share this behavior.

For instance, if $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories so is $F^\natural : \mathcal{C}^\natural \to \mathcal{D}^\natural$, but the converse is not always true. It is known that $F^\natural$ is an equivalence if, and only if, F is fully faithful and surjective up to retracts, i.e. every $D \in \mathcal{D}$ is a retract of FC, for some $C \in \mathcal{C}$. A functor F such that $F^\natural$ is an equivalence is sometimes called *equivalence up to retracts* in the literature. A similar situation happens for a (co)reflection. If F is a (co)reflection so is $F^\natural$, but the converse is not true in general. In [5] we have defined a *(co)reflection up to retracts* to be a functor F whose completion $F^\natural$ is a (co)reflection. Since bireflections are particular (co)reflections, we have also introduced the stronger notion of *bireflection up to retracts*, which identifies a functor whose idempotent completion is a bireflection. These notions are freely referred to as *conditions up to retracts*. Clearly, any coreflection (resp., reflection, bireflection, equivalence) is a coreflection up to retracts (resp., reflection up to retracts, bireflection up to retracts, equivalence up to retracts), see Lemma 5.3. The converse holds in some cases, for instance if the source category of the functor is idempotent complete (see Proposition 5.9). In Proposition 5.10 we revise from [5] a characterization of (co)reflections up to retracts as part of a semiadjunction by means of semifull faithfulness. Explicitly, given a semiadjunction $F \dashv_{\mathrm{s}} G$ with unit η and counit ϵ, if F (resp., G) is a functor, then it is a (co)reflection up to retracts if, and only if, G (resp., F) is semifully faithful. By this characterization we show that, given a category $\mathcal{C}$ and an idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$, the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$ onto the coidentifier category is a (co)reflection up to retracts, whence a bireflection up to retracts (Theorem 5.14). In Theorem 5.18 we find out that the (co)comparison functor attached to an adjunction whose associated (co)monad is (co)separable is a coreflection up to retracts (resp., reflection up to retracts). As a consequence, in Theorem 5.19 we prove that, given a functor $G : \mathcal{D} \to \mathcal{C}$ with a left adjoint F, then G is semiseparable if, and only if, the associated monad GF is separable and the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ is a bireflection up to retracts. A dual statement still holds for a left adjoint functor. In case G is separable, we retrieve [31, Proposition 3.5] where K_{GF} is shown to be an equivalence up to retracts. Moreover, in Proposition 5.31 we prove that, in case G is a semiseparable functor with associated idempotent e, then there is an equivalence up to retracts $(K_{GF})_e : \mathcal{D}_e \to \mathcal{C}_{GF}$ such that $(K_{GF})_e \circ H = K_{GF}$ and $U_{GF} \circ (K_{GF})_e = G_e$, where $U_{GF} : \mathcal{C}_{GF} \to \mathcal{C}$ is the forgetful functor and H is the quotient functor. As a consequence of this, in Proposition 5.36 we show that, when G is semiseparable, the idempotent completions of the Kleisli category associated to the monad GF, of the coidentifier $\mathcal{D}_e$ and of the Eilenberg-Moore category $\mathcal{C}_{GF}$ are equivalent categories. We apply these results in the context of pre-triangulated categories, obtaining in Theorem 5.49 an analogue of P. Balmer's [12, Theorem 4.1]. More explicitly, after defining a *stably semiseparable* functor by adapting [12, Definition 3.7], we see how, given a stably semiseparable right adjoint $G : \mathcal{D} \to \mathcal{C}$ with associated idempotent natural transformation e, under the relevant assumptions, one can transfer the pre-triangulation from $\mathcal{C}$ to the coidentifier category $\mathcal{D}_e$. Then, in Corollary 5.50 we provide conditions for the Eilenberg-Moore category $\mathcal{C}_{GF}$ of modules to inherit the pre-triangulation from the base category $\mathcal{C}$. As new outcomes of this thesis, we show that similar results hold for the Eilenberg-Moore category $\mathcal{D}^{FG}$ of comodules (Corollary 5.53) and for the Kleisli category GF-$\mathrm{Free}_{\mathcal{C}}$ (Corollary 5.55).

Variants of semiseparability

In [3, Definition 2.4] *relative* notions of faithfulness, (natural) fullness, separability have been investigated. In the separable case, this notion is somehow related to the separability of the second kind [26], see [3, Proposition 2.15]. In this thesis, we propose a relative notion for semiseparability as well, obtaining in Proposition 6.3 a “relative” version of Proposition 2.5. Explicitly, given functors $U : \mathcal{D} \to \mathcal{B}$, $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{E} \to \mathcal{D}$, we say that U is (F, G)-*semiseparable* if $\mathcal{F}^U_{F,G}$, given on components by $\left(\mathcal{F}^U_{F,G}\right)_{X,Y} = \mathcal{F}^U_{FX,GY}$, for every $X \in \mathcal{C}$, $Y \in \mathcal{E}$, is a regular natural transformation. When both F and G are the identity functors, one recovers the classical definition of semiseparable functor. We observe that a functor $U : \mathcal{D} \to \mathcal{B}$ is $(\mathrm{Id}_{\mathcal{D}}, G)$-semiseparable (resp., $(F, \mathrm{Id}_{\mathcal{D}})$-semiseparable) if, and only if, U is (F, G)-semiseparable for every $F : \mathcal{C} \to \mathcal{D}$ (resp., for every $G : \mathcal{E} \to \mathcal{D}$), see Corollary 6.6. Moreover, if U is semiseparable, then U is (F, G)-semiseparable for all functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{E} \to \mathcal{D}$ (Corollary 6.7). Then, some of the results for semiseparable functors can be shown in this context. For instance, in Proposition 6.9 we attach suitable idempotent natural transformations to $(\mathrm{Id}_{\mathcal{D}}, G)$-semiseparable (resp., $(F, \mathrm{Id}_{\mathcal{D}})$-semiseparable) functors, which provide a criterion to establish when they are $(\mathrm{Id}_{\mathcal{D}}, G)$-separable (resp., $(F, \mathrm{Id}_{\mathcal{D}})$-separable). In Theorem 6.14, we prove a Rafael-type Theorem which characterizes the $(\mathrm{Id}_{\mathcal{D}}, G)$-semiseparability (resp., $(F, \mathrm{Id}_{\mathcal{D}'})$-semiseparability) of a left (resp., right) adjoint functor.

On the other hand, a stronger notion of separability has been recently introduced in [11] under the name of *heavily separable functor*. We recall that an *heavily separable* functor is a separable functor through a natural transformation $\mathcal{P}$ which is multiplicative. In Definition 6.17 we call a functor $F : \mathcal{C} \to \mathcal{D}$ *heavily semiseparable* if it is semiseparable through a natural transformation $\mathcal{P}$ which is multiplicative. We prove the main properties concerning composition (Lemma 6.20) and closure under natural isomorphims (Lemma 6.21) for heavily semiseparable functors. We obtain a stronger Rafael-type Theorem (Theorem 6.23): explicilty, given an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$ with unit η and counit ϵ, then F is h-semiseparable if, and only if, η is regular through a natural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\nu\nu = \nu \circ G\epsilon F$, while G is h-semiseparable if, and only if, ϵ is regular through a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\gamma\gamma = F\eta G \circ \gamma$.

It is known that, given an R-coring $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$, an element $g \in \mathcal{C}$ is said to be *semi-grouplike* provided $\Delta_{\mathcal{C}}(g) = g \otimes_R g$, and g is *grouplike* if g is semi-grouplike and $\varepsilon_{\mathcal{C}}(g) = 1_R$. In [65] the notion of a grouplike element in an R-coring $\mathcal{C}$ has been extended to the notion of *grouplike morphism* for a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to \perp$, where $\perp$ is a comonad on a category $\mathcal{D}$. Dually, see e.g. [59, Section 4], one can define the notion of an *augmentation* of a monad $\top : \mathcal{D} \to \mathcal{D}$. Interestingly, in [11, Corollary 2.7] a characterization of h-separability of a right (resp., left) adjoint functor has been given in terms of the existence of a grouplike morphism (resp., an augmentation) of the associated comonad (resp., monad). After defining what we have called a *semi-grouplike morphism* and a *semi-augmentation*, as a consequence of the Rafael-type Theorem, we show a characterization for the heavy semiseparability of a right (resp., left) adjoint functor involving these notions, see Proposition 6.27. As an application of the previous result, in Theorem 6.29 we prove that, given an R-coring $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$, the h-semiseparability of the induction functor $G = (-) \otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}^{\mathcal{C}}$ can be completely described in terms of the existence of an invariant semi-grouplike element $z \in \mathcal{C}$ such that $\varepsilon_{\mathcal{C}}(z)c = c$, for every $c \in \mathcal{C}$. Looking at the extension of scalars functor φ^* attached to a morphism of rings $\varphi : R \to S$, in Proposition 6.31 we show that it is h-semiseparable if, and only if, there is a morphism $E : S \to R$ of R-bimodules which is multiplicative and such that $\varphi \circ E \circ \varphi = \varphi$. We exhibit a couple of examples considered with respect to semiseparability that fulfill the requirements of heavy semiseparability (Example 6.30, Example 6.33).

Outline of the thesis. In Chapter 1 we recall the main properties of separable and naturally full functors, their characterizations for functors that are part of an adjunction, their behavior with respect to Frobenius functors, and connections with (co)separable (co)monads. We remind the main tools we will use in the sequel, such as Eilenberg-Moore and Kleisli categories. We look at adjoint functors attached to ring and coalgebra morphisms, corings and bimodules, recalling known results concerning their separability and natural fullness.

In Chapter 2 we present the notion of semiseparable functor and its main properties. We show results concerning the semiseparability for functors which are part of an adjunction. We characterize the semiseparability of a right (resp., left) adjoint functor in terms of properties of the (co)comparison functor and of the forgetful functor attached to the Eilenberg-Moore category of (co)modules. We look at semiseparable (co)reflections, and we provide a factorization of a semiseparable functor as a bireflection followed by a separable functor when its associated idempotent natural transformation splits.

In Chapter 3 we describe semiseparability for functors traditionally connected with the study of separability.

In Chapter 4 we report the results concerning the semifunctorial notions corresponding to functorial fullness, full faithfulness, (semi)separability, natural fullness. We characterize these properties for semifunctors that are part of a semiadjunction in terms of semisplitting conditions for the unit and counit attached to the semiadjunction.

In Chapter 5 we discuss semiseparability with respect to functors whose idempotent completion admits a fully faithful right (left) adjoint, that is, (co)reflections up to retracts and bireflections up to retracts. We present an application of the results achieved in the context of pre-triangulated categories.

The results contained in Chapter 6 have not appeared in the literature: we investigate new notions connected to semiseparability, namely relative semiseparable functors and heavily semiseparable functors.

Notations and conventions. Given an object X in a category $\mathcal{C}$, the identity morphism on X will be denoted either by Id_X or X for short. For categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ just means a covariant functor. By $\mathrm{Id}_{\mathcal{C}}$ we denote the identity functor on $\mathcal{C}$. The image of an object $X \in \mathcal{C}$ through a (semi)functor $F : \mathcal{C} \to \mathcal{D}$ is written $F(X)$ or simply FX; the image of a morphism $f : X \to Y$ in $\mathcal{C}$ is written $F(f)$ or just Ff. For any functor (or semifunctor) $F : \mathcal{C} \to \mathcal{D}$, we denote $\mathrm{Id}_F : F \to F$ the natural transformation defined by $(\mathrm{Id}_F)_X := \mathrm{Id}_{FX}$.

Given a category $\mathcal{C}$, we denote by $\mathcal{C}^{\mathrm{op}}$ the opposite category of $\mathcal{C}$. An object X and a morphism $f : X \to Y$ in $\mathcal{C}$ are denoted by X^{op} and $f^{\mathrm{op}} : Y^{\mathrm{op}} \to X^{\mathrm{op}}$ respectively when regarded as an object and a morphism in $\mathcal{C}^{\mathrm{op}}$. Given a functor $F : \mathcal{C} \to \mathcal{D}$, one defines its opposite functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ by setting $F^{\mathrm{op}}X^{\mathrm{op}} = (FX)^{\mathrm{op}}$ and $F^{\mathrm{op}}f^{\mathrm{op}} = (Ff)^{\mathrm{op}}$.

By a ring we mean a unital associative ring. Given rings R, S, for an (R,S)-bimodule M we often write ${}_RM$, M_S, ${}_RM_S$ to specify the left R-module, the right S-module, the (R,S)-bimodule structure, respectively, and morphisms in the corresponding categories are denoted by ${}_R\mathrm{Hom}(-,-)$, $\mathrm{Hom}_S(-,-)$ and ${}_R\mathrm{Hom}_S(-,-)$, respectively. We denote by ${}_R\mathcal{M}$ (resp., $\mathcal{M}_R$) the category of left (resp., right) R-modules and by ${}_R\mathcal{M}_S$ the category of (R,S)-bimodules. The category of vector spaces over a field $\Bbbk$ is usually denoted by $\mathfrak{M}_{\Bbbk}$, simply by $\mathfrak{M}$ when the field is clear from the context, and the tensor product over $\Bbbk$ by the unadorned $\otimes$.

Chapter 1

Separable and naturally full functors

In this chapter we recall the notions of separable and naturally full functors. The definition of separable functor, due to C. Năstăsescu, M. Van den Bergh and F. Van Oystaeyen, was introduced in [72] in order to reinterpret categorically the theory of separable field extensions, and of separable algebras. The terminology was indeed inspired by the result that the restriction of scalars functor associated with a morphism of rings is a separable functor if, and only if, the corresponding ring extension is separable in the classical sense, cf. Subsection 1.4.1. Naturally full functors have been defined in [7] as a dual version of separable functors. We review here the main results concerning these functors that we will use in the sequel. We remind the known characterizations for the separability and natural fullness of several pairs of adjoint functors attached to homomorphisms of rings, corings and bimodules. The last part of this chapter is devoted to some background on (co)monads, Eilenberg-Moore and Kleisli categories, and their interaction with separability.

1.1 Separable functors

Given a functor $F : \mathcal{C} \to \mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$, we consider the functors

$$\mathrm{Hom}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set}, \quad \mathrm{Hom}_{\mathcal{D}}(F-,F-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set}$$

and the natural transformation

$$\mathcal{F} : \mathrm{Hom}_{\mathcal{C}}(-,-) \to \mathrm{Hom}_{\mathcal{D}}(F-,F-), \tag{1.1}$$

given by $\mathcal{F}_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X,Y) \to \mathrm{Hom}_{\mathcal{D}}(FX,FY)$, $f \mapsto Ff$, for all objects X, Y in $\mathcal{C}$.

If $\mathcal{F}_{X,Y}$ is injective, surjective, bijective for every pair of objects $X, Y \in \mathcal{C}$, then F is a *faithful*, *full*, *fully faithful* functor, respectively, see e.g. [19, Definition 1.5.1].

Definition 1.1. [72, Section 1] A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *separable* if, for all objects X, Y in $\mathcal{C}$, there are maps $\mathcal{P}_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(FX,FY) \to \mathrm{Hom}_{\mathcal{C}}(X,Y)$, satisfying the following conditions:

i) for any $f : X \to Y$ in $\mathcal{C}$, $(\mathcal{P}_{X,Y} \circ \mathcal{F}_{X,Y})(f) = \mathcal{P}_{X,Y}(Ff) = f$;

ii) given $f : X \to Y$, $f' : X' \to Y'$ in $\mathcal{C}$ and $g : FX \to FX'$, $g' : FY \to FY'$ in $\mathcal{D}$, if $g' \circ Ff = Ff' \circ g$, then the diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathcal{P}_{X,X'}(g)} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow[\mathcal{P}_{Y,Y'}(g')]{} & Y' \end{array}$$

is commutative, i.e., $\mathcal{P}_{Y,Y'}(g') \circ f = f' \circ \mathcal{P}_{X,X'}(g)$.

As observed in [78], one can restate Definition 1.1 as follows. A functor $F : \mathcal{C} \to \mathcal{D}$ is *separable* if there exists a natural transformation $\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ such that

$$\mathcal{P} \circ \mathcal{F} = \mathrm{Id}_{\mathrm{Hom}_{\mathcal{C}}(-,-)}.$$

When needed we will denote $\mathcal{F}$, $\mathcal{P}$ by $\mathcal{F}^F$, $\mathcal{P}^F$, respectively, to make explicit the functor F we are considering.

Remark 1.2. Since $\mathcal{F}^F_{X,Y} = \mathcal{F}^{F^{\mathrm{op}}}_{Y^{\mathrm{op}},X^{\mathrm{op}}}$ for every $X, Y \in \mathcal{C}$, a functor $F : \mathcal{C} \to \mathcal{D}$ is separable if, and only if, so is $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$.

Lemma 1.3. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If F is separable, then it is faithful.*

Proof. If F is a separable functor, then the map $\mathcal{F}_{X,Y}$ has a left inverse, for every $X, Y \in \mathcal{C}$, hence it is injective, so F is faithful. □

The following lemma describes how separable functors behave with respect to composition.

Lemma 1.4. [72, Lemma 1.1] *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors.*

i) If F and G are separable, then so is the composite $G \circ F : \mathcal{C} \to \mathcal{E}$.

ii) If $G \circ F : \mathcal{C} \to \mathcal{E}$ is separable, then so is F.

Proof. i). For every $X, Y \in \mathcal{C}$, define $\mathcal{P}^{GF}_{X,Y} := \mathcal{P}^F_{X,Y} \circ \mathcal{P}^G_{FX,FY}$.
ii). For every $X, Y \in \mathcal{C}$, define $\mathcal{P}^F_{X,Y} := \mathcal{P}^{GF}_{X,Y} \circ \mathcal{F}^G_{FX,FY}$. □

Separable functors can be seen as the categorical counterpart of separable field extensions and of separable algebras, that we review in the next two examples.

Example 1.5. We recall from [45] the classical notion of separable field extensions. Let $\Bbbk$ be a field and let $f(X)$ be a non-zero polynomial in the polynomial ring $\Bbbk[X]$, with indeterminate X. A non-constant polynomial $f(X) \in \Bbbk[X]$ is called *separable* if each irreducible factor of $f(X)$ has only simple roots in its splitting field. If $\mathrm{char}(\Bbbk) = 0$, then any $f(X) \in \Bbbk[X]$ is separable. An algebraic field extension $\Bbbk \subseteq \mathbb{L}$ (also denoted by $\mathbb{L}/\Bbbk$) is said to be *separable* if, for every $\alpha \in \mathbb{L}$, the minimal polynomial $p(X) \in \Bbbk[X]$ of α over $\Bbbk$ is separable. If the only elements of an algebraic field extension $\mathbb{L}/\Bbbk$ which are separable are the elements of $\Bbbk$, then $\mathbb{L}/\Bbbk$ is said to be *purely inseparable*. Denoting by $\mathfrak{M}_{\Bbbk}$ (resp., $\mathfrak{M}_{\mathbb{L}}$) the category of vector spaces over the field $\Bbbk$ (resp., $\mathbb{L}$), as observed in [26, Remark 2.2], a finite extension of fields $\Bbbk \subseteq \mathbb{L}$ is separable if, and only if, the forgetful functor $F : \mathfrak{M}_{\mathbb{L}} \to \mathfrak{M}_{\Bbbk}$ is separable. Indeed, let $\alpha \in \mathbb{L}$ be a primitive element (i.e., $\mathbb{L} = \Bbbk(\alpha)$) and consider the minimal polynomial $p(X) \in \Bbbk[X]$ of α, given by $p(X) = X^n - \sum_{i=0}^{n-1} c_i X^i$. Then, the natural transformation $\mathcal{P}$ defined, for any $M, N \in \mathfrak{M}_{\mathbb{L}}$, by

$$\mathcal{P}_{M,N} : \mathrm{Hom}_{\mathfrak{M}_{\Bbbk}}(M,N) \to \mathrm{Hom}_{\mathfrak{M}_{\mathbb{L}}}(M,N)$$

$$\mathcal{P}_{M,N}(f)(m) := p'(\alpha)^{-1} \sum_{i=0}^{n-1} \alpha^{-i-1} \left(\sum_{j=0}^{i} c_j \alpha^j \right) f(\alpha^i m),$$

for every $f \in \mathrm{Hom}_{\mathfrak{M}_{\Bbbk}}(M,N)$ and $m \in M$, where $p'(X) = nX^{n-1} - \sum_{i=1}^{n-1} i c_i X^{i-1}$, splits $\mathcal{F} : \mathrm{Hom}_{\mathfrak{M}_{\mathbb{L}}}(M,N) \to \mathrm{Hom}_{\mathfrak{M}_{\Bbbk}}(FM, FN)$, $f \mapsto Ff$.

In fact, since $\mathbb{L} = \Bbbk(\alpha)$, the map $\mathcal{P}_{M,N}(f)$ is a morphism in $\mathfrak{M}_{\mathbb{L}}$, and $\mathcal{P}$ is a natural transformation as, for every $h : M' \to M$, $k : N \to N'$ in $\mathfrak{M}_{\mathbb{L}}$ and for every $f : M \to N$ in $\mathfrak{M}_{\Bbbk}$, one has

$$\begin{aligned}
\mathcal{P}_{M',N'}(Fk \circ f \circ Fh)(m') &= p'(\alpha)^{-1} \sum_{i=0}^{n-1} \alpha^{-i-1} \left(\sum_{j=0}^{i} c_j \alpha^j \right) kfh(\alpha^i m') \\
&= k \left(p'(\alpha)^{-1} \sum_{i=0}^{n-1} \alpha^{-i-1} \left(\sum_{j=0}^{i} c_j \alpha^j \right) f(\alpha^i h(m')) \right) = (k \circ \mathcal{P}_{M,N}(f) \circ h)(m'),
\end{aligned}$$

for every $m' \in M'$. Moreover, for every $f \in \mathrm{Hom}_{\mathfrak{M}_{\mathbb{L}}}(M,N)$, we have that

$$\begin{aligned}
\mathcal{P}_{M,N}\mathcal{F}_{M,N}(f)(m) &= \mathcal{P}_{M,N}(Ff)(m) = p'(\alpha)^{-1} \sum_{i=0}^{n-1} \alpha^{-i-1} \left(\sum_{j=0}^{i} c_j \alpha^j \right) Ff(\alpha^i m) \\
&= p'(\alpha)^{-1} \sum_{i=0}^{n-1} \alpha^{-i-1} \left(\sum_{j=0}^{i} c_j \alpha^j \right) \alpha^i f(m) = \frac{\sum_{i=0}^{n-1} \alpha^{-1} \left(\sum_{j=0}^{i} c_j \alpha^j \right)}{n\alpha^{n-1} - \sum_{i=1}^{n-1} i c_i \alpha^{i-1}} f(m) \\
&= \frac{\sum_{i=0}^{n-1} \sum_{j=0}^{i} c_j \alpha^{j-1}}{n\alpha^{n-1} - \sum_{i=1}^{n-1} i c_i \alpha^{i-1}} f(m) = \frac{\sum_{j=0}^{n-1} \sum_{i=j}^{n-1} c_j \alpha^{j-1}}{n\alpha^{n-1} - \sum_{i=1}^{n-1} i c_i \alpha^{i-1}} f(m) \\
&= \frac{\sum_{j=0}^{n-1} (n-j) c_j \alpha^{j-1}}{n\alpha^{n-1} - \sum_{i=1}^{n-1} i c_i \alpha^{i-1}} f(m) = \frac{\alpha \sum_{j=0}^{n-1} (n-j) c_j \alpha^{j-1}}{\alpha(n\alpha^{n-1} - \sum_{i=1}^{n-1} i c_i \alpha^{i-1})} f(m) \\
&= \frac{\sum_{j=0}^{n-1} (n-j) c_j \alpha^{j}}{n\alpha^{n} - \sum_{i=1}^{n-1} i c_i \alpha^{i}} f(m) = \frac{\sum_{j=0}^{n-1} (n-j) c_j \alpha^{j}}{n\sum_{i=0}^{n-1} c_i \alpha^i - \sum_{i=1}^{n-1} i c_i \alpha^{i}} f(m) \\
&= \frac{\sum_{j=0}^{n-1} (n-j) c_j \alpha^{j}}{nc_0 + \sum_{j=1}^{n-1} (n-j) c_j \alpha^{j}} f(m) = \frac{\sum_{j=0}^{n-1} (n-j) c_j \alpha^{j}}{\sum_{j=0}^{n-1} (n-j) c_j \alpha^{j}} f(m) = f(m),
\end{aligned}$$

for every $m \in M$.

Example 1.6. Let A be an algebra over a commutative ring R. We recall from [34, Proposition 1.1, page 40] (see also [28, Theorem 3]) that A is said to be a *separable algebra* if there exists an element $e = \sum_i a_i \otimes_R b_i \in A \otimes_R A$ such that $ae = \sum_i a a_i \otimes_R b_i = \sum_i a_i \otimes_R b_i a = ea$, for all $a \in A$, satisfying the normalizing separability condition $\sum_i a_i b_i = 1_A$. The element e is usually called *separability idempotent*. Moreover, see e.g. [28, Theorem 27 and Section 8.1], A is a separable algebra if, and only if, the multiplication map $m_A : A \otimes_R A \to A$ splits in the category ${}_A\mathcal{M}_A$ of A-bimodules, i.e., denoting ${}_A\mathcal{M}$ (resp., $\mathcal{M}_A$) the category of left (resp., right) A-modules, the restriction of scalars functor ${}_A\mathcal{M} \to {}_R\mathcal{M}$ (resp., $\mathcal{M}_A \to \mathcal{M}_R$), is separable.

A crucial result in classical representation theory is the so-called "Maschke's Theorem", stating that a finite group algebra $\Bbbk G$ over a field $\Bbbk$ is semisimple as a ring if, and only if, the characteristic of $\Bbbk$ does not divide the order of G (see e.g. [48, page 26]) if, and only if, $\Bbbk G$ is a separable algebra (see e.g. [26]). Among the main properties of separable functors, it is worth to mention that any separable functor satisfies the following functorial version of Maschke's Theorem.

Proposition 1.7. (Maschke's Theorem) (See [72, Proposition 1.2], [28, Proposition 47]) *Let $F : \mathcal{C} \to \mathcal{D}$ be a separable functor, and let $f : X \to Y$ be a morphism in $\mathcal{C}$. If $F(f)$ has a left (resp., right, two-sided) inverse g in $\mathcal{D}$, then f has a left (resp., right, two-sided) inverse in $\mathcal{C}$, namely $\mathcal{P}_{Y,X}(g)$.*

For a separable functor between abelian categories, the previous result can be restated in terms of split exact sequences. First, we recall the following definitions which are well known.

Definition 1.8. (See e.g. [76, Corollary 7.4, page 48]) A short exact sequence

$$0 \longrightarrow X' \xrightarrow{f} X \xrightarrow{g} X'' \longrightarrow 0 \tag{1.2}$$

in an abelian category $\mathcal{C}$ is said to be *split exact* if the following equivalent conditions hold true:

(*i*) f is a *section* in $\mathcal{C}$, i.e., there exists a morphism $f' : X \to X'$ in $\mathcal{C}$ such that $f' \circ f = \mathrm{Id}_{X'}$;

(*ii*) g is a *retraction* in $\mathcal{C}$, i.e., there exists a morphism $g' : X'' \to X$ in $\mathcal{C}$ such that $g \circ g' = \mathrm{Id}_{X''}$;

(*iii*) X' is a direct summand of X.

In this case, X is canonically isomorphic to the direct sum of X' and X''.

Definition 1.9. (See [26, Section 3]) A functor $F : \mathcal{C} \to \mathcal{D}$ between abelian categories is called *semisimple* if, given a short exact sequence (1.2) in $\mathcal{C}$, if the sequence

$$0 \longrightarrow FX' \xrightarrow{Ff} FX \xrightarrow{Fg} FX'' \longrightarrow 0$$

is split exact in $\mathcal{D}$, then the sequence (1.2) is also split exact.

Example 1.10. An abelian category $\mathcal{C}$ is called *semisimple* [49, Definition 5.1] if every short exact sequence in $\mathcal{C}$ is split exact. Thus, a functor between semisimple categories is semisimple. For instance, the categories ${}_R\mathcal{M}$ (resp., $\mathcal{M}_R$) of left (resp., right) modules over a semisimple[1] ring R is semisimple. In particular, the category $\mathfrak{M}$ of vector spaces over a field $\Bbbk$ is semisimple.

For a separable functor between abelian categories, Proposition 1.7 can then be reformulated as follows.

Corollary 1.11. [28, Corollary 5] *Let $F : \mathcal{C} \to \mathcal{D}$ be a separable functor between abelian categories. Then, F is semisimple.*

[1]Recall from [1, Proposition 13.9] that a ring R is *semisimple* if, and only if, every short exact sequence of left (or right) R-modules is split exact.

We recall some terminologies. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. An object M in $\mathcal{C}$ is called *relative injective* if, for every morphism $i : C \to C'$ in $\mathcal{C}$ such that Fi is split-mono, then the map $\mathrm{Hom}_{\mathcal{C}}(i, M) : \mathrm{Hom}_{\mathcal{C}}(C', M) \to \mathrm{Hom}_{\mathcal{C}}(C, M), f \mapsto f \circ i$, is surjective. A *relative projective* object in $\mathcal{C}$ is defined dually. A functor $F : \mathcal{C} \to \mathcal{D}$ is called a *Maschke functor* if it reflects split-monomorphisms, i.e., for every morphism i in $\mathcal{C}$ such that Fi is split-mono, then i is split-mono, and F is called a *dual Maschke functor* if it reflects split-epimorphisms. These notions are equivalent to [26, Definition 3.1], where F is called a Maschke functor if every object in $\mathcal{C}$ is relative injective, while F is called dual Maschke if every object in $\mathcal{C}$ is relative projective. As shown in [26, Proposition 3.7], a semisimple functor between abelian categories is Maschke (resp., dual Maschke) if it reflects monomorphisms (resp., epimorphisms). A functor is said to be *conservative* if it reflects isomorphisms.

Remark 1.12. A functor which is both Maschke and dual Maschke is conservative. By Proposition 1.7 a separable functor is both Maschke and dual Maschke, thus conservative.

On the other hand, a Maschke (resp., dual Maschke) functor is not necessarily separable, as the next example shows.

Example 1.13. [26, Example 3.6] Let $\Bbbk \subseteq \mathbb{L}$ be a finite purely inseparable field extension. The restriction of scalars functor $F : \mathfrak{M}_{\mathbb{L}} \to \mathfrak{M}_{\Bbbk}$ is a Maschke and a dual Maschke functor, since every vector space over $\mathbb{L}$ is an injective and projective object in $\mathfrak{M}_{\mathbb{L}}$, but F is not separable, since $\mathbb{L}/\Bbbk$ is not separable.

However, in some cases the converse holds true, as for instance in the following Maschke's Theorem for Hopf algebras, a restatement of the Sweedler's results [85].

Proposition 1.14. [26, Proposition 4.7] *Let H be a Hopf algebra over a commutative ring $\Bbbk$, and let $G : \mathcal{M}_H \to \mathcal{M}_{\Bbbk}$ be the restriction of scalars functor. Then, the following assertions are equivalent:*

- (i) *G is a dual Maschke functor;*
- (ii) *G is a Maschke functor;*
- (iii) *G is a semisimple functor;*
- (iv) *there exists a right integral[2] $t \in H$ with $\varepsilon(t) = 1$;*
- (v) *G is a separable functor.*

1.1.1 Separability and adjunctions

Given functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$, the pair (F, G) is an *adjunction* (usually denoted by $F \dashv G$) if, for every $C \in \mathcal{C}$ and $D \in \mathcal{D}$, there is a bijection

$$\Phi_{C,D} : \mathrm{Hom}_{\mathcal{D}}(FC, D) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(C, GD), \tag{1.3}$$

which is natural both in C and D, i.e. for every morphism $f : C' \to C$ in $\mathcal{C}$, $g : D \to D'$ in $\mathcal{D}$, the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(FC, D) & \xrightarrow{\Phi_{C,D}} & \mathrm{Hom}_{\mathcal{C}}(C, GD) \\ \downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{D}}(Ff,g)} & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathcal{C}}(f,Gg)} \\ \mathrm{Hom}_{\mathcal{D}}(FC', D') & \xrightarrow[\Phi_{C',D'}]{} & \mathrm{Hom}_{\mathcal{C}}(C', GD') \end{array}$$

[2] Given a Hopf algebra $(H, m, u, \Delta, \varepsilon, S)$ over a commutative ring $\Bbbk$, $t \in H$ is a right (resp., left) integral in H if $th = \varepsilon(h)t$ (resp., $ht = \varepsilon(h)t$), for all $h \in H$.

commutes. On elements, given $h : FC \to D$, one has $\Phi_{C',D'}(g \circ h \circ Ff) = Gg \circ \Phi_{C,D}(h) \circ f$. In particular, if $F \dashv G : \mathcal{D} \to \mathcal{C}$ is an adjunction, then F is a *left adjoint* of G and G is a *right adjoint* of F. Equivalently, cf. [19, Theorem 3.1.5], $F \dashv G : \mathcal{D} \to \mathcal{C}$ if, and only if, there are natural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ (called the *unit* and *counit* of the adjunction, respectively) that fulfill the *triangle identities*

$$G\epsilon \circ \eta G = \mathrm{Id}_G \quad \text{and} \quad \epsilon F \circ F\eta = \mathrm{Id}_F. \tag{1.4}$$

Explicitly, the natural isomorphism $\Phi_{C,D}$ in (1.3) is given, for any $f : FC \to D$ by

$$\Phi_{C,D} : \mathrm{Hom}_{\mathcal{D}}(FC, D) \to \mathrm{Hom}_{\mathcal{C}}(C, GD), \quad \Phi_{C,D}(f) = Gf \circ \eta_C, \tag{1.5}$$

with inverse

$$\Psi_{C,D} := \Phi^{-1}_{C,D} : \mathrm{Hom}_{\mathcal{C}}(C, GD) \to \mathrm{Hom}_{\mathcal{D}}(FC, D), \quad \Psi_{C,D}(g) = \epsilon_D \circ Fg, \tag{1.6}$$

for any $g : C \to GD$ in $\mathcal{C}$. It is well-known that the left or right adjoint of a functor is unique up to natural isomorphism. We recall that, given two adjunctions $(F \dashv G : \mathcal{D} \to \mathcal{C}, \eta, \epsilon)$ and $(F' \dashv G' : \mathcal{D}' \to \mathcal{C}', \eta', \epsilon')$, a *map of adjunctions* [61, IV.7] is defined to be a pair (S, T) of functors $S : \mathcal{D}' \to \mathcal{D}$, $T : \mathcal{C}' \to \mathcal{C}$, such that both squares

$$\begin{array}{ccc} \mathcal{D}' & \xrightarrow{\;S\;} & \mathcal{D} \\ F' \big\uparrow \dashv \big\downarrow G' & & F \big\uparrow \dashv \big\downarrow G \\ \mathcal{C}' & \xrightarrow{\;T\;} & \mathcal{C} \end{array}$$

are commutative, i.e., $F \circ T = S \circ F'$ and $T \circ G' = G \circ S$, and such that the diagram of hom-sets

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{D}'}(F'C', D') & \xrightarrow{\mathcal{F}^S_{F'C',D'}} & \mathrm{Hom}_{\mathcal{D}}(SF'C', SD') & = & \mathrm{Hom}_{\mathcal{D}}(FTC', SD') \\ \big\downarrow \Phi_{C',D'} & & & & \big\downarrow \Phi_{TC',SD'} \\ \mathrm{Hom}_{\mathcal{C}'}(C', G'D') & \xrightarrow{\mathcal{F}^T_{C',G'D'}} & \mathrm{Hom}_{\mathcal{C}}(TC', TG'D') & = & \mathrm{Hom}_{\mathcal{C}}(TC', GSD') \end{array} \tag{1.7}$$

is commutative.

Remark 1.15. By [61, Proposition 1, page 99] the commutativity of (1.7) is equivalent to $T\eta' = \eta T$ and also to $\epsilon S = S\epsilon'$.

The following properties will be useful in the sequel.

Proposition 1.16. (See e.g. [28, Proposition 10]) *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ. Then, we have the following isomorphisms:*

$$\mathrm{Nat}(\mathrm{Id}_{\mathcal{C}}, GF) \cong \mathrm{Nat}(G, G) \cong \mathrm{Nat}(F, F) \cong \mathrm{Nat}(FG, \mathrm{Id}_{\mathcal{D}}).$$

Proof. We just exhibit the isomorphism $\mathrm{Nat}(\mathrm{Id}_{\mathcal{C}}, GF) \cong \mathrm{Nat}(G, G)$. Given a natural transformation $\theta : \mathrm{Id}_{\mathcal{C}} \to GF$, define $\alpha : G \to G$ by $\alpha_D = G\epsilon_D \circ \theta_{GD}$, for every $D \in \mathcal{D}$. On the other hand, given a natural transformation $\alpha : G \to G$, define $\theta : \mathrm{Id}_{\mathcal{C}} \to GF$ by $\theta_C = \alpha_{FC} \circ \eta_C$, for every $C \in \mathcal{C}$. □

Proposition 1.17. (See e.g. [28, Proposition 11]) *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ. Then, we have the following isomorphisms:*

$$\mathrm{Nat}(GF, \mathrm{Id}_{\mathcal{C}}) \cong \mathrm{Nat}(\mathrm{Hom}_{\mathcal{D}}(F-, F-), \mathrm{Hom}_{\mathcal{C}}(-,-)), \tag{1.8}$$

$$\mathrm{Nat}(\mathrm{Id}_{\mathcal{D}}, FG) \cong \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(G-, G-), \mathrm{Hom}_{\mathcal{D}}(-,-)). \tag{1.9}$$

Proof. Let $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ be a natural transformation. Define $\theta : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ by

$$\theta_{C,C'}(g) = \nu_{C'} \circ Gg \circ \eta_C, \tag{1.10}$$

for any $g : FC \to FC'$ in $\mathcal{D}$. From the naturality of η and ν, for any $h : X \to Y$, $l : Z \to T$ in $\mathcal{C}$, and $k : FY \to FZ$ in $\mathcal{D}$, we have $\theta_{X,T}(Fl \circ k \circ Fh) = \nu_T \circ G(Fl \circ k \circ Fh) \circ \eta_X = (\nu_T \circ GFl) \circ Gk \circ (GFh \circ \eta_X) = l \circ (\nu_Z \circ Gk \circ \eta_Y) \circ h = l \circ \theta_{Y,Z}(k) \circ h$, thus $\theta : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ is a natural transformation. Note that $\theta_{GFC,C}(\epsilon_{FC}) = \nu_C \circ G\epsilon_{FC} \circ \eta_{GFC} = \nu_C \circ \mathrm{Id}_{GFC} = \nu_C$, for every $C \in \mathcal{C}$.

Conversely, given a natural transformation $\theta : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$, define $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ by

$$\nu_C = \theta_{GFC,C}(\epsilon_{FC}) : GFC \to C, \tag{1.11}$$

for every $C \in \mathcal{C}$. The naturality of ν follows from the one of θ. By naturality of θ, for any C, C' in $\mathcal{C}$ and $g : FC \to FC'$ in $\mathcal{D}$, we have

$$\begin{aligned} \nu_{C'} \circ Gg \circ \eta_C = \theta_{GFC',C'}(\epsilon_{FC'}) \circ Gg \circ \eta_C = \theta_{C,C'}(\epsilon_{FC'} \circ FGg \circ F\eta_C) \\ = \theta_{C,C'}(g \circ \epsilon_{FC} \circ F\eta_C) = \theta_{C,C'}(g \circ \mathrm{Id}_{FC}) = \theta_{C,C'}(g), \end{aligned} \tag{1.12}$$

thus the correspondence between θ and ν is bijective. The proof of (1.9) follows from (1.8) by dual arguments. Explicitly, given a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$, define $\psi : \mathrm{Hom}_{\mathcal{C}}(G-, G-) \to \mathrm{Hom}_{\mathcal{D}}(-,-)$ by

$$\psi_{D,D'}(f) = \epsilon_{D'} \circ Ff \circ \gamma_D, \tag{1.13}$$

for every $f : GD \to GD'$ in $\mathcal{C}$. On the other hand, given a natural transformation $\psi : \mathrm{Hom}_{\mathcal{C}}(G-, G-) \to \mathrm{Hom}_{\mathcal{D}}(-,-)$, define $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ by

$$\gamma_D = \psi_{D,FGD}(\eta_{GD}) : D \to FGD, \tag{1.14}$$

for every $D \in \mathcal{D}$. □

A key result for separable functors is provided by the so-called *Rafael Theorem* [78] which characterizes the separability for functors that have an adjoint.

Theorem 1.18. [78, Theorem 1.2] (Rafael Theorem) *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ. Then,*

i) F is separable if, and only if, there exists a natural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\nu \circ \eta = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}}$;

ii) G is separable if, and only if, there exists a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\epsilon \circ \gamma = \mathrm{Id}_{\mathrm{Id}_{\mathcal{D}}}$.

We include a Rafael-type Theorem for Maschke and dual Maschke functors.

Theorem 1.19. [26, Theorem 3.4] *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ. Then,*

i) F is Maschke if, and only if, η_C is a split-mono in $\mathcal{C}$, for every $C \in \mathcal{C}$, i.e. there exists a morphism $\nu_C : GFC \to C$ in $\mathcal{C}$ such that $\nu_C \circ \eta_C = \mathrm{Id}_{GFC}$;

ii) G is dual Maschke if, and only if, ϵ_D is a split-epi in $\mathcal{D}$, for every $D \in \mathcal{D}$, i.e. there exists a morphism $\gamma_D : D \to FGD$ in $\mathcal{D}$ such that $\epsilon_D \circ \gamma_D = \mathrm{Id}_{FGD}$.

1.1.2 Separability of the second kind

We recall a generalization of separable functors introduced in [26], known as *separability of the second kind.*

Definition 1.20. [26, Definition 2.1] Let $F : \mathcal{C} \to \mathcal{D}$ and $H : \mathcal{C} \to \mathcal{E}$ be functors. Then, F is called *H-separable* if there exists a natural transformation

$$\mathcal{P}^{F,H} : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{E}}(H-, H-)$$

such that $\mathcal{P}^{F,H} \circ \mathcal{F}^F = \mathcal{F}^H$, i.e., $\mathcal{F}^H$ factors through $\mathcal{F}^F$ as a natural transformation and the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(-,-) & \xrightarrow{\mathcal{F}^F} & \mathrm{Hom}_{\mathcal{D}}(F-, F-) \\ {\scriptstyle \mathcal{F}^H}\downarrow & \swarrow{\scriptstyle \mathcal{P}^{F,H}} & \\ \mathrm{Hom}_{\mathcal{E}}(H-, H-) & & \end{array}$$

is commutative.

A $\mathrm{Id}_{\mathcal{C}}$-separable functor coincides with a separable functor. Note also that any functor $F : \mathcal{C} \to \mathcal{D}$ is F-separable. The fact that $\mathcal{P}^{F,H}$ is natural means that, for any $u : X \to Y$, $v : Z \to T$ in $\mathcal{C}$ and $g : FY \to FZ$ in $\mathcal{D}$, one has

$$\mathcal{P}^{F,H}_{X,T}(Fv \circ g \circ Fu) = Hv \circ \mathcal{P}^{F,H}_{Y,Z}(g) \circ Hu.$$

Most properties of separable functors can be extended to H-separable functors.

Proposition 1.21. [28, Proposition 51] *Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{D}'$ and $H : \mathcal{C} \to \mathcal{E}$ be functors.*

i) If $G \circ F$ is a H-separable functor, then F is H-separable.

ii) If F is H-separable and G is separable, then $G \circ F$ is H-separable.

A Maschke-type Theorem holds for H-separable functors.

Proposition 1.22. [28, Proposition 52], [26, Proposition 2.4] *Let $F : \mathcal{C} \to \mathcal{D}$, $H : \mathcal{C} \to \mathcal{E}$ be functors and assume that F is H-separable. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. If $F(f)$ has a left (resp., right, two-sided) inverse g in $\mathcal{D}$, then $H(f)$ has a left (resp., right, two-sided) inverse in $\mathcal{E}$, namely $\mathcal{P}^{F,H}_{Y,X}(g)$.*

Corollary 1.23. [28, Corollary 8] *Let $F : \mathcal{C} \to \mathcal{D}$, $H : \mathcal{C} \to \mathcal{E}$ be functors between abelian categories. Assume that F is H-separable. If a short exact sequence in $\mathcal{C}$ is split exact after applying F, then it becomes split exact also after applying H.*

Necessary and sufficient conditions can be given to describe the H-separability of functors that are part of an adjunction.

Theorem 1.24. [28, Theorem 25] (Rafael-type Theorem for H-separability) *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ. Let $H : \mathcal{C} \to \mathcal{E}$, $K : \mathcal{D} \to \mathcal{E}$ be functors. Then,*

i) F is H-separable if, and only if, there exists a natural transformation $\nu : HGF \to H$ such that $\nu \circ H\eta = \mathrm{Id}_H$;

ii) G is K-separable if, and only if, there exists a natural transformation $\gamma : K \to KFG$ such that $K\epsilon \circ \gamma = \mathrm{Id}_K$.

1.2 Naturally full functors

The condition defining a separable functor can be thought as a natural version of faithfulness. In a somehow dual way *naturally full* functors have been introduced in [7] by requiring that the natural transformation $\mathcal{F}$ associated with a functor has a right inverse. Here we recall the definition and the main properties of natural fullness.

Definition 1.25. [7, Definition 2.1] A functor $F : \mathcal{C} \to \mathcal{D}$ is called *naturally full* if there exists a natural transformation $\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ such that $\mathcal{F} \circ \mathcal{P} = \mathrm{Id}$.

Remark 1.26. Since $\mathcal{F}^{F}_{X,Y} = \mathcal{F}^{F^{\mathrm{op}}}_{Y^{\mathrm{op}},X^{\mathrm{op}}}$ for every $X, Y \in \mathcal{C}$, a functor $F : \mathcal{C} \to \mathcal{D}$ is naturally full if, and only if, so is $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$.

Remark 1.27. It is clear that:

i) A naturally full functor is full. The converse is not true in general, see e.g. [7, Example 3.2].

ii) A functor is fully faithful if, and only if, it is both separable and naturally full.

Naturally full functors are stable under composition.

Proposition 1.28. [7, Proposition 2.3] *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors.*

i) If F and G are naturally full, then the functor $G \circ F$ is naturally full.

ii) If $G \circ F$ is naturally full and G is faithful, then F is naturally full.

Proof. i). If F and G are naturally full through $\mathcal{P}^F$, $\mathcal{P}^G$, respectively, then $G \circ F$ is naturally full with respect to $\mathcal{P}^{GF} := \mathcal{P}^F \circ \mathcal{P}^G$.
ii). If $G \circ F$ is naturally full with respect to $\mathcal{P}^{GF}$, then $\mathcal{P}^{GF} \circ \mathcal{F}^G$ is a right inverse for $\mathcal{F}^F$. □

The following is a Rafael-type Theorem for naturally full functors.

Theorem 1.29. [7, Theorem 2.6] (Rafael-type Theorem for natural fullness) *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ. Then,*

i) F is naturally full if, and only if, there exists a natural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\eta \circ \nu = \mathrm{Id}_{GF}$;

ii) G is naturally full if, and only if, there exists a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\gamma \circ \epsilon = \mathrm{Id}_{FG}$.

A similar characterization can be given for faithfulness and fullness of adjoint functors.

Proposition 1.30. [7, Proposition 2.5] *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ. Then,*

i) F is faithful if, and only if, η_C is a monomorphism in $\mathcal{C}$, for every $C \in \mathcal{C}$;

ii) F is full if, and only if, η_C is a split-epi in $\mathcal{C}$, for every $C \in \mathcal{C}$, i.e. there exists a morphism $\nu_C : GFC \to C$ in $\mathcal{C}$ such that $\eta_C \circ \nu_C = \mathrm{Id}_{GFC}$;

iii) G is faithful if, and only if, ϵ_D is an epimorphism in $\mathcal{D}$, for every $D \in \mathcal{D}$;

iv) *G is full if, and only if, ϵ_D is a split-mono in $\mathcal{D}$, for every $D \in \mathcal{D}$, i.e. there exists a morphism $\gamma_D : D \to FGD$ in $\mathcal{D}$ such that $\gamma_D \circ \epsilon_D = \mathrm{Id}_{FGD}$.*

Corollary 1.31. [19, Cf. Proposition 3.4.1] *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ. Then,*

i) *F is fully faithful if, and only if, η is a natural isomorphism;*

ii) *G is fully faithful if, and only if, ϵ is a natural isomorphism.*

1.3 Adjoint triples and Frobenius functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An *adjoint triple* $F \dashv G \dashv H : \mathcal{C} \to \mathcal{D}$ of functors is a triple of functors $F, H : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \dashv G$ and $G \dashv H$. Given an adjoint triple $F \dashv G \dashv H : \mathcal{C} \to \mathcal{D}$, the separability and natural fullness properties pass from F to H and viceversa, cf. [4, Proposition 2.19].

Proposition 1.32. *Let $F \dashv G \dashv H : \mathcal{C} \to \mathcal{D}$ be an adjoint triple. Then, F is separable (resp., naturally full) if, and only if, so is H.*

Proof. We denote by η^l, ϵ^l and η^r, ϵ^r the unit and the counit of the adjunction $F \dashv G$ and of the adjunction $G \dashv H$, respectively. We observe that to a natural transformation $\nu^l : GF \to \mathrm{Id}_\mathcal{C}$ we can attach the natural transformation

$$\gamma^r := GH\nu^l \circ G\eta^r F \circ \eta^l : \mathrm{Id}_\mathcal{C} \to GH. \tag{1.15}$$

On the other hand, to a natural transformation $\gamma^r : \mathrm{Id}_\mathcal{C} \to GH$ we can attach the natural transformation

$$\nu^l := \epsilon^r \circ G\epsilon^l H \circ GF\gamma^r : GF \to \mathrm{Id}_\mathcal{C}. \tag{1.16}$$

If F is separable, by Theorem 1.18 *i*) there exists a natural transformation $\nu^l : GF \to \mathrm{Id}_\mathcal{C}$ such that $\nu^l \circ \eta^l = \mathrm{Id}$. Define $\gamma^r : \mathrm{Id}_\mathcal{C} \to GH$ as in (1.15). By naturality of ϵ^r, we have $\epsilon^r \circ \gamma^r = \epsilon^r \circ GH\nu^l \circ G\eta^r F \circ \eta^l = \nu^l \circ \epsilon^r GF \circ G\eta^r F \circ \eta^l = \nu^l \circ \eta^l = \mathrm{Id}$, so that H is separable by Theorem 1.18 *ii*). On the other hand, if H is separable, by Theorem 1.18 *ii*) there is a natural transformation $\gamma^r : \mathrm{Id}_\mathcal{C} \to GH$ such that $\epsilon^r \circ \gamma^r = \mathrm{Id}$. Define $\nu^l : GF \to \mathrm{Id}_\mathcal{C}$ as in (1.16). By naturality of η^l, we have $\nu^l \circ \eta^l = \epsilon^r \circ G\epsilon^l H \circ GF\gamma^r \circ \eta^l = \epsilon^r \circ G\epsilon^l H \circ \eta^l GH \circ \gamma^r = \epsilon^r \circ \gamma^r = \mathrm{Id}$, hence F is separable by Theorem 1.18 *i*).

If F is naturally full, by Theorem 1.29 *i*) there exists a natural transformation $\nu^l : GF \to \mathrm{Id}_\mathcal{C}$ such that $\eta^l \circ \nu^l = \mathrm{Id}_{GF}$. Define $\gamma^r : \mathrm{Id}_\mathcal{C} \to GH$ as in (1.15). Observe that, from $\eta^l G \circ \nu^l G = \mathrm{Id}_{GFG}$ and $G\epsilon^l \circ \eta^l G = \mathrm{Id}_G$, it follows that $\nu^l G = (\eta^l G)^{-1} = G\epsilon^l$. Then, by naturality of γ^r and η^r we have $\gamma^r \circ \epsilon^r = GH\epsilon^r \circ \gamma^r GH = GH\epsilon^r \circ GH\nu^l GH \circ G\eta^r FGH \circ \eta^l GH = GH\epsilon^r \circ GHG\epsilon^l H \circ G\eta^r FGH \circ \eta^l GH = GH\epsilon^r \circ G(HG\epsilon^l \circ \eta^r FG)H \circ \eta^l GH = GH\epsilon^r \circ G\eta^r H \circ G\epsilon^l H \circ \eta^l GH = \mathrm{Id}_{GH} \circ \mathrm{Id}_{GH} = \mathrm{Id}_{GH}$. Conversely, assume H is naturally full. By Theorem 1.29 *ii*) there exists a natural transformation $\gamma^r : \mathrm{Id}_\mathcal{C} \to GH$ such that $\gamma^r \circ \epsilon^r = \mathrm{Id}_{GH}$. Define $\nu^l : GF \to \mathrm{Id}_\mathcal{C}$ as in (1.16). Observe that, from $\gamma^r G \circ \epsilon^r G = \mathrm{Id}_{GHG}$ and $\epsilon^r G \circ G\eta^r = \mathrm{Id}_G$, it follows that $\gamma^r G = (\epsilon^r G)^{-1} = G\eta^r$. Then, by naturality of ν^l and ϵ^l, we have $\eta^l \circ \nu^l = \nu^l GF \circ GF\eta^l = \epsilon^r GF \circ G\epsilon^l HGF \circ GF\gamma^r GF \circ GF\eta^l = \epsilon^r GF \circ G(\epsilon^l HG \circ F\gamma^r G)F \circ GF\eta^l = \epsilon^r GF \circ G(\epsilon^l HG \circ FG\eta^r)F \circ GF\eta^l = \epsilon^r GF \circ G\eta^r F \circ G\epsilon^l F \circ GF\eta^l = \mathrm{Id}_{GF} \circ \mathrm{Id}_{GF} = \mathrm{Id}_{GF}$, hence F is naturally full. □

Remark 1.33. Alternatively, the "if" part of the statement follows also from the proof of the "only if" part, by considering the adjoint triple $H^{\mathrm{op}} \dashv G^{\mathrm{op}} \dashv F^{\mathrm{op}}$ together with Remark 1.2 and Remark 1.26.

Corollary 1.34. (See e.g. [19, Proposition 3.4.2]) *Let $F \dashv G \dashv H : \mathcal{C} \to \mathcal{D}$ be an adjoint triple. Then, F is fully faithful if, and only if, so is H.*

Proof. It follows from Remark 1.27 *ii*) and Proposition 1.32. □

Adjoint triples $F \dashv G \dashv H$ where F and H are fully faithful are usually called *fully faithful adjoint triples*. Now, we recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is called *Frobenius* [28] if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ which is both a right and left adjoint of F. Thus, a Frobenius functor $F : \mathcal{C} \to \mathcal{D}$ fits into an adjoint triple $G \dashv F \dashv G : \mathcal{D} \to \mathcal{C}$ where the right and left adjoint G are equal. In this case, the pair (F, G) is called a *Frobenius pair*. This notion is symmetric in F and G: if (F, G) is a Frobenius pair, then (G, F) is a Frobenius pair. As noted in [28], the property that defines a Frobenius functor was first introduced by K. Morita [69] under the name of *strongly adjoint pair*. The terminology "Frobenius" is inspired by the fact that, for a ring homomorphism $R \to S$, the restriction of scalars functor is Frobenius if, and only if, the extension S/R is Frobenius in the classical sense, i.e., S is finitely generated and projective as a right R-module and there is an isomorphism $\mathrm{Hom}_R(S, R) \cong S$ of (R, S)-bimodules, see [70].

If (F, G) is a Frobenius pair, then both F and G preserve limits and colimits. Moreover, the composition of Frobenius functors is a Frobenius functor, see [28, Proposition 43, Proposition 44]. From Proposition 1.16 we have the following.

Corollary 1.35. [28, Corollary 6] *Let (F, G) is a Frobenius pair of functors. Then, the isomorphisms*

$$\mathrm{Nat}(F, F) \cong \mathrm{Nat}(G, G) \cong \mathrm{Nat}(\mathrm{Id}_{\mathcal{C}}, GF) \cong \mathrm{Nat}(FG, \mathrm{Id}_{\mathcal{D}}) \cong \mathrm{Nat}(GF, \mathrm{Id}_{\mathcal{C}}) \cong \mathrm{Nat}(\mathrm{Id}_{\mathcal{D}}, FG)$$

hold true.

As a consequence of Theorem 1.18 and Theorem 1.29, necessary and sufficient conditions can be provided for the separability and natural fullness of a Frobenius functor that is part of an adjunction. Let (F, G) be a Frobenius pair. We denote by η^l, ϵ^l and by η^r, ϵ^r the unit and the counit of the adjunctions $F \dashv G$ and $G \dashv F$, respectively.

Proposition 1.36. (Cf. [28, Proposition 49]) *Let $F : \mathcal{C} \to \mathcal{D}$ be a Frobenius functor, with left and right adjoint $G : \mathcal{D} \to \mathcal{C}$. Then, the following assertions are equivalent:*

(*i*) *F is separable;*

(*ii*) *there exists a natural transformation $\alpha : G \to G$ such that $\epsilon^r \circ \alpha F \circ \eta^l = \mathrm{Id}$;*

(*iii*) *there exists a natural transformation $\beta : F \to F$ such that $\epsilon^r \circ G\beta \circ \eta^l = \mathrm{Id}$;*

Proposition 1.37. [7, Proposition 2.7] *Let $F : \mathcal{C} \to \mathcal{D}$ be a Frobenius functor, with left and right adjoint $G : \mathcal{D} \to \mathcal{C}$. Then, the following assertions are equivalent:*

(*i*) *F is naturally full;*

(*ii*) *there exists a natural transformation $\alpha : G \to G$ such that $\eta^l \circ \epsilon^r \circ \alpha F = \mathrm{Id}_{GF}$;*

(*iii*) *there exists a natural transformation $\beta : F \to F$ such that $\eta^l \circ \epsilon^r \circ G\beta = \mathrm{Id}_{GF}$;*

(*iv*) *there exists a natural transformation $\alpha' : G \to G$ such that $\alpha' F \circ \eta^l \circ \epsilon^r = \mathrm{Id}_{GF}$;*

(*v*) *there exists a natural transformation $\beta' : F \to F$ such that $G\beta' \circ \eta^l \circ \epsilon^r = \mathrm{Id}_{GF}$.*

In this case, $\eta^l \circ \epsilon^r : GF \to GF$ *is a natural isomorphism with inverse* $\alpha F = G\beta = \alpha' F = G\beta'$*.*

A Rafael-type criterion can be also given for the H-separability of Frobenius functors, cf. [26, Proposition 2.9].

Proposition 1.38. *Let* $H : \mathcal{C} \to \mathcal{E}$ *be a functor and let* $F : \mathcal{C} \to \mathcal{D}$ *be a Frobenius functor, with left and right adjoint* $G : \mathcal{D} \to \mathcal{C}$*. Then, the following assertions are equivalent:*

(i) *F is H-separable;*

(ii) *there exists a natural transformation* $\alpha : HG \to HG$ *such that* $H\epsilon^r \circ \alpha F \circ H\eta^l = \mathrm{Id}_H$*;*

(iii) *there exists a natural transformation* $\beta : HF \to HF$ *such that* $H\epsilon^r \circ \beta G \circ H\eta^r = \mathrm{Id}_H$*.*

Proof. $(i) \Leftrightarrow (ii)$. From Theorem 1.24 applied to the adjunction (F, G) we have that F is H-separable if, and only if, there exists a natural transformation $\nu : HGF \to H$ such that $\nu \circ H\eta^l = \mathrm{Id}_H$. Since $\mathrm{Nat}(HGF, H) \cong \mathrm{Nat}(HG, HG)$, given $\nu : HGF \to H$, one can define a natural transformation $\alpha : HG \to HG$ by $\alpha = \nu G \circ HG\eta^r$. On the other hand, given $\alpha : HG \to HG$ one can define $\nu : HGF \to H$ by $\nu = H\epsilon^r \circ \alpha F$. Thus, F is H-separable if, and only if, there exists a natural transformation $\alpha : HG \to HG$ such that $H\epsilon^r \circ \alpha F \circ H\eta^l = \mathrm{Id}_H$.

$(i) \Leftrightarrow (iii)$. From Theorem 1.24 applied to the adjunction (G, F) we have that F is H-separable if, and only if, there exists a natural transformation $\gamma : H \to HFG$ such that $H\epsilon^r \circ \gamma = \mathrm{Id}_H$. Since $\mathrm{Nat}(H, HFG) \cong \mathrm{Nat}(HF, HF)$, given $\gamma : H \to HFG$, one can define a natural transformation $\beta : HF \to HF$ by $\beta = HF\epsilon^r \circ \gamma F$. On the other hand, given $\beta : HF \to HF$, one can define $\gamma : H \to HFG$ by $\gamma = \beta G \circ H\eta^r$. Thus, F is H-separable if, and only if, there exists a natural transformation $\beta : HF \to HF$ such that $H\epsilon^r \circ \beta G \circ H\eta^r = \mathrm{Id}_H$. □

1.4 Examples and applications

In this section we consider examples of pairs of adjoint functors attached to ring and coalgebra morphisms, corings and bimodules, and we provide characterizations of their separability and natural fullness known in the literature. In Chapter 3 we will return on these examples with respect to semiseparability.

1.4.1 Extension and restriction of scalars

Let R, S be unital rings, and let ${}_R\mathcal{M}$ (resp., ${}_S\mathcal{M}$) be the categories of left modules over R (resp., S). A morphism of rings $\varphi : R \to S$ induces the restriction of scalars functor $\varphi_* : {}_S\mathcal{M} \to {}_R\mathcal{M}$, the extension of scalars (or induction) functor $\varphi^* := S \otimes_R (-) : {}_R\mathcal{M} \to {}_S\mathcal{M}$ and the coinduction functor $\varphi^! := {}_R\mathrm{Hom}(S, -) : {}_R\mathcal{M} \to {}_S\mathcal{M}$. All together these functors form an adjoint triple

$$\begin{array}{ccc} & \overset{\varphi^* = S\otimes_R(-)}{\curvearrowright} \ \bot & \\ {}_R\mathcal{M} & \xleftarrow{\ \varphi_*\ } & {}_S\mathcal{M}. \\ & \bot \ \underset{\varphi^! = {}_R\mathrm{Hom}(S,-)}{\curvearrowright} & \end{array} \tag{1.17}$$

The unit η and the counit ϵ of the adjunction $\varphi^* \dashv \varphi_*$, are respectively defined by

$$\eta_M = \varphi \otimes_R M : M \to S \otimes_R M,\ m \mapsto 1_S \otimes_R m, \text{ for every } M \in {}_R\mathcal{M},$$

$\epsilon_N : S \otimes_R N \to N,\ s \otimes_R n \mapsto sn$, for every $N \in {}_S\mathcal{M}$,

while the unit $\eta^!$ and the counit $\epsilon^!$ of the adjunction $\varphi_* \dashv \varphi^!$, are defined by

$\eta^!_N : N \to {}_R\mathrm{Hom}(S, N),\ n \mapsto [s \mapsto sn]$, for every $N \in {}_S\mathcal{M}$,

$\epsilon^!_M : {}_R\mathrm{Hom}(S, M) \to M,\ f \mapsto f(1_S)$, for every $M \in {}_R\mathcal{M}$.

Remark 1.39. It is known that there is a bijective correspondence $\mathrm{Nat}(\varphi_*\varphi^*, \mathrm{Id}_{{}_R\mathcal{M}}) \cong {}_R\mathrm{Hom}_R(S, R)$, see e.g. [28, Theorem 27].

The following results characterize the separability and natural fullness for the functors φ_*, φ^*. By Proposition 1.32 we know that φ^* is separable (resp., naturally full) if, and only if, so is $\varphi^!$.

Proposition 1.40. [72, Proposition 1.3]

i) *The functor φ_* is separable if, and only if, the extension S/R is separable, i.e. the multiplication $m_S : S \otimes_R S \to S,\ s \otimes_R s' \mapsto ss'$, splits as an S-bimodule map.*

ii) *The functor φ^* is separable if, and only if, φ is split-mono as an R-bimodule map, i.e. if there is $E \in {}_R\mathrm{Hom}_R(S, R)$ such that $E \circ \varphi = \mathrm{Id}_R$, i.e. S/R is a* split extension. *The latter condition is also equivalent to the existence of a conditional expectation $E \in {}_R\mathrm{Hom}_R(S, R)$ such that $E(1_S) = 1_R$, see [28, Theorem 27].*

Remark 1.41. In case R is commutative, the separability of φ_* is equivalent to S/R being a separable extension in the sense of [34], recalled in Example 1.6.

Proposition 1.42. [7, Proposition 3.1] (Cf. [81, Proposition XI.1.2])

i) *The functor φ_* is naturally full if, and only if, it is full if, and only if, it is fully faithful if, and only if, φ is an epimorphism in the category of rings.*

ii) *The functor φ^* is naturally full if, and only if, φ is split-epi as an R-bimodule map, i.e. if there is $E \in {}_R\mathrm{Hom}_R(S, R)$ such that $\varphi \circ E = \mathrm{Id}_S$. The latter condition is also equivalent to the existence of a central idempotent z of R such that $S \cong Rz$ and $\varphi : R \to S \cong Rz$ is the projection $\varphi(r) = rz$.*

1.4.2 Coinduction and corestriction of coscalars

Let $\Bbbk$ be a field and let $\otimes$ denote the tensor product over $\Bbbk$. A $\Bbbk$-*coalgebra* C is a vector space C over $\Bbbk$ equipped with two $\Bbbk$-linear maps, $\Delta_C : C \to C \otimes C$ and $\varepsilon_C : C \to \Bbbk$, such that Δ_C is coassociative and counital, i.e. the diagrams

$$\begin{array}{ccc} C & \xrightarrow{\Delta_C} & C \otimes C \\ {\scriptstyle \Delta_C}\downarrow & & \downarrow{\scriptstyle C\otimes\Delta_C} \\ C\otimes C & \xrightarrow[\Delta_C\otimes C]{} & C \otimes C \otimes C \end{array} \qquad\qquad \begin{array}{ccc} C & \xrightarrow{\Delta_C} & C \otimes C \\ {\scriptstyle \Delta_C}\downarrow & & \downarrow{\scriptstyle C\otimes\varepsilon_C} \\ C\otimes C & \xrightarrow[\varepsilon_C\otimes C]{} & C \end{array}$$

commute. A *right C-comodule* M is a $\Bbbk$-vector space together with a $\Bbbk$-linear map $\rho_M : M \to M \otimes C$, called the *coaction*, that is coassociative and right counital, i.e.

$$(\rho_M \otimes C) \circ \rho_M = (M \otimes \Delta_C) \circ \rho_M \quad \text{and} \quad (M \otimes \varepsilon_C) \circ \rho_M = M$$

hold true, respectively. A coalgebra C can be seen as a right C-comodule with $\rho_C = \Delta_C$. Both for Δ_C and ρ_M we adopt the usual Sweedler notations $\Delta_C(c) = \sum c_1 \otimes c_2$ and

$\rho_M(m) = \sum m_0 \otimes m_1$, for every $c \in C, m \in M$. A *morphism of right C-comodules* (or a *C-colinear* morphism) is a $\Bbbk$-linear map $f : M \to N$ between right C-comodules such that $\rho_N \circ f = (f \otimes C) \circ \rho_M$. The category of right C-comodules and their morphisms is denoted by $\mathfrak{M}^C$. Symmetrically, one can define the category ${}^C\mathfrak{M}$ of left C-comodules and their morphisms.

Recall from [87] that, given a right C-comodule M and a left C-comodule N, the *cotensor product* $M\Box_C N$ is the kernel of the $\Bbbk$-linear map

$$\rho_M \otimes N - M \otimes \lambda_N : M \otimes N \to M \otimes C \otimes N,$$

where ρ_M and λ_N are the right and the left C-comodule structures of M and N, respectively.

Now, let $\psi : C \to D$ be a *morphism of coalgebras*, i.e., a $\Bbbk$-linear map $\psi : C \to D$ such that $\Delta_D \circ \psi = (\psi \otimes \psi) \circ \Delta_C$ and $\varepsilon_D \circ \psi = \varepsilon_C$. Since any right C-comodule M with coaction $\rho_M : M \to M \otimes C$ can be viewed as a right D-comodule with coaction $(M \otimes \psi) \circ \rho_M : M \to M \otimes D$ and C can be considered as a (D, C)-bicomodule, ψ induces

- the corestriction of coscalars functor $\psi_* : \mathfrak{M}^C \to \mathfrak{M}^D$,
- the coinduction functor $\psi^* := (-)\Box_D C : \mathfrak{M}^D \to \mathfrak{M}^C$,

which form an adjunction $\psi_* \dashv \psi^* : \mathfrak{M}^D \to \mathfrak{M}^C$, with unit $\eta : \mathrm{Id}_{\mathfrak{M}^C} \to \psi^*\psi_*$ and counit $\epsilon : \psi_*\psi^* \to \mathrm{Id}_{\mathfrak{M}^D}$, given by

$$\eta_M : M \to M\Box_D C,\ m \mapsto \sum m_0 \Box_D m_1, \quad \text{and} \quad \epsilon_N : N\Box_D C \to N,\ n\Box_D c \mapsto n\varepsilon_C(c),$$

for any $M \in \mathfrak{M}^C$ and $N \in \mathfrak{M}^D$, see [24, 11.10]. Note that, in the definition of ψ^*, for any right D-comodule N, the cotensor product $N\Box_D C$ is regarded as a right C-comodule via $\rho_{N\Box_D C} : N\Box_D C \to (N\Box_D C) \otimes C, n\Box_D c \mapsto \sum(n\Box_D c_1) \otimes c_2$. Furthermore, the coaction ρ_M of M as a right C-comodule induces a morphism of right C-comodules $\bar{\rho}_M = \eta_M : M \to M\Box_D C$ such that $\rho_M = i \circ \bar{\rho}_M$, where $i : M\Box_D C \to M \otimes C$ is the canonical inclusion. In particular, if $M = C$ then $\bar{\rho}_C = \bar{\Delta}_C = \eta_C : C \to C\Box_D C$.

For the functors ψ_* and ψ^* the separability and the natural fullness can be characterized as follows.

Proposition 1.43. [29, Theorem 2.4, Theorem 2.7] *The functor ψ_* is separable if, and only if, the canonical morphism $\bar{\Delta}_C : C \to C\Box_D C$ is split-mono as a C-bicomodule map. The functor ψ^* is separable if, and only if, ψ is split-epi as a D-bicomodule map.*

Proposition 1.44. [7, Examples 3.23 (1)] *The functor ψ_* is naturally full if, and only if, $\bar{\Delta}_C$ is split-epi as a C-bicomodule map. The functor ψ^* is naturally full if, and only if, ψ is split-mono as a D-bicomodule map.*

We recall that a coalgebra C is said to be *coseparable* (see e.g. [24, 3.28]) if $\Delta_C : C \to C \otimes C$ splits as a (C, C)-bicomodule map, i.e. there exists a map $\pi : C \otimes C \to C$ such that $(\mathrm{Id}_C \otimes \pi) \circ (\Delta_C \otimes \mathrm{Id}_C) = \Delta_C \circ \pi = (\pi \otimes \mathrm{Id}_C) \circ (\mathrm{Id}_C \otimes \Delta_C)$ and $\pi \circ \Delta_C = \mathrm{Id}_C$. The coseparability of C can be described equivalently either by the separability of the forgetful functor $\mathfrak{M}^C \to \mathfrak{M}$, or of the forgetful functor ${}^C\mathfrak{M} \to \mathfrak{M}$, see [24, 3.29], or in terms of the existence of a *coseparability idempotent*, that is a $\Bbbk$-linear map $\sigma : C \otimes C \to \Bbbk$ such that $\sigma \circ \Delta_C = \varepsilon_C$ and $(\mathrm{Id}_C \otimes \sigma) \circ (\Delta_C \otimes \mathrm{Id}_C) = (\sigma \otimes \mathrm{Id}_C) \circ (\mathrm{Id}_C \otimes \Delta_C)$.

1.4.3 Corings

The notion of coring was introduced by Sweedler in [86] as a generalization of coalgebras. Indeed, given a ring R, an R-coring is a coalgebra in the monoidal category $({}_R\mathcal{M}_R, \otimes_R, R)$ of R-bimodules. More explicitly, an *R-coring* [86] is an R-bimodule $\mathcal{C}$ together with R-bimodule maps $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_R \mathcal{C}$ and $\varepsilon_{\mathcal{C}} : \mathcal{C} \to R$, called the *comultiplication* (or *coproduct*) and the *counit*, respectively, such that $\Delta_{\mathcal{C}}$ is *coassociative* and *counital*, i.e. such that the diagrams

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\Delta_{\mathcal{C}}} & \mathcal{C} \otimes_R \mathcal{C} \\
{\scriptstyle \Delta_{\mathcal{C}}} \downarrow & & \downarrow {\scriptstyle \mathcal{C} \otimes_R \Delta_{\mathcal{C}}} \\
\mathcal{C} \otimes_R \mathcal{C} & \xrightarrow[\Delta_{\mathcal{C}} \otimes_R \mathcal{C}]{} & \mathcal{C} \otimes_R \mathcal{C} \otimes_R \mathcal{C}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\Delta_{\mathcal{C}}} & \mathcal{C} \otimes_R \mathcal{C} \\
{\scriptstyle \Delta_{\mathcal{C}}} \downarrow & \searrow^{\mathrm{Id}_{\mathcal{C}}} & \downarrow {\scriptstyle \mathcal{C} \otimes_R \varepsilon_{\mathcal{C}}} \\
\mathcal{C} \otimes_R \mathcal{C} & \xrightarrow[\varepsilon_{\mathcal{C}} \otimes_R \mathcal{C}]{} & \mathcal{C}
\end{array}
$$

are commutative, respectively. Given two R-corings $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$, $(\mathcal{C}', \Delta_{\mathcal{C}'}, \varepsilon_{\mathcal{C}'})$, an (R, R)-bilinear map $f : \mathcal{C} \to \mathcal{C}'$ is a *coring morphism* provided $\Delta_{\mathcal{C}'} \circ f = (f \otimes_R f) \circ \Delta_{\mathcal{C}}$ and $\varepsilon_{\mathcal{C}'} \circ f = \varepsilon_{\mathcal{C}}$. We refer to [24, 24.1] for the definition of a morphism of corings over different rings.

Example 1.45. A ring R is an R-coring with the canonical isomorphism $R \mapsto R \otimes_R R$ as a coproduct and the identity map $\mathrm{Id}_R : R \to R$ as a counit, and it is known as the *trivial R-coring.* We will consider it e.g. in Proposition 3.38.

Example 1.46. Let $R \to S$ be a ring extension. Then, $\mathcal{C} := S \otimes_R S$ is an S-coring with comultiplication $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_S \mathcal{C} = S \otimes_R S \otimes_R S$, $s \otimes_R s' \mapsto s \otimes_R 1_S \otimes_R s'$ and counit $\varepsilon_{\mathcal{C}} : \mathcal{C} \to S$, $\varepsilon_{\mathcal{C}}(s \otimes_R s') = ss'$. The coring $\mathcal{C}$ is called the *Sweedler's canonical S-coring* associated with the extension $R \to S$.

Example 1.47. See [24, 17.6]. Let M be an (R, S)-bimodule which is finitely generated and projective as a right S-module. Then, the map

$$N \otimes_S M^* \to \mathrm{Hom}_S(M, N), \; n \otimes_S f \mapsto [m \mapsto nf(m)], \tag{1.18}$$

where $M^* = \mathrm{Hom}_S(M, S)$, is an isomorphism natural in N, for every right S-module N, see e.g. [1, Proposition 20.10]. The S-bimodule $M^* \otimes_R M$ is an S-coring, called the *comatrix coring*, with coproduct

$$\begin{aligned} \Delta : M^* \otimes_R M &\to (M^* \otimes_R M) \otimes_S (M^* \otimes_R M) \cong M^* \otimes_R \mathrm{End}_S(M) \otimes_R M, \\ f \otimes_R m &\mapsto f \otimes_R \mathrm{Id}_M \otimes_R m, \end{aligned}$$

and counit $\varepsilon : M^* \otimes_R M \to S$, $f \otimes_R m \mapsto f(m)$. Comatrix corings have been introduced in [37] and they generalize the Sweedler's canonical coring.

Example 1.48. We recall from [24, 28.1] that, given an R-coring $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$, an element $g \in \mathcal{C}$ is said to be *semi-grouplike* provided $\Delta_{\mathcal{C}}(g) = g \otimes_R g$, and g is a *grouplike element* if g is semi-grouplike and $\varepsilon_{\mathcal{C}}(g) = 1_R$. Note that every coring has a semi-grouplike element (indeed, take $g = 0$) and, if $R \to S$ is a ring extension, then $g = 1_S \otimes_R 1_S$ is a grouplike element in the Sweedler S-coring $S \otimes_R S$. Then, corings with a grouplike element can be viewed as a generalisation of the Sweedler's canonical coring, as well. Furthermore, [24, 28.2] an R-coring has a grouplike element if, and only if, R is a (right or left) $\mathcal{C}$-comodule.

We remind that, given an R-coring $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$, a *right $\mathcal{C}$-comodule* is a right R-module M together with a right R-linear map $\rho_M : M \to M \otimes_R \mathcal{C}$, called the *coaction*, that is coassociative and right counital, i.e. such that the diagrams

$$\begin{array}{ccc} M & \xrightarrow{\rho_M} & M \otimes_R \mathcal{C} \\ {\scriptstyle \rho_M}\downarrow & & \downarrow{\scriptstyle M\otimes_R \Delta_{\mathcal{C}}} \\ M \otimes_R \mathcal{C} & \xrightarrow[\rho_M \otimes_R \mathcal{C}]{} & M \otimes_R \mathcal{C} \otimes_R \mathcal{C} \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{\rho_M} & M \otimes_R \mathcal{C} \\ & {\scriptstyle \mathrm{Id}_M}\searrow & \downarrow{\scriptstyle M\otimes_R \varepsilon_{\mathcal{C}}} \\ & & M \end{array}$$

commute. A *morphism between right $\mathcal{C}$-comodules* $f : M \to N$ is defined as a right R-module map $f : M \to N$ respecting the coactions, i.e., such that $\rho_N \circ f = (f \otimes_R \mathcal{C}) \circ \rho_M$. Let $\mathrm{Hom}^{\mathcal{C}}(M, N)$ denote the set of morphisms of right $\mathcal{C}$-comodules from M to N and let $\mathcal{M}^{\mathcal{C}}$ denote the category of right $\mathcal{C}$-comodules. As for coalgebras, we adopt the usual Sweedler notations $\Delta_{\mathcal{C}}(c) = \sum c_1 \otimes_R c_2$ and $\rho_M(m) = \sum m_0 \otimes_R m_1$ for every $c \in \mathcal{C}$, $M \in \mathcal{M}^{\mathcal{C}}$, and $m \in M$. Symmetrically, one can define the category ${}^{\mathcal{C}}\mathcal{M}$ of left $\mathcal{C}$-comodules. An (R, R)-bimodule M that is a right $\mathcal{C}$-comodule by $\rho_M : M \to M \otimes_R \mathcal{C}$ and a left $\mathcal{C}$-comodule by $\lambda_M : M \to \mathcal{C} \otimes_R M$ is called a *$(\mathcal{C}, \mathcal{C})$-bicomodule* provided it satisfies the compatibility condition $(\mathcal{C} \otimes_R \rho_M) \circ \lambda_M = (\lambda_M \otimes_R \mathcal{C}) \circ \rho_M$. A *morphism of bicomodules* $f : M \to N$ is a map that is both left and right $\mathcal{C}$-colinear, and the R-module of all these maps is denoted by ${}^{\mathcal{C}}\mathrm{Hom}^{\mathcal{C}}(M, N)$. Clearly, $\mathcal{C}$ is a $(\mathcal{C}, \mathcal{C})$-bicomodule by the structure map $\Delta_{\mathcal{C}}$.

Consider the induction functor

$$G := (-) \otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}^{\mathcal{C}}, \quad M \mapsto M \otimes_R \mathcal{C}, \quad f \mapsto f \otimes_R \mathcal{C},$$

which is the right adjoint of the forgetful functor $F : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_R$, see e.g. [23, Lemma 3.1]. The right $\mathcal{C}$-comodule structure of $M \otimes_R \mathcal{C}$ is given by $M \otimes_R \Delta_{\mathcal{C}}$. The unit and counit of the adjunction are given by $\eta_M = \rho_M : M \to M \otimes_R \mathcal{C}$, for every $M \in \mathcal{M}^{\mathcal{C}}$, and $\epsilon_N = N \otimes_R \varepsilon_{\mathcal{C}} : N \otimes_R \mathcal{C} \to N$, $\epsilon_N(n \otimes_R c) = n\varepsilon_{\mathcal{C}}(c)$, for every $N \in \mathcal{M}_R$, $n \in N$, $c \in \mathcal{C}$, respectively.

The following results characterize the separability and natural fullness of F and G. We denote by $\mathcal{C}^R = \{c \in \mathcal{C} \mid rc = cr, \forall r \in R\}$ the set of invariant elements in $\mathcal{C}$. An R-coring $\mathcal{C}$ is said to be *coseparable* [24, Section 26] if $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_R \mathcal{C}$ splits as a $(\mathcal{C}, \mathcal{C})$-bicomodule map, that is, if there exists an (R, R)-bimodule map $\pi : \mathcal{C} \otimes_R \mathcal{C} \to \mathcal{C}$ such that $(\mathrm{Id}_{\mathcal{C}} \otimes_R \pi) \circ (\Delta_{\mathcal{C}} \otimes_R \mathrm{Id}_{\mathcal{C}}) = \Delta_{\mathcal{C}} \circ \pi = (\pi \otimes_R \mathrm{Id}_{\mathcal{C}}) \circ (\mathrm{Id}_{\mathcal{C}} \otimes_R \Delta_{\mathcal{C}})$ and $\pi \circ \Delta_{\mathcal{C}} = \mathrm{Id}_{\mathcal{C}}$.

Proposition 1.49. [23, Theorem 3.3, Corollary 3.6] *The functor F is separable if, and only if, the coring $\mathcal{C}$ is coseparable. The functor G is separable if, and only if, there exists an invariant element $z \in \mathcal{C}^R$ such that $\varepsilon_{\mathcal{C}}(z) = 1_R$.*

In case the functor G is separable, the coring $\mathcal{C}$ is said to be *cosplit* [24, 26.12], and the corresponding invariant $z \in \mathcal{C}^R$ is called a *normalised integral in $\mathcal{C}$*.

Proposition 1.50. [7, Proposition 3.13] *The functor F is naturally full if, and only if, $\Delta_{\mathcal{C}}$ is surjective if, and only if, $c\varepsilon_{\mathcal{C}}(d) = \varepsilon_{\mathcal{C}}(c)d$, for all $c, d \in \mathcal{C}$. The functor G is naturally full if, and only if, there exists an invariant element $z \in \mathcal{C}^R$ such that $c = \varepsilon_{\mathcal{C}}(c)z$ for every $c \in \mathcal{C}$ if, and only if, $\varepsilon_{\mathcal{C}}$ splits in ${}_R\mathcal{M}_R$, i.e., there is $\xi : R \to \mathcal{C}$ in ${}_R\mathcal{M}_R$ such that $\xi \circ \varepsilon_{\mathcal{C}} = \mathrm{Id}_{\mathcal{C}}$.*

Coinduction and corestriction functors of comodules over corings

The cotensor product for comodules over coalgebras (cf. Subsection 1.4.2) can be extended to the cotensor product of comodules over corings. We remind from [42, 2.4] that, for $M \in \mathcal{M}^{\mathcal{C}}$ and $N \in {}^{\mathcal{C}}\mathcal{M}$, the *cotensor product* $M\square_{\mathcal{C}}N$ is defined as the kernel of the $(\mathcal{C},\mathcal{C})$-bicomodule map

$$M \otimes_R N \xrightarrow{\omega_{M,N}:=\rho_M \otimes_R N - M\otimes_R \lambda_N} M \otimes_R \mathcal{C} \otimes_R N,$$

where ρ_M and λ_N are the right and the left $\mathcal{C}$-comodule structures of M and N, respectively. For any any $M \in \mathcal{M}^{\mathcal{C}}$, $N \in {}^{\mathcal{C}}\mathcal{M}$, there are $\mathcal{C}$-comodule isomorphisms $M \cong M\square_{\mathcal{C}}\mathcal{C}$ and $N \cong \mathcal{C}\square_{\mathcal{C}}N$, see [24, 22.4]. Any $M \in \mathcal{M}^{\mathcal{C}}$ induces a covariant functor

$$\begin{gathered} M\square_{\mathcal{C}}- \,:\, {}^{\mathcal{C}}\mathcal{M} \to \mathcal{M}_R, \\ N \mapsto M\square_{\mathcal{C}}N, \quad [f : N \to N'] \mapsto [M\square_{\mathcal{C}}f : M\square_{\mathcal{C}}N \to M\square_{\mathcal{C}}N']. \end{gathered} \tag{1.19}$$

Notice that in general, for $M \in \mathcal{M}^{\mathcal{C}}$ and $N \in {}^{\mathcal{C}}\mathcal{M}$, $M\square_{\mathcal{C}}N$ is just an R-module. If M is a $(\mathcal{D},\mathcal{C})$-bicomodule, then $M\square_{\mathcal{C}}N$ is a left $\mathcal{D}$-comodule provided the map $\omega_{M,N}$ is $\mathcal{D}$-pure in ${}_R\mathcal{M}$, see [24, 22.3]. The pureness conditions required are satified for example if $\mathcal{C}$ is flat as left R-module. In this case, given a morphism $\psi : \mathcal{C} \to \mathcal{D}$ of R-corings, the *coinduction functor* $\psi^* := -\square_{\mathcal{D}}\mathcal{C} : \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$, $N \mapsto N\square_{\mathcal{D}}\mathcal{C}$, is the right adjoint of the *corestriction functor* $\psi_* : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}^{\mathcal{D}}$, $(M, \rho_M) \mapsto (M, \rho^{\psi}_M)$, see [24, 22.12], where $\rho^{\psi}_M = (\mathrm{Id}_M \otimes_R \psi)\rho_M$. The unit is given by $\eta_M : M \to M\square_{\mathcal{D}}\mathcal{C}$, $m \mapsto \sum m_0\square_{\mathcal{D}}m_1$, for every $M \in \mathcal{M}^{\mathcal{C}}$, and the counit is $\epsilon_N : N\square_{\mathcal{D}}\mathcal{C} \to N$, $n\square_{\mathcal{D}}c \mapsto n\varepsilon_{\mathcal{C}}(c)$, for every $N \in \mathcal{M}^{\mathcal{D}}$.

We denote $\mathcal{C}^* = \mathrm{Hom}_R(\mathcal{C}, R)$, ${}^*\mathcal{C} = {}_R\mathrm{Hom}(\mathcal{C}, R)$, respectively.

Definition 1.51. An R-coring $\mathcal{C}$ is said to *satisfy the left α-condition* [24, 19.2] if the map $\alpha^l_N : N \otimes_R \mathcal{C} \to \mathrm{Hom}_R({}^*\mathcal{C}, N)$, $n \otimes_R c \mapsto [f \mapsto nf(c)]$, is injective for every $N \in \mathcal{M}_R$.

Remark 1.52. By [24, 19.2, 42.10] $\mathcal{C}$ satisfies the left α-condition if, and only if, $\mathcal{C}$ is locally projective as a left R-module if, and only if, for $N \in \mathcal{M}_R$ and $u \in N\otimes_R\mathcal{C}$, $(\mathrm{Id}_N\otimes_R f)(u) = 0$ for all $f \in {}^*\mathcal{C}$, implies $u = 0$.

The left α-condition implies that $\mathcal{M}^{\mathcal{C}}$ is a full subcategory of ${}_{^*\mathcal{C}}\mathcal{M}$, see [24, 19.3]. The *right α-condition* can be formulated symmetrically.

Definition 1.53. An R-coring $\mathcal{C}$ is said to *satisfy the right α-condition* if the map

$$\alpha^r_N : \mathcal{C} \otimes_R N \to {}_R\mathrm{Hom}(\mathcal{C}^*, N), \quad c \otimes_R n \mapsto [f \mapsto f(c)n],$$

is injective for every $N \in {}_R\mathcal{M}$.

Proposition 1.54. [24, 26.15] *Let $\psi : \mathcal{C} \to \mathcal{D}$ be a morphism of R-corings and assume that ${}_R\mathcal{C}$ is flat and that $\mathcal{D}$ satisfies the right α-condition. Then, ψ^* is separable if, and only if, there exists a $(\mathcal{D},\mathcal{D})$-colinear map $\chi : \mathcal{D} \to \mathcal{C}$ such that $\psi \circ \chi = \mathrm{Id}_{\mathcal{D}}$.*

1.4.4 Bimodules

Let R, S be rings. An (R, S)-*bimodule* M is a left R-module and a right S-module such that $(rx)s = r(xs)$, for $r \in R$, $x \in M$, $s \in S$. Let ${}_R\mathcal{M}_S$ denote the category of (R, S)-bimodules. We recall that, given bimodules ${}_RL_S$ and ${}_SM_{R'}$, one gets an (R, R')-bimodule $L \otimes_S M$ with $r(x \otimes_S y)r' = (rx) \otimes_S (yr')$, for $r \in R$, $x \in L$, $y \in M$ and $r' \in R'$. Moreover, see e.g. [81, Proposition 9.2, page 32], there is a natural isomorphism

$$\mathrm{Hom}_S(L \otimes_R M, N) \cong \mathrm{Hom}_R(L, \mathrm{Hom}_S(M, N)),$$

for modules L_S, ${}_RM_S$, N_R, that yields the well-known tensor-hom adjunction. In fact, every $M \in {}_R\mathcal{M}_S$ defines an adjunction $\sigma^* \dashv \sigma_* : \mathcal{M}_S \to \mathcal{M}_R$, formed by the induction functor $\sigma^* := (-) \otimes_R M : \mathcal{M}_R \to \mathcal{M}_S$, and the coinduction functor $\sigma_* := \mathrm{Hom}_S(M, -) : \mathcal{M}_S \to \mathcal{M}_R$, with unit η and counit ϵ given by

$$\eta_X : X \to \mathrm{Hom}_S(M, X \otimes_R M),\ x \mapsto [m \mapsto x \otimes_R m],$$

$$\epsilon_Y : \mathrm{Hom}_S(M, Y) \otimes_R M \to Y,\ f \otimes_R m \mapsto f(m),$$

respectively, for all $X \in \mathcal{M}_R$ and $Y \in \mathcal{M}_S$. Given bimodules ${}_RM_S$ and ${}_{R'}N_S$, where R' is a ring, the abelian group $\mathrm{Hom}_S(M, N)$ is an (R', R)-bimodule via the multiplication defined by

$$(r'fr)(m) := r'f(rm), \quad \text{for every } f \in \mathrm{Hom}_S(M, N), r \in R, m \in M, r' \in R'.$$

In particular, the endomorphism ring $\mathcal{E} := \mathrm{End}_S(M)$ belongs to the category ${}_R\mathcal{M}_R$. We will consider the ring homomorphism

$$\varphi : R \to \mathcal{E}, \quad r \mapsto r\mathrm{Id}_M. \tag{1.20}$$

We denote by

$${}^*M = {}_R\mathrm{Hom}(M, R) \quad \text{and} \quad M^* = \mathrm{Hom}_S(M, S)$$

the left dual and the right dual of M, respectively, which both belong to ${}_S\mathcal{M}_R$.

Given an (R, S)-bimodule M, in [83] R is said to be *M-separable over S* if the evaluation map

$$\mathrm{ev}_M : M \otimes_S {}^*M \to R, \quad \mathrm{ev}_M(m \otimes_S f) = f(m), \tag{1.21}$$

is a split epimorphism of R-bimodules. Hereafter, we consider the corresponding right version of this definition. Explicitly, we say that S is *M-separable over R*, if the evaluation map

$$\mathrm{ev}_M : M^* \otimes_R M \to S, \quad \mathrm{ev}_M(f \otimes_R m) = f(m), \tag{1.22}$$

is a split epimorphism of S-bimodules. This means that there is a central element $\sum_i f_i \otimes_R m_i \in (M^* \otimes_R M)^S$ such that $\sum_i f_i(m_i) = 1_S$. The set $\{f_i, m_i\}$ is usually called a *system of M-separability.*

Given an (R, S)-bimodule M, concerning the separability and natural fullness of $\sigma_* = \mathrm{Hom}_S(M, -)$ and $\sigma^* = (-) \otimes_R M$, the following results hold true (the quoted references show the results for the functors between the respective categories of left modules).

Proposition 1.55. [28, Theorem 34] *The functor σ_* is separable if, and only if, S is M-separable over R.*

Proposition 1.56. [78, Proposition 2.5 1.] *If the functor σ^* is separable, then there is $E \in {}_R\mathrm{Hom}_R(\mathcal{E}, R)$ such that $E \circ \varphi = \mathrm{Id}_R$, i.e. φ^* is separable, where $\varphi : R \to \mathcal{E}$ is the map in* (1.20).

As observed in [28, Remark 5], there is no algebraic interpretation for the functor σ^* to be separable, unless M_S is finitely generated and projective. If we assume that the bimodule ${}_RM_S$ is finitely generated and projective as right S-module, then, as already observed in Example 1.47, the map $N \otimes_S M^* \to \mathrm{Hom}_S(M, N)$, $n \otimes_S f \mapsto [m \mapsto nf(m)]$, is an isomorphism natural in N, for every right S-module N. In particular, the endomorphism ring $\mathcal{E} = \mathrm{End}_S(M)$ is isomorphic to $M \otimes_S M^*$. As a consequence, the right adjoint of σ^* can be chosen to be $\sigma_* = (-) \otimes_S M^* : \mathcal{M}_S \to \mathcal{M}_R$. Given a finite dual basis $\{e_i^*, e_i\} \subseteq M^* \times M$, the unit η and the counit ϵ of the adjunction $\sigma^* \dashv \sigma_*$ become

$$\eta_X : X \to X \otimes_R M \otimes_S M^*,\ x \mapsto \sum_i x \otimes_R e_i \otimes_S e_i^*,$$
$$\epsilon_Y : Y \otimes_S M^* \otimes_R M \to Y,\ y \otimes_S f \otimes_R m \mapsto yf(m),$$

for all $X \in \mathcal{M}_R$, $Y \in \mathcal{M}_S$.

In this setting, it is known that the converse of Proposition 1.56 holds true, see e.g. [4, Remark 3.28]. We give here an explicit proof.

Proposition 1.57. *Assume that M_S is finitely generated and projective. Let $\varphi : R \to \mathcal{E}$ as in* (1.20)*. Then, σ^* is separable if, and only if, φ^* is separable, i.e., there is $E \in {}_R\mathrm{Hom}_R(\mathcal{E}, R)$ such that $E \circ \varphi = \mathrm{Id}_R$.*

Proof. In case M_S is finitely generated and projective, it is known that there is a bijection $\mathrm{Nat}(\sigma_*\sigma^*, \mathrm{Id}_{{}_R\mathcal{M}}) \cong {}_R\mathrm{Hom}_R(\mathcal{E}, R)$ (see [7, Proposition 3.6] for right modules). Explicitly, for $E \in {}_R\mathrm{Hom}_R(\mathcal{E}, R)$, the corresponding natural transformation $\nu : \sigma_*\sigma^* \to \mathrm{Id}_{{}_R\mathcal{M}}$ is given by $\nu_X : \sigma_*\sigma^*X = X \otimes_R M \otimes_S M^* \to X$, $x \otimes_R m \otimes_S f \mapsto xE(mf)$. In fact, it is well-defined as, for every $a \in R$, $s \in S$, we have $\nu_X(xa \otimes_R m \otimes_S g) = xaE(mg) = xE(amg) = \nu_X(x \otimes_R am \otimes_S g)$ and $\nu_X(x \otimes_R ms \otimes_S g) = xE(msg) = \nu_X(x \otimes_R m \otimes_S sg)$. For every $f : X \to Y$ in $\mathcal{M}_R$, one has $(\nu_Y \circ \sigma_*\sigma^* f)(x \otimes_R m \otimes_S g) = \nu_Y(f(x) \otimes_R m \otimes_S g) = f(x)E(mg) = f(xE(mg)) = (f \circ \nu_X)(x \otimes_R m \otimes_S g)$, so ν is natural. Given a finite dual basis $\{e_i^*, e_i\} \subseteq M^* \times M$, one has $\sum_i e_i e_i^*(m) = m = \mathrm{Id}_M(m) = \varphi(1_R)(m)$, for every $m \in M$, so $\sum_i e_i e_i^* = \varphi(1_R)$, hence we get $(\nu_X \circ \eta_X)(x) = \nu_X(\sum_i x \otimes_R e_i \otimes_S e_i^*) = xE(\sum_i e_i e_i^*) = xE\varphi(1_R) = x1_R = x = \mathrm{Id}_X(x)$, for every $X \in \mathcal{M}_R$, $x \in X$. Thus, by Theorem 1.18 σ^* is separable. The converse holds by Proposition 1.56. □

The next result characterizes the natural fullness of σ_* and σ^*.

Proposition 1.58. [7, Theorem 3.8] *The functor σ_* is naturally full if, and only if, there is $\sum_i f_i \otimes_R m_i \in (M^* \otimes_R M)^S$ satisfying $\mathrm{Id}_M \otimes_R m = \sum_i m f_i(-) \otimes_R m_i$, for all $m \in M$.*

Assume that $M \in {}_R\mathcal{M}_S$ is finitely generated and projective as a right S-module. Let $\varphi : R \to \mathcal{E}$ be the map as in (1.20)*. Then, σ^* is naturally full if, and only if, φ^* is naturally full, i.e. if, and only if, there is $E \in {}_R\mathrm{Hom}_R(\mathcal{E}, R)$ such that $\varphi \circ E = \mathrm{Id}_{\mathcal{E}}$.*

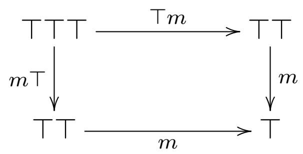
Remark 1.59. Given a morphism of rings $\varphi : R \to S$ we can consider the (R, S)-bimodule $M := {}_RS_S$, with left action induced by φ, which is trivially finitely generated and projective as a right S-module. In this case, $\sigma^* = (-) \otimes_R S = \varphi^* : \mathcal{M}_S \to \mathcal{M}_R$ is the extension of scalars functor, so its right adjoint σ_* is isomorphic to the restriction of scalars functor $\varphi_* : \mathcal{M}_S \to \mathcal{M}_R$. As a consequence, Proposition 1.40 and Proposition 1.42 are special cases of Proposition 1.57 and Proposition 1.58, respectively. In particular, it follows that S is S-separable over R if, and only if, φ_* is separable, i.e. if, and only if, the extension S/R is separable.

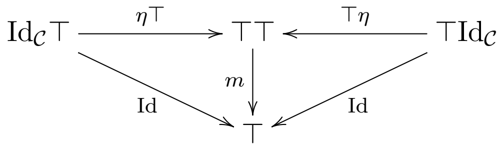
1.5 (Co)monads and separability

In this section we recall how separable functors and (co)separable (co)monads are related. We start by reminding the definitions of (co)monads, Eilenberg-Moore categories [36] and the Kleisli category [57], see also [61, VI].

1.5.1 Monads and comonads

A *monad* (or *triple* [41]), on a category $\mathcal{C}$ is a tern $\mathbb{T} = (\top, m, \eta)$ where $\top : \mathcal{C} \to \mathcal{C}$ is a functor, $m : \top\top \to \top$ and $\eta : \mathrm{Id}_{\mathcal{C}} \to \top$ are natural transformations satisfying the *associative* law and the *left* and *right unit* laws, i.e. such that the diagrams





commute. Let $\phi : \top \to \top'$ be a natural transformation between functors $\top, \top'$. Then, the horizontal composition (also called the *Godement product*) $\phi\top' \circ \top\phi = \top'\phi \circ \phi\top$ is usually denoted by $\phi\phi : \top\top \to \top'\top'$. A *morphism between two monads* $\mathbb{T} = (\top, m, \eta)$ and $\mathbb{T}' = (\top', m', \eta')$ on a category $\mathcal{C}$ is a natural transformation $\phi : \top \to \top'$ such that $\phi \circ m = m' \circ \phi\phi$, and $\phi \circ \eta = \eta'$.

Remark 1.60. The category $\mathrm{End}(\mathcal{C})$ of endofunctors of a category $\mathcal{C}$ is a (strict) monoidal category, with composition of endofunctors for monoidal product and the identity functor $\mathrm{Id}_{\mathcal{C}}$ for unit object. A monad on $\mathcal{C}$ is just a monoid (or algebra) in the category $\mathrm{End}(\mathcal{C})$.

Remark 1.61. [61, page 138] Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and counit $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$. Then, $(GF, G\epsilon F, \eta)$ is a monad on $\mathcal{C}$.

Example 1.62. Let $\varphi : R \to S$ be a morphism of unital rings, cf. Subsection 1.4.1. The R-bimodule structure on S is given by

$$r \cdot s = \varphi(r)s, \qquad s \cdot r = s\varphi(r),$$

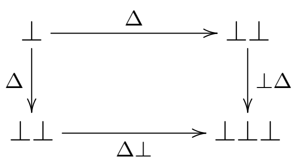
for every $r \in R$ and $s \in S$. Consider the multiplication $m_S : S \otimes_R S \to S$, $s \otimes_R s' \mapsto ss'$, which is a morphism of S-bimodules such that $m_S \circ (S \otimes_R m_S) = m_S \circ (m_S \otimes_R S)$. Let $l_- : R \otimes_R - \to \mathrm{Id}_{{}_R\mathcal{M}}$ be the functorial isomorphism given, for any left R-module M, by $l_M : R \otimes_R M \to M$, $r \otimes_R x \mapsto r \cdot x$, with inverse $l_M^{-1} : M \to R \otimes_R M$, $l_M^{-1}(x) = 1_R \otimes_R x$, and consider

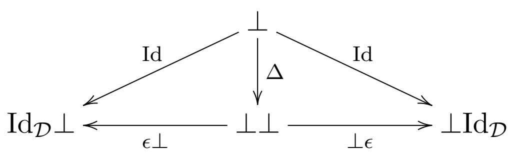
$$\top := S \otimes_R - : {}_R\mathcal{M} \to {}_R\mathcal{M}, \quad m := m_S \otimes_R - : S \otimes_R S \otimes_R - \to S \otimes_R -,$$

$$\eta := (\eta_S \otimes_R -) \circ l_-^{-1} : \mathrm{Id}_{{}_R\mathcal{M}} \to S \otimes_R -,$$

where $\eta_S = \varphi \otimes_R - : R \otimes_R - \to S \otimes_R -$. Then, $\mathbb{T} = (\top, m, \eta)$ is a monad on the category ${}_R\mathcal{M}$ of left R-modules.

Dually, a *comonad* on a category $\mathcal{D}$ is a triple $\mathbb{C} = (\bot, \Delta, \epsilon)$ where $\bot : \mathcal{D} \to \mathcal{D}$ is a functor, $\Delta : \bot \to \bot\bot$ and $\epsilon : \bot \to \mathrm{Id}_{\mathcal{D}}$ are natural transformations satisfying the *coassociative* law and the *left* and *right counit* laws, i.e. such that the diagrams





commute. Let $\phi : \bot \to \bot'$ be a natural transformation between functors $\bot, \bot'$, and denote by $\phi\phi : \bot\bot \to \bot'\bot'$ the horizontal composition $\phi\bot' \circ \bot\phi = \bot'\phi \circ \phi\bot$. A *morphism between two comonads* $\mathbb{C} = (\bot, \Delta, \epsilon)$ and $\mathbb{C}' = (\bot', \Delta', \epsilon')$ on a category $\mathcal{D}$ is a natural transformation $\phi : \bot \to \bot'$ such that $\Delta' \circ \phi = \phi\phi \circ \Delta$ and $\epsilon' \circ \phi = \epsilon$. Any adjunction (F, G, η, ϵ) defines a comonad $(FG, F\eta G, \epsilon)$ on $\mathcal{D}$.

Example 1.63. Given a ring R, let $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ be an R-coring. Denoting

$$\perp := - \otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}_R, \qquad \Delta := - \otimes_R \Delta_{\mathcal{C}} : - \otimes_R \mathcal{C} \to - \otimes_R \mathcal{C} \otimes_R \mathcal{C},$$

$$\epsilon := r_- \circ (- \otimes_R \varepsilon_{\mathcal{C}}) : - \otimes_R \mathcal{C} \to \mathrm{Id}_{\mathcal{M}_R},$$

where $r_- : - \otimes_R R \to \mathrm{Id}_{\mathcal{M}_R}$ is the functorial isomorphism given by $r_M : M \otimes_R R \to M$, $x \otimes_R r \mapsto x \cdot r$, we have that $\mathbb{C} = (\perp, \Delta, \epsilon)$ is a comonad on the category $\mathcal{M}_R$ of right R-modules. Indeed, see [24, 18.28 (1)], the following are equivalent for an (R, R)-bimodule $\mathcal{C}$:

(i) $\mathcal{C}$ is an R-coring;

(ii) the functor $- \otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}_R$ is a comonad;

(iii) the functor $\mathcal{C} \otimes_R - : {}_R\mathcal{M} \to {}_R\mathcal{M}$ is a comonad.

1.5.2 Eilenberg-Moore categories

Given a monad $\mathbb{T} = (\top, m, \eta)$ on a category $\mathcal{C}$, in [36] Eilenberg and Moore constructed the category $\mathcal{C}_\top$ of $\top$-modules and an adjunction $V_\top \dashv U_\top : \mathcal{C}_\top \to \mathcal{C}$, whose associated monad is given by $\mathbb{T}$.

Definition 1.64. Let $\mathbb{T} = (\top, m, \eta)$ be a monad on a category $\mathcal{C}$. An *action* of $\top$ on an object X of $\mathcal{C}$ is a morphism $\mu_X : \top X \to X$ in $\mathcal{C}$ such that the associative and unital laws hold, i.e. the diagrams

$$\begin{array}{ccc} \top\top X & \xrightarrow{\top\mu_X} & \top X \\ {\scriptstyle m_X}\downarrow & & \downarrow{\scriptstyle \mu_X} \\ \top X & \xrightarrow[\mu_X]{} & X \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{\eta_X} & \top X \\ & {\scriptstyle \mathrm{Id}_X}\searrow \quad \swarrow{\scriptstyle \mu_X} & \\ & X & \end{array}$$

commute. The pair (X, μ_X) is called a $\top$-*module* (or $\top$-*algebra*) in $\mathcal{C}$. A morphism f between two $\top$-modules (X, μ_X) and $(X', \mu_{X'})$ is a morphism $f : X \to X'$ in $\mathcal{C}$ such that $\mu_{X'} \circ \top f = f \circ \mu_X$.

The category of $\top$-modules and their morphisms is called the *Eilenberg-Moore category* of the monad $\mathbb{T}$, and it is denoted by $\mathcal{C}_\top$. The forgetful functor $U_\top : \mathcal{C}_\top \to \mathcal{C}$,

$$(X, \mu_X : \top X \to X) \mapsto X, \quad f \mapsto f,$$

has a left adjoint, the so-called *free functor* $V_\top : \mathcal{C} \to \mathcal{C}_\top$,

$$X \mapsto (\top X, m_X), \qquad f \mapsto \top(f).$$

The unit $\mathrm{Id}_{\mathcal{C}} \to U_\top V_\top = \top$ is exactly η, while the counit $\epsilon_\top : V_\top U_\top \to \mathrm{Id}_{\mathcal{C}_\top}$ is determined by the equality $U_\top \epsilon_{\top(X,\mu_X)} = \mu_X$, for every object (X, μ_X) in $\mathcal{C}_\top$. The monad on $\mathcal{C}$ attached to the *Eilenberg-Moore adjunction* $V_\top \dashv U_\top$, is the given monad $\mathbb{T} = (\top, m, \eta)$.

Let us give some known examples.

Example 1.65. Let $\mathbb{T} = (\top, m, \eta)$ be the monad of Example 1.62 on the category ${}_R\mathcal{M}$ of left R-modules. Let (X, μ_X) be an object in $({}_R\mathcal{M})_\top$. This means that $\mu_X : \top X =$

$S \otimes_R X \to X$ is a morphism in ${}_R\mathcal{M}$ such that $\mu_X \circ \top\mu_X = \mu_X \circ m_X$ and $\mu_X \circ \eta_X = \mathrm{Id}_X$. Thus, for every $x \in X$, $s \in S$, writing $sx = \mu_X(s \otimes_R x)$, we get

$$(\mu_X \circ (S \otimes_R \mu_X))(s' \otimes_R s \otimes_R x) = \mu_X(s' \otimes_R sx) = s'(sx)$$
$$(\mu_X \circ m_X)(s' \otimes_R s \otimes_R x) = \mu_X(ss' \otimes_R x) = (s's)x$$
$$1_S x = \mu_X(1_S \otimes_R x) = (\mu_X \circ \eta_X)(x) = x.$$

Then, the category $({}_R\mathcal{M})_\top$ is isomorphic to the category ${}_S\mathcal{M}$ of left S-modules through the assignment

$$(X, \mu_X) \mapsto (X, \mu_X \circ \tau)$$

where $\tau : S \times X \to S \otimes_R X$, $(s, x) \mapsto s \otimes_R x$.

Example 1.66. A monoid M with identity element 1_M determines a functor $\top = M \times - :$ $\mathsf{Set} \to \mathsf{Set}$ and natural transformations $m : \top\top \to \top$ and $\eta : \mathrm{Id}_{\mathsf{Set}} \to \top$ defined by

$$m_X : \top\top X = M \times M \times X \to M \times X = \top X : (m, n, x) \mapsto (mn, x),$$

$$\eta_X : X \to M \times X = \top X : x \mapsto (1_M, x).$$

Then, $(\top, m, \eta)$ is a monad on Set. Let (X, μ_X) be an object in $\mathsf{Set}_\top$, i.e. $X \in \mathsf{Set}$ and $\mu_X : M \times X \to X$ is a map such that $\mu_X \circ (M \times \mu_X) = \mu_X \circ m_X$ and $\mu_X \circ \eta_X = \mathrm{Id}_X$. Thus, (X, μ_X) is an M-set, and $\mathsf{Set}_\top$ is the category of M-sets and their morphisms.

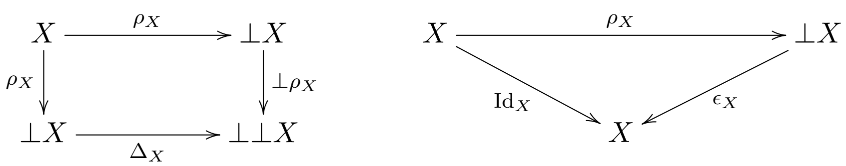
Example 1.67. An algebra A over a field $\Bbbk$ determines a functor $\top = A \otimes - : \mathfrak{M} \to \mathfrak{M}$, where $\mathfrak{M}$ is the category of vector spaces over $\Bbbk$ and the unadorned $\otimes$ is the tensor product over $\Bbbk$, and natural transformations $m : \top\top \to \top$ and $\eta : \mathrm{Id}_{\mathfrak{M}} \to \top$ defined by

$$m_V : \top\top V = A \otimes A \otimes V \to A \otimes V = \top V : a \otimes b \otimes v \mapsto ab \otimes v,$$

$$\eta_V : V \to A \otimes V = \top V : v \mapsto 1_A \otimes v.$$

Then, $(\top, m, \eta)$ is a monad on $\mathfrak{M}$, and $\mathfrak{M}_\top$ identifies with the category ${}_A\mathcal{M}$ of left A-modules.

Definition 1.68. Let $\mathbb{C} = (\bot, \Delta, \epsilon)$ be a comonad on a category $\mathcal{D}$. A *coaction* of $\bot$ on an object X of $\mathcal{D}$ is a morphism $\rho_X : X \to \bot X$ in $\mathcal{D}$ such that the coassociative and counital laws hold, i.e. the diagrams



commute. The pair (X, ρ_X) is called a $\bot$-*comodule* (or $\bot$-*coalgebra*) in $\mathcal{D}$. A morphism f between two $\bot$-comodules (X, ρ_X) and $(X', \rho_{X'})$ is a morphism $f : X \to X'$ in $\mathcal{D}$ such that $\rho_{X'} \circ f = \bot f \circ \rho_X$.

The category of $\bot$-comodules and their morphisms is denoted by $\mathcal{D}^\bot$. The forgetful functor $U^\bot : \mathcal{D}^\bot \to \mathcal{D}$ has a right adjoint, namely the *cofree functor*

$$V^\bot : \mathcal{D} \to \mathcal{D}^\bot, \qquad X \mapsto (\bot X, \Delta_X), \qquad f \mapsto \bot(f).$$

The unit $\eta^\bot : \mathrm{Id}_{\mathcal{D}^\bot} \to V^\bot U^\bot$ is completely determined by the equality $U^\bot \eta^\bot_{(X,\rho_X)} = \rho_X$, for every object (X, ρ_X) in $\mathcal{D}^\bot$, while the counit $U^\bot V^\bot = \bot \to \mathrm{Id}_{\mathcal{D}}$ is exactly ϵ.

Example 1.69. We come back to Example 1.63. If $\mathcal{C}$ is an R-coring, then the category $(\mathcal{M}_R)^{-\otimes_R \mathcal{C}}$ (resp., $({}_R\mathcal{M})^{\mathcal{C}\otimes_R -}$) of comodules over the comonad $-\otimes_R \mathcal{C}$ (resp., $\mathcal{C}\otimes_R -$) is isomorphic to the category $\mathcal{M}^{\mathcal{C}}$ (resp., ${}^{\mathcal{C}}\mathcal{M}$) of right (resp., left) $\mathcal{C}$-comodules, see e.g. [24, 18.28 (2)].

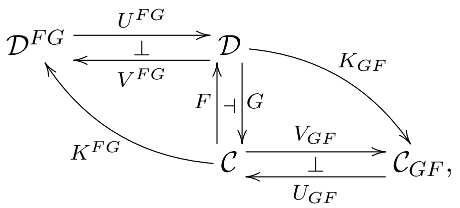
Given an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$, with unit η and counit ϵ, we can consider the monad $(GF, G\epsilon F, \eta)$ on $\mathcal{C}$ and the comonad $(FG, F\eta G, \epsilon)$ on $\mathcal{D}$. We have the *comparison functor*

$$K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}, \qquad D \mapsto (GD, G\epsilon_D), \qquad f \mapsto G(f),$$

and the *cocomparison functor*

$$K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}, \qquad C \mapsto (FC, F\eta_C), \qquad f \mapsto F(f),$$

the forgetful functors U_{GF} and U^{FG}, the free functor V_{GF} and the cofree functor V^{FG}, that fit into the diagram



(1.23)

where $U_{GF} \circ K_{GF} = G$, $K_{GF} \circ F = V_{GF}$, $U^{FG} \circ K^{FG} = F$ and $K^{FG} \circ G = V^{FG}$.

Definition 1.70. *i)* An adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$ is called *monadic* (*tripleable* in Beck's terminology [15]) (resp., *premonadic*) whenever the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ is an equivalence of categories (resp., fully faithful).

ii) A functor G is called *monadic* (resp., *premonadic*) if G has a left adjoint F such that the adjunction (F, G) is monadic (resp., premonadic).

iii) An adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$ is called *comonadic* (resp., *precomonadic*) whenever the cocomparison functor $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$ is an equivalence of categories (resp., fully faithful).

iv) A functor F is called *comonadic* (resp., *precomonadic*) if F has a right adjoint G such that the adjunction (F, G) is comonadic (resp., precomonadic).

Example 1.71. Let M be a monoid. Then, the forgetful functor $G : M\text{-}\mathsf{Set} \to \mathsf{Set}$ is monadic. The left adjoint F of G is given by $F : \mathsf{Set} \to M\text{-}\mathsf{Set}$, $S \mapsto (M \times S, \mu_M \times \mathrm{Id}_S : M \times M \times S \to M \times S)$, $f \mapsto \mathrm{Id}_M \times f$. The monad associated with this adjunction is given as in Example 1.66. Thus, the Eilenberg-Moore category Set_{GF} is the category M-Set of M-sets. The comparison functor is an equivalence and G is monadic.

Example 1.72. Let R, S be rings. Let ${}_RM_S$ be a bimodule which is a finitely generated and projective right S-module, and let $\sigma^* \dashv \sigma_* : \mathcal{M}_S \to \mathcal{M}_R$ be the adjunction considered in Subsection 1.4.4. Consider an object $(X, \mu : \sigma_*\sigma^* X \to X)$ in $(\mathcal{M}_R)_{\sigma_*\sigma^*}$. Note that $\sigma_*\sigma^*(X) = X \otimes_R M \otimes_S M^* = X \otimes_R \mathcal{E}$, where $\mathcal{E} := \mathrm{End}_S(M) \cong M \otimes_S M^*$ is the endomorphism ring with canonical morphism $\varphi : R \to \mathcal{E}$, $\varphi(r)(m) = rm$, for all $r \in R$ and $m \in M$. Then, see e.g. [64, page 30], the Eilenberg-Moore category $(\mathcal{M}_R)_{\sigma_*\sigma^*}$ is equivalent to the category $\mathcal{M}_{\mathcal{E}}$ of right $\mathcal{E}$-modules, but the comparison functor $K_{\sigma_*\sigma^*} : \mathcal{M}_S \to \mathcal{M}_{\mathcal{E}}$, $Y \mapsto Y \otimes_S M^*$ (the latter is a right $\mathcal{E}$-module through the evaluation map), needs not to be an equivalence unless M is a generator in $\mathcal{M}_S$, see e.g. [34, Lemma 3.2, Proposition 3.3], [81, Proposition IV. 10.7, page 108]. Then, the forgetful and free functors result to be $U_{\sigma_*\sigma^*} = \varphi_*$, $V_{\sigma_*\sigma^*} = \varphi^*$, respectively. Dually, see e.g. [64, page 36], the Eilenberg-Moore category $(\mathcal{M}_S)^{\sigma^*\sigma_*}$ is equivalent to the category $\mathcal{M}^{\mathcal{C}}$ of right comodules over the comatrix S-coring $\mathcal{C} := M^* \otimes_R M$ (cf. Example 1.47). In this case, the diagram (1.23) becomes:

$$\begin{array}{ccccc} (\mathcal{M}_S)^{\sigma^*\sigma_*} \cong \mathcal{M}^{\mathcal{C}} & \underset{G=(-)\otimes_S \mathcal{C}}{\overset{F}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} & \mathcal{M}_S & \xrightarrow{K_{\sigma_*\sigma^*}} & \\ \uparrow K^{\sigma^*\sigma_*} & & \sigma^*=(-)\otimes_R M \uparrow \dashv \downarrow \sigma_*=(-)\otimes_S M^* & & \\ & & \mathcal{M}_R & \underset{\varphi_*}{\overset{\varphi^*=(-)\otimes_R \mathcal{E}}{\underset{\leftarrow}{\overset{\rightarrow}{\bot}}}} & \mathcal{M}_{\mathcal{E}} \cong (\mathcal{M}_R)_{\sigma_*\sigma^*}, \end{array} \tag{1.24}$$

where $K^{\sigma^*\sigma_*}$ is the cocomparison functor, and $F \dashv G$ is the adjunction given by the forgetful functor $F : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_S$ and the induction functor $G := (-) \otimes_S \mathcal{C} : \mathcal{M}_S \to \mathcal{M}^{\mathcal{C}}$ (cf. Subsection 1.4.3). Thus, we have

$$\varphi_* \circ K_{\sigma_*\sigma^*} = \sigma_*, \qquad F \circ K^{\sigma^*\sigma_*} = \sigma^*, \qquad K_{\sigma_*\sigma^*} \circ \sigma^* = \varphi^*, \qquad K^{\sigma^*\sigma_*} \circ \sigma_* = G.$$

1.5.3 Kleisli category

Let $(\top, m, \eta)$ be a monad on a category $\mathcal{C}$. A $\top$-module is said to be *free* (see e.g. [20, Definition 4.1.5][61, VI. 5]) when it is isomorphic to one of the form $V_\top C = (\top C, m_C)$, for some object $C \in \mathcal{C}$, and the full subcategory of $\mathcal{C}_\top$ generated by the free $\top$-modules is equivalent to the so-called *Kleisli category* $\top\text{-Free}_{\mathcal{C}}$ *of free* $\top$*-modules* (see [57]). Explicitly, the objects of $\top\text{-Free}_{\mathcal{C}}$ are those of $\mathcal{C}$ and a morphism $f : C \nrightarrow D$ in $\top\text{-Free}_{\mathcal{C}}$ is a morphism $f : C \to \top(D)$ in $\mathcal{C}$; the composite of two morphisms $f : C \nrightarrow D$, $g : D \nrightarrow E$ in $\top\text{-Free}_{\mathcal{C}}$, that we denote by $g \circ' f$, is given by the composite

$$C \xrightarrow{f} \top(D) \xrightarrow{\top(g)} \top\top(E) \xrightarrow{m_E} \top(E)$$

in $\mathcal{C}$, and the identity $C \nrightarrow C$ on an object C of $\top\text{-Free}_{\mathcal{C}}$ is the unit $\eta_C : C \to \top(C)$ in $\mathcal{C}$. There is (see [20, Proposition 4.1.6]) a fully faithful functor

$$J_\top : \top\text{-Free}_{\mathcal{C}} \to \mathcal{C}_\top, \quad C \mapsto (\top C, m_C), \quad [f : C \nrightarrow D] \mapsto m_D \circ \top(f),$$

that fits into the following diagram

$$\begin{array}{ccccc} & & \mathcal{C} & & \\ & U'_\top \nearrow \swarrow V'_\top & & U_\top \nwarrow \searrow V_\top & \\ \top\text{-Free}_{\mathcal{C}} & & \xhookrightarrow{\quad J_\top \quad} & & \mathcal{C}_\top, \end{array} \tag{1.25}$$

where the adjunction $V_\top \dashv U_\top$ restricts to an adjunction $V'_\top \dashv U'_\top$ (called the *Kleisli adjunction*) between $\mathcal{C}$ and $\top\text{-Free}_{\mathcal{C}}$, that is, $U'_\top = U_\top \circ J_\top$ and $J_\top \circ V'_\top = V_\top$ (see [20, Corollary 4.1.7]). Explicitly, $U'_\top$ and $V'_\top$ are given by

$$U'_\top : \top\text{-Free}_{\mathcal{C}} \to \mathcal{C}, \quad C \mapsto \top(C), \quad [f : C \nrightarrow D] \mapsto m_D \circ \top(f), \tag{1.26}$$

$$V'_\top : \mathcal{C} \to \top\text{-Free}_{\mathcal{C}}, \quad C \mapsto C, \quad [f : C \to D] \mapsto \eta_D \circ f. \tag{1.27}$$

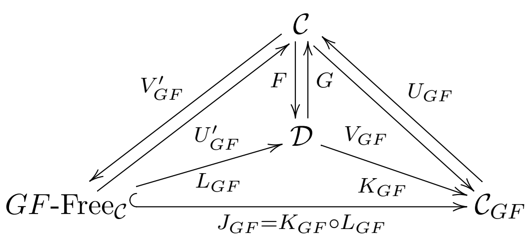
The unit of $V'_\top \dashv U'_\top$ is η, while the counit $\beta' : V'_\top U'_\top \to \mathrm{Id}_{\top\text{-Free}_\mathcal{C}}$ is given by $\beta'_Y = \mathrm{Id}_{TY} : \top Y \nrightarrow Y$, for every $Y \in \top\text{-Free}_\mathcal{C}$. Indeed, by employing the composition law for the Kleisli category, β' is a natural transformation as, for every $f : X \nrightarrow Y$ in $\top\text{-Free}_\mathcal{C}$, $\beta'_Y \circ' \top f = m_Y \circ \top\beta'_Y \circ \top f = m_Y \circ \top\mathrm{Id}_{\top X} \circ \top f = m_Y \circ \top f = m_Y \circ \top f \circ \mathrm{Id}_{\top X} = m_Y \circ \top f \circ \beta'_X = f \circ' \beta'_X$; we have $U'_\top \beta'_Y \circ \eta_{U'_\top Y} = m_Y \circ \top\mathrm{Id}_{\top Y} \circ \eta_{\top Y} = m_Y \circ \eta_{\top Y} = \mathrm{Id}_{\top Y} = \mathrm{Id}_{U'_\top Y}$, for every $Y \in \mathcal{C}$, and $\beta'_{V'_\top X} \circ' V'_\top \eta_X = m_X \circ \top\beta'_X \circ V'_\top \eta_X = m_X \circ \top\mathrm{Id}_{\top X} \circ \eta_{\top X} \circ \eta_X = m_X \circ \eta_{\top X} \circ \eta_X = \mathrm{Id}_{\top X} \circ \eta_X = \eta_X = \mathrm{Id}_{V'_\top X}$, for every $X \in \mathcal{C}$, so the triangle identities hold true.

Remark 1.73. See [20, Proposition 4.2.2]. Let $\mathbb{T} = (\top, m, \eta)$ be a monad on a category $\mathcal{C}$. Then, as in the case of the Eilenberg-Moore adjunction $V_\top \dashv U_\top$, the monad associated with the Kleisli adjunction $V'_\top \dashv U'_\top$ is the given monad $\mathbb{T}$. More precisely, see also [61, Theorem VI 5.3], the category of adjunctions which define the monad $\mathbb{T}$ on $\mathcal{C}$ and morphisms between them, has the Kleisli adjunction as initial object, and the Eilenberg-Moore adjunction as terminal object.

Now, let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ, and consider the diagram (1.25) for the associated monad $(GF, G\epsilon F, \eta)$. Then, (see [20, Proposition 4.2.1]) there is the so-called *Kleisli comparison functor*

$$L_{GF} : GF\text{-Free}_\mathcal{C} \to \mathcal{D}, \quad C \mapsto F(C), \quad [f : C \nrightarrow D] \mapsto \epsilon_{FD} \circ F(f),$$

such that $K_{GF} \circ L_{GF}$ is the functor $J_{GF} : GF\text{-Free}_\mathcal{C} \to \mathcal{C}_{GF}$, $C \mapsto (GFC, G\epsilon_{FC})$, $f \mapsto G\epsilon_{FD} \circ GF(f)$, and the following diagram



(1.28)

is commutative. In particular, we have $G \circ L_{GF} = U'_{GF}$ and $L_{GF} \circ V'_{GF} = F$ where, as in (1.26) and (1.27),

$$U'_{GF} : GF\text{-Free}_\mathcal{C} \to \mathcal{C}, \quad C \mapsto GF(C), \quad [f : C \nrightarrow D] \mapsto G\epsilon_{FD} \circ GF(f), \tag{1.29}$$
$$V'_{GF} : \mathcal{C} \to GF\text{-Free}_\mathcal{C}, \quad C \mapsto C, \quad [f : C \to D] \mapsto \eta_D \circ f. \tag{1.30}$$

Remark 1.74. Cf. [20, Proposition 4.2.1]. Since $K_{GF} \circ L_{GF} = J_{GF}$ and the functor J_{GF} is fully faithful, then the functor L_{GF} is faithful. Moreover, L_{GF} is actually fully faithful. In fact, a morphism $h : F(C) \to F(D)$ in $\mathcal{D}$ corresponds by adjunction with the morphism $f := Gh \circ \eta_C : C \to GF(D)$ in $\mathcal{C}$, i.e. a morphism $f : C \nrightarrow D$ in $GF\text{-Free}_\mathcal{C}$ such that $L_{GF}(f) = h$, hence L_{GF} is full as well.

1.5.4 (Co)separable (co)monads

In [22] the notion of separable algebra was extended to monads. Explicitly, a monad $(\top, m : \top\top \to \top, \eta : \mathrm{Id}_\mathcal{C} \to \top)$ on a category $\mathcal{C}$ is said to be *separable* if there exists a natural transformation $\sigma : \top \to \top\top$ such that $m \circ \sigma = \mathrm{Id}_\top$ and $\top m \circ \sigma\top = \sigma \circ m = m\top \circ \top\sigma$; moreover, a separable monad is a monad satisfying the equivalent conditions of [22, Proposition 6.3].

Dually, a comonad $(\perp, \Delta : \perp \to \perp\perp, \epsilon : \perp \to \mathrm{Id}_{\mathcal{C}})$ on a category $\mathcal{C}$ is said to be *coseparable* if there exists a natural transformation $\tau : \perp\perp \to \perp$ satisfying $\tau \circ \Delta = \mathrm{Id}_{\perp}$ and $\perp\tau \circ \Delta\perp = \Delta \circ \tau = \tau\perp \circ \perp\Delta$.

The (co)separability of a (co)monad is equivalent to the separability of the forgetful functor from the Eilenberg-Moore category of (co)modules over the (co)monad.

Lemma 1.75. [17, 2.9] *Let $\mathcal{C}$ be a category. Then,*

i) a monad $(\top, m : \top\top \to \top, \eta : \mathrm{Id}_{\mathcal{C}} \to \top)$ on $\mathcal{C}$ is separable if, and only if, the forgetful functor $U_{\top} : \mathcal{C}_{\top} \to \mathcal{C}$ is separable;

ii) a comonad $(\perp, \Delta : \perp \to \perp\perp, \epsilon : \perp \to \mathrm{Id}_{\mathcal{C}})$ on $\mathcal{C}$ is coseparable if, and only if, the forgetful functor $U^{\perp} : \mathcal{C}^{\perp} \to \mathcal{C}$ is separable.

Remark 1.76. Let $\mathcal{C}$ be a category. If the forgetful functor $U_{\top} : \mathcal{C}_{\top} \to \mathcal{C}$ is separable, then so is $U'_{\top}$. In fact, since $U'_{\top} = U_{\top} \circ J_{\top}$ and $J_{\top}$ is fully faithful, then $U'_{\top}$ is separable by Lemma 1.4 *i)* if so is $U_{\top}$.

Furthermore, an *idempotent monad* is a monad $(\top, m, \eta)$ on a category $\mathcal{C}$ whose multiplication m is an isomorphism or, equivalently, such that the forgetful functor $U_{\top} : \mathcal{C}_{\top} \to \mathcal{C}$ is fully faithful, see [20, Proposition 4.2.3]. Dually, an *idempotent comonad* is a comonad $(\perp, \Delta, \epsilon)$ on a category $\mathcal{C}$ whose comultiplication Δ is an isomorphism or, equivalently, such that the forgetful functor $U^{\perp} : \mathcal{C}^{\perp} \to \mathcal{C}$ is fully faithful, see [2, Section 6].

An adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$ with unit $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and counit $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ is said to be an *idempotent adjunction* if the monad $(GF, G\epsilon F, \eta)$ is idempotent, or equivalently if the comonad $(FG, F\eta G, \epsilon)$ is idempotent, see e.g. [32, Subsection 3.4]. The first hint of *idempotent adjunctions* can be found in [62] under the name of *idempotent constructions.* By [60, Proposition 2.8] the idempotence of an adjunction (F, G, η, ϵ) is equivalent to any of the natural transformations ϵF, $G\epsilon$, $F\eta$ and ηG being an isomorphism.

Remark 1.77. An idempotent (co)monad on a category $\mathcal{C}$ is always (co)separable with splitting given by the inverse of the (co)multiplication. Alternatively, the forgetful functor $U_{\top} : \mathcal{C}_{\top} \to \mathcal{C}$ (resp., $U^{\perp} : \mathcal{C}^{\perp} \to \mathcal{C}$) is both separable and naturally full whenever it is fully faithful.

Remark 1.78. [4, Remark 2.5] Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and counit $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$.

i) The adjunctions (F, G) and (V_{GF}, U_{GF}) have the same associated monad $(GF, G\epsilon F, \eta)$, whereas the adjunctions (F, G) and (U^{FG}, V^{FG}) have the same associated comonad $(FG, F\eta G, \epsilon)$.

ii) By *i)*, (F, G) is idempotent if, and only if, (V_{GF}, U_{GF}) is idempotent if, and only if, (U^{FG}, V^{FG}) is idempotent.

iii) The counit of an adjunction coincides with the counit of the associated comonad. Thus, by *ii)*, the adjunctions (F, G) and (U^{FG}, V^{FG}) have the same counit. As a consequence, G is separable (resp., naturally full, fully faithful) if, and only if, so is V^{FG}, in view of the corresponding Rafael-type Theorems (Theorem 1.18, Theorem 1.29 and Theorem 1.31, respectively).

iv) Similarly, the adjunctions (F, G) and (V_{GF}, U_{GF}) have the same unit and hence F is separable (resp., naturally full, fully faithful) if, and only if, so is V_{GF}.

In Proposition 1.57 we have shown that the reverse implication of Proposition 1.56 holds true if M_S is finitely generated and projective. An alternative way to see this fact (see [4, Remark 3.28]) follows by the previous remark. In fact, from Example 1.72 we know that $K_{\sigma_*\sigma^*} \circ \sigma^* = \varphi^*$. Hence, by Remark 1.78 *iv*) we get that σ^* is separable if, and only if, $V_{\sigma_*\sigma^*} = \varphi^*$ is separable. By Proposition 1.40 this is equivalent to φ being split-mono as an R-bimodule map, i.e. to the existence of $E \in {}_R\mathrm{Hom}_R(\mathcal{E}, R)$ such that $E \circ \varphi = \mathrm{Id}$.

The next lemma relates separable functors that are part of an adjunction to (co)separable (co)monads.

Lemma 1.79. [31, Cf. Lemma 3.1] *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ.*

i) If F is separable, then the associated comonad $(FG, F\eta G, \epsilon)$ is coseparable.

ii) If G is separable, then the associated monad $(GF, G\epsilon F, \eta)$ is separable.

Proof. i). If F is separable, then by Theorem 1.18 there is a natural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\nu \circ \eta = \mathrm{Id}$. Set $\tau := F\nu G : F\mathrm{Id}_{\mathcal{C}}G \to FGFG$. We have that $\tau \circ F\eta G = F\nu G \circ F\eta G = F(\nu \circ \eta)G = \mathrm{Id}_{FG}$. From naturality of η and ν it follows that $FG\tau \circ F\eta GFG = FGF\nu G \circ F\eta GFG = F\eta G \circ F\nu G = F\nu GFG \circ FGF\eta G = \tau FG \circ FGF\eta G$, hence $(FG, F\eta G, \epsilon)$ is coseparable.
ii). It follows dually. □

In the next chapter we will show that a weakening of the assumptions on the right (resp., left) adjoint in Lemma 1.79 still implies the (co)separability of the associated (co)monad.

Chapter 2

The notion of semiseparability

In this chapter we deal with the notion of semiseparable functor, introduced in [4] by requiring a regularity condition (in the sense of Von Neumann) on its hom-components, and we review its main properties. To any semiseparable functor we attach an invariant which controls the separability of the functor and allows a characterization of separable functors in terms of semiseparability and (dual) Maschke or conservative functors. We show that any semiseparable functor admits a canonical factorization as a naturally full functor followed by a separable one. Then, we study semiseparability for functors that are part of an adjunction and we prove a Rafael-type Theorem. In the context of Eilenberg-Moore categories, we characterize the semiseparability of a right (resp., left) adjoint in terms of the (co)separability of the associated (co)monad and the natural fullness of the corresponding (co)comparison functor. Finally, we look at the notions of (co)reflection, bireflection and Frobenius functor with respect to semiseparability.

2.1 Semiseparable functors

Given functors $F, F' : \mathcal{C} \to \mathcal{D}$, we call a natural transformation $\alpha : F \to F'$ *regular* if there exists a natural transformation $\beta : F' \to F$ such that $\alpha \circ \beta \circ \alpha = \alpha$.

Remark 2.1. A natural transformation $\alpha : F \to F'$ is regular if, and only if, there exists a natural transformation $\beta : F' \to F$ such that $\alpha \circ \beta \circ \alpha = \alpha$ and $\beta \circ \alpha \circ \beta = \beta$. In fact, if α is regular through $\gamma : F' \to F$, then it is enough to set $\beta := \gamma \circ \alpha \circ \gamma$. This is the analogue of the fact that every regular element has an inverse in semigroup theory, see e.g. [54, page 51].

Definition 2.2. [4, Definition 1.1] We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is **semiseparable** if the natural transformation $\mathcal{F} : \mathrm{Hom}_{\mathcal{C}}(-,-) \to \mathrm{Hom}_{\mathcal{D}}(F-, F-)$, $f \mapsto Ff$, is regular. This means that there exists a natural transformation $\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ such that

$$\mathcal{F} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{F}.$$

Remark 2.3. Since $\mathcal{F}^F_{X,Y} = \mathcal{F}^{F^{\mathrm{op}}}_{Y^{\mathrm{op}}, X^{\mathrm{op}}}$ for every $X, Y \in \mathcal{C}$, a functor $F : \mathcal{C} \to \mathcal{D}$ is semiseparable if and only if so is $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$.

Remark 2.4. From Remark 2.1 it follows that a functor $F : \mathcal{C} \to \mathcal{D}$ is semiseparable if, and only if, there exists a natural transformation $\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ such that $\mathcal{F} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{F}$ and $\mathcal{P} \circ \mathcal{F} \circ \mathcal{P} = \mathcal{P}$.

We know that a separable functor is faithful and that a naturally full functor is full. If we add the assumption of semiseparability, also the viceversa holds true.

Proposition 2.5. [4, Proposition 1.3] *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then,*

i) F is separable if, and only if, F is semiseparable and faithful;

ii) F is naturally full if, and only if, F is semiseparable and full.

Proof. *i*). Assume that F is separable. From $\mathcal{P} \circ \mathcal{F} = \mathrm{Id}$ it follows that $\mathcal{F} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{F} \circ \mathrm{Id} = \mathcal{F}$, i.e. F is semiseparable, and by Lemma 1.3 F is faithful. Conversely, if F is semiseparable, then there exists a natural transformation $\mathcal{P}$ such that $\mathcal{F} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{F}$. If F is faithful, then $\mathcal{F}$ is injective on components, so $\mathcal{P} \circ \mathcal{F} = \mathrm{Id}$.
ii). If F is naturally full, then F is semiseparable, as $(\mathcal{F} \circ \mathcal{P}) \circ \mathcal{F} = \mathrm{Id} \circ \mathcal{F} = \mathcal{F}$, and by Remark 1.27 *i*) F is full. Conversely, if F is semiseparable, then there is a natural transformation $\mathcal{P}$ such that $\mathcal{F} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{F}$. So, if F is full, then $\mathcal{F} \circ \mathcal{P} = \mathrm{Id}$, as $\mathcal{F}$ is surjective on components. □

In view of Proposition 2.5, both separable and naturally full functors are instances of semiseparable functors. The first difference, with respect to the separable and naturally full cases, is that semiseparable functors are not closed under composition (we will see an instance in Example 3.4). In the following result we show that the closeness is available in some cases.

Lemma 2.6. [4, Lemma 1.12] *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors and consider the composite $G \circ F : \mathcal{C} \to \mathcal{E}$.*

i) If F is semiseparable and G is separable, then $G \circ F$ is semiseparable.

ii) If F is naturally full and G is semiseparable, then $G \circ F$ is semiseparable.

Proof. *i*). If F is semiseparable with respect to $\mathcal{P}^F$ and G is separable with respect to $\mathcal{P}^G$, then we have

$$\begin{aligned}\mathcal{F}^{GF}_{X,Y}\mathcal{P}^{F}_{X,Y}\mathcal{P}^{G}_{FX,FY}\mathcal{F}^{GF}_{X,Y} &= \mathcal{F}^{GF}_{X,Y}\mathcal{P}^{F}_{X,Y}\mathcal{P}^{G}_{FX,FY}\mathcal{F}^{G}_{FX,FY}\mathcal{F}^{F}_{X,Y}\\ = \mathcal{F}^{GF}_{X,Y}\mathcal{P}^{F}_{X,Y}\mathcal{F}^{F}_{X,Y} = \mathcal{F}^{G}_{FX,FY}\mathcal{F}^{F}_{X,Y}\mathcal{P}^{F}_{X,Y}\mathcal{F}^{F}_{X,Y} &= \mathcal{F}^{G}_{FX,FY}\mathcal{F}^{F}_{X,Y} = \mathcal{F}^{GF}_{X,Y},\end{aligned}$$

for every X, Y in $\mathcal{C}$, hence $G \circ F$ is semiseparable through $\mathcal{P}^{GF}_{X,Y} := \mathcal{P}^{F}_{X,Y}\mathcal{P}^{G}_{FX,FY}$.

ii). It follows similarly. Indeed, if F is naturally full with respect to $\mathcal{P}^F$ and G is semiseparable with respect to $\mathcal{P}^G$, then we have

$$\begin{aligned}\mathcal{F}^{GF}_{X,Y}\mathcal{P}^{F}_{X,Y}\mathcal{P}^{G}_{FX,FY}\mathcal{F}^{GF}_{X,Y} &= \mathcal{F}^{G}_{FX,FY}\mathcal{F}^{F}_{X,Y}\mathcal{P}^{F}_{X,Y}\mathcal{P}^{G}_{FX,FY}\mathcal{F}^{GF}_{X,Y}\\ = \mathcal{F}^{G}_{FX,FY}\mathcal{P}^{G}_{FX,FY}\mathcal{F}^{GF}_{X,Y} = \mathcal{F}^{G}_{FX,FY}\mathcal{P}^{G}_{FX,FY}\mathcal{F}^{G}_{FX,FY}\mathcal{F}^{F}_{X,Y} &= \mathcal{F}^{G}_{FX,FY}\mathcal{F}^{F}_{X,Y} = \mathcal{F}^{GF}_{X,Y},\end{aligned}$$

for every X, Y in $\mathcal{C}$, hence $G \circ F$ is semiseparable with respect to $\mathcal{P}^{GF}_{X,Y} := \mathcal{P}^{F}_{X,Y}\mathcal{P}^{G}_{FX,FY}$. □

Corollary 2.7. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors. If F is naturally full and G is separable, then the composite $G \circ F : \mathcal{C} \to \mathcal{E}$ is semiseparable.*

Proof. By Lemma 2.6 *i*), the composition $G \circ F$, of a naturally full (whence semiseparable) functor F followed by a separable functor G, is semiseparable. □

Lemma 2.8. [4, Lemma 1.13] *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors. If $G \circ F$ is semiseparable and G is faithful, then F is semiseparable.*

Proof. If $G \circ F$ is semiseparable through $\mathcal{P}^{GF}$, then $\mathcal{F}^{GF}_{X,Y} \circ \mathcal{P}^{GF}_{X,Y} \circ \mathcal{F}^{GF}_{X,Y} = \mathcal{F}^{GF}_{X,Y}$, i.e. $\mathcal{F}^{G}_{FX,FY} \circ \mathcal{F}^{F}_{X,Y} \circ \mathcal{P}^{GF}_{X,Y} \circ \mathcal{F}^{G}_{FX,FY} \circ \mathcal{F}^{F}_{X,Y} = \mathcal{F}^{G}_{FX,FY} \circ \mathcal{F}^{F}_{X,Y}$, for every $X, Y \in \mathcal{C}$. Since G is faithful, we have that $\mathcal{F}^{F}_{X,Y} \circ \mathcal{P}^{GF}_{X,Y} \circ \mathcal{F}^{G}_{FX,FY} \circ \mathcal{F}^{F}_{X,Y} = \mathcal{F}^{F}_{X,Y}$, for every X, Y in $\mathcal{C}$, so F is semiseparable through $\mathcal{P}^{F}_{X,Y} := \mathcal{P}^{GF}_{X,Y} \circ \mathcal{F}^{G}_{FX,FY}$. □

In Proposition 2.58 we will prove that, under stronger assumptions on F, the functor G results to be semiseparable whenever $G \circ F$ is. In the next proposition we show that semiseparable functors are closed under isomorphisms (cf. [4, Corollary 1.11]).

Proposition 2.9. *A functor naturally isomorphic to a semiseparable functor is semiseparable.*

Proof. Let $\alpha : F \to G$ be a natural isomorphism of functors, where $G : \mathcal{C} \to \mathcal{D}$ is semiseparable with respect to $\mathcal{P}^G$. Consider the map $\varsigma_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(FX, FY) \to \mathrm{Hom}_{\mathcal{D}}(GX, GY)$ defined by $\varsigma_{X,Y}(f) = \alpha_Y \circ f \circ \alpha_X^{-1}$, cf. the proof of [11, Lemma 1.7]. Note that $\mathcal{F}^{F}_{X,Y} = \varsigma^{-1}_{X,Y} \circ \mathcal{F}^{G}_{X,Y}$ and then $\mathcal{F}^{G}_{X,Y} = \varsigma_{X,Y} \circ \mathcal{F}^{F}_{X,Y}$, for every $X, Y \in \mathcal{C}$. In fact, $\mathcal{F}^{F}_{X,Y}(f) = Ff = \mathrm{Id}_{FY} \circ Ff = \alpha_Y^{-1} \circ \alpha_Y \circ Ff = \alpha_Y^{-1} \circ Gf \circ \alpha_X = \varsigma^{-1}_{X,Y}(Gf) = (\varsigma^{-1}_{X,Y} \circ \mathcal{F}^{G}_{X,Y})(f)$ and, by composing the latter by $\varsigma_{X,Y}$, the equality $\mathcal{F}^{G}_{X,Y} = \varsigma_{X,Y} \circ \mathcal{F}^{F}_{X,Y}$ follows. We show that F results to be semiseparable with respect to $\mathcal{P}^{F}_{X,Y} := \mathcal{P}^{G}_{X,Y} \circ \varsigma_{X,Y}$. Indeed, we have $\mathcal{F}^{F}_{X,Y} \circ \mathcal{P}^{F}_{X,Y} \circ \mathcal{F}^{F}_{X,Y} = \varsigma^{-1}_{X,Y} \circ \mathcal{F}^{G}_{X,Y} \circ \mathcal{P}^{G}_{X,Y} \circ \varsigma_{X,Y} \circ \mathcal{F}^{F}_{X,Y} = \varsigma^{-1}_{X,Y} \circ \mathcal{F}^{G}_{X,Y} \circ \mathcal{P}^{G}_{X,Y} \circ \mathcal{F}^{G}_{X,Y} = \varsigma^{-1}_{X,Y} \circ \mathcal{F}^{G}_{X,Y} = \mathcal{F}^{F}_{X,Y}$. □

We recall here the following lemma, inspired by [10, Lemma 2.9], that will be useful in the sequel.

Lemma 2.10. [5, Lemma 3.9] *Let $F \dashv G : \mathcal{C} \to \mathcal{D}$ be an adjunction of functors and let $S : \mathcal{C}' \to \mathcal{C}$ and $T : \mathcal{D}' \to \mathcal{D}$ be fully faithful functors. Assume that there exist functors $F' : \mathcal{D}' \to \mathcal{C}'$ and $G' : \mathcal{C}' \to \mathcal{D}'$ such that both squares*

$$\begin{array}{ccc} \mathcal{C}' & \xrightarrow{\quad S \quad} & \mathcal{C} \\ {\scriptstyle F'}\uparrow\ \downarrow{\scriptstyle G'} & & {\scriptstyle F}\uparrow \dashv \downarrow{\scriptstyle G} \\ \mathcal{D}' & \xrightarrow{\quad T \quad} & \mathcal{D} \end{array}$$

are commutative, i.e. $F \circ T = S \circ F'$ and $T \circ G' = G \circ S$. Then, (F', G') is an adjunction in a unique way such that the pair of functors (S, T) is a map of adjunctions (cf. Subsection 1.1.1). Moreover, if G (respectively, F) is (semi)separable, then also G' (respectively, F') is (semi)separable.

Proof. Consider $D' \in \mathcal{D}'$, $C' \in \mathcal{C}'$. The composition of natural isomorphisms yields the natural isomorphism $\varphi_{D',C'} := (\mathcal{F}^{T}_{D',G'C'})^{-1} \circ \Phi_{TD',SC'} \circ \mathcal{F}^{S}_{F'D',C'}$, where Φ is the natural isomorphism (1.3). By construction the diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{C}'}(F'D', C') & \xrightarrow{\mathcal{F}^{S}_{F'D',C'}} & \mathrm{Hom}_{\mathcal{C}}(SF'D', SC') & = & \mathrm{Hom}_{\mathcal{C}}(FTD', SC') \\ \downarrow{\scriptstyle \varphi_{D',C'}} & & & & \downarrow{\scriptstyle \Phi_{TD',SC'}} \\ \mathrm{Hom}_{\mathcal{D}'}(D', G'C') & \xrightarrow{\mathcal{F}^{T}_{D',G'C'}} & \mathrm{Hom}_{\mathcal{D}}(TD', TG'C') & = & \mathrm{Hom}_{\mathcal{D}}(TD', GSC') \end{array}$$

commutes and this means that the pair of functors (S, T) is a map of adjunctions. Finally, assume that G is semiseparable. Since S is fully faithful, by Lemma 2.6 *ii)* $G \circ S$ is semiseparable, and then $T \circ G'$ is semiseparable, hence, since T is faithful, by Lemma 2.8

it follows that also G' is semiseparable. If G is separable, the proof follows analogously. The case with F and F' is similar. $\square$

2.1.1 An invariant associated with semiseparability

In this subsection we show that it is possible to attach, in a canonical way, a suitable idempotent natural transformation to any semiseparable functor. Recall that an endomorphism $e_X : X \to X$ in a category $\mathcal{C}$ is *idempotent* if $e_X^2 := e_X \circ e_X = e_X$. A natural transformation $\alpha : F \to F'$ is *idempotent* if $\alpha_X : FX \to F'X$ is idempotent for every $X \in \mathcal{C}$.

Proposition 2.11. [4, Proposition 1.4] *Let $F : \mathcal{C} \to \mathcal{D}$ be a semiseparable functor. Then, there is a unique idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ such that $Fe = \mathrm{Id}_F$, with the following universal property: if $f, g : X \to Y$ are morphisms in $\mathcal{C}$, then*

$$Ff = Fg \quad \text{if, and only if,} \quad e_Y \circ f = e_Y \circ g. \tag{2.1}$$

Proof. Since F is semiseparable, there is a natural transformation $\mathcal{P}$ such that $\mathcal{F} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{F}$. Set $e_X := \mathcal{P}_{X,X}(\mathrm{Id}_{FX})$. Note that $Fe_X = F\mathcal{P}_{X,X}(\mathrm{Id}_{FX}) = \mathcal{F}_{X,X}\mathcal{P}_{X,X}\mathcal{F}_{X,X}(\mathrm{Id}_X) = \mathcal{F}_{X,X}(\mathrm{Id}_X) = \mathrm{Id}_{FX}$. Thus,

$$e_X \circ e_X = \mathcal{P}_{X,X}(\mathrm{Id}_{FX}) \circ e_X = \mathcal{P}_{X,X}(\mathrm{Id}_{FX} \circ Fe_X) = \mathcal{P}_{X,X}(\mathrm{Id}_{FX}) = e_X,$$

hence e_X is idempotent, for every $X \in \mathcal{C}$. Moreover, for every morphism $f : X \to Y$ in $\mathcal{C}$ we have $f \circ e_X = f \circ \mathcal{P}_{X,X}(\mathrm{Id}_{FX}) = \mathcal{P}_{X,Y}(Ff \circ \mathrm{Id}_{FX}) = \mathcal{P}_{X,Y}(\mathrm{Id}_{FY} \circ Ff) = \mathcal{P}_{Y,Y}(\mathrm{Id}_{FY}) \circ f = e_Y \circ f$, so that $f \circ e_X = e_Y \circ f$, i.e., $e = (e_X)_{X \in \mathcal{C}} : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ is an idempotent natural transformation such that $Fe = \mathrm{Id}_F$. Now, consider morphisms $f, g : X \to Y$ in $\mathcal{C}$. If $Ff = Fg$, then $\mathcal{P}_{X,Y}(Ff) = \mathcal{P}_{X,Y}(Fg)$, i.e. $\mathcal{P}_{Y,Y}(\mathrm{Id}_{FY}) \circ f = \mathcal{P}_{Y,Y}(\mathrm{Id}_{FY}) \circ g$, i.e. $e_Y \circ f = e_Y \circ g$. Conversely, from $e_Y \circ f = e_Y \circ g$ we get $Fe_Y \circ Ff = Fe_Y \circ Fg$, and hence $Ff = Fg$ as $Fe = \mathrm{Id}_F$. Finally, let $e' : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be an idempotent natural transformation such that, if $f, g : X \to Y$ are morphisms in $\mathcal{C}$, then $Ff = Fg$ if and only if $e'_Y \circ f = e'_Y \circ g$. From $e'_X \circ e'_X = e'_X \circ \mathrm{Id}_X$ we get $Fe'_X = F\mathrm{Id}_X$ (whence $Fe' = \mathrm{Id}_F$). From (2.1) we have $e_X \circ e'_X = e_X \circ \mathrm{Id}_X$, i.e. $e_X \circ e'_X = e_X$. If we interchange the roles of e and e', in a similar way we get $e'_X \circ e_X = e'_X$. By naturality, for every $X \in \mathcal{C}$, we have $e_X = e_X \circ e'_X = e'_X \circ e_X = e'_X$, hence $e = e'$. $\square$

Thus, to any semiseparable functor we have attached an idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ defined on components by $e_X := \mathcal{P}_{X,X}(\mathrm{Id}_{FX})$, for every $X \in \mathcal{C}$, where $\mathcal{P}$ is any natural transformation such that $\mathcal{F} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{F}$. We call e the **associated idempotent natural transformation**. As particular cases, any separable and any naturally full functor admits the associated idempotent natural transformation. Moreover, the associated idempotent provides a criterion to establish if a semiseparable functor is separable.

Corollary 2.12. [4, Corollary 1.7] *Let $F : \mathcal{C} \to \mathcal{D}$ be a semiseparable functor and let $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be the associated idempotent natural transformation. Then, F is separable if, and only if, $e = \mathrm{Id}$.*

Proof. If F is separable, then $\mathcal{P} \circ \mathcal{F} = \mathrm{Id}$ and hence $e_X = \mathcal{P}_{X,X}(\mathrm{Id}_{FX}) = \mathcal{P}_{X,X}\mathcal{F}_{X,X}(\mathrm{Id}_X) = \mathrm{Id}_X$, for every $X \in \mathcal{C}$. Conversely, suppose $e = \mathrm{Id}$. For every $f : X \to Y$ in $\mathcal{C}$, we have $\mathcal{P}_{X,Y}(Ff) = \mathcal{P}_{X,Y}(Ff \circ \mathrm{Id}_{FX}) = f \circ \mathcal{P}_{X,X}(\mathrm{Id}_{FX}) = f \circ e_X = f$, so that $\mathcal{P} \circ \mathcal{F} = \mathrm{Id}$ and F is separable. $\square$

The following result highlights a connection between semiseparable functors and separability of the second kind, recalled in Subsection 1.1.2.

Proposition 2.13. [4, Cf. Proposition 1.10] *Let $H : \mathcal{C} \to \mathcal{E}$ be a semiseparable functor with associated idempotent natural transformation e and let $F : \mathcal{C} \to \mathcal{D}$ be a H-separable functor. If $Fe = \mathrm{Id}_F$ (e.g. $\mathcal{P}^{F,H}$ is injective on components), then F is semiseparable, with associated idempotent natural transformation e.*

Proof. By definition $\mathcal{P}^{F,H} \circ \mathcal{F}^F = \mathcal{F}^H$. Since H is semiseparable, there exists a natural transformation $\mathcal{P}^H : \mathrm{Hom}_\mathcal{E}(H-, H-) \to \mathrm{Hom}_\mathcal{C}(-,-)$ such that $\mathcal{F}^H \circ \mathcal{P}^H \circ \mathcal{F}^H = \mathcal{F}^H$. Set $\mathcal{P}^F := \mathcal{P}^H \circ \mathcal{P}^{F,H}$, for every X, Y in $\mathcal{C}$. Then, $\mathcal{P}^F \circ \mathcal{F}^F = \mathcal{P}^H \circ \mathcal{P}^{F,H} \circ \mathcal{F}^F = \mathcal{P}^H \circ \mathcal{F}^H$. Thus, for every $f : FX \to FY$, we have $\mathcal{P}^F_{X,Y}\mathcal{F}^F_{X,Y}(f) = \mathcal{P}^H_{X,Y}\mathcal{F}^H_{X,Y}(f) = \mathcal{P}^H_{X,Y}(Hf) = f \circ \mathcal{P}^H_{X,X}(\mathrm{Id}_{HX}) = f \circ e_X$, and hence $\mathcal{F}^F_{X,Y}\mathcal{P}^F_{X,Y}\mathcal{F}^F_{X,Y}(f) = F(f \circ e_X) = Ff \circ Fe_X = Ff = \mathcal{F}^F_{X,Y}(f)$ so that $\mathcal{F}^F_{X,Y}\mathcal{P}^F_{X,Y}\mathcal{F}^F_{X,Y} = \mathcal{F}^F_{X,Y}$, i.e. F is semiseparable. If $\mathcal{P}^{F,H}$ is injective on components, then from $\mathcal{P}^{F,H}_{X,X}(Fe_X) = \mathcal{P}^{F,H}_{X,X}\mathcal{F}^F_{X,X}(e_X) = \mathcal{F}^H_{X,X}(e_X) = He_X = H\mathrm{Id}_X = \mathcal{F}^H_{X,X}(\mathrm{Id}_X) = \mathcal{P}^{F,H}_{X,X}(F\mathrm{Id}_X)$ we infer $Fe_X = \mathrm{Id}_{FX}$. Note that F has the same idempotent natural transformation associated with H. Indeed, if e' is the idempotent natural transformation associated with F, then, for every $X \in \mathcal{C}$, we have that $e'_X = \mathcal{P}^F_{X,X}(\mathrm{Id}_{FX}) = \mathcal{P}^H_{X,X}\mathcal{P}^{F,H}_{X,X}(\mathrm{Id}_{FX}) = \mathcal{P}^H_{X,X}\mathcal{P}^{F,H}_{X,X}\mathcal{F}^F_{X,X}(\mathrm{Id}_X) = \mathcal{P}^H_{X,X}\mathcal{F}^H_{X,X}(\mathrm{Id}_X) = \mathcal{P}^H_{X,X}(\mathrm{Id}_{HX}) = e_X$. □

Corollary 2.14. [4, Corollary 1.11] *Let $H : \mathcal{C} \to \mathcal{E}$ be a semiseparable functor with associated idempotent natural transformation e and assume H is a retract of a functor $F : \mathcal{C} \to \mathcal{D}$. If $Fe = \mathrm{Id}_F$, then F is semiseparable, with associated idempotent natural transformation e.*

Proof. Since H is a retract of F, there are natural transformations $\varphi : F \to H$ and $\psi : H \to F$ such that $\varphi \circ \psi = \mathrm{Id}_H$. Define $\mathcal{P}^{F,H}$ by setting $\mathcal{P}^{F,H}_{X,Y}(g) := \varphi_Y \circ g \circ \psi_X$, for every $g : FX \to FY$, and note that $\mathcal{P}^{F,H} \circ \mathcal{F}^F = \mathcal{F}^H$, so that $F : \mathcal{C} \to \mathcal{D}$ is H-separable. Thus, by Proposition 2.13, if $Fe = \mathrm{Id}_F$, the functor F is semiseparable. □

Remark 2.15. Note that Lemma 2.8 follows also from Proposition 2.13 by setting, for every X, Y in $\mathcal{C}$, $\mathcal{P}^{F,GF}_{X,Y} := \mathcal{F}^G_{FX,FY}$, which is injective, as G is faithful. Moreover, Lemma 2.9 is a consequence of Corollary 2.14.

As a particular case of Corollary 2.14, the next result gives a sufficient condition for the semiseparability of a functor, extending [31, Lemma 2.1 (4)] for the separable case. Cf. also [67, Proposition 2.20].

Corollary 2.16. *Let $F : \mathcal{C} \to \mathcal{C}$ be a functor on a category $\mathcal{C}$. If there exist natural transformations $\alpha : \mathrm{Id}_\mathcal{C} \to F$, $\alpha' : F \to \mathrm{Id}_\mathcal{C}$ such that $F(\alpha' \circ \alpha) = \mathrm{Id}_F$, then $F : \mathcal{C} \to \mathcal{C}$ is semiseparable.*

Proof. Set $e := \alpha' \circ \alpha : \mathrm{Id}_\mathcal{C} \to \mathrm{Id}_\mathcal{C}$. Then, $Fe = F(\alpha' \circ \alpha) = \mathrm{Id}_F$. Since $e \circ e = \alpha' \circ \alpha \circ e = \alpha' \circ Fe \circ \alpha = \alpha' \circ \alpha = e$ and $\mathrm{Id}_\mathcal{C}$ is semiseparable, we conclude by Corollary 2.14. □

The existence of the associated idempotent natural transformation allows to describe separable functors in terms of reflectivity conditions.

Corollary 2.17. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then, the following assertions are equivalent:*

(*i*) *F is separable;*

(*ii*) *F is semiseparable and reflects monomorphisms;*

(*iii*) *F is semiseparable and reflects epimorphisms.*

Proof. By Proposition 2.5 i), a separable functor is semiseparable and faithful. Moreover, a faithful functor reflects monomorphisms (resp., epimorphisms), see [19, Proposition 1.7.6] (resp., [19, Proposition 1.8.4]). Conversely, if F is semiseparable through associated idempotent natural transformation e such that $Fe = \mathrm{Id}_F$, then Fe_X is an isomorphism, whence a monomorphism and an epimorphism, for every $X \in \mathcal{C}$. Thus, if F reflects monomorphisms (resp., epimorphisms), then e_X is a monomorphism (resp., an epimorphism), for every $X \in \mathcal{C}$. From $e_X \circ e_X = e_X = e_X \circ \mathrm{Id}_X$ (resp., $e_X \circ e_X = e_X = \mathrm{Id}_X \circ e_X$) we get $e_X = \mathrm{Id}_X$, so that F is separable by Corollary 2.12. □

A further characterization of separable functors can be also given in terms of Maschke, dual Maschke and conservative functors, that have been recalled in Section 1.1.

Corollary 2.18. [4, Corollary 1.9] *The following assertions are equivalent for a functor $F : \mathcal{C} \to \mathcal{D}$:*

(*i*) *F is separable;*

(*ii*) *F is semiseparable and Maschke;*

(*iii*) *F is semiseparable and dual Maschke;*

(*iv*) *F is semiseparable and conservative.*

Proof. $(i) \Rightarrow (ii), (iii), (iv)$. By Proposition 2.5 i), a separable functor is semiseparable. Moreover, by Remark 1.12, a separable functor is both Maschke and dual Maschke, whence conservative.

$(ii), (iii), (iv) \Rightarrow (i)$. Since F is semiseparable, we can consider its associated idempotent natural transformation e such that $Fe_X = \mathrm{Id}_{FX}$, for every object X in $\mathcal{C}$. Thus Fe_X is split-mono, split-epi and iso. Depending on whether F is either Maschke, dual Maschke or conservative, we get that e_X is either split-mono, split-epi or iso. Since e_X is idempotent, we get $e_X = \mathrm{Id}_X$, so that F is separable by Corollary 2.12. □

Example 2.19. The restriction of scalars functor $F : \mathfrak{M}_{\mathbb{L}} \to \mathfrak{M}$ in Example 1.13 is an instance of a Maschke (and dual Maschke) functor that is not semiseparable.

We recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is said to *reflect coequalizers* if, when

$$X \underset{v}{\overset{u}{\rightrightarrows}} Y \xrightarrow{f} Z \tag{2.2}$$

is a diagram such that $FX \underset{Fv}{\overset{Fu}{\rightrightarrows}} FY \xrightarrow{Ff} FZ$ is a coequalizer, then (2.2) is a coequalizer as well. Similarly, F is said to *reflect equalizers*. As a consequence of Corollary 2.18, the notions of separability and semiseparability coincide in these cases, as well.

Lemma 2.20. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories with (co)equalizers. If F reflects (co)equalizers, then F is conservative.*

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories with coequalizers that reflects coequalizers. Let $f : X \to Y$ be a morphism in $\mathcal{C}$ such that $Ff : FX \to FY$ is an isomorphism in $\mathcal{D}$ with inverse $h : FY \to FX$. Then, Ff is the coequalizer of the parallel pair $(\mathrm{Id}_{FX}, \mathrm{Id}_{FX})$. Indeed, $Ff \circ \mathrm{Id}_{FX} = Ff \circ \mathrm{Id}_{FX}$ trivially holds and, for every $g : FX \to Z$ in $\mathcal{D}$, by setting $t := g \circ h$ one has $g \circ \mathrm{Id}_{FX} = g \circ \mathrm{Id}_{FX}$ and $t \circ Ff = g \circ h \circ Ff = g \circ \mathrm{Id}_{FX} = g$. Since F reflects coequalizers, f is the coequalizer of $(\mathrm{Id}_X, \mathrm{Id}_X)$. Then, there exists $k : Y \to X$ such that $k \circ f = \mathrm{Id}_X$. Moreover, since f is an epimorphism, from $f \circ k \circ f = f$ we get $f \circ k = \mathrm{Id}_Y$, hence f is an isomorphism. The case for a functor between categories with equalizers that reflects equalizers follows by duality. □

Corollary 2.21. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories with (co)equalizers. If F reflects (co)equalizers, then F is separable if, and only if, F is semiseparable.*

Proof. If F is separable, then it is semiseparable by Proposition 2.5 i). Conversely, if F reflects (co)equalizers, by Lemma 2.20 F is conservative, so, if F is semiseparable, by Corollary 2.18 F is separable. □

In the next results we show other situations in which the notion of semiseparable functor collapses into the one of separable functor. For instance, this is the case when there exists a suitable type of generator within a category $\mathcal{C}$.

We recall, from [47, Definition 7], that a morphism $k : X \to Y$ in a category $\mathcal{C}$ is *constant* if, for each object Z in $\mathcal{C}$ and for each pair of morphisms $g, h : Z \to X$, one has $k \circ g = k \circ h$. A category $\mathcal{C}$ is said to be *constant-generated* [47, Definition 8] provided that, for each pair of objects (X, Y) in $\mathcal{C}$,

i) $\mathrm{Hom}_{\mathcal{C}}(X, Y) \neq \emptyset$,

ii) if $f, g : X \to Y$ are distinct morphisms in $\mathcal{C}$, then there exist an object K and a constant morphism $k : K \to X$ in $\mathcal{C}$ such that $f \circ k \neq g \circ k$.

Several categories are constant-generated, e.g. the categories of nonempty sets, nonempty ordered sets, nonempty topological spaces, nonempty Hausdorff spaces etc. Here, in the definition of constant-generated category, we do not require condition i).

Proposition 2.22. [4, Proposition 1.17] *Let $\mathcal{C}$ be a constant-generated category. Then,* $\mathrm{Nat}(\mathrm{Id}_{\mathcal{C}}, \mathrm{Id}_{\mathcal{C}}) = \{\mathrm{Id}\}$. *As a consequence, a functor $F : \mathcal{C} \to \mathcal{D}$ is semiseparable if, and only if, it is separable.*

Proof. Let $e \in \mathrm{Nat}(\mathrm{Id}_{\mathcal{C}}, \mathrm{Id}_{\mathcal{C}})$ and suppose that $e_X \neq \mathrm{Id}_X$, for some object X in $\mathcal{C}$. Since $\mathcal{C}$ is constant-generated, there are an object K and a constant morphism $k : K \to X$ such that $e_X \circ k \neq \mathrm{Id}_X \circ k$. By naturality of e and since k is constant, we have $e_X \circ k = k \circ e_K = k \circ \mathrm{Id}_K = \mathrm{Id}_X \circ k$, a contradiction. Therefore $e_X = \mathrm{Id}_X$, and hence $e = \mathrm{Id}$. We conclude by Corollary 2.12. □

A set $\{\mathfrak{G}_i\}_i$ of objects in a category $\mathcal{C}$ is said to be a *set of generators of* $\mathcal{C}$ [76, page 5] if, for every pair of objects (X, Y) in $\mathcal{C}$ and for every pair of distinct morphisms $f, g : X \to Y$ in $\mathcal{C}$, there is an index i_0 and a morphism $p : \mathfrak{G}_{i_0} \to X$ such that $f \circ p \neq g \circ p$. In particular, a category has a *generator* if it has a set of generators consisting of a single object. For instance, in the category Set of sets the one-point set is a generator, in the category Ab of abelian groups $\mathbb{Z}$ is a generator, etc. Any Grothendieck category has a generator by definition, see e.g. [81, Chapter V]. If the domain of a functor is a category with a generator, instead of a constant-generated category, it is not obvious that semiseparability and separability coincide. By adding suitable assumptions, it is possible to retrieve the same conclusion.

A first example in this direction is given by a *well-pointed* category, i.e. a category that has a generator which is at the same time a terminal object. For instance, the categories Set of sets, Top of topological spaces, Comp of compact Hausdorff spaces are well-pointed. In fact, the singleton $\{*\}$ is both a terminal object and a generator in all of these categories, see [19, 2.3.2.a, 2.1.7g, 4.5.17.a, 4.5.17.f and 4.5.17.g].

Corollary 2.23. [4, Corollary 1.18] *If $\mathcal{C}$ is a well-pointed category, then it is constant-generated. As a consequence, a functor $F : \mathcal{C} \to \mathcal{D}$ is semiseparable if, and only if, it is separable.*

Proof. Let $\mathfrak{G} \in \mathcal{C}$ be a generator which is a terminal object. Given morphisms $f, g : X \to Y$ in $\mathcal{C}$ such that $f \neq g$, since $\mathfrak{G}$ is a generator there is a morphism $k : \mathfrak{G} \to X$ such that $f \circ k \neq g \circ k$. Moreover, if $h, h' : Z \to \mathfrak{G}$ are any morphisms in $\mathcal{C}$, then, since $\mathfrak{G}$ is a terminal object, they are necessarily equal, whence $k \circ h = k \circ h'$, so k is constant. Thus, $\mathcal{C}$ is constant-generated. We conclude by Proposition 2.22. □

Remark 2.24. In view of Proposition 2.22 and Corollary 2.23, we get that in a well-pointed category $\mathcal{C}$ there is no non-trivial idempotent natural transformation on $\mathrm{Id}_{\mathcal{C}}$. This result already appeared in [39, Corollary 21].

In the following proposition we give another additional condition that guarantees the equivalence between the semiseparability and the separability of a functor. By a *central idempotent endomorphism* of an object X in a category $\mathcal{C}$, we mean a central idempotent in the monoid $(\mathrm{End}(X), \circ, \mathrm{Id}_X)$, i.e. a morphism $g : X \to X$ in $\mathcal{C}$ such that $g \circ g = g$ and $g \circ f = f \circ g$, for every morphism $f : X \to X$ in $\mathcal{C}$.

Proposition 2.25. [4, Proposition 1.21] *Let $\mathcal{C}$ be a category with a generator $\mathfrak{G}$ and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Assume there is no central idempotent endomorphism $g \neq \mathrm{Id}_{\mathfrak{G}} : \mathfrak{G} \to \mathfrak{G}$ such that $Fg = \mathrm{Id}_{F\mathfrak{G}}$. Then, F is semiseparable if, and only if, it is separable.*

Proof. Consider an idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ such that $Fe = \mathrm{Id}_F$. Then, $e_{\mathfrak{G}}$ is a central idempotent endomorphism of $\mathfrak{G}$ such that $Fe_{\mathfrak{G}} = \mathrm{Id}_{F\mathfrak{G}}$, and hence $e_{\mathfrak{G}} = \mathrm{Id}_{\mathfrak{G}}$ by assumption. Let X be an object in $\mathcal{C}$ and suppose that $e_X \neq \mathrm{Id}_X$. Since $\mathfrak{G}$ is a generator, there is a morphism $p : \mathfrak{G} \to X$ such that $e_X \circ p \neq \mathrm{Id}_X \circ p$ but, by naturality of e, we have $e_X \circ p = p \circ e_{\mathfrak{G}} = p \circ \mathrm{Id}_{\mathfrak{G}} = p$, so that we are led to a contradiction. Therefore $e_X = \mathrm{Id}_X$, and hence $e = \mathrm{Id}$. We conclude by Corollary 2.12. □

We exhibit an application of Proposition 2.25 to the category ${}_R\mathcal{M}$ of left R-modules.

Lemma 2.26. [4, Lemma 1.22] *Let R be a ring. Then, $g : R \to R$ is a central idempotent endomorphism of left R-modules if, and only if, $g = z\mathrm{Id}_R$ for a central idempotent $z \in R$, namely $z = g(1)$.*

Proof. For every $r \in R$, consider the morphism of left R-modules $f_r : R \to R, f_r(x) := xr$. Assume that g is a central idempotent endomorphism of left R-modules. Then, $g(r) = rg(1) = rz$. From $g \circ f_r = f_r \circ g$ we get $zr = f_r(z) = f_r(g(1)) = g(f_r(1)) = g(r) = rz$, so that z is in the center of R. Moreover, since g is left R-linear and idempotent, we get $zz = g(z) = g(g(1)) = g(1) = z$. Conversely, it is clear that $z\mathrm{Id}_R : R \to R$ is a central idempotent endomorphism of left R-modules in case z is a central idempotent in R. □

Corollary 2.27. [4, Corollary 1.23] *Let R be a ring with no non-trivial central idempotent (e.g. R is a domain). A functor $F : {}_R\mathcal{M} \to \mathcal{D}$ such that $F0 \neq \mathrm{Id}_{FR}$ is semiseparable if, and only if, it is separable.*

Proof. Let $g : R \to R$ be a central idempotent endomorphism of left R-modules such that $Fg = \mathrm{Id}_{FR}$. By Lemma 2.26, we have that $g = z\mathrm{Id}_R$ for a central idempotent $z \in R$. By assumption z is trivial, i.e. $z = 1$ or $z = 0$, and hence we get either $g = \mathrm{Id}_R$ or $g = 0$. Since $Fg = \mathrm{Id}_{FR}$, we must have $g = \mathrm{Id}_R$. Since R is a generator in ${}_R\mathcal{M}$, by Proposition 2.25, we conclude. □

2.1.2 A canonical factorization

To any semiseparable functor it is possible to attach a canonical factorization through its associated coidentifier category. We recall this notion defined as in [39, Example 17].

Given a category $\mathcal{C}$ and an idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$, consider the quotient category $\mathcal{C}_e := \mathcal{C}/\sim$ of $\mathcal{C}$, where $\sim$ is the congruence relation on the hom-sets defined, for all $f, g : X \to Y$ in $\mathcal{C}$, by setting

$$f \sim g \quad \text{if, and only if,} \quad e_Y \circ f = e_Y \circ g.$$

Thus, $\mathrm{Ob}(\mathcal{C}_e) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}_e}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)/\sim$. The category $\mathcal{C}_e$ is called the *coidentifier category*. We denote by $\overline{f}$ the class of $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ in $\mathrm{Hom}_{\mathcal{C}_e}(X, Y)$. We have the quotient functor

$$H : \mathcal{C} \to \mathcal{C}_e, \quad X \mapsto X, \quad f \mapsto \overline{f},$$

that satisfies the following properties. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *essentially surjective on objects* (*eso* for short) if, for every $D \in \mathcal{D}$, there is $C \in \mathcal{C}$ such that $D \cong F(C)$.

Lemma 2.28. *Given a category $\mathcal{C}$ and an idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$, consider the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$. Then,*

i) H is naturally full and e is its associated idempotent natural transformation. In particular, $He = \mathrm{Id}_H$.

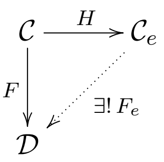
ii) H is essentially surjective on objects.

Proof. i). Consider the map $\mathcal{P}_{X,Y} : \mathrm{Hom}_{\mathcal{C}_e}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$, given by $\mathcal{P}_{X,Y}(\overline{f}) = e_Y \circ f$. It is a natural transformation as for any $h : X' \to X$, $k : Y \to Y'$ in $\mathcal{C}$, $\overline{f} : X \to Y$ in $\mathcal{C}_e$, from naturality of e we have that $k \circ \mathcal{P}_{X,Y}(\overline{f}) \circ h = k \circ (e_Y \circ f) \circ h = e_{Y'} \circ (k \circ f \circ h) = \mathcal{P}_{X',Y'}(\overline{k \circ f \circ h}) = \mathcal{P}_{X',Y'}(\overline{k} \circ \overline{f} \circ \overline{h}) = \mathcal{P}_{X',Y'}(H(k) \circ \overline{f} \circ H(h))$. Moreover, $\mathcal{F}_{X,Y}\mathcal{P}_{X,Y}(\overline{f}) = \mathcal{F}_{X,Y}(e_Y \circ f) = \overline{e_Y \circ f} = \overline{f} = \mathrm{Id}_{\mathrm{Hom}_{\mathcal{C}_e}(X,Y)}(\overline{f})$, for every $\overline{f} : X \to Y$ in $\mathcal{C}_e$, hence H is naturally full. Since $\mathcal{P}_{X,X}(\overline{\mathrm{Id}_X}) = e_X \circ \mathrm{Id}_X = e_X$, for every $X \in \mathcal{C}$, the idempotent natural transformation associated with H is exactly e. Thus, $He = \mathrm{Id}_H$.
ii). Since every object $X \in \mathcal{C}_e$ is an object in $\mathcal{C}$ and H acts as the identity on objects, we have that $H(X) = X$, hence H is essentially surjective on objects. □

Remark 2.29. Let $\mathcal{C}$ be a category and consider the idempotent natural transformation $e = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}} : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$. Then, the congruence relation $\sim$ becomes the identity relation, so the quotient category $\mathcal{C}_{\mathrm{Id}} := \mathcal{C}/\sim$ of $\mathcal{C}$ can be identified with $\mathcal{C}$ and the quotient functor $H : \mathcal{C} \to \mathcal{C}_{\mathrm{Id}}$ results to be the identity functor.

The following is the universal property of the coidentifier that can be deduced from the dual version of [39, Definition 14(1)], cf. [4, Lemma 1.14 (1)].

Lemma 2.30. (Universal property) *A functor $F : \mathcal{C} \to \mathcal{D}$ satisfies $Fe = \mathrm{Id}_F$ if, and only if, there is a functor $F_e : \mathcal{C}_e \to \mathcal{D}$ (necessarily unique) such that the triangle*



commutes, i.e. $F = F_e \circ H$.

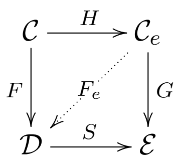
Given $F, F' : \mathcal{C} \to \mathcal{D}$ such that $Fe = \mathrm{Id}_F$ and $F'e = \mathrm{Id}_{F'}$, and a natural transformation $\beta : F \to F'$, there is a unique natural transformation $\beta_e : F_e \to F'_e$ such that $\beta = \beta_e H$.

Proof. Assume $Fe = \mathrm{Id}_F$. Given $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we have that $\overline{f} = \overline{g}$ implies $e_Y \circ f = e_Y \circ g$ and hence $Fe_Y \circ Ff = Fe_Y \circ Fg$, i.e. $Ff = Fg$ as $Fe_Y = \mathrm{Id}_{FY}$. Thus we can define the functor

$$F_e : \mathcal{C}_e \to \mathcal{D}, \quad X \to FX, \quad \overline{f} \mapsto Ff.$$

A direct computation shows that $F_e \circ H = F$. Conversely, given a functor $G : \mathcal{C}_e \to \mathcal{D}$ such that $G \circ H = F$, we have $Fe = GHe = G\mathrm{Id}_H = \mathrm{Id}_{GH} = \mathrm{Id}_F$, so that $Fe = \mathrm{Id}_F$ and we can consider F_e. Moreover, $GX = GHX = FX = F_eX$ and $G\overline{f} = GHf = Ff = F_e\overline{f}$, so that $G = F_e$. For every object X in $\mathcal{C}_e$, we have $(\beta_e)_X := \beta_X$. Given $\overline{f} : X \to Y$ in $\mathcal{C}_e$, the naturality of β_e follows from $(\beta_e)_Y \circ F_e\overline{f} = F'_e\overline{f} \circ (\beta_e)_X$, i.e. $\beta_Y \circ Ff = F'f \circ \beta_X$, which is true by naturality of β. If $\alpha : F_e \to F'_e$ is a natural transformation such that $\beta = \alpha H$ then, for every $X \in \mathcal{C}_e$, $\alpha_X = \alpha_{HX} = \beta_X = (\beta_e)_X$, hence $\alpha = \beta_e$. □

Lemma 2.31. [4, Cf. Lemma 1.14 (2)] *Let $\mathcal{C}$ be a category, let $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be an idempotent natural transformation and let $H : \mathcal{C} \to \mathcal{C}_e$ be the quotient functor. Then, H is orthogonal to any faithful functor $S : \mathcal{D} \to \mathcal{E}$, i.e., given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C}_e \to \mathcal{E}$ such that $S \circ F = G \circ H$, then there is a unique functor $F_e : \mathcal{C}_e \to \mathcal{D}$ such that the diagram*



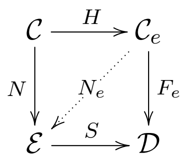
commutes, i.e., $F_e \circ H = F$ and $S \circ F_e = G$.

Proof. For every X in $\mathcal{C}$, we compute $SFe_X = GHe_X = G\mathrm{Id}_{HX} = \mathrm{Id}_{GHX} = \mathrm{Id}_{SFX} = S\mathrm{Id}_{FX}$ so that, since S is faithful, we get that $Fe_X = \mathrm{Id}_{FX}$, and hence $Fe = \mathrm{Id}_F$. Thus, by Lemma 2.30 there is a unique functor $F_e : \mathcal{C}_e \to \mathcal{D}$, such that $F_e \circ H = F$, which acts as F on objects and maps the class $\overline{f}$ into Ff. Moreover, $SF_eX = SF_eHX = SFX = GHX = GX$ and $SF_e\overline{f} = SF_eHf = SFf = GHf = G\overline{f}$, so that $S \circ F_e = G$. □

Remark 2.32. The fact that $H : \mathcal{C} \to \mathcal{C}_e$ is orthogonal to any faithful functor $S : \mathcal{D} \to \mathcal{E}$ follows also since there is an (eso and full, faithful) factorization system in the 2-category Cat of categories, see e.g. [35, Example 7.9], and H is eso and (naturally) full by Lemma 2.28.

In the following result, we show that any semiseparable functor admits a special type of (eso and full, faithful) factorization. In fact, any semiseparable functor $F : \mathcal{C} \to \mathcal{D}$ arises as the composition of a naturally full functor followed by a separable functor: we have $F = F_e \circ H$, where H is eso and (naturally) full, while F_e is separable whence faithful.

Theorem 2.33. [4, Theorem 1.15] *Let $F : \mathcal{C} \to \mathcal{D}$ be a semiseparable functor and let $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be the associated idempotent natural transformation. Then, there is a unique functor $F_e : \mathcal{C}_e \to \mathcal{D}$ (necessarily separable) such that $F = F_e \circ H$, where $H : \mathcal{C} \to \mathcal{C}_e$ is the quotient functor. Furthermore, if F also factors as $S \circ N$, where $S : \mathcal{E} \to \mathcal{D}$ is a separable functor and $N : \mathcal{C} \to \mathcal{E}$ is a naturally full functor, then there is a unique functor $N_e : \mathcal{C}_e \to \mathcal{E}$ (necessarily fully faithful) such that $N_e \circ H = N$ and $S \circ N_e = F_e$, and e is also the idempotent natural transformation associated to N.*



Proof. By Lemma 2.30, there is a unique functor $F_e : \mathcal{C}_e \to \mathcal{D}$ such that $F = F_e \circ H$, where $H : \mathcal{C} \to \mathcal{C}_e$ is the quotient functor. If $F_e\overline{f} = F_e\overline{g}$, then $Ff = Fg$ so that, by Proposition 2.11, we get $e_B \circ f = e_B \circ g$ which means $\overline{f} = \overline{g}$. Thus, F_e is faithful. Moreover, $\mathcal{F}^F_{X,Y} \circ \mathcal{P}^F_{X,Y} \circ \mathcal{F}^F_{X,Y} = \mathcal{F}^F_{X,Y}$ rewrites as $\mathcal{F}^{F_e}_{X,Y} \circ \mathcal{F}^H_{X,Y} \circ \mathcal{P}^F_{X,Y} \circ \mathcal{F}^{F_e}_{X,Y} \circ \mathcal{F}^H_{X,Y} = \mathcal{F}^{F_e}_{X,Y} \circ \mathcal{F}^H_{X,Y}$. Since $\mathcal{F}^{F_e}_{X,Y}$ is injective and $\mathcal{F}^H_{X,Y}$ is surjective (as H is indeed naturally full by Lemma 2.28 i)), we get $\mathcal{F}^H_{X,Y} \circ \mathcal{P}^F_{X,Y} \circ \mathcal{F}^{F_e}_{X,Y} = \mathrm{Id}$, thus F_e is separable.

Concerning the last sentence, since S is separable, then it is faithful. By Lemma 2.31 H is orthogonal to S, so that there is a unique functor $N_e : \mathcal{C}_e \to \mathcal{E}$ such that $N_e \circ H = N$ and $S \circ N_e = F_e$. Since H acts as the identity on objects and N is full, from $N_e \circ H = N$ we deduce that N_e is full. Since $S \circ N_e = F_e$ and F_e is faithful, we deduce that N_e is faithful. Thus, N_e is fully faithful.

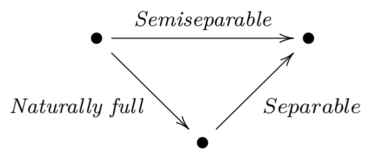
We show that F and N share the same associated idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$. Indeed, since N is naturally full (hence semiseparable), it has an associated idempotent natural transformation $e' : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ and by definition we have $e'_X := \mathcal{P}^N_{X,X}(\mathrm{Id}_{NX})$, for any $X \in \mathcal{C}$. Since $F = S \circ N$, by the proof of Lemma 2.6 i), we can choose $\mathcal{P}^F_{X,Y} := \mathcal{P}^N_{X,Y} \circ \mathcal{P}^S_{NX,NY}$, so that $e_X = \mathcal{P}^F_{X,X}(\mathrm{Id}_{FX}) = \mathcal{P}^N_{X,X}(\mathcal{P}^S_{NX,NX}(\mathrm{Id}_{SNX})) = \mathcal{P}^N_{X,X}(\mathrm{Id}_{NX}) = e'_X$, whence $e = e'$. □

Remark 2.34. i) By applying the first part of Theorem 2.33 to the naturally full functor $N : \mathcal{C} \to \mathcal{E}$, there is a unique separable functor $N_e : \mathcal{C}_e \to \mathcal{E}$ such that $N = N_e \circ H$, which is exactly the functor such that also $S \circ N_e = F_e$ holds true, obtained in the second part of the proof.

ii) If $F : \mathcal{C} \to \mathcal{D}$ is a separable functor, then by Corollary 2.12 its associated idempotent natural transformation is $e = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}}$. By Remark 2.29 we know that $\mathcal{C}_{\mathrm{Id}} = \mathcal{C}$ and $H : \mathcal{C} \to \mathcal{C}_{\mathrm{Id}}$ is the identity functor. Then, by Theorem 2.33, there is a unique functor $F_{\mathrm{Id}} : \mathcal{C}_{\mathrm{Id}} \to \mathcal{D}$ (necessarily separable) such that $F = F_{\mathrm{Id}} \circ H$, so $F_{\mathrm{Id}} = F$.

We can further characterize the semiseparability of a functor as follows.

Corollary 2.35. [4, Corollary 1.16] *A functor is semiseparable if, and only if, it factors as $S \circ N$, where S is a separable functor and N is a naturally full functor.*



Proof. If a functor is semiseparable, it factors as a naturally full functor followed by a separable one by Theorem 2.33. The converse is Corollary 2.7. □

We will apply Theorem 2.33 and Corollary 2.35 in the next section, in the context of Eilenberg-Moore categories.

2.2 Semiseparability and adjunctions

This section collects results on semiseparability for functors which are part of an adjunction. We prove a Rafael-type theorem for semiseparable functors in terms of regularity conditions for the unit and counit of the adjunction. In Subsection 2.2.2, we study the behavior of semiseparable adjoint functors with respect to (co)monads and associated (co)comparison functors.

2.2.1 Rafael-type Theorem and Frobenius functors

The next result extends (Rafael) Theorem 1.18 to semiseparable functors.

Theorem 2.36. [4, Theorem 2.1] (Rafael-type Theorem for semiseparability) *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction, with unit $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and counit $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$. Then,*

i) F is semiseparable if, and only if, η is regular;

ii) G is semiseparable if, and only if, ϵ is regular.

Proof. i). Assume that F is semiseparable and let $\mathcal{P}$ be a natural transformation such that $\mathcal{F} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{F}$. By Proposition 1.17 we define $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ on components by setting $\nu_X := \mathcal{P}_{GFX,X}(\epsilon_{FX}) : GFX \to X$, for any object X in $\mathcal{C}$. The naturality of ν_X in X follows from the one of $\mathcal{P}$. The idempotent natural transformation e associated with F is defined by $e_X := \mathcal{P}_{X,X}(\mathrm{Id}_{FX})$, for every $X \in \mathcal{C}$, hence from (1.12) we have $e_X = \nu_X \circ G\mathrm{Id}_{FX} \circ \eta_X = \nu_X \circ \eta_X$, so that $e = \nu \circ \eta$. We compute $\eta \circ \nu \circ \eta = \eta \circ e = GFe \circ \eta = G\mathrm{Id}_F \circ \eta = \eta$. Thus, η is regular.

Conversely, assume η is regular, i.e. there exists a natural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\eta \circ \nu \circ \eta = \eta$. For any $f : FX \to FY$ in $\mathcal{D}$, define $\mathcal{P}_{X,Y}(f) := \nu_Y \circ Gf \circ \eta_X$. By Proposition 1.17 we know that $\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ is a natural transformation and the correspondence between $\mathcal{P}$ and ν is bijective. For every $f : X \to Y$ in $\mathcal{C}$, we have that

$$\begin{aligned}(\mathcal{F}_{X,Y} \circ \mathcal{P}_{X,Y} \circ \mathcal{F}_{X,Y})(f) &= F(\mathcal{P}_{X,Y}(F(f))) = F(\nu_Y \circ GF(f) \circ \eta_X) = F(\nu_Y \circ \eta_Y \circ f)\\ &= \mathrm{Id}_{FY} \circ F\nu_Y \circ F\eta_Y \circ Ff = \epsilon_{FY} \circ F\eta_Y \circ F\nu_Y \circ F\eta_Y \circ Ff\\ &= \epsilon_{FY} \circ F\eta_Y \circ Ff = \mathrm{Id}_{FY} \circ \mathcal{F}_{X,Y}(f) = \mathcal{F}_{X,Y}(f),\end{aligned}$$

hence F is semiseparable.
ii). It follows by duality. □

Remark 2.37. The previous Rafael-type Theorem can be also proved by means of Yoneda Lemma, as in [61, Theorem 1, page 90].

The regularity condition of the unit and counit can be equivalently formulated as follows.

Lemma 2.38. [4, Lemma 2.2] *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction, with unit η and counit ϵ.*

i) The following equalities are equivalent for a natural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$:

(1) $\eta \circ \nu \circ \eta = \eta$ *(i.e., η is regular);*
(2) $F\nu \circ F\eta = \mathrm{Id}_F$*;*
(3) $\nu G \circ \eta G = \mathrm{Id}_G$*.*

ii) The following equalities are equivalent for a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$:

(1) $\epsilon \circ \gamma \circ \epsilon = \epsilon$ *(i.e., ϵ is regular);*
(2) $G\epsilon \circ G\gamma = \mathrm{Id}_G$*;*
(3) $\epsilon F \circ \gamma F = \mathrm{Id}_F$*.*

Proof. We just prove *i)* as *ii)* follows dually.
(1) $\Rightarrow$ (2). From $\eta \circ \nu \circ \eta = \eta$ we have

$$F\nu \circ F\eta = \mathrm{Id}_F \circ F\nu \circ F\eta = \epsilon F \circ F\eta \circ F\nu \circ F\eta = \epsilon F \circ F\eta = \mathrm{Id}_F.$$

(2) $\Rightarrow$ (1). By naturality of η, we have

$$\eta \circ \nu \circ \eta = \eta \circ (\nu \circ \eta) = GF(\nu \circ \eta) \circ \eta = G(F\nu \circ F\eta) \circ \eta = G\mathrm{Id}_F \circ \eta = \eta.$$

(1) $\Rightarrow$ (3). From $\eta \circ \nu \circ \eta = \eta$ we have

$$\nu G \circ \eta G = \mathrm{Id}_G \circ \nu G \circ \eta G = G\epsilon \circ \eta G \circ \nu G \circ \eta G = G\epsilon \circ \eta G = \mathrm{Id}_G.$$

(3) $\Rightarrow$ (1). By naturality of $\nu \circ \eta$, we have

$$\eta \circ \nu \circ \eta = \eta \circ (\nu \circ \eta) = (\nu \circ \eta)GF \circ \eta = (\nu G \circ \eta G)F \circ \eta = \mathrm{Id}_{GF} \circ \eta = \eta. \quad \square$$

As a consequence of the previous lemma and Corollary 2.16 we have the following result, cf. Lemma 2.6 *i)*.

Corollary 2.39. *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ.*

i) If F is semiseparable, then also the composite functor $GF : \mathcal{C} \to \mathcal{C}$ is semiseparable.

ii) If G is semiseparable, then also the composite functor $FG : \mathcal{D} \to \mathcal{D}$ is semiseparable.

Proof. i). If F is semiseparable, then by Lemma 2.38 *i)* and Theorem 2.36 $F\nu \circ F\eta = \mathrm{Id}_F$. Therefore, we have $GF\nu \circ GF\eta = \mathrm{Id}_{GF}$. By Corollary 2.16 also the composite functor $G \circ F$ is semiseparable.
ii). If G is semiseparable, then by Lemma 2.38 *ii)* and Theorem 2.36, $G\epsilon \circ G\gamma = \mathrm{Id}_G$. Therefore, we have $FG\epsilon \circ FG\gamma = \mathrm{Id}_{FG}$. By Corollary 2.16 also the composite functor $F \circ G$ is semiseparable. $\square$

For semiseparable functors that are part of an adjunction, the associated idempotent natural transformations can be written explicitly in terms of the unit and counit.

Lemma 2.40. *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ. Then,*

i) if F is semiseparable, i.e. there is a natural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\eta \circ \nu \circ \eta = \eta$, then its associated idempotent natural transformation is

$$e = \nu \circ \eta : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}};$$

ii) if G is semiseparable, i.e. there is a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\epsilon \circ \gamma \circ \epsilon = \epsilon$, then its associated idempotent natural transformation is

$$e = \epsilon \circ \gamma : \mathrm{Id}_{\mathcal{D}} \to \mathrm{Id}_{\mathcal{D}}.$$

Proof. i). It follows from the proof of Theorem 2.36. Indeed, $\mathcal{P}_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(FX, FY) \to \mathrm{Hom}_{\mathcal{C}}(X, Y)$ is defined by setting $\mathcal{P}_{X,Y}(f) := \nu_Y \circ Gf \circ \eta_X$, for every morphism $f : FX \to FY$. By definition, we have $e_X := \mathcal{P}_{X,X}(\mathrm{Id}_{FX}) = \nu_X \circ \eta_X$ for every $X \in \mathcal{C}$, so that $e = \nu \circ \eta$.
ii). It is dual to i). □

In Proposition 1.32 we have recalled that, given an adjoint triple $F \dashv G \dashv H : \mathcal{C} \to \mathcal{D}$ of functors $F, H : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$, then F is separable (resp., naturally full) if, and only if, so is H. A similar behavior holds with respect to the semiseparability condition.

Proposition 2.41. [4, Proposition 2.19] *Let $F \dashv G \dashv H : \mathcal{C} \to \mathcal{D}$ be an adjoint triple. Then, F is semiseparable if, and only if, so is H.*

Proof. We denote by η^l, ϵ^l and η^r, ϵ^r the unit and the counit of the adjunction $F \dashv G$ and of the adjunction $G \dashv H$, respectively. We just prove the "only if" part of the statement. For the "if" part consider the adjoint triple $H^{\mathrm{op}} \dashv G^{\mathrm{op}} \dashv F^{\mathrm{op}}$ together with Remark 2.3. Assume F is semiseparable. By Theorem 2.36 i), there exists a natural transformation $\nu^l : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\eta^l \circ \nu^l \circ \eta^l = \eta^l$. Define $\gamma^r : \mathrm{Id}_{\mathcal{C}} \to GH$ as in (1.15). By naturality of ϵ^r, we have $\epsilon^r \circ \gamma^r \circ \epsilon^r = \epsilon^r \circ GH\nu^l \circ G\eta^r F \circ \eta^l \circ \epsilon^r = \nu^l \circ \eta^l \circ \epsilon^r = \nu^l \circ \epsilon^r GF \circ G\eta^r F \circ \eta^l \circ \epsilon^r = \nu^l \circ \eta^l \circ \epsilon^r = \epsilon^r \circ \nu^l GH \circ \eta^l GH = \epsilon^r$, where the last equality follows from (1) $\Leftrightarrow$ (3) of Lemma 2.38 i). Thus, by Theorem 2.36 ii) H is separable. □

We have recalled in Section 1.3 that a functor $F : \mathcal{C} \to \mathcal{D}$ is Frobenius if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ which is both a left and a right adjoint to F. As a consequence of Theorem 2.36 and Lemma 2.38, the following result extends Proposition 1.36 and Proposition 1.37 to semiseparable Frobenius functors.

Theorem 2.42. [4, Cf. Proposition 2.21] (Rafael-type Theorem for semiseparable Frobenius functors) *Let $F : \mathcal{C} \to \mathcal{D}$ be a Frobenius functor, with left and right adjoint $G : \mathcal{D} \to \mathcal{C}$. Denote by η^l, ϵ^l and by η^r, ϵ^r the unit and the counit of the adjunctions $F \dashv G$ and $G \dashv F$, respectively. Then, the following assertions are equivalent.*

(i) F is semiseparable.

(ii) There exists a natural transformation $\alpha : G \to G$ such that one of the following equivalent conditions holds:

(1) $\eta^l \circ \epsilon^r \circ \alpha F \circ \eta^l = \eta^l$;

(2) $\epsilon^r \circ \alpha' F \circ \eta^l \circ \epsilon^r = \epsilon^r$;

(3) $F\epsilon^r \circ F\alpha F \circ F\eta^l = \mathrm{Id}_F$;

(4) $\epsilon^r G \circ \alpha F G \circ \eta^l G = \mathrm{Id}_G$.

(*iii*) *There exists a natural transformation $\beta : F \to F$ such that one of the following equivalent conditions holds:*

(1) $\eta^l \circ \epsilon^r \circ G\beta \circ \eta^l = \eta^l$;

(2) $\epsilon^r \circ G\beta' \circ \eta^l \circ \epsilon^r = \epsilon^r$;

(3) $F\epsilon^r \circ FG\beta \circ F\eta^l = \mathrm{Id}_F$;

(4) $\epsilon^r G \circ G\beta G \circ \eta^l G = \mathrm{Id}_G$.

Proof. By Corollary 1.35 for any natural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ there are unique natural transformations $\alpha : G \to G$, $\beta : F \to F$ such that

$$\epsilon^r \circ \alpha F = \nu = \epsilon^r \circ G\beta, \tag{2.3}$$

and for any natural transformation $\gamma : \mathrm{Id}_{\mathcal{C}} \to GF$ there are unique natural transformations $\alpha' : G \to G$, $\beta' : F \to F$ such that

$$\alpha' F \circ \eta^l = \gamma = G\beta' \circ \eta^l. \tag{2.4}$$

By Theorem 2.36 applied to the adjunction (F, G), F is semiseparable if, and only if, there exists a natural transformation $\nu^l : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\eta^l \circ \nu^l \circ \eta^l = \eta^l$, hence from (2.3) and Lemma 2.38 the latter is equivalent to any of (1), (3), (4) in (*i*) and it is also equivalent to (1) in (*ii*).
By Theorem 2.36 applied to the adjunction (G, F), F is semiseparable if, and only if, there exists a natural transformation $\gamma^r : \mathrm{Id}_{\mathcal{C}} \to GF$ such that $\epsilon^r \circ \gamma^r \circ \epsilon^r = \epsilon^r$, hence from (2.4) and Lemma 2.38 the latter is equivalent to any of (2), (3), (4) in (*ii*) and it is also equivalent to (2) in (*i*). ☐

We observe that a semiseparable functor is not necessarily Frobenius. Indeed, from [28, Example 18, item 6] let G be a finite group and consider the group algebra $A = \Bbbk G$ over a field $\Bbbk$. Thus, A is a Frobenius $\Bbbk$-algebra and then the restriction of scalars functor $\varphi_* : {}_A\mathcal{M} \to \mathfrak{M}$ is Frobenius, cf. [28, Theorem 28, item 3]. However, if char($\Bbbk$) divides $|G|$, the extension $A/\Bbbk$ is not separable, so that φ_* is not separable by Maschke Theorem, hence not even semiseparable as φ_* is faithful.

2.2.2 (Co)separable (co)monads and (co)comparisons

Now, let (F, G, η, ϵ) be an adjunction. If the right (resp., left) adjoint is separable, then by Lemma 1.79, the associated monad $(GF, G\epsilon F, \eta)$ (resp., comonad $(FG, F\eta G, \epsilon)$) is separable (resp., coseparable). We show that the semiseparability condition on the right (left) adjoint is sufficient to obtain the (co)separability of the associated (co)monad. The proof is similar to the separable case but uses Lemma 2.38.

Lemma 2.43. [4, Lemma 2.6] *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ.*

i) *If G is semiseparable, then the associated monad $(GF, G\epsilon F, \eta)$ is separable.*

ii) *If F is semiseparable, then the associated comonad $(FG, F\eta G, \epsilon)$ is coseparable.*

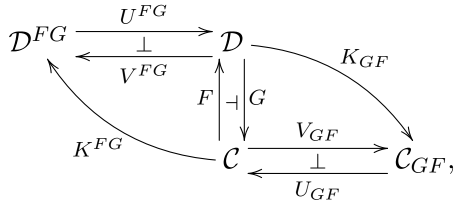
Proof. i). Assume G is semiseparable. Then, by Theorem 2.36 and Lemma 2.38 ii), there is a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $G\epsilon \circ G\gamma = \mathrm{Id}_G$. Set $\sigma := G\gamma F : G\mathrm{Id}_{\mathcal{D}}F \to GFGF$. It follows that $G\epsilon F \circ \sigma = G\epsilon F \circ G\gamma F = \mathrm{Id}_{GF}$. Moreover, from the naturality of ϵ and γ, we have $\gamma \circ \epsilon = \epsilon FG \circ FG\gamma$ and $\gamma \circ \epsilon = FG\epsilon \circ \gamma FG$, respectively, hence $GFG\epsilon F \circ \sigma GF = GFG\epsilon F \circ G\gamma FGF = G\gamma F \circ G\epsilon F = G\epsilon FGF \circ GFG\gamma F = G\epsilon FGF \circ GF\sigma$. Therefore, the monad $(GF, G\epsilon F, \eta)$ is separable.

ii). The proof is dual by using Lemma 2.38 i). □

Remark 2.44. By Corollary 2.39, in the monad $(GF, G\epsilon F, \eta)$ associated to a semiseparable left adjoint F, the endofunctor GF is semiseparable; in the comonad $(FG, F\eta G, \epsilon)$ associated to a semiseparable right adjoint G, the endofunctor FG is semiseparable.

Remark 2.45. (Cf. [4, Remark 2.7]) We have recalled in Lemma 1.79 that, if the right adjoint G of an adjunction (F, G) is separable, then the associated monad $(GF, G\epsilon F, \eta)$ is separable. The converse is not true in general. For instance, see [31, Example 3.7(2)]. Explicitly, let $\mathcal{C}$ and $\mathcal{C}'$ be two nontrivial additive categories, and consider the product category $\mathcal{D} := \mathcal{C} \times \mathcal{C}'$. Let $F : \mathcal{C} \to \mathcal{D}$ be the canonical functor sending an object C to $(C, 0)$ and a morphism f to $(f, 0)$. Its right adjoint is the projection functor $G : \mathcal{D} \to \mathcal{C}$. Then, the associated monad GF equals the identity monad on $\mathcal{C}$, which is separable, but G is not separable, as it is not faithful. Nevertheless, G results to be semiseparable. Indeed, let $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ be the counit of the adjunction given for any $D = (C, C')$ in $\mathcal{D}$ by $\epsilon_D = (\mathrm{Id}_C, \varphi^I_{C'}) : FGD \to D$, where $\varphi^I_{C'}$ is the unique map from the zero object 0 of $\mathcal{C}$ to C'. Consider the natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$, given for any $D = (C, C')$ in $\mathcal{D}$ by $\gamma_D = (\mathrm{Id}_C, \varphi^T_{C'}) : D \to FGD$, where $\varphi^T_{C'}$ is the unique map from C' to 0. Then, from $\gamma_D \circ \epsilon_D = (\mathrm{Id}_C, \varphi^T_{C'}) \circ (\mathrm{Id}_C, \varphi^I_{C'}) = (\mathrm{Id}_C, \mathrm{Id}_0) = \mathrm{Id}_{FGD}$, it follows that G is naturally full by Theorem 1.29, hence in particular semiseparable. However, the fact that the associated (co)monad is (co)separable does not imply that the right (left) adjoint is semiseparable in general, i.e. the converse of Lemma 2.43 is not necessarily true. Indeed, if (F, G) is an adjunction with G (resp., F) fully faithful, then the associated monad (resp., comonad) is always idempotent (this will be proved in Corollary 2.51, resp., Corollary 2.53) whence separable (resp., coseparable). However F (resp., G) needs not to be semiseparable in this case. For instance, we consider the usual adjunction (φ^*, φ_*) attached to a ring homomorphism $\varphi : R \to S$ (recalled in Subsection 1.4.1). In [7, Example 3.3] it is shown an example of a ring epimorphism $\varphi : R \to S$ (thus φ_* is fully faithful) such that the extension of scalars functor φ^* is full, but not naturally full, hence φ^* is not semiseparable by Proposition 2.5.

The following result characterizes the semiseparability of a right (resp., left) adjoint functor in terms of properties of the (co)comparison functor and of the forgetful functor from the Eilenberg-Moore category of (co)modules over the associated (co)monad. Given an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$, we recall the diagram (1.23)



where $U_{GF} \circ K_{GF} = G$, $K_{GF} \circ F = V_{GF}$, $U^{FG} \circ K^{FG} = F$ and $K^{FG} \circ G = V^{FG}$.

Remark 2.46. By Remark 1.78 iii) and iv) and in view of Theorem 2.36, it follows that G (resp., F) is semiseparable if, and only if, so is V^{FG} (resp., V_{GF}).

Theorem 2.47. [4, Cf. Theorem 2.9 and Theorem 2.14] *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ. Then,*

i) G is semiseparable if, and only if, the forgetful functor $U_{GF} : \mathcal{C}_{GF} \to \mathcal{C}$ is separable (equivalently, the monad $(GF, G\epsilon F, \eta)$ is separable) and the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ is naturally full;

ii) F is semiseparable if, and only if, the forgetful functor $U^{FG} : \mathcal{D}^{FG} \to \mathcal{D}$ is separable (equivalently, the comonad $(FG, F\eta G, \epsilon)$ is coseparable) and the cocomparison functor $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$ is naturally full.

Proof. i). Set $U := U_{GF}$ and $K := K_{GF}$. Assume G is semiseparable. By Theorem 2.36 *ii)* and Lemma 2.38 (*ii*), there is a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $G\epsilon \circ G\gamma = \mathrm{Id}_G$. By Lemma 2.43 *i*), the monad $(GF, G\epsilon F, \eta)$ is separable, and by Lemma 1.75 *i)* its separability is equivalent to the separability of U. We now prove that $K : \mathcal{D} \to \mathcal{C}_{GF}$ is naturally full. Let $h : KX \to KY$ be a morphism in $\mathcal{C}_{GF}$, so that $G\epsilon_Y \circ GFUh = Uh \circ G\epsilon_X$. Set $h' := \epsilon_Y \circ FUh \circ \gamma_X$. Then, since $U \circ K = G$, which is semiseparable by assumption, we obtain

$$UKh' = G(\epsilon_Y \circ FUh \circ \gamma_X) = (G\epsilon_Y \circ GFUh) \circ G\gamma_X = Uh \circ G\epsilon_X \circ G\gamma_X = Uh,$$

hence $Kh' = h$, thus K is full. Moreover, since U is faithful and UK is semiseparable, by Lemma 2.8 K is semiseparable. By Proposition 2.5 *ii)* we have that K is naturally full. Conversely, if U is separable and K is naturally full, then by Corollary 2.35 $G = U \circ K$ is semiseparable.
ii). The proof is dual to *i*). □

As a consequence of Theorem 2.47 we can now recover similar characterizations for separable, naturally full and fully faithful adjoint functors. Cf. the proof of [31, Proposition 3.5] and [8, Proposition 2.16].

Corollary 2.48. [4, Corollary 2.11 and Corollary 2.15] *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ. Then,*

i) G is separable if, and only if, the forgetful functor $U_{GF} : \mathcal{C}_{GF} \to \mathcal{C}$ is separable (equivalently, the monad $(GF, G\epsilon F, \eta)$ is separable) and the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ is fully faithful (i.e., G is premonadic).

ii) F is separable if, and only if, the forgetful functor $U^{FG} : \mathcal{D}^{FG} \to \mathcal{D}$ is separable (equivalently, the comonad $(FG, F\eta G, \epsilon)$ is coseparable) and the cocomparison functor $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$ is fully faithful (i.e., F is precomonadic).

Proof. i). By Proposition 2.5 *i*), G is separable if and only if it is semiseparable and faithful. By Theorem 2.47 *i*), G is semiseparable if and only if U_{GF} is separable and K_{GF} is naturally full. Since $G = U_{GF} \circ K_{GF}$ and U_{GF} is faithful, we get that G is faithful if and only if K_{GF} is faithful. So we get that G is separable if, and only if, U_{GF} is separable and K_{GF} is both naturally full and faithful. The latter means that K_{GF} is fully faithful, i.e. G is premonadic.
ii). The proof is similar to *i*). □

Remark 2.49. We recall from the *Weak Tripleability Theorem* of Beck, see [15, Theorem 1], that given an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$, if $\mathcal{D}$ has all coequalizers, then the comparison functor K_{GF} has a left adjoint L. If in addition G reflects all coequalizers, then the counit of the adjunction $L \dashv K_{GF}$ is an isomorphism, i.e., K_{GF} is fully faithful. By Corollary 2.21 we know that a functor that reflects coequalizers is semiseparable if, and only if, it is separable. Hence, by Corollary 2.48, G is semiseparable if, and only if, the monad $(GF, G\epsilon F, \eta)$ is separable.

The natural fullness of a right adjoint functor can be characterized in terms of idempotence of its adjunction (and of its associated monad) and natural fullness of the comparison functor as follows.

Corollary 2.50. [4, Corollary 2.12] *The following assertions are equivalent for an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$ with unit η and counit ϵ.*

(i) *G is naturally full.*

(ii) *The adjunction (F, G) is idempotent and G is semiseparable.*

(iii) *The forgetful functor $U_{GF} : \mathcal{C}_{GF} \to \mathcal{C}$ is fully faithful (i.e., the monad $(GF, G\epsilon F, \eta)$ is idempotent) and the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ is naturally full.*

Proof. $(i) \Rightarrow (ii)$. If G is naturally full, by Theorem 1.29 ii), there is a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\gamma \circ \epsilon = \mathrm{Id}_{FG}$. Thus, $G\gamma \circ G\epsilon = \mathrm{Id}_{GFG}$. From the triangular identity $G\epsilon \circ \eta G = \mathrm{Id}_G$, we have that $G\epsilon$ is invertible, and hence (F, G) is idempotent. Moreover, G is semiseparable by Proposition 2.5 ii).

$(ii) \Rightarrow (iii)$. It follows from the definition of an idempotent adjunction and from Theorem 2.47 i).

$(iii) \Rightarrow (i)$. Since $G = U_{GF} \circ K_{GF}$, by Proposition 1.28 i) we have that G is naturally full as it is composition of naturally full functors. □

From Corollary 2.48 i) and Corollary 2.50, we recover the characterization for a fully faithful right adjoint.

Corollary 2.51. [4, Corollary 2.13] *The following assertions are equivalent for an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$ with unit η and counit ϵ.*

(i) *G is fully faithful.*

(ii) *The forgetful functor $U_{GF} : \mathcal{C}_{GF} \to \mathcal{C}$ is fully faithful (i.e., the monad $(GF, G\epsilon F, \eta)$ is idempotent) and the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ is fully faithful (i.e., G is premonadic).*

(iii) *The adjunction (F, G) is idempotent and the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ is an equivalence (i.e., G is monadic).*

Proof. $(i) \Leftrightarrow (ii)$. It follows from Corollary 2.48 and Corollary 2.50.

$(i) \Leftrightarrow (iii)$. This follows by [8, Proposition 2.5]. □

We state the dual results of Corollaries 2.50 and 2.51 for the left adjoint functor of an adjunction, whose proofs follow from similar arguments.

Corollary 2.52. [4, Corollary 2.16] *The following assertions are equivalent for an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$ with unit η and counit ϵ.*

(i) *F is naturally full.*

(ii) *The adjunction (F, G) is idempotent and F is semiseparable.*

(iii) *The forgetful functor $U^{FG} : \mathcal{D}^{FG} \to \mathcal{D}$ is fully faithful (i.e., the comonad $(FG, F\eta G, \epsilon)$ is idempotent) and the cocomparison functor $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$ is naturally full.*

Corollary 2.53. [4, Corollary 2.17] *The following assertions are equivalent for an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$ with unit η and counit ϵ.*

(i) *F is fully faithful.*

(ii) *The comonad $(FG, F\eta G, \epsilon)$ is idempotent and the cocomparison functor $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$ is fully faithful (i.e., F is precomonadic).*

(iii) *The adjunction (F, G) is idempotent and the cocomparison functor $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$ is an equivalence (i.e., F is comonadic).*

As a consequence of Corollaries 2.50 and 2.52, we have the following.

Corollary 2.54. [4, Corollary 2.18] *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an idempotent adjunction. Then, G (resp., F) is semiseparable if, and only if, it is naturally full.*

Proof. It follows by $(i) \Leftrightarrow (ii)$ in Corollary 2.50 (resp., Corollary 2.52). □

2.3 (Co)reflections, bireflections and semiseparability

In this section we show how functors admitting a fully faithful adjoint behave with respect to semiseparability. As in [16], we call *coreflection* any functor admitting a fully faithful left adjoint, while we call *reflection* any functor with a fully faithful right adjoint. The adjoint of the inclusion of a (co)reflective subcategory is an example of (co)reflection.

Remark 2.55. By [19, Proposition 3.4.1] for a right adjoint adjunction being a coreflection (respectively, a reflection) is equivalent to the fact that the unit (respectively, the counit) of the corresponding adjunction is an isomorphism.

Remark 2.56. (Co)reflections are closed under composition. In fact, if $G : \mathcal{D} \to \mathcal{C}$, $G' : \mathcal{E} \to \mathcal{D}$ are (co)reflections with fully faithful left (right) adjoints $F : \mathcal{C} \to \mathcal{D}$ and $F' : \mathcal{D} \to \mathcal{E}$ respectively, then $G \circ G'$ is a (co)reflection with fully faithful left (right) adjoint $F' \circ F$ (by [61, IV.8, Theorem 1]).

Lemma 2.57. *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor.*

i) *If G is a coreflection, denoting by F the left adjoint of G, by η, ϵ the unit and the counit of the adjunction (F, G), then $(F\eta)^{-1} = \epsilon F$ and $(\eta G)^{-1} = G\epsilon$.*

ii) *If G is a reflection, denoting by H the right adjoint of G, by η', ϵ' the unit and the counit of the adjunction (G, H), then $(\epsilon' G)^{-1} = G\eta'$ and $(H\epsilon')^{-1} = \eta' H$.*

Proof. *i)*. Assume that G is a coreflection. Since F is fully faithful, by Remark 2.55 η is invertible. Therefore, from $\epsilon F \circ F\eta = \mathrm{Id}_F$ and $G\epsilon \circ \eta G = \mathrm{Id}_G$, we get $(F\eta)^{-1} = \epsilon F$ and $(\eta G)^{-1} = G\epsilon$.
ii). Assume that G is a reflection. Since H is fully faithful, by Remark 2.55 ϵ is invertible. Therefore, from $\epsilon' G \circ G\eta' = \mathrm{Id}_G$ and $H\epsilon' \circ \eta' H = \mathrm{Id}_H$, we get $(\epsilon' G)^{-1} = G\eta'$ and $(H\epsilon')^{-1} = \eta' H$. □

In Lemma 2.8 we proved that if $H \circ G$ is semiseparable and H is faithful, then G is semiseparable. In the following proposition we see that also the functor H comes out to be semiseparable whenever $H \circ G$ is and G is assumed to be a (co)reflection, cf. [7, Proposition 2.4] for the naturally full case.

Proposition 2.58. [4, Proposition 2.23] *Let $G : \mathcal{D} \to \mathcal{C}$, $H : \mathcal{C} \to \mathcal{E}$ be functors, and assume that G is a (co)reflection. If $H \circ G : \mathcal{D} \to \mathcal{E}$ is semiseparable, then H is semiseparable.*

Proof. Assume that G is a coreflection with a fully faithful left adjoint F. If $H \circ G$ is semiseparable, since F is fully faithful (whence naturally full), then HGF is semiseparable by Lemma 2.6 *ii*). The unit $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ of the adjunction (F, G) is an isomorphism, so that $H\eta : H \to HGF$ is an isomorphism. By Proposition 2.9 H is semiseparable. If G is a reflection, the proof is similar. □

In [39, Definition 8] a *bireflective* subcategory $\mathcal{B}$ of a category $\mathcal{C}$ is defined to be a subcategory of $\mathcal{C}$ such that the inclusion functor $J : \mathcal{B} \to \mathcal{C}$ has left and right adjoints $S : \mathcal{C} \to \mathcal{B}$ equal, such that $\eta \circ \epsilon' = \mathrm{Id}$, where η is the unit of the adjunction $S \dashv J$ and ϵ' is the counit of the adjunction $J \dashv S$. Thus, a stronger notion of (co)reflection involving both left and right adjoint can be given as follows.

Definition 2.59. [4, Section 2.3] A functor $G : \mathcal{D} \to \mathcal{C}$ is called *bireflection* if it has a left and right adjoint equal, say $F : \mathcal{C} \to \mathcal{D}$, which is fully faithful and satisfies the coherent condition $\gamma \circ \epsilon = \mathrm{Id}$, where $\epsilon : FG \to \mathrm{Id}$ is the counit of $F \dashv G$ while $\gamma : \mathrm{Id} \to FG$ is the unit of $G \dashv F$.

Example 2.60. The inclusion functor $J : \mathcal{B} \to \mathcal{C}$ where $\mathcal{B}$ is a bireflective subcategory of a category $\mathcal{C}$ is an example of bireflection.

Remark 2.61. Bireflections are closed under composition. Indeed, if $G : \mathcal{D} \to \mathcal{C}$, $G' : \mathcal{E} \to \mathcal{D}$ are bireflections with fully faithful left and right adjoints F and F' respectively, satisfying the coherent conditions $\eta^r \circ \epsilon^l = \mathrm{Id}$ and $\bar{\eta}^r \circ \bar{\epsilon}^l = \mathrm{Id}$, where $\epsilon^l : FG \to \mathrm{Id}$ is the counit of $F \dashv G$, $\bar{\epsilon}^l : F'G' \to \mathrm{Id}$ is the counit of $F' \dashv G'$, while $\eta^r : \mathrm{Id} \to FG$ is the unit of $G \dashv F$ and $\bar{\eta}^r : \mathrm{Id} \to F'G'$ is the unit of $G' \dashv F'$, then $G \circ G'$ is a bireflection with fully faithful left and right adjoint $F' \circ F$, satisfying the coherent condition $F'\eta^r G' \circ \bar{\eta}^r \circ \bar{\epsilon}^l \circ F'\epsilon^l G' = \mathrm{Id}$.

Remark 2.62.

i) Any bireflection is a Frobenius functor.

ii) An equivalence is clearly a fully faithful bireflection, and conversely a fully faithful bireflection is an equivalence as the unit and counit of the corresponding adjunction are both invertible (see [19, Proposition 3.4.3]).

By requiring the semiseparability condition on a (co)reflection we retrieve the notion of bireflection.

Proposition 2.63. *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Then, G is a semiseparable (co)reflection if, and only if, G is a bireflection.*

Proof. Assume that G is a semiseparable coreflection. Denote by F, η, ϵ, the left adjoint of G, the unit and the counit of the adjunction (F, G), respectively. By Theorem 2.36 *ii*) there is a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\epsilon \circ \gamma \circ \epsilon = \epsilon$. By Lemma 2.38, we have $\epsilon F \circ \gamma F = \mathrm{Id}_F$ and $G\epsilon \circ G\gamma = \mathrm{Id}_G$, so that from Lemma 2.57 it follows that

$$\begin{aligned} F\left(\eta^{-1}\right) \circ \gamma F &= (F\eta)^{-1} \circ \gamma F = \epsilon F \circ \gamma F = \mathrm{Id}_F, \\ \eta^{-1} G \circ G\gamma &= (\eta G)^{-1} \circ G\gamma = G\epsilon \circ G\gamma = \mathrm{Id}_G. \end{aligned}$$

This means that (G, F) is an adjunction with unit $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ and counit $\eta^{-1} : GF \to \mathrm{Id}_{\mathcal{C}}$. The equality $\eta^{-1}G = G\epsilon$ implies the coherent condition $\gamma \circ \epsilon = \mathrm{Id}$. Indeed, since $G\gamma = \eta G$ we have that $\gamma \circ \epsilon = \epsilon FG \circ FG\gamma = \epsilon FG \circ F\eta G = \mathrm{Id}_{FG}$, thus G is a bireflection.

On the other hand, assume that G is a bireflection. Then, G is both a coreflection and a reflection, and it is also a Frobenius functor. Consider the unit $\eta' : \mathrm{Id}_{\mathcal{D}} \to FG$ and the counit $\epsilon' : GF \to \mathrm{Id}_{\mathcal{C}}$ of the adjunction (G, F). Set $\sigma := \epsilon' G \circ \eta G : G \to G$ and note that $\sigma \circ G\epsilon = \epsilon' G \circ \eta G \circ G\epsilon = \epsilon' G \circ \eta G \circ (\eta G)^{-1} = \epsilon' G$. If we set $\gamma := F\sigma \circ \eta'$, from naturality of η' we obtain

$$\gamma \circ \epsilon = F\sigma \circ \eta' \circ \epsilon = F\sigma \circ FG\epsilon \circ \eta' FG = F(\sigma \circ G\epsilon) \circ \eta' FG = F\epsilon' G \circ \eta' FG = \mathrm{Id}_{FG}.$$

By Theorem 1.29 *ii*) we conclude that G is naturally full, hence semiseparable. The case with G a semiseparable reflection follows dually. □

As a consequence, semiseparable (co)reflections can be further characterized as follows.

Corollary 2.64. [4, Theorem 2.24] *Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Then, the following assertions are equivalent.*

(i) *G is a naturally full coreflection.*

(ii) *G is a semiseparable coreflection.*

(iii) *G is a bireflection.*

(iv) *G is a Frobenius coreflection.*

(v) *G is a naturally full reflection.*

(vi) *G is a semiseparable reflection.*

(vii) *G is a Frobenius reflection.*

Proof. We prove the equivalence between $(i), (ii), (iii)$ and (iv). Denote by F, η, ϵ, the left adjoint of G, the unit and the counit of the adjunction (F, G), respectively.

$(i) \Leftrightarrow (ii)$. Since η is invertible, the adjunction (F, G) is idempotent and Corollary 2.54 applies.

$(ii) \Leftrightarrow (iii)$. It follows from Proposition 2.63.

$(iii) \Rightarrow (iv)$. It is obvious.

$(iv) \Rightarrow (i)$. It follows from the proof of the "if" part in Proposition 2.63.

The implications $(v) \Leftrightarrow (vi) \Leftrightarrow (iii) \Rightarrow (vii) \Rightarrow (v)$ follow dually. □

Remark 2.65. [4, Remark 2.25] A conservative (co)reflection is an equivalence, see e.g. [16, Remark 1.4]. In fact, consider an adjunction (F, G, η, ϵ). If G is a coreflection, then F is fully faithful and hence η is invertible. From the triangular identity $G\epsilon \circ \eta G = \mathrm{Id}_G$ we get that $G\epsilon$ is invertible. Thus, if G is also conservative, we have that ϵ is invertible, whence G is an equivalence. By Remark 1.12 we know that separable functors are conservative, thus one recovers that any separable (co)reflection is actually an equivalence, see e.g. in [79, Proposition 2.4]. Then, Corollary 2.64 can be seen as a semi-analogue of this result.

Remark 2.66. [4, Remark 3.34] It is known (see e.g. [43, A1.2]) that a faithful functor from a balanced category (i.e. a category where every monomorphism which is an epimorphism is necessarily an isomorphism) is always conservative. As a consequence, a faithful (co)reflection from a balanced category is always an equivalence. For instance, since the category ${}_R\mathcal{M}$ of left modules over a ring R is abelian, it is in particular balanced, thus any faithful (co)reflection from ${}_R\mathcal{M}$ is always an equivalence.

Recall from [82, Definition 1.1] that a monad $(\top : \mathcal{C} \to \mathcal{C}, m, \eta)$ is *Frobenius* if it is equipped with a natural transformation $\zeta : \top \to \mathrm{Id}_{\mathcal{C}}$ such that there exists a natural transformation $\rho : \mathrm{Id}_{\mathcal{C}} \to \top\top$ satisfying

$$\top m \circ \rho\top = m\top \circ \top\rho \quad \text{and} \quad \top\zeta \circ \rho = \eta = \zeta\top \circ \rho.$$

For a Frobenius monad as above there is an adjunction $\top \dashv \top$ with counit $\sigma := \zeta \circ m : \top\top \to \mathrm{Id}_{\mathcal{C}}$ and unit $\rho : \mathrm{Id}_{\mathcal{C}} \to \top\top$, see [82, Lemma 1.3] . As shown in [82, Theorem 1.6], the monad $(\top : \mathcal{C} \to \mathcal{C}, m, \eta)$ equipped with a natural transformation $\zeta : \top \to \mathrm{Id}_{\mathcal{C}}$ is Frobenius if, and only if, there exists a comonad $(\top, \Delta, \zeta)$ such that $\top m \circ \Delta\top = \Delta \circ m = m\top \circ \top\Delta$ if, and only if, the natural transformation $\sigma := \zeta \circ m : \top\top \to \mathrm{Id}_{\mathcal{C}}$ is a counit for an adjunction $\top \dashv \top$. Thus, the notion of Frobenius monad results to be self-dual, in the sense that it is the same as a comonad $(\bot, \Delta, \epsilon)$ with a natural transformation $\eta : \mathrm{Id}_{\mathcal{C}} \to \bot$ such that $\Delta \circ \eta : \mathrm{Id}_{\mathcal{C}} \to \bot\bot$ is a unit for an adjunction $\bot \dashv \bot$.

Corollary 2.67. *Let $F : \mathcal{C} \to \mathcal{D}$ be a semiseparable reflection (i.e., F is a bireflection). Denote by G, η, ϵ, the right adjoint of F, the unit and the counit of the adjunction (F, G), respectively. Then, the associated monad $(GF, G\epsilon F, \eta)$ is Frobenius.*

Proof. If $F : \mathcal{C} \to \mathcal{D}$ is a semiseparable reflection, then by Corollary 2.64 it is in particular Frobenius and it has a fully faithful right (and left) adjoint $G : \mathcal{D} \to \mathcal{C}$, hence also G is Frobenius as well. By [79, Proposition 2.5] we know that G is a Frobenius functor if, and only if, the monad $(GF, G\epsilon F, \eta)$ is Frobenius. □

The following result will be useful in Subsection 3.5.

Proposition 2.68. [4, Proposition 2.26] *Let $F \dashv G \dashv H : \mathcal{C} \to \mathcal{D}$ be an adjoint triple with G fully faithful. Denote by η^l, ϵ^l and η^r, ϵ^r the unit and the counit of the adjunction $F \dashv G$ and of the adjunction $G \dashv H$, respectively. Consider the natural transformation $\sigma : H \to F$ defined by $\sigma := F\epsilon^r \circ (\epsilon^l H)^{-1} : H \to F$. Then, F is semiseparable if, and only if, H is semiseparable if, and only if, σ is split-mono if, and only if, σ is invertible.*

Proof. By Proposition 2.41 F is semiseparable if, and only if, so is H. Since G is fully faithful, then H is a coreflection so that, by Corollary 2.64, it is semiseparable if, and only if, it is naturally full if, and only if, it is Frobenius, i.e. $F \cong H$. By [79, Proposition 2.2], the condition $F \cong H$ is equivalent to the invertibility of σ. We now prove that G is naturally full if and only if σ is split-mono. We have a bijective correspondence $\mathrm{Nat}(F, H) \cong \mathrm{Nat}(\mathrm{Id}_{\mathcal{C}}, GH)$. Explicitly, for any natural transformation $\tau : F \to H$ there is a unique natural transformation $\gamma : \mathrm{Id}_{\mathcal{C}} \to GH$ given by $\gamma := G\tau \circ \eta^l$. Then, $\gamma \circ \epsilon^r = G\tau \circ \eta^l \circ \epsilon^r = G\tau \circ GF\epsilon^r \circ \eta^l GH = G\tau \circ G(\sigma \circ \epsilon^l H) \circ \eta^l GH = G(\tau \circ \sigma) \circ G\epsilon^l H \circ \eta^l GH = G(\tau \circ \sigma)$, so that $\gamma \circ \epsilon^r = G(\tau \circ \sigma)$. Thus, $\gamma \circ \epsilon^r = \mathrm{Id}_{GH}$ if, and only if, $G(\tau \circ \sigma) = \mathrm{Id}_{GH}$ if, and only if, $\tau \circ \sigma = \mathrm{Id}_H$, as G is faithful. By Theorem 1.29 from $\gamma \circ \epsilon^r = \mathrm{Id}_{GH}$ it follows that H is naturally full. □

2.3.1 Factorization by a bireflection

Recall that an idempotent $f : X \to X$ in a category $\mathcal{C}$ *splits* if there exist $g : X \to Y$ and $h : Y \to X$ such that $h \circ g = f$ and $g \circ h = \mathrm{Id}_Y$. The splitting is unique up to isomorphism. A category $\mathcal{C}$ is said to be *idempotent complete* if all idempotents split. We will focus on this notion in Section 4.1.1.

In [39] an endo-natural transformation whose components are all split idempotents is called a *split-idempotent natural transformation.* It is known that bireflective subcategories correspond bijectively to split-idempotent natural transformations $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ with specified splitting, [39, Theorem 13]. This is related to the fact that the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$ into the coidentifier category results to be a bireflection in meaningful cases, e.g. when $\mathcal{C}$ is idempotent complete.

Proposition 2.69. [4, Proposition 2.27] *Let $\mathcal{C}$ be a category and let $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be an idempotent natural transformation. Then, the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$ is a bireflection if, and only if, e splits.*

Proof. Assume that $H : \mathcal{C} \to \mathcal{C}_e$ is a bireflection. Then, H has left and right adjoint functors equal, say $L : \mathcal{C}_e \to \mathcal{C}$, which is fully faithful, and such that the coherence condition $\eta \circ \epsilon' = \mathrm{Id}_{HL}$ is satisfied, where $\eta : \mathrm{Id} \to HL$ is the unit of adjunction $L \dashv H$, and $\epsilon' : HL \to \mathrm{Id}$ is the counit of $H \dashv L$. Denote by $\eta' : \mathrm{Id} \to LH$ and $\epsilon : LH \to \mathrm{Id}$ the unit and counit of the adjunctions $H \dashv L$ and $L \dashv H$, respectively. Since L is fully faithful, η is an isomorphism and hence, from the coherence condition and the triangle identity $L\epsilon' \circ \eta' L = \mathrm{Id}_L$, we get that $L\eta = (L\epsilon')^{-1} = \eta' L$. Therefore, by naturality of η', we have $\eta' \circ \epsilon = LH\epsilon \circ \eta' LH = LH\epsilon \circ L\eta H = L\mathrm{Id}_H = \mathrm{Id}_{LH}$. Similarly, from the latter condition and the triangular identity $H\epsilon \circ \eta H = \mathrm{Id}_H$, it follows that $H\eta' = (H\epsilon)^{-1} = \eta H$. Then, $H(\epsilon \circ \eta') = H\epsilon \circ H\eta' = \mathrm{Id}_H = H\mathrm{Id}$. Thus, for all $X \in \mathcal{C}$, we have $e_X = e_X \circ \epsilon_X \circ \eta'_X$. Now, from Lemma 2.28 *i*) we have that $He = \mathrm{Id}_H$ as e is the idempotent natural transformation associated to H. Then, $e_X \circ \epsilon_X = \epsilon_X \circ LHe_X = \epsilon_X \circ LH\mathrm{Id}_X = \epsilon_X$, so that the equality $e_X = e_X \circ \epsilon_X \circ \eta'_X$ simplifies as $e_X = \epsilon_X \circ \eta'_X$ and hence e splits.

Conversely, assume that e splits. Since we know by Lemma 2.28 *i*) that H is naturally full, it is in particular semiseparable. In order to conclude, by Corollary 2.64, it is enough to check that H is a coreflection. Choose a splitting $\mathrm{Id}_{\mathcal{C}} \overset{\pi}{\twoheadrightarrow} P \overset{\epsilon}{\hookrightarrow} \mathrm{Id}_{\mathcal{C}}$ of the idempotent e such that $\pi \circ \epsilon = \mathrm{Id}_P$. Note that $Pe = \mathrm{Id}_P$, as $Pe = Pe \circ \pi \circ \epsilon \overset{\text{nat.}\pi}{=} \pi \circ e \circ \epsilon = \pi \circ \epsilon \circ \pi \circ \epsilon = \mathrm{Id}_P$. By Lemma 2.30, there is a unique functor $P_e : \mathcal{C}_e \to \mathcal{C}$ such that $P_e \circ H = P$. It is now straightforward to check that $P_e \dashv H$ with counit ϵ and invertible unit $\eta : \mathrm{Id}_{\mathcal{C}_e} \to HP_e$ defined by the equality $\eta H = H\pi$, i.e. by setting $\eta_X := (\overline{\pi_X})_{X \in \mathcal{C}}$. Thus, H is a coreflection. □

Remark 2.70. Alternatively, the "if" part of Proposition 2.69 follows from the proof of the dual of [39, Proof of Theorem 13].

As a consequence of Theorem 2.33 and Proposition 2.69, we have the following corollary.

Corollary 2.71. [4, Corollary 2.28] *A functor $F : \mathcal{C} \to \mathcal{D}$ factors as a bireflection followed by a separable functor if, and only if, it is semiseparable and the associated natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ splits. Moreover, any such a factorization is the same given by the coidentifier within Theorem 2.33, up to a category equivalence.*

Proof. Assume that $F = S \circ N$, where $N : \mathcal{C} \to \mathcal{E}$ is a bireflection and $S : \mathcal{E} \to \mathcal{D}$ is a separable functor. Since N is in particular naturally full, we get that $F = S \circ N$ is semiseparable and hence, by Theorem 2.33, there is a unique functor $N_e : \mathcal{C}_e \to \mathcal{E}$ (necessarily fully faithful) such that $N_e \circ H = N$ and $S \circ N_e = F_e$. If we denote by $L : \mathcal{E} \to \mathcal{C}$ the left adjoint of N, then it is fully faithful and hence, since the unit $\eta : \mathrm{Id} \to NL$ is an isomorphism, we have that $\mathrm{Id} \cong N \circ L = N_e \circ H \circ L$. Therefore, N_e is essentially surjective on objects. Since it is also fully faithful, we get N_e is an equivalence of categories and, from $\mathrm{Id} \cong N_e \circ H \circ L$, it has quasi-inverse $H \circ L$. Thus, $(H \circ L) \circ N_e \cong \mathrm{Id}$ and hence $(H \circ L) \circ N = (H \circ L \circ N_e) \circ H \cong \mathrm{Id} \circ H = H$, from which it follows that H is a bireflection as N is. Thus, by Proposition 2.69, the idempotent e splits. Moreover, the factorization

$F = S \circ N$, up to the category equivalence N_e, is the same given by the coidentifier within Theorem 2.33.

Conversely, if the natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ attached to the semiseparable functor F splits, then by Proposition 2.69 the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$ results to be a bireflection. Thus, since by Theorem 2.33 the semiseparable functor F factors as $H : \mathcal{C} \to \mathcal{C}_e$ followed by a separable functor $F_e : \mathcal{C}_e \to \mathcal{D}$, we achieve the desired factorization of F into a bireflection followed by a separable functor. □

Remark 2.72. By Theorem 2.33, any semiseparable functor $F : \mathcal{C} \to \mathcal{D}$ factors as $H : \mathcal{C} \to \mathcal{C}_e$ followed by a separable functor $F_e : \mathcal{C}_e \to \mathcal{D}$, where e is the associated idempotent natural transformation. Assume that e splits. Then, by Proposition 2.69 $H : \mathcal{C} \to \mathcal{C}_e$ is a bireflection. In particular, H is a coreflection and F_e is conservative. This is what is called an *image-factorization* of F in [16, Definition 1.1]. Image-factorizations are unique up to an equivalence of categories, see [16, Lemma 1.2]. As a consequence, if we can write $F = S \circ N$ where $S : \mathcal{E} \to \mathcal{D}$ is conservative (e.g. separable) and $N : \mathcal{C} \to \mathcal{E}$ is a coreflection (e.g. a bireflection), then there is an equivalence $\mathcal{E} \cong \mathcal{C}_e$.

We now provide some examples of (co)reflections, cf. [4, Section 3.6]. The next is an example of a coreflection which is neither full nor faithful.

Example 2.73. [4, Example 3.33] Let $\Bbbk$ be a field, let Coalg be the category of coalgebras over $\Bbbk$ and let Set be the category of sets. The functor $G : \mathsf{Coalg} \to \mathsf{Set}$ that associates to a coalgebra C the set $G(C)$ of grouplike elements in C, is a coreflection. In fact, it has a fully faithful left adjoint F that takes a set S to the group-like coalgebra $\Bbbk S$. The unit and counit components are given by the canonical bijection $\eta_S : S \to GFS = G(\Bbbk S)$ and the canonical injection $\epsilon_C : FGC = \Bbbk G(C) \hookrightarrow C$, respectively. We show that G is not full, neither faithful. Let D be the matrix coalgebra $\mathrm{M}_2^c(\Bbbk)$. Note that $GD = G(\mathrm{M}_2^c(\Bbbk)) = \emptyset$ which is the initial object in Set. If we denote by 0 the zero coalgebra, then we also have $G0 = \emptyset$, so that $GD = G0$. If G is full, then there is a coalgebra map $f : D \to 0$. In particular, we have $\varepsilon_D = \varepsilon_0 \circ f = 0$, a contradiction as ε_D is the map that assigns to a matrix its trace. Thus, G is not full. Let us check it is not even faithful. Otherwise, by Proposition 1.30 *iii)* ϵ_C would be an epimorphism in Coalg, for every coalgebra C, but by [71, Theorem 3.1] an epimorphism in Coalg is necessarily surjective, whence ϵ_C would be invertible and hence every coalgebra C would be isomorphic to $\Bbbk G(C)$, a contradiction.

In the next, we exhibit a (full) coreflection which is not a bireflection.

Example 2.74. [4, Example 3.36] Let Set be the category of sets and, given an arbitrary field $\Bbbk$, let $\mathsf{Coalg}_{\mathrm{pt}}$ be the full subcategory of Coalg whose objects are pointed coalgebras over $\Bbbk$, i.e. coalgebras whose coradical is a grouplike coalgebra. We consider the functor $G : \mathsf{Coalg}_{\mathrm{pt}} \to \mathsf{Set}$, that associates to a coalgebra C the set $G(C)$ of grouplike elements in C. It has a fully faithful left adjoint F that takes a set S to the group-like coalgebra $\Bbbk S$, so it is a coreflection. The unit and counit components are given by the canonical bijection $\eta_S : S \to GFS = G(\Bbbk S)$ and the canonical injection $\epsilon_C : FGC = \Bbbk G(C) \hookrightarrow C$, respectively. By the dual Wedderburn-Malcev Theorem [68, Theorem 5.4.2], since C is pointed, there exists a coalgebra projection $\pi : C \to C_0 = \Bbbk G(C)$ such that $\pi \circ \epsilon_C = \mathrm{Id}$. Thus, ϵ_C is a split monomorphism for each pointed coalgebra C, and hence by Proposition 1.30 *iv)* G is full. Therefore, G is a full coreflection, but it is not a bireflection in general. Otherwise, G would be Frobenius and hence from $F \dashv G$ we should deduce $G \dashv F$. Consider the Sweedler's 4-dimensional Hopf algebra $H = \Bbbk\langle g, x \mid g^2 = 1, x^2 = 0, gx + xg = 0\rangle$ with coalgebra structure given by $\Delta(g) = g \otimes g$ and $\Delta(x) = x \otimes 1 + g \otimes x$ and set

$S := G(H) = \{1, g\}$. We have

$$\mathrm{Hom}_{\mathsf{Set}}(S, S) = \mathrm{Hom}_{\mathsf{Set}}(G(H), S) \cong \mathrm{Hom}_{\mathsf{Coalg}_{\mathrm{pt}}}(H, F(S)) = \mathrm{Hom}_{\mathsf{Coalg}}(H, \Bbbk S).$$

Since $\mathrm{Hom}_{\mathsf{Set}}(S, S)$ has cardinality 4, we get that $\mathrm{Hom}_{\mathsf{Coalg}}(H, \Bbbk S)$ must contain exactly 4 elements. For every $k \in \Bbbk$ define $f_k : H \to \Bbbk S$ by setting $f_k(1) = 1, f_k(g) = g, f_k(x) = k(1 - g) = f_k(xg)$. Then, f_k is a coalgebra map. By linear independence of grouplike elements, we have that $f_k \neq f_l$, for every $k, l \in \Bbbk$ such that $k \neq l$. Since $\mathrm{Hom}_{\mathsf{Coalg}}(H, \Bbbk S)$ contains 4 elements we deduce that the field $\Bbbk$ has at most 4 elements, a contradiction. Thus, G is not a bireflection, hence by Proposition 2.63 it is a not semiseparable coreflection.

Chapter 3

Applications and examples of semiseparable functors

In this chapter we test the notion of semiseparability on functors traditionally connected to the study of separability, recalled in Section 1.4 and studied mainly in [4, Section 3]. The first functors we look at are the restriction of scalars functor $\varphi_* : {}_S\mathcal{M} \to {}_R\mathcal{M}$, the extension of scalars functor $\varphi^* = S \otimes_R (-) : {}_R\mathcal{M} \to {}_S\mathcal{M}$ and the coinduction functor $\varphi^! = {}_R\mathrm{Hom}(S, -) : {}_R\mathcal{M} \to {}_S\mathcal{M}$ associated to a ring morphism $\varphi : R \to S$. These functors form an adjoint triple $\varphi^* \dashv \varphi_* \dashv \varphi^!$. Since φ_* is faithful, its semiseparability falls back to its separability, while the semiseparability of $\varphi^!$ is equivalent to the one of φ^* by Proposition 2.41. We characterize the latter in Proposition 3.1 in terms of the regularity of φ as a morphism of R-bimodules and in Proposition 3.8 in terms of the existence of a suitable central idempotent element $z \in R$. Then, Subsection 3.1.1 is devoted to the (semi)separability of φ^* restricted to the full subcategory ${}_R\mathcal{M}_f$ of ${}_R\mathcal{M}$ consisting of free left R-modules, which is an example of a not idempotent complete category. Since φ^* preserves free modules, we consider what we call the *free induction functor* $S \otimes_R (-) : {}_R\mathcal{M}_f \to {}_S\mathcal{M}_f$ between the categories of free left modules and its right adjoint, that we call the *free restriction of scalars functor*, see [5, Section 3].

As a new investigation of this thesis, in Subsection 3.1.2 we look at categories of (left) firm modules ${}_R\overline{\mathcal{M}}$ over a possibly non-unital ring R, and we extend Proposition 3.1 to the case of functors between these categories. Moreover, we show that a result similar to [78, Proposition 2.2] holds with respect to the (semi)separability of a right adjoint functor whose source category is a full subcategory of ${}_R\overline{\mathcal{M}}$ containing the firm ring R.

In Proposition 3.21 we describe the semiseparability of the corestriction of coscalars functor $\psi_* : \mathcal{M}^C \to \mathcal{M}^D$, and the coinduction functor $\psi^* = (-)\square_D C : \mathcal{M}^D \to \mathcal{M}^C$, attached to a coalgebra map $\psi : C \to D$. In Proposition 3.29 we extend this result to the case of a coring morphism, that we have not considered in [4].

Then, in Theorem 3.24 we show that the induction functor $(-) \otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}^{\mathcal{C}}$, attached to an R-coring $\mathcal{C}$, is semiseparable if, and only if, the coring counit $\varepsilon_{\mathcal{C}} : \mathcal{C} \to R$ is a regular morphism of R-bimodules.

Next, for an (R, S)-bimodule M, we consider the coinduction functor $\sigma_* = \mathrm{Hom}_S(M, -) : \mathcal{M}_S \to \mathcal{M}_R$ together with its left adjoint $\sigma^* = (-) \otimes_R M : \mathcal{M}_R \to \mathcal{M}_S$. We present a semiseparable version of M-separability over R for the ring S, given by Sugano and recalled in Subsection 1.4.4. In Theorem 3.31 we show that the semiseparability of σ_* can be completely rewritten both in terms of the *M-semiseparability of S over R*, and in terms of regularity of the evaluation map plus a mild condition that is redundant when M is projective as a right S-module. In Corollary 3.34 we prove that S is M-separable over R

if, and only if, S is M-semiseparable over R and M is a generator in $\mathcal{M}_S$. A different characterization of M-semiseparability of S over R is obtained in Proposition 3.36 in terms of the existence of a central idempotent in S. In Example 3.37 we exhibit an example where S is M-semiseparable but not M-separable over R.

Then, if one adds the assumption that M is finitely generated and projective as a right S-module, further characterizations of the semiseparability of σ_* and σ^* can be given (Proposition 3.39, Proposition 3.40). As a particular case, in Corollary 3.41 and Corollary 3.42 we apply these results to the (R,S)-bimodule $M := {}_RS_S$, with left action induced by a morphism of rings $\varphi : R \to S$.

It is worth noticing that the above functors φ^*, $(-) \otimes_R \mathcal{C}$, and σ_* have sources which are idempotent complete categories so that, by Corollary 2.71, they always admit a factorization as a bireflection followed by a separable functor, when they are semiseparable. In Proposition 3.6, Corollary 3.27, and Proposition 3.38, we explicitly provide such factorizations.

Finally, Section 3.5 concerns the semiseparability of the coinvariant functor $(-)^{\mathrm{co}B}$: $\mathfrak{M}_B^B \to \mathfrak{M}$, from the category of right Hopf modules over a $\Bbbk$-bialgebra B to the category of $\Bbbk$-vector spaces over a field $\Bbbk$. In Theorem 3.44 we show that it is semiseparable if, and only if, B is a right Hopf algebra with anti-multiplicative and anti-comultiplicative right antipode.

3.1 Extension and restriction of scalars

Let R, S be unital rings. We recall from Section 1.4.1 that a morphism of rings $\varphi : R \to S$ induces the restriction of scalars functor $\varphi_* : {}_S\mathcal{M} \to {}_R\mathcal{M}$, the extension of scalars functor $\varphi^* := S \otimes_R (-) : {}_R\mathcal{M} \to {}_S\mathcal{M}$, and the coinduction functor $\varphi^! := {}_R\mathrm{Hom}(S, -)$: ${}_R\mathcal{M} \to {}_S\mathcal{M}$, which form an adjoint triple $\varphi^* \dashv \varphi_* \dashv \varphi^!$.

From Proposition 2.41 we know that $\varphi^!$ is semiseparable (resp., separable, naturally full) if, and only if, so is φ^*. Since φ_* is faithful, we have that φ_* is semiseparable if, and only if, φ_* is separable, that is, S/R is separable. We point out that naming S/R "semiseparable" whenever φ_* is semiseparable retrieves the notion of separable extension of rings (see Proposition 1.40 *i*)).

In the next results we provide characterizations for the semiseparability of the functor φ^*, and hence of $\varphi^!$.

Proposition 3.1. [4, Proposition 3.1] *Let $\varphi : R \to S$ be a morphism of rings. Then, the extension of scalars functor $\varphi^* = S \otimes_R (-) : {}_R\mathcal{M} \to {}_S\mathcal{M}$ is semiseparable if, and only if, φ is a regular morphism of R-bimodules, i.e., there is $E \in {}_R\mathrm{Hom}_R(S, R)$ such that $\varphi \circ E \circ \varphi = \varphi$, i.e., such that $\varphi E(1_S) = 1_S$.*

Proof. By Theorem 2.36, φ^* is semiseparable if, and only if, there exists a natural transformation $\nu \in \mathrm{Nat}(\varphi_*\varphi^*, \mathrm{Id}_{{}_R\mathcal{M}})$ such that $\eta \circ \nu \circ \eta = \eta$. By Remark 1.39 there is a bijective correspondence $\mathrm{Nat}(\varphi_*\varphi^*, \mathrm{Id}_{{}_R\mathcal{M}}) \cong {}_R\mathrm{Hom}_R(S, R)$. So, given ν for φ^*, we consider the corresponding $E \in {}_R\mathrm{Hom}_R(S, R)$, $E(s) := \nu_R(s \otimes_R 1_R)$, for every $s \in S$. Then, for every $r \in R$, we get $(\varphi \circ E \circ \varphi)(r) = \varphi(E(\varphi(r))) = \varphi(\nu_R(\varphi(r) \otimes_R 1_R)) = \varphi(\nu_R(\eta_R(r))) = r_S\eta_R(\nu_R(\eta_R(r))) = r_S\eta_R(r) = \varphi(r)$ where $r_S : S \otimes_R R \to S, s \otimes_R r \mapsto s\varphi(r)$, is the canonical isomorphism. Conversely, given $E \in {}_R\mathrm{Hom}_R(S, R)$ such that $\varphi \circ E \circ \varphi = \varphi$, define

$\nu_M : S \otimes_R M \to M$, $\nu_M(s \otimes_R m) = E(s)m$, for every $M \in {}_R\mathcal{M}$, $m \in M$ and $s \in S$. Then,

$$\begin{aligned}(\eta_M \circ \nu_M \circ \eta_M)(m) &= \eta_M(\nu_M(1_S \otimes_R m)) = \eta_M(\nu_M(\varphi(1_R) \otimes_R m) = \eta_M(E(\varphi(1_R))m) \\ &= 1_S \otimes_R E(\varphi(1_R))m = 1_S E(\varphi(1_R)) \otimes_R m = \varphi(E(\varphi(1_R))) \otimes_R m \\ &\overset{\varphi E \varphi = \varphi}{=} \varphi(1_R) \otimes_R m = 1_S \otimes_R m = \eta_M(m).\end{aligned}$$

Since E is a morphism of R-bimodules, we get $(\varphi \circ E \circ \varphi)(r) = \varphi(E(\varphi(r))) = \varphi(E(r1_S)) = \varphi(rE(1_S)) = \varphi(r)\varphi E(1_S)$. As a consequence, the condition $(\varphi \circ E \circ \varphi)(r) = \varphi(r)$ is equivalent to $\varphi E(1_S) = 1_S$. □

Remark 3.2. The result of Proposition 3.1 is left-right symmetric as there is also a bijective correspondence $\mathrm{Nat}(\varphi_*\varphi^*, \mathrm{Id}_{\mathcal{M}_R}) \cong {}_R\mathrm{Hom}_R(S, R)$, for $\varphi^* := (-) \otimes_R S \dashv \varphi_* : \mathcal{M}_S \to \mathcal{M}_R$. Thus, $- \otimes_R S : \mathcal{M}_R \to \mathcal{M}_S$ is semiseparable if, and only if, so is $S \otimes_R - : {}_R\mathcal{M} \to {}_S\mathcal{M}$.

The following is an example of a semiseparable functor which is neither separable nor naturally full.

Example 3.3. [4, Example 3.2] Let $\varphi : R \to S$ and $\psi : Q \to R$ be morphisms of rings whose induction functors φ^* and ψ^* are separable and naturally full, respectively. This means that there is $E \in {}_R\mathrm{Hom}_R(S, R)$ such that $E \circ \varphi = \mathrm{Id}_R$ (in particular φ is injective) and there is $D \in {}_Q\mathrm{Hom}_Q(R, Q)$ such that $\psi \circ D = \mathrm{Id}_R$ (in particular ψ is surjective). By Corollary 2.35, the composition $\varphi^* \circ \psi^* \cong (\varphi \circ \psi)^*$ is semiseparable. The map corresponding to $\varphi \circ \psi$ via Proposition 3.1 is $D \circ E \in {}_Q\mathrm{Hom}_Q(S, Q)$. Note that, if $\varphi \circ \psi$ is neither injective nor surjective, we can conclude that $(\varphi \circ \psi)^*$ is neither separable nor naturally full. For instance, let $\varphi : \mathbb{Q} \to \mathbb{Q}[X]$ be the canonical injection of the field of rational numbers into the polynomial ring over it and let $\psi : \mathbb{Q} \times \mathbb{Z} \to \mathbb{Q}$ be given by $\psi((q, z)) = q$. Then, we can define D by setting $D(q) = (q, 0)$ and E to be the evaluation at 0 of the given polynomial. So, $(\varphi \circ \psi)^*$ is semiseparable but it is neither separable nor naturally full.

In a similar way, the following example shows that semiseparable functors are not closed under composition.

Example 3.4. [4, Example 3.3] Let $\varphi : R \to S$ and $\psi : S \to Q$ be morphisms of rings whose induction functors φ^* and ψ^* are separable and naturally full respectively (in particular, both semiseparable by Proposition 2.5). This means there is $E \in {}_R\mathrm{Hom}_R(S, R)$ such that $E \circ \varphi = \mathrm{Id}_R$ and there is $D \in {}_S\mathrm{Hom}_S(Q, S)$ such that $\psi \circ D = \mathrm{Id}_Q$. The results we have proved so far do not allow to conclude that the composition $\psi^* \circ \varphi^* \cong (\psi \circ \varphi)^*$ is semiseparable. Indeed, we can provide a specific example where this is not true. Let $\varphi : \mathbb{Z} \to \mathbb{Q} \times \mathbb{Z}, z \mapsto (z, z)$, and let $\psi : \mathbb{Q} \times \mathbb{Z} \to \mathbb{Q}$ be given by $\psi((q, z)) = q$. Then, we can define D by setting $D(q) = (q, 0)$ and E by setting $E((q, z)) = z$. In this way we get that φ^* and ψ^* are separable and naturally full, respectively. Let us show that $(\psi \circ \varphi)^*$ is not semiseparable. Otherwise, by Proposition 3.1 there exists $E' \in {}_\mathbb{Z}\mathrm{Hom}_\mathbb{Z}(\mathbb{Q}, \mathbb{Z})$ such that $\psi\varphi(E'(1_\mathbb{Q})) = 1_\mathbb{Q}$. Since ${}_\mathbb{Z}\mathrm{Hom}_\mathbb{Z}(\mathbb{Q}, \mathbb{Z}) = \{0\}$, this means $0_\mathbb{Z} = 1_\mathbb{Z}$, a contradiction.

We show that all morphisms of rings $\varphi : R \to S$ whose induction functor $\varphi^* = S \otimes_R (-) : {}_R\mathcal{M} \to {}_S\mathcal{M}$ is semiseparable are of the form given in Example 3.3. More precisely, we show that φ^* factors as a bireflection followed by a separable functor. First we need the following observation.

Remark 3.5. Let $\varphi : R \to S$ be an epimorphism in the category of rings. By [81, Proposition 1.2] the faithful functor φ_* is also full, and hence its left adjoint $\varphi^* = S \otimes_R (-)$ is a reflection, whereas its right adjoint $\varphi^! = {}_R\mathrm{Hom}(S, -)$ is a coreflection. Thus, Corollary 2.64 applies in this case to get that φ^* is naturally full if, and only if, it is semiseparable if, and only if, it is Frobenius, that is, in the same way $\varphi^!$ is naturally full if, and only if, it is semiseparable if, and only if, it is Frobenius. In particular, in this case φ^* and $\varphi^!$ are isomorphic bireflections.

Proposition 3.6. [4, Proposition 3.6] *Let $\varphi : R \to S$ be a morphism of rings. Write $\varphi = \iota \circ \overline{\varphi}$ where $\iota : \varphi(R) \to S$ is the canonical inclusion and $\overline{\varphi} : R \to \varphi(R)$ is the corestriction of φ to its image $\varphi(R)$. Then, the induction functor $\varphi^* := S \otimes_R (-) : {}_R\mathcal{M} \to {}_S\mathcal{M}$ is semiseparable if, and only if, ι^* is separable and $\overline{\varphi}^*$ is a bireflection.*

Proof. If φ^* is semiseparable, by Proposition 3.1, there is $E \in {}_R\mathrm{Hom}_R(S, R)$ such that $\varphi \circ E \circ \varphi = \varphi$, i.e. $\iota \circ \overline{\varphi} \circ E \circ \iota \circ \overline{\varphi} = \iota \circ \overline{\varphi}$. Since ι is injective and $\overline{\varphi}$ is surjective, we get $\overline{\varphi} \circ E \circ \iota = \mathrm{Id}_{\varphi(R)}$ which implies that ι^* is separable. On the other hand, $\overline{\varphi}^*$ is a bireflection in view of Remark 3.5 and surjectivity of $\overline{\varphi}$. Conversely, if ι^* is separable and $\overline{\varphi}^*$ is a bireflection, whence naturally full, then the composition $\iota^* \circ \overline{\varphi}^* \cong (\iota \circ \overline{\varphi})^* = \varphi^*$ is semiseparable by Corollary 2.35. □

Remark 3.7. In the proof of Proposition 3.6 we obtained a factorization $\iota^* \circ \overline{\varphi}^* \cong \varphi^*$ in case φ^* is semiseparable. In view of Corollary 2.71, this factorization is the same given by the coidentifier within Theorem 2.33, up to a category equivalence.

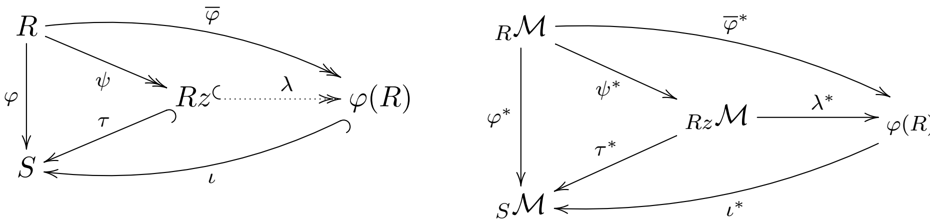
A further characterization of the semiseparability of φ^* can be proved in terms of the existence of a suitable central idempotent element $z \in R$. First recall that, given a central idempotent element z in a ring R, then $zRz = Rz$ is a ring with addition and multiplication those of R restricted to zR and with identities $0_{Rz} = 0_R z = 0_R$ and $1_{Rz} = 1_R z = z$, and there is a surjective ring homomorphism $R \to Rz, r \mapsto rz$, see [1, 1.16].

Proposition 3.8. [4, Proposition 3.6] *Let $\varphi : R \to S$ be a ring homomorphism. Then, the induction functor $\varphi^* = S \otimes_R (-) : {}_R\mathcal{M} \to {}_S\mathcal{M}$ is semiseparable if, and only if, there is a central idempotent $z \in R$ (necessarily unique) such that $\varphi(z) = 1_S$ and the ring map $\tau := \varphi_{|Rz} : Rz \to S$ is split-mono as an Rz-bimodule map.*

Proof. By Proposition 3.1, the functor φ^* is semiseparable if, and only if, there exists $E \in {}_R\mathrm{Hom}_R(S, R)$ such that $\varphi E(1_S) = 1_S$. Assume that there is E as above and set $z := E(1_S) \in R$. Clearly $\varphi(z) = \varphi E(1_S) = 1_S$, i.e. $\varphi(z) = 1_S$. For any $r \in R$ we have $rz = rE(1_S) = E(\varphi(r)1_S) = E(\varphi(r))$ and similarly $zr = E(\varphi(r))$ so that $rz = zr$, i.e. z is central. Taking $z = r$ in the computation above, we get $zz = E(\varphi(z)) = E(1_S) = z$, thus z is an idempotent. Concerning the ring map $\tau := \varphi_{|Rz} : Rz \to S$, consider the canonical projection $\psi : R \to Rz, r \mapsto rz$. Since z is central, we get that ψ is R-bilinear so that the map $\pi := \psi \circ E : S \to Rz, s \mapsto E(s)z$ is R-bilinear as a composition of bilinear maps. In particular, π is Rz-bilinear. Then, for any $r \in R$ we have $\pi\tau(rz) = \pi\varphi(rz) = \pi(\varphi(r)\varphi(z)) = \pi(r1_S) = r\pi(1_S) = rE(1_S)z = rzz = rz$, hence $\pi \circ \tau = \mathrm{Id}_{Rz}$, i.e. τ is a split-monomorphism of Rz-bimodules.

Conversely, assume there is a central idempotent $z \in S$ such that $\varphi(z) = 1_S$ and the ring map $\tau := \varphi_{|Rz} : Rz \to S$ is a split-mono through an Rz-bimodule map $\pi : S \to Rz$. Set $E : S \to R, s \mapsto \pi(s)$. Then, $rs = \varphi(r)s = \varphi(r)\varphi(z)s = \varphi(rz)s = (rz)s$, so that $E(rs) = E((rz)s) = \pi((rz)s) = rz\pi(s) = r\pi(s) = rE(s)$, where the second-last equality follows from the fact that $\pi(s) \in Rz$ and z is a central idempotent. Similarly one gets $E(sr) = E(s)r$ so that E is R-bilinear. Finally, we have $E(1_S) = \pi(1_S) = \pi(\varphi(z)) = \pi(\tau(z)) = z$ and hence $\varphi E(1_S) = \varphi(z) = 1_S$. Assume there is another idempotent $z' \in R$ as in the statement. Then, $zz' = E(1_S)z' = E(1_S z') = E(\varphi(z')) = E(1_S) = z$. By exchanging the roles of z and z' we get $z'z = z'$, and hence $z = z'$. □

Remark 3.9. [4, Remark 3.7] In the proof of Proposition 3.8 we considered the maps $\tau := \varphi_{|Rz} : Rz \to S$ and $\psi : R \to Rz, r \mapsto rz$. Since $\varphi(z) = 1_S$, we get the equality $\varphi = \tau \circ \psi$ which provides the factorization $\varphi^* \cong \tau^* \circ \psi^*$. On the other hand, in Proposition 3.6 we obtained the equality $\varphi = \iota \circ \overline{\varphi}$, where $\iota : \varphi(R) \to S, s \mapsto s$, and $\overline{\varphi} : R \to \varphi(R), r \mapsto \varphi(r)$, which yields the factorization $\varphi^* \cong \iota^* \circ \overline{\varphi}^*$. Define the morphism $\lambda : Rz \to \varphi(R), rz \mapsto \varphi(r)$. Then, the diagrams



commute. The first diagram entails that λ is both injective and surjective whence bijective. As a consequence, the given factorizations are the same up to the equivalence λ^*.

3.1.1 The free induction and restriction functors

We now focus on the (semi)separability of the induction functor and of the restriction of scalars functor, restricted to categories of free modules. Let R be a unital ring and let ${}_R\mathcal{M}_f$ denote the full subcategory of ${}_R\mathcal{M}$ consisting of free left R-modules. Given a ring morphism $\varphi : R \to S$, the induction functor $\varphi^* = S \otimes_R (-) : {}_R\mathcal{M} \to {}_S\mathcal{M}$, preserves free modules as $S \otimes_R R^{(B)} \cong (S \otimes_R R)^{(B)} \cong S^{(B)}$, giving rise to the functor

$$\varphi_f^* = S \otimes_R (-) : {}_R\mathcal{M}_f \to {}_S\mathcal{M}_f,$$

that we call the **free induction functor**.

We have the following result.

Proposition 3.10. [5, Proposition 3.10] *Let $\varphi : R \to S$ be a ring morphism. The following assertions are equivalent:*

(i) *the free induction functor $\varphi_f^* : {}_R\mathcal{M}_f \to {}_S\mathcal{M}_f$ has a right adjoint φ_{*f};*

(ii) *S is free as a left R-module;*

(iii) *the restriction of scalars functor $\varphi_* : {}_S\mathcal{M} \to {}_R\mathcal{M}$ preserves free modules.*

In case the above equivalent conditions hold, then φ_{*f} is induced by φ_* and the unit and counit of $(\varphi_f^*, \varphi_{*f})$ are the restrictions of the ones of (φ^*, φ_*). Moreover, if $S \neq 0$, then φ is injective and φ_f^* is faithful.

We call the functor φ_{*f} the **free restriction of scalars functor**.

Proof. (i) $\Rightarrow$ (ii). Assume that φ_f^* has a right adjoint $G : {}_S\mathcal{M}_f \to {}_R\mathcal{M}_f$. Then, we have the following isomorphisms of left R-modules: $S \cong {}_S\mathrm{Hom}({}_SS, {}_SS) \cong {}_S\mathrm{Hom}(S \otimes_R R, {}_SS) = {}_S\mathrm{Hom}(\varphi_f^*(R), {}_SS) \cong {}_R\mathrm{Hom}({}_RR, {}_RG(S)) \cong {}_RG(S)$. Since ${}_RG(S)$ is a free left R-module, then so is S.

(ii) $\Rightarrow$ (iii). Assume that S is a free left R-module. Then, $S \cong R^{(J)}$. If X is a free left S-module (i.e. $X \cong S^{(A)}$), then it can be regarded as a left R-module where the action of R is given by $R \times X \to X$, $(r, x) \mapsto \varphi(r)x$. Then $\varphi_*(X) = {}_RX \cong ({}_RS)^{(A)} \cong (R^{(J)})^{(A)} \cong R^{(A \times J)}$ is a free left R-module.

$(iii) \Rightarrow (i)$. If φ_* preserves free modules, it induces $\varphi_{*f} : {}_S\mathcal{M}_f \to {}_R\mathcal{M}_f$. Since the inclusion functors $i_S : {}_S\mathcal{M}_f \hookrightarrow {}_S\mathcal{M}$ and $i_R : {}_R\mathcal{M}_f \hookrightarrow {}_R\mathcal{M}$ are fully faithful, then the assumptions of Lemma 2.10 are satisfied and $(\varphi_f^*, \varphi_{*f})$ results to be an adjunction. Indeed, the square

$$\begin{array}{ccc} {}_S\mathcal{M}_f & \xrightarrow{i_S} & {}_S\mathcal{M} \\ \varphi_f^* \uparrow \downarrow \varphi_{*f} & & \varphi^* \uparrow \dashv \downarrow \varphi_* \\ {}_R\mathcal{M}_f & \xrightarrow{i_R} & {}_R\mathcal{M} \end{array}$$

is commutative, i.e. $i_R \circ \varphi_{*f} = \varphi_* \circ i_S$ and $i_S \circ \varphi_f^* = \varphi^* \circ i_R$, since φ_{*f} and φ_f^* have been defined as the restrictions of φ_* and φ^*, respectively. Since the pair (i_S, i_R) constitute a morphism of adjunctions, by Remark 1.15 we know that the unit η_f and counit ϵ_f of $(\varphi_f^*, \varphi_{*f})$ are related to the unit η and counit ϵ of (φ^*, φ_*) by the equalities $\eta i_R = i_R \eta_f$ and $\epsilon i_S = i_S \epsilon_f$. This means that η_f and ϵ_f are just the restrictions of η and ϵ, respectively. Explicitly, the unit is defined as $(\eta_f)_M : M \to S \otimes_R M$, $m \mapsto 1_S \otimes_R m$, for any $M \in {}_R\mathcal{M}_f$. Note that $(\eta_f)_M = (\varphi \otimes_R M) \circ l_M^{-1}$, where $l_M : R \otimes_R M \to M$ is the canonical isomorphism. Assume $S \neq 0$. Since M is a free left R-module, then it is flat, so that $(\eta_f)_M$ is injective as so is φ since $\mathrm{Ker}(\varphi) \subseteq \mathrm{Ann}_R(S)$ and the annihilator is zero as every non-trivial free left R-module is faithful. Then, φ_f^* is faithful. □

The free restriction of scalars functor φ_{*f} is a faithful functor, so by Proposition 2.5 it is semiseparable if, and only if, it is separable. Assuming that $S \neq 0$ is free as a left R-module, then by Proposition 3.10 the functor φ_f^* is faithful, hence again by Proposition 2.5 it is semiseparable if, and only if, it is separable. It remains to check when φ_{*f} and φ_f^* are separable functors.

Proposition 3.11. [5, Proposition 3.11] *Let $\varphi : R \to S$ be a morphism of rings, with S a free left R-module.*

i) The free induction functor $\varphi_f^* = S \otimes_R (-) : {}_R\mathcal{M}_f \to {}_S\mathcal{M}_f$ is separable if, and only if, φ is a split-mono as an R-bimodule map.

ii) The free restriction of scalars functor $\varphi_{*f} : {}_S\mathcal{M}_f \to {}_R\mathcal{M}_f$ is separable if, and only if, S/R is separable.

Proof. i). Assume that φ_f^* is separable. Then, by Theorem 1.18 there exists a natural transformation $\nu \in \mathrm{Nat}(\varphi_{*f}\varphi_f^*, \mathrm{Id}_{{}_R\mathcal{M}_f})$ such that $\nu \circ \eta = \mathrm{Id}$, where η is the unit of $(\varphi_f^*, \varphi_{*f})$, i.e. $\eta_M : M \to S \otimes_R M$, $m \mapsto 1_S \otimes_R m$, for any $M \in {}_R\mathcal{M}_f$. Now, since R is a free R-module, we consider $E \in {}_R\mathrm{Hom}_R(S, R)$ defined by setting $E(s) := \nu_R(s \otimes_R 1_R)$, for every $s \in S$. We note that the right R-linearity of E descends from the naturality of ν. Indeed, for any $s \in S$, $r \in R$, we have that $E(s)r = \nu_R(s \otimes_R 1_R)r = (f_r \circ \nu_R)(s \otimes_R 1_R) = (\nu_R \circ (S \otimes_R f_r))(s \otimes_R 1_R) = \nu_R(s \otimes_R r) = \nu_R(sr \otimes_R 1_R) = E(sr)$, where $f_r : R \to R$ is the left R-module map $r' \mapsto r'r$. Then, for every $r \in R$, we get $(E \circ \varphi)(r) = E(\varphi(r)) = \nu_R(\varphi(r) \otimes_R 1_R) = \nu_R(\eta_R(r)) = r$. Thus, $E \circ \varphi = \mathrm{Id}$. Conversely, if φ is a split-mono as an R-bimodule map, we mentioned that φ^* is separable. By Lemma 2.10, so is φ_f^*.
ii). Assume now that φ_{f*} is separable. Then, by Rafael Theorem, there exists a natural transformation $\gamma \in \mathrm{Nat}(\mathrm{Id}_{{}_S\mathcal{M}_f}, \varphi_f^*\varphi_{*f})$ such that $\epsilon \circ \gamma = \mathrm{Id}$, where ϵ is the counit of $(\varphi_f^*, \varphi_{*f})$, i.e. $\epsilon_N : S \otimes_R N \to N$, $s \otimes n \mapsto sn$, for any $N \in_S \mathcal{M}_f$. Now, since S is a free S-module, we consider $\gamma_S \in {}_S\mathrm{Hom}_S(S, S \otimes_R S)$ (note that the right S-linearity of γ_S descends from the naturality of γ). Since $\epsilon_S \circ \gamma_S = \mathrm{Id}$, we conclude that the multiplication

$m_S = \epsilon_S : S \otimes_R S \to S$ splits as an S-bimodule map so that S/R is separable. Conversely, if S/R is separable, we mentioned that φ_* is separable. By Lemma 2.10, so is φ_{f*}. □

Example 3.12. [5, Example 3.12]

i) Consider the morphism of rings $\varphi : \mathbb{R} \to \mathbb{R}\times\mathbb{R}, r \mapsto (r, r)$. The $\mathbb{R}$-bimodule structure induced on $\mathbb{R}\times\mathbb{R}$ via φ is the canonical one, so that it is free. The canonical projection $E : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (a, b) \mapsto a$, is a morphism of $\mathbb{R}$-bimodules such that $E \circ \varphi = \mathrm{Id}$. By Proposition 3.11, the free induction functor $\varphi^*_f = \mathbb{R}^2 \otimes_{\mathbb{R}} (-) : {}_{\mathbb{R}}\mathfrak{M}_f \to {}_{\mathbb{R}^2}\mathfrak{M}_f$ is separable.

ii) Let R be a ring and let $\varphi : R \to \mathrm{M}_n(R)$ be the canonical inclusion into the ring of $n \times n$ matrices over R. It is well-known that $\mathrm{M}_n(R)/R$ is separable (see e.g. [34, Example II]) and clearly $\mathrm{M}_n(R) \cong R^{n^2}$ is free as a left R-module. By Proposition 3.11, the free restriction of scalars functor $\varphi_{*f} : {}_{\mathrm{M}_n(R)}\mathcal{M}_f \to {}_R\mathcal{M}_f$ is separable.

3.1.2 Firm modules

Inspired by [18, Lemma 2.1] and by the central role played by firm modules in the Morita Theory for non-unital rings, see e.g. [25] and [44, 63] (where firm rings are named regular rings), we investigate here how Proposition 3.1 can be extended to the case where R is a firm ring and S is any ring, possibly non-unital, possibly non-firm. We plan to continue exploring semiseparability in this context in future research.

The notion of firm module goes back to D. Quillen [77] and it allows to develop a module theory over non-unital rings. For an arbitrary ring R, a left R-module M is called *firm* if the morphism

$$\mu_{R,M} : R \otimes_R M \to M, \quad \mu_{R,M}(r \otimes_R m) = rm,$$

is an isomorphism with inverse $\mathrm{d}_{R,M} : M \to R \otimes_R M$, $\mathrm{d}_{R,M}(m) = r \otimes_R m^r$, where the summation is understood. We denote ${}_R\overline{\mathcal{M}}$ the category of all firm left R-modules with left R-linear maps between them. If $M = R$, then $\mu_R := \mu_{R,M} = \mu_{M,R}$, so $R \in {}_R\overline{\mathcal{M}}$ if, and only if, $R \in \overline{\mathcal{M}}_R$. In this case, R is said to be a *firm* ring. Examples of firm rings are rings with unit, rings with local units and coseparable corings.

We first show that [78, Proposition 2.2] can be extended to the case of firm modules. The arguments used in the proof are similar to the ones of the unital case.

Proposition 3.13. *Let R be a firm ring. Let $\mathcal{C}$ be an additive category and let $\mathcal{D}$ be a full subcategory of ${}_R\overline{\mathcal{M}}$ containing ${}_RR$ and let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction of additive functors, with unit η and counit ϵ. Then, G is separable if, and only if, ϵ_R is a split-epi of R-bimodules.*

Proof. Assume that G is separable. By Theorem 1.18 there is a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\epsilon \circ \gamma = \mathrm{Id}$. Hence γ_R is a right inverse of ϵ_R, and it is a morphism of R-bimodules, cf. [78, Lemma 2.1]. Indeed, FGR has an R-bimodule structure obtained from the ring homomorphism $R^{\mathrm{op}} \to \mathrm{End}_{\mathcal{D}}(R) \to \mathrm{End}_{\mathcal{D}}(FGR)$, $r \mapsto \varphi(r) \mapsto FG(\varphi(r))$, where $\varphi(r) : R \to R$, $\varphi(r)(t) = tr$, is left R-linear, so FGR has the right R-module structure $dr = FG(\varphi(r))(d)$, for every $d \in FGR$, $r \in R$, hence by naturality of γ, for every $r, r' \in R$, we get $\gamma_R(rr') = \gamma_R(\varphi(r')(r)) = FG(\varphi(r'))(\gamma_R(r)) = \gamma_R(r)r'$.

Conversely, assume that ϵ_R is a split-epi of R-bimodules with right inverse $\gamma_R : R \to FGR$ such that $\epsilon_R \circ \gamma_R = \mathrm{Id}_R$. Let M be an object in $\mathcal{D}$. Consider the map

$$M \xrightarrow[\mathrm{d}_{R,M}]{} R \otimes_R M \xrightarrow[\gamma_R \otimes_R \mathrm{Id}_M]{} FGR \otimes_R M \xrightarrow[\psi_M]{} FGM, \tag{3.1}$$

where $\psi_M(u \otimes_R m) = FG(f_m)(u)$, for $f_m : R \to M$, $r \mapsto rm$, which is a map in $\mathrm{Hom}_{\mathcal{D}}(R, M)$ as $\mathcal{D}$ is a full subcategory of ${}_R\overline{\mathcal{M}}$. We observe that $\mathrm{d}_{R,-}$ and $\gamma_R \otimes_R \mathrm{Id}_-$ are natural transformations as, for every $f : M \to M'$ in ${}_R\overline{\mathcal{M}}$, we have $(R \otimes_R f)\mathrm{d}_{R,M}(m) = (R \otimes_R f)(r \otimes_R m^r) = r \otimes_R f(m^r) = r \otimes_R f(m)^r = \mathrm{d}_{R,M'}f(m)$ since $\mu_{R,M'}(r \otimes_R f(m^r)) = \mu_{R,M'}(r \otimes_R f(m)^r)$ so $r \otimes_R f(m^r) = r \otimes_R f(m)^r$, and $(FGR \otimes_R f)(\gamma_R \otimes_R \mathrm{Id}_M)(r \otimes_R m) = \gamma_R(r) \otimes_R f(m) = (\gamma_R \otimes_R \mathrm{Id}_{M'})(R \otimes_R f)(r \otimes_R m)$. We show that ψ_- is a natural transformation as well. In fact, for every $f : M \to M'$ in $\mathcal{D}$, we have $(FGf \circ \psi_M)(u \otimes_R m) = FGf(FG(f_m)(u)) = FG(f \circ f_m)(u) = FG(f_{f(m)})(u) = \psi_{M'}(u \otimes_R f(m)) = \psi_{M'}(\mathrm{Id}_{FGR} \otimes_R f)(u \otimes_R m)$, hence $FGf \circ \psi_M = \psi_{M'} \circ (\mathrm{Id}_{FGR} \otimes_R f)$. Thus, $\gamma'_- := \psi_-(\gamma_R \otimes_R \mathrm{Id}_-)\mathrm{d}_{R,-}$ is a natural tranformation. We prove that $\epsilon \circ \gamma' = \mathrm{Id}$. Indeed, for every $m \in M$,

$$\begin{aligned}\epsilon_M\gamma'_M(m) &= \epsilon_M\psi_M(\gamma_R \otimes_R \mathrm{Id}_M)\mathrm{d}_{R,M}(m) = \epsilon_M\psi_M(\gamma_R \otimes_R \mathrm{Id}_M)(r \otimes_R m^r)\\ &= \epsilon_M\psi_M(\gamma_R(r) \otimes_R m^r) = (\epsilon_M \circ FG(f_{m^r}))(\gamma_R(r)) = (f_{m^r} \circ \epsilon_R)(\gamma_R(r))\\ &= f_{m^r}(\epsilon_R\gamma_R(r)) = f_{m^r}(r) = rm^r = m.\end{aligned}$$

Then, by Theorem 1.18 G is separable. ☐

In case the map ψ_M is surjective, we prove a similar result that characterizes the semiseparability of a right adjoint functor.

Proposition 3.14. *Let R be a firm ring. Let $\mathcal{D}$ be a full subcategory of ${}_R\overline{\mathcal{M}}$ containing ${}_RR$ and let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction of functors between additive categories, with unit η and counit ϵ. If G is semiseparable, then ϵ_R is a regular morphism of R-bimodules. On the other hand, if ϵ_R is a regular morphism of R-bimodules and the map*

$$\psi_M : FGR \otimes_R M \to M,\ \psi_M(u \otimes_R m) = FG(f_m)(u), \tag{3.2}$$

for $f_m : R \to M$, $r \mapsto rm$, is surjective, for every $M \in \mathcal{D}$, then G is semiseparable.

Proof. Assume that G is semiseparable. By Theorem 2.36 there is a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\epsilon \circ \gamma \circ \epsilon = \epsilon$, hence $\epsilon_R\gamma_R\epsilon_R = \epsilon_R$. As in Proposition 3.13, γ_R is a morphism of R-bimodules. Conversely, assume that ϵ_R is a regular morphism of R-bimodules, i.e., there is a morphism $\gamma_R : R \to FGR$ of R-bimodules such that $\epsilon_R \circ \gamma_R \circ \epsilon_R = \epsilon_R$. Let M be an object in $\mathcal{D}$. Consider the map (3.1). By the proof of Proposition 3.13 we know that $\gamma'_- := \psi_-(\gamma_R \otimes_R \mathrm{Id}_-)\mathrm{d}_{R,-}$ is a natural tranformation. We prove that $\epsilon \circ \gamma' \circ \epsilon = \epsilon$. We observe that on generators $\epsilon_M\psi_M(u \otimes_R m) = \epsilon_M(FG(f_m)(u)) = f_m\epsilon_R(u) = \epsilon_R(u)m = \mu_{R,M}(\epsilon_R \otimes_R \mathrm{Id}_M)(u \otimes_R m)$, so that $\epsilon_M\psi_M = \mu_{R,M}(\epsilon_R \otimes_R \mathrm{Id}_M)$. Then, we get

$$\begin{aligned}\epsilon_M\gamma'_M\epsilon_M\psi_M &= \epsilon_M\gamma'_M\mu_{R,M}(\epsilon_R \otimes_R \mathrm{Id}_M) = \epsilon_M\psi_M(\gamma_R \otimes_R \mathrm{Id}_M)\mathrm{d}_{R,M}\mu_{R,M}(\epsilon_R \otimes_R \mathrm{Id}_M)\\ &= \epsilon_M\psi_M(\gamma_R\epsilon_R \otimes_R \mathrm{Id}_M) = \mu_{R,M}(\epsilon_R \otimes_R \mathrm{Id}_M)(\gamma_R\epsilon_R \otimes_R \mathrm{Id}_M)\\ &= \mu_{R,M}(\epsilon_R\gamma_R\epsilon_R \otimes_R \mathrm{Id}_M) = \mu_{R,M}(\epsilon_R \otimes_R \mathrm{Id}_M) = \epsilon_M\psi_M.\end{aligned}$$

Since ψ_M is surjective, we obtain $\epsilon_M\gamma'_M\epsilon_M = \epsilon_M$, for every $M \in \mathcal{D}$. Thus, by Theorem 2.36 G is semiseparable. ☐

We now show that Proposition 3.1 can be extended to the case where R is a firm ring and S is any ring, possibly non-unital, possibly non-firm. The following is the left version of [18, Lemma 2.1].

Lemma 3.15. (Cf. [18, Lemma 2.1]) *Let $\varphi : R \to S$ be a morphism of rings where S is any ring (possibly non-unital, possibly non-firm) and R is a firm ring. Then, the functor $S \otimes_R - : {}_R\overline{\mathcal{M}} \to {}_S\overline{\mathcal{M}}$ has a right adjoint given by $R \otimes_R - : {}_S\overline{\mathcal{M}} \to {}_R\overline{\mathcal{M}}$. The unit and the counit are given by*

$$\alpha_M : M \to R \otimes_R S \otimes_R M,\ m \mapsto t \otimes_R \varphi(r^t) \otimes_R m^r,\ \textit{for every } M \in {}_R\overline{\mathcal{M}},$$
$$\beta_N : S \otimes_R R \otimes_R N \to N,\ s \otimes_R r \otimes_R n \mapsto s\varphi(r)n,\ \textit{for every } N \in {}_S\overline{\mathcal{M}},$$

respectively.

Remark 3.16. In case R is a firm ring, we observe that $R \otimes_R R$ is a firm left R-module. In fact, consider $\mu_{R,R\otimes_R R} : R \otimes_R R \otimes_R R \to R \otimes_R R$, $r \otimes_R q \otimes_R r' \mapsto rq \otimes_R r'$, and $\mathrm{d}_{R,R\otimes_R R} : R \otimes_R R \to R \otimes_R R \otimes_R R$, $r \otimes_R r' \mapsto \xi \otimes_R r^\xi \otimes_R r'$. We have that $\mu_{R,R\otimes_R R}$ is well-defined as, for every $a \in R$, $\mu_{R,R\otimes_R R}(ra \otimes_R r' \otimes_R q) = rar' \otimes_R q = \mu_{R,R\otimes_R R}(r \otimes_R ar' \otimes_R q)$ and $\mu_{R,R\otimes_R R}(r \otimes_R r' \otimes_R aq) = rr' \otimes_R aq = rr'a \otimes_R q = \mu_{R,R\otimes_R R}(r \otimes_R r'a \otimes_R q)$; $\mathrm{d}_{R,R\otimes_R R}$ is well-defined as $t \otimes_R (ra)^t \otimes_R q = t \otimes_R r^t \otimes_R aq$, since $\mu_{R,R}(t \otimes_R r^t a) = tr^t a = ra = t(ra)^t = \mu_{R,R}(t \otimes_R (ra)^t)$. Moreover, we have $\mu_{R,R\otimes_R R}\mathrm{d}_{R,R\otimes_R R}(r \otimes_R r') = \mu_{R,R\otimes_R R}(\xi \otimes_R r^\xi \otimes_R r') = \xi r^\xi \otimes_R r' = r \otimes_R r'$, and $\mathrm{d}_{R,R\otimes_R R}\mu_{R,R\otimes_R R}(r \otimes_R q \otimes_R r') = \mathrm{d}_{R,R\otimes_R R}(rq \otimes_R r') = \mathrm{d}_{R,R\otimes_R R}(r \otimes_R qr') = \xi \otimes_R r^\xi \otimes_R qr' = \xi \otimes_R r^\xi q \otimes_R r' = \xi r^\xi \otimes_R q \otimes_R r' = r \otimes_R q \otimes_R r'$.

Proposition 3.17. *Let R be a firm ring and let S be an arbitrary ring (possibly non-unital, possibly non-firm). Consider a morphism of rings $\varphi : R \to S$. Then, the following assertions are equivalent:*

(*i*) *the extension of scalars functor $S \otimes_R (-) : {}_R\overline{\mathcal{M}} \to {}_S\overline{\mathcal{M}}$ is semiseparable;*

(*ii*) *the map $\hat{\varphi} : R \to R \otimes_R S \otimes_R R$, $r \mapsto t \otimes_R \varphi(q) \otimes (r^t)^q$ (equivalently, the component α_R of the unit α), is a regular morphism of R-bimodules, i.e. there is $E \in {}_R\mathrm{Hom}_R(R \otimes_R S \otimes_R R, R)$ such that $\hat{\varphi} \circ E \circ \hat{\varphi} = \hat{\varphi}$.*

Remark 3.18. By Remark 3.16 we know that $R \otimes_R R \in {}_R\overline{\mathcal{M}}$, so from $\mu_{R,R\otimes_R R}(t \otimes_R q^t \otimes_R r^q) = tq^t \otimes_R r^q = q \otimes_R r^q = t \otimes_R r^t = t \otimes_R q(r^t)^q = tq \otimes_R (r^t)^q = \mu_{R,R\otimes_R R}(t \otimes_R q \otimes_R (r^t)^q)$ we have $t \otimes_R q^t \otimes_R r^q = t \otimes_R q \otimes_R (r^t)^q$, hence $\hat{\varphi} = \alpha_R$.

Proof. From Lemma 3.15 we know that $S \otimes_R -$ is a left adjoint of $R \otimes_R -$.

We show that there is a bijective correspondence

$$\mathrm{Nat}(R \otimes_R S \otimes_R -, \mathrm{Id}_{{}_R\overline{\mathcal{M}}}) \cong {}_R\mathrm{Hom}_R(R \otimes_R S \otimes_R R, R).$$

Given $E \in {}_R\mathrm{Hom}_R(R \otimes_R S \otimes_R R, R)$, define $\nu_M : R \otimes_R S \otimes_R M \to M$, $\nu_M(r \otimes_R s \otimes_R m) = E(r \otimes_R s \otimes_R t)m^t$, for every $M \in {}_R\overline{\mathcal{M}}$, $m \in M$, $r \in R$, and $s \in S$. Since, for every $a \in R$, we have $\nu_M(ra \otimes_R s \otimes_R m) = E(ra \otimes_R s \otimes_R t)m^t = E(r \otimes_R as \otimes_R t)m^t = \nu_M(r \otimes_R as \otimes_R m)$ and $\nu_M(r \otimes_R sa \otimes_R m) = E(r \otimes_R sa \otimes_R t)m^t = E(r \otimes_R s \otimes_R at)m^t = E(r \otimes_R s \otimes_R t)(am)^t = \nu_M(r \otimes_R s \otimes_R am)$, where the second-last equality follows since $\mu_{R,M}(t \otimes_R (am)^t) = t(am)^t = am = atm^t = \mu_{R,M}(at \otimes_R m^t)$, hence $E(r \otimes_R s \otimes_R at)mt = \mu_{R,M}(E \otimes_R \mathrm{Id}_M)(r \otimes_R s \otimes_R at \otimes_R mt) = \mu_{R,M}(E \otimes_R \mathrm{Id}_M)(r \otimes_R s \otimes_R t \otimes_R (am)^t) = E(r \otimes_R s \otimes_R t)(am)^t$, then ν_M is well-defined.

Given a morphism $f : M \to M'$ in ${}_R\overline{\mathcal{M}}$, we observe that, since $M' \in {}_R\overline{\mathcal{M}}$, from $\mu_{R,M'}(r \otimes_R f(m)^r) = rf(m)^r = f(m) = f(rm^r) = \mu_{R,M'}(r \otimes_R f(m^r))$ we get that $r \otimes_R$

$f(m)^r = r\otimes_R f(m^r)$. Thus, for every $f : M \to M'$ in ${}_R\overline{\mathcal{M}}$, we have $(f\circ\nu_M)(r\otimes_R s\otimes_R m) = f(E(r\otimes_R s\otimes_R t)m^t) = E(r\otimes_R s\otimes_R t)f(m^t) = E(r\otimes_R s\otimes_R t)f(m)^t = \nu_{M'}(r\otimes_R s\otimes_R f(m)) = (\nu_{M'} \circ (R\otimes_R S\otimes_R f))(r\otimes_R s\otimes_R m)$, for every $r \in R$, $s \in S$, $m \in M$, so ν is a natural transformation.

Conversely, given $\nu : R\otimes_R S\otimes_R - \to \mathrm{Id}_{{}_R\overline{\mathcal{M}}}$, define $E := \nu_R : R\otimes_R S\otimes_R R \to R$. We know that E is left R-linear as so is ν_R. We show that E is also right R-linear. Indeed, by naturality of ν we have $f_r \circ \nu_R = \nu_R \circ (R\otimes_R S\otimes_R f_r)$, where $f_r : R \to R$, $r' \mapsto r'r$, is a morphism of left R-modules, so that $E(r\otimes_R s\otimes_R qr') = \nu_R(r\otimes_R s\otimes_R qr') = \nu_R(r\otimes_R s\otimes_R f_{r'}(q)) = f_{r'}\nu_R(r\otimes_R s\otimes_R q) = \nu_R(r\otimes_R s\otimes_R q)r' = E(r\otimes_R s\otimes_R q)r'$, hence $E \in {}_R\mathrm{Hom}_R(R\otimes_R S\otimes_R R, R)$. For every $s \in S$, $m \in M$, we have $E(r\otimes_R s\otimes_R t)m^t = f_{m^t}E(r\otimes_R s\otimes_R t) = f_{m^t}\nu_R(r\otimes_R s\otimes_R t) = \nu_M(r\otimes_R s\otimes_R tm^t) = \nu_M(r\otimes_R s\otimes_R m)$, where $f_{m^t} : R \to M$, $r \mapsto rm^t$, is in ${}_R\overline{\mathcal{M}}$, so the correspondence is bijective.

Now, by Theorem 2.36, $S\otimes_R -$ is semiseparable if, and only if, there exists a natural transformation $\nu \in \mathrm{Nat}(R\otimes_R S\otimes_R -, \mathrm{Id}_{{}_R\overline{\mathcal{M}}})$ such that $\alpha\circ\nu\circ\alpha = \alpha$. Given ν such that $\alpha\circ\nu\circ\alpha = \alpha$, consider the corresponding $E \in {}_R\mathrm{Hom}_R(R\otimes_R S\otimes_R R, R)$, $E := \nu_R$. Then, since by Remark 3.18 $\hat{\varphi} = \alpha_R$, we have that $\alpha_R E\alpha_R = \alpha_R\nu_R\alpha_R = \alpha_R$, so that α_R is regular. Conversely, given $E \in {}_R\mathrm{Hom}_R(R\otimes_R S\otimes_R R, R)$ such that $\hat{\varphi}\circ E\circ\hat{\varphi} = \hat{\varphi}$, define $\nu_M : R\otimes_R S\otimes_R M \to M$, $\nu_M(r\otimes_R s\otimes_R m) = E(r\otimes_R s\otimes_R t)m^t$, for every $r \in R$, $s \in S$, $m \in M \in {}_R\overline{\mathcal{M}}$. For every $r, q \in R$ and $s \in S$, we have $\nu_R(r\otimes_R s\otimes_R q) = E(r\otimes_R s\otimes_R t)q^t = E(r\otimes_R s\otimes_R tq^t) = E(r\otimes_R s\otimes_R q)$, hence $\alpha_R\nu_R\alpha_R = \alpha_R$. Define $\beta_M := \alpha_M\nu_M\alpha_M : M \to R\otimes_R S\otimes_R M$, for every $M \in {}_R\overline{\mathcal{M}}$, thus by assumption we have $\beta_R := \alpha_R\nu_R\alpha_R = \alpha_R$. Then, by naturality of β we get $\beta_M(m) = \beta_M(tm^t) = \beta_M(f_{m^t}(t)) = (R\otimes_R S\otimes_R f_{m^t})\beta_R(t) = (R\otimes_R S\otimes_R f_{m^t})\alpha_R(t) = (R\otimes_R S\otimes_R f_{m^t})(q\otimes_R \varphi(r^q)\otimes_R t^r) = q\otimes_R \varphi(r^q)\otimes_R t^r m^t = q\otimes_R \varphi(t^q)\otimes_R m^t = \alpha_M(m)$, where the second-last equality follows since $r\otimes_R t^r m^t = t\otimes_R m^t$, so $q\otimes_R\varphi(r^q)\otimes_R t^r m^t = (\mathrm{Id}_R\otimes_R\varphi\otimes_R\mathrm{Id}_M)(\mathrm{d}_{R,R}\otimes_R\mathrm{Id}_M)(rt^r\otimes_R m^t) = (\mathrm{Id}_R\otimes_R\varphi\otimes_R\mathrm{Id}_M)(\mathrm{d}_{R,R}\otimes_R\mathrm{Id}_M)(t\otimes_R m^t) = (q\otimes_R\varphi(t^q)\otimes_R m^t)$. Thus, by Theorem 2.36 $S\otimes_R -$ is semiseparable. □

Remark 3.19. A module M of a unital ring R is firm if, and only if, it is unital (i.e., $1_R \cdot m = m$, for every $m \in M$). Thus, in case R is unital, the adjunction of Lemma 3.15 becomes $S\otimes_R - =: F \dashv G := R\otimes_R - : {}_S\overline{\mathcal{M}} \to {}_R\mathcal{M}$ with unit α and counit β given by

$$\alpha_M : M \to R\otimes_R S\otimes_R M,\ m \mapsto 1_R\otimes_R \varphi(1_R)\otimes_R m,\ \text{for every } M \in {}_R\mathcal{M},$$
$$\beta_N : S\otimes_R R\otimes_R N \to N,\ s\otimes_R r\otimes_R n \mapsto s\varphi(r)n,\ \text{for every } N \in {}_S\overline{\mathcal{M}}.$$

In case R is unital and S is an arbitrary ring, the following can be seen as an intermediate step between Proposition 3.1 and Proposition 3.17.

Proposition 3.20. *Let R be a unital ring and let S be a possibly non-unital ring. Consider a morphism of rings $\varphi : R \to S$. Then, the following assertions are equivalent:*

(i) *the extension of scalars functor $F = S\otimes_R (-) : {}_R\mathcal{M} \to {}_S\overline{\mathcal{M}}$ is semiseparable;*

(ii) *φ is a regular morphism of R-bimodules, i.e. there is $E \in {}_R\mathrm{Hom}_R(S, R)$ such that $\varphi\circ E\circ\varphi = \varphi$;*

(iii) *there is $E \in {}_R\mathrm{Hom}_R(S, R)$ such that $(\varphi\circ E\circ\varphi)(1_R) = \varphi(1_R)$.*

3.2 Coinduction and corestriction of coscalars

Let $\Bbbk$ be a field and let $\psi : C \to D$ be a morphism of coalgebras over $\Bbbk$. Recall from Subsection 1.4.2 that ψ induces the adjunction $\psi_* \dashv \psi^* : \mathfrak{M}^D \to \mathfrak{M}^C$, where $\psi_* :$

$\mathfrak{M}^C \to \mathfrak{M}^D$ is the corestriction of coscalars functor and $\psi^* = (-)\square_D C : \mathfrak{M}^D \to \mathfrak{M}^C$ is the coinduction functor, with unit $\eta : \mathrm{Id}_{\mathcal{M}^C} \to \psi^*\psi_*$ and counit $\epsilon : \psi_*\psi^* \to \mathrm{Id}_{\mathcal{M}^D}$, given by

$$\eta_M : M \to M\square_D C,\ m \mapsto \sum m_0 \square_D m_1, \quad \text{and} \quad \epsilon_N : N\square_D C \to N,\ n\square_D c \mapsto n\varepsilon_C(c),$$

for any $M \in \mathfrak{M}^C$ and $N \in \mathfrak{M}^D$. Since ψ_* is faithful, then ψ_* is semiseparable if, and only if, it is separable.

In the following result we study the semiseparability of ψ^*.

Proposition 3.21. [4, Proposition 3.8] *Let $\psi : C \to D$ be a morphism of coalgebras. Then, the coinduction functor $\psi^* = (-)\square_D C : \mathfrak{M}^D \to \mathfrak{M}^C$ is semiseparable if, and only if, ψ is a regular morphism of D-bicomodules if, and only if, there is a D-bicomodule morphism $\chi : D \to C$ such that $\varepsilon_C \circ \chi \circ \psi = \varepsilon_C$.*

Proof. Assume that ψ^* is semiseparable. By Theorem 2.36, there exists a natural transformation $\gamma : \mathrm{Id}_{\mathfrak{M}^D} \to \psi_*\psi^*$ such that $\epsilon_N \circ \gamma_N \circ \epsilon_N = \epsilon_N$, for any $N \in \mathfrak{M}^D$. Since D is a right D-comodule, consider the right D-comodule map $\gamma_D : D \to D\square_D C$ and define the map $\chi : D \to C$ as $\chi := l_C \circ \gamma_D$, where $l_C : D\square_D C \to C, \sum_i d_i \otimes c_i \mapsto \sum_i \varepsilon_D(d_i)c_i$, is the canonical isomorphism. Note that ψ is a morphism of D-bicomodules and $\psi = \epsilon_D \circ l_C^{-1}$. We show that χ is a morphism of D-bicomodules. For any $f \in D^* = \mathrm{Hom}_{\Bbbk}(D, \Bbbk)$, consider the morphism of right D-comodules $\hat{f} : D \to D$, $d \mapsto \sum f(d_1)d_2$. In fact, $\Delta_D\hat{f}(d) = \Delta_D(\sum f(d_1)d_2) = \sum f(d_1)\Delta(d_2) = \sum f(d_1)d_2 \otimes d_3 = \sum f(d_{1_1})d_{1_2} \otimes d_2 = (\hat{f} \otimes D)(\sum d_1 \otimes d_2) = (\hat{f} \otimes D)\Delta_D(d)$. For any $d \in D$, denote $\gamma_D(d) := \sum_i d_i \otimes c_i \in D\square_D C$. Then, by naturality of γ, we have that $l_C\gamma_D\hat{f}(d) = \chi\hat{f}(d) = \sum f(d_1)\chi(d_2)$ is equal to $l_C(\hat{f}\square_D C)\gamma_D(d) = l_C(\hat{f}\square_D C)(\sum_i d_i \otimes c_i) = l_C(\sum_i f(d_{i_1})d_{i_2} \otimes c_i) = \sum_i f(d_i)c_i$. Since f is arbitrary, it follows that for all $d \in D$, $\gamma_D(d) = \sum_i d_i \otimes c_i = \sum d_1 \otimes \chi(d_2)$. Moreover, $\sum_i d_i \otimes c_i = \gamma_D(d) = l_C^{-1}l_C\gamma_D(d) = l_C^{-1}\chi(d) = (\psi\otimes C)\Delta_C(\chi(d)) = (\psi\otimes C)(\chi(d)_1 \otimes \chi(d)_2) = \psi(\chi(d)_1) \otimes \chi(d)_2$, then $\sum d_1 \otimes \chi(d_2) = \psi(\chi(d)_1) \otimes \chi(d)_2$, so χ is a morphism of left D-comodules, whence of D-bicomodules. We have $\psi\circ\chi\circ\psi = (\epsilon_D \circ l_C^{-1}) \circ (l_C \circ \gamma_D) \circ (\epsilon_D \circ l_C^{-1}) = \epsilon_D \circ \gamma_D \circ \epsilon_D \circ l_C^{-1} = \epsilon_D \circ l_C^{-1} = \psi$, hence χ is a regular morphism of D-bicomodules.

Assume that ψ is a regular morphism of D-bicomodules, i.e. there is a D-bicomodule morphism $\chi : D \to C$ such that $\psi\circ\chi\circ\psi = \psi$. Then $\varepsilon_C\circ\chi\circ\psi = \varepsilon_D\circ\psi\circ\chi\circ\psi = \varepsilon_D\circ\psi = \varepsilon_C$.

Assume now there is a D-bicomodule morphism $\chi : D \to C$ such that $\varepsilon_C \circ \chi \circ \psi = \varepsilon_C$ and let us prove that ψ^* is semiseparable. For any $N \in \mathfrak{M}^D$ define $\gamma_N : N \to N\square_D C$ as $\gamma_N(n) = \sum n_0 \otimes \chi(n_1)$, for every $n \in N$. Using that χ is a left D-comodule morphism, one easily checks that the image of γ_N is really contained in $N\square_D C$. Moreover, γ_N comes out to be a right D-comodule morphism, since χ is a morphism of right D-comodules, and natural in N. For every $n \in N$, $c \in C$, we have $\gamma_N\epsilon_N(n\square_D c) = \gamma_N(n\varepsilon_C(c)) = \gamma_N(n)\varepsilon_C(c) = \sum n_0 \otimes \chi(n_1)\varepsilon_C(c) = \sum n \otimes \chi\psi(c_1)\varepsilon_C(c_2) = \sum n \otimes \chi\psi(c)$, where in the second-last equality we used that $n\square_D c$ belongs to $N\square_D C$. Thus $\epsilon_N\gamma_N\epsilon_N(n\square_D c) = \epsilon_N(\sum n \otimes \chi\psi(c)) = \sum n\varepsilon_C\chi\psi(c) = n\varepsilon_C(c) = \epsilon_N(n\square_D c)$. Therefore, by Theorem 2.36, ψ^* is semiseparable. □

Example 3.22. [4, Example 3.9] It is known that the Axiom of Choice is equivalent to require that, for any function $f : A \to B$, there is a function $g : B \to A$ such that $f \circ g \circ f = f$. Consider the group-like coalgebras $\Bbbk A$ and $\Bbbk B$ and the coalgebra map $\psi := \Bbbk f : \Bbbk A \to \Bbbk B$ defined by setting $\psi(a) = f(a)$, for every $a \in A$. Define the linear map $\chi : \Bbbk B \to \Bbbk A$ by setting $\chi(b) = g(b)$ if $b \in \mathrm{Im}(f)$ and $\chi(b) = 0$ otherwise, for all $b \in B$. It is easy to check that χ is a $\Bbbk B$-bicomodule morphism such that $\varepsilon_{\Bbbk A}\circ\chi\circ\psi = \varepsilon_{\Bbbk A}$. Thus, by Proposition 3.21 the functor $\psi^* = (-)\square_{\Bbbk B}\Bbbk A : \mathfrak{M}^{\Bbbk B} \to \mathfrak{M}^{\Bbbk A}$ is semiseparable. However, it is neither separable nor naturally full in general. Indeed, if ψ^* is separable, then ψ is split-epi whence surjective. In this case, f is surjective too. Similarly, if ψ^* is naturally full, then ψ is split-mono whence injective. In this case, f is injective as well.

3.3 Corings

Let R be a ring and let $\mathcal{C}$ be an R-coring. Recall from Subsection 1.4.3 that the induction functor $G := (-) \otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}^{\mathcal{C}}$, $M \mapsto M \otimes_R \mathcal{C}$, $f \mapsto f \otimes_R \mathcal{C}$, is the right adjoint of the forgetful functor $F : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_R$, with unit and counit given by $\eta_M = \rho_M : M \to M \otimes_R \mathcal{C}$, for every $M \in \mathcal{M}^{\mathcal{C}}$, and $\epsilon_N = N \otimes_R \varepsilon_{\mathcal{C}} : N \otimes_R \mathcal{C} \to N$, $\epsilon_N(n \otimes_R c) = n\varepsilon_{\mathcal{C}}(c)$, for every $N \in \mathcal{M}_R$, $n \in N$, $c \in \mathcal{C}$, respectively.

Remark 3.23. Since $F : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_R$ is faithful, by Proposition 2.5 i), it is semiseparable if, and only if, it is separable so, defining the notion of "semicoseparable coring" whenever F is semiseparable retrieves the notion of coseparable coring. Analogously, naming a coalgebra "semicoseparable" whenever the forgetful functor $\mathfrak{M}^C \to \mathfrak{M}$ is semiseparable, would get back the notion of coseparable coalgebra, recalled in Subsection 1.4.2.

We now study when the induction functor $G = (-) \otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}^{\mathcal{C}}$ is semiseparable. In this case, we say that the R-coring $\mathcal{C}$ is **semicosplit**. Note that, since separable functors are in particular semiseparable, it follows that cosplit corings (i.e., G is separable) are in particular semicosplit.

Theorem 3.24. [4, Theorem 3.10] *Let $\mathcal{C}$ be an R-coring. Then, the following are equivalent:*

(i) *$\mathcal{C}$ is semicosplit;*

(ii) *the coring counit $\varepsilon_{\mathcal{C}} : \mathcal{C} \to R$ is regular as a morphism of R-bimodules;*

(iii) *there exists an invariant element $z \in \mathcal{C}^R = \{c \in \mathcal{C} \mid rc = cr, \text{ for all } r \in R\}$ such that*

$$\varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(c)$$

(equivalently, such that $\varepsilon_{\mathcal{C}}(z)c = c$), for every $c \in \mathcal{C}$.

Proof. (i) $\Rightarrow$ (ii). Assume that $\mathcal{C}$ is semicosplit, i.e. the induction functor $G = (-) \otimes_R \mathcal{C}$ is semiseparable. Then, by Theorem 2.36, there exists a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\epsilon \circ \gamma \circ \epsilon = \epsilon$. Consider the canonical isomorphism $l_{\mathcal{C}} : R \otimes_R \mathcal{C} \to \mathcal{C}$. Since R is a right R-module, consider the right R-linear map $\gamma_R : R \to R \otimes_R \mathcal{C}$. Let us check it is also left R-linear. For any $r \in R$ define the morphism $f_r : R \to R$ by $f_r(r') = rr'$. Since γ_R is natural, we have

$$\gamma_R(rr') = (\gamma_R \circ f_r)(r') = ((f_r \otimes_R \mathcal{C}) \circ \gamma_R)(r') = r\gamma_R(r').$$

Thus, γ_R is a morphism of R-bimodules. Define the R-bimodule map $\alpha := l_{\mathcal{C}} \circ \gamma_R : R \to \mathcal{C}$. By noting that $\varepsilon_{\mathcal{C}} = \epsilon_R \circ l_{\mathcal{C}}^{-1}$, we get

$$\varepsilon_{\mathcal{C}} \circ \alpha \circ \varepsilon_{\mathcal{C}} = (\epsilon_R \circ l_{\mathcal{C}}^{-1}) \circ (l_{\mathcal{C}} \circ \gamma_R) \circ (\epsilon_R \circ l_{\mathcal{C}}^{-1}) = \epsilon_R \circ \gamma_R \circ \epsilon_R \circ l_{\mathcal{C}}^{-1} = \epsilon_R \circ l_{\mathcal{C}}^{-1} = \varepsilon_{\mathcal{C}},$$

so that $\varepsilon_{\mathcal{C}}$ is a regular morphism of R-bimodules.

(ii) $\Rightarrow$ (iii). Assuming the regularity of $\varepsilon_{\mathcal{C}}$, i.e. the existence of an R-bimodule map α such that $\varepsilon_{\mathcal{C}} \circ \alpha \circ \varepsilon_{\mathcal{C}} = \varepsilon_{\mathcal{C}}$, we can set $z = \alpha(1_R) \in \mathcal{C}$. For $r \in R$, we have $rz = r\alpha(1_R) = \alpha(r) = \alpha(1_R)r = zr$, so that z is in $\mathcal{C}^R$. Moreover, from $\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}\alpha\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}\alpha(1_R\varepsilon_{\mathcal{C}}(c)) = \varepsilon_{\mathcal{C}}\alpha(1_R)\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c)$ it follows that $\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c)$, for every $c \in \mathcal{C}$.

$(iii) \Rightarrow (i)$. Suppose there exists $z \in \mathcal{C}^R$ such that $\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c)$, for every $c \in \mathcal{C}$. For any $N \in \mathcal{M}_R$ define $\gamma_N : N \to N \otimes_R \mathcal{C}$, $\gamma_N(n) = n \otimes_R z$, for every $n \in N$. Since $z \in \mathcal{C}^R$, for every $n \in N$, $r \in R$, we have $\gamma_N(nr) = nr \otimes_R z = n \otimes_R rz = n \otimes_R zr = \gamma_N(n)r$, so γ_N is a right R-module morphism, and it is also natural in N: indeed, for any morphism $f : N \to M$ in $\mathcal{M}_R$, $(\gamma_M \circ f)(n) = f(n) \otimes_R z = ((f \otimes_R \mathcal{C}) \circ \gamma_N)(n)$. Moreover, for every $n \in N$, $c \in \mathcal{C}$, we have

$$\begin{aligned}(\epsilon_N \circ \gamma_N \circ \epsilon_N)(n \otimes_R c) &= \epsilon_N \gamma_N(n\varepsilon_{\mathcal{C}}(c)) = \epsilon_N(n \otimes_R \varepsilon_{\mathcal{C}}(c)z) \\ &= n\varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c) = n\varepsilon_{\mathcal{C}}(c) \\ &= \epsilon_N(n \otimes_R c),\end{aligned}$$

where the second-last equality follows from the assumption $\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c)$. Therefore, by Theorem 2.36 G is semiseparable and $\mathcal{C}$ is semicosplit.

Finally, assume that $\varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(c)$, for every $c \in \mathcal{C}$. Then,

$$\varepsilon_{\mathcal{C}}(z)c = \varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c_{(1)})c_{(2)} = \varepsilon_{\mathcal{C}}(c_{(1)})c_{(2)} = c,$$

and hence $\varepsilon_{\mathcal{C}}(z)c = c$. Conversely, if $\varepsilon_{\mathcal{C}}(z)c = c$, for every $c \in \mathcal{C}$, then $\varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(\varepsilon_{\mathcal{C}}(z)c) = \varepsilon_{\mathcal{C}}(c)$. □

Remark 3.25. [4, Remark 3.11] In Proposition 1.50 we recalled that the functor $G := (-) \otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}^{\mathcal{C}}$ is naturally full if, and only if, there exists $z \in \mathcal{C}^R$ such that $c = \varepsilon_{\mathcal{C}}(c)z$, for every $c \in \mathcal{C}$. This characterization is a particular case of Theorem 3.24. Indeed, if there exists $z \in \mathcal{C}^R$ such that $c = z\varepsilon_{\mathcal{C}}(c)$ for every $c \in \mathcal{C}$, then $\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(z\varepsilon_{\mathcal{C}}(c)) = \varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c)$, for every $c \in \mathcal{C}$, and equivalently, $\varepsilon_{\mathcal{C}}(z)c = \varepsilon_{\mathcal{C}}(z)z\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(z\varepsilon_{\mathcal{C}}(c))z = \varepsilon_{\mathcal{C}}(c)z = c$. Moreover, G is separable if, and only if, there exists an invariant element $z \in \mathcal{C}^R$ such that $\varepsilon_{\mathcal{C}}(z) = 1_R$ and hence the equality $\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(z)\varepsilon_{\mathcal{C}}(c)$ trivially holds true in this case.

We already mentioned that a cosplit coring is semicosplit. We now give an example of a semicosplit coring which is not cosplit.

Example 3.26. [4, Example 3.14]

1) Let $\varphi : R \to S$ be a morphism of rings such that the induction functor $\varphi^* = S \otimes_R (-)$ is naturally full. As recalled in Proposition 1.42, there exists $\varepsilon \in {}_R\mathrm{Hom}_R(S, R)$ such that $\varphi \circ \varepsilon = \mathrm{Id}_S$. Since, in particular, $\varphi : R \to S$ is an epimorphism in the category of rings, by [81, Proposition XI.1.2], the multiplication $m : S \otimes_R S \to S$ is bijective and hence we can set $\Delta := m^{-1}$ so that $\Delta(s) = s \otimes_R 1_S = 1_S \otimes_R s$. We compute

$$\begin{aligned}(\varepsilon \otimes_R S)\Delta(s) &= (\varepsilon \otimes_R S)(1_S \otimes_R s) = \varepsilon(1_S) \otimes_R s \\ &= 1_R \otimes_R \varepsilon(1_S)s = 1_R \otimes_R \varphi\varepsilon(1_S)s = 1_R \otimes_R s,\end{aligned}$$

and similarly $(S \otimes_R \varepsilon)\Delta(s) = (S \otimes_R \varepsilon)(s \otimes_R 1_S) = s \otimes_R 1_S$. As a consequence, (S, Δ, ε) is an R-coring. Now $\varepsilon(1_S)s = \varphi\varepsilon(1_S)s = 1_S s = s$, so that $z := 1_S \in S^R$ fulfills the conditions of Theorem 3.24 guaranteeing that the functor $G := (-) \otimes_R S : \mathcal{M}_R \to \mathcal{M}^S$ is semiseparable and hence S is a semicosplit R-coring. Nevertheless, S is not cosplit in general. In fact, if G is separable, there exists $w \in S^R$ such that $1_R = \varepsilon(w)$ and hence, for every $r \in R$, we have $r = r1_R = r\varepsilon(w) = \varepsilon(rw)$, so that ε is surjective which, together with the condition $\varphi \circ \varepsilon = \mathrm{Id}_S$, implies that φ and ε are mutual inverses.

2) To get an example of 1) with φ not invertible, consider S and T rings, set $R := S \times T$, take $\varphi : R \to S, (s, t) \mapsto s$ and $\varepsilon : S \to R, s \mapsto (s, 0)$. Then, S is a semicosplit but not cosplit R-coring.

When $G = (-) \otimes_R \mathcal{C}$ is semiseparable we can provide an explicit factorization of it as a bireflection followed by a separable functor. By Corollary 2.71, this factorization amounts to the one given by the coidentifier, up to a category equivalence.

Corollary 3.27. [4, Corollary 3.12] *Let $\mathcal{C}$ be an R-coring. Then, $\mathcal{C}$ is semicosplit if, and only if, the induction functor $G = (-) \otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}^{\mathcal{C}}$ factors up to isomorphism as $\psi^* \circ G'$, where $\psi^* = (-)\square_I \mathcal{C} : \mathcal{M}^I \to \mathcal{M}^{\mathcal{C}}$ is separable and $G' = (-) \otimes_R I : \mathcal{M}_R \to \mathcal{M}^I$ is a bireflection for some morphism of corings $\psi : \mathcal{C} \to I$.*

Proof. Assume $\mathcal{C}$ is semicosplit, i.e., $G = (-) \otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}^{\mathcal{C}}$ is semiseparable. Then, by Theorem 3.24, there exists an invariant element $z_{\mathcal{C}} \in \mathcal{C}^R$ such that $\varepsilon_{\mathcal{C}}(c) = \varepsilon_{\mathcal{C}}(z_{\mathcal{C}})\varepsilon_{\mathcal{C}}(c)$, for every $c \in \mathcal{C}$. We observe that, since $\varepsilon_{\mathcal{C}}$ is a morphism of bimodules, $I := \mathrm{Im}(\varepsilon_{\mathcal{C}})$ is an ideal of R with multiplicative identity $z := \varepsilon_{\mathcal{C}}(z_{\mathcal{C}})$. Indeed, for any $r \in I$ there is $c \in \mathcal{C}$ such that $r = \varepsilon_{\mathcal{C}}(c)$ and hence $rz = \varepsilon_{\mathcal{C}}(c)\varepsilon_{\mathcal{C}}(z_{\mathcal{C}}) = \varepsilon_{\mathcal{C}}(c) = r$. Therefore the morphism $\varphi : R \to I, r \mapsto rz$, is a ring epimorphism (in fact it is surjective) and hence the map $m_I : I \otimes_R I \to I$ is bijective, see [81, Proposition XI.1.2]. Thus we can consider $\Delta_I = m_I^{-1} : I \to I \otimes_R I$, $\Delta_I(i) = i \otimes_R z = z \otimes_R i$, so that $(I, \Delta_I, \varepsilon_I)$ is an R-coring, where the counit $\varepsilon_I : I \hookrightarrow R$ is the canonical inclusion. Note that $\psi : \mathcal{C} \to I, c \mapsto \varepsilon_{\mathcal{C}}(c)$, is a morphism of corings and we can consider the corresponding coinduction functor $\psi^* = (-)\square_I \mathcal{C} : \mathcal{M}^I \to \mathcal{M}^{\mathcal{C}}$. Consider also the induction functor $G' := (-) \otimes_R I : \mathcal{M}_R \to \mathcal{M}^I$. We prove that G factors as $G \cong \psi^* \circ G'$, ψ^* is separable and G' is a bireflection. First we check that $G \cong \psi^* \circ G'$. In fact, for every $T \in \mathcal{M}_R$,

$$(\psi^* \circ G')(T) = \psi^*(T \otimes_R I) = (T \otimes_R I)\square_I \mathcal{C} \cong T \otimes_R (I \square_I \mathcal{C}) \cong T \otimes_R \mathcal{C} = G(T),$$

where the second-last isomorphism follows e.g. from [24, 22.5], once observed that $\lambda_{\mathcal{C}}(c) = \psi(c_{(1)}) \otimes_R c_{(2)} = z \otimes_R \varepsilon_{\mathcal{C}}(c_{(1)})c_{(2)} = z \otimes_R c$, and hence $\omega_{I,\mathcal{C}}(i \otimes_R c) = \rho_I(i) \otimes_R c - i \otimes_R \lambda_{\mathcal{C}}(c) = i \otimes_R z \otimes_R c - i \otimes_R z \otimes_R c = 0$, so that $\omega_{I,\mathcal{C}}$ is the zero map whence trivially T-*pure* [24, 40.13]. Let us check that G' is a bireflection. To this aim, first note that, since $i = \varepsilon_I(i)z$, for every $i \in I$, then G' is naturally full by Proposition 1.50.

The functor G' is right adjoint of the forgetful functor F' and the unit is $\eta'_M = \rho_M : M \to M \otimes_R I$ for every (M, ρ_M) in $\mathcal{M}^I$. Since $I = Rz$, for every $m \in M$ there is $m' \in M$ such that $\rho_M(m) = m' \otimes_R z$ and hence $m = \sum m_0 \varepsilon_I(m_1) = \sum m_0 m_1 = m'z$. As a consequence, $\rho_M(m) = m' \otimes_R z = m' \otimes_R zz = m'z \otimes_R z = m \otimes_R z$ for every $m \in M$. Now, given $w \in M \otimes_R I$, there is $m \in M$ such that $w = m \otimes_R z = \rho_M(m)$ and hence ρ_M is surjective. Since it is also split-mono via $r_M \circ (M \otimes_R \varepsilon_I)$, where $r_M : M \otimes_R R \to M$ is the canonical isomorphism, we get that $\eta'_M = \rho_M$ is invertible and hence F' is fully faithful. Hence G' is a naturally full coreflection, thus a bireflection by Corollary 2.64.

It remains to check that ψ^* is separable. If we see $\mathcal{C}$ as an I-bicomodule with left structure $\lambda_{\mathcal{C}} : \mathcal{C} \to I \otimes_R \mathcal{C}, c \mapsto z \otimes c$, and right structure $\rho_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_R I, c \mapsto c \otimes z$, the map $\nu : I \to \mathcal{C}, i \mapsto iz_{\mathcal{C}} = z_{\mathcal{C}}i$, is an I-bicomodule morphism which satisfies $\psi \circ \nu = \mathrm{Id}_I$. Indeed, $\psi(\nu(i)) = \psi(iz_{\mathcal{C}}) = \varepsilon_{\mathcal{C}}(iz_{\mathcal{C}}) = i\varepsilon_{\mathcal{C}}(z_{\mathcal{C}}) = iz = i$. The existence of ν implies that $\psi^* : (-)\square_I \mathcal{C} : \mathcal{M}^I \to \mathcal{M}^C$ is separable by [42, Theorem 5.8] in case $A = B = R$ and $\mathcal{D} = I$ once we have checked its hypothesis, namely that both ${}_R R$ and ${}_R \mathcal{C}$ preserve the equalizer of $(\rho_M \otimes_R \mathcal{C}, M \otimes_R \lambda_{\mathcal{C}})$ for every (M, ρ_M) in $\mathcal{M}^I$. By the foregoing, for such an (M, ρ_M), one has $\rho_M(m) = m \otimes z$, so that $\omega_{M,\mathcal{C}}(m \otimes_R c) = \rho_M(m) \otimes_R c - m \otimes_R \lambda_{\mathcal{C}}(c) = m \otimes_R z \otimes_R c - m \otimes_R z \otimes_R c = 0$, hence $\omega_{M,\mathcal{C}} = \rho_M \otimes_R \mathcal{C} - M \otimes_R \lambda_{\mathcal{C}}$ is the zero map. Thus, both ${}_R R$ and ${}_R \mathcal{C}$ trivially preserve the equalizer of $(\rho_M \otimes_R \mathcal{C}, M \otimes_R \lambda_{\mathcal{C}})$ for every (M, ρ_M) in $\mathcal{M}^I$ as desired. □

By construction, the R-coring I of Corollary 3.27 is also a ring with unit z. Since the comultiplication Δ_I of I is invertible, then I is a coseparable R-coring. Thus, by

[18, Proposition 2.17] there is a category isomorphism between the category $\mathcal{M}^I$ of right comodules over the coring I and the category $\mathcal{M}_I$ of right modules over the ring I.

Example 3.28. [4, Example 3.15] Let R be a commutative ring and consider an idempotent ideal I of R, assumed to be a pure right R-submodule. We recall that a submodule N of an R-module M is said to be *pure* [24, 40.13] if the inclusion $N \hookrightarrow M$ remains injective after tensoring by any right R-module. Since I is pure, we get that the multiplication $m_I : I \otimes_R I \to I$, $m_I(a \otimes_R a') = aa'$, is injective as it is obtained by the composition $I \otimes_R I \overset{I\otimes_R\varepsilon_I}{\to} I \otimes_R R \overset{\cong}{\to} I$, where $\varepsilon_I : I \to R$ is the canonical inclusion. Since I is idempotent, i.e. $I^2 = I$, we get that m_I is also surjective whence bijective. Thus, we can consider $\Delta_I = m_I^{-1} : I \to I \otimes_R I$ and write $\Delta_I(a) = \sum a_1 \otimes_R a_2$ by means of Sweedler's notation. Then, $\sum \varepsilon_I(a_1)a_2 = \sum a_1a_2 = m_I(\sum a_1 \otimes_R a_2) = m_I(\Delta_I(a)) = a$ and similarly $\sum a_1\varepsilon_I(a_2) = a$, so that $(I, \Delta_I, \varepsilon_I)$ is an R-coring. Now, the condition *(iii)* in Theorem 3.24 for this coring is the existence of an element $z \in I^R$ such that $c = \varepsilon_I(z)c$, i.e., by definition of ε_I, the existence of $z \in I$ such that $c = zc = cz$, for every $c \in I$. This means that z is the multiplicative identity in I. This goes back to a particular case of the ideal I constructed in Corollary 3.27 by taking $\mathcal{C} = I$ and noting that $\mathrm{Im}(\varepsilon_I) = I$. Moreover, in Example 3.26 2) we can identify S with the idempotent ideal $I = S \times \{0\}$ of the ring $R = S \times S$, through the isomorphism $S \overset{\cong}{\to} I : s \mapsto (s, 0)$. In this case, we can take $z = (1, 0)$ (note that $z \neq (1, 1) = 1_R$) and $\Delta_I(x) := x \otimes_R z = z \otimes_R x$.

We now show that Proposition 3.21 can be extended to the case of a morphism of R-corings, which has not appeared in the literature.

Proposition 3.29. *Let $\psi : \mathcal{C} \to \mathcal{D}$ be a morphism of R-corings. Assume that $\mathcal{C}$ is flat as left R-module and that $\mathcal{D}$ satisfies the right α-condition (see Definition 1.53). Then, the coinduction functor $\psi^* = (-)\Box_{\mathcal{D}}\mathcal{C} : \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ is semiseparable if, and only if, ψ is a regular morphism of $\mathcal{D}$-bicomodules if, and only if, there is a $\mathcal{D}$-bicomodule morphism $\chi : \mathcal{D} \to \mathcal{C}$ such that $\varepsilon_{\mathcal{C}} \circ \chi \circ \psi = \varepsilon_{\mathcal{C}}$.*

Proof. From Subsection 1.4.3 we know that $\psi_* \dashv \psi^*$ is an adjunction, where $\psi_* : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}^{\mathcal{D}}$ is the corestriction functor. The unit is given by $\eta_M : M \to M\Box_{\mathcal{D}}\mathcal{C}$, $m \mapsto \sum m_0\Box_{\mathcal{D}}m_1$, for every $M \in \mathcal{M}^{\mathcal{C}}$, and the counit is $\epsilon_N : N\Box_{\mathcal{D}}\mathcal{C} \to N$, $n\Box_{\mathcal{D}}c \mapsto n\varepsilon_{\mathcal{C}}(c)$, for every $N \in \mathcal{M}^{\mathcal{D}}$. Assume that ψ^* is semiseparable. By Theorem 2.36, there exists a natural transformation $\gamma : \mathrm{Id}_{\mathcal{M}^{\mathcal{D}}} \to \psi_*\psi^*$ such that $\epsilon_N \circ \gamma_N \circ \epsilon_N = \epsilon_N$, for every $N \in \mathcal{M}^{\mathcal{D}}$. Since $\mathcal{D}$ is a right $\mathcal{D}$-comodule, consider the right $\mathcal{D}$-comodule map $\gamma_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}\Box_{\mathcal{D}}\mathcal{C}$ and define the map $\chi : \mathcal{D} \to \mathcal{C}$ as $\chi := l_{\mathcal{C}} \circ \gamma_{\mathcal{D}}$, where $l_{\mathcal{C}} : \mathcal{D}\Box_{\mathcal{D}}\mathcal{C} \to \mathcal{C}$, $\sum_i d_i \otimes_R c_i \mapsto \sum_i \varepsilon_{\mathcal{D}}(d_i)c_i$, is the canonical isomorphism, see [24, 22.4]. Since $\Delta_{\mathcal{D}}\psi = (\psi \otimes_R \psi)\Delta_{\mathcal{C}} = (\mathrm{Id}_{\mathcal{D}} \otimes_R \psi)(\psi \otimes_R \mathrm{Id}_{\mathcal{C}})\Delta_{\mathcal{C}}$ and $\Delta_{\mathcal{D}}\psi = (\psi \otimes_R \mathrm{Id}_{\mathcal{D}})(\mathrm{Id}_{\mathcal{C}} \otimes_R \psi)\Delta_{\mathcal{C}}$, then ψ is a morphism of $\mathcal{D}$-bicomodules. Note that $\psi = \epsilon_{\mathcal{D}} \circ l_{\mathcal{C}}^{-1}$ as $\epsilon_{\mathcal{D}}(l_{\mathcal{C}}^{-1}(c)) = \epsilon_{\mathcal{D}}(\psi \otimes_R \mathrm{Id}_{\mathcal{C}})\Delta_{\mathcal{C}}(c) = \epsilon_{\mathcal{D}}(\psi(c_1) \otimes_R c_2) = \psi(c_1)\varepsilon_{\mathcal{C}}(c_2) = \psi(c_1\varepsilon(c_2)) = \psi(c)$. We show that χ is a morphism of $\mathcal{D}$-bicomodules. Since χ is a morphism of right $\mathcal{D}$-bicomodules, it is enough to show that χ is a morphism of left $\mathcal{D}$-bicomodules. For any $f \in \mathcal{D}^*$ (see [24, 17.8]), consider the morphism of right $\mathcal{D}$-comodules $\hat{f} : \mathcal{D} \to \mathcal{D}$, $d \mapsto \sum f(d_1)d_2$. In fact, $\Delta_{\mathcal{D}}\hat{f}(d) = \Delta_{\mathcal{D}}(\sum f(d_1)d_2) = \sum f(d_1)\Delta_{\mathcal{D}}(d_2) = \sum f(d_1)(d_2 \otimes_R d_3) = \sum f(d_{1_1})d_{1_2} \otimes_R d_2 = (\hat{f} \otimes_R \mathcal{D})(d_1 \otimes_R d_2) = (\hat{f} \otimes_R \mathcal{D})\Delta_{\mathcal{D}}(d)$. We observe that χ is left R-linear. Indeed, for $r \in R$, consider $f_r : \mathcal{D} \to \mathcal{D}$, $d \mapsto rd$, which is a morphism of right $\mathcal{D}$-comodules as $f_r(dr') = rdr' = f_r(d)r'$, for every $d \in \mathcal{D}, r' \in R$, and $\Delta_{\mathcal{D}}f_r(d) = \Delta_{\mathcal{D}}(rd) = r\Delta_{\mathcal{D}}(d) = rd_1 \otimes_R d_2 = f_r(d_1) \otimes d_2 = (f_r \otimes_R \mathcal{D})\Delta_{\mathcal{D}}(d)$, for every $d \in \mathcal{D}$. For any $d \in \mathcal{D}$, denote $\gamma_{\mathcal{D}}(d) := \sum_i d_i \otimes_R c_i \in \mathcal{D}\Box_{\mathcal{D}}\mathcal{C}$. By naturality of γ, we have that $\gamma_{\mathcal{D}}(rd) = \gamma_{\mathcal{D}}(f_r(d)) = (f_r\Box_{\mathcal{D}}\mathcal{C})(\gamma_{\mathcal{D}}(d)) = f_r(d_i) \otimes_R c_i = rd_i \otimes_R c_i = r\gamma_{\mathcal{D}}(d)$, so that $\chi(rd) = l_{\mathcal{C}}\gamma_{\mathcal{D}}(rd) =$

$l_{\mathcal{C}}(r\gamma_{\mathcal{D}}(d)) = l_{\mathcal{C}}(rd_i \otimes_R c_i) = \sum \varepsilon_{\mathcal{D}}(rd_i)c_i = \sum r\varepsilon_{\mathcal{D}}(d_i)c_i = rl_{\mathcal{C}}(\sum d_i \otimes_R c_i) = rl_{\mathcal{C}}\gamma_{\mathcal{D}}(d) = r\chi(d)$. Then, again by naturality of γ, we have that $\sum f(d_1)\chi(d_2) = \chi(\sum f(d_1)d_2) = \chi\hat{f}(d) = l_{\mathcal{C}}\gamma_{\mathcal{D}}\hat{f}(d) = l_{\mathcal{C}}(\hat{f}\Box_{\mathcal{D}}\mathcal{C})\gamma_{\mathcal{D}}(d) = l_{\mathcal{C}}(\hat{f}\Box_{\mathcal{D}}\mathcal{C})(\sum_i d_i \otimes_R c_i) = l_{\mathcal{C}}(\sum_i f(d_{i_1})d_{i_2} \otimes_R c_i) = \sum_i f(d_{i_1})\varepsilon_{\mathcal{D}}(d_{i_2})c_i = \sum_i f(d_{i_1}\varepsilon_{\mathcal{D}}(d_{i_2}))c_i = \sum_i f(d_i)c_i$, so $\sum f(d_1)\chi(d_2) = \sum_i f(d_i)c_i$. Since $\mathcal{D}$ satisfies the right α-condition, that is, the map $\alpha^r_N : \mathcal{D} \otimes_R N \to {}_R\mathrm{Hom}(\mathcal{D}^*, N)$, $d \otimes_R n \mapsto [f \mapsto f(d)n]$ is injective for every $N \in {}_R\mathcal{M}$, we get that $\sum d_1 \otimes_R \chi(d_2) = \sum_i d_i \otimes_R c_i$. Moreover, $\sum_i d_i \otimes_R c_i = \gamma_{\mathcal{D}}(d) = l_{\mathcal{C}}^{-1} l_{\mathcal{C}}\gamma_{\mathcal{D}}(d) = l_{\mathcal{C}}^{-1}\chi(d) = (\psi \otimes_R \mathcal{C})\Delta_{\mathcal{C}}(\chi(d)) = (\psi \otimes_R \mathcal{C})(\chi(d)_1 \otimes_R \chi(d)_2) = \psi(\chi(d)_1) \otimes_R \chi(d)_2$, then $\sum d_1 \otimes_R \chi(d_2) = \psi(\chi(d)_1) \otimes_R \chi(d)_2$, so χ is a morphism of left $\mathcal{D}$-comodules, whence of $\mathcal{D}$-bicomodules. Furthermore, we have $\psi \circ \chi \circ \psi = (\epsilon_{\mathcal{D}} \circ l_{\mathcal{C}}^{-1}) \circ (l_{\mathcal{C}} \circ \gamma_{\mathcal{D}}) \circ (\epsilon_{\mathcal{D}} \circ l_{\mathcal{C}}^{-1}) = \epsilon_{\mathcal{D}} \circ \gamma_{\mathcal{D}} \circ \epsilon_{\mathcal{D}} \circ l_{\mathcal{C}}^{-1} = \epsilon_{\mathcal{D}} \circ l_{\mathcal{C}}^{-1} = \psi$, hence χ is a regular morphism of $\mathcal{D}$-bicomodules.

Assume now that ψ is a regular morphism of $\mathcal{D}$-bicomodules, i.e. there is a $\mathcal{D}$-bicomodule morphism $\chi : \mathcal{D} \to \mathcal{C}$ such that $\psi \circ \chi \circ \psi = \psi$. Then, $\varepsilon_{\mathcal{C}} \circ \chi \circ \psi = \varepsilon_{\mathcal{D}} \circ \psi \circ \chi \circ \psi = \varepsilon_{\mathcal{D}} \circ \psi = \varepsilon_{\mathcal{C}}$.

Finally, assume that there is a $\mathcal{D}$-bicomodule morphism $\chi : \mathcal{D} \to \mathcal{C}$ such that $\varepsilon_{\mathcal{C}} \circ \chi \circ \psi = \varepsilon_{\mathcal{C}}$ and let us prove that ψ^* is semiseparable. For any $N \in \mathcal{M}^{\mathcal{D}}$ define $\gamma_N : N \to N\Box_{\mathcal{D}}\mathcal{C}$ by $\gamma_N(n) = \sum n_0 \otimes_R \chi(n_1)$, for every $n \in N$. Since χ is a left $\mathcal{D}$-comodule morphism, the image of γ_N is really contained in $N\Box_{\mathcal{D}}\mathcal{C}$ as $(\rho_N \otimes_R \mathcal{C})(\sum n_0 \otimes_R \chi(n_1)) = \sum n_{0_0} \otimes_R n_{0_1} \otimes_R \chi(n_1)$ and $(N \otimes_R \rho^{\psi}_{\mathcal{C}})(\sum n_0 \otimes_R \chi(n_1)) = \sum n_0 \otimes_R (\psi \otimes_R \mathcal{C})\Delta_{\mathcal{C}}(\chi(n_1)) = \sum n_0 \otimes_R (\mathcal{D} \otimes_R \chi)\Delta_{\mathcal{D}}(n_1) = \sum n_0 \otimes_R n_{1_1} \otimes_R \chi(n_{1_2})$. Since χ is a morphism of right $\mathcal{D}$-comodules, then γ_N results to be a right $\mathcal{D}$-comodule morphism as, for all $n \in N$, we have $(\gamma_N \otimes_R \mathcal{D})\rho_N(n) = (\gamma_N \otimes_R \mathcal{D})(\sum n_0 \otimes_R n_1) = \sum n_0 \otimes_R \chi(n_1) \otimes_R n_2 = \sum n_0 \otimes_R \chi(n_{1_1}) \otimes_R n_{1_2} = \sum n_0 \otimes_R \chi(n_1)_1 \otimes_R \psi(\chi(n_1)_2) = \rho_{N\Box_{\mathcal{D}}\mathcal{C}}(\sum n_0 \otimes_R \chi(n_1)) = \rho_{N\Box_{\mathcal{D}}\mathcal{C}}\gamma_N(n)$. For any morphism $f : N \to M$ in $\mathcal{M}^{\mathcal{D}}$, $(\gamma_M \circ f)(n) = \sum f(n)_0 \otimes_R \chi(f(n)_1) = \sum f(n_0) \otimes_R \chi(n_1) = (f\Box_{\mathcal{D}}\mathcal{C})(\sum n_0 \otimes_R \chi(n_1)) = ((f\Box_{\mathcal{D}}\mathcal{C}) \circ \gamma_N)(n)$, so γ_N is natural in N. For every $n \in N$, $c \in \mathcal{C}$, we have $\gamma_N\epsilon_N(n\Box_{\mathcal{D}}c) = \gamma_N(n\varepsilon_{\mathcal{C}}(c)) = \gamma_N(n)\varepsilon_{\mathcal{C}}(c) = \sum n_0 \otimes_R \chi(n_1)\varepsilon_{\mathcal{C}}(c) = \sum n \otimes_R \chi\psi(c_1)\varepsilon_{\mathcal{C}}(c_2) = \sum n \otimes_R \chi\psi(c)$, where in the second-last equality we used that $n\Box_{\mathcal{D}}c$ belongs to $N\Box_{\mathcal{D}}\mathcal{C}$. Thus, $\epsilon_N\gamma_N\epsilon_N(n\Box_{\mathcal{D}}c) = \epsilon_N(\sum n \otimes_R \chi\psi(c)) = \sum n\varepsilon_{\mathcal{C}}\chi\psi(c) = n\varepsilon_{\mathcal{C}}(c) = \epsilon_N(n\Box_{\mathcal{D}}c)$. Therefore, by Theorem 2.36, ψ^* is semiseparable. □

3.4 Bimodules

Let R and S be rings. For an (R,S)-bimodule M we consider the adjunction $\sigma^* \dashv \sigma_*$ recalled in Subsection 1.4.4 given by the induction functor $\sigma^* = (-) \otimes_R M : \mathcal{M}_R \to \mathcal{M}_S$, and the coinduction functor $\sigma_* = \mathrm{Hom}_S(M, -) : \mathcal{M}_S \to \mathcal{M}_R$. The unit η and counit ϵ are given by

$$\eta_X : X \to \mathrm{Hom}_S(M, X \otimes_R M),\ x \mapsto [m \mapsto x \otimes_R m],$$
$$\epsilon_Y : \mathrm{Hom}_S(M, Y) \otimes_R M \to Y,\ f \otimes_R m \mapsto f(m),$$

respectively, for all $X \in \mathcal{M}_R$ and $Y \in \mathcal{M}_S$.

We investigate the semiseparability of σ_* and, in the finitely generated and projective case, the one of σ^*. We first introduce the following definition, which is a semiseparable version of M-separability of S over R.

Definition 3.30. [4, Definition 3.17] Let R, S be rings and M an (R,S)-bimodule. We say that S is M**-semiseparable over** R if there exists an element $\sum_i f_i \otimes_R m_i \in (M^* \otimes_R M)^S$ such that $\sum_i m f_i(m_i) = m$ for every $m \in M$. We call $\{f_i, m_i\}$ a *system of M-semiseparability*. In a similar way, it is possible to define R to be M**-semiseparable over** S.

Theorem 3.31. [4, Theorem 3.18] *Let R, S be rings and M an (R, S)-bimodule. Then, the following are equivalent:*

(i) the functor $\sigma_* = \mathrm{Hom}_S(M, -) : \mathcal{M}_S \to \mathcal{M}_R$ is semiseparable;

(ii) the evaluation map $\mathrm{ev}_M : M^* \otimes_R M \to S$ is regular as a morphism of S-bimodules and $M \otimes_S \mathrm{ev}_M$ is surjective;

(iii) S is M-semiseparable over R.

Proof. It is known (see e.g. the right version of [28, Lemma 11]) that there is a bijective correspondence

$$\mathrm{Nat}(\mathrm{Id}_{\mathcal{M}_S}, \sigma^*\sigma_*) \cong (M^* \otimes_R M)^S. \tag{3.3}$$

Explicitly, a natural transformation $\gamma : \mathrm{Id}_{\mathcal{M}_S} \to \sigma^*\sigma_*$ is mapped to $\gamma_S(1_S) \in (M^* \otimes_R M)^S$, while an element $\sum_i f_i \otimes_R m_i \in (M^* \otimes_R M)^S$ is mapped to a natural transformation $\gamma : \mathrm{Id}_{\mathcal{M}_S} \to \sigma^*\sigma_*$ given, for every $Y \in \mathcal{M}_S$, by

$$\gamma_Y : Y \to \mathrm{Hom}_S(M, Y) \otimes_R M, \quad \gamma_Y(y) = \sum_i y f_i(-) \otimes_R m_i. \tag{3.4}$$

$(i) \Rightarrow (ii)$. If the functor σ_* is semiseparable, then by Theorem 2.36 there exists a natural transformation $\gamma : \mathrm{Id}_{\mathcal{M}_S} \to \sigma^*\sigma_*$ such that $\epsilon \circ \gamma \circ \epsilon = \epsilon$. Consider the right S-module map $\gamma_S : S \to M^* \otimes_R M$ and, for every $s \in S$, the right S-module map $f_s : S \to S, s' \mapsto ss'$. From naturality of γ we have $\gamma_S(ss') = (\gamma_S \circ f_s)(s') = ((\mathrm{Hom}_S(M, f_s) \otimes_R M) \circ \gamma_S)(s') = s\gamma_S(s')$ so that γ_S is S-bilinear. Since $\epsilon_S = \mathrm{ev}_M$, from $\epsilon \circ \gamma \circ \epsilon = \epsilon$ we get $\mathrm{ev}_M \circ \gamma_S \circ \mathrm{ev}_M = \mathrm{ev}_M$ and hence ev_M is regular as a morphism of S-bimodules. Note that any $m \in M$ is of the form $m = \mathrm{Id}_M(m) = \epsilon_M(\mathrm{Id}_M \otimes_R m)$ so that ϵ_M is surjective. Thus, from $\epsilon_M \circ \gamma_M \circ \epsilon_M = \epsilon_M$, we get $\epsilon_M \circ \gamma_M = \mathrm{Id}_M$. From (3.3) we have that γ_M is defined by (3.4) for $Y = M$, where $\sum_i f_i \otimes_R m_i = \gamma_S(1_S) \in (M^* \otimes_R M)^S$. Then,

$$m = \mathrm{Id}_M(m) = (\epsilon_M \circ \gamma_M)(m) = \sum_i m f_i(m_i) = r_M(M \otimes_S \mathrm{ev}_M)\left(\sum_i m \otimes_S f_i \otimes_R m_i\right),$$

where $r_M : M \otimes_S S \to M$ is the canonical isomorphism. Thus, $r_M \circ (M \otimes_S \mathrm{ev}_M)$ is surjective and hence also $M \otimes_S \mathrm{ev}_M$ is surjective.
$(ii) \Rightarrow (iii)$. Assume that ev_M is regular as a morphism of S-bimodules, i.e. that there is an S-bimodule map $\gamma_S : S \to M^* \otimes_R M$ such that $\mathrm{ev}_M \circ \gamma_S \circ \mathrm{ev}_M = \mathrm{ev}_M$. Thus, $(M \otimes_S \mathrm{ev}_M) \circ (M \otimes_S \gamma_S) \circ (M \otimes_S \mathrm{ev}_M) = (M \otimes_S \mathrm{ev}_M)$. If $M \otimes_S \mathrm{ev}_M$ is surjective, we get $(M \otimes_S \mathrm{ev}_M) \circ (M \otimes_S \gamma_S) = \mathrm{Id}_{M \otimes_S S}$. Now set $\sum_i f_i \otimes_R m_i = \gamma_S(1_S) \in (M^* \otimes_R M)^S$. Thus, S is M-semiseparable over R as

$$m = r_M \mathrm{Id}_{M \otimes_S S}(m \otimes_S 1_S) = r_M(M \otimes_S \mathrm{ev}_M)(M \otimes_S \gamma_S)(m \otimes_S 1_S) = \sum_i m f_i(m_i).$$

$(iii) \Rightarrow (i)$. Assume S is M-semiseparable over R. By definition, there exists an element $\sum_i f_i \otimes_R m_i \in (M^* \otimes_R M)^S$ such that $\sum_i f_i(m_i) m = m$ for every $m \in M$ and the corresponding natural transformation $\gamma : \mathrm{Id}_{\mathcal{M}_S} \to \sigma^*\sigma_*$ is given as in (3.3). Moreover, for every $Y \in \mathcal{M}_S$, $m \in M$, $f \in \mathrm{Hom}_S(M, Y)$, we have $\epsilon_Y \gamma_Y \epsilon_Y(f \otimes_R m) = \epsilon_Y \gamma_Y(f(m)) = \epsilon_Y(\sum_i f(m) f_i(-) \otimes_R m_i) = \sum_i f(m) f_i(m_i) = f(\sum_i m f_i(m_i)) = f(m) = \epsilon_Y(f \otimes_R m)$. Thus, ϵ is regular and by Theorem 2.36 *ii)* σ_* is semiseparable. □

Remark 3.32. Given an (R, S)-bimodule M, the equivalence between (i) and (iii) in the previous result is the semiseparable counterpart of Proposition 1.55.

Remark 3.33. In the setting of Theorem 3.31, assume further that M_S is projective. Then, the requirement that $M \otimes_S \mathrm{ev}_M$ is surjective is superfluous. Indeed, there is a dual basis formed by elements $m_i \in M, f_i \in M^*$, with $i \in I$, such that, for every $m \in M$, we have $m = \sum_{i\in I} m_i f_i(m)$. By definition, $f_i(m) = 0$ for almost all i. Thus there is a finite subset $I(m)$ of I such that $m = \sum_{i\in I(m)} m_i f_i(m) = r_M(M \otimes_S \mathrm{ev}_M)(\sum_{i\in I(m)} m_i \otimes_S f_i \otimes_R m)$, whence $M \otimes_S \mathrm{ev}_M$ is surjective.

As a consequence of Theorem 3.31, we have the following characterization of M-separability, for an (R, S)-bimodule M, which extends some known results, see e.g. [83, Theorem 1], [78, Corollary 2.4] and [6, Proposition 4.3].

Corollary 3.34. [4, Corollary 3.22] *Let R, S be rings and M an (R, S)-bimodule. Then, S is M-separable over R if, and only if, S is M-semiseparable over R and M_S is a generator.*

Proof. By Proposition 1.55, S is M-separable over R if, and only if, $\sigma_* = \mathrm{Hom}_S(M, -) : \mathcal{M}_S \to \mathcal{M}_R$ is a separable functor. By Proposition 2.5 *i*), this is equivalent to require that σ_* is semiseparable and faithful. The semiseparability of σ_* is equivalent to S being M-semiseparable over R, by Theorem 3.31. Since the forgetful functor $U : \mathcal{M}_R \to \mathsf{Set}$ is faithful, the faithfulness of σ_* is equivalent to the faithfulness of the composition $U \circ \sigma_* = \mathrm{Hom}_S(M, -) : \mathcal{M}_S \to \mathsf{Set}$, i.e. to M_S being a generator, see e.g. [81, Section 6]. □

Remark 3.35. [4, Remark 3.21] Let R, S be rings and M an (R, S)-bimodule. If M_S is a generator and $\varphi : R \to \mathcal{E} = \mathrm{End}_S(M)$ is a ring epimorphism, then by the corresponding right version of [7, Proposition 3.11], the functor σ_* is fully faithful, i.e. σ^* is a reflection. Thus, by Corollary 2.64, σ^* results to be semiseparable if, and only if, it is naturally full if, and only if, it is Frobenius.

We now give a different characterization of M-semiseparability of S over R, for an (R, S)-bimodule M. As a consequence, in Example 3.37 we exhibit an example where S is M-semiseparable but not M-separable over R.

Proposition 3.36. [4, Proposition 3.22] *Let R, S be rings and let M be an (R, S)-bimodule. Then, S is M-semiseparable over R if, and only if, there is a central idempotent $z \in S$ (necessarily unique) such that M is obtained by restriction of scalars from an (R, Sz)-bimodule N and Sz is N-separable over R, via $\varphi : S \to Sz, s \mapsto sz$. Furthermore, S is M-separable over R if, and only if, $z = 1_S$.*

Proof. Assume that S is M-semiseparable over R, i.e. that there is a central element $\sum_i f_i \otimes_R m_i \in (M^* \otimes_R M)^S$ such that $\sum_i m f_i(m_i) = m$, for every $m \in M$. Set $z := \sum_i f_i(m_i) \in S$ so that $mz = m$, for every $m \in M$. Since $\mathrm{ev}_M : M^* \otimes_R M \to S$ is a morphism of S-bimodules, it induces a morphism $\mathrm{ev}_M^S : (M^* \otimes_R M)^S \to S^S$ so that $z = \mathrm{ev}_M(\sum_i f_i \otimes_R m_i) \in S^S$, i.e. z is central. Moreover $zz = \sum_i f_i(m_i)z = \sum_i f_i(m_i z) = \sum_i f_i(m_i) = z$, so that z is idempotent. Since for every $m \in M$ one has $mz = m$, then M becomes a right Sz-module, via $\mu_M : M \times Sz \to M, (m, sz) \mapsto ms$. Let us write N for M regarded as an (R, Sz)-bimodule so that $M = \varphi_* N$ where $\varphi : S \to Sz, s \mapsto sz$. Set $N^* := \mathrm{Hom}_{Sz}(N, Sz)$. Then, $\sum_i \varphi f_i \otimes_R m_i \in (N^* \otimes_R N)^{Sz}$ and $\sum_i \varphi f_i(m_i) = \varphi(z) = zz = z = 1_{Sz}$, so that Sz is N-separable over R. Conversely, assume there is a central idempotent $z \in S$ such that such that M is obtained by restriction of scalars from an (R, Sz)-bimodule N and Sz is N-separable over R, via $\varphi : S \to Sz, s \mapsto sz$. This implies $mz = m$ for every $m \in M$. Since Sz is N-separable over R, there is $\sum_i g_i \otimes_R m_i \in (N^* \otimes_R N)^S$ such that $\sum_i g_i(m_i) = 1_{Sz} = z$. Let $j : Sz \to S$ be the canonical injection. Then $f_i := j \circ g_i \in M^*$ and $\sum_i f_i \otimes_R m_i \in (M^* \otimes_R M)^S$. Moreover, $\sum_i m f_i(m_i) = \sum_i m j g_i(m_i) = mj(z) = mz = m$ so that S is M-semiseparable over R. Assume there is another central idempotent $z' \in S$ such that $M = \varphi_* N'$ for some (R, Sz')-bimodule N' and Sz is N'-separable over R via the ring homomorphism $\varphi' : S \to Sz', s \mapsto sz'$. Then, $zz' = \sum_i f_i(m_i)z' = \sum_i f_i(m_i z') = \sum_i f_i(m_i) = z$. Exchanging the roles of z and z', we also get $z'z = z'$, so that $z = z'$. Let $z \in S$ be a central idempotent such that $M = \varphi_* N$ where Sz is N-separable over R via $\varphi : S \to Sz, s \mapsto sz$. If $z = 1_S$, then S is N-separable over R and $\varphi = \mathrm{Id}_S$ so that S is $M = \varphi_* N$-separable over R as well. Conversely, if S is M-separable over R, then $z = 1_S$ is an idempotent as in the statement, whence the unique one. □

The following is an instance of an (R, S)-bimodule M such that S is M-semiseparable but not M-separable over R.

Example 3.37. [4, Example 3.23] Let $\varphi : S \to T$ be a ring homomorphism and assume that there is $E \in {}_S\mathrm{Hom}_S(T, S)$ such that $\varphi \circ E = \mathrm{Id}_T$. If we set $z := E(1_T) \in S$, then z is a central idempotent in S, the map $\varphi_{|Sz} : Sz \to T$ is a ring isomorphism and $\varphi : S \to T \cong Sz$ is the projection $s \mapsto sz$, see Proposition 1.42 *ii*). By Proposition 3.36, if N is a (R, T)-bimodule such that T is N-separable over R, then $M := \varphi_* N$ is an (R, S)-bimodule such that S is M-semiseparable over R. Moreover, if S is also M-separable over R, then $z = 1_S$, whence φ is bijective. As a consequence, S is not M-separable over R unless $\varphi : S \to T$ is bijective. As an example, let $\psi : \mathbb{Q} \times \mathbb{Z} \to \mathbb{Q}, (q, z) \mapsto q$ and $D : \mathbb{Q} \to \mathbb{Q} \times \mathbb{Z}, q \mapsto (q, 0)$ be as in Example 3.3. Then, if N is a $(R, \mathbb{Q})$-bimodule such that $\mathbb{Q}$ is N-separable over R, then the $(R, \mathbb{Q} \times \mathbb{Z})$-bimodule $M := \psi_* N$ is such that $\mathbb{Q} \times \mathbb{Z}$ is M-semiseparable but not M-separable over R. For instance, consider the $\mathbb{Q}$-vector space $N := \mathbb{Q}^n$, with $n > 1$, and take $R := \mathbb{Q}$. Let us check that N is a $(\mathbb{Q}, \mathbb{Q})$-bimodule such that $\mathbb{Q}$ is N-separable over $\mathbb{Q}$, with $nq = qn$ for all $n \in N, q \in \mathbb{Q}$. Since N is a free left $\mathbb{Q}$-module, then it is a generator. Moreover, $\mathcal{E} = \mathrm{End}_{\mathbb{Q}}(N) = \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}^n) \cong \mathrm{M}_n(\mathrm{End}_{\mathbb{Q}}(\mathbb{Q})) \cong \mathrm{M}_n(\mathbb{Q})$ is a separable $\mathbb{Q}$-algebra. Therefore, by the corresponding of [83, Theorem 1(1)] for right modules (see also Proposition 3.39 below), $\mathbb{Q}$ is N-separable over $\mathbb{Q}$. Thus, the $(\mathbb{Q}, \mathbb{Q} \times \mathbb{Z})$-bimodule $M := \psi_* N = \mathbb{Q}^n$ is such that $\mathbb{Q} \times \mathbb{Z}$ is M-semiseparable but not M-separable over $\mathbb{Q}$. For a direct computation by means of Definition 3.30, set $m := (1, 0, \ldots, 0)$ and define $f \in M^* = \mathrm{Hom}_{\mathbb{Q} \times \mathbb{Z}}(\mathbb{Q}^n, \mathbb{Q} \times \mathbb{Z})$ by $f(q_1, \ldots, q_n) := (q_1, 0)$. Then, $f \otimes_{\mathbb{Q}} m \in (M^* \otimes_{\mathbb{Q}} M)^{\mathbb{Q} \times \mathbb{Z}}$ and for every $m' \in M$ one has $m' f(m) = m'(1, 0) = m' \psi(1, 0) = m'$.

In the next result we provide an explicit factorization as a bireflection followed by a separable functor for the coinduction functor σ_* attached to an (R, S)-bimodule M in case it is semiseparable. By Corollary 2.71, this factorization is the one given by the coidentifier, up to a category equivalence.

Proposition 3.38. [4, Proposition 3.24] *Let M be an (R, S)-bimodule. The coinduction functor $\sigma_* = \mathrm{Hom}_S(M, -) : \mathcal{M}_S \to \mathcal{M}_R$ is semiseparable if, and only if, there is an S-coring I with a grouplike element $z \in I^S$ such that $\sigma_* \cong \tilde{\sigma}_* \circ G_I$, where $\tilde{\sigma}_* := \mathrm{Hom}^I(M, -) : \mathcal{M}^I \to \mathcal{M}_R$ is separable and the induction functor $G_I := (-) \otimes_S I : \mathcal{M}_S \to \mathcal{M}^I$ is a bireflection. Here M is in $\mathcal{M}^I$ via $\rho_M(m) = m \otimes_S z$.*

Proof. Assume that σ_* is semiseparable. Then, by Theorem 3.31 S is M-separable over R through some $c := \sum_i f_i \otimes_R m_i \in (M^* \otimes_R M)^S$. Since $\mathrm{ev}_M : M^* \otimes_R M \to S$ is a morphism of S-bimodules, then $I := \mathrm{Im}(\mathrm{ev}_M)$ is an ideal of S with multiplicative identity $z := \mathrm{ev}_M(c) = \sum_i f_i(m_i)$. Indeed, for all $s \in S$, $zs = \mathrm{ev}_M(c)s = \mathrm{ev}_M(cs) = \mathrm{ev}_M(sc) = s\mathrm{ev}_M(c) = sz$ and hence $z \in I^S$. For all $m \in M, f \in M^*$, we have $zf(m) = \sum_i f(m)f_i(m_i) = f(\sum_i m f_i(m_i)) = f(m)$ and hence $zi = i$, for every $i \in I$. Moreover, since the morphism $\varphi : S \to I, s \mapsto sz$, is a ring epimorphism, the map $m_I : I \otimes_S I \to I$

is bijective. Thus, we can consider $\Delta_I := m_I^{-1} : I \to I \otimes_S I$, $\Delta_I(i) = i \otimes_S z = z \otimes_S i$, so that $(I, \Delta_I, \varepsilon_I)$ becomes an S-coring, where $\varepsilon_I : I \hookrightarrow S$ is the canonical inclusion. By the foregoing $z \in I^S$ and, for every $i \in I$, we have $\varepsilon_I(i)z = iz = i$. By what we recalled in Proposition 1.50, the induction functor $G_I := (-) \otimes_S I : \mathcal{M}_S \to \mathcal{M}^I$ is naturally full. Consider its left adjoint, the forgetful functor $F_I : \mathcal{M}^I \to \mathcal{M}_S$, and the corresponding unit η defined on each N in $\mathcal{M}^I$ by $\eta_N := \rho_N : N \to N \otimes_S I$. Given $n \in N$ write $\rho_N(n) = \sum_t n_t \otimes_S i_t$. By applying $N \otimes_S \varepsilon_I$ we get $n = \sum_t n_t i_t$. Thus, $\rho_N(n) = \sum_t n_t \otimes_S i_t = \sum_t n_t \otimes_S i_t z = \sum_t n_t i_t \otimes_S z = n \otimes_S z$. We have so proved that $\rho_N(n) = n \otimes_S z$, for every $n \in N$. By applying $N \otimes_S \varepsilon_I$ to this equality we get $n = nz$, for every $n \in N$. Therefore, ρ_N is invertible with inverse given by $n \otimes_S i \mapsto ni$, and then the unit η of $F_I \dashv G_I$ is invertible, i.e. G_I is a coreflection. By Corollary 2.64, G_I is a bireflection. As in [42, Example 4.3], we can consider the functor $\tilde{\sigma}^* := (-) \otimes_R M : \mathcal{M}_R \to \mathcal{M}^I$. By [24, 18.10.2] we have that $\tilde{\sigma}^* \dashv \tilde{\sigma}_* = \mathrm{Hom}^I(M, -)$, with unit and counit given by

$$\tilde{\eta}_X : X \to \mathrm{Hom}^I(M, X \otimes_R M),\ x \mapsto [m \mapsto x \otimes_R m],$$
$$\tilde{\epsilon}_Y : \mathrm{Hom}^I(M, Y) \otimes_R M \to Y,\ f \otimes_R m \mapsto f(m),$$

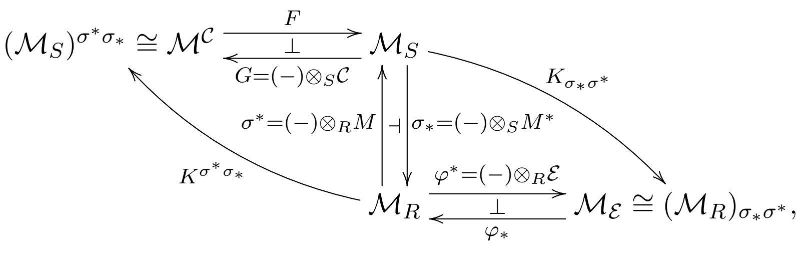
respectively. Thus, by (Rafael) Theorem 1.18, $\tilde{\sigma}_*$ is separable if, and only if, there is a natural transformation $\tilde{\gamma} : \mathrm{Id} \to \tilde{\sigma}^* \tilde{\sigma}_*$ such that $\tilde{\epsilon} \circ \tilde{\gamma} = \mathrm{Id}$. For Y in $\mathcal{M}^I$, define $\tilde{\gamma}_Y : Y \to \mathrm{Hom}^I(M, Y) \otimes_R M$, $y \mapsto \sum_i y f_i(-) \otimes_R m_i$. It is easy to check it defines a natural transformation $\tilde{\gamma} : \mathrm{Id} \to \tilde{\sigma}^* \tilde{\sigma}_*$. Moreover, $\tilde{\epsilon}_Y \tilde{\gamma}_Y(y) = \sum_i y f_i(m_i) = yz$ but we already proved that $yz = y$, hence $\tilde{\epsilon} \circ \tilde{\gamma} = \mathrm{Id}$, so $\tilde{\sigma}_*$ is separable. Let us check that $G \cong \tilde{\sigma}_* \circ G_I$. Note that $\varphi \circ \varepsilon_I = \mathrm{Id}_I$ and both φ and ε_I are left S-linear. As a consequence, I is projective, whence flat, as a left S-module. Thus, by [24, 22.12] applied in case $\mathcal{D}$ is the S-coring S, for every N in $\mathcal{M}_R$ we have a functorial isomorphism of abelian groups

$$\tilde{\sigma}_* G_I(N) = \mathrm{Hom}^I(M, N \otimes_S I) \to \mathrm{Hom}_S(M, N) = \sigma_*(N), \quad f \mapsto (N \otimes_S \varepsilon_I) \circ f.$$

This isomorphism is easily checked to be right R-linear. Thus, it yields $\tilde{\sigma}_* \circ G_I \cong \sigma_*$ as desired. Conversely, if $\sigma_* \cong \tilde{\sigma}_* \circ G_I$, where G_I is a bireflection, whence naturally full by Corollary 2.64, and $\tilde{\sigma}_*$ is separable, then σ_* is semiseparable in view of Lemma 2.6 *ii*). □

3.4.1 Projective and finitely generated bimodules

Given an (R, S)-bimodule M, in order to characterize the semiseparability of the induction functor $\sigma^* = (-) \otimes_R M : \mathcal{M}_R \to \mathcal{M}_S$ we need, as in the separable case, the further assumption that M_S is finitely generated and projective, see Proposition 1.57. We consider the setting of Example 1.72 and the diagram (1.24)



where the right adjoint of σ^* is given by $\sigma_* = (-) \otimes_S M^* : \mathcal{M}_S \to \mathcal{M}_R$, and $\mathcal{C}$ is the comatrix S-coring $M^* \otimes_R M$. In the next result we obtain a further characterization for the semiseparability of σ_*.

Proposition 3.39. [4, Proposition 3.26] *In the setting of Example 1.72, the following assertions are equivalent:*

(*i*) *S is M-semiseparable over R;*

(*ii*) *$\sigma_* = (-) \otimes_S M^* : \mathcal{M}_S \to \mathcal{M}_R$ is semiseparable;*

(*iii*) *the comatrix S-coring $\mathcal{C}$ is semicosplit;*

(*iv*) *there exists an invariant element $z \in \mathcal{C}^S$ such that for every $c \in \mathcal{C}$, $c = \varepsilon_{\mathcal{C}}(z)c$, where $\varepsilon_{\mathcal{C}}$ is the counit of the comatrix S-coring $\mathcal{C}$;*

(*v*) *$\varphi_* : \mathcal{M}_{\mathcal{E}} \to \mathcal{M}_R$ is separable (that is, $\mathcal{E}/R$ is separable) and $K_{\sigma_*\sigma^*}$ is naturally full.*

Proof. $(i) \Leftrightarrow (ii)$. It is Theorem 3.31.
$(ii) \Leftrightarrow (iii)$. By Remark 1.78 *iii*), σ_* is semiseparable if and only if so is $V^{\sigma^*\sigma_*} = G$.
$(iii) \Leftrightarrow (iv)$. It follows by Theorem 3.24.
$(ii) \Leftrightarrow (v)$. It follows by Theorem 2.47 *ii*) applied to the adjunction (σ^*, σ_*). □

We now obtain the announced characterization of the semiseparability of σ^*.

Proposition 3.40. [4, Proposition 3.27] *In the setting of Example 1.72, the following assertions are equivalent:*

(*i*) *$\sigma^* = (-) \otimes_R M : \mathcal{M}_R \to \mathcal{M}_S$ is semiseparable;*

(*ii*) *$\varphi^* = (-) \otimes_R \mathcal{E} : \mathcal{M}_R \to \mathcal{M}_{\mathcal{E}}$ is semiseparable;*

(*iii*) *there exists an $E \in {}_R\mathrm{Hom}_R(\mathcal{E}, R)$ such that $\varphi E(1_{\mathcal{E}}) = 1_{\mathcal{E}}$;*

(*iv*) *the forgetful functor $F : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_S$ is separable (i.e., $\mathcal{C}$ is coseparable) and $K^{\sigma^*\sigma_*}$ is naturally full.*

Proof. $(i) \Leftrightarrow (ii)$. By Remark 1.78 *iv*), σ^* is semiseparable if and only if so is $V_{\sigma_*\sigma^*} = \varphi^*$.
$(ii) \Leftrightarrow (iii)$. It follows by Proposition 3.1.
$(i) \Leftrightarrow (iv)$. It follows by Theorem 2.47 *ii*) applied to the adjunction (σ^*, σ_*). □

As a particular case of Example 1.72, given a morphism of rings $\varphi : R \to S$, the (R, S)-bimodule $M := {}_RS_S$, with left action induced by φ, is trivially finitely generated and projective as a right S-module, see Remark 1.59. In this case $\sigma^* = (-) \otimes_R S = \varphi^* : \mathcal{M}_S \to \mathcal{M}_R$ is the extension of scalars functor. As a consequence, the right adjoint σ_* of σ^* is isomorphic to the restriction of scalars functor $\varphi_* : \mathcal{M}_S \to \mathcal{M}_R$ and since it is faithful, it follows that S is S-semiseparable over R if, and only if, S is S-separable over R. Moreover, in this case we have that the comatrix S-coring $\mathcal{C}$ is the Sweedler coring $S \otimes_R S$, $\mathcal{E} = \mathrm{End}_S(M) \cong M \otimes_S M^* \cong S$, $K_{\varphi_*\varphi^*} = \mathrm{Id}_{\mathcal{M}_S}$, i.e. φ_* is *strictly monadic*, and $K^{\varphi^*\varphi_*} = (-) \otimes_R S : \mathcal{M}_R \to \mathcal{M}^{\mathcal{C}}$. Consider the induction functor $G = (-) \otimes_S \mathcal{C} : \mathcal{M}_S \to \mathcal{M}^{\mathcal{C}}$ and the forgetful functor $F : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_S$. In this setting, as a consequence of Proposition 3.39 and Proposition 3.40 the following corollaries relate the functors φ_*, φ^*, F, G and the Sweedler coring $\mathcal{C}$.

Corollary 3.41. [4, Corollary 3.29] *In the above setting, the following assertions are equivalent:*

(*i*) *S is S-separable over R;*

(*ii*) *$\varphi_* : \mathcal{M}_S \to \mathcal{M}_R$ is separable, i.e. S/R is separable;*

(*iii*) *the Sweedler S-coring $S \otimes_R S$ is semicosplit;*

(*iv*) *S/R has a separability idempotent.*

Corollary 3.42. [4, Corollary 3.30] *In the above setting, the following assertions are equivalent:*

(*i*) *φ^* is semiseparable;*

(*ii*) *there exists an $E \in {}_R\mathrm{Hom}_R(S,R)$ such that $\varphi E(1_S) = 1_S$;*

(*iii*) *F is separable (that is, the Sweedler S-coring $S \otimes_R S$ is coseparable) and $K^{\varphi^*\varphi_*}$ is naturally full.*

Remark 3.43. In Corollary 3.41 the equivalence between (i), (ii) and (iv) is Proposition 1.40 i). Since the coring counit $\varepsilon_{\mathcal{C}}$ is the multiplication $S \otimes_R S \to S$ and we can choose $c = 1_S \otimes_R 1_S \in \mathcal{C}$, the existence of $z \in \mathcal{C}^S$ such that $c = \varepsilon_{\mathcal{C}}(z)c$, for every $c \in \mathcal{C}$, is equivalent to the existence of $z \in \mathcal{C}^S$ such that $1_S = \varepsilon_{\mathcal{C}}(z)$, i.e. of a separability idempotent of S/R. In Corollary 3.42 the equivalence between (i) and (ii) is Proposition 3.1 ii), while the equivalence between (i) and (iii) is Theorem 2.47 ii) applied to the adjunction (φ^*, φ_*).

3.5 Right Hopf algebras

Let B be a bialgebra over a field $\Bbbk$, let $\mathfrak{M}$ denote the category of vector spaces over $\Bbbk$ and let $\mathfrak{M}_B^B$ denote the category of right Hopf modules over B. Consider the coinvariant functor $(-)^{\mathrm{co}B} : \mathfrak{M}_B^B \to \mathfrak{M}$ which is defined, for every object M in $\mathfrak{M}_B^B$, by setting $M^{\mathrm{co}B} := \{m \in M \mid \rho_M(m) = m \otimes 1_B\}$. It is known that it fits into an adjoint triple $\overline{(-)}^B \dashv (-) \otimes B \dashv (-)^{\mathrm{co}B}$, see e.g. [79, Section 3], where $\overline{M}^B = \frac{M}{MB^+}$ and $B^+ = \ker(\varepsilon_B)$. The unit and counit are given by

$$\eta_M : M \to \overline{M}^B \otimes B,\ m \mapsto \sum \overline{m_0} \otimes m_1, \qquad \epsilon_V : \overline{(V \otimes B)}^B \xrightarrow{\cong} V,\ \overline{v \otimes b} \mapsto v\varepsilon_B(b)$$

$$\nu_V : V \xrightarrow{\cong} (V \otimes B)^{\mathrm{co}B},\ v \mapsto v \otimes 1_B, \qquad \theta_M : M^{\mathrm{co}B} \otimes B \to M,\ m \otimes b \mapsto mb.$$

By Proposition 2.41, the functor $(-)^{\mathrm{co}B}$ is semiseparable (resp., separable, naturally full) if and only if so is $\overline{(-)}^B$. Moreover, by [19, Proposition 3.4.1], the functor $(-) \otimes B$ is fully faithful so that $(-)^{\mathrm{co}B}$ is a coreflection. Thus, by Corollary 2.64 it follows that $(-)^{\mathrm{co}B}$ is semiseparable if, and only if, it is naturally full if, and only if, it is Frobenius.

We now characterize the semiseparability of $(-)^{\mathrm{co}B}$. Note that there is a natural transformation $\sigma : (-)^{\mathrm{co}B} \to \overline{(-)}^B$ defined on components by $\sigma_M : M^{\mathrm{co}B} \to \overline{M}^B, m \mapsto \overline{m} := m + MB^+$, see [79, Section 3].

Theorem 3.44. [4, Theorem 3.31] *Let B be a bialgebra over a field $\Bbbk$ and consider the coinvariant functor $(-)^{\mathrm{co}B} : \mathfrak{M}_B^B \to \mathfrak{M}$. The following assertions are equivalent:*

(*i*) *$(-)^{\mathrm{co}B}$ is semiseparable;*

(*ii*) *B is a right Hopf algebra with anti-multiplicative and anti-comultiplicative right antipode;*

(*iii*) *the canonical natural transformation $\sigma : (-)^{\mathrm{co}B} \to \overline{(-)}^B$ is invertible;*

(*iv*) *the canonical natural transformation $\sigma : (-)^{\mathrm{co}B} \to \overline{(-)}^B$ is split-mono.*

Proof. $(i) \Leftrightarrow (ii)$. We noticed that $(-)^{\mathrm{co}B}$ is semiseparable if and only if it is Frobenius. Moreover, $(-)^{\mathrm{co}B}$ is Frobenius if and only if the natural transformation σ is invertible, c.f. [79, Lemma 2.3] applied to the adjoint triple $\overline{(-)}^B \dashv (-) \otimes B \dashv (-)^{\mathrm{co}B}$.
$(ii) \Leftrightarrow (iii)$. The equivalence follows by [79, Theorem 3.7].
$(i) \Leftrightarrow (iii) \Leftrightarrow (iv)$. It follows from Proposition 2.68. □

Remark 3.45. The equivalence $(i) \Leftrightarrow (ii)$ in the previous result is an analogue of $(1) \Leftrightarrow (6)$ in [79, Theorem 3.13] for semiseparability.

Remark 3.46. [4, Remark 3.32] The functor $(-)^{\mathrm{co}B} : \mathfrak{M}_B^B \to \mathfrak{M}$ fits into an adjoint triple $\overline{(-)}^B \dashv (-) \otimes B \dashv (-)^{\mathrm{co}B}$. Thus, $(-)^{\mathrm{co}B}$ is Frobenius if, and only if, $(-)^{\mathrm{co}B} \dashv (-) \otimes B$ if, and only if, $\overline{(-)}^B \cong (-)^{\mathrm{co}B}$. Note that there are bialgebras B which are not right Hopf algebras and hence $(-)^{\mathrm{co}B}$ needs not to be a Frobenius functor in general. For instance, let G be a monoid and consider the monoid algebra $B = \Bbbk G$ over a field $\Bbbk$. If B is a right Hopf algebra, then it has a right antipode $S_B : B \to B$ and hence, for every $x \in G$, one has $xS_B(x) = \sum x_{(1)} S_B\left(x_{(2)}\right) = \varepsilon_B(x) 1_B = 1_G$. In particular, each element in G is right invertible and hence G must be a group, which is not always the case. Moreover, see [79, Example 3.9], there are bialgebras B satisfying the equivalent conditions of Theorem 3.44 that are not Hopf algebras, i.e. such that the coreflection $(-)^{\mathrm{co}B}$ is semiseparable but not separable. Indeed, B is a Hopf algebra if, and only if, $(-)^{\mathrm{co}B}$ is an equivalence if, and only if, it is separable, cf. Remark 2.65.

Chapter 4

Semifunctors and semifullness

The notion of semifunctor between categories, due to S. Hayashi (1985), is defined as a functor that does not necessarily preserve identities. In this chapter we present the results investigated in [21]. We show how several properties of functors, such as fullness, full faithfulness, (semi)separability, natural fullness, can be formulated for semifunctors. Since a full semifunctor is actually a functor, we introduce a notion of semifullness (and then semifull faithfulness and natural semifullness) for semifunctors. We derive these conditions from requirements on the hom-set components associated with a semifunctor, that we refer to as "semisplitting properties" for seminatural transformations and we investigate the corresponding properties for morphisms whose source or target is the image of a semifunctor. We characterize these properties for semifunctors that are part of a semiadjunction in terms of semisplitting conditions for the unit and counit of the semiadjunction. We provide examples of semifunctors studied with respect to these notions.

4.1 Semifunctors and semiadjunctions

The notion of semifunctor between categories was investigated by S. Hayashi in [46], in order to develop a categorical semantics for non-extensional typed lambda calculus. This notion also appeared in [38] under the name of *weak functor* and in [40, 1.284] under the name of *prefunctor*.

In this section we recall mainly from [46] and [52] the notions of semifunctor, seminatural transformation, natural semi-isomorphism and semiadjunction.

Definition 4.1. [46, Definition 1.1] Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *semifunctor* $F : \mathcal{C} \to \mathcal{D}$ is the datum of an object map $\mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$, $X \mapsto F(X)$, between the classes of objects of $\mathcal{C}$ and $\mathcal{D}$, and of a morphism map $\mathcal{F}_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$, $f \mapsto F(f)$, for every pair of objects X, Y in $\mathcal{C}$, preserving compositions, i.e. $F(g \circ f) = F(g) \circ F(f)$, for every pair of composable morphisms $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$.

The image of an identity morphism Id_X through a semifunctor $F : \mathcal{C} \to \mathcal{D}$ is an idempotent morphism in $\mathcal{D}$ as $F(\mathrm{Id}_X) = F(\mathrm{Id}_X \circ \mathrm{Id}_X) = F(\mathrm{Id}_X) \circ F(\mathrm{Id}_X)$.

Remark 4.2. The composite of a semifunctor with a functor is a semifunctor.

There is a related notion of morphism between semifunctors. As in the functorial case, a *natural transformation* $\alpha : F \to F'$ between semifunctors $F, F' : \mathcal{C} \to \mathcal{D}$ is defined as a family $(\alpha_X : FX \to F'X)_{X \in \mathcal{C}}$ of morphisms in $\mathcal{D}$ such that $\alpha_{X'} \circ Ff = F'f \circ \alpha_X$ for any morphism $f : X \to X'$ in $\mathcal{C}$. Given a semifunctor $F : \mathcal{C} \to \mathcal{D}$, there is a natural transformation $F\mathrm{Id} : F \to F$ with components $F\mathrm{Id}_X : FX \to FX$ and a natural transformation $\mathrm{Id}_F : F \to F$ with components $\mathrm{Id}_{FX} : FX \to FX$. Note that $F\mathrm{Id} \neq \mathrm{Id}_F$ in general, unless F is a functor.

Definition 4.3. [52, Definition 2.4] A *seminatural transformation* $\alpha : F \to F'$ between semifunctors $F, F' : \mathcal{C} \to \mathcal{D}$ is a natural transformation with the additional property that, for every X in $\mathcal{C}$, $\alpha_X \circ F\mathrm{Id}_X = \alpha_X$.

Remark 4.4. Note that, by naturality of α, the condition $\alpha_X \circ F\mathrm{Id}_X = \alpha_X$ in Definition 4.3 is equivalent to $F'\mathrm{Id}_X \circ \alpha_X = \alpha_X$. If (at least) one of the semifunctors F, F' is a functor, then the notions of natural transformation and seminatural transformation coincide, see [52, Theorem 2.5].

Remark 4.5. Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be semifunctors. We observe that for any natural transformation $\alpha : GF \to \mathrm{Id}_{\mathcal{C}}$ with codomain the identity functor (it is indeed a seminatural transformation), we have $\alpha \circ G\mathrm{Id}_F = \alpha \circ GF\mathrm{Id} \circ G\mathrm{Id}_F = \alpha \circ G(F\mathrm{Id} \circ \mathrm{Id}_F) = \alpha \circ GF\mathrm{Id} = \alpha$. Analogously, for any (semi)natural transformation $\alpha : \mathrm{Id}_{\mathcal{C}} \to GF$ with domain the identity functor, we have $G\mathrm{Id}_F \circ \alpha = G\mathrm{Id}_F \circ GF\mathrm{Id} \circ \alpha = G(\mathrm{Id}_F \circ F\mathrm{Id}) \circ \alpha = GF\mathrm{Id} \circ \alpha = \alpha$.

Semifunctors $F, F' : \mathcal{C} \to \mathcal{D}$ are said to be *naturally semi-isomorphic* if, and only if, there are natural transformations $\alpha : F \to F'$ and $\beta : F' \to F$ such that

i) $\alpha \circ F\mathrm{Id} = \alpha$; *ii)* $\beta \circ F'\mathrm{Id} = \beta$; *iii)* $\alpha \circ \beta = F'\mathrm{Id}$; *iv)* $\beta \circ \alpha = F\mathrm{Id}$.

In this case, α is said to be a *natural semi-isomorphism* [52, Subsection 2.2] and it is denoted by $F \cong_{\mathrm{s}} F'$. Since its semi-inverse β is uniquely determined by α, it will be usually written as α^{-1}.

Definition 4.6. [21, Section 1.1] Let $\alpha : F \to F'$ be a seminatural transformation between semifunctors $F, F' : \mathcal{C} \to \mathcal{D}$. We call α a

i) **natural semisplit-mono** if there exists a seminatural transformation $\beta : F' \to F$ such that $\beta \circ \alpha = F\mathrm{Id}$;

ii) **natural semisplit-epi** if there exists a seminatural transformation $\beta : F' \to F$ such that $\alpha \circ \beta = F'\mathrm{Id}$.

Lemma 4.7. [21, Lemma 1.2] *A seminatural transformation $\alpha : F \to F'$ between semifunctors F, F' is a natural semi-isomorphism if, and only if, α is both a natural semisplit-mono and a natural semisplit-epi.*

Proof. If α is a natural semi-isomorphism, then it is trivially both a natural semisplit-mono and a natural semisplit-epi. Conversely, if α is a natural semisplit-mono and a natural semisplit-epi, then there exists a seminatural transformation $\beta : F' \to F$ such that $\beta \circ \alpha = F\mathrm{Id}$ and there is a seminatural transformation $\beta' : F' \to F$ such that $\alpha \circ \beta' = F'\mathrm{Id}$. Note that $\beta = \beta \circ F'\mathrm{Id} = \beta \circ \alpha \circ \beta' = F\mathrm{Id} \circ \beta' = \beta'$, thus α is a natural semi-isomorphism. □

Moreover, α is a *natural split-mono* (resp., *natural split-epi*), if there exists a seminatural transformation $\beta : F' \to F$ such that $\beta \circ \alpha = \mathrm{Id}_F$ (resp., $\alpha \circ \beta = \mathrm{Id}_{F'}$); α is a *natural isomorphism* if there exists a seminatural transformation $\beta : F' \to F$ such that $\beta \circ \alpha = \mathrm{Id}_F$ and $\alpha \circ \beta = \mathrm{Id}_{F'}$.

Remark 4.8. Let $\alpha : F \to F'$ be a seminatural transformation between semifunctors $F, F' : \mathcal{C} \to \mathcal{D}$. If F is a functor, then α is a natural semisplit-mono if, and only if, α is a natural split-mono; if F' is a functor, then α is a natural semisplit-epi if, and only if, α is a natural split-epi. If both F and F' are functors, then $\alpha : F \to F'$ is a natural semi-isomorphism if, and only if, α is a natural isomorphism.

In Section 4.2 we will describe the corresponding "semisplitting properties" for the component morphisms of a seminatural transformation.

4.1.1 Idempotent completion of a category

In this subsection we remind the idempotent completion construction which provides a canonical way to turn semifunctors into functors.

An idempotent morphism $e : X \to X$ in $\mathcal{C}$ *splits* if there exist two morphisms $h : X \to Y$ and $k : Y \to X$ in $\mathcal{C}$ such that $e = k \circ h$ and $h \circ k = \mathrm{Id}_Y$; the category $\mathcal{C}$ is said to be *idempotent complete* or *Cauchy complete* if all idempotents split.

Example 4.9. Any category equipped with (co)equalizers is idempotent complete, see [52, Theorem 2.15], [19, Proposition 6.5.4].

Definition 4.10. The *idempotent completion* $\mathcal{C}^\natural$ (also known under the names of *Cauchy completion* [58] or *Karoubi envelope* [56]) of a category $\mathcal{C}$ is the category whose objects are pairs (X, e), where X is an object in $\mathcal{C}$ and $e : X \to X$ is an idempotent morphism in $\mathcal{C}$, and a morphism $f : (X, e) \to (X', e')$ in $\mathcal{C}^\natural$ is a morphism $f : X \to X'$ in $\mathcal{C}$ such that $f = e' \circ f \circ e$ (or equivalently, such that $e' \circ f = f = f \circ e$).

Remark 4.11. *i*) Given $(X, e) \in \mathcal{C}^\natural$, $\mathrm{Id}_{(X,e)} \neq \mathrm{Id}_X$ but $\mathrm{Id}_{(X,e)} = e : (X, e) \to (X, e)$.
ii) The category $\mathcal{C}^\natural$ is idempotent complete.

There is a canonical functor

$$\iota_\mathcal{C} : \mathcal{C} \to \mathcal{C}^\natural, \quad X \mapsto (X, \mathrm{Id}_X), \quad [f : X \to Y] \mapsto [f : (X, \mathrm{Id}_X) \to (Y, \mathrm{Id}_Y)],$$

which is fully faithful; $\iota_\mathcal{C}$ is an equivalence if, and only if, $\mathcal{C}$ is idempotent complete. Any semifunctor $F : \mathcal{C} \to \mathcal{D}$ induces a functor $F^\natural : \mathcal{C}^\natural \to \mathcal{D}^\natural$ such that

$$F^\natural (X, e) = (FX, Fe), \quad F^\natural f = Ff.$$

In fact, $F^\natural \mathrm{Id}_{(X,e)} = Fe = \mathrm{Id}_{(FX,Fe)} = \mathrm{Id}_{F^\natural(X,e)}$, as observed in [46, Definition 1.3]. Note that in general $\iota_\mathcal{D} \circ F \neq F^\natural \circ \iota_\mathcal{C}$, unless F is a functor. On the other hand, there is a semifunctor $\upsilon_\mathcal{C} : \mathcal{C}^\natural \to \mathcal{C}$ which maps an object (C, c) in $\mathcal{C}^\natural$ to the underlying object C and a morphism $f : (C, c) \to (C', c')$ to the underlying morphism $\upsilon_\mathcal{C} f : C \to C'$ such that $c' \circ \upsilon_\mathcal{C} f \circ c = \upsilon_\mathcal{C} f$. If $G : \mathcal{C}^\natural \to \mathcal{D}^\natural$ is a functor, then there is a semifunctor $F : \mathcal{C} \to \mathcal{D}$ given as in the following lemma that will be helpful afterwards.

Lemma 4.12. (Cf. [53, proof of Theorem 1]) *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. Consider the functor $\iota_\mathcal{C} : \mathcal{C} \to \mathcal{C}^\natural$ and the semifunctor $\upsilon_\mathcal{D} : \mathcal{D}^\natural \to \mathcal{D}$ as above.*

i) *For every functor $G : \mathcal{C}^\natural \to \mathcal{D}^\natural$, then $F := \upsilon_\mathcal{D} \circ G \circ \iota_\mathcal{C} : \mathcal{C} \to \mathcal{D}$ is a semifunctor such that $F^\natural \cong G$.*

ii) *Given semifunctors $F, G : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\alpha : F^\natural \to G^\natural$, then $\beta := \upsilon_\mathcal{D} \circ \alpha \circ \iota_\mathcal{C} : F \to G$ is a seminatural transformation such that $\beta^\natural = \alpha$.*

Lemma 4.13. (Cf. [52, Theorem 7.3]) *A (semi)natural transformation $\alpha : F \to F'$ of semifunctors $F, F' : \mathcal{C} \to \mathcal{D}$ induces the natural transformation $\alpha^\natural : F^\natural \to (F')^\natural$ with components $\alpha^\natural_{(X,e)} := \alpha_X \circ Fe = F'e \circ \alpha_X$.*

Proof. For any morphism $f : (X, e) \to (X', e')$ in $\mathcal{C}^\natural$ we have that $F'^\natural f \circ \alpha^\natural_{(X,e)} = F' f \circ F'e \circ \alpha_X = F'(f \circ e) \circ \alpha_X = F'(e' \circ f) \circ \alpha_X = F'e' \circ F' f \circ \alpha_X = F'e' \circ \alpha_{X'} \circ Ff = \alpha_{X'} \circ Fe' \circ Ff = \alpha^\natural_{(X',e')} \circ F^\natural f$. □

We observe the following fact.

Proposition 4.14. *Let $F, F' : \mathcal{C} \to \mathcal{D}$ be semifunctors.*

i) If $\alpha : F \to F'$ is a natural semisplit-mono, then $\alpha^\natural$ is a natural split-mono; on the other hand, if $\gamma : F^\natural \to F'^\natural$ is a natural split-mono, then there is a natural semisplit-mono $\gamma' : F \to F'$ such that $\gamma = (\gamma')^\natural$.

ii) If $\alpha : F \to F'$ is a natural semisplit-epi, then $\alpha^\natural$ is a natural split-epi; on the other hand, if $\gamma : F^\natural \to F'^\natural$ is a natural split-epi, then there is a natural semisplit-epi $\gamma' : F \to F'$ such that $\gamma = (\gamma')^\natural$.

Proof. i). Assume that α is a natural semisplit-mono, i.e. there exists a seminatural transformation $\beta : F' \to F$ such that $\beta \circ \alpha = F\mathrm{Id}$. By Lemma 4.13 consider the natural transformations $\alpha^\natural$, $\beta^\natural$ given, for every $(X, e) \in \mathcal{C}^\natural$, by $\alpha^\natural_{(X,e)} := \alpha_X \circ Fe = F'e \circ \alpha_X$ and $\beta^\natural_{(X,e)} := \beta_X \circ F'e = Fe \circ \beta_X$, respectively. Then, for every $(X, e) \in \mathcal{C}^\natural$, we have

$$\beta^\natural_{(X,e)} \circ \alpha^\natural_{(X,e)} = \beta_X \circ F'e \circ \alpha_X \circ Fe = \beta_X \circ \alpha_X \circ Fe \circ Fe = F\mathrm{Id}_X \circ Fe = Fe = \mathrm{Id}_{(FX,Fe)}.$$

On the other hand, if $\gamma : F^\natural \to F'^\natural$ is a natural split-mono (i.e., there exists a natural transformation $\xi : F'^\natural \to F^\natural$ such that $\xi \circ \gamma = \mathrm{Id}_{F^\natural}$), then define $\gamma' : F \to F'$ by $\gamma'_X := \gamma_{(X,\mathrm{Id}_X)} : FX \to F'X$ and $\xi' : F' \to F$ by $\xi'_X := \xi_{(X,\mathrm{Id}_X)} : F'X \to FX$, for every $X \in \mathcal{C}$. We have that

$$\xi'_X \circ \gamma'_X = \xi_{(X,\mathrm{Id}_X)} \circ \gamma_{(X,\mathrm{Id}_X)} = \mathrm{Id}_{(FX,F\mathrm{Id}_X)} = F\mathrm{Id}_X,$$

for every $X \in \mathcal{C}$, hence $\xi' \circ \gamma' = F\mathrm{Id}$. Moreover, $\gamma'_X \circ F\mathrm{Id}_X = \gamma_{(X,\mathrm{Id}_X)} \circ \mathrm{Id}_{(FX,F\mathrm{Id}_X)} = \gamma_{(X,\mathrm{Id}_X)} = \gamma'_X$ and $\xi'_X \circ F'\mathrm{Id}_X = \xi_{(X,\mathrm{Id}_X)} \circ \mathrm{Id}_{(F'X,F'\mathrm{Id}_X)} = \xi_{(X,\mathrm{Id}_X)} = \xi'_X$, for every $X \in \mathcal{C}$, hence γ', ξ' are seminatural. Note that $\gamma' = \upsilon_{\mathcal{D}}\gamma\iota_{\mathcal{C}}$, so by Lemma 4.12 we have $\gamma = (\gamma')^\natural$.
ii). It follows dually from *i)*. ☐

Corollary 4.15. [52, Theorem 2.12] *Given semifunctors $F, F' : \mathcal{C} \to \mathcal{D}$, $F \cong_{\mathrm{s}} F'$ is a natural semi-isomorphism if, and only if, $F^\natural \cong (F')^\natural$ is a natural isomorphism of functors.*

Example 4.16. (Cf. [5, Example 3.3]) Let R be a ring and let ${}_R\mathcal{M}$ be the category of left R-modules. Denote by ${}_R\mathcal{M}_f$ and ${}_R\mathcal{M}_p$ the full subcategories of ${}_R\mathcal{M}$ whose objects are free left R-modules and projective left R-modules, respectively. Let $\Psi : {}_R\mathcal{M}_f \to {}_R\mathcal{M}_p$ be the inclusion functor. By [56, Theorem 6.12, page 30], the functor Ψ induces an equivalence $\Psi' : ({}_R\mathcal{M}_f)^\natural \to {}_R\mathcal{M}_p$, $(F, e) \mapsto \mathrm{Im}(e)$. This fact is well-known and, in the finitely generated case, it is written explicitly in [56, Theorem 6.16]. For sake of completeness we include here a proof. Let $\Psi : ({}_R\mathcal{M}_f)^\natural \to {}_R\mathcal{M}_p$ be the functor defined by $(F, e) \mapsto \mathrm{Im}(e)$, where F is a free left R-module and $e : F \to F$ is an idempotent, which splits as $e = i \circ p$, where $p : F \to \mathrm{Im}(e)$ is the canonical projection and the inclusion map $i : \mathrm{Im}(e) \to F$ is the induced section, i.e. $p \circ i = \mathrm{Id}_{\mathrm{Im}(e)}$; to a given morphism $f : (F, e) \to (F', e')$ in $({}_R\mathcal{M}_f)^\natural$, Ψ assigns the composite morphism $\Psi(f) = p' \circ f \circ i : \mathrm{Im}(e) \to \mathrm{Im}(e')$ of projective left R-modules, where the idempotent arrow $e' : F' \to F'$ splits as the canonical projection $p' : F' \to \mathrm{Im}(e')$ followed by the inclusion $i' : \mathrm{Im}(e') \to F'$. It results that Ψ is an equivalence of categories. Indeed, given a morphism $h : \mathrm{Im}(e) \to \mathrm{Im}(e')$, we set $f := i'hp$. Then, $e'fe = i'p'i'hpip = i'hp = f$, so that we get a morphism $f : (F, e) \to (F', e')$ in $({}_R\mathcal{M}_f)^\natural$. Moreover, $h = p'i'hpi = p'fi = \Psi(f)$, so that Ψ is full. If $f, g : (F, e) \to (F', e')$ are such that $\Psi(f) = \Psi(g)$, then $p'fi = p'gi$ and hence $f = e'fe = i'p'fip = i'p'gip = e'ge = g$, thus Ψ is faithful. Given P projective, the canonical projection $\pi : R^{(P)} \to P$, $(r_p)_{p\in P} \mapsto \sum_{p\in P} r_p p$ splits via some morphism $\sigma : P \to R^{(P)}$ as P is projective. Then, $\Psi(R^{(P)}, \sigma\pi) = \mathrm{Im}(\sigma\pi) = \mathrm{Im}(\sigma) \cong P$, so that Ψ is essentially surjective on objects. By [61, Theorem 1, page 93] this proves that Ψ is an equivalence of categories.

The category $\mathsf{Cat}_{\mathrm{s}}$ with categories as objects, semifunctors as arrows, and seminatural transformations as 2-cells is a 2-category [52, Theorem 7.2]. Since any functor is in particular a semifunctor, there is an inclusion of the 2-category Cat of categories, functors and natural transformations, in $\mathsf{Cat}_{\mathrm{s}}$. Conversely, the idempotent completion is a canonical way to transform semifunctors into functors. In fact, the *Karoubi envelope functor* $\kappa : \mathsf{Cat}_{\mathrm{s}} \to \mathsf{Cat}$, defined by $\kappa(\mathcal{C}) = \mathcal{C}^{\natural}$, $\kappa(F) = F^{\natural}$, for any category $\mathcal{C}$ and any semifunctor $F : \mathcal{C} \to \mathcal{D}$, is the right adjoint of the inclusion functor $i : \mathsf{Cat} \to \mathsf{Cat}_{\mathrm{s}}$ (see [52, Theorem 2.10]). Moreover, as shown in [52, Theorem 7.3], κ is a 2-functor, sending any seminatural transformation α into $\alpha^{\natural}$, and it defines an equivalence of 2-categories between $\mathsf{Cat}_{\mathrm{s}}$ and the full 2-subcategory of Cat having idempotent complete categories as objects [53, Theorem 1]. We recall the following.

Lemma 4.17. (See [50, Lemma 23], [52, Lemma 7.5]) *For categories $\mathcal{C}$, $\mathcal{D}$, semifunctors $F, G : \mathcal{C} \to \mathcal{D}$, seminatural transformations $\alpha, \beta : F \to G$, the Karoubi envelope functor fulfills the following properties:*

i) $\kappa(\mathcal{C}) = \kappa(\mathcal{D})$*, then* $\mathcal{C} = \mathcal{D}$*;*

ii) $\kappa(F) = \kappa(G)$*, then* $F = G$*;*

iii) $\kappa(\alpha) = \kappa(\beta)$*, then* $\alpha = \beta$*.*

Many standard properties for functors can be extended to semifunctors, as for instance the notion of adjunction.

4.1.2 Semiadjunctions

Given the opposite category $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$ and a semifunctor $F : \mathcal{C} \to \mathcal{D}$, consider the semifunctor

$$\mathrm{Hom}_{\mathcal{D}}(F-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathsf{Set},$$

$$(C, D) \mapsto \mathrm{Hom}_{\mathcal{D}}(FC, D), \quad (f : C' \to C, g : D \to D') \mapsto \mathrm{Hom}_{\mathcal{D}}(Ff, g)(-) = g \circ - \circ Ff.$$

Since in general, for any morphism $h : FC \to D$ in $\mathcal{D}$, $\mathrm{Hom}_{\mathcal{D}}(F\mathrm{Id}_C, \mathrm{Id}_D)(h) = \mathrm{Id}_D \circ h \circ F\mathrm{Id}_C = h \circ F\mathrm{Id}_C \neq h$, then $\mathrm{Hom}_{\mathcal{D}}(F-, -)$ is really a semifunctor. Analogously, for a semifunctor $G : \mathcal{D} \to \mathcal{C}$ one can consider the semifunctor $\mathrm{Hom}_{\mathcal{C}}(-, G-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathsf{Set}$.

Definition 4.18. [52, Definition 3.1] A *semiadjunction* is a triple $(F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}, \tau)$, where F, G are semifunctors and $\tau : \mathrm{Hom}_{\mathcal{D}}(F-, -) \to \mathrm{Hom}_{\mathcal{C}}(-, G-)$ is a natural semi-isomorphism.

Equivalently, by [52, Theorem 3.10] a semiadjunction (F, G, η, ϵ) is the datum of semifunctors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ equipped with natural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ (unit) and $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ (counit) such that the "semitriangle" identities

$$G\epsilon \circ \eta G = G\mathrm{Id} \quad \text{and} \quad \epsilon F \circ F\eta = F\mathrm{Id} \tag{4.1}$$

hold true, see also [50, Definition 22]. In particular, η and ϵ are indeed seminatural transformations. We usually denote a semiadjunction (F, G, η, ϵ) by $F \dashv_{\mathrm{s}} G$. As in the functorial case, cf. (1.5), the natural semi-isomorphism τ is given, for all $C \in \mathcal{C}$, $D \in \mathcal{D}$, by $\tau_{C,D} : \mathrm{Hom}_{\mathcal{D}}(FC, D) \to \mathrm{Hom}_{\mathcal{C}}(C, GD)$,

$$\tau_{C,D}(h) = G(h) \circ \eta_C, \tag{4.2}$$

for any morphism $h : FC \to D$ in $\mathcal{D}$.

Remark 4.19. The semi-inverse $\sigma : \mathrm{Hom}_{\mathcal{C}}(-, G-) \to \mathrm{Hom}_{\mathcal{D}}(F-, -)$ of τ is given by

$$\sigma_{C,D}(g) = \epsilon_D \circ F(g), \tag{4.3}$$

for any $g : C \to GD$ in $\mathcal{C}$. In fact, for any $C \in \mathcal{C}$, $D \in \mathcal{D}$, we have that $(\tau_{C,D} \circ \mathrm{Hom}_{\mathcal{D}}(F\mathrm{Id}_C, \mathrm{Id}_D))(h) = \tau_{C,D}(\mathrm{Id}_D \circ h \circ F\mathrm{Id}_C) = \tau_{C,D}(h \circ F\mathrm{Id}_C) = Gh \circ GF\mathrm{Id}_C \circ \eta_C = Gh \circ \eta_C = \tau_{C,D}(h)$, for every $h : FC \to D$ in $\mathcal{D}$. Similarly, $(\sigma_{C,D} \circ \mathrm{Hom}_{\mathcal{C}}(\mathrm{Id}_C, G\mathrm{Id}_D))(g) = \sigma_{C,D}(G\mathrm{Id}_D \circ g \circ \mathrm{Id}_C) = \epsilon_D \circ FG\mathrm{Id}_D \circ Fg = \epsilon_D \circ Fg = \sigma_{C,D}(g)$, for every $g : C \to GD$ in $\mathcal{C}$. Moreover, for every $g : C \to GD$ in $\mathcal{C}$ we have that $\tau_{C,D}\sigma_{C,D}(g) = \tau_{C,D}(\epsilon_D \circ Fg) = G(\epsilon_D \circ Fg) \circ \eta_C = G\epsilon_D \circ GFg \circ \eta_C = G\epsilon_D \circ \eta_{GD} \circ g = G\mathrm{Id}_D \circ g = G\mathrm{Id}_D \circ g \circ \mathrm{Id}_C = \mathrm{Hom}_{\mathcal{C}}(\mathrm{Id}_C, G\mathrm{Id}_D)(g)$, and for every morphism $h : FC \to D$ in $\mathcal{D}$ we have that $\sigma_{C,D}\tau_{C,D}(h) = \sigma_{C,D}(Gh \circ \eta_C) = \epsilon_D \circ FGh \circ F\eta_C = h \circ \epsilon_{FC} \circ F\eta_C = h \circ F\mathrm{Id}_C = \mathrm{Id}_D \circ h \circ F\mathrm{Id}_C = \mathrm{Hom}_{\mathcal{D}}(F\mathrm{Id}_C, \mathrm{Id}_D)(h)$. It is easy to see that τ and σ are natural.

Remark 4.20. Any adjunction of functors is trivially a semiadjunction, and if (F, G, η, ϵ) is a semiadjunction, then $(F^\natural, G^\natural, \eta^\natural, \epsilon^\natural)$ is an adjunction of functors [46, Theorem 1.9], with unit and counit given on components respectively by

$$\eta^\natural_{(C,c)} = \eta_C \circ c : (C, c) \to (GFC, GFc)\,, \quad \epsilon^\natural_{(D,d)} = d \circ \epsilon_D : (FGD, FGd) \to (D, d)\,.$$

More precisely, we state the following result.

Theorem 4.21. [52, Theorem 3.5] *Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be semifunctors. Then, $F \dashv_{\mathrm{s}} G$ if, and only if, $F^\natural \dashv G^\natural$.*

We observe the following fact.

Lemma 4.22. *Let $F : \mathcal{C} \to \mathcal{D}$, $K : \mathcal{C} \to \mathcal{C}$ be semifunctors. If there exists a (semi)natural transformation $\alpha : K \to \mathrm{Id}_{\mathcal{C}}$ such that $F\alpha$ is either a split-epi, or a split-mono, then F is a functor.*

Proof. Let $\alpha : K \to \mathrm{Id}_{\mathcal{C}}$ be a (semi)natural transformation such that $F\alpha$ is split-epi, i.e. there exists a (semi)natural transformation $\beta : F \to FK$ such that $F\alpha \circ \beta = \mathrm{Id}_F$. Then, for every $X \in \mathcal{C}$, we have $F\mathrm{Id}_X = F\mathrm{Id}_X \circ \mathrm{Id}_{FX} = F\mathrm{Id}_X \circ F\alpha_X \circ \beta_X = F\alpha_X \circ \beta_X = \mathrm{Id}_{FX}$, hence F is a functor. Similarly, if $F\alpha$ is a split-mono, i.e. there exists a (semi)natural transformation $\beta : F \to FK$ such that $\beta \circ F\alpha = \mathrm{Id}_F$. Then, for every $X \in \mathcal{C}$, we have $F\mathrm{Id}_X = \mathrm{Id}_{FX} \circ F\mathrm{Id}_X = \beta_X \circ F\alpha_X \circ F\mathrm{Id}_X = \beta_X \circ F\alpha_X = \mathrm{Id}_{FX}$, hence F is a functor. ☐

As a particular case, we have the next corollary, which we will use in Chapter 5.

Corollary 4.23. *Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be semifunctors.*

i) If there exist (semi)natural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$, $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ such that $G\epsilon \circ \eta G = \mathrm{Id}_G$, then G is a functor.

ii) If there exist (semi)natural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$, $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ such that $\epsilon F \circ F\eta = \mathrm{Id}_F$, then F is a functor.

An analogue of Proposition 1.17 can be shown for semifunctors.

Proposition 4.24. *Let $F \dashv_{\mathrm{s}} G : \mathcal{D} \to \mathcal{C}$ be a semiadjunction with unit η and counit ϵ. Then, we have the following isomorphisms:*

$$\mathrm{Nat}(GF, \mathrm{Id}_{\mathcal{C}}) \cong \mathrm{Nat}(\mathrm{Hom}_{\mathcal{D}}(F-, F-), \mathrm{Hom}_{\mathcal{C}}(-,-)), \tag{4.4}$$

$$\mathrm{Nat}(\mathrm{Id}_{\mathcal{D}}, FG) \cong \mathrm{Nat}(\mathrm{Hom}_{\mathcal{C}}(G-, G-), \mathrm{Hom}_{\mathcal{D}}(-,-)). \tag{4.5}$$

Remark 4.25. The natural transformations as in (4.4), (4.5) are actually seminatural transformations.

Proof. Let $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ be a (semi)natural transformation. Define $\theta : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ by

$$\theta_{C,C'}(g) = \nu_{C'} \circ Gg \circ \eta_C, \tag{4.6}$$

for any $g : FC \to FC'$ in $\mathcal{D}$. The naturality of θ follows as in the functorial case. For every $C \in \mathcal{C}$, we have $\theta_{GFC,C}(\epsilon_{FC}) = \nu_C \circ G\epsilon_{FC} \circ \eta_{GFC} = \nu_C \circ G\mathrm{Id}_{FC} = \nu_C$, where the last equality follows from Remark 4.5. Conversely, given a natural transformation $\theta : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$, define $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ by

$$\nu_C = \theta_{GFC,C}(\epsilon_{FC}) : GFC \to C, \tag{4.7}$$

for every $C \in \mathcal{C}$. The naturality of ν follows from the naturality of θ. For any C, C' in $\mathcal{C}$ and $g : FC \to FC'$ in $\mathcal{D}$, by naturality of θ, we have

$$\begin{aligned}\nu_{C'} \circ Gg \circ \eta_C &= \theta_{GFC',C'}(\epsilon_{FC'}) \circ Gg \circ \eta_C = \theta_{C,C'}(\epsilon_{FC'} \circ FGg \circ F\eta_C) \\ &= \theta_{C,C'}(g \circ \epsilon_{FC} \circ F\eta_C) = \theta_{C,C'}(g \circ F\mathrm{Id}_C) = \theta_{C,C'}(g) \circ \mathrm{Id}_C = \theta_{C,C'}(g),\end{aligned}$$

thus the correspondence between θ and ν is bijective.

The proof of (4.5) follows from (4.4) by duality arguments. □

It is known that semiadjoint semifunctors are not unique up to isomorphism, but they are unique up to natural semi-isomorphism, cf. [52, Theorem 3.6]. We include a proof for completeness sake. Cf. e.g. [28, Proof of Proposition 9] for the case of functors.

Proposition 4.26. [21, Proposition 1.4]

i) Let $F \dashv_{\mathrm{s}} G$, $F \dashv_{\mathrm{s}} G'$ be semiadjunctions of semifunctors. Then, G and G' are naturally semi-isomorphic.

ii) Let $F \dashv_{\mathrm{s}} G$, $F' \dashv_{\mathrm{s}} G$ be semiadjunctions of semifunctors. Then, F and F' are naturally semi-isomorphic.

Proof. i). Let $F \dashv_{\mathrm{s}} G$, $F \dashv_{\mathrm{s}} G'$ be semiadjunctions with units η, η', and counits ϵ, ϵ', respectively. Consider $\gamma := G'\epsilon \circ \eta' G : G \to G'$ and $\gamma' := G\epsilon' \circ \eta G' : G' \to G$. Note that $\gamma \circ G\mathrm{Id} = G'\epsilon \circ \eta' G \circ G\mathrm{Id} = G'\epsilon \circ G'FG\mathrm{Id} \circ \eta' G = G'(\epsilon \circ FG\mathrm{Id}) \circ \eta' G = G'\epsilon \circ \eta' G = \gamma$, and $\gamma' \circ G'\mathrm{Id} = G\epsilon' \circ \eta G' \circ G'\mathrm{Id} = G\epsilon' \circ GFG'\mathrm{Id} \circ \eta G' = G(\epsilon' \circ FG'\mathrm{Id}) \circ \eta G' = G\epsilon' \circ \eta G' = \gamma'$. Moreover, γ and γ' are natural as they are composition of natural transformations, so they are seminatural transformations. From naturality of η it follows that $\eta G' \circ G'\epsilon = GFG'\epsilon \circ \eta G' FG$ and $\eta G' FG \circ \eta' G = GF\eta' G \circ \eta G$, and from naturality of ϵ' it follows that $G\epsilon \circ G\epsilon' FG = G\epsilon' \circ GFG'\epsilon$. Then, we obtain

$$\begin{aligned}\gamma' \circ \gamma &= G\epsilon' \circ \eta G' \circ G'\epsilon \circ \eta' G = G\epsilon' \circ GFG'\epsilon \circ \eta G' FG \circ \eta' G \\ &= G\epsilon \circ G\epsilon' FG \circ GF\eta' G \circ \eta G = G\epsilon \circ GF\mathrm{Id}_G \circ \eta G = G\epsilon \circ \eta G = G\mathrm{Id}.\end{aligned}$$

Similarly, from naturality of η' and ϵ, we have

$$\begin{aligned}\gamma \circ \gamma' &= G'\epsilon \circ \eta' G \circ G\epsilon' \circ \eta G' = G'\epsilon \circ G'FG\epsilon' \circ \eta' GFG' \circ \eta G' \\ &= G'\epsilon' \circ G'\epsilon FG' \circ G'F\eta G' \circ \eta' G' = G'\epsilon' \circ G'F\mathrm{Id}_{G'} \circ \eta' G' = G'\epsilon' \circ \eta' G' = G'\mathrm{Id}.\end{aligned}$$

ii). It follows dually from *i*). □

In [52] the terminology of *right* (resp., *left*) *semiadjoint* is used to denote a semifunctor G (resp., F) that is part of a semiadjunction $F \dashv_{\mathrm{s}} G$, that is, both semitriangle identities (4.1) hold true. In the following definition we adopt the same terminology with a weaker meaning, inspired by [66, Definition 1.3] for functors.

Definition 4.27. [21, Definition 1.5] We say that:

i) a semifunctor $G : \mathcal{D} \to \mathcal{C}$ is a *right semiadjoint* if there exist a semifunctor $F : \mathcal{C} \to \mathcal{D}$ and seminatural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$, such that $G\epsilon \circ \eta G = G\mathrm{Id}$;

ii) a semifunctor $F : \mathcal{C} \to \mathcal{D}$ is a *left semiadjoint* if there exist a semifunctor $G : \mathcal{D} \to \mathcal{C}$ and seminatural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$, such that $\epsilon F \circ F\eta = F\mathrm{Id}$.

Remark 4.28. In a semiadjunction $F \dashv_{\mathrm{s}} G$, F is a left semiadjoint and G is a right semiadjoint.

Now we show that a right (resp., left) semiadjoint is actually part of a semiadjunction. In particular, we have the following characterization of left and right semiadjoints.

Proposition 4.29. [21, Proposition 1.7]

i) *A semifunctor $G : \mathcal{D} \to \mathcal{C}$ is a right semiadjoint if, and only if, there is a semifunctor $F' : \mathcal{C} \to \mathcal{D}$ (unique up to natural semi-isomorphism), such that $F' \dashv_{\mathrm{s}} G$ is a semiadjunction.*

ii) *A semifunctor $F : \mathcal{C} \to \mathcal{D}$ is a left semiadjoint if, and only if, there is a semifunctor $G' : \mathcal{D} \to \mathcal{C}$ (unique up to natural semi-isomorphism), such that $F \dashv_{\mathrm{s}} G'$ is a semiadjunction.*

Proof. i). If $F' \dashv_{\mathrm{s}} G$ is a semiadjunction, then by Remark 4.28 the semifunctor G is a right semiadjoint and $F' : \mathcal{C} \to \mathcal{D}$ is a left semiadjoint. Conversely, if G is a right semiadjoint, then there exist a semifunctor $F : \mathcal{C} \to \mathcal{D}$ and seminatural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$, such that $G\epsilon \circ \eta G = G\mathrm{Id}$. Set $e := \epsilon F \circ F\eta : F \to F$, which is an idempotent seminatural transformation. Indeed, it is natural as it is composition of natural transformations; for any $X \in \mathcal{C}$ we have $e_X \circ F\mathrm{Id}_X = \epsilon_{FX} \circ F\eta_X \circ F\mathrm{Id}_X = \epsilon_{FX} \circ F(\eta_X \circ \mathrm{Id}_X) = \epsilon_{FX} \circ F\eta_X = e_X$ and, cf. e.g. [66, Lemma 1.4(2)], $e \circ e = \epsilon F \circ F\eta \circ \epsilon F \circ F\eta = \epsilon F \circ \epsilon FGF \circ FGF\eta \circ F\eta = \epsilon F \circ FG\epsilon F \circ F\eta GF \circ F\eta = \epsilon F \circ FG\mathrm{Id}_F \circ F\eta = \epsilon F \circ F\eta = e$. Then, there is a semifunctor $F' : \mathcal{C} \to \mathcal{D}$ given by

$$F'(X) = FX, \quad F'(f) = Ff \circ e_X = e_Y \circ Ff,$$

for every $X \in \mathcal{C}$, $f : X \to Y$ in $\mathcal{C}$. Indeed, for every $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$, we have $F'g \circ F'f = Fg \circ e_Y \circ Ff \circ e_X = Fg \circ Ff \circ e_X \circ e_X = F(g \circ f) \circ e_X = F'(g \circ f)$, so that F' is a semifunctor. Now we show that $(F', G, \eta', \epsilon')$ is a semiadjunction where $\eta'_C := \eta_C$ and $\epsilon'_D := \epsilon_D$, for every object $C \in \mathcal{C}$ and $D \in \mathcal{D}$. Note that by the assumption $G\epsilon \circ \eta G = G\mathrm{Id}$, we get $\epsilon_D \circ e_{GD} = \epsilon_D \circ \epsilon_{FGD} \circ F\eta_{GD} = \epsilon_D \circ FG\epsilon_D \circ F\eta_{GD} = \epsilon_D \circ F\left(G\epsilon_D \circ \eta_{GD}\right) = \epsilon_D \circ FG\mathrm{Id}_D = \epsilon_D$, for every $D \in \mathcal{D}$, so

$$\epsilon \circ eG = \epsilon. \tag{4.8}$$

For every $D \in \mathcal{D}$, we have $\epsilon'_D \circ F'G\mathrm{Id}_D = \epsilon_D \circ FG\mathrm{Id}_D \circ e_{GD} \overset{(4.8)}{=} \epsilon_D \circ e_{GD} \circ FG\mathrm{Id}_D \circ e_{GD} = \epsilon_D \circ e_{GD} \circ e_{GD} \circ FG\mathrm{Id}_D = \epsilon_D \circ e_{GD} \circ FG\mathrm{Id}_D \overset{(4.8)}{=} \epsilon_D \circ FG\mathrm{Id}_D = \epsilon_D = \epsilon'_D$, and for every morphism $f : D \to D'$ in $\mathcal{D}$ we have $\epsilon'_{D'} \circ F'Gf = \epsilon_{D'} \circ FGf \circ e_{GD} = f \circ \epsilon_D \circ e_{GD} \overset{(4.8)}{=} f \circ \epsilon_D = f \circ \epsilon'_D$, so that $\epsilon' := (\epsilon_D)_{D\in\mathcal{D}} : F'G \to \mathrm{Id}_{\mathcal{D}}$ is indeed a seminatural transformation. For every object C in $\mathcal{C}$, it holds $\eta'_C \circ \mathrm{Id}_{\mathcal{C}}(\mathrm{Id}_C) = \eta'_C \circ \mathrm{Id}_C = \eta'_C$, and for every morphism $f : X \to Y$ in $\mathcal{C}$ we have

$$\begin{aligned} GF'f \circ \eta'_X &= G\left(Ff \circ e_X\right) \circ \eta_X = G\left(e_Y \circ Ff\right) \circ \eta_X = Ge_Y \circ GFf \circ \eta_X \\ &= G(\epsilon_{FY} \circ F\eta_Y) \circ GFf \circ \eta_X = G\epsilon_{FY} \circ (GF\eta_Y \circ \eta_Y) \circ f = G\epsilon_{FY} \circ \eta_{GFY} \circ \eta_Y \circ f \\ &= G\mathrm{Id}_{FY} \circ GFf \circ \eta_X = G(\mathrm{Id}_{FY} \circ Ff) \circ \eta_X = GFf \circ \eta_X = \eta_Y \circ f = \eta'_Y \circ f, \end{aligned}$$

so that $\eta' := (\eta_C)_{C\in\mathcal{C}} : \mathrm{Id}_{\mathcal{C}} \to GF'$ is indeed a seminatural transformation. Thus, from $G\epsilon'_D \circ \eta'_{GD} = G\epsilon_D \circ \eta_{GD} = G\mathrm{Id}_D$ and $\epsilon'_{F'C} \circ F'\eta'_C = \epsilon_{FC} \circ F'\eta_C = \epsilon_{FC} \circ F\eta_C \circ e_C = e_C \circ e_C = e_C = F\mathrm{Id}_C \circ e_C = F'\mathrm{Id}_C$, it follows that $(F', G, \eta', \epsilon')$ is a semiadjunction. By Proposition 4.26 *ii*), F' is unique up to natural semi-isomorphism.

ii). It is dual to *i*). ☐

As a particular case, in the functorial case we retrieve the following [5, Lemma 2.16].

Lemma 4.30. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors endowed with natural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$.*

i) If $G\epsilon \circ \eta G = \mathrm{Id}_G$, then there is a semifunctor $F' : \mathcal{C} \to \mathcal{D}$, that acts as F on objects, such that (F', G) is a semiadjunction.

ii) If $\epsilon F \circ F\eta = \mathrm{Id}_F$, then there is a semifunctor $G' : \mathcal{D} \to \mathcal{C}$, that acts as G on objects, such that (F, G') is a semiadjunction.

We observe that any functor whose completion has an adjoint is part of a semiadjunction.

Lemma 4.31. [5, Lemma 2.15] *The following assertions hold true.*

i) Any functor G whose completion has a left adjoint is part of a semiadjunction (F, G).

ii) Any functor F whose completion has a right adjoint is part of a semiadjunction (F, G).

Proof. i). Let $G : \mathcal{D} \to \mathcal{C}$ be a functor whose completion $G^\natural : \mathcal{D}^\natural \to \mathcal{C}^\natural$ has a left adjoint $L : \mathcal{C}^\natural \to \mathcal{D}^\natural$. From Lemma 4.12, there exists a semifunctor $F : \mathcal{C} \to \mathcal{D}$ such that $F^\natural \cong L$, hence $F^\natural \dashv G^\natural$. Thus, by Theorem 4.21 it follows that (F, G) is a semiadjunction.
ii). It is proved similarly. ☐

The notion of right (resp., left) semiadjoint is stable under composition.

Proposition 4.32. [21, Proposition 1.8]

i) Given two right semiadjoints $G : \mathcal{D} \to \mathcal{C}$ and $G' : \mathcal{E} \to \mathcal{D}$, then the composite semifunctor $G \circ G' : \mathcal{E} \to \mathcal{C}$ is a right semiadjoint.

ii) Given two left semiadjoints $F : \mathcal{C} \to \mathcal{D}$ and $F' : \mathcal{D} \to \mathcal{E}$, then the composite semifunctor $F' \circ F : \mathcal{C} \to \mathcal{E}$ is a left semiadjoint.

Proof. i). If $G : \mathcal{D} \to \mathcal{C}$ and $G' : \mathcal{E} \to \mathcal{D}$ are right semiadjoints, then by definition there exist a semifunctor $F : \mathcal{C} \to \mathcal{D}$ and seminatural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$, such that $G\epsilon \circ \eta G = G\mathrm{Id}$, and there exist a semifunctor $F' : \mathcal{D} \to \mathcal{E}$ and seminatural transformations $\eta' : \mathrm{Id}_{\mathcal{D}} \to G'F'$ and $\epsilon' : F'G' \to \mathrm{Id}_{\mathcal{E}}$, such that $G'\epsilon' \circ \eta' G' = G'\mathrm{Id}$, respectively. Set $\bar{\eta} := G\eta' F \circ \eta$ and $\bar{\epsilon} := \epsilon' \circ F'\epsilon G'$. We now show that the composite $G \circ G' : \mathcal{E} \to \mathcal{C}$ is a right semiadjoint through the semifunctor $F' \circ F : \mathcal{C} \to \mathcal{E}$ and the seminatural transformations $\bar{\eta} : \mathrm{Id}_{\mathcal{C}} \to GG'F'F$ and $\bar{\epsilon} : F'FGG' \to \mathrm{Id}_{\mathcal{E}}$. Indeed, we have

$$\begin{aligned} GG'\bar{\epsilon} \circ \bar{\eta} GG' &= GG'\epsilon' \circ GG'F'\epsilon G' \circ G\eta' FGG' \circ \eta GG' = G(G'\epsilon' \circ G'F'\epsilon G' \circ \eta' FGG') \circ \eta GG' \\ &= G(G'\epsilon' \circ \eta' G' \circ \epsilon G') \circ \eta GG' = G(G'\mathrm{Id} \circ \epsilon G') \circ \eta GG' = GG'\mathrm{Id} \circ G\epsilon G' \circ \eta GG' \\ &= GG'\mathrm{Id} \circ (G\epsilon \circ \eta G)G' = GG'\mathrm{Id} \circ G\mathrm{Id}_{G'} = G(G'\mathrm{Id} \circ \mathrm{Id}_{G'}) = GG'\mathrm{Id}. \end{aligned}$$

ii). At the same way, if $F : \mathcal{C} \to \mathcal{D}$ and $F' : \mathcal{D} \to \mathcal{E}$ are left semiadjoints, then by definition there exist a semifunctor $G : \mathcal{D} \to \mathcal{C}$ and seminatural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$, such that $\epsilon F \circ F\eta = F\mathrm{Id}$, and there exist a semifunctor $G' : \mathcal{E} \to \mathcal{D}$ and seminatural transformations $\eta' : \mathrm{Id}_{\mathcal{D}} \to G'F'$ and $\epsilon' : F'G' \to \mathrm{Id}_{\mathcal{E}}$, such that $\epsilon' F' \circ F'\eta' = F'\mathrm{Id}$, respectively. By setting again $\bar{\eta} := G\eta' F \circ \eta$ and $\bar{\epsilon} := \epsilon' \circ F'\epsilon G'$, it holds that $\bar{\epsilon} F'F \circ F'F\bar{\eta} = F'F\mathrm{Id}$, hence $F' \circ F : \mathcal{C} \to \mathcal{E}$ is a left semiadjoint. In fact, we have that

$$\begin{aligned} \bar{\epsilon} F'F \circ F'F\bar{\eta} &= \epsilon' F'F \circ F'\epsilon G'F'F \circ F'FG\eta' F \circ F'F\eta = (\epsilon' F' \circ F'\epsilon G'F' \circ F'FG\eta')F \circ F'F\eta \\ &= (\epsilon' F' \circ F'\eta' \circ F'\epsilon)F \circ F'F\eta = (F'\mathrm{Id} \circ F'\epsilon)F \circ F'F\eta = F'\epsilon F \circ F'F\eta = F'F\mathrm{Id}. \end{aligned}$$

□

Similarly to the case of adjunctions of functors (cf. [61, IV.8, Theorem 1]), as pointed out in [53, page 4], semiadjunctions remain stable under composition.

Corollary 4.33. [21, Corollary 1.9] *Given semiadjunctions $(F \dashv_{\mathrm{s}} G : \mathcal{D} \to \mathcal{C}, \eta, \epsilon)$ and $(F' \dashv_{\mathrm{s}} G' : \mathcal{E} \to \mathcal{D}, \eta', \epsilon')$, then also $(F'F \dashv_{\mathrm{s}} GG' : \mathcal{E} \to \mathcal{C}, G\eta' F \circ \eta, \epsilon' \circ F'\epsilon G')$ is a semiadjunction.*

Proof. By Remark 4.28 G and G' are right semiadjoints through F, η, ϵ and F', η', ϵ', respectively, and F, F' are left semiadjoints through G, η, ϵ and G', η', ϵ', respectively. Then, by the proof of Proposition 4.32 we know that $GG'\bar{\epsilon} \circ \bar{\eta} GG' = GG'\mathrm{Id}$ and $\bar{\epsilon} F'F \circ F'F\bar{\eta} = F'F\mathrm{Id}$, where $\bar{\eta} := G\eta' F \circ \eta : \mathrm{Id}_{\mathcal{C}} \to GG'F'F$ and $\bar{\epsilon} := \epsilon' \circ F'\epsilon G' : F'FGG' \to \mathrm{Id}_{\mathcal{E}}$, thus $F'F \dashv_{\mathrm{s}} GG'$ is a semiadjunction with unit $\bar{\eta}$ and counit $\bar{\epsilon}$. □

An idempotent (semi)natural transformation on the identity functor on a category allows to obtain a canonical semiadjunction of semifunctors.

Proposition 4.34. [21, Proposition 1.10] *Given a category $\mathcal{C}$, any idempotent (semi)natural transformation $e = (e_X)_{X \in \mathcal{C}} : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ defines an idempotent endosemifunctor $E^e : \mathcal{C} \to \mathcal{C}$, which is self-semiadjoint, i.e. $E^e \dashv_{\mathrm{s}} E^e$. Conversely, any semifunctor which is self-semiadjoint defines an idempotent seminatural transformation.*

Proof. Given the idempotent seminatural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$, consider the assignment

$$X \mapsto X, \quad [f : X \to Y] \mapsto f \circ e_X = e_Y \circ f,$$

for any object $X \in \mathcal{C}$ and for any morphism f in $\mathcal{C}$. It defines a semifunctor $E^e : \mathcal{C} \to \mathcal{C}$. In fact, given morphisms $f : X \to Y$, $g : Y \to Z$ in $\mathcal{C}$, we have that $E^e(g \circ f) = g \circ f \circ e_X =$

$g \circ (f \circ e_X) \circ e_X = g \circ e_Y \circ f \circ e_X = E^e(g) \circ E^e(f)$ but $E^e(\mathrm{Id}_X) = \mathrm{Id}_X \circ e_X = e_X$, which is not necessarily Id_X. Note that E^e is idempotent as for every $X \in \mathcal{C}$, $f : X \to Y$ in $\mathcal{C}$ one has $E^eE^e(X) = X = E^e(X)$ and $E^eE^e(f) = E^e(f \circ e_X) = f \circ e_X \circ e_X = f \circ e_X = E^e(f)$. We show that $E^e \dashv_s E^e$ is a semiadjunction with unit $\eta^e : \mathrm{Id}_\mathcal{C} \to E^eE^e$, $\eta^e_X = e_X$, and counit $\epsilon^e : E^eE^e \to \mathrm{Id}_\mathcal{C}$, $\epsilon^e_X = e_X$, for every $X \in \mathcal{C}$. Indeed, we have $E^e\epsilon^e_X \circ \eta^e_{E^eX} = E^e(e_X) \circ \eta^e_X = e_X \circ e_X \circ e_X = e_X = E^e\mathrm{Id}_X$, and $\epsilon^e_{E^eX} \circ E^e\eta^e_X = e_X \circ \eta^e_X \circ e_X = e_X \circ e_X \circ e_X = e_X = E^e\mathrm{Id}_X$. Conversely, if $E : \mathcal{C} \to \mathcal{C}$ is a self-semiadjoint semifunctor, then there exist seminatural transformations $\eta : \mathrm{Id}_\mathcal{C} \to EE$ and $\epsilon : EE \to \mathrm{Id}_\mathcal{C}$, such that $E\epsilon \circ \eta E = E\mathrm{Id}$ and $\epsilon E \circ E\eta = E\mathrm{Id}$. As in the proof of Proposition 4.29, set $e := \epsilon E \circ E\eta : E \to E$, which is an idempotent seminatural transformation. Indeed, it is natural as it is composition of natural transformations; for every $X \in \mathcal{C}$, we have $e_X \circ E(\mathrm{Id}_X) = \epsilon_{EX} \circ E\eta_X \circ E(\mathrm{Id}_X) = \epsilon_{EX} \circ E(\eta_X \circ \mathrm{Id}_X) = \epsilon_{EX} \circ E\eta_X = e_X$ and $e \circ e = E\mathrm{Id} \circ E\mathrm{Id} = E\mathrm{Id} = e$. □

Definition 4.35. [21, Definition 1.11] We call the semifunctor E^e given as in Proposition 4.34 the **canonical semifunctor** attached to an idempotent seminatural transformation $e = (e_X)_{X\in\mathcal{C}} : \mathrm{Id}_\mathcal{C} \to \mathrm{Id}_\mathcal{C}$ on a category $\mathcal{C}$.

Example 4.36. *i)* Given a category $\mathcal{C}$, if $e = \mathrm{Id}_{\mathrm{Id}_\mathcal{C}} : \mathrm{Id}_\mathcal{C} \to \mathrm{Id}_\mathcal{C}$, then $E^e : \mathcal{C} \to \mathcal{C}$ is the identity functor on $\mathcal{C}$.

ii) We recall from [75, Section 1.7] that a category $\mathcal{C}$ is called a *category with zero morphisms* if there is a family $\{0_{XY} \in \mathrm{Hom}_\mathcal{C}(X, Y), \text{ for all } X, Y \in \mathcal{C}\}$ such that $f \circ 0_{XY} = 0_{XZ}$ and $0_{YZ} \circ g = 0_{XZ}$, for every $f : Y \to Z$, $g : X \to Y$ in $\mathcal{C}$. In particular, a category with a *zero object* (i.e., an object which is both initial and terminal) is a category with zero morphisms. Let $\mathcal{C}$ be a category with zero morphisms. Then, $e := (0_{XX})_{X\in\mathcal{C}}$, where 0_{XX} is the zero morphism on X, is an idempotent natural transformation. Indeed, e_X is clearly idempotent and for every $f : X \to Y$ in $\mathcal{C}$ we have $f \circ e_X = f \circ 0_{XX} = 0_{XY} = 0_{YY} \circ f = e_Y \circ f$. In this case, the semifunctor $E^e : \mathcal{C} \to \mathcal{C}$ is given by $X \mapsto X$, $f \mapsto 0_{XY}$, for every $X \in \mathcal{C}$ and $f : X \to Y$ in $\mathcal{C}$.

As a consequence of Corollary 4.33 and Proposition 4.34, given a (semi)adjunction of (semi)functors and an idempotent seminatural transformation, we can obtain another semiadjunction of semifunctors as follows, cf. [21, Corollary 1.13].

Corollary 4.37. *Let $F \dashv_s G : \mathcal{D} \to \mathcal{C}$ be a semiadjunction with unit η and counit ϵ, and let $e : \mathrm{Id}_\mathcal{C} \to \mathrm{Id}_\mathcal{C}$, $e' : \mathrm{Id}_\mathcal{D} \to \mathrm{Id}_\mathcal{D}$ be idempotent seminatural transformations on the categories $\mathcal{C}$, $\mathcal{D}$, respectively.*

i) Consider the canonical semifunctor $E^e : \mathcal{C} \to \mathcal{C}$. Then, $F' := FE^e : \mathcal{C} \to \mathcal{D}$ and $G' := E^eG : \mathcal{D} \to \mathcal{C}$ form a semiadjunction $F' \dashv_s G'$.

ii) Consider the canonical semifunctor $E^{e'} : \mathcal{D} \to \mathcal{D}$. Then, $F'' := E^{e'}F : \mathcal{C} \to \mathcal{D}$ and $G'' := GE^{e'} : \mathcal{D} \to \mathcal{C}$ form a semiadjunction $F'' \dashv_s G''$.

In Subsection 4.7.4 we will consider an instance of a semiadjunction as in the previous corollary, constructed out of a morphism of rings. Let us make some further observations about the semifunctor E^e.

Example 4.38. Given a category $\mathcal{C}$ and an idempotent natural transformation $e : \mathrm{Id}_\mathcal{C} \to \mathrm{Id}_\mathcal{C}$, in Subsection 2.1.2 we have seen that there is a canonical functor $H : \mathcal{C} \to \mathcal{C}_e$ into the coidentifier category $\mathcal{C}_e$, acting as the identity on objects and as the canonical projection on morphisms. Consider the semifunctor $L : \mathcal{C}_e \to \mathcal{C}$ defined by

$$L(X) = X, \quad L(\bar{f}) = e_Y \circ f : X \to Y,$$

for every $X \in \mathcal{C}_e$, $\bar{f} : X \to Y$ in $\mathcal{C}_e$. Note that it is really a semifunctor as $L\overline{\mathrm{Id}}_X = e_X \circ \mathrm{Id}_X = e_X \neq \mathrm{Id}_{LX} = \mathrm{Id}_X$ and it is well defined as $\bar{f} = \bar{g}$ if, and only if, $e_Y \circ f = e_Y \circ g$. We observe that $E^e = L \circ H$. Indeed, $LH(X) = X$ and $LH(f) = L(\bar{f}) = f \circ e_X$, for every $X \in \mathcal{C}$, $f : X \to Y$ in $\mathcal{C}$. In [5, Theorem 3.1] it is shown that L, H form a semiadjunction $L \dashv_s H$ with unit $\eta : \mathrm{Id}_{\mathcal{C}_e} \to HL$, $\eta_X = \overline{\mathrm{Id}}_X : X \to HLX = X$, for every $X \in \mathcal{C}_e$, and counit $\epsilon : LH \to \mathrm{Id}_{\mathcal{C}}$, $\epsilon_Y := e_Y : LHY = Y \to Y$, for every $Y \in \mathcal{C}$. Explicitly, for every $\bar{f} : X \to Y$ in $\mathcal{C}_e$, we have $HL\bar{f} \circ \eta_X = H(e_Y \circ f) \circ \overline{\mathrm{Id}}_X = He_Y \circ Hf \circ H\mathrm{Id}_X = \mathrm{Id}_{HY} \circ Hf \circ \mathrm{Id}_{HX} = \overline{\mathrm{Id}}_Y \circ \bar{f} = \eta_Y \circ \bar{f}$, thus η is a seminatural transformation. The same holds for ϵ, as $\epsilon_Y \circ LHf = e_Y \circ L\bar{f} = e_Y \circ e_Y \circ f = e_Y \circ f = f \circ e_X = f \circ \epsilon_X$. Moreover, for every $X \in \mathcal{C}$ and $Y \in \mathcal{C}_e$, we have the identities $\epsilon_{LX} \circ L\eta_X = e_{LX} \circ L\overline{\mathrm{Id}}_X = e_X \circ L\overline{\mathrm{Id}}_X = e_X \circ e_X \circ \mathrm{Id}_X = e_X \circ \mathrm{Id}_X = L\overline{\mathrm{Id}}_X$ and $H\epsilon_Y \circ \eta_{HY} = He_Y \circ \mathrm{Id}_{HY} = He_Y = \mathrm{Id}_{HY} = H\mathrm{Id}_Y$. So $L \dashv_s H$ is a semiadjunction with unit η and counit ϵ.

Now we show that $H \dashv_s L : \mathcal{C}_e \to \mathcal{C}$ is a semiadjunction as well. In fact, consider the seminatural transformations $\eta' : \mathrm{Id}_{\mathcal{C}} \to LH$, $\epsilon' : HL \to \mathrm{Id}_{\mathcal{C}_e}$, given by $\eta'_X := e_X$, for every $X \in \mathcal{C}$, and $\epsilon'_Y := \overline{\mathrm{Id}}_Y$, for every $Y \in \mathcal{C}_e$. For every $\bar{f} : Y \to Y'$ in $\mathcal{C}_e$, we have $\epsilon'_{Y'} \circ HL\bar{f} = \overline{\mathrm{Id}}_{Y'} \circ H(e_{Y'} \circ f) = H\mathrm{Id}_{Y'} \circ He_{Y'} \circ Hf = H\mathrm{Id}_{Y'} \circ H\mathrm{Id}_{Y'} \circ Hf = Hf = Hf \circ H\mathrm{Id}_Y = \bar{f} \circ \overline{\mathrm{Id}}_Y$, and for every $f : X \to X'$ in $\mathcal{C}$, $LHf \circ \eta'_X = L\bar{f} \circ e_X = e_{X'} \circ f \circ e_X = e_{X'} \circ e_{X'} \circ f = e_{X'} \circ f = \eta_{X'} \circ f$, hence ϵ' and η' are natural. Moreover, for every $X \in \mathcal{C}$ and $Y \in \mathcal{C}_e$, we have $\epsilon'_{HX} \circ H\eta'_X = \mathrm{Id}_{HX} \circ He_X = He_X = \mathrm{Id}_{HX} = H\mathrm{Id}_X$, and $L\epsilon'_Y \circ \eta'_{LY} = L\overline{\mathrm{Id}}_Y \circ e_{LY} = e_Y \circ \mathrm{Id}_Y \circ e_Y = e_Y = L\overline{\mathrm{Id}}_Y$, hence $H \dashv_s L : \mathcal{C}_e \to \mathcal{C}$ is a semiadjunction with unit η' and counit ϵ'.

The previous example leads us to define the following notions for semifunctors. We say that $F \dashv_s G \dashv_s H : \mathcal{C} \to \mathcal{D}$ is a *semiadjoint triple* if it is a triple of semifunctors $F, H : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $F \dashv_s G$ and $G \dashv_s H$ are semiadjunctions. We call a semifunctor $F : \mathcal{C} \to \mathcal{D}$ *Frobenius* if there exists a semifunctor $G : \mathcal{D} \to \mathcal{C}$ such that both $F \dashv_s G$ and $G \dashv_s F$ are semiadjunctions. Thus, a Frobenius semifunctor $F : \mathcal{C} \to \mathcal{D}$ fits into a semiadjoint triple of the form $G \dashv_s F \dashv_s G : \mathcal{D} \to \mathcal{C}$. Note that if F is a Frobenius semifunctor, also G is a Frobenius semifunctor. Thus, $L : \mathcal{C}_e \to \mathcal{C}$ and $H : \mathcal{C} \to \mathcal{C}_e$ in Example 4.38 are Frobenius semifunctors.

Remark 4.39. Any Frobenius functor is a Frobenius semifunctor. The converse does not hold in general. For instance, the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$ is a Frobenius semifunctor, but in general it is not a Frobenius functor (as we will see in Remark 5.15). By Proposition 2.69 it is when e.g. e splits.

4.2 Semisplitting properties for morphisms

In this section we study semisplitting properties for morphisms whose source or target is the image of a semifunctor. For semifunctors $F : \mathcal{C} \to \mathcal{D}$, $F' : \mathcal{C}' \to \mathcal{D}$ and objects $C \in \mathcal{C}$, $C' \in \mathcal{C}'$, we recall from [21, Section 2] the notions of F_C-semisplit-mono, F_C-semisplit-epi, $(F_C, F'_{C'})$-semisplit-mono, $(F_C, F'_{C'})$-semisplit-epi, $(F_C, F'_{C'})$-semi-isomorphism.

Definition 4.40. Given a semifunctor $F : \mathcal{C} \to \mathcal{D}$, we say that

- a morphism $f : FC \to D$ in $\mathcal{D}$ is an F_C**-semisplit-mono** if there exists a morphism $g : D \to FC$ in $\mathcal{D}$ such that $g \circ f = F\mathrm{Id}_C$;

- a morphism $f : D \to FC$ in $\mathcal{D}$ is an F_C**-semisplit-epi** if there exists a morphism $g : FC \to D$ in $\mathcal{D}$ such that $f \circ g = F\mathrm{Id}_C$.

Remark 4.41. *i*) For every object C in $\mathcal{C}$ we have that $F\mathrm{Id}_C \circ F\mathrm{Id}_C = F\mathrm{Id}_C$, so $F\mathrm{Id}_C : FC \to FC$ is both an F_C-semisplit-mono and an F_C-semisplit-epi.

ii) If $f : FC \to D$ is an F_C-semisplit-mono, then the morphism $g : D \to FC$ in $\mathcal{D}$ such that $g \circ f = F\mathrm{Id}_C$ is an F_C-semisplit-epi. On the other hand, if $f : D \to FC$ is an F_C-semisplit-epi through $g : FC \to D$, then g is an F_C-semisplit-mono. Note that in case F is a functor, then $f : FC \to D$ is an F_C-semisplit-mono if, and only if, it is a split-mono, i.e. there exists a morphism $g : D \to FC$ in $\mathcal{D}$ such that $g \circ f = \mathrm{Id}_{FC}$; analogously, in case F is a functor, f is an F_C-semisplit-epi if, and only if, it is a split-epi, i.e. there exists a morphism $g : FC \to D$ in $\mathcal{D}$ such that $f \circ g = \mathrm{Id}_{FC}$.

Proposition 4.42. [21, Proposition 2.5] *Let $F : \mathcal{C} \to \mathcal{D}$, $F' : \mathcal{C}' \to \mathcal{D}$ be semifunctors.*

i) *If $f : FC \to D$ is an F_C-semisplit-mono and $f' : F'C' \to FC$ is an $F'_{C'}$-semisplit-mono such that $F\mathrm{Id}_C \circ f' = f'$, then the composite $f \circ f' : F'C' \to D$ is an $F'_{C'}$-semisplit-mono.*

ii) *Let $f : FC \to D$, $g : D \to D'$ be morphisms in $\mathcal{D}$. If the composite $g \circ f$ is an F_C-semisplit-mono, then f is an F_C-semisplit-mono.*

iii) *If $f : D \to FC$ is an F_C-semisplit-epi and $f' : FC \to F'C'$ is an $F'_{C'}$-semisplit-epi such that $f' \circ F\mathrm{Id}_C = f'$, then the composite $f' \circ f : D \to F'C'$ is an $F'_{C'}$-semisplit-epi.*

iv) *Let $f : D \to FC$, $g : D' \to D$ be morphisms in $\mathcal{D}$. If the composite $f \circ g$ is an F_C-semisplit-epi, then f is an F_C-semisplit-epi.*

Proof. We prove only *i*) and *ii*), as *iii*) and *iv*) follow similarly.
i). If $f : FC \to D$ is an F_C-semisplit-mono, then there exists a morphism $g : D \to FC$ in $\mathcal{D}$ such that $g \circ f = F\mathrm{Id}_C$. Assume that $f' : F'C' \to FC$ is an $F'_{C'}$-semisplit-mono, i.e. there exists a morphism $g' : FC \to F'C'$ in $\mathcal{D}$ such that $g' \circ f' = F'\mathrm{Id}_{C'}$, and assume that $F\mathrm{Id}_C \circ f' = f'$. Consider the composite $g' \circ g : D \to F'C'$. We have $g' \circ g \circ f \circ f' = g' \circ F\mathrm{Id}_C \circ f' = g' \circ f' = F'\mathrm{Id}_{C'}$, thus $f \circ f'$ is an $F'_{C'}$-semisplit-mono.
ii). If the composite $g \circ f : FC \to D'$ is an F_C-semisplit-mono, then there exists a morphism $h : D' \to FC$ in $\mathcal{D}$ such that $h \circ g \circ f = F\mathrm{Id}_C$, thus f is an F_C-semisplit-mono. □

A stronger notion of semisplit-mono (resp., semisplit-epi) can be defined as follows.

Definition 4.43. [21, Definition 2.6] Given semifunctors $F : \mathcal{C} \to \mathcal{D}$, $F' : \mathcal{C}' \to \mathcal{D}$, we say that a morphism $f : FC \to F'C'$ in $\mathcal{D}$ is an

- $(F_C, F'_{C'})$**-semisplit-mono** if $f \circ F\mathrm{Id}_C = f$ and there exists a morphism $g : F'C' \to FC$ in $\mathcal{D}$ such that
$$g \circ f = F\mathrm{Id}_C \quad \text{and} \quad g \circ F'\mathrm{Id}_{C'} = g;$$

- $(F_C, F'_{C'})$**-semisplit-epi** if $F'\mathrm{Id}_{C'} \circ f = f$ and there exists a morphism $g : F'C' \to FC$ in $\mathcal{D}$ such that
$$f \circ g = F'\mathrm{Id}_{C'} \quad \text{and} \quad F\mathrm{Id}_C \circ g = g.$$

Remark 4.44. It is clear that:

i) Any $(F_C, F'_{C'})$-semisplit-mono is an F_C-semisplit-mono and any $(F_C, F'_{C'})$-semisplit-epi is an $F'_{C'}$-semisplit-epi.

ii) Given a seminatural transformation $\alpha : F \to F'$ of semifunctors, if α is a natural semisplit-mono (resp., natural semisplit-epi), then every component morphism $\alpha_C : FC \to F'C$ is an (F_C, F'_C)-semisplit-mono (resp., (F_C, F'_C)-semisplit-epi).

The following properties hold true.

Proposition 4.45. [21, Proposition 2.8] *Let $F : \mathcal{C} \to \mathcal{D}$, $F' : \mathcal{C}' \to \mathcal{D}$, $F'' : \mathcal{C}'' \to \mathcal{D}$ be semifunctors.*

i) *For every object C in $\mathcal{C}$, the morphism $F\mathrm{Id}_C : FC \to FC$ is both an (F_C, F_C)-semisplit-mono and an (F_C, F_C)-semisplit-epi.*

ii) *Given an $(F_C, F'_{C'})$-semisplit-mono $f : FC \to F'C'$ and an $(F'_{C'}, F''_{C''})$-semisplit-mono $f' : F'C' \to F''C''$, then the composite $f' \circ f : FC \to F''C''$ is an $(F_C, F''_{C''})$-semisplit-mono.*

iii) *Given an $(F_C, F'_{C'})$-semisplit-epi $f : FC \to F'C'$ and an $(F'_{C'}, F''_{C''})$-semisplit-epi $f' : F'C' \to F''C''$, then the composite $f' \circ f : FC \to F''C''$ is an $(F_C, F''_{C''})$-semisplit-epi.*

Proof. *i*). It is clear.
ii). If $f : FC \to F'C'$ is an $(F_C, F'_{C'})$-semisplit-mono, then $f \circ F\mathrm{Id}_C = f$ and there exists a morphism $g : F'C' \to FC$ in $\mathcal{D}$ such that $g \circ f = F\mathrm{Id}_C$ and $g \circ F'\mathrm{Id}_{C'} = g$. If $f' : F'C' \to F''C''$ is an $(F'_{C'}, F''_{C''})$-semisplit-mono, then $f' \circ F'\mathrm{Id}_{C'} = f'$ and there exists a morphism $g' : F''C'' \to F'C'$ in $\mathcal{D}$ such that $g' \circ f' = F'\mathrm{Id}_{C'}$ and $g' \circ F''\mathrm{Id}_{C''} = g'$. Consider the composite $g \circ g' : F''C'' \to FC$. We have $g \circ g' \circ f' \circ f = g \circ F'\mathrm{Id}_{C'} \circ f = g \circ f = F\mathrm{Id}_C$. Moreover, $f' \circ f \circ F\mathrm{Id}_C = f' \circ f$ and $g \circ g' \circ F''\mathrm{Id}_{C''} = g \circ g'$, thus $f' \circ f$ is an $(F_C, F''_{C''})$-semisplit-mono.
iii). It is dual to *ii*). ☐

As in [21, Definition 2.1] one can define the notions of F_C-semi-monomorphism and F_C-semi-epimorphism. We refer to [21] for further results involving these notions. We now define an (F_C, F'_C)-semi-isomorphism in $\mathcal{D}$, see [21, Definition 2.9].

Definition 4.46. [21, Definition 2.9] Given semifunctors $F : \mathcal{C} \to \mathcal{D}$, $F' : \mathcal{C}' \to \mathcal{D}$, we say that a morphism $f : FC \to F'C'$ in $\mathcal{D}$ is an $(F_C, F'_{C'})$**-semi-isomorphism** if $f \circ F\mathrm{Id}_C = f$ and there exists a morphism $g : F'C' \to FC$ in $\mathcal{D}$ such that

i) $g \circ f = F\mathrm{Id}_C$;

ii) $f \circ g = F'\mathrm{Id}_{C'}$.

We call a morphism $g : F'C' \to FC$ in $\mathcal{D}$ which satisfies *i*) and *ii*) the $(F_C, F'_{C'})$**-semi-inverse** of f if $F\mathrm{Id}_C \circ g = g$ holds true in addition.

In case both F and F' are functors, then $f : FC \to F'C'$ is an $(F_C, F'_{C'})$-semi-isomorphism if, and only if, it is an isomorphism in $\mathcal{D}$.

We have the following properties.

Lemma 4.47. [21, Lemma 2.10] *Let $F : \mathcal{C} \to \mathcal{D}$, $F' : \mathcal{C}' \to \mathcal{D}$ be semifunctors and let $f : FC \to F'C'$ be an $(F_C, F'_{C''})$-semi-isomorphism in $\mathcal{D}$. Then, $f = f \circ F\mathrm{Id}_C$ is equivalent to $F'\mathrm{Id}_{C'} \circ f = f$. Moreover, if a morphism $g : F'C' \to FC$ in $\mathcal{D}$ satisfies i) and ii) as in Definition 4.46, then $F\mathrm{Id}_C \circ g = g$ is equivalent to $g \circ F'\mathrm{Id}_{C'} = g$.*

Proof. Let $f : FC \to F'C'$ be an $(F_C, F'_{C'})$-semi-isomorphism. Then, $f \circ F\mathrm{Id}_C = f \circ g \circ f = F'\mathrm{Id}_{C'} \circ f$, so $f = f \circ F\mathrm{Id}_C$ is equivalent to $f = F'\mathrm{Id}_{C'} \circ f$. Analogously, by interchanging the roles of f and g, $F\mathrm{Id}_C \circ g = g$ is equivalent to $g \circ F'\mathrm{Id}_{C'} = g$. □

Lemma 4.48. [21, Lemma 2.11] *Let $F : \mathcal{C} \to \mathcal{D}$, $F' : \mathcal{C}' \to \mathcal{D}$ be semifunctors and let $f : FC \to F'C'$ be an $(F_C, F'_{C'})$-semi-isomorphism in $\mathcal{D}$. Then, f admits a unique $(F_C, F'_{C'})$-semi-inverse.*

Proof. If $f : FC \to F'C'$ is an $(F_C, F'_{C'})$-semi-isomorphism, then $f \circ F\mathrm{Id}_C = f$ (which is equivalent to $F'\mathrm{Id}_{C'} \circ f = f$ by Lemma 4.47) and there exists a morphism $g : F'C' \to FC$ in $\mathcal{D}$ such that $g \circ f = F\mathrm{Id}_C$ and $f \circ g = F'\mathrm{Id}_{C'}$. Consider the composite $g' := g \circ f \circ g : F'C' \to FC$. Then, $g' \circ f = g \circ f \circ g \circ f = F\mathrm{Id}_C \circ F\mathrm{Id}_C = F\mathrm{Id}_C$ and $f \circ g' = f \circ g \circ f \circ g = F'\mathrm{Id}_{C'} \circ F'\mathrm{Id}_{C'} = F'\mathrm{Id}_{C'}$. Moreover, we have that $F\mathrm{Id}_C \circ g' = F\mathrm{Id}_C \circ g \circ f \circ g = g \circ f \circ g \circ f \circ g = g \circ F'\mathrm{Id}_{C'} \circ f \circ g = g \circ f \circ g = g'$, so g' is an $(F_C, F'_{C'})$-semi-inverse of f. Assume that there exists another $(F_C, F'_{C'})$-semi-inverse $h : F'C' \to FC$ in $\mathcal{D}$ that satisfies conditions $i)$, $ii)$ in Definition 4.46 and such that $h = F\mathrm{Id}_C \circ h$. Then, we have $h = F\mathrm{Id}_C \circ h = g' \circ f \circ h = g' \circ F'\mathrm{Id}_{C'} = g'$, thus the $(F_C, F'_{C'})$-semi-inverse of f is unique. □

We show the following characterization for a natural semi-isomorphism.

Lemma 4.49. *Let F, F' be semifunctors. Then, α is a natural semi-isomorphism if, and only if, $\alpha : F \to F'$ is a natural transformation between semifunctors and α_C is an (F_C, F'_C)-semi-isomorphism for every $C \in \mathcal{C}$.*

Proof. If $\alpha : F \to F'$ is a natural semi-isomorphism, then every component morphism $\alpha_C : FC \to F'C$ is an (F_C, F'_C)-semi-isomorphism in $\mathcal{D}$. On the other hand, if α_C is a (F_C, F'_C)-semi-isomorphism for every $C \in \mathcal{C}$, then by Lemma 4.48 α_C admits a unique (F_C, F'_C)-semi-inverse β_C. Thus, $F\mathrm{Id}_C \circ \beta_C = \beta_C$, $\beta_C \circ \alpha_C = F\mathrm{Id}_C$ and $\alpha_C \circ \beta_C = F'\mathrm{Id}_C$. By naturality of α, for every $f : C \to C'$ in $\mathcal{C}$ we have $Ff \circ \beta_C = F\mathrm{Id}_{C'} \circ Ff \circ \beta_C = \beta_{C'} \circ \alpha_{C'} \circ Ff \circ \beta_C = \beta_{C'} \circ F'f \circ \alpha_C \circ \beta_C = \beta_{C'} \circ F'f \circ F'\mathrm{Id}_C = \beta_{C'} \circ F'f$, hence $(\beta_C)_{C \in \mathcal{C}}$ is a seminatural transformation and it is the semi-inverse of α. □

Proposition 4.50. [21, Proposition 2.12] *Let $F : \mathcal{C} \to \mathcal{D}$, $F' : \mathcal{C}' \to \mathcal{D}$ be semifunctors. A morphism $f : FC \to F'C'$ in $\mathcal{D}$ is an $(F_C, F'_{C'})$-semi-isomorphism if, and only if, f is both an $(F_C, F'_{C'})$-semisplit-mono and an $(F_C, F'_{C'})$-semisplit-epi.*

Proof. If $f : FC \to F'C'$ is an $(F_C, F'_{C'})$-semi-isomorphism in $\mathcal{D}$ with $(F_C, F'_{C'})$-semi-inverse g', then it is trivially an $(F_C, F'_{C'})$-semisplit-mono and an $(F_C, F'_{C'})$-semisplit-epi. On the other hand, if f is an $(F_C, F'_{C'})$-semisplit-mono, then $f \circ F\mathrm{Id}_C = f$ and there exists a morphism $g : F'C' \to FC$ in $\mathcal{D}$ such that $g \circ f = F\mathrm{Id}_C$ and $g \circ F'\mathrm{Id}_{C'} = g$. If f is an $(F_C, F'_{C'})$-semisplit-epi, then $F'\mathrm{Id}_{C'} \circ f = f$ and there exists a morphism $g' : F'C' \to FC$ in $\mathcal{D}$ such that $f \circ g' = F'\mathrm{Id}_{C'}$ and $F\mathrm{Id}_C \circ g' = g'$. Since $g = g \circ F'\mathrm{Id}_{C'} = g \circ f \circ g' = F\mathrm{Id}_C \circ g' = g'$, we have that f is an $(F_C, F'_{C'})$-semi-isomorphism in $\mathcal{D}$ with $(F_C, F'_{C'})$-semi-inverse $g = g'$. □

Remark 4.51. From Proposition 4.45 $i)$ and Proposition 4.50 it is clear that $F\mathrm{Id}_C : FC \to FC$ is an (F_C, F_C)-semi-isomorphism, for every object C in $\mathcal{C}$.

Lemma 4.52. *Let $F : \mathcal{C} \to \mathcal{D}$, $F' : \mathcal{C}' \to \mathcal{D}$ be semifunctors.*

i) If $f : FC \to F'C'$ is a monomorphism and $f \circ F\mathrm{Id}_C = f$, then $F\mathrm{Id}_C = \mathrm{Id}_{FC}$.

ii) If $f : FC \to F'C'$ is an epimorphism and $F'\mathrm{Id}_{C'} \circ f = f$, then $F'\mathrm{Id}_{C'} = \mathrm{Id}_{F'C'}$.

Proof. *i*). If f is a monomorphism, from $f \circ F\mathrm{Id}_C = f = f \circ \mathrm{Id}_{FC}$ we have that $F\mathrm{Id}_C = \mathrm{Id}_{FC}$.
ii). If f is an epimorphism, from $F'\mathrm{Id}_{C'} \circ f = f = \mathrm{Id}_{F'C'} \circ f$ we have that $F'\mathrm{Id}_{C'} = \mathrm{Id}_{F'C'}$. □

Corollary 4.53. *Let $F : \mathcal{C} \to \mathcal{D}$, $F' : \mathcal{C}' \to \mathcal{D}$ be semifunctors.*

i) If $f : FC \to F'C'$ is both a $(F_C, F'_{C'})$-semisplit-mono and a monomorphism, then f is a split-mono.

ii) If $f : FC \to F'C'$ is both a $(F_C, F'_{C'})$-semisplit-epi and an epimorphism, then f is a split-epi.

iii) If $f : FC \to F'C'$ is a $(F_C, F'_{C'})$-semi-isomorphism and both a monomorphism and an epimorphism, then f is an isomorphism.

Proof. *i*). If $f : FC \to F'C'$ is a $(F_C, F'_{C'})$-semisplit-mono, then $f \circ F\mathrm{Id}_C = f$ and there exists a morphism $g : F'C' \to FC$ in $\mathcal{D}$ such that $g \circ f = F\mathrm{Id}_C$ and $g \circ F'\mathrm{Id}_{C'} = g$; if f is a monomorphism, then by Lemma 4.52 *i*) we have $g \circ f = F\mathrm{Id}_C = \mathrm{Id}_{FC}$.
ii). If $f : FC \to F'C'$ is a $(F_C, F'_{C'})$-semisplit-epi, then $F'\mathrm{Id}_{C'} \circ f = f$ and there exists a morphism $g : F'C' \to FC$ in $\mathcal{D}$ such that $f \circ g = F'\mathrm{Id}_{C'}$ and $F\mathrm{Id}_C \circ g = g$; if f is an epimorphism by Lemma 4.52 *ii*) we have $f \circ g = F'\mathrm{Id}_{C'} = \mathrm{Id}_{F'C'}$.
iii). It follows from Proposition 4.50 and *i*) + *ii*). □

Proposition 4.54. (Cf. [21, Proposition 2.13 (3),(4)]) *Let $F : \mathcal{C} \to \mathcal{D}$, $F' : \mathcal{C}' \to \mathcal{D}$ be semifunctors.*

i) If $f : FC \to F'C'$ is both an $(F_C, F'_{C'})$-semisplit-epi and a monomorphism in $\mathcal{D}$, then f is an $(F_C, F'_{C'})$-semi-isomorphism.

ii) If $f : FC \to F'C'$ is both an $(F_C, F'_{C'})$-semisplit-mono and an epimorphism in $\mathcal{D}$, then f is an $(F_C, F'_{C'})$-semi-isomorphism.

Proof. *i*). If $f : FC \to F'C'$ is an $(F_C, F'_{C'})$-semisplit-epi, then $F'\mathrm{Id}_{C'} \circ f = f$ and there exists a morphism $g : F'C' \to FC$ in $\mathcal{D}$ such that $f \circ g = F'\mathrm{Id}_{C'}$ and $F\mathrm{Id}_C \circ g = g$. Thus, we have $f \circ g \circ f = F'\mathrm{Id}_{C'} \circ f = f$. If f is a monomorphism, we get $g \circ f = \mathrm{Id}_{FC}$, hence $g \circ f = F\mathrm{Id}_C \circ g \circ f = F\mathrm{Id}_C \circ \mathrm{Id}_{FC} = F\mathrm{Id}_C$, so f is an $(F_C, F'_{C'})$-semi-isomorphism.
ii). If $f : FC \to F'C'$ is an $(F_C, F'_{C'})$-semisplit-mono, then $f \circ F\mathrm{Id}_C = f$ and there exists a morphism $g : F'C' \to FC$ in $\mathcal{D}$ such that $g \circ f = F\mathrm{Id}_C$ and $g \circ F'\mathrm{Id}_{C'} = g$. Thus, we have $f \circ g \circ f = f \circ F\mathrm{Id}_C = f$. If f is an epimorphism, we get $f \circ g = \mathrm{Id}_{F'C'}$, hence $f \circ g = f \circ g \circ F'\mathrm{Id}_{C'} = \mathrm{Id}_{F'C'} \circ F'\mathrm{Id}_{C'} = F'\mathrm{Id}_{C'}$, so f is an $(F_C, F'_{C'})$-semi-isomorphism. □

Moreover, we have the following.

Lemma 4.55. [21, Lemma 2.14] *Let $F : \mathcal{C} \to \mathcal{D}$, $F' : \mathcal{C}' \to \mathcal{D}$ be semifunctors. Any semifunctor $H : \mathcal{D} \to \mathcal{E}$ preserves $(F_C, F'_{C'})$-semisplit-monos, $(F_C, F'_{C'})$-semisplit-epis, $(F_C, F'_{C'})$-semi-isomorphisms.*

Proof. Let $f : FC \to F'C'$ be an $(F_C, F'_{C'})$-semisplit-mono in $\mathcal{D}$. Then, $f \circ F\mathrm{Id}_C = f$ and there exists a morphism $g : F'C' \to FC$ in $\mathcal{D}$ such that $g \circ f = F\mathrm{Id}_C$ and $g \circ F'\mathrm{Id}_{C'} = g$. We have that $Hf \circ HF\mathrm{Id}_C = H(f \circ F\mathrm{Id}_C) = Hf$, $Hg \circ Hf = H(g \circ f) = HF\mathrm{Id}_C$ and $Hg \circ HF'\mathrm{Id}_{C'} = H(g \circ F'\mathrm{Id}_{C'}) = Hg$, thus Hf is an $(HF_C, HF'_{C'})$-semisplit-mono. If $f : FC \to F'C'$ is an $(F_C, F'_{C'})$-semisplit-epi in $\mathcal{D}$, then $F'\mathrm{Id}_{C'} \circ f = f$ and there exists a morphism $g : F'C' \to FC$ in $\mathcal{D}$ such that $f \circ g = F'\mathrm{Id}_{C'}$ and $F\mathrm{Id}_C \circ g = g$. We have that

$HF'\mathrm{Id}_{C'} \circ Hf = Hf$, $Hf \circ Hg = H(f \circ g) = HF'\mathrm{Id}_{C'}$ and $HF\mathrm{Id}_C \circ Hg = Hg$, thus Hf is an $(HF_C, HF'_{C'})$-semisplit-epi. If $f : FC \to F'C'$ is an $(F_C, F'_{C'})$-semi-isomorphism in $\mathcal{D}$, then as in the previous cases Hf is an $(HF_C, HF'_{C'})$-semi-isomorphism. □

Example 4.56. [21, Example 2.16] (See also [52, Section 2.4]) Let $F : \mathcal{C} \to \mathsf{Set}$ be a semifunctor and let $\overline{FX} = \{x \in F(X) | F(\mathrm{Id}_X)(x) = x\}$ denote the subset of FX of fixpoints of $F(\mathrm{Id}_X)$. For any morphism $f : X \to Y$ in $\mathcal{C}$, if $x \in \overline{FX}$, then $F(f)(x) \in \overline{FY}$, hence the function $F(f) : FX \to FY$ restricts to a function $\overline{F(f)} : \overline{FX} \to \overline{FY}$. In fact, $F(\mathrm{Id}_Y)(F(f)(x)) = F(\mathrm{Id}_Y \circ f)(x) = F(f)(x)$, for every $x \in X$. Thus, we have a functor $\overline{F} : \mathcal{C} \to \mathsf{Set}$, $X \mapsto \overline{FX}$, $f \mapsto \overline{F(f)}$, which is naturally semi-isomorphic to F. Indeed, let $\alpha : F \to \overline{F}$, $\alpha = (\alpha_X : FX \to \overline{F}X)_{X \in \mathcal{C}}$, be given by $\alpha_X(p) = F(\mathrm{Id}_X)(p)$, for every $p \in FX$. Note that $F(\mathrm{Id}_X)(p) \in \overline{FX}$ as $F(\mathrm{Id}_X)(F(\mathrm{Id}_X)(p)) = F(\mathrm{Id}_X \circ \mathrm{Id}_X)(p) = F(\mathrm{Id}_X)(p)$. We have that, for any morphism $f : X \to Y$ in $\mathcal{C}$ and any $p \in FX$, $(\alpha_Y \circ Ff)(p) = (F\mathrm{Id}_Y \circ Ff)(p) = Ff(p) = (Ff \circ F\mathrm{Id}_X)(p) = (\overline{F}f \circ \alpha_X)(p)$, thus α is a natural transformation. Since $\overline{F}$ is a functor, we have that α is seminatural. Let $\beta : \overline{F} \to F$, $(\beta_X : \overline{F}X \to FX)_{X \in \mathcal{C}}$, be given by the canonical inclusion $\beta_X(q) = q$, for every $q \in \overline{F}X = \overline{FX}$. We have that, for any $f : X \to Y$ in $\mathcal{C}$, $(Ff \circ \beta_X)(q) = Ff(q) = (\beta_Y \circ \overline{F}f)(q)$ for every $q \in \overline{F}X$ and, since $\overline{F}$ is a functor, $\beta \circ \overline{F}\mathrm{Id} = \beta$ holds true. Finally, $\alpha \circ \beta = \overline{F}\mathrm{Id}$ and $\beta \circ \alpha = F\mathrm{Id}$, as for every $X \in \mathcal{C}$, $p \in FX$ and $q \in \overline{F}X$, we have $\alpha_X \beta_X(q) = \alpha_X(q) = F(\mathrm{Id}_X)(q) = q = \overline{F}\mathrm{Id}_X(q)$ and $\beta_X \alpha_X(p) = \beta_X(F(\mathrm{Id}_X)(p)) = F\mathrm{Id}_X(p)$, respectively. Hence any component morphism α_X is an $(F_X, \overline{F}_X)$-semi-isomorphism in Set, and any β_X is an $(\overline{F}_X, F_X)$-semi-isomorphism in Set.

When the semifunctors $F : \mathcal{C} \to \mathcal{D}$, $F' : \mathcal{C}' \to \mathcal{D}$ are clear from the context, we will often write C-semisplit-mono (resp., (C, C')-semisplit-mono, C-semisplit-epi, (C, C')-semisplit-epi, (C, C')-semi-isomorphism) instead of F_C-semisplit-mono (resp., $(F_C, F'_{C'})$-semisplit-mono, F_C-semisplit-epi, $(F_C, F'_{C'})$-semisplit-epi, $(F_C, F'_{C'})$-semi-isomorphism).

4.3 The notion of semifull semifunctor

Let $F : \mathcal{C} \to \mathcal{D}$ be a semifunctor and consider the associated natural transformation

$$\mathcal{F} : \mathrm{Hom}_{\mathcal{C}}(-,-) \to \mathrm{Hom}_{\mathcal{D}}(F-, F-), \tag{4.9}$$

given by $\mathcal{F}_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(FX, FY)$, $\mathcal{F}_{X,Y}(f) = F(f)$, for any morphism $f : X \to Y$ in $\mathcal{C}$. Note that the codomain $\mathrm{Hom}_{\mathcal{D}}(F-, F-)$ is a semifunctor as, given $h : FX \to FY$ in $\mathcal{D}$, $\mathrm{Hom}_{\mathcal{D}}(F\mathrm{Id}_X, F\mathrm{Id}_Y)(h) = F\mathrm{Id}_Y \circ h \circ F\mathrm{Id}_X \neq h$ in general, while the domain $\mathrm{Hom}_{\mathcal{C}}(-,-)$ is a functor, so $\mathcal{F}$ is actually a seminatural transformation. When needed we will denote $\mathcal{F}$ by $\mathcal{F}^F$ to make explicit the semifunctor F we are considering.

As in the functorial case, if $\mathcal{F}_{X,Y}$ is injective, surjective, bijective for every pair of objects $X, Y \in \mathcal{C}$, then F is a *faithful*, *full*, *fully faithful* semifunctor, respectively. As we will show in Proposition 4.58, a full (and hence a fully faithful) semifunctor is actually a functor. Motivated by the following example, we investigate a weaker notion of fullness for semifunctors that we call *semifullness*.

Example 4.57. See [52, Example 2.1], [21, Example 3.1]. Let Set be the category of sets and functions, and consider the semifunctor $F : \mathsf{Set} \to \mathsf{Set}$, defined on objects A by $F(A) = A \times A$, where $A \times A$ is the cartesian product of A by itself, and on morphisms $f : A \to B$ by $F(f) : A \times A \to B \times B$, $F(f)((a, a')) = (f(a), f(a))$, for every $a, a' \in A$. In particular, if $a, a' \in A$ and $a \neq a'$, then $F(\mathrm{Id}_A)((a, a')) = (a, a)$, whereas $\mathrm{Id}_{FA}((a, a')) =$

(a, a'), hence F is really a semifunctor. Note that F is faithful as if $F(f) = F(f')$ for morphisms $f, f' : A \to B$ in Set, then for every $a \in A$ we get $(f(a), f(a)) = F(f)((a, a')) = F(f')((a, a')) = (f'(a), f'(a))$, thus $f(a) = f'(a)$ for every $a \in A$, hence $f = f'$. Moreover, F is not full, as there is no $f : A \to A$ in Set such that $F(f) = \mathrm{Id}_{FA}$. Indeed, if such f exists, then for all $a, a' \in A$ we have $F(f)((a, a')) = \mathrm{Id}_{FA}((a, a')) = (a, a')$, but this cannot happen if $a \neq a'$, as $F(f)((a, a')) = (f(a), f(a)) \neq (a, a')$. A deeper look at the semifunctor F allows to highlight the following property. Let $\psi_B : B \times B \to B$ be the canonical projection on the first factor of the cartesian product $B \times B$, and let $\Delta_A : A \to A \times A$ be the diagonal arrow of A, given by $\Delta_A(a) = (a, a)$, for every $a \in A$. For any morphism $f = \langle f_1, f_2 \rangle : A \times A \to B \times B$ in Set, where $f_1, f_2 : A \times A \to B$, consider the morphism

$$g := \psi_B \circ f \circ \Delta_A = f_1 \circ \Delta_A : A \to B$$

in Set. We note that $F(g) = F\mathrm{Id}_B \circ f \circ F\mathrm{Id}_A$. Indeed, for all $a, a' \in A$, we have $F(g)\,((a, a')) = (g(a), g(a)) = (f_1((a, a)), f_1((a, a))) = F\mathrm{Id}_B(f((a, a))) = (F\mathrm{Id}_B \circ f \circ F\mathrm{Id}_A)((a, a'))$.

We say that a semifunctor $F : \mathcal{C} \to \mathcal{D}$ is **semifull** if, for every morphism $f : FX \to FY$ in $\mathcal{D}$, there exists a morphism $g : X \to Y$ in $\mathcal{C}$ such that

$$F(g) = F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X.$$

Thus, the semifunctor F considered in Example 4.57 is semifull. The following result shows how semifullness is related to the traditional notion of full functor.

Proposition 4.58. [21, Proposition 3.3] *Let $F : \mathcal{C} \to \mathcal{D}$ be a semifunctor. Then, F is full if, and only if, it is semifull and* $\mathrm{Id}_F = F\mathrm{Id}$.

Proof. Assume F is full. Then, since for any $f : FX \to FY$ in $\mathcal{D}$ there exists a morphism $g : X \to Y$ in $\mathcal{C}$ such that $f = F(g)$, we have that $F(g) = F(\mathrm{Id}_Y \circ g \circ \mathrm{Id}_X) = F\mathrm{Id}_Y \circ F(g) \circ F\mathrm{Id}_X = F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X$, thus F is semifull. In particular, for every $X \in \mathcal{C}$, $\mathrm{Id}_{FX} = F(h)$ for some $h : X \to X$ in $\mathcal{C}$, hence $\mathrm{Id}_{FX} = F(\mathrm{Id}_X \circ h) = F\mathrm{Id}_X \circ F(h) = F\mathrm{Id}_X \circ \mathrm{Id}_{FX} = F\mathrm{Id}_X$. On the other hand, assume that $\mathrm{Id}_F = F\mathrm{Id}$. If F is semifull, then for any morphism $f : FX \to FY$ in $\mathcal{D}$ there exists $g : X \to Y$ in $\mathcal{C}$ such that $F(g) = F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X = \mathrm{Id}_{FY} \circ f \circ \mathrm{Id}_{FX} = f$, thus F is full. ☐

Remark 4.59. From $\mathrm{Id}_F = F\mathrm{Id}$ it follows that a full semifunctor is actually a functor.

We show that semifull semifunctors are stable under composition.

Proposition 4.60. [21, Proposition 3.5] *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be semifunctors.*

i) If F and G are semifull, then the semifunctor $G \circ F$ is semifull.

ii) If $G \circ F$ is semifull and G is faithful, then F is semifull.

Proof. i). Let F and G be semifull semifunctors. Then, for any morphism $f : GFX \to GFY$ in $\mathcal{E}$, since G is semifull, there exists a morphism $g : FX \to FY$ in $\mathcal{D}$ such that $G(g) = G\mathrm{Id}_{FY} \circ f \circ G\mathrm{Id}_{FX}$. Now, since F is semifull, there exists a morphism $h : X \to Y$ in $\mathcal{C}$ such that $F(h) = F\mathrm{Id}_Y \circ g \circ F\mathrm{Id}_X$. Then, we have that $GF(h) = GF\mathrm{Id}_Y \circ G(g) \circ GF\mathrm{Id}_X = GF\mathrm{Id}_Y \circ G\mathrm{Id}_{FY} \circ f \circ G\mathrm{Id}_{FX} \circ GF\mathrm{Id}_X = GF\mathrm{Id}_Y \circ f \circ GF\mathrm{Id}_X$, so $G \circ F$ is semifull.
ii). Assume that $G \circ F$ is semifull. Then, for any morphism $f : FX \to FY$ in $\mathcal{D}$, there exists a morphism $h : X \to Y$ in $\mathcal{C}$ such that $GF(h) = GF\mathrm{Id}_Y \circ G(f) \circ GF\mathrm{Id}_X$, so $G(F(h)) = G(F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X)$. If G is faithful, we have that $F(h) = F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X$, hence F is semifull. ☐

We say that a semifunctor $F : \mathcal{C} \to \mathcal{D}$ is **semifully faithful** if F is a faithful and semifull semifunctor.

Remark 4.61. A fully faithful semifunctor, which is actually a functor by Remark 4.59, is in particular semifully faithful. From Proposition 4.58 it follows that a semifunctor F is fully faithful if, and only if, it is semifully faithful and $\mathrm{Id}_F = F\mathrm{Id}$.

Example 4.62. The semifunctor F in Example 4.57 is faithful and semifull, hence semifully faithful, but clearly not fully faithful.

Now we see how the semifull and semifully faithful conditions can be derived from requirements on the hom-set components associated with a semifunctor. In the second item of the next result we improve [21, Proposition 3.6 (ii)] showing that also the "if part" holds true.

Proposition 4.63. *Let $F : \mathcal{C} \to \mathcal{D}$ be a semifunctor and consider the associated natural transformation $\mathcal{F} : \mathrm{Hom}_{\mathcal{C}}(-,-) \to \mathrm{Hom}_{\mathcal{D}}(F-,F-)$.*

i) If, for every $X, Y \in \mathcal{C}$, $\mathcal{F}_{X,Y}$ is a $\mathrm{Hom}_{\mathcal{D}}(F-,F-)_{(X,Y)}$-semisplit-epi (or (X,Y)-semisplit-epi for short), then F is semifull.

ii) $\mathcal{F}$ is a natural semi-isomorphism if, and only if, F is semifully faithful.

Proof. i). If $\mathcal{F}_{X,Y}$ is an (X,Y)-semisplit-epi for every $X, Y \in \mathcal{C}$, then there exists a map $\mathcal{G}_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(FX,FY) \to \mathrm{Hom}_{\mathcal{C}}(X,Y)$ such that $\mathcal{F}_{X,Y} \circ \mathcal{G}_{X,Y} = \mathrm{Hom}_{\mathcal{D}}(F\mathrm{Id}_X, F\mathrm{Id}_Y)$, i.e. for any morphism $g : FX \to FY$ in $\mathcal{D}$, $(\mathcal{F}_{X,Y} \circ \mathcal{G}_{X,Y})(g) = F\mathrm{Id}_Y \circ g \circ F\mathrm{Id}_X$. Thus, for any morphism $g : FX \to FY$ in $\mathcal{D}$, we have that $F(\mathcal{G}_{X,Y}(g)) = F\mathrm{Id}_Y \circ g \circ F\mathrm{Id}_X$, where $\mathcal{G}_{X,Y}(g) : X \to Y$ is a morphism in $\mathcal{C}$, hence F is semifull.
ii). If $\mathcal{F}_{X,Y}$ is an $((X,Y),(X,Y))$-semi-isomorphism for every $X, Y \in \mathcal{C}$, then there exists a map $\mathcal{G}_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(FX,FY) \to \mathrm{Hom}_{\mathcal{C}}(X,Y)$ such that $\mathcal{G}_{X,Y} \circ \mathcal{F}_{X,Y} = \mathrm{Hom}_{\mathcal{C}}(\mathrm{Id}_X, \mathrm{Id}_Y)$ and $\mathcal{F}_{X,Y} \circ \mathcal{G}_{X,Y} = \mathrm{Hom}_{\mathcal{D}}(F\mathrm{Id}_X, F\mathrm{Id}_Y)$. By i) F is semifull. Moreover, for any morphism $h, k : X \to Y$ in $\mathcal{C}$ such that $F(h) = F(k)$, we have that $\mathcal{G}_{X,Y}(F(h)) = \mathcal{G}_{X,Y}(F(k))$, hence from $\mathcal{G}_{X,Y} \circ \mathcal{F}_{X,Y} = \mathrm{Hom}_{\mathcal{C}}(\mathrm{Id}_X, \mathrm{Id}_Y)$ it follows that $h = k$. Thus, F is semifully faithful. On the other hand, assume that F is semifully faithful. Then, we can define the map $\mathcal{G}_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(FX,FY) \to \mathrm{Hom}_{\mathcal{C}}(X,Y)$ which assigns to every $h : FX \to FY$ in $\mathcal{D}$ the unique $g : X \to Y$ in $\mathcal{C}$ such that $F(g) = F\mathrm{Id}_Y \circ h \circ F\mathrm{Id}_X$. We have that $\mathcal{G}_{X,Y}\mathcal{F}_{X,Y}(f) = \mathcal{G}_{X,Y}(Ff) = f = \mathrm{Hom}_{\mathcal{C}}(\mathrm{Id}_X, \mathrm{Id}_Y)(f)$ and $\mathcal{F}_{X,Y}\mathcal{G}_{X,Y}(h) = \mathcal{F}_{X,Y}(g) = F(g) = F\mathrm{Id}_Y \circ h \circ F\mathrm{Id}_X = \mathrm{Hom}_{\mathcal{D}}(F\mathrm{Id}_X, F\mathrm{Id}_Y)(h)$. Since $\mathcal{F}$ is natural, by Lemma 4.49 $\mathcal{F}$ is a natural semi-isomorphism. □

It is known that fully faithful functors reflect isomorphisms, i.e. they are conservative. In the next result we show a similar behavior for semifully faithful semifunctors.

Proposition 4.64. [21, Proposition 3.8] *Let $F : \mathcal{C} \to \mathcal{D}$, $F' : \mathcal{C}' \to \mathcal{D}$ be semifunctors and let $H : \mathcal{D} \to \mathcal{E}$ be a semifully faithful semifunctor. Then, H reflects $(F_C, F'_{C'})$-semisplit-monos, $(F_C, F'_{C'})$-semisplit-epis, $(F_C, F'_{C'})$-semi-isomorphisms.*

Proof. Let $f : FC \to F'C'$ be a morphism in $\mathcal{D}$ and assume that $H(f) : HFC \to HF'C'$ is a $(HF_C, HF'_{C'})$-semisplit-mono in $\mathcal{E}$. Then, $H(f) \circ HF\mathrm{Id}_C = H(f)$ and there exists a morphism $h : HF'C' \to HFC$ in $\mathcal{E}$ such that $h \circ H(f) = HF\mathrm{Id}_C$ and $h \circ HF'\mathrm{Id}_{C'} = h$. From $H(f \circ F\mathrm{Id}_C) = H(f)$ we obtain $f \circ F\mathrm{Id}_C = f$, as H is faithful. Since H is semifull, there is a morphism $g : F'C' \to FC$ in $\mathcal{D}$ such that $H(g) = H\mathrm{Id}_{FC} \circ h \circ H\mathrm{Id}_{F'C'}$. Thus, from $h \circ H(f) = HF\mathrm{Id}_C$ we have $H(g \circ f) = H(g) \circ H(f) = H\mathrm{Id}_{FC} \circ h \circ H\mathrm{Id}_{F'C'} \circ H(f) = H\mathrm{Id}_{FC} \circ h \circ H(f) = H\mathrm{Id}_{FC} \circ HF\mathrm{Id}_C = HF\mathrm{Id}_C$, and then, since H is faithful, we get

$g \circ f = F\mathrm{Id}_C$. Moreover, we have $H(g \circ F'\mathrm{Id}_{C'}) = H(g) \circ HF'\mathrm{Id}_{C'} = H\mathrm{Id}_{FC} \circ h \circ H\mathrm{Id}_{F'C'} \circ HF'\mathrm{Id}_{C'} = H\mathrm{Id}_{FC} \circ h \circ HF'\mathrm{Id}_{C'} \circ H\mathrm{Id}_{F'C'} = H\mathrm{Id}_{FC} \circ h \circ H\mathrm{Id}_{F'C'} = H(g)$, hence $g \circ F'\mathrm{Id}_{C'} = g$ as H is faithful. Then, f is an $(F_C, F'_{C'})$-semisplit-mono in $\mathcal{D}$. For $(F_C, F'_{C'})$-semisplit-epis and $(F_C, F'_{C'})$-semi-isomorphisms the proof is similar. □

Inspired by [7, Proposition 2.5], we provide a characterization of faithfulness and semifullness for semifunctors that are part of a semiadjunction.

Proposition 4.65. [21, Proposition 3.9] *Let $F \dashv_s G : \mathcal{D} \to \mathcal{C}$ be a semiadjunction with unit η and counit ϵ. Then,*

i) F is faithful if, and only if, η_C is a monomorphism in $\mathcal{C}$, for every $C \in \mathcal{C}$;

ii) F is semifull if, and only if, η_C is a GF_C-semisplit-epi in $\mathcal{C}$, for every $C \in \mathcal{C}$;

iii) G is faithful if, and only if, ϵ_D is an epimorphism in $\mathcal{D}$, for every $D \in \mathcal{D}$;

iv) G is semifull if, and only if, ϵ_D is a FG_D-semisplit-mono in $\mathcal{D}$, for every $D \in \mathcal{D}$.

Proof. We prove only *i)* and *ii)*, as *iii)* and *iv)* follow by duality. For any C, C' in $\mathcal{C}$, consider the composition

$$\tau_{C,FC'} \circ \mathcal{F}^F_{C,C'} : \mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{C}}(C, GFC')$$

where τ is defined on components as in (4.2), thus $\tau_{C,FC'}(Ff) = GFf \circ \eta_C = \eta_{C'} \circ f$, for any morphism $f : C \to C'$ in $\mathcal{C}$.
i). Assume that F is faithful. Let $f, f' : C \to C'$ be morphisms in $\mathcal{C}$ such that $\eta_{C'} \circ f = \eta_{C'} \circ f'$, i.e. $\tau_{C,FC'}(Ff) = \tau_{C,FC'}(Ff')$. Then, by composing the latter equality with $\sigma_{C,FC'}$ defined as in (4.3), we get $\sigma_{C,FC'}\tau_{C,FC'}(Ff) = \sigma_{C,FC'}\tau_{C,FC'}(Ff')$, i.e. $Ff \circ F\mathrm{Id}_C = Ff' \circ F\mathrm{Id}_C$, so that $Ff = Ff'$. Since F is faithful we have $f = f'$, thus $\eta_{C'}$ is a monomorphism. Conversely, suppose that η_C is a monomorphism for every $C \in \mathcal{C}$. Let $f, f' : C \to C'$ be morphisms in $\mathcal{C}$ such that $Ff = Ff'$. Then, $\eta_{C'} \circ f = GFf \circ \eta_C = GFf' \circ \eta_C = \eta_{C'} \circ f'$, thus $f = f'$ as $\eta_{C'}$ is a monomorphism. Hence F is faithful.
ii). Assume that F is semifull. Then, for any $f : FC \to FC'$ in $\mathcal{D}$ there exists $g : C \to C'$ in $\mathcal{C}$ such that $F(g) = F\mathrm{Id}_{C'} \circ f \circ F\mathrm{Id}_C$. In particular, for $\epsilon_{FC} : FGFC \to FC$, there exists $\nu_C : GFC \to C$ such that $F(\nu_C) = F\mathrm{Id}_C \circ \epsilon_{FC} \circ F\mathrm{Id}_{GFC}$. Then, for every $C \in \mathcal{C}$, we have $\eta_C \circ \nu_C = GF\nu_C \circ \eta_{GFC} = \tau_{GFC,FC}(F\nu_C) = \tau_{GFC,FC}(F\mathrm{Id}_C \circ \epsilon_{FC} \circ F\mathrm{Id}_{GFC}) = GF\mathrm{Id}_C \circ G\epsilon_{FC} \circ GF\mathrm{Id}_{GFC} \circ \eta_{GFC} = GF\mathrm{Id}_C \circ G\epsilon_{FC} \circ \eta_{GFC} = GF\mathrm{Id}_C \circ G\mathrm{Id}_{FC} = GF\mathrm{Id}_C$, thus η_C is a GF_C-semisplit-epi, for every C in $\mathcal{C}$. Conversely, suppose that for every $C \in \mathcal{C}$ η_C is a GF_C-semisplit-epi in $\mathcal{C}$, i.e. there exists a morphism $\nu_C : GFC \to C$ in $\mathcal{C}$ such that $\eta_C \circ \nu_C = GF\mathrm{Id}_C$. Let $f : FC \to FC'$ be a morphism in $\mathcal{D}$. Consider the composite morphism $\nu_{C'} \circ Gf \circ \eta_C : C \to C'$ in $\mathcal{C}$. Then, we have $F(\nu_{C'} \circ Gf \circ \eta_C) = F(\mathrm{Id}_{C'} \circ \nu_{C'} \circ Gf \circ \eta_C) = F\mathrm{Id}_{C'} \circ F\nu_{C'} \circ FGf \circ F\eta_C = \epsilon_{FC'} \circ (F\eta_{C'} \circ F\nu_{C'}) \circ FGf \circ F\eta_C = \epsilon_{FC'} \circ FGF\mathrm{Id}_{C'} \circ FGf \circ F\eta_C = \epsilon_{FC'} \circ FG(F\mathrm{Id}_{C'} \circ f) \circ F\eta_C = F\mathrm{Id}_{C'} \circ f \circ \epsilon_{FC} \circ F\eta_C = F\mathrm{Id}_{C'} \circ f \circ F\mathrm{Id}_C$, thus F is semifull. □

Remark 4.66. Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be semifunctors. We observe that for any natural transformation $\alpha : \mathrm{Id}_{\mathcal{C}} \to GF$ with domain the identity functor, which is indeed a seminatural transformation, any component morphism $\alpha_X : X \to GFX$ in $\mathcal{C}$ is an X-semisplit-epi if, and only if, it is an (X, X)-semisplit-epi. Indeed, by Remark 4.44 *i)* any (X, X)-semisplit-epi is an X-semisplit-epi. If α_X is an X-semisplit-epi, then there is $\beta_X : GFX \to X$ in $\mathcal{C}$ such that $\alpha_X \circ \beta_X = GF\mathrm{Id}_X$. Moreover, $\mathrm{Id}_X \circ \beta_X = \beta_X$ and from seminaturality of α it follows that $GF\mathrm{Id}_X \circ \alpha_X = \alpha_X$. Analogously, for any seminatural transformation $\alpha : GF \to \mathrm{Id}_{\mathcal{C}}$ with codomain the identity functor, any component morphism $\alpha_X : GFX \to X$ is an X-semisplit-mono if, and only if, it is an (X, X)-semisplit-mono. Thus, in the statement $ii)$ of Proposition 4.65 η_C is actually a (C, C)-semisplit-epi, and in the statement $iv)$ ϵ_D is a (D, D)-semisplit-mono.

From Proposition 4.65 it follows the next characterization of semifull faithfulness, which is a semifunctorial analogue of [19, Proposition 3.4.1].

Corollary 4.67. [21, Corollary 3.11] *Let $F \dashv_{\mathrm{s}} G : \mathcal{D} \to \mathcal{C}$ be a semiadjunction with unit η and counit ϵ. Then,*

i) F is semifully faithful if, and only if, η is a natural semi-isomorphism;

ii) G is semifully faithful if, and only if, ϵ is a natural semi-isomorphism.

Proof. We show only $i)$ as $ii)$ follows dually. If F is semifully faithful, then by Proposition 4.65 η_C is a monomorphism and a (C, C)-semisplit-epi in $\mathcal{C}$ (see Remark 4.66) for every $C \in \mathcal{C}$. Thus, by Proposition 4.54 $i)$ η_C is a (C, C)-semi-isomorphism. Since η is natural, by Lemma 4.49 η is a natural semi-isomorphism.

Conversely, if η is a natural semi-isomorphism, then by Lemma 4.49 η_C is a (C, C)-semi-isomorphism in $\mathcal{C}$ for every $C \in \mathcal{C}$, so by Proposition 4.50 η_C is a $(\mathrm{Id}_{\mathcal{C}C}, GF_C)$-semisplit-epi (hence a GF_C-semisplit-epi) and a $(\mathrm{Id}_{\mathcal{C}C}, GF_C)$-semiplit-mono (hence a split-mono, and then a monomorphism) for every C in $\mathcal{C}$, thus again by Proposition 4.65 F is semifull and faithful. □

4.4 The notion of natural semifullness

Let $F : \mathcal{C} \to \mathcal{D}$ be a semifunctor and consider its associated natural transformation $\mathcal{F}$. We say that F is a **naturally semifull** semifunctor if there is a natural transformation $\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-, -)$ such that for every object X, Y in $\mathcal{C}$,

$$\mathcal{F}_{X,Y} \circ \mathcal{P}_{X,Y} = \mathrm{Hom}_{\mathcal{D}}(F\mathrm{Id}_X, F\mathrm{Id}_Y) \tag{4.10}$$

i.e., for any morphism $f : FX \to FY$ in $\mathcal{D}$, one has $(\mathcal{F}_{X,Y} \circ \mathcal{P}_{X,Y})(f) = F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X$.

When needed we denote $\mathcal{P}$, which is actually a seminatural transformation, by $\mathcal{P}^F$.

Remark 4.68. If $F : \mathcal{C} \to \mathcal{D}$ is a functor, then we recover the definition of naturally full functor. Moreover, since a full semifunctor is actually a functor, if we require the natural fullness condition as in Section 1.2 (which implies fullness) on the natural transformation $\mathcal{F}$ associated with a semifunctor, that we then call *naturally full*, we retrieve the notion of naturally full functor.

Lemma 4.69. [21, Lemma 5.2] *Let $F : \mathcal{C} \to \mathcal{D}$ be a semifunctor. If F is naturally semifull, then it is semifull.*

Proof. If F is naturally semifull, then for any morphism $f : FX \to FY$ in $\mathcal{D}$ there exists a morphism $\mathcal{P}_{X,Y}(f) : X \to Y$ in $\mathcal{C}$ such that $F(\mathcal{P}_{X,Y}(f)) = F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X$, hence F is semifull. □

Similarly to Proposition 4.58, the next result shows how the notions of naturally semifull semifunctor and naturally full functor are related.

Proposition 4.70. [21, Proposition 5.3] *Let $F : \mathcal{C} \to \mathcal{D}$ be a semifunctor. Then, F is naturally full if, and only if, F is naturally semifull and* $\mathrm{Id}_F = F\mathrm{Id}$.

Proof. If F is naturally full, then it is trivially full and by Proposition 4.58 it holds $\mathrm{Id}_F = F\mathrm{Id}$. Since $F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X = \mathrm{Id}_{FY} \circ f \circ \mathrm{Id}_{FX} = f = \mathcal{F}_{X,Y}\mathcal{P}_{X,Y}(f)$, F is also naturally semifull. Conversely, if F is naturally semifull, then it is semifull by Lemma 4.69. Thus, if $\mathrm{Id}_F = F\mathrm{Id}$, by Proposition 4.58 F is full, so for any $f : FX \to FY$ in $\mathcal{D}$ there exists a morphism $g : X \to Y$ in $\mathcal{C}$ such that $f = F(g)$. Since F is naturally semifull, we have $\mathcal{F}_{X,Y}(\mathcal{P}_{X,Y}(f)) = F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X = F\mathrm{Id}_Y \circ Fg \circ F\mathrm{Id}_X = Fg = f$, hence F is naturally full. □

Example 4.71. (Cf. [21, Example 4.2, Example 5.4]) Let $F : \mathsf{Set} \to \mathsf{Set}$ be the semifunctor considered in Example 4.57. We define

$$\begin{aligned} \mathcal{P}_{A,B} : \mathrm{Hom}_{\mathsf{Set}}(FA, FB) = \mathrm{Hom}_{\mathsf{Set}}(A \times A, B \times B) &\to \mathrm{Hom}_{\mathsf{Set}}(A, B) \\ \mathcal{P}_{A,B}(g) := \psi_B \circ g \circ \Delta_A, & \end{aligned} \tag{4.11}$$

for every map $g : A \times A \to B \times B$, where $\psi_B : B \times B \to B$ is the canonical projection on the first factor of the cartesian product $B \times B$, and $\Delta_A : A \to A \times A$, $\Delta_A(a) = (a, a)$, is the diagonal arrow of A. For any map $h : A \to B$, $g : B \times B \to C \times C$, $g(x) = \langle g_1(x), g_2(x)\rangle$, with $g_1, g_2 : B \times B \to C$, and $k : C \to D$, by definition of $\mathcal{P}_{A,D}$, we have:

$$\begin{aligned} (\mathcal{P}_{A,D}(Fk \circ g \circ Fh))(a) &= (\psi_D \circ (Fk \circ g \circ Fh) \circ \Delta_A)(a) = (\psi_D \circ Fk \circ g \circ Fh)((a, a)) \\ &= (\psi_D \circ Fk \circ g)((h(a), h(a))) = (\psi_D \circ Fk)((g_1((h(a), h(a))), g_2((h(a), h(a))))) \\ &= \psi_D((k(g_1((h(a), h(a)))), k(g_1((h(a), h(a)))))) = k(g_1((h(a), h(a)))) \\ &= k(\psi_C((g_1((h(a), h(a))), g_2((h(a), h(a)))))) = (k \circ \psi_C \circ g)((h(a), h(a))) \\ &= (k \circ \psi_C \circ g \circ \Delta_B \circ h)(a) = (k \circ \mathcal{P}_{B,C}(g) \circ h)(a), \end{aligned}$$

for every $a \in A$, thus $\mathcal{P} : \mathrm{Hom}_{\mathsf{Set}}(F-, F-) \to \mathrm{Hom}_{\mathsf{Set}}(-, -)$ is a natural transformation. Note that for any map $f = \langle f_1, f_2\rangle : A \times A \to B \times B$, with $f_1, f_2 : A \times A \to B$, and for every $(a, a') \in A \times A$, we have

$$\begin{aligned} (\mathcal{F}_{A,B}(\mathcal{P}_{A,B}(f)))((a, a')) &= F(\mathcal{P}_{A,B}(f))((a, a')) = F(\psi_B \circ f \circ \Delta_A)((a, a')) \\ &= (F(\psi_B) \circ F(f) \circ F(\Delta_A))((a, a')) = F(\psi_B)(F(f)((a, a), (a, a))) \\ &= F(\psi_B)(f((a, a)), f((a, a))) = (\psi_B f((a, a)), \psi_B f((a, a))) \\ &= (f_1((a, a)), f_1((a, a))) = F(\mathrm{Id}_B)((f_1((a, a)), f_2((a, a)))) \\ &= F(\mathrm{Id}_B)(f((a, a))) = (F\mathrm{Id}_B \circ f)((a, a)) = (F\mathrm{Id}_B \circ f \circ F\mathrm{Id}_A)((a, a')), \end{aligned}$$

hence F is naturally semifull with respect to such $\mathcal{P}$.

In the next proposition we describe the behavior of naturally semifull semifunctors with respect to composition, cf. Proposition 1.28 for the naturally full functor case.

Proposition 4.72. [21, Proposition 5.5] *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be semifunctors.*

i) If F and G are naturally semifull, then the semifunctor $G \circ F$ is naturally semifull.

ii) If $G \circ F$ is naturally semifull and G is faithful, then F is naturally semifull.

Proof. *i*). Let F and G be naturally semifull semifunctors with respect to $\mathcal{P}^F$ and $\mathcal{P}^G$, respectively. Then, $G \circ F$ is naturally semifull with respect to $\mathcal{P}^{GF}_{X,Y} := \mathcal{P}^F_{X,Y} \circ \mathcal{P}^G_{FX,FY}$. Indeed, for any $g : GFX \to GFY$ in $\mathcal{E}$, we have

$$\begin{aligned}(\mathcal{F}^{GF}_{X,Y} \circ \mathcal{P}^{GF}_{X,Y})(g) &= (\mathcal{F}^G_{FX,FY} \circ \mathcal{F}^F_{X,Y} \circ \mathcal{P}^F_{X,Y} \circ \mathcal{P}^G_{FX,FY})(g)\\ &= \mathcal{F}^G_{FX,FY}(\mathcal{F}^F_{X,Y}(\mathcal{P}^F_{X,Y}(\mathcal{P}^G_{FX,FY}(g))))\\ &= \mathcal{F}^G_{FX,FY}(F\mathrm{Id}_Y \circ \mathcal{P}^G_{FX,FY}(g) \circ F\mathrm{Id}_X)\\ &= GF\mathrm{Id}_Y \circ G\mathcal{P}^G_{FX,FY}(g) \circ GF\mathrm{Id}_X\\ &= GF\mathrm{Id}_Y \circ (G\mathrm{Id}_{FY} \circ g \circ G\mathrm{Id}_{FX}) \circ GF\mathrm{Id}_X\\ &= G(F\mathrm{Id}_Y \circ \mathrm{Id}_{FY}) \circ g \circ G(\mathrm{Id}_{FX} \circ F\mathrm{Id}_X) = GF\mathrm{Id}_Y \circ g \circ GF\mathrm{Id}_X.\end{aligned}$$

ii). Assume that $G \circ F$ is naturally semifull with respect to $\mathcal{P}^{GF}$. Then, for any $f : FX \to FY$ in $\mathcal{D}$, we have $\mathcal{F}^G_{FX,FY}(\mathcal{F}^F_{X,Y}(\mathcal{P}^{GF}_{X,Y}(\mathcal{F}^G_{FX,FY}(f)))) = GF\mathcal{P}^{GF}_{X,Y}(Gf) = GF\mathrm{Id}_Y \circ Gf \circ GF\mathrm{Id}_X = G(F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X) = \mathcal{F}^G_{FX,FY}(F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X)$, and if G is faithful, it follows that $(\mathcal{F}^F_{X,Y} \circ \mathcal{P}^{GF}_{X,Y} \circ \mathcal{F}^G_{FX,FY})(f) = F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X$, thus F is naturally semifull with respect to $\mathcal{P}^F := \mathcal{P}^{GF} \circ \mathcal{F}^G$. □

Remark 4.73. If $F : \mathcal{C} \to \mathcal{D}$ is a full functor which is not naturally full (see e.g. [7, Example 3.3]) and $G : \mathcal{D} \to \mathcal{E}$ is a semifully faithful semifunctor, then the composite $G \circ F : \mathcal{C} \to \mathcal{E}$ is a semifull semifunctor which is not naturally semifull. In fact, by Proposition 4.60 *i*) $G \circ F$ is semifull. If $G \circ F$ were naturally semifull, then by Proposition 4.72 *ii*) it would follow that F is a naturally semifull functor, i.e. a naturally full functor, and this contradicts our assumption.

4.4.1 Naturally semifull semiadjoints

The next result is a Rafael-type Theorem for naturally semifull semifunctors. A characterization of natural semifullness similar to Theorem 1.29 can be given for semifunctors that are part of a semiadjunction in terms of semisplitting properties for the unit and the counit.

Theorem 4.74. [21, Theorem 5.9] *Let $F \dashv_s G : \mathcal{D} \to \mathcal{C}$ be a semiadjunction with unit η and counit ϵ. Then,*

i) F is naturally semifull if, and only if, η is a natural semisplit-epi, i.e. there exists a seminatural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\eta \circ \nu = GF\mathrm{Id}$;

ii) G is naturally semifull if, and only if, ϵ is a natural semisplit-mono, i.e. there exists a seminatural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\gamma \circ \epsilon = FG\mathrm{Id}$.

Proof. We prove only *i*) as *ii*) follows by duality. Assume that F is a naturally semifull semifunctor and let $\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ be the associated natural transformation such that for any $f : FX \to FY$ in $\mathcal{D}$,

$$(\mathcal{F}_{X,Y} \circ \mathcal{P}_{X,Y})(f) = F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X.$$

By Proposition 4.24 we define a seminatural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ given for any object X in $\mathcal{C}$ by $\nu_X := \mathcal{P}_{GFX,X}(\epsilon_{FX}) : GFX \to X$. Then, for every $X \in \mathcal{C}$, by naturality of η, we get

$$\begin{aligned}\eta_X \circ \nu_X = GF\nu_X \circ \eta_{GFX} &= GF\mathcal{P}_{GFX,X}(\epsilon_{FX}) \circ \eta_{GFX} = G(F\mathrm{Id}_X \circ \epsilon_{FX} \circ F\mathrm{Id}_{GFX}) \circ \eta_{GFX}\\ &= GF\mathrm{Id}_X \circ G\epsilon_{FX} \circ GF\mathrm{Id}_{GFX} \circ \eta_{GFX} = GF\mathrm{Id}_X \circ G\epsilon_{FX} \circ \eta_{GFX}\\ &= GF\mathrm{Id}_X \circ G\mathrm{Id}_{FX} = G(F\mathrm{Id}_X \circ \mathrm{Id}_{FX}) = GF\mathrm{Id}_X,\end{aligned}$$

hence $\eta \circ \nu = GF\mathrm{Id}$. Conversely, suppose that there exists a seminatural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\eta \circ \nu = GF\mathrm{Id}$. By Proposition 4.24 define, for any $f \in \mathrm{Hom}_{\mathcal{D}}(FX, FY)$, $\mathcal{P}_{X,Y}(f) := \nu_Y \circ Gf \circ \eta_X$. For any $f \in \mathrm{Hom}_{\mathcal{D}}(FX, FY)$, we have

$$\begin{aligned}(\mathcal{F}_{X,Y} \circ \mathcal{P}_{X,Y})(f) &= F(\mathcal{P}_{X,Y}(f)) = F(\nu_Y \circ Gf \circ \eta_X) = F(\mathrm{Id}_Y \circ \nu_Y \circ Gf \circ \eta_X)\\ &= F\mathrm{Id}_Y \circ F\nu_Y \circ FGf \circ F\eta_X = \epsilon_{FY} \circ (F\eta_Y \circ F\nu_Y) \circ FGf \circ F\eta_X\\ &= \epsilon_{FY} \circ FGF\mathrm{Id}_Y \circ FGf \circ F\eta_X = \epsilon_{FY} \circ FG(F\mathrm{Id}_Y \circ f) \circ F\eta_X\\ &= F\mathrm{Id}_Y \circ f \circ \epsilon_{FX} \circ F\eta_X = F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X,\end{aligned}$$

so F is a naturally semifull semifunctor. □

4.5 Separable semifunctors

In this section we study the property of separability for semifunctors, see [21, Section 4]. We define a semifunctor $F : \mathcal{C} \to \mathcal{D}$ to be separable by requiring the same condition as in Definition 1.1 on the associated natural transformation $\mathcal{F}$. We discuss general properties, such as a Maschke-type Theorem and a Rafael-type Theorem for separable semifunctors.

We say that a semifunctor $F : \mathcal{C} \to \mathcal{D}$ is **separable** if there is a natural transformation $\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ such that

$$\mathcal{P} \circ \mathcal{F} = \mathrm{Id}_{\mathrm{Hom}_{\mathcal{C}}(-,-)}, \tag{4.12}$$

i.e., for any morphism $f : X \to Y$ in $\mathcal{C}$, one has $(\mathcal{P}_{X,Y} \circ \mathcal{F}_{X,Y})(f) = f$.

Remark 4.75. *i*) A separable functor is a separable semifunctor.
ii) A separable semifunctor is faithful.

Example 4.76. We come back to Example 4.71. We know that the semifunctor $F : \mathsf{Set} \to \mathsf{Set}$, $A \mapsto F(A) = A \times A$, $[f : A \to B] \mapsto F(f) : A \times A \to B \times B$, $F(f)((a, a')) = (f(a), f(a))$ is naturally semifull with respect to $\mathcal{P}_{A,B} : \mathrm{Hom}_{\mathsf{Set}}(FA, FB) = \mathrm{Hom}_{\mathsf{Set}}(A \times A, B \times B) \to \mathrm{Hom}_{\mathsf{Set}}(A, B)$, $\mathcal{P}_{A,B}(g) = \psi_B \circ g \circ \Delta_A$. For any morphism $f : A \to B$ in Set and for every $a \in A$, we have $(\mathcal{P}_{A,B}(\mathcal{F}_{A,B}(f)))(a) = (\mathcal{P}_{A,B}(Ff))(a) = (\psi_B \circ Ff \circ \Delta_A)(a) = (\psi_B \circ Ff)((a, a)) = \psi_B((f(a), f(a))) = f(a)$, hence F results to be also a separable semifunctor.

The behavior of separable semifunctors with respect to composition is the same as in the functorial case, cf. Lemma 1.4.

Lemma 4.77. [21, Lemma 4.3] *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be semifunctors.*

i) If F and G are separable, then so is the composite $G \circ F : \mathcal{C} \to \mathcal{E}$.

ii) If $G \circ F : \mathcal{C} \to \mathcal{E}$ is separable, then so is F.

Proof. For *i*) define $\mathcal{P}^{GF}_{X,Y}(g) = \mathcal{P}^{F}_{X,Y}\mathcal{P}^{G}_{FX,FY}(g)$, for any morphism g in $\mathrm{Hom}_{\mathcal{E}}(GFX, GFY)$. For *ii*) define $\mathcal{P}^{F}_{X,Y}(f) = \mathcal{P}^{GF}_{X,Y}(Gf)$, for every $f \in \mathrm{Hom}_{\mathcal{D}}(FX, FY)$. □

Since separable functors satisfy a functorial version of Maschke Theorem, see Proposition 1.7, we show that a similar behavior holds for separable semifunctors.

Theorem 4.78. [21, Theorem 4.5] (Maschke-type Theorem) *Let $F : \mathcal{C} \to \mathcal{D}$ be a separable semifunctor. For any morphism $f : C \to C'$ in $\mathcal{C}$, consider the morphism $F(f) : FC \to FC'$ in $\mathcal{D}$.*

i) *If $F(f)$ is a C-semisplit-mono, then f is a split-mono;*

ii) *if $F(f)$ is a C'-semisplit-epi, then f is a split-epi;*

iii) *if $F(f)$ is a (C, C')-semi-isomorphism, then f is an isomorphism.*

Proof. We show only *i)*, as *ii)* is analogous and *iii)* follows from *i)*+*ii)*. Assume that $F(f)$ is a C-semisplit-mono, i.e. there exists a morphism $g : FC' \to FC$ such that $g \circ F(f) = F\mathrm{Id}_C$, and that F is separable through a natural transformation $\mathcal{P}$. Then, by naturality of $\mathcal{P}$, we get that $\mathcal{P}_{C',C}(g) \circ f = \mathcal{P}_{C,C}(g \circ F(f)) = \mathcal{P}_{C,C}(F\mathrm{Id}_C) = \mathcal{P}_{C,C}\mathcal{F}_{C,C}(\mathrm{Id}_C) = \mathrm{Id}_C$, hence f is a split-mono. □

It is known that a functor is fully faithful if, and only if, it is separable and naturally full, see Remark 1.27 *ii)*. A similar characterization in terms of separable and naturally semifull semifunctors holds for semifully faithful semifunctors.

Proposition 4.79. [21, Proposition 5.8] *Let $F : \mathcal{C} \to \mathcal{D}$ be a semifunctor. Then, F is semifully faithful if, and only if, it is separable and naturally semifull.*

Proof. If F is separable and naturally semifull, then it is trivially semifully faithful by Remark 4.75 *ii)* and Lemma 4.69. Conversely, assume that F is semifully faithful. Since F is semifull, for any morphism $f : FX \to FY$ in $\mathcal{D}$ there exists a morphism $g : X \to Y$ in $\mathcal{C}$ such that $F(g) = F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X$, and by faithfulness of F, g is unique. This assignment defines a mapping

$$\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-, -)$$

such that for any $f : FX \to FY$ in $\mathcal{D}$, with X, Y in $\mathcal{C}$, $\mathcal{P}_{X,Y}(f) = g$, where $g : X \to Y$ in $\mathcal{C}$ is such that $F(g) = F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X$. We show that such $\mathcal{P}$ is actually a natural transformation. For any $h : X \to Y$ in $\mathcal{C}$, $k : FY \to FZ$ in $\mathcal{D}$, $l : Z \to T$ in $\mathcal{C}$, we have that there is a morphism $g : X \to T$ in $\mathcal{C}$ such that $F(g) = F\mathrm{Id}_T \circ Fl \circ k \circ Fh \circ F\mathrm{Id}_X = Fl \circ k \circ Fh$ and $\mathcal{P}_{X,T}(Fl \circ k \circ Fh) = g$. Then, we get $\mathcal{F}_{X,T}(\mathcal{P}_{X,T}(Fl \circ k \circ Fh)) = F(g) = Fl \circ k \circ Fh = Fl \circ (F\mathrm{Id}_Z \circ k \circ F\mathrm{Id}_Y) \circ Fh = Fl \circ F(\mathcal{P}_{Y,Z}(k)) \circ Fh = \mathcal{F}_{X,T}(l \circ \mathcal{P}_{Y,Z}(k) \circ h)$, hence since F is faithful it follows that $\mathcal{P}_{X,T}(Fl \circ k \circ Fh) = l \circ \mathcal{P}_{Y,Z}(k) \circ h$ and so $\mathcal{P}$ is a natural transformation. Since for any $f : FX \to FY$ in $\mathcal{D}$ there is a morphism $g : X \to Y$ in $\mathcal{C}$ such that $F(g) = F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X$, we have that $F(\mathcal{P}_{X,Y}(f)) = F(g) = F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X$, thus F is naturally semifull. Moreover, for any $f : X \to Y$ in $\mathcal{C}$, $\mathcal{P}_{X,Y}(F(f)) = h$, for some $h : X \to Y$ in $\mathcal{C}$ such that $F(h) = F\mathrm{Id}_Y \circ F(f) \circ F\mathrm{Id}_X = F(f)$, but since F is faithful we achieve $h = f$, and hence F is separable as $\mathcal{P}_{X,Y}(F(f)) = f$. □

4.5.1 Separable semiadjoints

We obtain a characterization of separability for semifunctors that are part of a semi-adjunction, extending (Rafael) Theorem 1.18 to semifunctors.

Theorem 4.80. [21, Theorem 4.7] (Rafael-type Theorem for separable semifunctors) *Let $F \dashv_{\mathrm{s}} G : \mathcal{D} \to \mathcal{C}$ be a semiadjunction, with unit η and counit ϵ. Then,*

i) *F is separable if, and only if, η is a natural split-mono, i.e. there exists a seminatural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\nu \circ \eta = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}}$;*

ii) *G is separable if, and only if, ϵ is a natural split-epi, i.e. there exists a seminatural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\epsilon \circ \gamma = \mathrm{Id}_{\mathrm{Id}_{\mathcal{D}}}$.*

Proof. We prove only *i*) as *ii*) follows by duality. Assume that F is a separable semifunctor and let $\mathcal{P}$ be the associated natural transformation such that $\mathcal{P} \circ \mathcal{F} = \mathrm{Id}_{\mathrm{Hom}_{\mathcal{C}}(-,-)}$. By Proposition 4.24 we define a seminatural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ given for any object X in $\mathcal{C}$ by

$$\nu_X := \mathcal{P}_{GFX,X}(\epsilon_{FX}) : GFX \to X.$$

We have

$$\begin{aligned}\nu_X \circ \eta_X &= \mathcal{P}_{GFX,X}(\epsilon_{FX}) \circ \eta_X = \mathcal{P}_{X,X}(\epsilon_{FX} \circ F\eta_X) \\ &= \mathcal{P}_{X,X}(F\mathrm{Id}_X) = \mathcal{P}_{X,X}\mathcal{F}_{X,X}(\mathrm{Id}_X) = \mathrm{Id}_X,\end{aligned}$$

where the last equality follows from the separability of F. Conversely, suppose that there exists a seminatural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\nu \circ \eta = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}}$. By Proposition 4.24 define, for every $g : FX \to FY$ in $\mathcal{D}$,

$$\mathcal{P}_{X,Y}(g) := \nu_Y \circ Gg \circ \eta_X.$$

For every $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ it holds

$$(\mathcal{P}_{X,Y} \circ \mathcal{F}_{X,Y})(f) = \mathcal{P}_{X,Y}(F(f)) = \nu_Y \circ GF(f) \circ \eta_X = \nu_Y \circ \eta_Y \circ f = \mathrm{Id}_Y \circ f = f,$$

hence F is separable. □

Given an idempotent (semi)natural transformation $e = (e_X)_{X\in\mathcal{C}} : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ on a category $\mathcal{C}$, consider the canonical semifunctor $E^e : \mathcal{C} \to \mathcal{C}$. As a consequence of Theorem 4.80 we have the following.

Proposition 4.81. [21, Proposition 4.8] *Let $e = (e_X)_{X\in\mathcal{C}} : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be an idempotent (semi)natural transformation. Then, the canonical semifunctor $E^e : \mathcal{C} \to \mathcal{C}$ is separable if, and only if, $e_X = \mathrm{Id}_X$, for every $X \in \mathcal{C}$.*

Proof. From the proof of Proposition 4.34 E^e is self-semiadjoint with unit and counit given on components by $e_X : X \to X$, for any $X \in \mathcal{C}$. By Theorem 4.80 E^e is separable if, and only if, there exists a seminatural transformation $\nu = (\nu_X : X \to X)_{X\in\mathcal{C}}$ such that $\nu_X \circ e_X = \mathrm{Id}_X$ for any $X \in \mathcal{C}$. Then, $e_X = \mathrm{Id}_X \circ e_X = \nu_X \circ e_X \circ e_X = \nu_X \circ e_X = \mathrm{Id}_X$. □

Remark 4.82. By Corollary 4.37 *i*) we know that, given a semiadjunction $F \dashv_{\mathrm{s}} G : \mathcal{D} \to \mathcal{C}$ and the canonical semifunctor $E^e : \mathcal{C} \to \mathcal{C}$, then $F' := FE^e : \mathcal{C} \to \mathcal{D}$ and $G' := E^eG : \mathcal{D} \to \mathcal{C}$ form a semiadjunction $F' \dashv_{\mathrm{s}} G'$. If F' is separable, then by Lemma 4.77 *ii*) E^e is separable, hence by Proposition 4.81 $F' = F$, so F is separable. If G' is separable, then so is G again by Lemma 4.77 *ii*).

4.6 Semiseparable semifunctors

In this section we present the notion of semiseparability for semifunctors, see [21, Section 6]. We say that a semifunctor $F : \mathcal{C} \to \mathcal{D}$ is **semiseparable** if there exists a natural transformation $\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ such that

$$\mathcal{F} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{F}. \tag{4.13}$$

In the following proposition we characterize separable and naturally semifull semifunctors in terms of faithful and semifull semifunctors, respectively. In the functor case we retrieve Proposition 2.5.

Proposition 4.83. [21, Proposition 6.1] *Let $F : \mathcal{C} \to \mathcal{D}$ be a semifunctor. Then,*

i) F is separable if, and only if, F is semiseparable and faithful;

ii) F is naturally semifull if, and only if, F is semiseparable and semifull.

Proof. i). The proof is analogous to that for Proposition 2.5 *i)*.
ii). If F is naturally semifull, then for any $f : X \to Y$ in $\mathcal{C}$, we have that $(\mathcal{F}_{X,Y} \circ \mathcal{P}_{X,Y} \circ \mathcal{F}_{X,Y})(f) = \mathcal{F}_{X,Y}(\mathcal{P}_{X,Y}(Ff)) = F\mathrm{Id}_Y \circ Ff \circ F\mathrm{Id}_X = F(\mathrm{Id}_Y \circ f \circ \mathrm{Id}_X) = Ff = \mathcal{F}_{X,Y}(f)$, hence F is semiseparable. Moreover, a naturally semifull semifunctor is semifull by Lemma 4.69. Conversely, assume that F is semiseparable. Then, there is a natural transformation $\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ such that $\mathcal{F} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{F}$. If F is semifull, then for any $f : FX \to FY$ in $\mathcal{D}$ there exists $h : X \to Y$ in $\mathcal{C}$ such that $F(h) = F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X : FX \to FY$, so by naturality of $\mathcal{P}_{X,Y}$ we get that $\mathcal{F}_{X,Y}(\mathcal{P}_{X,Y}(f)) = F(\mathrm{Id}_Y \circ \mathcal{P}_{X,Y}(f) \circ \mathrm{Id}_X) = F\mathcal{P}_{X,Y}(F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X) = F\mathcal{P}_{X,Y}(F(h)) = F(h) = F\mathrm{Id}_Y \circ f \circ F\mathrm{Id}_X$, hence F is a naturally semifull semifunctor. ☐

Semiseparable semifunctors satisfy the following properties with respect to composition, as in the functor case, cf. Lemma 2.6 and Lemma 2.8.

Lemma 4.84. [21, Lemma 6.3] *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be semifunctors and consider the composite semifunctor $G \circ F : \mathcal{C} \to \mathcal{E}$.*

i) If F is semiseparable and G is separable, then $G \circ F$ is semiseparable.

ii) If F is naturally semifull and G is semiseparable, then $G \circ F$ is semiseparable.

iii) If $G \circ F$ is semiseparable and G is faithful, then F is semiseparable.

Proof. i). If F is semiseparable with respect to $\mathcal{P}^F$ and G is separable with respect to $\mathcal{P}^G$, then $G \circ F$ is semiseparable with respect to $\mathcal{P}^{GF}_{X,Y} := \mathcal{P}^F_{X,Y}\mathcal{P}^G_{FX,FY}$, for every X, Y in $\mathcal{C}$, and the proof is the same of that for functors.
ii). If F is naturally semifull with respect to $\mathcal{P}^F$ and G is semiseparable with respect to $\mathcal{P}^G$, then for every $f : X \to Y$ in $\mathcal{C}$ we have

$$\begin{aligned} \mathcal{F}^{GF}_{X,Y}\mathcal{P}^F_{X,Y}\mathcal{P}^G_{FX,FY}\mathcal{F}^{GF}_{X,Y}(f) &= \mathcal{F}^G_{FX,FY}(\mathcal{F}^F_{X,Y}\mathcal{P}^F_{X,Y}(\mathcal{P}^G_{FX,FY}(GFf))) \\ = \mathcal{F}^G_{FX,FY}(F\mathrm{Id}_Y \circ \mathcal{P}^G_{FX,FY}(GFf) \circ F\mathrm{Id}_X) &= GF\mathrm{Id}_Y \circ G\mathcal{P}^G_{FX,FY}(GFf) \circ GF\mathrm{Id}_X \\ = GF\mathrm{Id}_Y \circ GFf \circ GF\mathrm{Id}_X &= GF(\mathrm{Id}_Y \circ f \circ \mathrm{Id}_X) = GFf = \mathcal{F}^{GF}_{X,Y}(f), \end{aligned}$$

hence $G \circ F$ is semiseparable with respect to $\mathcal{P}^{GF}_{X,Y} := \mathcal{P}^F_{X,Y}\mathcal{P}^G_{FX,FY}$.
iii). If $G \circ F$ is semiseparable through $\mathcal{P}^{GF}$, then $\mathcal{F}^{GF}_{X,Y} \circ \mathcal{P}^{GF}_{X,Y} \circ \mathcal{F}^{GF}_{X,Y} = \mathcal{F}^{GF}_{X,Y}$, i.e. $\mathcal{F}^G_{FX,FY} \circ \mathcal{F}^F_{X,Y} \circ \mathcal{P}^{GF}_{X,Y} \circ \mathcal{F}^G_{FX,FY} \circ \mathcal{F}^F_{X,Y} = \mathcal{F}^G_{FX,FY} \circ \mathcal{F}^F_{X,Y}$, for every $X, Y \in \mathcal{C}$. Since G is faithful, we have that $\mathcal{F}^F_{X,Y} \circ \mathcal{P}^{GF}_{X,Y} \circ \mathcal{F}^G_{FX,FY} \circ \mathcal{F}^F_{X,Y} = \mathcal{F}^F_{X,Y}$, for every X, Y in $\mathcal{C}$, so F is semiseparable through $\mathcal{P}^F_{X,Y} := \mathcal{P}^{GF}_{X,Y} \circ \mathcal{F}^G_{FX,FY}$. ☐

Remark 4.85. If $F : \mathcal{C} \to \mathcal{D}$ is a semiseparable functor which is neither separable, nor naturally full (see e.g. Example 3.3) and $G : \mathcal{D} \to \mathcal{E}$ is a separable semifunctor, then the composite $G \circ F : \mathcal{C} \to \mathcal{E}$ is a semiseparable semifunctor by Lemma 4.84 *i)*, which is not separable, neither naturally semifull. In fact, if $G \circ F$ were separable, then by Lemma 4.77 *ii)* F would be separable, and if $G \circ F$ were naturally semifull, then by Proposition 4.72 *ii)* F would be a naturally semifull functor, i.e. a naturally full functor, contradicting our assumptions.

We now show that the properties considered so far are stable under natural semi-isomorphisms. Cf. [21, Proposition 6.2], [21, Proposition 4.4], [21, Proposition 5.7] for the semiseparable, separable, naturally semifull case, respectively.

Proposition 4.86. *A semifunctor naturally semi-isomorphic to a semifull (resp., faithful, semifully faithful, semiseparable, separable, naturally semifull) semifunctor is semifull (resp., faithful, semifully faithful, semiseparable, separable, naturally semifull).*

Proof. Consider a natural semi-isomorphism $\alpha : F \to G$ of semifunctors, and define $\varsigma : \mathrm{Hom}_{\mathcal{D}}(F-,F-) \to \mathrm{Hom}_{\mathcal{D}}(G-,G-)$ by $\varsigma_{X,Y}(f) = \alpha_Y \circ f \circ \alpha_X^{-1}$, for every $f : FX \to FY$ in $\mathcal{D}$. It is a natural semi-isomorphism with semi-inverse $\varsigma^{-1} : \mathrm{Hom}_{\mathcal{D}}(G-,G-) \to \mathrm{Hom}_{\mathcal{D}}(F-,F-)$ given by $\varsigma^{-1}_{X,Y}(g) = \alpha_Y^{-1} \circ g \circ \alpha_X$, for every $g : GX \to GY$ in $\mathcal{D}$. Indeed, from naturality of α and α^{-1}, for every $h : X' \to X$, $k : Y \to Y'$ in $\mathcal{C}$ and $f : FX \to FY$ in $\mathcal{D}$, we have that $(\mathrm{Hom}_{\mathcal{D}}(Gh,Gk)\circ\varsigma_{X,Y})(f) = Gk\circ(\alpha_Y\circ f\circ\alpha_X^{-1})\circ Gh = \alpha_{Y'}\circ Fk\circ f\circ Fh\circ \alpha_{X'}^{-1} = \alpha_{Y'}\circ\mathrm{Hom}_{\mathcal{D}}(Fh,Fk)(f)\circ\alpha_{X'}^{-1} = (\varsigma_{X',Y'}\circ\mathrm{Hom}_{\mathcal{D}}(Fh,Fk))(f)$, hence ς is natural. It is also seminatural as $(\varsigma_{X,Y}\circ\mathrm{Hom}_{\mathcal{D}}(F\mathrm{Id}_X,F\mathrm{Id}_Y))(f) = \varsigma_{X,Y}\circ(F\mathrm{Id}_Y\circ f\circ F\mathrm{Id}_X) = \alpha_Y\circ F\mathrm{Id}_Y\circ f\circ F\mathrm{Id}_X\circ\alpha_X^{-1} = \alpha_Y\circ f\circ\alpha_X^{-1} = \varsigma_{X,Y}(f)$. Similarly, for ς^{-1} we have $(\mathrm{Hom}_{\mathcal{D}}(Fh,Fk)\circ\varsigma^{-1}_{X,Y})(f) = Fk\circ(\alpha_Y^{-1}\circ f\circ\alpha_X)\circ Fh = \alpha_{Y'}^{-1}\circ Gk\circ f\circ Gh\circ\alpha_{X'} = \alpha_{Y'}^{-1}\circ\mathrm{Hom}_{\mathcal{D}}(Gh,Gk)(f)\circ\alpha_{X'} = (\varsigma^{-1}_{X',Y'}\circ\mathrm{Hom}_{\mathcal{D}}(Gh,Gk))(f)$, hence ς^{-1} is natural, and $(\varsigma^{-1}_{X,Y}\circ\mathrm{Hom}_{\mathcal{D}}(G\mathrm{Id}_X,G\mathrm{Id}_Y))(g) = \varsigma^{-1}_{X,Y}\circ(G\mathrm{Id}_Y\circ g\circ G\mathrm{Id}_X) = \alpha_Y^{-1}\circ G\mathrm{Id}_Y\circ g\circ G\mathrm{Id}_X\circ\alpha_X = \alpha_Y^{-1}\circ g\circ\alpha_X = \varsigma^{-1}_{X,Y}(g)$. Furthermore, $\varsigma^{-1}_{X,Y}(\varsigma_{X,Y}(f)) = \alpha_Y^{-1}\circ\varsigma_{X,Y}(f)\circ\alpha_X = \alpha_Y^{-1}\circ\alpha_Y\circ f\circ\alpha_X^{-1}\circ\alpha_X = F\mathrm{Id}_Y\circ f\circ F\mathrm{Id}_X = \mathrm{Hom}_{\mathcal{D}}(F\mathrm{Id}_X,F\mathrm{Id}_Y)(f)$ and $\varsigma_{X,Y}(\varsigma^{-1}_{X,Y}(g)) = \alpha_Y\circ\varsigma^{-1}_{X,Y}(g)\circ\alpha_X^{-1} = \alpha_Y\circ\alpha_Y^{-1}\circ g\circ\alpha_X\circ\alpha_X^{-1} = G\mathrm{Id}_Y\circ g\circ G\mathrm{Id}_X = \mathrm{Hom}_{\mathcal{D}}(G\mathrm{Id}_X,G\mathrm{Id}_Y)(g)$.
Note that

$$\mathcal{F}^F_{X,Y} = \varsigma^{-1}_{X,Y}\circ\mathcal{F}^G_{X,Y} \quad\text{and}\quad \mathcal{F}^G_{X,Y} = \varsigma_{X,Y}\circ\mathcal{F}^F_{X,Y}.$$

In fact, $\mathcal{F}^F_{X,Y}(f) = Ff = F\mathrm{Id}_Y\circ Ff = \alpha_Y^{-1}\circ\alpha_Y\circ Ff = \alpha_Y^{-1}\circ Gf\circ\alpha_X = \varsigma^{-1}_{X,Y}(Gf) = (\varsigma^{-1}_{X,Y}\circ\mathcal{F}^G_{X,Y})(f)$ and $\mathcal{F}^G_{X,Y}(f) = Gf = Gf\circ G\mathrm{Id}_X = Gf\circ\alpha_X\circ\alpha_X^{-1} = \alpha_Y\circ Ff\circ\alpha_X^{-1} = \varsigma_{X,Y}(Ff) = (\varsigma_{X,Y}\circ\mathcal{F}^F_{X,Y})(f)$, for every $f : X \to Y$ in $\mathcal{C}$.
Moreover, by naturality of ς, for every $X, Y \in \mathcal{C}$ we have

$$\begin{aligned}\varsigma^{-1}_{X,Y}\circ\mathrm{Hom}_{\mathcal{D}}(G\mathrm{Id}_X,G\mathrm{Id}_Y)\circ\varsigma_{X,Y} &= \varsigma^{-1}_{X,Y}\circ\varsigma_{X,Y}\circ\mathrm{Hom}_{\mathcal{D}}(F\mathrm{Id}_X,F\mathrm{Id}_Y)\\ &= \varsigma^{-1}_{X,Y}\circ\varsigma_{X,Y} = \mathrm{Hom}_{\mathcal{D}}(F\mathrm{Id}_X,F\mathrm{Id}_Y),\end{aligned}$$

Assume that $G : \mathcal{C} \to \mathcal{D}$ is semifull and consider $f : FX \to FY$ in $\mathcal{D}$. Thus, there exists $h : X \to Y$ in $\mathcal{C}$ such that $G(h) = G\mathrm{Id}_Y\circ\varsigma_{X,Y}(f)\circ G\mathrm{Id}_X$. Then, from $\mathcal{F}^F_{X,Y} = \varsigma^{-1}_{X,Y}\circ\mathcal{F}^G_{X,Y}$ we have

$$\mathrm{Hom}_{\mathcal{D}}(F\mathrm{Id}_X,F\mathrm{Id}_Y)(f) = (\varsigma^{-1}_{X,Y}\circ\mathrm{Hom}_{\mathcal{D}}(G\mathrm{Id}_X,G\mathrm{Id}_Y))(\varsigma_{X,Y}(f)) = \varsigma^{-1}_{X,Y}(G(h)) = F(h),$$

so F is semifull.
Assume now that $G : \mathcal{C} \to \mathcal{D}$ is faithful. If $Fh = Fk$, for $h, k : X \to Y$ in $\mathcal{C}$, then $\varsigma_{X,Y}(Fh) = \varsigma_{X,Y}(Fk)$, that is $Gh = Gk$. Since by assumption G is faithful, we have $h = k$, hence F is faithful as well. The semifully faithful case follow from the previous two cases by Proposition 4.79.
Assume that $G : \mathcal{C} \to \mathcal{D}$ is a semiseparable (resp., separable, naturally semifull) semifunctor with respect to $\mathcal{P}^G$. We show that F results to be semiseparable (resp., separable, naturally semifull) with respect to $\mathcal{P}^F_{X,Y} := \mathcal{P}^G_{X,Y}\circ\varsigma_{X,Y}$, for every $X, Y \in \mathcal{C}$.

Indeed, in the semiseparable case, for every $X, Y \in \mathcal{C}$, we have

$$\begin{aligned}\mathcal{F}^F_{X,Y} \circ \mathcal{P}^F_{X,Y} \circ \mathcal{F}^F_{X,Y} &= \varsigma^{-1}_{X,Y} \circ \mathcal{F}^G_{X,Y} \circ \mathcal{P}^G_{X,Y} \circ \varsigma_{X,Y} \circ \mathcal{F}^F_{X,Y} \\ &= \varsigma^{-1}_{X,Y} \circ \mathcal{F}^G_{X,Y} \circ \mathcal{P}^G_{X,Y} \circ \mathcal{F}^G_{X,Y} = \varsigma^{-1}_{X,Y} \circ \mathcal{F}^G_{X,Y} = \mathcal{F}^F_{X,Y}.\end{aligned}$$

In the separable case, for every $X, Y \in \mathcal{C}$, we have

$$\mathcal{P}^F_{X,Y} \circ \mathcal{F}^F_{X,Y} = \mathcal{P}^G_{X,Y} \circ \varsigma_{X,Y} \circ \mathcal{F}^F_{X,Y} = \mathcal{P}^G_{X,Y} \circ \mathcal{F}^G_{X,Y} = \mathrm{Id}_{\mathrm{Hom}_{\mathcal{C}}(X,Y)}.$$

Finally, we consider the naturally semifull case. For every $X, Y \in \mathcal{C}$, we have

$$\begin{aligned}&\mathcal{F}^F_{X,Y} \circ \mathcal{P}^F_{X,Y} = \mathcal{F}^F_{X,Y} \circ (\mathcal{P}^G_{X,Y} \circ \varsigma_{X,Y}) = \varsigma^{-1}_{X,Y} \circ \mathcal{F}^G_{X,Y} \circ \mathcal{P}^G_{X,Y} \circ \varsigma_{X,Y} = \\ &\varsigma^{-1}_{X,Y} \circ \mathrm{Hom}_{\mathcal{D}}(G\mathrm{Id}_X, G\mathrm{Id}_Y) \circ \varsigma_{X,Y} \overset{(*)}{=} \varsigma^{-1}_{X,Y} \circ \varsigma_{X,Y} = \mathrm{Hom}_{\mathcal{D}}(F\mathrm{Id}_X, F\mathrm{Id}_Y)\end{aligned}$$

where $(*)$ follows since ς^{-1} is seminatural, as shown above. Alternatively, the separable and naturally semifull cases follow from Proposition 4.83 by combining the semiseparable with the faithful and semifull cases, respectively. □

Similarly to Proposition 2.11, we can attach to any semiseparable semifunctor a canonical idempotent (semi)natural transformation, that we call *the associated idempotent*, which controls when the semifunctor is separable.

Proposition 4.87. [21, Proposition 6.5] *Let $F : \mathcal{C} \to \mathcal{D}$ be a semiseparable semifunctor. Then, there is a unique idempotent (semi)natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ such that $Fe = F\mathrm{Id}$ with the following universal property: if $f, g : X \to Y$ are morphisms in $\mathcal{C}$, then $Ff = Fg$ if, and only if, $e_Y \circ f = e_Y \circ g$. Moreover, $e = \mathrm{Id} : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ if, and only if, F is separable.*

Proof. Since F is semiseparable, there is a natural transformation $\mathcal{P}$ such that $\mathcal{F} \circ \mathcal{P} \circ \mathcal{F} = \mathcal{F}$. Set $e_X := \mathcal{P}_{X,X}(F\mathrm{Id}_X)$, for every $X \in \mathcal{C}$. Note that for all $X \in \mathcal{C}$, $Fe_X = F\mathcal{P}_{X,X}(F\mathrm{Id}_X) = \mathcal{F}_{X,X}\mathcal{P}_{X,X}\mathcal{F}_{X,X}(\mathrm{Id}_X) = \mathcal{F}_{X,X}(\mathrm{Id}_X) = F\mathrm{Id}_X$. Then, by naturality of $\mathcal{P}$, we have $e_X \circ e_X = \mathcal{P}_{X,X}(F\mathrm{Id}_X) \circ e_X = \mathcal{P}_{X,X}(F\mathrm{Id}_X \circ Fe_X) = \mathcal{P}_{X,X}(Fe_X) = \mathcal{P}_{X,X}(F\mathrm{Id}_X) = e_X$ and hence e_X is idempotent. Moreover, for every morphism $f : X \to Y$ in $\mathcal{C}$ we have $f \circ e_X = f \circ \mathcal{P}_{X,X}(F\mathrm{Id}_X) = \mathcal{P}_{X,Y}(Ff \circ F\mathrm{Id}_X) = \mathcal{P}_{X,Y}(F\mathrm{Id}_Y \circ Ff) = \mathcal{P}_{Y,Y}(F\mathrm{Id}_Y) \circ f = e_Y \circ f$, so that $f \circ e_X = e_Y \circ f$. Thus, $e = (e_X)_{X \in \mathcal{C}} : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ is an idempotent (semi)natural transformation such that $Fe = F\mathrm{Id}$. Now, consider morphisms $f, g : X \to Y$ in $\mathcal{C}$. If $Ff = Fg$, then $\mathcal{P}_{X,Y}(Ff) = \mathcal{P}_{X,Y}(Fg)$, i.e. $\mathcal{P}_{Y,Y}(F\mathrm{Id}_Y) \circ f = \mathcal{P}_{Y,Y}(F\mathrm{Id}_Y \circ Ff) = \mathcal{P}_{Y,Y}(F\mathrm{Id}_Y \circ Fg) = \mathcal{P}_{Y,Y}(F\mathrm{Id}_Y) \circ g$, i.e. $e_Y \circ f = e_Y \circ g$. Conversely, from $e_Y \circ f = e_Y \circ g$ we get $Fe_Y \circ Ff = Fe_Y \circ Fg$ and hence $Ff = Fg$ as $Fe_Y = F\mathrm{Id}_Y$. Finally, let $e' : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be an idempotent (semi)natural transformation such that, if $f, g : X \to Y$ are morphisms in $\mathcal{C}$, then $Ff = Fg$ if, and only if, $e'_Y \circ f = e'_Y \circ g$. From $e'_X \circ e'_X = e'_X \circ \mathrm{Id}_X$ we get $Fe'_X = F\mathrm{Id}_X$, hence $Fe' = F\mathrm{Id}$. From the property of e we have $e_X \circ e'_X = e_X \circ \mathrm{Id}_X$, i.e. $e_X \circ e'_X = e_X$. By interchanging the roles of e and e', similarly we get $e'_X \circ e_X = e'_X$, and by naturality we have $e_X \circ e'_X = e'_X \circ e_X$, hence $e_X = e'_X$, i.e. $e = e'$.

Now, if F is separable then there is a natural transformation $\mathcal{P}$ such that $\mathcal{P} \circ \mathcal{F} = \mathrm{Id}$ and hence $e_X = \mathcal{P}_{X,X}(F\mathrm{Id}_X) = \mathcal{P}_{X,X}\mathcal{F}_{X,X}(\mathrm{Id}_X) = \mathrm{Id}_X$ for all $X \in \mathcal{C}$. Conversely, let $F : \mathcal{C} \to \mathcal{D}$ be a semiseparable semifunctor through $\mathcal{P}$ and suppose $e = \mathrm{Id}$ is the associated idempotent. Then, for every $f : X \to Y$ in $\mathcal{C}$, we have $\mathcal{P}_{X,Y}(Ff) = \mathcal{P}_{X,Y}(Ff \circ F\mathrm{Id}_X) = f \circ \mathcal{P}_{X,X}(F\mathrm{Id}_X) = f \circ e_X = f \circ \mathrm{Id}_X = f$ so that $\mathcal{P} \circ \mathcal{F} = \mathrm{Id}$, hence F is separable. □

Corollary 4.88. *Let F be a semiseparable semifunctor with associated idempotent natural transformation e. Then, $F = FE^e$, where E^e is the canonical semifunctor attached to e.*

Proof. Consider the composite $F = FE^e$. Then, for every $X \in \mathcal{C}$, $f : X \to Y$ in $\mathcal{C}$, we have $FE^eX = FX$ and $FE^e(f) = F(f \circ e_X) = Ff \circ Fe_X = Ff \circ \mathrm{Id}_{FX} = Ff$, as $Fe_X = \mathrm{Id}_{FX}$. □

Corollary 4.89. *Let $F \dashv_{\mathrm{s}} G$ be a semiadjunction.*

i) If F is semiseparable with associated idempotent $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$, then $G \cong_{\mathrm{s}} E^eG$.

ii) If G is semiseparable with associated idempotent $e : \mathrm{Id}_{\mathcal{D}} \to \mathrm{Id}_{\mathcal{D}}$, then $F \cong_{\mathrm{s}} E^eF$.

Proof. *i*). By Corollary 4.88 we have that $FE^e \dashv_{\mathrm{s}} G$. By Corollary 4.37 *i*) we know that $FE^e \dashv_{\mathrm{s}} E^eG$ is a semiadjunction. Thus, by Proposition 4.26 $G \cong_{\mathrm{s}} E^eG$.
ii). It follows similarly from Corollary 4.37 *ii*). □

Example 4.90. Cf. [21, Example 7.3]. We come back to Example 4.38. Let $\mathcal{C}$ be a category and consider an idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$. Since for every object $X \in \mathcal{C}_e$, $HL(X) = X$, and for every morphism $\bar{f}$ in $\mathcal{C}_e$, $HL\bar{f} = H(e_Y \circ f) = He_Y \circ Hf = \mathrm{Id}_{HY} \circ \bar{f} = \bar{f}$, we have $HL = \mathrm{Id}_{\mathcal{C}_e}$, hence $\eta = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}_e}}$. Thus, there exists a seminatural transformation $\nu = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}_e}} : HL \to \mathrm{Id}_{\mathcal{C}_e}$ such that $\nu \circ \eta = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}_e}}$ and $\eta \circ \nu = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}_e}} = HL\mathrm{Id}$. Then, by Theorem 4.80 and Theorem 4.74, L is a separable and naturally semifull semifunctor, whence semifully faithful.

Recall that $E^e = LH$. We know that L is semifully faithful and H is naturally full by Lemma 2.28, hence in particular they are naturally semifull semifunctors. Thus, by Proposition 4.72 the semifunctor E^e is naturally semifull, whence semiseparable. By Proposition 4.87 its associated idempotent $\alpha : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ such that $E^e\alpha = E^e\mathrm{Id}$ is given for any X in $\mathcal{C}$ by $\alpha_X = \mathcal{P}_{X,X}(E^e\mathrm{Id}_X) = \mathcal{P}_{X,X}(e_X) : X \to X$, and then we have that $\alpha_X = \mathcal{P}_{X,X}(e_X) = \mathcal{P}_{X,X}(E^e e_X \circ e_X) = e_X \circ \mathcal{P}_{X,X}(e_X) = e_X \circ \alpha_X = E^e\alpha_X = E^e\mathrm{Id}_X = e_X$. Moreover, from Proposition 4.83 E^e results to be separable if, and only if, it is faithful. By Proposition 4.81 we know that E^e is separable if, and only if, $e = \mathrm{Id}$. Thus, E^e is (semifully) faithful if, and only if, it is the identity functor on $\mathcal{C}$.

4.6.1 Semiseparable semiadjoints

The following Rafael-type Theorem characterizes semiseparability for semifunctors that are part of a semiadjunction.

Theorem 4.91. [21, Theorem 6.6] *Let $F \dashv_{\mathrm{s}} G : \mathcal{D} \to \mathcal{C}$ be a semiadjunction, with unit η and counit ϵ. Then,*

i) F is semiseparable if, and only if, there exists a natural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ which satisfies one of the following equivalent conditions:

(1) $\eta \circ \nu \circ \eta = \eta$;

(2) $F\nu \circ F\eta = F\mathrm{Id}$.

ii) G is semiseparable if, and only if, there exists a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ which satisfies one of the following equivalent conditions:

(1) $\epsilon \circ \gamma \circ \epsilon = \epsilon$;

(2) $G\epsilon \circ G\gamma = G\mathrm{Id}$.

Proof. We just prove *i*) as *ii*) follows by duality. Assume that F is semiseparable and let $\mathcal{P}$ be the associated natural transformation such that $\mathcal{F}\circ\mathcal{P}\circ\mathcal{F}=\mathcal{F}$. By Proposition 4.24 we define $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ by $\nu_X := \mathcal{P}_{GFX,X}(\epsilon_{FX}) : GFX \to X$. For every $X \in \mathcal{C}$, we have

$$\begin{aligned}\eta_X\circ\nu_X\circ\eta_X &= \eta_X\circ\mathcal{P}_{GFX,X}(\epsilon_{FX})\circ\eta_X = \eta_X\circ\mathcal{P}_{X,X}(\epsilon_{FX}\circ F\eta_X) = \eta_X\circ\mathcal{P}_{X,X}(F\mathrm{Id}_X)\\ &= GF\mathcal{P}_{X,X}(F\mathrm{Id}_X)\circ\eta_X = G((\mathcal{F}_{X,X}\circ\mathcal{P}_{X,X}\circ\mathcal{F}_{X,X})(\mathrm{Id}_X))\circ\eta_X\\ &\overset{(4.13)}{=} G\mathcal{F}_{X,X}(\mathrm{Id}_X)\circ\eta_X = GF(\mathrm{Id}_X)\circ\eta_X = \eta_X,\end{aligned}$$

hence condition (1) is satisfied. Conversely, assume that there exists a natural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\eta\circ\nu\circ\eta = \eta$, and for any $f \in \mathrm{Hom}_{\mathcal{D}}(FX, FY)$ define $\mathcal{P}_{X,Y}(f) := \nu_Y \circ Gf \circ \eta_X$ as in Proposition 4.24. For every $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, we have

$$\begin{aligned}(\mathcal{F}_{X,Y}\circ\mathcal{P}_{X,Y}\circ\mathcal{F}_{X,Y})(f) &= F(\mathcal{P}_{X,Y}(F(f))) = F(\nu_Y\circ GF(f)\circ\eta_X) = F(\nu_Y\circ\eta_Y\circ f)\\ &= F(\mathrm{Id}_Y\circ\nu_Y\circ\eta_Y\circ f) = F\mathrm{Id}_Y\circ F(\nu_Y\circ\eta_Y\circ f) = \epsilon_{FY}\circ F\eta_Y\circ F(\nu_Y\circ\eta_Y\circ f)\\ &= \epsilon_{FY}\circ F(\eta_Y\circ\nu_Y\circ\eta_Y\circ f) = \epsilon_{FY}\circ F(\eta_Y\circ f) = \epsilon_{FY}\circ F\eta_Y\circ Ff\\ &= F\mathrm{Id}_Y\circ Ff = Ff = \mathcal{F}_{X,Y}(f),\end{aligned}$$

so F is semiseparable. Finally, we prove that (1) and (2) are equivalent.
(1) $\Rightarrow$ (2). We have $F\nu\circ F\eta = F(\mathrm{Id}\circ\nu\circ\eta) = F\mathrm{Id}\circ F\nu\circ F\eta = \epsilon F\circ F\eta\circ F\nu\circ F\eta \overset{\text{(i)}}{=} \epsilon F\circ F\eta = F\mathrm{Id}$.
(2) $\Rightarrow$ (1). By naturality of η, we have $\eta\circ\nu\circ\eta = \eta\circ(\nu\circ\eta) = GF(\nu\circ\eta)\circ\eta = G(F\nu\circ F\eta)\circ\eta = GF\mathrm{Id}\circ\eta = \eta$. □

4.6.2 Idempotent completion and semiadjoint triples

The notions considered so far for a semifunctor F are related to the corresponding functorial notions for its completion $F^\natural$. The following result is a semifunctorial version of [5, Proposition 2.1 and Corollary 2.2].

Proposition 4.92. [21, Proposition 6.7] *Let $F : \mathcal{C} \to \mathcal{D}$ be a semifunctor. Then,*

i) *F is faithful if, and only if, $F^\natural$ is a faithful functor;*

ii) *F is semifull if, and only if, $F^\natural$ is a full functor;*

iii) *F is semiseparable if, and only if, $F^\natural$ is a semiseparable functor.*

Proof. *i*). It follows from the fact that $F^\natural f = Ff$, for any morphism $f : (C,c) \to (C',c')$ in $\mathcal{C}^\natural$, i.e. for any morphism $f : C \to C'$ in $\mathcal{C}$ such that $f = c'\circ f\circ c$.
ii). Assume that F is semifull. Let $f : F^\natural(C,c) \to F^\natural(C',c')$ be a morphism in $\mathcal{D}^\natural$, i.e. a morphism $f : FC \to FC'$ in $\mathcal{D}$ such that $f = Fc'\circ f\circ Fc$. Since F is semifull, there exists a morphism $g : C \to C'$ in $\mathcal{C}$ such that $F(g) = F\mathrm{Id}_{C'}\circ f\circ F\mathrm{Id}_C$. Set $g' := c'\circ g\circ c : C \to C'$. Note that $c'g'c = c'(c'gc)c = c'gc = g'$, hence $g' : (C,c) \to (C',c')$ is a morphism in $\mathcal{C}^\natural$. Then, $F^\natural(g') = F(g') = F(c'gc) = F(c')\circ F(g)\circ F(c) = Fc'\circ F\mathrm{Id}_{C'}\circ f\circ F\mathrm{Id}_C\circ Fc = Fc'\circ f\circ Fc = f$. Thus, $F^\natural$ is full. Conversely, assume that $F^\natural$ is full. Let $f : FC \to FC'$ be a morphism in $\mathcal{D}$. Consider the morphism $f' : (FC, F\mathrm{Id}_C) \to (FC', F\mathrm{Id}_{C'})$ in $\mathcal{D}^\natural$ given by $f' = F\mathrm{Id}_{C'}\circ f\circ F\mathrm{Id}_C$. Then, there is a morphism $g : (C,\mathrm{Id}_C) \to (C',\mathrm{Id}_{C'})$ in $\mathcal{C}^\natural$ (i.e. a morphism $g : C \to C'$ in $\mathcal{C}$) such that $F^\natural g = f'$. Thus, we have $F(g) = F^\natural(g) = f' = F\mathrm{Id}_{C'}\circ f\circ F\mathrm{Id}_C$, hence F is semifull.
iii). If F is semiseparable, then the proof of the fact that $F^\natural$ is semiseparable is the same as

in [5, Corollary 2.2] for the functorial case. Conversely, assume that $F^\natural$ is semiseparable. Then, there is a natural transformation $\mathcal{P}^{F^\natural} : \mathrm{Hom}_{\mathcal{D}^\natural}(F^\natural -, F^\natural -) \to \mathrm{Hom}_{\mathcal{C}^\natural}(-,-)$ such that $\mathcal{F}^{F^\natural}\mathcal{P}^{F^\natural}\mathcal{F}^{F^\natural} = \mathcal{F}^{F^\natural}$. Define $\mathcal{P}^F : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ by $\mathcal{P}^F_{C,C'}(g) = \mathcal{P}^{F^\natural}_{C,C'}(g)$, for every $g : FC \to FC'$ in $\mathcal{D}$, i.e. for every $g : (FC, \mathrm{Id}_{FC}) \to (FC', \mathrm{Id}_{FC'})$ in $\mathcal{D}^\natural$. Thus, $\mathcal{P}^F_{C,C'}(g)$ is a morphism in $\mathcal{C}$ and, by naturality of $\mathcal{P}^{F^\natural}$, also $\mathcal{P}^F$ is a natural transformation. Moreover, for any $f : C \to C'$ in $\mathcal{C}$, we have $\mathcal{F}^F_{C,C'}\mathcal{P}^F_{C,C'}\mathcal{F}^F_{C,C'}(f) = \mathcal{F}^{F^\natural}_{C,C'}\mathcal{P}^{F^\natural}_{C,C'}\mathcal{F}^{F^\natural}_{C,C'}(f) = \mathcal{F}^{F^\natural}_{C,C'}(f) = \mathcal{F}^F_{C,C'}(f)$. □

Corollary 4.93. [21, Corollary 6.8] *Let $F : \mathcal{C} \to \mathcal{D}$ be a semifunctor. Then,*

i) F is separable if, and only if, $F^\natural$ is a separable functor;

ii) F is naturally semifull if, and only if, $F^\natural$ is a naturally full functor;

iii) F is semifully faithful if, and only if, $F^\natural$ is a fully faithful functor.

Proof. It follows from Proposition 4.92 and Proposition 4.83. □

Remark 4.94. Proposition 4.79 can be seen as a consequence of Corollary 4.93. In fact, by [7, Remark 2.2 (3)] a functor is fully faithful if, and only if, it is separable and naturally full, so by Corollary 4.93 a semifunctor is semifully faithful if, and only if, it is separable and naturally semifull.

As a consequence, in the following corollary we retrieve [5, Proposition 2.1 and Corollary 2.2] which follow straightforwardly from Proposition 4.92 and Corollary 4.93 in case $F : \mathcal{C} \to \mathcal{D}$ is a functor.

Corollary 4.95. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then,*

i) F is faithful if, and only if, so is $F^\natural$;

ii) F is full if, and only if, so is $F^\natural$;

iii) F is fully faithful if, and only if, so is $F^\natural$;

iv) F is semiseparable if, and only if, so is $F^\natural$;

v) F is separable if, and only if, so is $F^\natural$;

vi) F is naturally full if, and only if, so is $F^\natural$.

Proof. *i*). It follows from Proposition 4.92 *i*).
ii). It follows from Proposition 4.58 and Proposition 4.92 *ii*).
iii). It follows from Corollary 4.93 *iii*) as a fully faithful functor is a semifully faithful semifunctor such that $F\mathrm{Id} = \mathrm{Id}_F$.
iv). It follows from Proposition 4.92 *iii*).
v). It follows from Corollary 4.93 *i*).
vi). It follows from Corollary 4.93 *ii*) and Proposition 4.70. □

Corollary 4.96. *Let $F : \mathcal{C} \to \mathcal{D}$ be a separable functor. Then, $F^\natural$ is Maschke, dual Maschke and conservative.*

Proof. If F is separable then $F^\natural$ is separable by Corollary 4.95, so by Corollary 2.18 $F^\natural$ is Maschke, dual Maschke and conservative. □

The following is a semifunctorial analogue of Proposition 2.41. In particular the semifully faithful case is a semifunctorial analogue of [19, Proposition 3.4.2].

Proposition 4.97. [21, Proposition 6.10] *Let $F \dashv_{\mathrm{s}} G \dashv_{\mathrm{s}} H : \mathcal{C} \to \mathcal{D}$ be a semiadjoint triple of semifunctors. Then, F is semiseparable (resp., separable, naturally semifull, semifully faithful) if, and only if, so is H.*

Proof. Given a semiadjoint triple $F \dashv_{\mathrm{s}} G \dashv_{\mathrm{s}} H : \mathcal{C} \to \mathcal{D}$, by Theorem 4.21 we obtain an adjoint triple $F^{\natural} \dashv G^{\natural} \dashv H^{\natural} : \mathcal{C}^{\natural} \to \mathcal{D}^{\natural}$ of functors. By Proposition 2.41 we know that $F^{\natural}$ is semiseparable (resp., separable, naturally full) if, and only if, so is $H^{\natural}$. Thus, by Proposition 4.92 and Corollary 4.93 we have that the semifunctor F is semiseparable (resp., separable, naturally semifull) if, and only if, so is H. As a consequence of Proposition 4.79, by combining the previous separable and naturally semifull cases, we get that F is semifully faithful if, and only if, so is H. □

4.6.3 (Co)separable semi(co)monads

In this subsection we show a semifunctorial version of Lemma 2.43. We recall from [52] the notions of *semimonad* and *semicomonad*. The theory of semi(co)monads finds applications e.g. in the categorical description of non-stable models of linear logic [51].

A *semimonad* $(\top, m : \top\top \to \top, \eta : \mathrm{Id}_{\mathcal{C}} \to \top)$ on a category $\mathcal{C}$ is the datum of a semifunctor $\top : \mathcal{C} \to \mathcal{C}$, a seminatural transformation $m : \top\top \to \top$, and a natural transformation $\eta : \mathrm{Id}_{\mathcal{C}} \to \top$, satisfying the following conditions for every $C \in \mathcal{C}$:

i) $m_C \circ \top(\eta_C) = m_C \circ \eta_{\top C} = \top(\mathrm{Id}_C)$,

ii) $m_C \circ \top(m_C) = m_C \circ m_{\top C}$.

A monad $(\top, m, \eta)$ on $\mathcal{C}$ is then a semimonad where $\top$ is a functor. Note that η is also a seminatural transformation. A *semimonad morphism* (F, μ) between semimonads $(\top, m : \top\top \to \top, \eta : \mathrm{Id}_{\mathcal{C}} \to \top)$ and $(\top', m' : \top'\top' \to \top', \eta' : \mathrm{Id}_{\mathcal{D}} \to \top')$ consists of a semifunctor $F : \mathcal{C} \to \mathcal{D}$ and a seminatural transformation $\mu : \top' F \to F\top$ such that, for every $C \in \mathcal{C}$, one has

i) $\mu_C \circ \eta'_{FC} = F\top(\mathrm{Id}_C) \circ F\eta_C$,

ii) $\mu_C \circ m'_{FC} = Fm_C \circ \mu_{\top C} \circ \top'\mu_C$.

The notions of *semicomonad* and *semicomonad morphism* are defined dually. As shown in [52, Theorem 6.9] any semiadjunction $F \dashv_{\mathrm{s}} G : \mathcal{D} \to \mathcal{C}$, with unit η and counit ϵ, gives rise to a semicomonad $(\bot : \mathcal{D} \to \mathcal{D}, \Delta, \epsilon)$ on $\mathcal{D}$ defined by $\bot = FG$, $\Delta = F\eta G \circ FG\mathrm{Id}$, and dually to a semimonad $(\top : \mathcal{C} \to \mathcal{C}, m, \eta)$ on $\mathcal{C}$ given by $\top = GF$, $m = GF\mathrm{Id} \circ G\epsilon F$.

Given a category $\mathcal{C}$, we say that $(\top, m : \top\top \to \top, \eta : \mathrm{Id}_{\mathcal{C}} \to \top)$ is a **separable semimonad** on $\mathcal{C}$ if there exists a natural transformation $\sigma : \top \to \top\top$ such that $m \circ \sigma = \top\mathrm{Id}$ and $\top m \circ \sigma\top = \sigma \circ m = m\top \circ \top\sigma$. Dually, $(\bot, \Delta : \bot \to \bot\bot, \epsilon : \bot \to \mathrm{Id}_{\mathcal{C}})$ is a **coseparable semicomonad** on a category $\mathcal{C}$ if there exists a natural transformation $\tau : \bot\bot \to \bot$ satisfying $\tau \circ \Delta = \bot\mathrm{Id}$ and $\bot\tau \circ \Delta\bot = \Delta \circ \tau = \tau\bot \circ \bot\Delta$.

Then, we can prove the announced result that extends Lemma 2.43 to semiseparable semifunctors that are part of a semiadjunction.

Lemma 4.98. *Let $F \dashv_{\mathrm{s}} G : \mathcal{D} \to \mathcal{C}$ be a semiadjunction with unit η and counit ϵ.*

i) If G is semiseparable, then $(GF, GF\mathrm{Id} \circ G\epsilon F, \eta)$ is a separable semimonad.

ii) If F is semiseparable, then $(FG, F\eta G \circ FG\mathrm{Id}, \epsilon)$ is a coseparable semicomonad.

Proof. *i)*. Assume G is a semiseparable semifunctor. Then, by Theorem 4.91 there is a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $G\epsilon \circ G\gamma = G\mathrm{Id}$. Set

$$\sigma := G\gamma F \circ GF\mathrm{Id} : GF \to GFGF.$$

It follows that $GF\mathrm{Id} \circ G\epsilon F \circ \sigma = GF\mathrm{Id} \circ G\epsilon F \circ G\gamma F \circ GF\mathrm{Id} = GF\mathrm{Id} \circ (G\epsilon \circ G\gamma)F \circ GF\mathrm{Id} = GF\mathrm{Id} \circ G\mathrm{Id}_F \circ GF\mathrm{Id} = GF\mathrm{Id}$. From the naturality of ϵ and γ, we have $\gamma \circ \epsilon = \epsilon FG \circ FG\gamma$ and $\gamma \circ \epsilon = FG\epsilon \circ \gamma FG$, respectively, hence

$$\begin{aligned}
&GF(GF\mathrm{Id} \circ G\epsilon F) \circ \sigma GF = GF(GF\mathrm{Id} \circ G\epsilon F) \circ (G\gamma F \circ GF\mathrm{Id})GF \\
&= GFGF\mathrm{Id} \circ GFG\epsilon F \circ G\gamma FGF \circ GF\mathrm{Id}_{GF} = GFGF\mathrm{Id} \circ G\gamma F \circ G\epsilon F \circ GF\mathrm{Id}_{GF} \\
&= GFGF\mathrm{Id} \circ G\epsilon FGF \circ GFG\gamma F \circ GF\mathrm{Id}_{GF} = GFGF\mathrm{Id} \circ G\epsilon FGF \circ GFG\gamma F \\
&= GFGF\mathrm{Id} \circ G\gamma F \circ G\epsilon F = GF\mathrm{Id}_{GF} \circ GFGF\mathrm{Id} \circ G\gamma F \circ G\epsilon F \\
&= GF\mathrm{Id}_{GF} \circ G\gamma F \circ GF\mathrm{Id} \circ G\epsilon F = GF\mathrm{Id}_{GF} \circ G\gamma F \circ G\epsilon F \circ GFGF\mathrm{Id} \\
&= GF\mathrm{Id}_{GF} \circ G\epsilon FGF \circ GFG\gamma F \circ GFGF\mathrm{Id} = (GF\mathrm{Id} \circ G\epsilon F)GF \circ GF(G\gamma F \circ GF\mathrm{Id}) \\
&= (GF\mathrm{Id} \circ G\epsilon F)GF \circ GF\sigma.
\end{aligned}$$

Thus, $GF(GF\mathrm{Id} \circ G\epsilon F) \circ \sigma GF = (GF\mathrm{Id} \circ G\epsilon F)GF \circ GF\sigma$. Moreover, we have $(GF\mathrm{Id} \circ G\epsilon F)GF \circ GF\sigma = (GF\mathrm{Id} \circ G\epsilon F)GF \circ GF(G\gamma F \circ GF\mathrm{Id}) = GF\mathrm{Id}_{GF} \circ G\epsilon FGF \circ GFG\gamma F \circ GFGF\mathrm{Id} = GF\mathrm{Id}_{GF} \circ G\gamma F \circ G\epsilon F \circ GFGF\mathrm{Id} = GF\mathrm{Id}_{GF} \circ G\gamma F \circ GF\mathrm{Id} \circ G\epsilon F = GF\mathrm{Id}_{GF} \circ GFGF\mathrm{Id} \circ G\gamma F \circ G\epsilon F = GFGF\mathrm{Id} \circ G\gamma F \circ G\epsilon F = G\gamma F \circ GF\mathrm{Id} \circ GF\mathrm{Id} \circ G\epsilon F = \sigma \circ GF\mathrm{Id} \circ G\epsilon F$. Then,

$$GF(GF\mathrm{Id} \circ G\epsilon F) \circ \sigma GF = (GF\mathrm{Id} \circ G\epsilon F)GF \circ GF\sigma = \sigma \circ GF\mathrm{Id} \circ G\epsilon F,$$

so the semimonad $(GF, GF\mathrm{Id} \circ G\epsilon F, \eta)$ is separable.
ii). It follows dually. □

4.7 Examples

In this section we provide examples of semifull, naturally semifull, (semi)separable and semifully faithful semifunctors, see [21, Section 7].

4.7.1 The forgetful semifunctor

See [21, Example 7.1], [5, Example 2.13]. Let $\mathcal{C}$ be a category with idempotent completion $\mathcal{C}^\natural$. Recall from Subsection 4.1.1 that the canonical functor $\iota_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^\natural$ is given by $X \mapsto (X, \mathrm{Id}_X)$, $[f : X \to X'] \mapsto [f : (X, \mathrm{Id}_X) \to (X', \mathrm{Id}_{X'})]$ and the forgetful semifunctor $\upsilon_{\mathcal{C}} : \mathcal{C}^\natural \to \mathcal{C}$ maps an object $(X, e) \in \mathcal{C}^\natural$ to the underlying object X and a morphism $f : (X, e) \to (X', e')$ to the underlying morphism $\upsilon_{\mathcal{C}} f : X \to X'$ in $\mathcal{C}$ such that $e' \circ \upsilon_{\mathcal{C}} f \circ e = \upsilon_{\mathcal{C}} f$. It is indeed a semifunctor as $\upsilon_{\mathcal{C}}(\mathrm{Id}_{(X,e)}) = e \neq \mathrm{Id}_X$ in general.

By [52, Theorem 2.10] the Karoubi envelope functor $\kappa : \mathsf{Cat}_{\mathrm{s}} \to \mathsf{Cat}$, defined by $\kappa(\mathcal{C}) = \mathcal{C}^\natural$, $\kappa(F) = F^\natural$, for any category $\mathcal{C}$ and any semifunctor $F : \mathcal{C} \to \mathcal{D}$, is the right adjoint of the inclusion functor $i : \mathsf{Cat} \to \mathsf{Cat}_{\mathrm{s}}$. Then, $\iota_{\mathcal{C}}$ and $\upsilon_{\mathcal{C}}$ result to be the $\mathcal{C}$-components of the unit and of the counit for the adjunction $i \dashv \kappa$, respectively. Moreover, $\upsilon_{\mathcal{C}} \dashv_{\mathrm{s}} \iota_{\mathcal{C}}$ and $\iota_{\mathcal{C}} \dashv_{\mathrm{s}} \upsilon_{\mathcal{C}}$ are semiadjunctions, cf. [50, Example 6], thus $\upsilon_{\mathcal{C}}$ and $\iota_{\mathcal{C}}$ are Frobenius semifunctors. The component $(\eta_\mathcal{C})_{(C,c)} : (C,c) \to \iota_\mathcal{C}\upsilon_\mathcal{C}(C,c) = (C, \mathrm{Id}_C)$ of the unit $\eta_\mathcal{C}$ of $\upsilon_\mathcal{C} \dashv_\mathrm{s} \iota_\mathcal{C}$ is defined by setting $\upsilon_\mathcal{C}((\eta_\mathcal{C})_{(C,c)}) := c : C \to C$ while the counit is $\epsilon_\mathcal{C} := \mathrm{Id}_{\mathrm{Id}_\mathcal{C}} : \upsilon_\mathcal{C}\iota_\mathcal{C} = \mathrm{Id}_\mathcal{C} \to \mathrm{Id}_\mathcal{C}$. The unit of $\iota_\mathcal{C} \dashv_\mathrm{s} \upsilon_\mathcal{C}$ is $\epsilon_\mathcal{C} := \mathrm{Id}_{\mathrm{Id}_\mathcal{C}} : \mathrm{Id}_\mathcal{C} \to \mathrm{Id}_\mathcal{C} = \upsilon_\mathcal{C}\iota_\mathcal{C}$ while the component $(\nu_\mathcal{C})_{(C,c)} : \iota_\mathcal{C}\upsilon_\mathcal{C}(C,c) = (C, \mathrm{Id}_C) \to (C,c)$ of the counit is given by $\upsilon_\mathcal{C}((\nu_\mathcal{C})_{(C,c)}) := c : C \to C$. Note that

$$\upsilon_\mathcal{C}\left((\nu_\mathcal{C})_{(C,c)} \circ (\eta_\mathcal{C})_{(C,c)}\right) = \upsilon_\mathcal{C}((\nu_\mathcal{C})_{(C,c)}) \circ \upsilon_\mathcal{C}((\eta_\mathcal{C})_{(C,c)}) = c \circ c = c = \upsilon_\mathcal{C}(\mathrm{Id}_{(C,c)}),$$

hence $\nu_\mathcal{C} \circ \eta_\mathcal{C} = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}^\natural}}$, thus by Theorem 4.80 it follows that $\upsilon_\mathcal{C}$ is a separable semifunctor. Moreover,

$$\upsilon_\mathcal{C}\left((\eta_\mathcal{C})_{(C,c)} \circ (\nu_\mathcal{C})_{(C,c)}\right) = \upsilon_\mathcal{C}((\eta_\mathcal{C})_{(C,c)}) \circ \upsilon_\mathcal{C}((\nu_\mathcal{C})_{(C,c)}) = c \circ c = c = \upsilon_\mathcal{C}\iota_\mathcal{C}\upsilon_\mathcal{C}\mathrm{Id}_{(C,c)},$$

so $(\eta_\mathcal{C})_{(C,c)} \circ (\nu_\mathcal{C})_{(C,c)} = \iota_\mathcal{C}\upsilon_\mathcal{C}\mathrm{Id}_{(C,c)}$ and hence it holds $\eta_\mathcal{C} \circ \nu_\mathcal{C} = \iota_\mathcal{C}\upsilon_\mathcal{C}\mathrm{Id}$. By Theorem 4.74 it follows that $\upsilon_\mathcal{C}$ is also a naturally semifull semifunctor, hence semifully faithful by Proposition 4.79.

4.7.2 Semi-product semifunctor

See [21, Example 7.2]. Recall from [52, Definition 4.3] that a binary *semi-product* of objects A, B in a category $\mathcal{C}$ consists of an object $A \times B$ in $\mathcal{C}$, an arrow $\pi_A : A \times B \to A$, an arrow $\pi_B : A \times B \to B$, an arrow $\langle f, g\rangle : C \to A \times B$, for each $f : C \to A$, $g : C \to B$ in $\mathcal{C}$, such that

i) $\pi_A \circ \langle f, g\rangle = f$,

ii) $\pi_B \circ \langle f, g\rangle = g$,

iii) $\langle f \circ h, g \circ h\rangle = \langle f, g\rangle \circ h$, for any morphism $h : D \to C$ in $\mathcal{C}$.

A binary semi-product is a binary product if, and only if, $\langle \pi_A, \pi_B\rangle = \mathrm{Id}_{A\times B}$. If $\mathcal{C}$ is a category with semi-products for all pairs of objects A, B, let $\times_\mathcal{C} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be the semifunctor given by

$$(A, B) \mapsto A \times B \quad \text{and} \quad (f, g) \mapsto f \times g := \langle f \circ \pi_A, g \circ \pi_B\rangle.$$

It is indeed a semifunctor as $\mathrm{Id}_A \times \mathrm{Id}_B = \langle \pi_A, \pi_B\rangle$. Moreover, it is a right semiadjoint of the functor $\Delta_\mathcal{C} : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$, $A \mapsto (A, A)$, $f \mapsto (f, f)$, for any morphism $f : A \to A'$ in $\mathcal{C}$. They actually form a semiadjunction $\Delta_\mathcal{C} \dashv_\mathrm{s} \times_\mathcal{C}$ with unit the seminatural transformation $\eta : \mathrm{Id}_\mathcal{C} \to \times_\mathcal{C}\Delta_\mathcal{C}$, given on components by $\eta_C = \langle \mathrm{Id}_C, \mathrm{Id}_C\rangle : C \to C \times C$, so that $\pi_C \circ \langle \mathrm{Id}_C, \mathrm{Id}_C\rangle = \mathrm{Id}_C$, for every C in $\mathcal{C}$, and counit the seminatural transformation $\epsilon : \Delta_\mathcal{C}\times_\mathcal{C} \to \mathrm{Id}_{\mathcal{C}\times\mathcal{C}}$, given on components by $\epsilon_{(A,B)} = (\pi_A, \pi_B) : (A \times B, A \times B) \to (A, B)$. We now study the properties of the semifunctor $\times_\mathcal{C}$. Note that in general $\times_\mathcal{C}$ is not faithful by Proposition 4.65 as the components $\epsilon_{(A,B)} = (\pi_A, \pi_B)$ of the counit are not epimorphisms for every (A, B) in $\mathcal{C} \times \mathcal{C}$. For instance, in Set consider a nonempty set B and the binary product $\emptyset \times B$, which is in particular a binary semi-product. Then, the map $\pi_B : \emptyset \times B = \emptyset \to B$ is the empty function from the emptyset into B, but it is not an epimorphism, cf. [19, page 40]. As a consequence, $\times_\mathcal{C}$ is not separable in general. By Proposition 4.65 we know that $\times_\mathcal{C}$ is semifull if, and only if, the component $\epsilon_{(A,B)} = (\pi_A, \pi_B)$ is an (A, B)-semisplit-mono for every (A, B) in $\mathcal{C} \times \mathcal{C}$, i.e. if, and only if, for every (A, B) in $\mathcal{C} \times \mathcal{C}$ there exists a morphism $\gamma_{(A,B)} = (\gamma_1, \gamma_2) : (A, B) \to (A \times B, A \times B)$ in $\mathcal{C} \times \mathcal{C}$ such that $\gamma_{(A,B)} \circ \epsilon_{(A,B)} = \Delta_\mathcal{C} \times_\mathcal{C} \mathrm{Id}_{(A,B)}$, i.e. such that $(\gamma_1\pi_A, \gamma_2\pi_B) = (\langle \pi_A, \pi_B\rangle, \langle \pi_A, \pi_B\rangle)$. The latter condition is not satisfied in general. For instance, in Set consider again the binary product $\emptyset \times B = \emptyset$ of the emptyset by a nonempty set B. Since $\emptyset$ is the initial object in Set we have $\pi_\emptyset = \mathrm{Id}_\emptyset$. Moreover, we have that $\langle \pi_\emptyset, \pi_B \rangle = \mathrm{Id}_{\emptyset \times B} = \mathrm{Id}_\emptyset$, so if $\times_{\mathsf{Set}}$ were semifull, then there would exist a morphism $\gamma_{(\emptyset,B)} = (\gamma_1, \gamma_2) : (\emptyset, B) \to (\emptyset, \emptyset)$ in $\mathsf{Set} \times \mathsf{Set}$ such that $(\gamma_1 \mathrm{Id}_\emptyset, \gamma_2 \pi_B) = (\mathrm{Id}_\emptyset, \mathrm{Id}_\emptyset) : \emptyset \to \emptyset$, but such γ_2 does not exist as there can be no map from a nonempty set to $\emptyset$.

4.7.3 The constant semifunctor

See [21, Example 7.2], [52, Subsection 2.3]. Let $\mathbf{1}$ be the category with only one object 1 and an identity morphism Id_1. Any semifunctor $F : \mathbf{1} \to \mathcal{C}$ determines an arrow $F(\mathrm{Id}_1) : F(1) \to F(1)$ which is idempotent. Conversely, any idempotent arrow $e : X \to X$ in $\mathcal{C}$ defines a semifunctor $F^e : \mathbf{1} \to \mathcal{C}$, given by $F^e(1) = X$, $F^e(\mathrm{Id}_1) = e$. In particular we have the functor $F^{\mathrm{Id}_X} : \mathbf{1} \to \mathcal{C}$ given by $F^{\mathrm{Id}_X}(1) = X$, $F^{\mathrm{Id}_X}(\mathrm{Id}_1) = \mathrm{Id}_X$. Hence $F^e = E^e \circ F^{\mathrm{Id}_X}$, where $E^e : \mathcal{C} \to \mathcal{C}$ is the canonical semifunctor. Now, we show that $F^e : \mathbf{1} \to \mathcal{C}$ is a separable semifunctor that is not naturally semifull in general. Note that $\mathrm{Hom}_{\mathbf{1}}(1,1) = \{\mathrm{Id}_1\}$. Consider the associated natural transformation $\mathcal{F}^{F^e}_{1,1} : \mathrm{Hom}_{\mathbf{1}}(1,1) \to \mathrm{Hom}_{\mathcal{C}}(F^e(1), F^e(1))$, $\mathcal{F}^{F^e}_{1,1}(\mathrm{Id}_1) = F^e(\mathrm{Id}_1) = e$, and the map $\mathcal{P}^{F^e}_{1,1} : \mathrm{Hom}_{\mathcal{C}}(F^e(1), F^e(1)) \to \mathrm{Hom}_{\mathbf{1}}(1,1)$, given by $\mathcal{P}^{F^e}_{1,1}(f) = \mathrm{Id}_1$, for any $f : F^e(1) \to F^e(1)$ in $\mathcal{C}$. We have that $\mathcal{P}^{F^e}_{1,1}$ is a natural transformation as for any $f : F^e(1) \to F^e(1)$ in $\mathcal{C}$, $\mathcal{P}^{F^e}_{1,1}(F^e\mathrm{Id}_1 \circ f \circ F^e\mathrm{Id}_1) = \mathcal{P}^{F^e}_{1,1}(e \circ f \circ e) = \mathrm{Id}_1 = \mathrm{Id}_1 \circ \mathcal{P}^{F^e}_{1,1}(f) \circ \mathrm{Id}_1$. Moreover, $(\mathcal{P}^{F^e}_{1,1} \circ \mathcal{F}^{F^e}_{1,1})(\mathrm{Id}_1) = \mathcal{P}^{F^e}_{1,1}(e) = \mathrm{Id}_1$, hence F^e is separable, and in particular semiseparable. Thus, F^e is naturally semifull if, and only if, it is semifull. If F^e were semifull, then it would follow that $e = e \circ f \circ e$ for any $f : F^e(1) \to F^e(1)$ in $\mathcal{C}$, and this does not happen in general (see e.g. Subsection 4.7.5). More generally, given two categories $\mathcal{C}$ and $\mathcal{D}$ and a fixed idempotent arrow $e_D : D \to D$ in $\mathcal{D}$ we can consider the *constant semifunctor* $K : \mathcal{C} \to \mathcal{D}$ given by $K(C) = D$, $K(f) = e_D$, for every object $C \in \mathcal{C}$ and for every morphism $f : C \to C'$ in $\mathcal{C}$. It is clearly not faithful and not even semifull. In fact, for any morphism $g : K(C) = D \to K(C') = D$ in $\mathcal{D}$ we have that $K\mathrm{Id}_{C'} \circ g \circ K\mathrm{Id}_C = e_D \circ g \circ e_D$. If K were semifull then there would exist a morphism $f : C \to C'$ such that $e_D = K(f) = e_D \circ g \circ e_D$, which is not true in general.

4.7.4 Semifunctors associated with a morphism of rings

See [21, Example 7.6]. Let R, S be unital rings and let ${}_R\mathcal{M}$, ${}_S\mathcal{M}$ be the categories of left R-modules and left S-modules, respectively. Consider a morphism of rings $\varphi : R \to S$, that induces the restriction of scalars functor $\varphi_* : {}_S\mathcal{M} \to {}_R\mathcal{M}$ and the extension of scalars functor $\varphi^* := S \otimes_R (-) : {}_R\mathcal{M} \to {}_S\mathcal{M}$. As recalled in Subsection 1.4.1, these functors form an adjunction $\varphi^* \dashv \varphi_*$, with unit η and counit ϵ given by

$$\eta_M = \varphi \otimes_R M : M \to S \otimes_R M,\ m \mapsto 1_S \otimes_R m \quad \text{and} \quad \epsilon_N : S \otimes_R N \to N,\ s \otimes_R n \mapsto sn,$$

for every $M \in {}_R\mathcal{M}$ and $N \in {}_S\mathcal{M}$, respectively. Given an idempotent (semi)natural transformation $e = (e_X)_{X \in {}_R\mathcal{M}} : \mathrm{Id}_{{}_R\mathcal{M}} \to \mathrm{Id}_{{}_R\mathcal{M}}$, by composing φ^* with the canonical semifunctor $E^e : {}_R\mathcal{M} \to {}_R\mathcal{M}$, we get the semifunctor

$$\varphi^*_e := \varphi^* \circ E^e : {}_R\mathcal{M} \to {}_S\mathcal{M}, \quad M \mapsto S \otimes_R M, \quad f \mapsto S \otimes_R f e_M$$

for any $f : M \to M'$ in ${}_R\mathcal{M}$. Consider the semifunctor

$$\varphi^e_* := E^e \circ \varphi_* : {}_S\mathcal{M} \to {}_R\mathcal{M}, \quad N \mapsto \varphi_*(N), \quad g \mapsto e_{\varphi_*(N')} \circ \varphi_*(g),$$

for any $g : N \to N'$ in ${}_S\mathcal{M}$. By Corollary 4.37 *i)* we know that φ_e^* and φ_*^e form a semiadjunction $\varphi_e^* \dashv_{\mathrm{s}} \varphi_*^e : {}_S\mathcal{M} \to {}_R\mathcal{M}$, with unit $\eta^e := E^e\eta_{E^e} \circ e$ and counit $\epsilon^e := \epsilon \circ \varphi^* e_{\varphi_*}$. By Theorem 4.80, φ_*^e is separable if, and only if, ϵ^e is a natural split-epi, i.e. there exists a seminatural transformation $\gamma : \mathrm{Id}_{{}_S\mathcal{M}} \to \varphi_e^*\varphi_*^e$ such that $\epsilon^e \circ \gamma = \mathrm{Id}_{\mathrm{Id}_{{}_S\mathcal{M}}}$, while φ_e^* is separable if, and only if, η^e is a natural split-mono, i.e. there exists a seminatural transformation $\nu : \varphi_*^e\varphi_e^* \to \mathrm{Id}_{{}_R\mathcal{M}}$ such that $\nu \circ \eta^e = \mathrm{Id}_{\mathrm{Id}_{{}_R\mathcal{M}}}$. In particular, if φ_*^e is separable, then by Lemma 4.77 *ii)* φ_* is separable. If φ_e^* is separable, then by Remark 4.82 it results to be $\varphi_e^* = \varphi^*$.

In Proposition 4.100 we study the natural semifullness of φ_*^e and φ_e^*, for which the following lemma will be useful.

Lemma 4.99. [21, Lemma 7.7] *Let $\varphi : R \to S$ be a morphism of rings and let $e = (e_X)_{X \in {}_R\mathcal{M}} : \mathrm{Id}_{{}_R\mathcal{M}} \to \mathrm{Id}_{{}_R\mathcal{M}}$ be an idempotent (semi)natural transformation. Consider the semifunctors φ_e^*, φ_*^e as above. Then,*

i) $1_S = e_{\varphi_*(S)}(\varphi(e_R(1_R)))$ *holds true if, and only if,* $\varphi(r) = e_{\varphi_*(S)}(\varphi(e_R(r)))$ *holds true for every $r \in R$ if, and only if,*

$$r_S^{-1} \circ \varphi = \varphi_*^e\varphi_e^*\mathrm{Id}_R \circ r_S^{-1} \circ \varphi \tag{4.14}$$

holds true, where $r_S : S \otimes_R R \to S$ is the canonical isomorphism $s \otimes_R r \mapsto s\varphi(r)$.

ii) $e_{\varphi_*(S)}$ *is a left S-module morphism if, and only if, $e_{\varphi_*(N)}$ is a left S-module morphism for every $N \in {}_S\mathcal{M}$. The latter condition means that there is an idempotent natural transformation $\alpha : \mathrm{Id}_{{}_S\mathcal{M}} \to \mathrm{Id}_{{}_S\mathcal{M}}$ such that $\varphi_*\alpha = e\varphi_*$. In this case, let $E^\alpha : {}_S\mathcal{M} \to {}_S\mathcal{M}$ be the canonical semifunctor attached to α. Then, $E^\alpha \circ \varphi^* = \varphi^* \circ E^e$ and $\varphi_* \circ E^\alpha = E^e \circ \varphi_*$.*

Proof. i). Assume that $1_S = e_{\varphi_*(S)}(\varphi(e_R(1_R)))$. Then, since e_R, $e_{\varphi_*(S)}$ are left R-module morphisms, we have that $e_{\varphi_*(S)}(\varphi(e_R(r))) = e_{\varphi_*(S)}(\varphi(re_R(1_R))) = e_{\varphi_*(S)}(\varphi(r)\varphi(e_R(1_R))) = \varphi(r)e_{\varphi_*(S)}(\varphi(e_R(1_R))) = \varphi(r)1_S = \varphi(r)$, for every $r \in R$. The converse implication is trivially satisfied. Now we show that the latter condition is also equivalent to (4.14). Indeed, we have that $r_S\varphi_*^e\varphi_e^*\mathrm{Id}_R r_S^{-1}\varphi(r) = r_S\varphi_*^e(S \otimes_R e_R)(\varphi(r) \otimes_R 1_R) = r_S e_{\varphi_*(S\otimes_R R)}\varphi_*(S \otimes_R e_R)(1_S \otimes_R r) = e_{\varphi_*(S)} r_S(1_S \otimes_R e_R(r)) = e_{\varphi_*(S)}(\varphi(e_R(r)))$, for every $r \in R$.
ii). If $e_{\varphi_*(N)}$ is a left S-module morphism for every $N \in {}_S\mathcal{M}$, then clearly $e_{\varphi_*(S)}$ is a left S-module morphism. On the other hand, assume that $e_{\varphi_*(S)}$ is a left S-module morphism. Consider a left S-module N and the left S-module morphism $f_n : S \to N$, $s \mapsto sn$, for $n \in N$. By naturality of e, we have that $e_{\varphi_*(N)}(sn) = (e_{\varphi_*(N)} \circ \varphi_*(f_n))(s) = (\varphi_*(f_n) \circ e_{\varphi_*(S)})(s) = e_{\varphi_*(S)}(s)n$. Since $e_{\varphi_*(S)}$ is left S-linear, we get $e_{\varphi_*(S)}(s)n = se_{\varphi_*(S)}(1_S)n = se_{\varphi_*(N)}(n)$, so $e_{\varphi_*(N)}$ is a left S-module morphism for every $N \in {}_S\mathcal{M}$, which means that there is $\alpha_N : N \to N$ in ${}_S\mathcal{M}$ such that $\varphi_*(\alpha_N) = e_{\varphi_*(N)}$. For any left S-module morphism $f : N \to N'$ we have that $\varphi_*(\alpha_{N'} \circ f) = \varphi_*(\alpha_{N'}) \circ \varphi_*(f) = e_{\varphi_*(N')} \circ \varphi_*(f) = \varphi_*(f) \circ e_{\varphi_*(N)} = \varphi_*(f) \circ \varphi_*(\alpha_N) = \varphi_*(f \circ \alpha_N)$, hence $\alpha_{N'} \circ f = f \circ \alpha_N$ as φ_* is faithful. Moreover, for any $N \in {}_S\mathcal{M}$, $\varphi_*(\alpha_N \circ \alpha_N) = \varphi_*(\alpha_N) \circ \varphi_*(\alpha_N) = e_{\varphi_*(N)} \circ e_{\varphi_*(N)} = e_{\varphi_*(N)} = \varphi_*(\alpha_N)$, thus $\alpha_N^2 = \alpha_N$, so we obtain an idempotent natural transformation $\alpha : \mathrm{Id}_{{}_S\mathcal{M}} \to \mathrm{Id}_{{}_S\mathcal{M}}$. Consider the canonical semifunctor $E^\alpha : {}_S\mathcal{M} \to {}_S\mathcal{M}$ attached to α. We show that $E^\alpha \circ \varphi^* = \varphi^* \circ E^e$ and $\varphi_* \circ E^\alpha = E^e \circ \varphi_*$. In fact, the semifunctor $E^\alpha \circ \varphi^* : {}_R\mathcal{M} \to {}_S\mathcal{M}$ maps $M \mapsto S \otimes_R M$, $[f : M \to M'] \mapsto \alpha_{S\otimes_R M'} \circ (S \otimes_R f) = (S \otimes_R f) \circ \alpha_{S\otimes_R M}$. By naturality of e, we have that $e_{\varphi_*(S\otimes_R M)}(1_S \otimes_R m) = 1_S \otimes_R e_M(m)$. Since $e_{\varphi_*(S\otimes_R M)}$ is a left S-module morphism, we get that $e_{\varphi_*(S\otimes_R M)}(s \otimes_R m) = se_{\varphi_*(S\otimes_R M)}(1_S \otimes_R m) = s(1_S \otimes_R e_M(m)) = s \otimes_R e_M(m) = \varphi_*(S \otimes_R e_M)(s \otimes_R m)$, so $e_{\varphi_*(S\otimes_R M)} = \varphi_*(S \otimes_R e_M)$, i.e. $\varphi_*(\alpha_{S\otimes_R M}) = \varphi_*(S \otimes_R e_M)$, i.e. $\alpha_{S\otimes_R M} = S \otimes_R e_M$ as φ_* is faithful. Thus, $E^\alpha \circ \varphi^* = \varphi^* \circ E^e$. The semifunctor

$\varphi_* \circ E^\alpha : {}_S\mathcal{M} \to {}_R\mathcal{M}$ maps $N \mapsto \varphi_*(N)$, $[f : N \to N'] \mapsto \varphi_*(f \circ \alpha_N) = \varphi_*(f) \circ \varphi_*(\alpha_N)$. Note that $\varphi_*(f) \circ \varphi_*(\alpha_N) = \varphi_*(f) \circ e_{\varphi_*(N)} = e_{\varphi_*(N')} \circ \varphi_*(f)$, hence $\varphi_* \circ E^\alpha = E^e \circ \varphi_*$. □

Proposition 4.100. [21, Proposition 7.8] *Let $\varphi : R \to S$ be a morphism of rings and let $e = (e_X)_{X \in {}_R\mathcal{M}} : \mathrm{Id}_{{}_R\mathcal{M}} \to \mathrm{Id}_{{}_R\mathcal{M}}$ be an idempotent (semi)natural transformation.*

i) If the semifunctor φ_*^e is semifull and $e_{\varphi_*(S)}$ is a left S-module morphism, then φ_*^e is naturally semifull.

ii) If the semifunctor φ_e^* is naturally semifull, then there is ψ in ${}_R\mathrm{Hom}_R(S, R)$ such that

$$\varphi \circ \psi = r_S \circ \varphi_*^e \varphi_e^* \mathrm{Id}_R \circ r_S^{-1}, \tag{4.15}$$

i.e. such that $r_S^{-1} \circ \varphi \circ \psi \circ r_S = \varphi_*^e \varphi_e^* \mathrm{Id}_R$, so $r_S^{-1} \circ \varphi : R \to \varphi_*^e \varphi_e^* R$ is an R-semisplit-epi (cf. Section 4.2) as an R-bimodule map; if in addition (4.14) holds true, then $r_S^{-1} \circ \varphi$ is an (R, R)-semisplit-epi.

On the other hand, assume that $e_{\varphi_*(S)}$ is a left S-module morphism. If there is ψ in ${}_R\mathrm{Hom}_R(S, R)$ such that (4.15) holds true and $1_S = e_{\varphi_*(S)}(\varphi(e_R(1_R)))$ is satisfied, i.e. $r_S^{-1} \circ \varphi$ is an (R, R)-semisplit-epi as an R-bimodule map through $\psi \circ r_S$, then φ_e^* is naturally semifull.

Proof. i). Assume that φ_*^e is semifull and $e_{\varphi_*(S)}$ is a morphism of left S-modules, i.e $e_{\varphi_*(N)}$ is a morphism of left S-modules for any $N \in {}_S\mathcal{M}$ by Lemma 4.99 *ii*). By Proposition 4.65, for every $N \in {}_S\mathcal{M}$, ϵ_N^e is an N-semisplit-mono, i.e. there exists $\gamma_N^e : N \to \varphi_e^* \varphi_*^e N = S \otimes_R \varphi_*(N)$ in ${}_S\mathcal{M}$ such that $\gamma_N^e \circ \epsilon_N^e = \varphi_e^* \varphi_*^e \mathrm{Id}_N$. Then, $\gamma_N^e \circ \epsilon_N \circ (S \otimes_R e_{\varphi_*(N)}) = \gamma_N^e \circ \epsilon_N^e = \varphi_e^* \varphi_*^e \mathrm{Id}_N = S \otimes_R (e_{\varphi_*(N)} \circ e_{\varphi_*(N)} \circ \varphi_*(\mathrm{Id}_N)) = S \otimes_R e_{\varphi_*(N)}$. Thus, for every $n \in N$, we have that $\gamma_N^e(e_{\varphi_*(N)}(n)) = (\gamma_N^e \circ \epsilon_N)(1_S \otimes_R e_{\varphi_*(N)}(n)) = (S \otimes_R e_{\varphi_*(N)})(1_S \otimes_R n) = 1_S \otimes_R e_{\varphi_*(N)}(n)$, so $\gamma_N^e(e_{\varphi_*(N)}(n)) = 1_S \otimes_R e_{\varphi_*(N)}(n)$, for every $n \in N$. By Lemma 4.99 *ii*) there is an idempotent natural transformation $\alpha : \mathrm{Id}_{{}_S\mathcal{M}} \to \mathrm{Id}_{{}_S\mathcal{M}}$ such that $\varphi_* \alpha = e \varphi_*$. So, we have that $\gamma_N^e(\alpha_N(n)) = \gamma_N^e(\varphi_*(\alpha_N)(n)) = \gamma_N^e(e_{\varphi_*(N)}(n)) = 1_S \otimes_R e_{\varphi_*(N)}(n) = 1_S \otimes_R \varphi_*(\alpha_N)(n) = 1_S \otimes_R \alpha_N(n)$, for every $n \in N$. Now, although γ_N^e is not natural a priori, still we can show that $\overline{\gamma}^e : \mathrm{Id}_{{}_S\mathcal{M}} \to \varphi_e^* \varphi_*^e$, defined for every $N \in {}_S\mathcal{M}$ by $\overline{\gamma}_N^e := \gamma_N^e \alpha_N : N \to \varphi_e^* \varphi_*^e N$, $n \mapsto 1_S \otimes_R \alpha_N(n)$, is a natural transformation. In fact, for any left S-module morphism $f : N \to N'$ we have that $(\overline{\gamma}_{N'}^e \circ f)(n) = 1_S \otimes_R \alpha_{N'}(f(n)) = 1_S \otimes_R f(\alpha_N(n)) = (S \otimes_R \varphi_*(f) \varphi_*(\alpha_N))(1_S \otimes_R \alpha_N(n)) = (1_S \otimes_R \varphi_*(f) e_{\varphi_*(N)})(1_S \otimes_R \alpha_N(n)) = (\varphi_e^* \varphi_*^e(f) \circ \overline{\gamma}_N^e)(n)$, for every $n \in N$. Moreover, $(\overline{\gamma}_N^e \circ \epsilon_N^e)(s \otimes_R n) = (\gamma_N^e \circ \alpha_N \circ \epsilon_N)(s \otimes_R e_{\varphi_*(N)}(n)) = (\gamma_N^e \circ \alpha_N \circ \epsilon_N)(s \otimes_R \varphi_*(\alpha_N)(n)) = \gamma_N^e(\alpha_N(s \alpha_N(n))) = s \gamma_N^e(\alpha_N(n)) = s \otimes_R \alpha_N(n) = s \otimes_R e_{\varphi_*(N)}(n) = \varphi_e^* \varphi_*^e \mathrm{Id}_N(s \otimes_R n)$, for every $s \in S$, $n \in N$, so φ_*^e is naturally semifull by Theorem 4.74 *ii*).

ii). Assume that φ_e^* is naturally semifull. Then, by Theorem 4.74 *i*), there exists a seminatural transformation $\nu^e : \varphi_*^e \varphi_e^* \to \mathrm{Id}_{{}_R\mathcal{M}}$ such that $\eta^e \circ \nu^e = \varphi_*^e \varphi_e^* \mathrm{Id}$. Consider the map ψ in ${}_R\mathrm{Hom}_R(S, R)$ given by $\psi(s) = (e_R \circ \nu_R^e \circ r_S^{-1})(s) = (e_R \circ \nu_R^e)(s \otimes_R 1_R)$, where $r_S : S \otimes_R R \to S$ is the canonical isomorphism $s \otimes_R r \mapsto s\varphi(r)$. The right R-linearity of ψ follows from the naturality of ν^e and e. Indeed, consider the left R-module map

$f_r : R \to R$, $r' \mapsto r'r$. For any $s \in S$, $r \in R$, we have that

$$\begin{aligned}\psi(s)r &= e_R(\nu^e_R(s \otimes_R 1_R))r = f_r(e_R(\nu^e_R(s \otimes_R 1_R))) = e_R(f_r(\nu^e_R(s \otimes_R 1_R)))\\ &= (e_R\nu^e_R\varphi^e_*\varphi^*_e(f_r))(s \otimes_R 1_R) = (e_R\nu^e_R e_{\varphi_*(S\otimes_R R)}\varphi_*(S \otimes_R f_r e_R))(s \otimes_R 1_R)\\ &= (e_R\nu^e_R e_{\varphi_*(S\otimes_R R)}\varphi_*(S \otimes_R e_R f_r))(s \otimes_R 1_R)\\ &= (e_R\nu^e_R e_{\varphi_*(S\otimes_R R)}\varphi_*(S \otimes_R e_R)\varphi_*(S \otimes_R f_r))(s \otimes_R 1_R)\\ &= (e_R\nu^e_R\varphi^e_*\varphi^*_e(e_R)\varphi_*(S \otimes_R f_r))(s \otimes_R 1_R) = (e_R e_R\nu^e_R\varphi_*(S \otimes_R f_r))(s \otimes_R 1_R)\\ &= (e_R\nu^e_R\varphi_*(S \otimes_R f_r))(s \otimes_R 1_R) = e_R(\nu^e_R(s \otimes_R r)) = e_R(\nu^e_R(sr \otimes_R 1_R)) = \psi(sr).\end{aligned}$$

Then, for every $s \in S$, we have that

$$\begin{aligned}(\varphi \circ \psi)(s) &= (r_S \circ \eta_R \circ e_R \circ \nu^e_R)(s \otimes_R 1_R) = (r_S \circ \eta^e_R \circ \nu^e_R)(s \otimes_R 1_R)\\ &= (r_S \circ \varphi^e_*\varphi^*_e\mathrm{Id}_R)(s \otimes_R 1_R) = (r_S \circ \varphi^e_*\varphi^*_e\mathrm{Id}_R \circ r_S^{-1})(s),\end{aligned}$$

thus $r_S^{-1} \circ \varphi \circ \psi \circ r_S = \varphi^e_*\varphi^*_e\mathrm{Id}_R$, so $r_S^{-1} \circ \varphi$ is an R-semisplit-epi. Moreover, if (4.14) is satisfied, then $r_S^{-1} \circ \varphi$ is an (R, R)-semisplit-epi.

Conversely, assume that $1_S = e_{\varphi_*(S)}(\varphi(e_R(1_R)))$ (which is equivalent to (4.14) by Lemma 4.99 *i*)) and that there is ψ in ${}_R\mathrm{Hom}_R(S, R)$ such that $\varphi \circ \psi = r_S \circ \varphi^e_*\varphi^*_e\mathrm{Id}_R \circ r_S^{-1}$, i.e. $r_S^{-1} \circ \varphi$ is an (R, R)-semisplit-epi as an R-bimodule map through $\psi \circ r_S$. Define for every M in ${}_R\mathcal{M}$, $\nu^e_M : \varphi^e_*\varphi^*_e M = \varphi_*(S \otimes_R M) \to M$ by $\nu^e_M(s \otimes_R m) = \psi(s)e_M(m)$, for any $m \in M$ and $s \in S$. Note that for any $M \in {}_R\mathcal{M}$, ν^e_M is a morphism of left R-modules as so is ψ. We get a natural transformation $\nu^e : \varphi^e_*\varphi^*_e \to \mathrm{Id}_{{}_R\mathcal{M}}$ as for any morphism $f : M \to M'$ in ${}_R\mathcal{M}$ we have that

$$\begin{aligned}(f \circ \nu^e_M)(s \otimes_R m) &= f(\psi(s)e_M(m)) = \psi(s)f(e_M(m)) = \psi(s)f(e_M(e_M(m)))\\ &= \psi(s)e_{M'}(f(e_M(m))) = \psi(s)e_{M'}(e_{M'}(f(e_M(m))))\\ &= e_{M'}(\psi(s)e_{M'}(f(e_M(m)))) = e_{M'}(\nu^e_{M'}(s \otimes_R f(e_M(m))))\\ &= (\nu^e_{M'} \circ e_{\varphi_*(S\otimes_R M')} \circ (S \otimes_R f e_M))(s \otimes_R m)\\ &= (\nu^e_{M'} \circ \varphi^e_*\varphi^*_e f)(s \otimes_R m),\end{aligned}$$

for every $m \in M$, $s \in S$. Note that $\varphi\psi(s) = r_S\varphi^e_*\varphi^*_e\mathrm{Id}_R(s \otimes_R 1_R) = r_S e_{\varphi_*(S\otimes_R R)}(s \otimes_R e_R(1_R)) = e_{\varphi_*(S)} r_S(s \otimes_R e_R(1_R)) = e_{\varphi_*(S)}(s\varphi(e_R(1_R)))$. If $e_{\varphi_*(S)}$ is a left S-module morphism, we have that $\varphi\psi(s) = se_{\varphi_*(S)}(\varphi(e_R(1_R))) = s1_S = s$, so $\varphi \circ \psi = \mathrm{Id}_S$ in this case. Then,

$$\begin{aligned}\eta^e_M\nu^e_M(s \otimes_R m) &= \eta^e_M(\psi(s)e_M(m)) = \psi(s)\eta^e_M(e_M(m)) = \psi(s)\eta_M(e_M(e_M(m)))\\ &= \psi(s)e_{\varphi_*(S\otimes_R M)}(\eta_M(e_M(m))) = \psi(s)e_{\varphi_*(S\otimes_R M)}(1_S \otimes_R e_M(m))\\ &= e_{\varphi_*(S\otimes_R M)}(\psi(s)1_S \otimes_R e_M(m)) = e_{\varphi_*(S\otimes_R M)}(\varphi\psi(s) \otimes_R e_M(m))\\ &= e_{\varphi_*(S\otimes_R M)}(s \otimes_R e_M(m)) = \varphi^e_*\varphi^*_e\mathrm{Id}_M(s \otimes_R m),\end{aligned}$$

for every $m \in M$, $s \in S$, so φ^*_e is naturally semifull. Alternatively, we observe that, since $\varphi \circ \psi = \mathrm{Id}_S$, by Proposition 1.42 *ii*), φ^* is naturally full, so since E^e is naturally semifull (cf. Example 4.90), φ^*_e results to be naturally semifull also by Proposition 4.72 *i*). □

4.7.5 Semifunctor on a monoid

See [21, Example 7.9]. Let $(M, \cdot_M)$ be a monoid. It can be viewed as a category with a single object, denoted by $*$ (see [19, Example 1.2.6.d]). Arrows $a : * \to *$ are the elements of the monoid, which are closed under composition and the identity element 1_M is the identity arrow $1_M : * \to *$. A monoid homomorphism $f : M \to N$ is a functor, as it preserves compositions of arrows and the identity arrow; it sends the unique object $*$ of M into the unique object $\star$ of N. A natural transformation $\alpha = (\alpha_*) : f \to g$ between functors $f, g : M \to N$ is an arrow $\alpha_* : f(*) \to g(*)$ in the category N (i.e. an element of the monoid N) such that, for all elements a of M, $g(a) \cdot_N \alpha_* = \alpha_* \cdot_N f(a)$, where $\cdot_N$ is the composition law of N. Any semigroup homomorphism between monoids which is not a monoid homomorphism is an example of semifunctor. For instance, let $(\mathbb{N}, \cdot, 1)$ be the monoid of natural numbers seen as a category with a single object. Consider the map $f : (\mathbb{N}, \cdot) \to (\mathbb{N}, \cdot)$, $n \mapsto 0$, which is a semigroup homomorphism but not a monoid homomorphism as $f(1) = 0 \neq 1$. Thus, f is a semifunctor which is not faithful. Consider the natural transformation $\mathcal{P}^f_{*,*} = \mathcal{F}^f_{*,*} : \mathrm{Hom}_{\mathbb{N}}(*, *) \to \mathrm{Hom}_{\mathbb{N}}(*, *)$, $\mathcal{P}^f_{*,*}(m) = 0$, for every $m \in \mathbb{N}$. For every $m \in \mathbb{N}$, one has $(\mathcal{F}^f_{*,*} \circ \mathcal{P}^f_{*,*})(m) = \mathcal{F}^f_{*,*}(0) = 0 = f(1) \cdot m \cdot f(1)$, hence f is naturally semifull. We observe that f is indeed the canonical semifunctor E^0 associated with the idempotent seminatural transformation $0 = (0_*)_{*\in\mathbb{N}} : \mathrm{Id}_{\mathbb{N}} \to \mathrm{Id}_{\mathbb{N}}$.

As another example, consider the direct product $M \times M$ of M by itself with componentwise multiplication $(a, b) \cdot_{M\times M} (a', b') = (a \cdot_M a', b \cdot_M b')$, which we also denote by $(a, b)(a', b') = (aa', bb')$ for shortness. Let $e \neq 1_M$ be an idempotent element of M. Consider the map

$$f_e = (e, -) : M \to M \times M, \quad f_e(b) = (e, b),$$

which is a semigroup homomorphism. Indeed, for any $b, c \in M$, $f_e(bc) = (e, bc) = (ee, bc) = (e, b)(e, c) = f_e(b)f_e(c)$, but it does not preserve the unit, as $f_e(1_M) = (e, 1_M) \neq (1_M, 1_M) = 1_{M\times M}$. Moreover, if we view $M \times M$ as the product category with a single object $\star$ and morphisms given by the pairs $(m, n) : \star \to \star$, where $m, n : * \to *$ are elements of M, and whose composition is that induced by the multiplication of $M \times M$, then $f_e : M \to M \times M$, $f_e(*) = \star$, $f_e(b) = (e, b)$ results to be a semifunctor as it preserves compositions, but not the identity arrow $1_M : * \to *$. Consider the associated natural transformation $\mathcal{F}^{f_e}_{*,*} : \mathrm{Hom}_M(*, *) \to \mathrm{Hom}_{M\times M}(\star, \star)$, $\mathcal{F}^{f_e}_{*,*}(m) = f_e(m) = (e, m)$, and the map $\mathcal{P}^{f_e}_{*,*} : \mathrm{Hom}_{M\times M}(\star, \star) \to \mathrm{Hom}_M(*, *)$, $\mathcal{P}^{f_e}_{*,*}((m, n)) = n$, which is a natural transformation as for any $m, n, b, c \in M$, $\mathcal{P}^{f_e}_{*,*}(f_e(b) \cdot_{M\times M} (m, n) \cdot_{M\times M} f_e(c)) = \mathcal{P}^{f_e}_{*,*}((e, b)(m, n)(e, c)) = \mathcal{P}^{f_e}_{*,*}((eme, bnc)) = bnc = b \cdot_M \mathcal{P}^{f_e}_{*,*}((m, n)) \cdot_M c$. Then, for any $m \in M$, we have $(\mathcal{P}^{f_e}_{*,*} \circ \mathcal{F}^{f_e}_{*,*})(m) = \mathcal{P}^{f_e}_{*,*}((e, m)) = m$, hence f_e is separable. Now, we show that the semifunctor f_e is not in general semifull, and hence not even naturally semifull. If f_e were semifull, then for any (m, n) in $M \times M$ there would exist an element $a \in M$ such that $(e, a) = (e, 1_M)(m, n)(e, 1_M) = (eme, n)$. But in general it is not true that $(e, a) = (eme, n)$. For instance, consider the commutative monoid $\{x, 1, e\}$ with the following multiplication $\cdot$ laws

$$x \cdot x = e, \quad x \cdot 1 = x, \quad x \cdot e = x, \quad 1 \cdot 1 = 1, \quad 1 \cdot e = e, \quad e \cdot e = e.$$

Fix the idempotent element e. Then $e \cdot x \cdot e = e \cdot (x \cdot e) = e \cdot x = x$ does not result to be equal to e. Thus, f_e is not semifull.

4.7.6 Semifunctor on a ring

See [21, Example 7.10]. Let A be a unital ring. Then, A is a preadditive category with a single object $*$ and $\mathrm{Hom}_A(*, *) = A$. The composition of morphisms in A is the multiplication of elements of A. The group structure of $\mathrm{Hom}_A(*, *)$ is that of the underlying additive group A. Given unital rings R, S, any homomorphism $R \to S$ of rings which possibly does not preserve the unit is a semifunctor which sends the single object of R into the single object of S. For instance, consider a morphism of rings $g : R \to S$ and let $z \neq 1_S \in S$ be an idempotent such that $z \in S^R = \{u \in S \mid ug(r) = g(r)u \text{ for every } r \in R\}$. Then, any morphism $f : R \to S$ given by $f(r) = g(r)z$ is a semifunctor.

As particular cases, we have the following.

- If $g = \mathrm{Id}_R$, where R is a ring with unit 1_R, consider an idempotent element $z \neq 0_R, 1_R$ in the center $Z(R) = \{r' \in R \mid r'r = rr' \text{ for every } r \in R\}$ of R. Then, let $f : R \to R$ be the map given by $f(x) = xz$, for $x \in R$. It is a non-unital endomorphism of rings, thus f defines a semifunctor $f : R \to R$ which is not faithful as $f(z) = z = f(1_R)$, but $z \neq 1_R$; for any $x \in R$ we have that $f(1_R)xf(1_R) = zxz = xzz = xz = f(x)$, hence f is semifull. Further, let $*$ be the single object of R and let $\mathcal{F}^f_{*,*} : \mathrm{Hom}_R(*,*) \to \mathrm{Hom}_R(*,*)$, $\mathcal{F}^f_{*,*}(x) = f(x) = xz$, for $x \in R$, be the natural transformation associated with f. Consider the map $\mathcal{P}^f_{*,*} = \mathcal{F}^f_{*,*} : \mathrm{Hom}_R(*,*) \to \mathrm{Hom}_R(*,*)$, $\mathcal{P}^f_{*,*}(x) = xz$, for every x in R. For every $x \in R$, we have $(\mathcal{F}^f_{*,*} \circ \mathcal{P}^f_{*,*})(x) = \mathcal{F}^f_{*,*}(xz) = xzz = zxz = f(1_R)xf(1_R)$, hence f is naturally semifull. As an instance, consider as ring the product $R \times S$ of unital rings R, S and the idempotent element $z = (0_R, 1_S) \in Z(R \times S)$. Then, $f : R \times S \to R \times S$, $(x, y) \mapsto (x, y)z = (0_R, y)$ is a semifunctor which is naturally semifull but not faithful.

- Let R be a ring with unit 1_R and consider the ring $\mathrm{M}_n(R)$ of square matrices of order $n \in \mathbb{N}$ with coefficients in R. Consider the canonical inclusion $g : R \to \mathrm{M}_n(R)$, $r \mapsto r\mathrm{I}_n$, where I_n is the identity matrix in $\mathrm{M}_n(R)$. Let $f : R \to \mathrm{M}_n(R)$ be the map given, for any $m \in R$, by
$$m \mapsto m\mathrm{E}_{ii},$$
where $\mathrm{E}_{ii} = (\delta_{ia}\delta_{ib})_{ab}$ is the matrix unit in which the entry ii, with $i \in \{1, \dots, n\}$, is the unique nonzero entry. It is a homomorphism of rings which does not preserve the unit, since $f(1_R) = \mathrm{E}_{ii} \neq 1_{\mathrm{M}_n(R)} = \mathrm{I}_n$. It defines a semifully faithful semifunctor $f : R \to \mathrm{M}_n(R)$. Indeed, it is clear that f is faithful, and it is semifull as, given a matrix $A = (a_{ij})$ in $\mathrm{M}_n(R)$, we have that
$$f(1_R)Af(1_R) = \mathrm{E}_{ii}A\mathrm{E}_{ii} = a_{ii}\mathrm{E}_{ii} = f(a_{ii}).$$
Thus, f is semifully faithful.

Chapter 5

Conditions up to retracts

In this chapter we present the notions of (co)reflections up to retracts, i.e. functors whose idempotent completion admits a fully faithful right (left) adjoint, and bireflections up to retracts, introduced and investigated in [5]. These notions naturally arise whenever one deals with semiseparable functors that are part of an adjunction. We indeed prove that a right (left) adjoint functor is semiseparable if, and only if, the associated (co)monad is separable and the (co)comparison functor is a bireflection up to retracts, extending a characterization obtained by X.-W. Chen in the separable case [31]. The notions of semifunctor and semiadjunctions are helpful tools in this setting. Given an adjunction, the semiseparability of the right adjoint provides an equivalence after idempotent completion between the Kleisli category of free modules over the associated monad and the Eilenberg-Moore category of modules over that monad. We describe an application of these results in the context of pre-triangulated categories. We provide conditions for the Eilenberg-Moore categories of (co)modules to inherit the pre-triangulation from the base category, obtaining a semi-analogue of a result shown by P. Balmer for separable functors [12]. We show a similar result also for the Kleisli category. Most of the results we revise in this chapter have been investigated in [5].

5.1 (Co)reflections and bireflections up to retracts

Recall that an object X in a category $\mathcal{C}$ is a *retract* of an object Y in $\mathcal{C}$ if there are morphisms $i : X \to Y$ and $p : Y \to X$ in $\mathcal{C}$ such that $p \circ i = \mathrm{Id}_X$. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *surjective up to retracts* if every object D in $\mathcal{D}$ is a retract of FC for some object C in $\mathcal{C}$, see [14, Definition 2.5]. In [31, page 47] a functor $F : \mathcal{C} \to \mathcal{D}$ is said to be an *equivalence up to retracts* if its completion $F^\natural : \mathcal{C}^\natural \to \mathcal{D}^\natural$ is an equivalence.

Example 5.1. The canonical functor $\iota_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^\natural$ is an equivalence up to retracts, see e.g. [55, Theorem A.6].

In the spirit of the latter property, we define the following notions.

Definition 5.2. [5, Definition 2.3] Consider a functor $F : \mathcal{C} \to \mathcal{D}$ and its completion $F^\natural : \mathcal{C}^\natural \to \mathcal{D}^\natural$. Then, F is a **reflection up to retracts** if $F^\natural$ is a reflection; F is a **coreflection up to retracts** if $F^\natural$ is a coreflection; F is a **bireflection up to retracts** if $F^\natural$ is a bireflection.

We freely refer to these notions as *conditions up to retracts*. The next lemma relates these conditions up to retracts to the notions of (co)reflection and bireflection, considered in Chapter 2.

Lemma 5.3. [5, Lemma 2.4] *The following assertions hold true.*

i) Any equivalence is an equivalence up to retracts.

ii) Any (co)reflection is a (co)reflection up to retracts.

iii) A functor is a bireflection up to retracts if, and only if, it is a semiseparable (co)reflection up to retracts if, and only if, it is a naturally full (co)reflection up to retracts.

iv) Any bireflection is a bireflection up to retracts.

v) A functor is an equivalence up to retracts if, and only if, it is fully faithful and surjective up to retracts if, and only if, it is a fully faithful bireflection up to retracts.

vi) An equivalence up to retracts is both a reflection up to retracts and a coreflection up to retracts.

Proof. From Remark 4.20 we know that an adjunction (F, G, η, ϵ) induces an adjunction $\left(F^\natural, G^\natural, \eta^\natural, \epsilon^\natural\right)$.
i). If F is an equivalence with quasi-inverse G, then in view of [19, Proposition 3.4.3] and Corollary 4.95 *iii*), $(F^\natural, G^\natural)$ is an equivalence, hence F is an equivalence up to retracts.
ii). If G is a coreflection, it has a fully faithful left adjoint F. Hence, $F^\natural \dashv G^\natural$ and $F^\natural$ is fully faithful by Corollary 4.95 *iii*). Thus, $G^\natural$ is a coreflection, i.e. G is a coreflection up to retracts. The proof for reflections is similar.
iii). Assume F is a semiseparable (resp., naturally full) (co)reflection up to retracts. By Corollary 4.95 *iv*) (resp., *vi*)), $F^\natural$ is a semiseparable (resp., naturally full) (co)reflection. Thus, by Corollary 2.64, $F^\natural$ is a bireflection, i.e. F is a bireflection up to retracts. Conversely, from Corollary 2.64 and Corollary 4.95, in a similar way one gets that a bireflection up to retracts is a semiseparable (resp., naturally full) (co)reflection up to retracts.
iv). A bireflection F is in particular a semiseparable (co)reflection by Corollary 2.64. As a consequence of *ii*) and *iii*), we get that F is a bireflection up to retracts.
v). It is known that F is an equivalence up to retracts if, and only if, it is fully faithful and surjective up to retracts, see e.g. [31, Lemma 3.4 (2)]. By Remark 2.62 it is also equivalent to F being a fully faithful bireflection up to retracts in view of Corollary 4.95 *iii*) and Corollary 2.64.
vi). If F is an equivalence up to retracts, its completion $F^\natural$ is an equivalence, hence in particular $F^\natural$ is a (co)reflection. This means that F is a (co)reflection up to retracts. □

Remark 5.4. From Remark 2.56 and Lemma 5.3, it follows that also (co)reflections up to retracts and bireflections up to retracts are closed under composition.

Example 5.5. Let R be a ring and ${}_R\mathcal{M}_f$, ${}_R\mathcal{M}_p$ be the full subcategories of ${}_R\mathcal{M}$ whose objects are free left R-modules and projective left R-modules, respectively. Then, the inclusion functor $\Psi : {}_R\mathcal{M}_f \to {}_R\mathcal{M}_p$ considered in Example 4.16 is an equivalence up to retracts by Lemma 5.3 *v*) as it is fully faithful and any projective module is a retract of a free module.

The following result provides conditions under which the viceversa of *i*), *ii*), *iv*) in Lemma 5.3 holds true: namely, in case a (co)reflection up to retracts has a right (resp., left) adjoint, then it is a (co)reflection.

Proposition 5.6. [5, Proposition 2.9] *The following assertions hold true.*

i) If a coreflection up to retracts has a left adjoint, then it is a coreflection.

ii) If a coreflection up to retracts has a right adjoint, then it is a reflection.

iii) If a reflection up to retracts has a right adjoint, then it is a reflection.

iv) If a reflection up to retracts has a left adjoint, then it is a coreflection.

v) If a bireflection up to retracts has an adjoint, then it is a bireflection.

vi) If an equivalence up to retracts has an adjoint, then it is an equivalence.

Proof. *i*). If G has a left adjoint F, then $F^\natural \dashv G^\natural$. If G is a coreflection up to retracts, then $G^\natural$ is a coreflection. Thus, $F^\natural$ is fully faithful, and hence so is F by Corollary 4.95 *iii*), i.e. G is a coreflection.
ii). If F has a right adjoint G, then $F^\natural \dashv G^\natural$. If F is a coreflection up to retracts, then $F^\natural$ is a coreflection. Thus, it has a fully faithful left adjoint. Then, also the right adjoint $G^\natural$ is fully faithful by [19, Proposition 3.4.2]. By Corollary 4.95 *iii*) G is fully faithful, i.e. F is a reflection.
iii). It is dual to *i*).
iv). It is dual to *ii*).
v). If F is a bireflection up to retracts, then by Lemma 5.3 F is a naturally full (co)reflection up to retracts. If F has a left adjoint, by *i*) it is a naturally full coreflection while if F has a right adjoint, by *iii*), it is a naturally full reflection. In both cases, by Corollary 2.64, F is a bireflection.
vi). By Lemma 5.3 an equivalence up to retracts is a fully faithful bireflection up to retracts. If it has an adjoint, by *v*), it is a fully faithful bireflection, i.e. an equivalence by Lemma 5.3. □

Remark 5.7. From Proposition 5.6 and Lemma 5.3, it follows that any coreflection up to retracts with a right adjoint is a reflection up to retracts and any reflection up to retracts with a left adjoint is a coreflection up to retracts.

Lemma 5.8. [5, Lemma 2.11] *Let $\mathcal{D}$ be an idempotent complete category. A functor $G : \mathcal{D} \to \mathcal{C}$ has a left (resp., right) adjoint if, and only if, so does $G^\natural$.*

Proof. If $F \dashv G$, from Remark 4.20 we know that $F^\natural \dashv G^\natural$. Conversely, assume that $G^\natural$ has a left adjoint $L : \mathcal{C}^\natural \to \mathcal{D}^\natural$. Since $\mathcal{D}$ is idempotent complete, the functor $\iota_\mathcal{D} : \mathcal{D} \to \mathcal{D}^\natural$ is an equivalence of categories and hence it has a left adjoint $V_\mathcal{D} : \mathcal{D}^\natural \to \mathcal{D}$. From $V_\mathcal{D} \dashv \iota_\mathcal{D}$ and $L \dashv G^\natural$, we get $V_\mathcal{D} L \dashv G^\natural \iota_\mathcal{D}$ and hence $V_\mathcal{D} L \dashv \iota_\mathcal{C} G$. Since $\iota_\mathcal{C}$ is fully faithful, this implies $V_\mathcal{D} L \iota_\mathcal{C} \dashv G$. Indeed, given an object C in $\mathcal{C}$ and an object D in $\mathcal{D}$, we have a chain of natural isomorphisms $\mathrm{Hom}_\mathcal{D}(V_\mathcal{D} L \iota_\mathcal{C} C, D) \cong \mathrm{Hom}_{\mathcal{C}^\natural}(\iota_\mathcal{C} C, \iota_\mathcal{C} G D) \cong \mathrm{Hom}_\mathcal{C}(C, GD)$ where the latter follows from the full faithfulness of $\iota_\mathcal{C}$. The case in which G has a right adjoint is proved similarly. □

Proposition 5.9. [5, Proposition 2.12] *Let $\mathcal{D}$ be an idempotent complete category. A functor $G : \mathcal{D} \to \mathcal{C}$ is a coreflection up to retracts (resp., reflection up to retracts, bireflection up to retracts, equivalence up to retracts) if, and only if, it is a coreflection (resp., reflection, bireflection, equivalence).*

Proof. If G is a coreflection up to retracts (resp., reflection up to retracts), then $G^\natural$ has a left (resp., right) adjoint so that, by Lemma 5.8, so does G. By Proposition 5.6 G is a coreflection (resp., reflection). The other implication is always true by Lemma 5.3. Similarly, one deals with the case of bireflection and equivalence. □

We now revise some results from [5] concerning a characterization of (co)reflections up to retracts as part of a semiadjunction, by means of properties of semifunctors seen in Chapter 4, such as semifull faithfulness.

Proposition 5.10. (Cf. [5, Proposition 2.17]) *Let (F, G, η, ϵ) be a semiadjunction.*

i) If G is a functor, then it is a coreflection up to retracts if, and only if, η is a natural semi-isomorphism if, and only if, F is semifully faithful.

ii) If F is a functor, then it is a reflection up to retracts if, and only if, ϵ is a natural semi-isomorphism if, and only if, G is semifully faithful.

Proof. i). Assume η is a natural semi-isomorphism, i.e. there is $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\eta \circ \nu = GF\mathrm{Id}$ and $\nu \circ \eta = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}}$. Recall from Remark 4.20 that $\eta^{\natural}_{(C,c)} = \eta_C \circ c$. Let us prove that $\eta^{\natural}$ is an isomorphism with inverse $\nu^{\natural}$ defined by $\nu^{\natural}_{(C,c)} := c \circ \nu_C$, so that $F^{\natural}$ is fully faithful, i.e. G is a coreflection up to retracts. Note that $c \circ (c \circ \nu_C) \circ GFc = c \circ c \circ \nu_C \circ GFc = c \circ c \circ c \circ \nu_C = c \circ \nu_C$, hence $\nu^{\natural}_{(C,c)} : (GFC, GFc) \to (C, c)$ is a morphism in $\mathcal{C}^{\natural}$. We compute $\nu^{\natural}_{(C,c)} \circ \eta^{\natural}_{(C,c)} = c \circ \nu_C \circ \eta_C \circ c = c \circ \mathrm{Id}_C \circ c = c \circ c = c = \mathrm{Id}_{(C,c)}$, and

$$\begin{aligned}\eta^{\natural}_{(C,c)} \circ \nu^{\natural}_{(C,c)} &= \eta_C \circ c \circ c \circ \nu_C = \eta_C \circ c \circ \nu_C = GFc \circ \eta_C \circ \nu_C \\ &= GFc \circ GF\mathrm{Id} = GFc = \mathrm{Id}_{(GFC,GFc)},\end{aligned}$$

so that $\eta^{\natural}_{(C,c)}$ is an isomorphism in $\mathcal{C}^{\natural}$. Conversely, assume that G is a coreflection up to retracts. Then, $G^{\natural}$ has a left adjoint $L \cong F^{\natural}$ which is fully faithful, so the unit $\eta^{\natural} : \mathrm{Id}_{\mathcal{C}^{\natural}} \to G^{\natural}F^{\natural}$ of the adjunction $(F^{\natural}, G^{\natural}, \eta^{\natural}, \epsilon^{\natural})$ is an isomorphism. By Lemma 4.12, there exists a seminatural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\nu^{\natural} = (\eta^{\natural})^{-1}$. Thus, we have

$$(\eta \circ \nu)^{\natural} = \eta^{\natural} \circ \nu^{\natural} = \mathrm{Id}_{G^{\natural}F^{\natural}} = (GF\mathrm{Id})^{\natural}$$

and

$$(\nu \circ \eta)^{\natural} = \nu^{\natural} \circ \eta^{\natural} = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}^{\natural}}} = (\mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}})^{\natural},$$

hence by Lemma 4.17 it follows that $\eta \circ \nu = GF\mathrm{Id}$ and $\nu \circ \eta = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}}$, respectively, so η is a natural semi-isomorphism. By Corollary 4.67 the latter holds if, and only if, F is semifully faithful.
ii). The proof follows by similar arguments. □

Motivated by the previous result, we can extend the definition of (co)reflection up to retracts to an arbitrary semifunctor, that is not necessarily part of a semiadjunction. We say that a semifunctor $F : \mathcal{C} \to \mathcal{D}$ is a *coreflection up to retracts* (resp., *reflection up to retracts*, *bireflection up to retracts*, *equivalence up to retracts*) if $F^{\natural}$ is a coreflection (resp., reflection, bireflection, equivalence). In the functor case we recover Definition 5.2.

Proposition 5.11. *Let $G : \mathcal{D} \to \mathcal{C}$ be a semifunctor. Then,*

i) G is a coreflection up to retracts if, and only if, G is part of a semiadjunction $F \dashv_{\mathrm{s}} G$, where $F : \mathcal{C} \to \mathcal{D}$ is a semifully faithful semifunctor;

ii) G is a reflection up to retracts if, and only if, G is part of a semiadjunction $G \dashv_{\mathrm{s}} F$, where $F : \mathcal{C} \to \mathcal{D}$ is a semifully faithful semifunctor.

Proof. *i*). If $G : \mathcal{D} \to \mathcal{C}$ is a coreflection up to retracts, i.e. $G^\natural$ is a coreflection, then there is a fully faithful functor $L : \mathcal{C}^\natural \to \mathcal{D}^\natural$ such that $L \dashv G^\natural$. By Lemma 4.12 $F := \upsilon_\mathcal{D} \circ L \circ \iota_\mathcal{C} : \mathcal{C} \to \mathcal{D}$ is a semifunctor such that $F^\natural \cong L \dashv G^\natural$, hence $F^\natural$ is fully faithful and by Theorem 4.21 $F \dashv_\mathrm{s} G$. By Corollary 4.93 *iii*) F is semifully faithful. On the other hand, if G is part of a semiadjunction $F \dashv_\mathrm{s} G$ where F is semifully faithful, then $F^\natural$ is fully faithful by Corollary 4.93 *iii*) and, by Theorem 4.21, $F^\natural \dashv G^\natural$. Thus, $G^\natural$ is a coreflection, i.e. G is a coreflection up to retracts.
ii). It follows by similar arguments. □

As a consequence of Proposition 5.11, we retrieve the characterization of (co)reflection up to retracts given in [5, Corollary 2.18] for the functor case.

Remark 5.12. We point out that any semifunctor required to satisfy the assumptions in Proposition 5.6 (e.g. to be a coreflection up to retracts with a left adjoint) results to be actually a functor in view of Corollary 4.23.

The following result follows from Proposition 5.11.

Corollary 5.13. (Cf. [5, Corollary 2.19]) *Any semifunctor which is a (co)reflection up to retracts is surjective up to retracts. Moreover, any fully faithful (co)reflection up to retracts is an equivalence up to retracts.*

Proof. Let $G : \mathcal{D} \to \mathcal{C}$ be a coreflection up to retracts. By Proposition 5.11 *i*), G is part of a semiadjunction (F, G, η, ϵ) where F is semifully faithful, then there is $\nu : GF \to \mathrm{Id}_\mathcal{C}$ such that $\nu \circ \eta = \mathrm{Id}_{\mathrm{Id}_\mathcal{C}}$. Given an object C in $\mathcal{C}$ we get $\nu_C \circ \eta_C = \mathrm{Id}_C$ and hence C is a retract of GFC, i.e. G is surjective up to retracts. Similarly, any reflection up to retracts is surjective up to retracts. As a consequence, if a (co)reflection up to retracts is fully faithful (which is actually a functor by Remark 4.59), then by Lemma 5.3 *v*) it is an equivalence up to retracts. □

5.2 Quotient and (co)comparison functor

In this section we exhibit two examples of (co)reflection up to retracts. We show that the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$ onto the coidentifier category defined in Subsection 2.1.2 is a (co)reflection up to retracts, and through Proposition 5.17 we prove that the (co)comparison functor attached to an adjunction whose associated (co)monad is (co)separable results to be a coreflection (reflection) up to retracts. As a consequence, we characterize a semiseparable right (left) adjoint in terms of (co)separability of the associated (co)monad and of bireflectivity up to retracts of the (co)comparison functor.

5.2.1 The quotient functor

Recall from Subsection 2.1.2 that, given a category $\mathcal{C}$ and an idempotent natural transformation $e : \mathrm{Id}_\mathcal{C} \to \mathrm{Id}_\mathcal{C}$, we have the quotient functor

$$H : \mathcal{C} \to \mathcal{C}_e, \quad X \mapsto X, \quad f \mapsto \overline{f}.$$

In the next result we prove that H is a (co)reflection up to retracts and that it reveals to be indeed a bireflection up to retracts. We revise the proof shown in [5] in view of Proposition 5.10.

Theorem 5.14. (Cf. [5, Theorem 3.1]) *Let $\mathcal{C}$ be a category, let $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be an idempotent natural transformation. Then, the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$ is a (co)reflection up to retracts, whence a bireflection up to retracts.*

Proof. In Example 4.90 we have shown that the semifunctor $L : \mathcal{C}_e \to \mathcal{C}$ defined as the identity on objects and by $[\bar{f} : X \to Y] \mapsto [e_Y \circ f : X \to Y]$ on morphisms is semifully faithful and (L, H, η, ϵ) is a semiadjunction. Thus, by Proposition 5.10 H results to be a coreflection up to retracts. Moreover, as seen in Example 4.38 H is also part of the semiadjunction $H \dashv_{\mathrm{s}} L$, hence H is a reflection up to retracts. Since by Lemma 2.28 *i*) H is also naturally full, then by Lemma 5.3 *iii*) H is indeed a bireflection up to retracts. □

Remark 5.15. By Proposition 2.69 the functor $H : \mathcal{C} \to \mathcal{C}_e$ is a bireflection if, and only if, the idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ splits. Thus, H is a bireflection up to retracts but not a bireflection in general. Morever, it is not even a Frobenius functor as otherwise it would have an adjoint and by Proposition 5.6 *v*) it would be a bireflection.

The next example provides an instance that shows that the functor $H : \mathcal{C} \to \mathcal{C}_e$ is a bireflection up to retracts but not a bireflection in general.

Example 5.16. [5, Example 3.3] We come back again to Example 4.16. Denote $\mathcal{C} := {}_R\mathcal{M}_f$ the category of free left modules over a ring R. Given a central idempotent element $z \in R$, with $z \neq 0, 1$, define the idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ by setting $e_M : M \to M, m \mapsto zm$, for every free left R-module M. If e splitted, then $e_R : R \to R$ would split in $\mathcal{C}$ and thus $zR = \mathrm{Im}(e_R)$ would be a free R-module. Since $0 \neq z \in zR$, we have $zR \neq 0$ and it is known that a nonzero free module is faithful, i.e. it has trivial annihilator. Hence $1 - z \in \mathrm{Ann}_R(zR) = 0$ and so $z = 1$, a contradiction. Therefore, e does not split, hence $H : \mathcal{C} \to \mathcal{C}_e$ is a bireflection up to retracts but not a bireflection in view of Remark 5.15. For example, take $R = \mathbb{R} \times \mathbb{R}$ and $z = (1, 0)$.

5.2.2 The (co)comparison functor

In this subsection we look at the (co)comparison functor attached to an adjunction. Recall that in Theorem 2.47 we have shown that in case the left (resp., right) adjoint functor is semiseparable, then the (co)comparison functor is naturally full. We now prove that the (co)comparison functor is a coreflection up to retracts (resp., reflection up to retracts), whenever the (co)monad associated to the adjunction is separable. First, we need the next result which provides sufficient conditions for a functor to be a (co)reflection up to retracts.

Proposition 5.17. [5, Proposition 2.20] *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors endowed with natural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$.*

i) *If there is a natural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\nu \circ \eta = \mathrm{Id}$ and $\nu G = G\epsilon$, then G is a coreflection up to retracts.*

ii) *If there is a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\epsilon \circ \gamma = \mathrm{Id}$ and $\gamma F = F\eta$, then F is a reflection up to retracts.*

Proof. We just prove *i*). Given ν as in the statement, note that $G\epsilon \circ \eta G = \nu G \circ \eta G = (\nu \circ \eta)\, G = \mathrm{Id}_G$, so we are in the setting of Lemma 4.30. Thus, there is a semifunctor $F' : \mathcal{C} \to \mathcal{D}$ such that $(F', G, \eta', \epsilon')$ is a semiadjunction, where $\eta'_C := \eta_C$ and $\epsilon'_D := \epsilon_D$. Recall that F' acts as F on objects and sends a morphism $f : X \to Y$ in $\mathcal{C}$ to $Ff \circ e_X$. Set $e := \epsilon F \circ F\eta$ and observe that $GF'\mathrm{Id}_X = GF\mathrm{Id}_X \circ Ge_X = Ge_X$, for every $X \in \mathcal{C}$. Define

$\nu'_X := \nu_X \circ Ge_X$, for every $X \in \mathcal{C}$. Then $\nu'_X \circ GF'(\mathrm{Id}_X) = \nu_X \circ Ge_X \circ Ge_X = \nu_X \circ Ge_X = \nu'_X$ and, for every morphism $f : X \to Y$ in $\mathcal{C}$, we have $\nu'_Y \circ GF'f = \nu_Y \circ Ge_Y \circ GFf \circ Ge_X = \nu_Y \circ GFf \circ Ge_X \circ Ge_X = f \circ \nu_X \circ Ge_X = f \circ \nu'_X$, so that we can define the seminatural transformation $\nu' := (\nu'_X)_{X \in \mathcal{C}} : GF' \to \mathrm{Id}_{\mathcal{C}}$. Note that, for every $X \in \mathcal{C}$, $Ge_X \circ \eta_X = G\epsilon_{FX} \circ GF\eta_X \circ \eta_X = G\epsilon_{FX} \circ \eta_{GFX} \circ \eta_X = (G\epsilon \circ \eta G)_{FX} \circ \eta_X = \mathrm{Id}_{GFX} \circ \eta_X = \eta_X$, thus $Ge \circ \eta = \eta$, and then

$$\nu'_X \circ \eta'_X = \nu_X \circ Ge_X \circ \eta_X = \nu_X \circ \eta_X = \mathrm{Id}_X.$$

From naturality of ν and the assumption $\nu G = G\epsilon$ it follows that, for every $X \in \mathcal{C}$,

$$\begin{aligned}\eta'_X \circ \nu'_X &= \eta_X \circ \nu_X \circ Ge_X = \nu_{GFX} \circ GF\eta_X \circ Ge_X = G\epsilon_{FX} \circ GF\eta_X \circ Ge_X \\ &= Ge_X \circ Ge_X = GF'\mathrm{Id}_X \circ GF'\mathrm{Id}_X = GF'\mathrm{Id}_X,\end{aligned}$$

hence we conclude by Proposition 5.10. □

Theorem 5.18. [5, Theorem 3.4] *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ.*

i) If the monad $(GF, G\epsilon F, \eta)$ is separable, then the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ is a coreflection up to retracts.

ii) If the comonad $(FG, F\eta G, \epsilon)$ is coseparable, then the cocomparison functor $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$ is a reflection up to retracts.

Proof. We just check *i*) as *ii*) follows by dual arguments. Set $K := K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$, $U := U_{GF} : \mathcal{C}_{GF} \to \mathcal{C}$, $V := V_{GF} : \mathcal{C} \to \mathcal{C}_{GF}$ and consider $\Lambda := FU : \mathcal{C}_{GF} \to \mathcal{D}$. Let us construct three natural transformations $\eta_1 : \mathrm{Id}_{\mathcal{C}_{GF}} \to K\Lambda$, $\epsilon_1 : \Lambda K \to \mathrm{Id}_{\mathcal{D}}$ and $\nu_1 : K\Lambda \to \mathrm{Id}_{\mathcal{C}_{GF}}$ that fulfill the requirements of Proposition 5.17, i.e. such that $\nu_1 \circ \eta_1 = \mathrm{Id}$ and $\nu_1 K = K\epsilon_1$. Since $\Lambda K = FUK = FG$ we define $\epsilon_1 := \epsilon$ as the counit of the adjunction (F, G). Since $K\Lambda = KFU = VU$ we can set $\nu_1 := \beta$ as the counit of the adjunction (V, U) which is defined by $U\beta_{(C,\mu)} = \mu$ for every object (C, μ) in $\mathcal{C}_{GF}$. Since the monad $(GF, G\epsilon F, \eta)$ is separable, then the functor U is separable and hence, by Theorem 1.18 there is a natural transformation $\eta_1 : \mathrm{Id}_{\mathcal{C}_{GF}} \to VU$ such that $\beta \circ \eta_1 = \mathrm{Id}$, i.e. $\nu_1 \circ \eta_1 = \mathrm{Id}$. Moreover, $U\beta_{KD} = U\beta_{(GD, G\epsilon D)} = G\epsilon D = UK\epsilon_1 D$, so that $\beta K = K\epsilon_1$, i.e. $\nu_1 K = K\epsilon_1$. □

Theorem 5.18 allows to obtain the following characterization improving Theorem 2.47.

Theorem 5.19. [5, Theorem 3.5] *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ.*

i) G is semiseparable if, and only if, the monad $(GF, G\epsilon F, \eta)$ is separable and the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ is a bireflection up to retracts.

ii) F is semiseparable if, and only if, the comonad $(FG, F\eta G, \epsilon)$ is coseparable and the cocomparison functor $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$ is a bireflection up to retracts.

Proof. We just prove *i*) as *ii*) follows by dual arguments. By Theorem 2.47 *i*), G is semiseparable if, and only if, the monad $(GF, G\epsilon F, \eta)$ is separable and K_{GF} is a naturally full. When $(GF, G\epsilon F, \eta)$ is separable, by Theorem 5.18 K_{GF} is a coreflection up to retracts, and hence it is naturally full if, and only if, it is a naturally full coreflection up to retracts if, and only if, it is a bireflection up to retracts by Lemma 5.3 *iii*). □

By Theorem 5.19 we recover the following known characterizations for separable adjoint functors, see [31], [84].

Corollary 5.20. [5, Corollary 3.6] *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ.*

i) [31, Proposition 3.5] *G is separable if, and only if, the monad $(GF, G\epsilon F, \eta)$ is separable and the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ is an equivalence up to retracts.*

ii) [84, Proposition 2.3] *F is separable if, and only if, the comonad $(FG, F\eta G, \epsilon)$ is coseparable and the cocomparison functor $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$ is an equivalence up to retracts.*

Proof. We just prove *i)* as *ii)* follows by dual arguments. By Theorem 5.19 *i)*, G is semiseparable if, and only if, the monad $(GF, G\epsilon F, \eta)$ is separable and K_{GF} is a bireflection up to retracts. Moreover, by Corollary 2.48 G is separable if, and only if, the monad $(GF, G\epsilon F, \eta)$ is separable and the comparison functor K_{GF} is fully faithful. Thus, G is separable if, and only if, $(GF, G\epsilon F, \eta)$ is separable and K_{GF} is a fully faithful bireflection up to retracts. By Lemma 5.3 *v)*, the latter requirements on K_{GF} mean that it is an equivalence up to retracts. □

Remark 5.21. Given an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$ with unit η and counit ϵ, in view of Theorem 5.19 and Corollary 5.20, the following assertions hold true.

i) If G is faithful and the monad $(GF, G\epsilon F, \eta)$ is separable, then the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ is an equivalence up to retracts if, and only if, it is a bireflection up to retracts.

ii) If F is faithful and the comonad $(FG, F\eta G, \epsilon)$ is coseparable, then the cocomparison functor $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$ is an equivalence up to retracts if, and only if, it is a bireflection up to retracts.

We observe that Theorem 5.19 and Corollary 5.20 apply even if the relevant categories are not idempotent complete. The following is an instance.

Example 5.22. We return to the example discussed in Subsection 3.1.1. Given a morphism of rings $\varphi : R \to S$, with $S \neq 0$ a free left R-module (i.e., $S \cong R^{(J)}$), we know that the free induction functor $\varphi_f^* = S \otimes_R (-) : {}_R\mathcal{M}_f \to {}_S\mathcal{M}_f$ is semiseparable if, and only if, it is separable if, and only if, φ is a split-mono as an R-bimodule map; the free restriction of scalars functor $\varphi_{*f} : {}_S\mathcal{M}_f \to {}_R\mathcal{M}_f$ is separable if, and only if, S/R is separable.

By Corollary 5.20, φ_{*f} is separable if, and only if, the monad $\varphi_{*f}\varphi_f^* = S \otimes_R (-) : {}_R\mathcal{M}_f \to {}_R\mathcal{M}_f$ is separable and the comparison functor

$$K_{\varphi_{*f}\varphi_f^*} : {}_S\mathcal{M}_f \to ({}_R\mathcal{M}_f)_{\varphi_{*f}\varphi_f^*},$$
$$N \cong S^{(A)} \mapsto (\varphi_{*f}(N), \varphi_{*f}(\epsilon_f)_N) \cong (R^{(A\times J)}, \varphi_{*f}(\epsilon_f)_{S^{(A)}}),$$

is an equivalence up to retracts; the free induction functor φ_f^* is separable if, and only if, the comonad $\varphi_f^*\varphi_{*f} = S \otimes_R {}_R(-) : {}_S\mathcal{M}_f \to {}_S\mathcal{M}_f$ is coseparable and the cocomparison functor $K^{\varphi_f^*\varphi_{*f}} : {}_R\mathcal{M}_f \to ({}_S\mathcal{M}_f)^{\varphi_f^*\varphi_{*f}}$, $M \cong R^{(B)} \mapsto (\varphi_f^*(M), \varphi_f^*(\eta_f)_M) \cong (S^{(B)}, \varphi_f^*(\eta_f)_{S^{(B)}})$, is an equivalence up to retracts. In view of Remark 5.21, in case the monad $\varphi_{*f}\varphi_f^*$ is separable, then the comparison functor $K_{\varphi_{*f}\varphi_f^*}$ is an equivalence up to retracts if, and only if, it is a bireflection up to retracts; since φ_f^* is semiseparable if, and only if, it is separable, in case the comonad $\varphi_f^*\varphi_{*f}$ is coseparable, then the cocomparison functor $K^{\varphi_f^*\varphi_{*f}}$ is an equivalence up to retracts if, and only if, it is a bireflection up to retracts.

As a consequence of the properties proved so far we have the following result.

Corollary 5.23. [5, Corollary 3.7] *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ and cocomparison functor $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$.*

i) *Assume G is semiseparable. If K_{GF} has a left adjoint, then K_{GF} is a bireflection.*

ii) *Assume F is semiseparable. If K^{FG} has a right adjoint, then K^{FG} is a bireflection.*

iii) (Cf. [74, Proposition, page 93] and [8, Proposition 2.16 (3)]) *Assume G is separable. If K_{GF} has a left adjoint, then K_{GF} is an equivalence (i.e., G is monadic)*

iv) (Cf. [64, Proposition 3.16]) *Assume F is separable. If K^{FG} has a right adjoint, then K^{FG} is an equivalence (i.e., F is comonadic).*

In case $\mathcal{D}$ (resp., $\mathcal{C}$) is idempotent complete, if G (resp., F) is (semi)separable, then K_{GF} (resp., K^{FG}) has a left (resp., right) adjoint, so the previous assertions apply.

Proof. We just prove *i*) and *iii*), as *ii*) and *iv*) follow by dual arguments. If G is semiseparable (resp., separable), by Theorem 5.19 (resp., Corollary 5.20) we know that K_{GF} is a bireflection up to retracts (resp., equivalence up to retracts). Then, if K_{GF} has a left adjoint, by Proposition 5.6 K_{GF} is a bireflection (resp., equivalence). By Proposition 5.9, if $\mathcal{D}$ is idempotent complete, then K_{GF} has a left adjoint as it is a bireflection up to retracts (resp., equivalence up to retracts). □

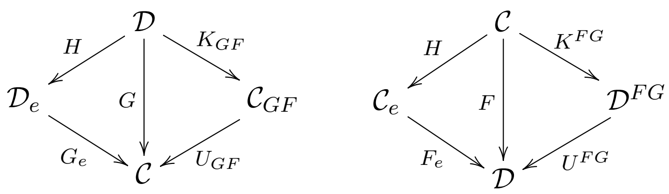
The next is an example of a coreflection (up to retracts) which is not an equivalence (up to retracts) and not even a bireflection (up to retracts).

Example 5.24. [5, Example 3.8] (See also [61, page 144]) Consider the forgetful functor $G : \mathsf{Top} \to \mathsf{Set}$ and its left adjoint $F : \mathsf{Set} \to \mathsf{Top}$ which assigns to each set X the topological space X equipped with the discrete topology (all subsets of X are open). This adjunction defines on Set the identity monad $\mathbb{I} = (\mathrm{Id}_{\mathsf{Set}}, \mathrm{Id}, \mathrm{Id})$. The Eilenberg-Moore category of modules over $\mathbb{I}$ is then Set, thus the comparison functor $K_{GF} : \mathsf{Top} \to \mathsf{Set}_{\mathbb{I}} = \mathsf{Set}$ is the given forgetful functor G. Since the identity monad $\mathbb{I}$ is separable, by Theorem 5.18 K_{GF} is a coreflection up to retracts and then a coreflection either by Proposition 5.9, as Top has equalizers and then it is an idempotent complete category (cf. Example 4.9), or by Proposition 5.6, as $K_{GF} = G$ has a left adjoint. Since K_{GF} is not an equivalence, again by Proposition 5.9 it follows that K_{GF} is not even an equivalence up to retracts. By Corollary 5.20 we have that G is not separable and, since G is faithful, G is not semiseparable by Proposition 2.5. Then, by Theorem 5.19 K_{GF} is not even a bireflection up to retracts, and hence not a bireflection by Proposition 5.9.

5.3 Canonical factorizations of a semiseparable adjoint

In this section we compare the two canonical factorizations attached to a semiseparable right adjoint $G : \mathcal{D} \to \mathcal{C}$, namely the one through the coidentifier category and the one through the Eilenberg-Moore category, showing they are connected by an equivalence up to retracts.

Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction. We have seen that if the right adjoint G is semiseparable, then it admits two canonical factorizations as a bireflection up to retracts followed by a separable functor, namely $G = G_e \circ H$ (cf. Theorem 2.33 and Theorem 5.14) and $G = U_{GF} \circ K_{GF}$ (cf. Theorems 2.47 and 5.19). Similar factorizations have been obtained also for the left adjoint F in case it is semiseparable, as the following diagrams show.



We now compare these factorizations. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ **lifts idempotents** whenever each idempotent morphism in $\mathcal{D}$ is of the form $F(q)$ for some idempotent morphism q in $\mathcal{C}$.

Lemma 5.25. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor that lifts idempotents. If $\mathcal{C}$ is idempotent complete, then so is $\mathcal{D}$.*

Proof. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor that lifts idempotents morphisms and assume that $\mathcal{C}$ is idempotent complete. If $d : D \to D$ is an idempotent morphism in $\mathcal{D}$, then there is an idempotent morphism $q : X \to X$ in $\mathcal{C}$ such that $d = F(q)$. Since $\mathcal{C}$ is idempotent complete we have that $q = i \circ p$, for some morphisms $p : X \to X'$ and $i : X' \to X$ such that $p \circ i = \mathrm{Id}_{X'}$. Then, $d = F(q) = F(i \circ p) = Fi \circ Fp$ and $Fp \circ Fi = F(p \circ i) = F(\mathrm{Id}_{X'}) = \mathrm{Id}_{FX'}$, so that the idempotent d in $\mathcal{D}$ splits. Thus, $\mathcal{D}$ is idempotent complete. □

Remark 5.26. Any fully faithful functor $F : \mathcal{C} \to \mathcal{D}$ lifts idempotents of the form $d : FX \to FX$. In fact, if $d : FX \to FX$ is an idempotent morphism in $\mathcal{D}$, then it is of the form $F(q)$ for some morphism $q : X \to X$ in $\mathcal{C}$, since F is full. We have $F(q) = d = d \circ d = F(q) \circ F(q) = F(q \circ q)$, hence $q = q \circ q$ as F is faithful.

Lemma 5.27. [5, Lemma 3.13] *Let $\mathcal{C}$ be a category and let $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be an idempotent natural transformation. Then, the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$ lifts idempotents. As a consequence, if $\mathcal{C}$ is idempotent complete so is the coidentifier category $\mathcal{C}_e$.*

Proof. Let $\overline{h} : C \to C$ be an idempotent morphism in $\mathcal{C}_e$. Then $\overline{h} \circ \overline{h} = \overline{h}$, i.e. $\overline{h \circ h} = \overline{h}$ and hence $e_C \circ h \circ h = e_C \circ h$. Set $q := e_C \circ h : C \to C$. Then, $q \circ q = e_C \circ h \circ e_C \circ h = e_C \circ e_C \circ h \circ h = e_C \circ h \circ h = e_C \circ h = q$, hence q is an idempotent morphism in $\mathcal{C}$. Moreover, $Hq = \overline{q} = \overline{e_C \circ h} = \overline{h}$. The last statement follows from Lemma 5.25. □

Lemma 5.28. [5, Lemma 3.14] *Let $G : \mathcal{D} \to \mathcal{C}$ and $U : \mathcal{C} \to \mathcal{C}'$ be functors.*

i) If G is a (co)reflection and U is conservative, then U is an equivalence if, and only if, $U \circ G$ is a (co)reflection.

ii) If G is a (co)reflection up to retracts and U is separable, then U is an equivalence up to retracts if, and only if, $U \circ G$ is a (co)reflection up to retracts.

Proof. Set $G' := U \circ G$.

i). Since U is conservative, if G' is a coreflection, by [9, Corollary 4.9], which is a consequence of [16, Lemma 1.2], we get that U is an equivalence. Conversely, if U is an equivalence then it is in particular a coreflection and hence, by Remark 2.56, G' is a coreflection as a composition of coreflections. The statement in case G is a reflection is proved dually.

ii). If U is separable, then by Corollary 4.96 $U^\natural$ is conservative, thus we have that $(G')^\natural = U^\natural \circ G^\natural$, where $G^\natural$ is a (co)reflection and $U^\natural$ is conservative. By *i*), we get that

$U^\natural$ is an equivalence (i.e., U is an equivalence up to retracts) if, and only if, $(G')^\natural$ is a (co)reflection (i.e., G' is a (co)reflection up to retracts). □

Proposition 5.29. [5, Proposition 3.15] *Let $F : \mathcal{C} \to \mathcal{D}$ be a bireflection up to retracts and let $e : \mathrm{Id}_\mathcal{C} \to \mathrm{Id}_\mathcal{C}$ be its associated idempotent natural transformation. Consider the corresponding factorization $F = F_e \circ H$, where $H : \mathcal{C} \to \mathcal{C}_e$ is the quotient functor onto the coidentifier category. Then, the unique functor $F_e : \mathcal{C}_e \to \mathcal{D}$ is an equivalence up to retracts. If $\mathcal{C}$ is idempotent complete, then F_e is an equivalence.*

Proof. If F is a bireflection up to retracts, it is a semiseparable (co)reflection up to retracts by Lemma 5.3 *iii*). In particular, F admits the associated idempotent natural transformation $e : \mathrm{Id}_\mathcal{C} \to \mathrm{Id}_\mathcal{C}$ by Proposition 2.11. By Theorem 2.33, there is a factorization $F = F_e \circ H$ for a unique functor $F_e : \mathcal{C}_e \to \mathcal{D}$ which is separable. Since H is a (co)reflection up to retracts by Theorem 5.14 and F_e is separable, then by Lemma 5.28 we get that F_e is an equivalence up to retracts.

If $\mathcal{C}$ is idempotent complete so is $\mathcal{C}_e$ by Lemma 5.27. Then F_e is an equivalence in view of Proposition 5.9. □

Example 5.30. [5, Example 3.16] Let $F : \mathcal{C} \to \mathcal{D}$ be a bireflection up to retracts. Thus, $F^\natural : \mathcal{C}^\natural \to \mathcal{D}^\natural$ is a bireflection. In particular, by Lemma 5.3 it is a bireflection up to retracts whose source category $\mathcal{C}^\natural$ is idempotent complete. By Proposition 5.29, $(F^\natural)_\alpha : (\mathcal{C}^\natural)_\alpha \to \mathcal{D}^\natural$ is an equivalence where $\alpha : \mathrm{Id}_{\mathcal{C}^\natural} \to \mathrm{Id}_{\mathcal{C}^\natural}$ is the idempotent natural transformation associated with $F^\natural$. By definition and from the proof of Proposition 4.92, we get that

$$\alpha_{(C,c)} = \mathcal{P}^{F^\natural}_{(C,c),(C,c)}(\mathrm{Id}_{F^\natural(C,c)}) = \mathcal{P}^{F^\natural}_{(C,c),(C,c)}(\mathrm{Id}_{(FC,Fc)}) = \mathcal{P}^{F}_{C,C}(Fc) = \mathcal{P}^{F}_{C,C}(\mathrm{Id}_{FC})\circ c = e_C \circ c$$

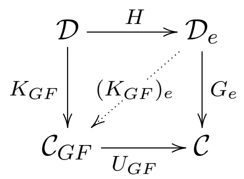
so that $\alpha = e^\natural$, where $e : \mathrm{Id}_\mathcal{C} \to \mathrm{Id}_\mathcal{C}$ is the idempotent natural transformation associated with F. This shows that $(F^\natural)_{e^\natural} : (\mathcal{C}^\natural)_{e^\natural} \to \mathcal{D}^\natural$ is an equivalence and hence $\mathcal{D}^\natural \cong (\mathcal{C}^\natural)_{e^\natural}$.

In particular, in Theorem 5.14 we proved that $H : \mathcal{C} \to \mathcal{C}_e$ is a bireflection up to retracts. By the foregoing, $(H^\natural)_{e^\natural} : (\mathcal{C}^\natural)_{e^\natural} \to (\mathcal{C}_e)^\natural$ is an equivalence and hence $(\mathcal{C}_e)^\natural \cong (\mathcal{C}^\natural)_{e^\natural}$.

We are now able to compare the two relevant factorizations.

Proposition 5.31. (Cf. [5, Proposition 3.17], [4, Remark 2.10]) *Consider an adjunction $F \dashv G : \mathcal{D} \to \mathcal{C}$.*

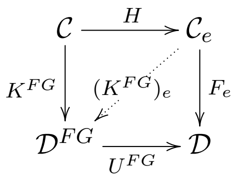
i) If G is semiseparable and $e : \mathrm{Id}_\mathcal{D} \to \mathrm{Id}_\mathcal{D}$ is the associated idempotent natural transformation, then there is a unique functor $(K_{GF})_e : \mathcal{D}_e \to \mathcal{C}_{GF}$ such that $(K_{GF})_e \circ H = K_{GF}$ and $U_{GF} \circ (K_{GF})_e = G_e$, i.e. the diagram



(5.1)

commutes, where $H : \mathcal{D} \to \mathcal{D}_e$ is the quotient functor onto the coidentifier category. Moreover, the functor $(K_{GF})_e$ is an equivalence up to retracts. If $\mathcal{D}$ is idempotent complete, then $(K_{GF})_e$ is an equivalence of categories.

ii) *If F is semiseparable and $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ is the associated idempotent natural transformation, then there is a unique functor $(K^{FG})_e : \mathcal{C}_e \to \mathcal{D}^{FG}$ such that $(K^{FG})_e \circ H = K^{FG}$ and $U^{FG} \circ (K^{FG})_e = F_e$, i.e. the diagram*



(5.2)

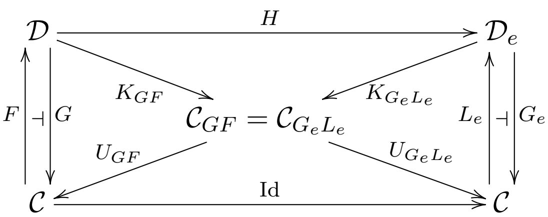
commutes, where $H : \mathcal{C} \to \mathcal{C}_e$ is the quotient functor onto the coidentifier category. Moreover, the functor $(K^{FG})_e$ is an equivalence up to retracts. If $\mathcal{C}$ is idempotent complete, then $(K^{FG})_e$ is an equivalence of categories.

Proof. We just prove *i)* as *ii)* follows by dual arguments. Assume G is semiseparable with associated idempotent natural transformation $e : \mathrm{Id}_{\mathcal{D}} \to \mathrm{Id}_{\mathcal{D}}$. Then, by Theorem 2.33 there is a unique functor $G_e : \mathcal{D}_e \to \mathcal{C}$ (necessarily separable) such that $G = G_e \circ H$, where $H : \mathcal{D} \to \mathcal{D}_e$ is the quotient functor onto the coidentifier category $\mathcal{D}_e$, which in turn is naturally full. By Theorem 2.47 *i)* G also factors as $U_{GF} \circ K_{GF}$, where U_{GF} is separable and K_{GF} is naturally full. These two factorizations of G as a naturally full functor followed by a separable one are related, in view of Theorem 2.33, by a unique functor $(K_{GF})_e : \mathcal{D}_e \to \mathcal{C}_{GF}$ (necessarily fully faithful) such that $(K_{GF})_e \circ H = K_{GF}$ and $U_{GF} \circ (K_{GF})_e = G_e$. Furthermore, by Theorem 2.33 G and K_{GF} have the same associated idempotent natural transformation e. Thus, the factorization $K_{GF} = (K_{GF})_e \circ H$ is necessarily the one of Proposition 5.29, once observed that K_{GF} is a bireflection up to retracts by Theorem 5.19. As a consequence $(K_{GF})_e$ is an equivalence up to retracts (an equivalence in case $\mathcal{D}$ is idempotent complete). □

We observe that it is possible to give a different proof of Theorem 5.19, by first showing that $(K_{GF})_e$ and $(K^{FG})_e$ are equivalences up to retracts. To this aim we need the following lemma.

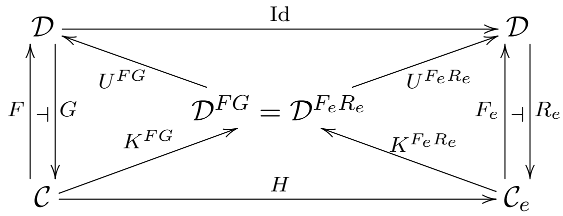
Lemma 5.32. [5, Lemma 3.18, Lemma 3.19]

i) *Given a category $\mathcal{D}$ and an idempotent natural transformation $e : \mathrm{Id}_{\mathcal{D}} \to \mathrm{Id}_{\mathcal{D}}$, consider the quotient functor $H : \mathcal{D} \to \mathcal{D}_e$. Let $G_e : \mathcal{D}_e \to \mathcal{C}$ be a functor. If $G := G_e \circ H : \mathcal{D} \to \mathcal{C}$ has a left adjoint F with unit η and counit ϵ, then $L_e := H \circ F : \mathcal{C} \to \mathcal{D}_e$ is a left adjoint of G_e with unit η_e and counit ϵ_e uniquely defined by the identities $\eta_e = \eta$ and $\epsilon_e H = H\epsilon$. Moreover, the adjunctions (L_e, G_e) and (F, G) have the same associated monad (whence $\mathcal{C}_{G_eL_e} = \mathcal{C}_{GF}$) and the respective comparison functors are related by the equality $K_{G_eL_e} \circ H = K_{GF}$.*



ii) *Given a category $\mathcal{C}$ and an idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$, consider the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$. Let $F_e : \mathcal{C}_e \to \mathcal{D}$ be a functor. If $F := F_e \circ H$:*

$\mathcal{C} \to \mathcal{D}$ has a right adjoint G with unit η and counit ϵ, then $R_e := H \circ G : \mathcal{D} \to \mathcal{C}_e$ is a right adjoint of F_e with unit η_e and counit ϵ_e uniquely defined by the identities $\eta_e H = H\eta$ and $\epsilon_e = \epsilon$. Moreover, the adjunctions (F_e, R_e) and (F, G) have the same associated comonad (whence $\mathcal{D}^{F_eR_e} = \mathcal{D}^{FG}$) and the respective cocomparison functors are related by the equality $K^{F_eR_e} \circ H = K^{FG}$.



Proof. We just prove $i)$ as $ii)$ follows by dual arguments. Given $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$ we have $H\epsilon : HFG \to H$. By the universal property of the coidentifier, since $(L_eG_e) \circ H = HFG$ and $\mathrm{Id}_{\mathcal{D}_e} \circ H = H$, we have $(HFG)_e = L_eG_e$ and $H_e = \mathrm{Id}_{\mathcal{D}_e}$ and hence there is a unique natural transformation $\epsilon_e : L_eG_e \to \mathrm{Id}_{\mathcal{D}_e}$ such that $\epsilon_e H = H\epsilon$ (see Lemma 2.30). Since $G_e \circ L_e = G_e \circ H \circ F = G \circ F$, we define $\eta_e := \eta$. Then, we have

$$G_e\epsilon_e H \circ \eta_e G_e H = G_e H\epsilon \circ \eta_e G_e H = G\epsilon \circ \eta G = \mathrm{Id}_G = \mathrm{Id}_{G_eH}.$$

Since H is the identity on objects, we deduce that $G_e\epsilon_e \circ \eta_e G_e = \mathrm{Id}_{G_e}$. Moreover

$$\epsilon_e L_e \circ L_e\eta_e = \epsilon_e HF \circ HF\eta_e = H\epsilon F \circ HF\eta = H\mathrm{Id}_F = \mathrm{Id}_{HF} = \mathrm{Id}_{L_e}.$$

Since $G_eL_e = GF$, $G_e\epsilon_e L_e = G_e\epsilon_e HF = G_eH\epsilon F = G\epsilon F$ and $\eta_e = \eta$ we have that the adjunctions (L_e, G_e) and (F, G) have the same associated monad. Thus $\mathcal{C}_{G_eL_e} = \mathcal{C}_{GF}$. Note that

$$\begin{aligned} K_{G_eL_e}HX &= (G_eHX, G_e\epsilon_{eHX}) = (G_eHX, G_eH\epsilon_X) = (GX, G\epsilon_X) = K_{GF}X, \\ K_{G_eL_e}Hf &= G_eHf = Gf = K_{GF}f, \end{aligned}$$

so that $K_{G_eL_e} \circ H = K_{GF}$. □

Proposition 5.33. *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction.*

i) If G is semiseparable with associated idempotent natural transformation $e : \mathrm{Id}_{\mathcal{D}} \to \mathrm{Id}_{\mathcal{D}}$, then the functor $(K_{GF})_e$ given as in Proposition 5.31 $i)$ is an equivalence up to retracts. As a consequence, K_{GF} is a bireflection up to retracts.

ii) If F is semiseparable with associated idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$, then the functor $(K^{FG})_e$ given as in Proposition 5.31 $ii)$ is an equivalence up to retracts. As a consequence, K^{FG} is a bireflection up to retracts.

Proof. We prove only $i)$ as $ii)$ follows by dual arguments. If G is semiseparable, then by Theorem 2.33 there is a unique functor $G_e : \mathcal{D}_e \to \mathcal{C}$ (necessarily separable) such that $G = G_e \circ H$, where $H : \mathcal{D} \to \mathcal{D}_e$ is the quotient functor onto the coidentifier category $\mathcal{D}_e$. By Lemma 5.32 the adjunctions $(L_e := H \circ F, G_e)$ and (F, G) have the same associated monad (whence $\mathcal{C}_{G_eL_e} = \mathcal{C}_{GF}$) and the respective comparison functors are related by the equality $K_{G_eL_e} \circ H = K_{GF}$. The functor $(K_{GF})_e : \mathcal{D}_e \to \mathcal{C}_{GF}$ of Proposition 5.31 is uniquely determined by the equality $(K_{GF})_e \circ H = K_{GF}$, so we get $(K_{GF})_e = K_{G_eL_e}$. Since G_e is separable, by Corollary 5.20 $i)$ the functor $K_{G_eL_e}$ is an equivalence up to retracts and then so is $(K_{GF})_e$. Since by Theorem 5.14 the functor H is a coreflection up to retracts, then from the equality $(K_{GF})_e \circ H = K_{GF}$ by Lemma 5.28 we conclude that K_{GF} is a coreflection up to retracts, whence a bireflection up to retracts, as by Theorem 2.47 *i*) K_{GF} is naturally full. □

As a consequence of Lemma 5.32, we have the following description of the separable functor G_e (resp., F_e) involved in the factorization $G = G_e H$ (resp., $F = F_e H$) of Theorem 2.33 for a semiseparable right (resp., left) adjoint functor G (resp., F).

Corollary 5.34. *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction.*

i) *Assume G is semiseparable with associated idempotent natural transformation $e : \mathrm{Id}_{\mathcal{D}} \to \mathrm{Id}_{\mathcal{D}}$ and consider the factorization $G = G_e \circ H$. Then, $G_e \cong_{\mathrm{s}} GL$, where $L : \mathcal{D} \to \mathcal{D}_e$ is the semifunctor of Example 4.38.*

ii) *Assume F is semiseparable with associated idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ and consider the factorization $F = F_e \circ H$. Then, $F_e \cong_{\mathrm{s}} FL$, where $L : \mathcal{C} \to \mathcal{C}_e$ is the semifunctor of Example 4.38.*

Proof. From Example 4.38 we know that $L \dashv_{\mathrm{s}} H \dashv_{\mathrm{s}} L : \mathcal{D}_e \to \mathcal{D}$ is a semiadjoint triple.
i). Assume G is semiseparable with associated idempotent natural transformation $e : \mathrm{Id}_{\mathcal{D}} \to \mathrm{Id}_{\mathcal{D}}$. By Corollary 4.33 from $F \dashv G$ and $H \dashv_{\mathrm{s}} L$ we get a semiadjunction $HF \dashv_{\mathrm{s}} GL$. By Lemma 5.32 *i*) we know that $HF \dashv G_e$, where G_e is the separable functor in the factorization $G = G_e \circ H$. By Proposition 4.26, we have $G_e \cong_{\mathrm{s}} GL$.
ii). It follows by dual arguments. □

5.4 Completion of Kleisli and Eilenberg-Moore categories

As an application of the results concerning conditions up to retracts, we now consider the Kleisli construction for a monad $(\top, m : \top\top \to \top, \eta : \mathrm{Id}_{\mathcal{C}} \to \top)$ on a category $\mathcal{C}$ recalled in Subsection 1.5.3. We prove that, given an adjunction, the semiseparability of the right adjoint provides an equivalence after idempotent completion between the associated Kleisli category $\top$-$\mathrm{Free}_{\mathcal{C}}$ and Eilenberg-Moore category $\mathcal{C}_\top$. As a consequence, these categories are also equivalent up to retracts to the coidentifier category attached to the semiseparable right adjoint.

Recall that the canonical functor

$$J_\top : \top\text{-}\mathrm{Free}_{\mathcal{C}} \to \mathcal{C}_\top, \quad C \mapsto (\top C, m_C), \quad [f : C \nrightarrow D] \mapsto m_D \circ \top(f),$$

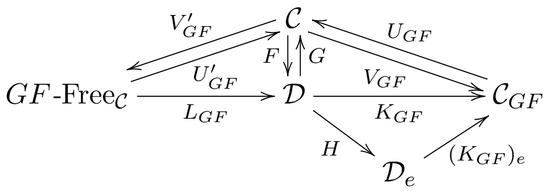
is fully faithful. The following result shows that, in case the monad $\top$ is separable, the functor $J_\top$ is indeed an equivalence up to retracts.

Proposition 5.35. [5, Proposition 3.20] *Let $(\top, m, \eta)$ be a separable monad on a category $\mathcal{C}$. Then, the canonical functor $J_\top : \top\text{-}\mathrm{Free}_{\mathcal{C}} \to \mathcal{C}_\top$ is an equivalence up to retracts. In particular, $\top\text{-}\mathrm{Free}_{\mathcal{C}}^\natural \cong \mathcal{C}_\top^\natural$.*

Proof. By Lemma 1.75 the separability of the monad $(\top, m, \eta)$ is equivalent to the separability of the forgetful functor $U_\top : \mathcal{C}_\top \to \mathcal{C}$, hence, by Theorem 1.18 this is also equivalent to the fact that the counit $\beta : V_\top U_\top \to \mathrm{Id}_{\mathcal{C}_\top}$ of the adjunction $(V_\top, U_\top)$ is a split natural epimorphism. Thus, we get that $V_\top$ is surjective up to retracts, and hence so is $J_\top$ in view of the equality $V_\top = J_\top \circ V'_\top$. Since $J_\top$ is also fully faithful, we have that it is an equivalence up to retracts by Lemma 5.3 *v*). □

The previous result applies to the particular case of a monad defined by an adjunction.

Proposition 5.36. [5, Proposition 3.21] *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction. Assume G is a semiseparable functor with associated idempotent natural transformation e. Consider the diagram*



(5.3)

where H is the quotient functor into the coidentifier category. Then, the composite functor $K_{GF} \circ L_{GF} : GF\text{-Free}_{\mathcal{C}} \to \mathcal{C}_{GF}$ is an equivalence up to retracts. Moreover, also the composite $H \circ L_{GF} : GF\text{-Free}_{\mathcal{C}} \to \mathcal{D}_e$ is an equivalence up to retracts, hence

$$GF\text{-Free}_{\mathcal{C}}^{\natural} \cong \mathcal{D}_e^{\natural} \cong \mathcal{C}_{GF}^{\natural}.$$

Proof. Since G is semiseparable, by Lemma 2.43 $i)$ the associated monad $(GF, G\epsilon F, \eta)$ is separable. By Proposition 5.35 we get that the composite functor $K_{GF} \circ L_{GF} = J_{GF}$: $GF\text{-Free}_{\mathcal{C}} \to \mathcal{C}_{GF}$ is an equivalence up to retracts.

Moreover, by Proposition 5.31 there is a unique functor $(K_{GF})_e : \mathcal{D}_e \to \mathcal{C}_{GF}$ such that $(K_{GF})_e \circ H = K_{GF}$ and $U_{GF} \circ (K_{GF})_e = G_e$, and in particular $(K_{GF})_e$ is an equivalence up to retracts, so the fact that $H \circ L_{GF}$ is an equivalence up to retracts follows from the equality $(K_{GF})_e^{\natural} \circ (H \circ L_{GF})^{\natural} = (K_{GF} \circ L_{GF})^{\natural}$. □

As a consequence of Proposition 5.36, we recover [13, Lemma 2.10], see also [12, Theorem 5.17 (d)] in the setting of idempotent complete suspended categories, and [33, Theorem 1.6] in the setting of idempotent complete triangulated categories.

Corollary 5.37. (Cf. [13, Lemma 2.10]) *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with G separable. Then, the functors $L_{GF} : GF\text{-Free}_{\mathcal{C}} \to \mathcal{D}$ and $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ are both equivalences up to retracts. Moreover, if $\mathcal{D}$ is idempotent complete, then G is monadic, i.e. $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ is an equivalence.*

Proof. Since G is a separable functor, then, by Corollary 2.12, the associated idempotent natural transformation $e : \mathrm{Id}_{\mathcal{D}} \to \mathrm{Id}_{\mathcal{D}}$ is the identity $\mathrm{Id}_{\mathrm{Id}_{\mathcal{D}}}$, and hence the quotient functor $H : \mathcal{D} \to \mathcal{D}_{\mathrm{Id}}$ is an equivalence (cf. Remark 2.29). Thus, by Proposition 5.36, L_{GF} : $GF\text{-Free}_{\mathcal{C}} \to \mathcal{D}$ results to be an equivalence up to retracts. Concerning K_{GF}, it is an equivalence up to retracts, in view of Corollary 5.20. Furthermore, by Corollary 5.23 it is an equivalence if $\mathcal{D}$ is idempotent complete. □

5.5 Pre-triangulated categories and semiseparability

In this section we extend to semiseparable functors a result shown by P. Balmer in [12] for separable functors in the context of pre-triangulated categories.

We recall from [12, Section 1] the following definitions. A *suspended category* $(\mathcal{C}, \Sigma)$ is an additive category $\mathcal{C}$ endowed with an autoequivalence $\Sigma : \mathcal{C} \to \mathcal{C}$, called the *suspension*. For simplicity we consider Σ as an isomorphism, i.e. $\Sigma^{-1} \circ \Sigma = \mathrm{Id}_{\mathcal{C}} = \Sigma \circ \Sigma^{-1}$.

Given a suspended category $(\mathcal{C}, \Sigma)$, a (*candidate*) *triangle* in $\mathcal{C}$ (with respect to Σ) is a diagram of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X.$$

where u, v, w are morphisms in $\mathcal{C}$. The morphism $u : X \to Y$ is called the *base* of the triangle. A morphism (f, g, h) of triangles in $\mathcal{C}$ is a commutative diagram of the form

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X', \end{array}$$

where each row is a triangle in $\mathcal{C}$. If the morphisms f, g and h are isomorphisms in $\mathcal{C}$, then the morphism (f, g, h) of triangles in $\mathcal{C}$ is called an *isomorphism of triangles*.

Definition 5.38. (See [73, Definition 1.1.2], [12, Definition 1.3]) A *pre-triangulated* category $\mathcal{C}$ is a suspended category $(\mathcal{C}, \Sigma)$ together with a class of triangles (with respect to Σ), called *distinguished triangles*, satisfying the following axioms:

T0) Any triangle which is isomorphic to a distinguished triangle is a distinguished triangle. The triangle

$$X \xrightarrow{\mathrm{Id}_X} X \longrightarrow 0 \longrightarrow \Sigma X$$

is distinguished.

T1) For any morphism $f : X \to Y$ in $\mathcal{C}$, there exists a distinguished triangle of the form

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow \Sigma X,$$

i.e., every morphism in $\mathcal{C}$ is the base of some distinguished triangle.

T2) Given two triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

and

$$Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y,$$

if one is a distinguished triangle, then so is the other.

T3) For any commutative diagram of the form

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X', \end{array}$$

where the rows are distinguished triangles, there is a morphism $h : Z \to Z'$ in $\mathcal{C}$, not necessarily unique, which makes the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \Sigma f} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X' \end{array}$$

commutative.

Remark 5.39. The suspension $\Sigma : \mathcal{C} \to \mathcal{C}$ is not required to be additive as Σ is part of an adjunction with $\mathcal{C}$ additive and, whenever $F \dashv G : \mathcal{D} \to \mathcal{C}$ is an adjunction with $\mathcal{C}$ and $\mathcal{D}$ additive, then both F and G are additive, see e.g. [76, Corollary 1.3, page 68].

Remark 5.40. Given a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$, then $v \circ u = 0$, $w \circ v = 0$ and $\Sigma u \circ w = 0$, see [73, Remark 1.1.3].

In Example 1.10 we have recalled the notion of semisimple category. By [49, Theorem 5.3] a category which is pre-triangulated and abelian results to be semisimple.

A functor between pre-triangulated categories is called *exact* if it commutes with the suspension and preserves distinguished triangles. It is known that an exact functor of pre-triangulated categories is additive, see e.g. [80, Lemma 05QY].

We recall some results concerning the coidentifier category.

Lemma 5.41. [5, Lemma 3.24] *Let $\mathcal{C}$ be a category and let $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be an idempotent natural transformation.*

i) If $\mathcal{C}$ is pointed (i.e., it has a zero object), so is the coidentifier $\mathcal{C}_e$.

ii) If $\mathcal{C}$ is (pre)additive, so is the coidentifier $\mathcal{C}_e$ and the functor $H : \mathcal{C} \to \mathcal{C}_e$ is an additive functor.

Proof. Recall from Subsection 2.1.2 that $\mathcal{C}_e$ is the quotient category $\mathcal{C}/\sim$ where $\sim$ is the congruence relation given, for all $f, g : X \to Y$ in $\mathcal{C}$ by $f \sim g$ if, and only if, $e_Y \circ f = e_Y \circ g$.

i). If 1 is a terminal object in $\mathcal{C}$, then the set $\mathrm{Hom}_{\mathcal{C}}(C, 1)$ is a singleton, for every object $C \in \mathcal{C}$. Since $\mathrm{Hom}_{\mathcal{C}_e}(C, 1) = \mathrm{Hom}_{\mathcal{C}}(C, 1)/\sim$, we have that $\mathrm{Hom}_{\mathcal{C}_e}(C, 1)$ is a singleton. Thus, 1 is terminal also in $\mathcal{C}_e$. Similarly, an initial object in $\mathcal{C}$ is initial also in $\mathcal{C}_e$. A zero object is both a terminal and an initial object, thus a zero object in $\mathcal{C}$ is also a zero object in $\mathcal{C}_e$.

ii). If $\mathcal{C}$ is (pre)additive, for any $X, Y \in \mathcal{C}$ the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is an abelian group via a binary operation $+$ and the composition of morphisms is bilinear. Note that $\sim$ is an additive congruence relation. In fact, for all $f, g, f', g' : X \to Y$ in $\mathcal{C}$, if $f \sim f'$ and $g \sim g'$, then $e_Y \circ f = e_Y \circ f'$ and $e_Y \circ g = e_Y \circ g'$ so that $e_Y \circ (f + g) = e_Y \circ f + e_Y \circ g = e_Y \circ f' + e_Y \circ g' = e_Y \circ (f' + g')$ and hence $(f + g) \sim (f' + g')$. It is known that the quotient category of a (pre)additive category modulo an additive congruence relation is also (pre)additive and the quotient functor H is an additive functor. □

Lemma 5.42. [5, Lemma 3.25] *Let $\mathcal{C}$ be a category and let $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be an idempotent natural transformation. If $\mathcal{C}$ has an endofunctor Σ such that $\Sigma e = e\Sigma$, then the coidentifier $\mathcal{C}_e$ has an endofunctor Σ_e such that $H \circ \Sigma = \Sigma_e \circ H$, where $H : \mathcal{C} \to \mathcal{C}_e$ is the quotient functor. Moreover, if $\mathcal{C}$ is a (pre)additive category, then Σ_e is an additive functor whenever Σ is.*

Proof. We have $H\Sigma e = He\Sigma = \mathrm{Id}_H \circ \Sigma = \mathrm{Id}_{H\Sigma}$ so that, by Lemma 2.30, there is a unique functor $\Sigma_e : \mathcal{C}_e \to \mathcal{C}_e$ such that $H \circ \Sigma = \Sigma_e \circ H$. Since H acts as the identity on objects, we get that Σ_e acts as Σ on objects. Moreover, $\Sigma_e \overline{f} = \Sigma_e H f = H\Sigma f = \overline{\Sigma f}$. Assume that $\mathcal{C}$ is a (pre)additive category and Σ is an additive functor. Since $\Sigma_e \left(\overline{f} + \overline{g}\right) = \Sigma_e \left(\overline{f + g}\right) = \overline{\Sigma (f + g)} = \overline{\Sigma f + \Sigma g} = \overline{\Sigma f} + \overline{\Sigma g} = \Sigma_e \overline{f} + \Sigma_e \overline{g}$, we get that Σ_e is additive. □

Corollary 5.43. *Let $\mathcal{C}$ be a suspended category and let $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ be an idempotent natural transformation which commutes with the suspension, i.e., $\Sigma e = e\Sigma$. Then, the coidentifier category $\mathcal{C}_e$ is suspended.*

Proof. Assume that $\mathcal{C}$ is suspended, through an autoequivalence Σ such that $\Sigma e = e\Sigma$. Then, by Lemma 5.41 the coidentifier $\mathcal{C}_e$ is additive, and the quotient functor $H : \mathcal{C} \to \mathcal{C}_e$ is additive. By Lemma 5.42 the coidentifier $\mathcal{C}_e$ has an endofunctor Σ_e such that $H \circ \Sigma = \Sigma_e \circ H$, and Σ_e is an additive functor as so is Σ. From $\Sigma e = e\Sigma$ we deduce $e\Sigma^{-1} = \Sigma^{-1} e$, so by Lemma 5.42 we also have an endofunctor Σ_e^{-1} such that $H \circ \Sigma^{-1} = \Sigma_e^{-1} \circ H$. We compute

$\Sigma_e \circ \Sigma_e^{-1} \circ H = \Sigma_e \circ H \circ \Sigma^{-1} = H \circ \Sigma \circ \Sigma^{-1} = H = \mathrm{Id}_{\mathcal{C}_e} \circ H$ and hence $\Sigma_e \circ \Sigma_e^{-1} = \mathrm{Id}_{\mathcal{C}_e}$ in view of Lemma 2.30. Similarly, $\Sigma_e^{-1} \circ \Sigma_e \circ H = \Sigma_e^{-1} \circ H \circ \Sigma = H \circ \Sigma^{-1} \circ \Sigma = H = \mathrm{Id}_{\mathcal{C}_e} \circ H$, so that $\Sigma_e^{-1} \circ \Sigma_e = \mathrm{Id}_{\mathcal{C}_e}$, thus Σ_e is an isomorphism. □

5.5.1 Stably semiseparable functors and stable (co)monads

Following [12, Section 2], if $\mathcal{C}$ and $\mathcal{D}$ are suspended categories, we say that $F \dashv G : \mathcal{D} \to \mathcal{C}$ is an *adjunction of functors commuting with the suspension* when both F and G commute with the suspension and we tacitly assume that the unit η and counit ϵ commute with the suspension as well. In this case, the monad $(GF, G\epsilon F, \eta)$ is *stable*, meaning that the functor $GF : \mathcal{C} \to \mathcal{C}$, the multiplication $G\epsilon F$ and the unit η commute with the suspension. More generally, a monad $(\top, m, \eta)$ on a category $\mathcal{C}$ is *stable* when the functor $\top : \mathcal{C} \to \mathcal{C}$, the multiplication m and the unit η commute with the suspension. Similarly, one can define a *stable* comonad on a category.

We adapt the notion of *stably separable* functor [12, Definition 3.7] to the semiseparable case. Let $(\mathcal{C}, \Sigma)$ and $(\mathcal{C}', \Sigma')$ be suspended categories. If a functor $G : \mathcal{C}' \to \mathcal{C}$ commutes with the suspension, i.e. $G \circ \Sigma' = \Sigma \circ G$, we say that G is **stably semiseparable** if it is semiseparable through some $\mathcal{P}^G_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(GX, GY) \to \mathrm{Hom}_{\mathcal{C}'}(X, Y)$ that commutes with the suspension, i.e. such that the diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{\mathcal{C}}(GX, GY) & & \xrightarrow{\mathcal{P}^G_{X,Y}} & & \mathrm{Hom}_{\mathcal{C}'}(X, Y) \\ \downarrow{\scriptstyle \mathcal{F}^{\Sigma}_{GX,GY}} & & & & \downarrow{\scriptstyle \mathcal{F}^{\Sigma'}_{X,Y}} \\ \mathrm{Hom}_{\mathcal{C}}(\Sigma GX, \Sigma GY) & = & \mathrm{Hom}_{\mathcal{C}}(G\Sigma' X, G\Sigma' Y) & \xrightarrow{\mathcal{P}^G_{\Sigma' X, \Sigma' Y}} & \mathrm{Hom}_{\mathcal{C}'}(\Sigma' X, \Sigma' Y) \end{array} \tag{5.4}$$

is commutative, that is, $\mathcal{P}^G_{\Sigma' X, \Sigma' Y} \mathcal{F}^{\Sigma}_{GX,GY} = \mathcal{F}^{\Sigma'}_{X,Y} \mathcal{P}^G_{X,Y}$, for every $X, Y \in \mathcal{C}'$. Analogously, one can define the notions of *stably naturally full* and *stably fully faithful* functor. We observe the following facts.

Remark 5.44. Any fully faithful functor $G : \mathcal{C}' \to \mathcal{C}$ which commutes with the suspension is stably fully faithful. Indeed, for every $f : GX \to GY$ in $\mathcal{C}$, we have $(\mathcal{F}^G_{\Sigma' X, \Sigma' Y} \circ \mathcal{P}^G_{\Sigma' X, \Sigma' Y} \circ \mathcal{F}^{\Sigma}_{GX,GY})(f) = \mathcal{F}^{\Sigma}_{GX,GY}(f) = \Sigma f = \Sigma \mathcal{F}^G_{X,Y}(\mathcal{P}^G_{X,Y}(f)) = \Sigma G \mathcal{P}^G_{X,Y}(f) = G\Sigma' \mathcal{P}^G_{X,Y}(f) = (\mathcal{F}^G_{\Sigma' X, \Sigma' Y} \circ \mathcal{F}^{\Sigma'}_{X,Y} \circ \mathcal{P}^G_{X,Y})(f)$, hence, since $\mathcal{F}^G$ is bijective, we get $\mathcal{P}^G_{\Sigma' X, \Sigma' Y} \circ \mathcal{F}^{\Sigma}_{GX,GY} = \mathcal{F}^{\Sigma'}_{X,Y} \circ \mathcal{P}^G_{X,Y}$, so that G is stably fully faithful.

Lemma 5.45. *Let $(\mathcal{C}, \Sigma)$, $(\mathcal{C}', \Sigma')$ and $(\mathcal{C}'', \Sigma'')$ be suspended categories. Consider functors $F : \mathcal{C}'' \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{C}$ commuting with the suspension.*

i) If F is stably semiseparable and G is stably separable, then GF is stably semiseparable.

ii) If F is stably naturally full and G is stably semiseparable, then GF is stably semiseparable.

Proof. In both items by Lemma 2.6 we know that $G \circ F$ is semiseparable through $\mathcal{P}^{GF}_{X,Y} := \mathcal{P}^F_{X,Y} \mathcal{P}^G_{FX,FY}$. Consider the diagram (5.4) for the composite functor $G \circ F : \mathcal{C}'' \to \mathcal{C}$. Then, in both cases the following diagram

$$
\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{C}}(GFX, GFY) & \xrightarrow{\mathcal{P}^G_{FX,FY}} & \mathrm{Hom}_{\mathcal{C}'}(FX, FY) & \xrightarrow{\mathcal{P}^F_{X,Y}} & \mathrm{Hom}_{\mathcal{C}''}(X, Y) \\
\downarrow{\scriptstyle \mathcal{F}^{\Sigma}_{GFX,GFY}} & & \downarrow{\scriptstyle \mathcal{F}^{\Sigma'}_{FX,FY}} & & \downarrow{\scriptstyle \mathcal{F}^{\Sigma''}_{X,Y}} \\
\mathrm{Hom}_{\mathcal{C}}(\Sigma GFX, \Sigma GFY) & & & & \\
\| & & & & \\
\mathrm{Hom}_{\mathcal{C}}(G\Sigma' FX, G\Sigma' FY) & \xrightarrow{\mathcal{P}^G_{\Sigma' FX,\Sigma' FY}} & \mathrm{Hom}_{\mathcal{C}'}(\Sigma' FX, \Sigma' FY) & & \\
\| & & \| & & \\
\mathrm{Hom}_{\mathcal{C}}(GF\Sigma'' X, GF\Sigma'' Y) & \xrightarrow{\mathcal{P}^G_{F\Sigma'' X,F\Sigma'' Y}} & \mathrm{Hom}_{\mathcal{C}'}(F\Sigma'' X, F\Sigma'' Y) & \xrightarrow{\mathcal{P}^F_{\Sigma'' X,\Sigma'' Y}} & \mathrm{Hom}_{\mathcal{C}''}(\Sigma'' X, \Sigma'' Y)
\end{array}
$$

commutes, as the inner rectangles are commutative, hence GF is stably semiseparable. □

Unless otherwise specified, hereafter we denote all suspensions by the same letter Σ.

Lemma 5.46. [5, Lemma 3.26] *Let $F : \mathcal{C} \to \mathcal{D}$ be a stably semiseparable functor. Then, the associated idempotent natural transformation commutes with the suspension.*

Proof. By definition, F is semiseparable through some $\mathcal{P}^F$ such that $\mathcal{P}^F_{\Sigma X,\Sigma Y} \circ \mathcal{F}^{\Sigma}_{FX,FY} = \mathcal{F}^{\Sigma}_{X,Y} \circ \mathcal{P}^F_{X,Y}$. Consider the associated idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ which is defined by setting $e_X := \mathcal{P}^F_{X,X}(\mathrm{Id}_{FX})$ for every X in $\mathcal{C}$. Then $\Sigma e_X = \mathcal{F}^{\Sigma}_{X,X}\mathcal{P}^F_{X,X}(\mathrm{Id}_{FX}) = \mathcal{P}^F_{\Sigma X,\Sigma X}\mathcal{F}^{\Sigma}_{FX,FY}(\mathrm{Id}_{FX}) = \mathcal{P}^F_{\Sigma X,\Sigma X}\Sigma(\mathrm{Id}_{FX}) = \mathcal{P}^F_{\Sigma X,\Sigma X}(\mathrm{Id}_{\Sigma FX}) = \mathcal{P}^F_{\Sigma X,\Sigma X}(\mathrm{Id}_{F\Sigma X}) = e_{\Sigma X}$ and hence $\Sigma e = e\Sigma$, i.e. e commutes with the suspension. □

Moreover, given a suspended category $\mathcal{C}$, a stable separable monad $(\top, m, \eta)$ on $\mathcal{C}$, through a section σ, is said to be *stably separable* [12, Definition 3.5] if σ commutes with the suspension. Dually, a stable coseparable comonad $(\perp, \Delta, \epsilon)$ on $\mathcal{C}$, through a retraction τ, is said to be *stably coseparable* if τ commutes with the suspension.

Lemma 5.47. *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction of functors between suspended categories commuting with the suspension.*

i) If G a is stably semiseparable functor, then the monad $(GF, G\epsilon F, \eta)$ is stably separable.

ii) If F a is stably semiseparable functor, then the comonad $(FG, F\eta G, \epsilon)$ is stably coseparable.

Proof. We only show *i)* as *ii)* follows by dual arguments. By Lemma 2.43, $(GF, G\epsilon F, \eta)$ is a separable monad through the section $\sigma := G\gamma F : GF \to GFGF$ where $\gamma : \mathrm{Id} \to FG$ is defined by $\gamma_X := \mathcal{P}_{X,FGX}(\eta_{GX})$. Moreover, GF is a stable monad. Thus,

$$
\begin{aligned}
\sigma_{\Sigma X} = G\gamma_{F\Sigma X} = G\mathcal{P}_{F\Sigma X,FGF\Sigma X}(\eta_{GF\Sigma X}) = G\mathcal{P}_{\Sigma FX,\Sigma FGFX}(\eta_{\Sigma GFX}) \\
= G\mathcal{P}_{\Sigma FX,\Sigma FGFX}(\Sigma\eta_{GFX}) = G\Sigma\mathcal{P}_{FX,FGFX}(\eta_{GFX}) = G\Sigma\gamma_{FX} = \Sigma G\gamma_{FX} = \Sigma\sigma_X,
\end{aligned}
$$

and hence σ commutes with the suspension, obtaining that it is a stably separable monad. □

We now recall Balmer's [12, Theorem 4.1] and we prove the announced analogue for semiseparable functors.

Theorem 5.48. [12, Theorem 4.1] *Let $\mathcal{C}$ be a pre-triangulated category and let $\mathcal{D}$ be an idempotent complete suspended category. Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction of functors commuting with the suspension. Assume that the stable monad $GF : \mathcal{C} \to \mathcal{C}$ is an exact functor and that G is stably separable. Then, $\mathcal{D}$ is pre-triangulated with distinguished triangles being exactly the ones whose image through the functor G is distinguished in $\mathcal{C}$. Moreover, with this pre-triangulation both functors F and G become exact.*

Theorem 5.49. [5, Theorem 3.27] *Let $\mathcal{C}$ be a pre-triangulated category and let $\mathcal{D}$ be an idempotent complete suspended category. Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction of functors commuting with the suspension. Suppose that the stable monad $GF : \mathcal{C} \to \mathcal{C}$ is an exact functor and that G is a stably semiseparable functor with associated idempotent natural transformation $e : \mathrm{Id}_{\mathcal{D}} \to \mathrm{Id}_{\mathcal{D}}$. Then, the coidentifier $\mathcal{D}_e$ is idempotent complete and pre-triangulated with distinguished triangles being exactly the ones whose image through the functor $G_e : \mathcal{D}_e \to \mathcal{C}$ (determined by the factorization $G = G_e \circ H$) is distinguished in $\mathcal{C}$. Moreover, with respect to this pre-triangulation, both functors $G_e : \mathcal{D}_e \to \mathcal{C}$ and its left adjoint $L_e : \mathcal{C} \to \mathcal{D}_e$ become exact.*

Proof. Since G is stably semiseparable, by Lemma 5.46 the associated idempotent natural transformation $e : \mathrm{Id}_{\mathcal{D}} \to \mathrm{Id}_{\mathcal{D}}$ commutes with the suspension, i.e., $e\Sigma = \Sigma e$. Then, by Corollary 5.43 the coidentifier category $\mathcal{D}_e$ is suspended through Σ_e such that $H \circ \Sigma = \Sigma_e \circ H$, where $H : \mathcal{D} \to \mathcal{D}_e$ is the quotient functor, and by Lemma 5.27 it is idempotent complete.

Since G is semiseparable, by Theorem 2.33 it factorizes as $G = G_e \circ H$ for a unique separable functor $G_e : \mathcal{D}_e \to \mathcal{C}$. Moreover, G_e is separable via $\mathcal{P}^{G_e}$ defined by $\mathcal{P}^{G_e}_{HX,HY} := \mathcal{F}^{H}_{X,Y} \circ \mathcal{P}^{G}_{X,Y}$ for all X, Y in $\mathcal{D}$. Since G commutes with the suspension, we have $G_e \circ \Sigma_e \circ H = G_e \circ H \circ \Sigma = G \circ \Sigma = \Sigma \circ G = \Sigma \circ G_e \circ H$ and hence, in view of Lemma 2.30, $G_e \circ \Sigma_e = \Sigma \circ G_e$, i.e. G_e commutes with the suspension as well. Now consider the composite functor $L_e = H \circ F : \mathcal{C} \to \mathcal{D}_e$, which is the left adjoint of G_e with unit η_e and counit ϵ_e given as in Lemma 5.32. Then, $\Sigma_e \circ L_e = \Sigma_e \circ H \circ F = H \circ \Sigma \circ F = H \circ F \circ \Sigma = L_e \circ \Sigma$, so that L_e commutes with the suspension as well. Note that $\epsilon_e \Sigma_e H = \epsilon_e H \Sigma = H \epsilon \Sigma = H \Sigma \epsilon = \Sigma_e H \epsilon = \Sigma_e \epsilon_e H$, so that $\epsilon_e \Sigma_e = \Sigma_e \epsilon_e$. Moreover, $\eta_e \Sigma = \eta \Sigma = \Sigma \eta = \Sigma \eta_e$. Thus, also the unit and counit of the adjunction (L_e, G_e) commute with the suspensions. Hence $L_e \dashv G_e$ is an adjunction of functors commuting with the suspension. By Remark 5.39 both G_e and L_e are additive. By Lemma 5.32, the adjunctions (L_e, G_e) and (F, G) have the same associated monad. As a consequence, we get that $G_e L_e = GF$ is a stable monad and an exact functor as by assumption so is GF. We have $\mathcal{F}^{\Sigma_e}_{HX,HY} \mathcal{P}^{G_e}_{HX,HY} = \mathcal{F}^{\Sigma_e}_{HX,HY} \mathcal{F}^{H}_{X,Y} \mathcal{P}^{G}_{X,Y} = \mathcal{F}^{\Sigma_e H}_{X,Y} \mathcal{P}^{G}_{X,Y} = \mathcal{F}^{H\Sigma}_{X,Y} \mathcal{P}^{G}_{X,Y} = \mathcal{F}^{H}_{\Sigma X,\Sigma Y} \mathcal{F}^{\Sigma}_{X,Y} \mathcal{P}^{G}_{X,Y} = \mathcal{F}^{H}_{\Sigma X,\Sigma Y} \mathcal{P}^{G}_{\Sigma X,\Sigma Y} \mathcal{F}^{\Sigma}_{GX,GY} = \mathcal{P}^{G_e}_{H\Sigma X,H\Sigma Y} \mathcal{F}^{\Sigma}_{G_e HX,G_e HY} = \mathcal{P}^{G_e}_{\Sigma_e HX,\Sigma_e HY} \mathcal{F}^{\Sigma}_{G_e HX,G_e HY}$, for all X, Y in $\mathcal{D}$. Since H is surjective on objects, this means that $\mathcal{F}^{\Sigma_e}_{X,Y} \mathcal{P}^{G_e}_{X,Y} = \mathcal{P}^{G_e}_{\Sigma_e X,\Sigma_e Y} \mathcal{F}^{\Sigma}_{G_e X,G_e Y}$ for all X, Y in $\mathcal{D}_e$, i.e., G_e is a stably separable functor. Then, we can apply Theorem 5.48 to the adjunction $L_e \dashv G_e : \mathcal{D}_e \to \mathcal{C}$. As a consequence, the coidentifier $\mathcal{D}_e$ is pre-triangulated with distinguished triangles Δ being exactly the ones whose image $G_e(\Delta)$ through the functor $G_e : \mathcal{D}_e \to \mathcal{C}$ is distinguished in $\mathcal{C}$. Moreover, with respect to this pre-triangulation, both functors $G_e : \mathcal{D}_e \to \mathcal{C}$ and $L_e : \mathcal{C} \to \mathcal{D}_e$ become exact. □

Pre-triangulation on the Eilenberg-Moore categories

In [12, Definition 2.4], it is claimed that, when $\mathcal{C}$ is a suspended category with the suspension Σ, and $\top$ is an additive stable monad on it, then the Eilenberg-Moore category $\mathcal{C}_\top$ of $\top$-modules inherits a structure of suspended category such that $V_\top \dashv U_\top : \mathcal{C}_\top \to \mathcal{C}$

is an adjunction of additive functors commuting with the suspension. Explicitly, the suspension $\Sigma_{\top} : \mathcal{C}_{\top} \to \mathcal{C}_{\top}$ is defined by

$$\begin{aligned} \Sigma_{\top}(C,\mu) &:= (\Sigma C, \Sigma\mu : \Sigma\top C = \top\Sigma C \to \Sigma C), \\ \Sigma_{\top}(f) &:= \Sigma f : (\Sigma C, \Sigma\mu) \to (\Sigma C', \Sigma\mu'), \end{aligned} \tag{5.5}$$

for every object (C,μ) and every morphism $f : (C,\mu) \to (C',\mu')$ in $\mathcal{C}_{\top}$.

Given a monad $\top$ on a triangulated category $\mathcal{C}$, in [33] the authors investigate whether the Eilenberg-Moore category $\mathcal{C}_{\top}$ inherits the structure of triangulated category from $\mathcal{C}$: this seems to rarely occur, and they quote [12] as a particular occurrence.

In the following result, as a consequence of Theorem 5.49, we show that $\mathcal{C}_{GF}$ inherits the structure of pre-triangulated category from $\mathcal{C}$.

Corollary 5.50. [5, Corollary 3.29] *Let $\mathcal{C}$ be a pre-triangulated category and let $\mathcal{D}$ be an idempotent complete suspended category. Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction of functors commuting with the suspension. Suppose that the stable monad $GF : \mathcal{C} \to \mathcal{C}$ is an exact functor and that G is a stably semiseparable functor with associated idempotent natural transformation $e : \mathrm{Id}_{\mathcal{D}} \to \mathrm{Id}_{\mathcal{D}}$. Then, the Eilenberg-Moore category $\mathcal{C}_{GF}$ is idempotent complete and pre-triangulated with distinguished triangles being exactly the ones whose image through the forgetful functor $U_{GF} : \mathcal{C}_{GF} \to \mathcal{C}$ is distinguished in $\mathcal{C}$. Moreover, with respect to this pre-triangulation, both the functor $U_{GF} : \mathcal{C}_{GF} \to \mathcal{C}$ and its left adjoint $V_{GF} : \mathcal{C} \to \mathcal{C}_{GF}$ become exact. Furthermore, there is a unique exact equivalence of categories $(K_{GF})_e : \mathcal{D}_e \to \mathcal{C}_{GF}$ such that $(K_{GF})_e \circ H = K_{GF}$ and $U_{GF} \circ (K_{GF})_e = G_e$.*

Proof. Assume that G is stably semiseparable with associated idempotent natural transformation $e : \mathrm{Id}_{\mathcal{D}} \to \mathrm{Id}_{\mathcal{D}}$. By Proposition 5.31 *i*), there is a unique functor $(K_{GF})_e : \mathcal{D}_e \to \mathcal{C}_{GF}$ such that $(K_{GF})_e \circ H = K_{GF}$ and $U_{GF} \circ (K_{GF})_e = G_e$. Moreover, since $\mathcal{D}$ is idempotent complete, then the functor $(K_{GF})_e$ is an equivalence of categories. By Lemma 5.27 $\mathcal{D}_e$ is idempotent complete, hence also $\mathcal{C}_{GF}$ becomes idempotent complete. Since $\mathcal{C}$ is pre-triangulated, it is suspended, and the monad $(GF, G\epsilon F, \eta)$ is stable as $F \dashv G : \mathcal{D} \to \mathcal{C}$ is an adjunction of functors commuting with the suspension. Moreover, the functor $GF : \mathcal{C} \to \mathcal{C}$ is additive being an exact functor between pre-triangulated categories. Thus, the Eilenberg-Moore category $\mathcal{C}_{GF}$ inherits a structure of suspended category through the suspension Σ_{GF} (see (5.5)) such that $V_{GF} \dashv U_{GF} : \mathcal{C}_{GF} \to \mathcal{C}$ is an adjunction of additive functors commuting with the suspension. Also the comparison functor $K_{GF} : \mathcal{D} \to \mathcal{C}_{GF}$ commutes with the suspension. Indeed,

$$\begin{aligned} (\Sigma_{GF} \circ K_{GF})(D) &= \Sigma_{GF}(GD, G\epsilon_D) = (\Sigma GD, \Sigma G\epsilon_D) = (G\Sigma D, G\Sigma\epsilon_D) \\ &= (G\Sigma D, G\epsilon_{\Sigma D}) = (K_{GF} \circ \Sigma)(D), \end{aligned}$$

and $(\Sigma_{GF} \circ K_{GF})(f) = \Sigma_{GF}(Gf) = \Sigma Gf = G\Sigma f = (K_{GF} \circ \Sigma)(f)$. Note that the monad $(GF, G\epsilon F, \eta)$ is stably separable in view of Lemma 5.47 *i*). By [12, Proposition 3.11], this means that $U_{GF} : \mathcal{C}_{GF} \to \mathcal{C}$ is a stably separable functor.

Then, Theorem 5.48 applied to the adjunction $V_{GF} \dashv U_{GF}$ yields a pre-triangulation on $\mathcal{C}_{GF}$, with distinguished triangles Δ being exactly the ones such that $U_{GF}(\Delta)$ is distinguished in $\mathcal{C}$. Moreover, with respect to this pre-triangulation, both functors U_{GF} and V_{GF} become exact. Note that $\Sigma_{GF} \circ (K_{GF})_e \circ H = \Sigma_{GF} \circ K_{GF} = K_{GF} \circ \Sigma = (K_{GF})_e \circ H \circ \Sigma = (K_{GF})_e \circ \Sigma_e \circ H$ and hence $\Sigma_{GF} \circ (K_{GF})_e = (K_{GF})_e \circ \Sigma_e$, i.e. the equivalence $(K_{GF})_e$ commutes with the suspension.

To check that $(K_{GF})_e$ is exact (hence additive), it remains to prove that it preserves distinguished triangles. Let Δ be a distinguished triangle in $\mathcal{D}_e$. Then, by Theorem 5.49,

$G_e(\Delta)$ is distinguished in $\mathcal{C}$. Since $U_{GF} \circ (K_{GF})_e = G_e$, we get that $U_{GF}\left((K_{GF})_e(\Delta)\right)$ is distinguished in $\mathcal{C}$. By definition of pre-triangulation on $\mathcal{C}_{GF}$ we obtain that $(K_{GF})_e(\Delta)$ is distinguished in $\mathcal{C}_{GF}$. Thus, $(K_{GF})_e$ is exact. □

As pointed out in [12, Remark 5.19], one can obtain similar results for the Eilenberg-Moore category of $\perp$-comodules over a comonad $\perp$, and we can also obtain the dual of Theorem 5.49 and Corollary 5.50. We include the proofs for the reader's sake. We first need the following dual of Theorem 5.48, whose proof follows by similar arguments.

Theorem 5.51. (Dual of [12, Theorem 4.1]) *Let $\mathcal{D}$ be a pre-triangulated category and let $\mathcal{C}$ be an idempotent complete suspended category. Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction of functors commuting with the suspension. Assume that the stable comonad $FG : \mathcal{D} \to \mathcal{D}$ is exact and that F is a stably separable functor. Then, $\mathcal{C}$ is pre-triangulated with distinguished triangles being exactly the ones whose image through F is distinguished in $\mathcal{D}$. Moreover, with this pre-triangulation both functors F and G become exact.*

Proof. Let us verify axioms T0)-T3) of Definition 5.38. We recall that by Remark 5.39 the functor F is additive.

T0). Let $\Delta := (\ X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X\ )$ be a triangle in $\mathcal{C}$ isomorphic to a distinguished triangle $\Delta' := (\ X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} \Sigma X'\ )$ in $\mathcal{C}$ (i.e., the triangle $F(\Delta')$ is distinguished in $\mathcal{D}$). Since $F(\Delta) \cong F(\Delta')$ and $\mathcal{D}$ is pre-triangulated, then $F(\Delta)$ is distinguished in $\mathcal{D}$, so Δ is distinguished in $\mathcal{C}$. Moreover, the triangle $X \xrightarrow{\mathrm{Id}_X} X \longrightarrow 0 \longrightarrow \Sigma X$ is distinguished in $\mathcal{C}$, as so is the triangle $FX \xrightarrow{\mathrm{Id}_{FX}} FX \longrightarrow F0 = 0 \longrightarrow F\Sigma X = \Sigma FX$ in $\mathcal{D}$.

T1). Let $f : X \to Y$ be a morphism in $\mathcal{C}$ and consider a distinguished triangle $\tilde{\Delta}$ in the pre-triangulated category $\mathcal{D}$ with base Ff:

$$\tilde{\Delta} = (\ FX \xrightarrow[Ff]{} FY \xrightarrow[\tilde{g}]{} \tilde{Z} \xrightarrow[\tilde{h}]{} \Sigma FX\ ).$$

By naturality of $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$, $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$, and by $\nu \circ \eta = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}}$, we have the diagram

$$G(\tilde{\Delta}): \qquad \begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \dashrightarrow & \bullet & \dashrightarrow & \Sigma X \\
\downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_Y} & & \vdots & & \downarrow{\scriptstyle \Sigma\eta_X} \\
GFX & \xrightarrow{GFf} & GFY & \xrightarrow{G\tilde{g}} & G\tilde{Z} & \xrightarrow{G\tilde{h}} & G\Sigma FX = \Sigma GFX \\
\downarrow{\scriptstyle \nu_X} & & \downarrow{\scriptstyle \nu_Y} & & \downarrow & & \downarrow{\scriptstyle \Sigma\nu_X} \\
X & \xrightarrow{f} & Y & \dashrightarrow & \bullet & \dashrightarrow & \Sigma X,
\end{array}$$

where the morphism $f : X \to Y$ is a direct summand of the morphism GFf in $\mathcal{C}$. Since FG is exact, then $FG(\tilde{\Delta})$ is distinguished in $\mathcal{D}$, so $G(\tilde{\Delta})$ is distinguished in $\mathcal{C}$. By [12, Lemma 1.6 (d)] there is an idempotent $e = (\eta_X\nu_X, \eta_Y\nu_Y, r) = e^2$ in $\mathcal{C}$ of the distinguished triangle $G(\tilde{\Delta})$:

$$\begin{array}{ccccccc}
GFX & \xrightarrow{GFf} & GFY & \xrightarrow{G\tilde{g}} & G\tilde{Z} & \xrightarrow{G\tilde{h}} & \Sigma GFX \\
\downarrow{\scriptstyle \eta_X\nu_X} & & \downarrow{\scriptstyle \eta_Y\nu_Y} & & \downarrow{\scriptstyle \exists\, r=r^2} & & \downarrow{\scriptstyle \Sigma\eta_X\nu_X} \\
GFX & \xrightarrow{GFf} & GFY & \xrightarrow{G\tilde{g}} & G\tilde{Z} & \xrightarrow{G\tilde{h}} & \Sigma GFX.
\end{array}$$

Since $\mathcal{C}$ is idempotent complete, [12, Proposition 1.10] gives a decomposition $G(\tilde{\Delta}) = \Delta \oplus \Delta'$, for triangles Δ and Δ' corresponding to the idempotents e and $\mathrm{Id} - e$, respectively. By construction, the summand Δ corresponding to e has the form $\Delta =$

$(X \xrightarrow{f} Y \xrightarrow{g} \mathrm{Im}(r) \xrightarrow{h} \Sigma X)$, where $g = rG(\tilde{g})\eta_Y$ and $h = \Sigma\nu_X G(\tilde{h})r$. Since FG is exact, the triangle $FG(\tilde{\Delta})$ is distinguished in $\mathcal{D}$. Since $FG(\tilde{\Delta}) = F(\Delta) \oplus F(\Delta')$ and by [73, Proposition 1.2.3] a direct summand of a distinguished triangle in the pre-triangulated category $\mathcal{D}$ is distinguished, we get that $F(\Delta)$ is distinguished in $\mathcal{D}$ and so Δ is distinguished in $\mathcal{C}$.

T2). Let $\Delta := (X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X)$ and $\Delta' := (Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y)$ be two triangles in $\mathcal{C}$. If Δ is a distinguished triangle in $\mathcal{C}$, then so is $F(\Delta)$ in $\mathcal{D}$. Since $\mathcal{D}$ is pre-triangulated, also the triangle

$$F(\Delta') = (FY \xrightarrow[-Fv=F(-v)]{} FZ \xrightarrow[-Fw=F(-w)]{} F\Sigma X \xrightarrow[-\Sigma Fu=F(-\Sigma u)]{} F\Sigma Y = \Sigma FY)$$

is distinguished in $\mathcal{D}$. Hence Δ' is distinguished in $\mathcal{C}$. Similarly, if Δ' is distinguished in $\mathcal{C}$, then $F(\Delta')$ is distinguished in $\mathcal{D}$, so is $F(\Delta)$. Hence Δ is distinguished in $\mathcal{C}$.

T3). Consider the commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X',
\end{array}$$

where the rows are distinguished triangles. By applying the functor F we get a similar diagram in $\mathcal{D}$, so from axiom T3) in $\mathcal{D}$ we get a fill-in map $\tilde{\gamma} : FZ \to FZ'$. Then, we have $\mathcal{P}_{Z,Z'}(\tilde{\gamma}) \circ v = \mathcal{P}_{Y,Z'}(\tilde{\gamma} \circ Fv) = \mathcal{P}_{Y,Z'}(Fv' \circ Fg) = \mathcal{P}_{Y,Z'}(F(v \circ g)) = v \circ g$, and $w' \circ \mathcal{P}_{Z,Z'}(\tilde{\gamma}) = \mathcal{P}_{Z,\Sigma X'}(Fw' \circ \tilde{\gamma}) = \mathcal{P}_{Z,\Sigma X'}(\Sigma Ff \circ Fw) = \mathcal{P}_{Z,\Sigma X'}(F\Sigma f \circ Fw) = \mathcal{P}_{Z,\Sigma X'}(F(\Sigma f \circ w)) = \Sigma f \circ w$, so the diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \mathcal{P}_{Z,Z'}(\tilde{\gamma})} & & \downarrow{\scriptstyle \Sigma f} \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X'
\end{array}$$

is commutative.

With this pre-triangulation both functors F and G become exact. □

We now show the dual of Theorem 5.49.

Theorem 5.52. *Let $\mathcal{D}$ be a pre-triangulated category and let $\mathcal{C}$ be an idempotent complete suspended category. Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction of functors commuting with the suspension. Suppose that the stable comonad $FG : \mathcal{D} \to \mathcal{D}$ is an exact functor and that F is a stably semiseparable functor with associated idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$. Then, the coidentifier $\mathcal{C}_e$ is idempotent complete and pre-triangulated with distinguished triangles being exactly the ones whose image through the functor $F_e : \mathcal{C}_e \to \mathcal{D}$ (determined by the factorization $F = F_e \circ H$) is distinguished in $\mathcal{D}$. Moreover, with respect to this pre-triangulation, both functors $F_e : \mathcal{C}_e \to \mathcal{D}$ and its right adjoint $R_e : \mathcal{D} \to \mathcal{C}_e$ become exact.*

Proof. Since F is stably semiseparable, by Lemma 5.46 the associated idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$ commutes with the suspension, i.e. $e\Sigma = \Sigma e$. Then, by Corollary 5.43 the coidentifier category $\mathcal{C}_e$ is suspended through Σ_e such that $H \circ \Sigma = \Sigma_e \circ H$, where $H : \mathcal{C} \to \mathcal{C}_e$ is the quotient functor, and by Lemma 5.27 it is idempotent complete. Since F is semiseparable, by Theorem 2.33 it factorizes as $F = F_e \circ H$ for a unique separable functor $F_e : \mathcal{C}_e \to \mathcal{D}$. Moreover, F_e is separable via $\mathcal{P}^{F_e}$ defined by $\mathcal{P}^{F_e}_{HX,HY} := \mathcal{F}^H_{X,Y} \circ \mathcal{P}^F_{X,Y}$

for all X, Y in $\mathcal{D}$. Since F commutes with the suspension, we have $F_e \circ \Sigma_e \circ H = F_e \circ H \circ \Sigma = F \circ \Sigma = \Sigma \circ F = \Sigma \circ F_e \circ H$ and hence $F_e \circ \Sigma_e = \Sigma \circ F_e$, i.e. F_e commutes with the suspension as well. Now consider the composite functor $R_e = H \circ G : \mathcal{D} \to \mathcal{C}_e$, which is the right adjoint of F_e with unit η_e and counit ϵ_e given as in Lemma 5.32 *ii*). Then, $\Sigma_e \circ R_e = \Sigma_e \circ H \circ G = H \circ \Sigma \circ G = H \circ G \circ \Sigma = R_e \circ \Sigma$ so that R_e commutes with the suspension as well. Note that $\epsilon_e \Sigma_e H = \epsilon_e H \Sigma = H \epsilon \Sigma = H \Sigma \epsilon = \Sigma_e H \epsilon = \Sigma_e \epsilon_e H$ so that $\epsilon_e \Sigma_e = \Sigma_e \epsilon_e$. Moreover, $\eta_e \Sigma = \eta \Sigma = \Sigma \eta = \Sigma \eta_e$. Thus, also the unit and counit of the adjunction (F_e, R_e) commute with the suspensions. Hence $F_e \dashv R_e$ is an adjunction of functors commuting with the suspension. By Lemma 5.32, the adjunctions (F_e, R_e) and (F, G) have the same associated comonad. As a consequence, we get that $F_e R_e = FG$ is a stable comonad and an exact functor by assumption. We have $\mathcal{F}^{\Sigma_e}_{HX,HY} \mathcal{P}^{F_e}_{HX,HY} = \mathcal{F}^{\Sigma_e}_{HX,HY} \mathcal{F}^{H}_{X,Y} \mathcal{P}^{F}_{X,Y} = \mathcal{F}^{\Sigma_e H}_{X,Y} \mathcal{P}^{F}_{X,Y} = \mathcal{F}^{H\Sigma}_{X,Y} \mathcal{P}^{F}_{X,Y} = \mathcal{F}^{H}_{\Sigma X,\Sigma Y} \mathcal{F}^{\Sigma}_{X,Y} \mathcal{P}^{F}_{X,Y} = \mathcal{F}^{H}_{\Sigma X,\Sigma Y} \mathcal{P}^{F}_{\Sigma X,\Sigma Y} \mathcal{F}^{\Sigma}_{FX,FY} = \mathcal{P}^{F_e}_{H\Sigma X,H\Sigma Y} \mathcal{F}^{\Sigma}_{F_e HX,F_e HY} = \mathcal{P}^{F_e}_{\Sigma_e HX,\Sigma_e HY} \mathcal{F}^{\Sigma}_{F_e HX,F_e HY}$, for all X, Y in $\mathcal{C}$. Since H is surjective on objects, this means $\mathcal{F}^{\Sigma_e}_{X,Y} \mathcal{P}^{F_e}_{X,Y} = \mathcal{P}^{F_e}_{\Sigma_e X,\Sigma_e Y} \mathcal{F}^{\Sigma}_{F_e X,F_e Y}$ for all X, Y in $\mathcal{C}_e$, i.e. F_e is a stably separable functor.

Then, we can apply Theorem 5.51 to the adjunction $F_e \dashv R_e : \mathcal{D} \to \mathcal{C}_e$. As a consequence, the coidentifier $\mathcal{C}_e$ is pre-triangulated with distinguished triangles Δ being exactly the ones whose image $F_e(\Delta)$ through the functor $F_e : \mathcal{C}_e \to \mathcal{D}$ is distinguished in $\mathcal{D}$. Moreover, with respect to this pre-triangulation, both functors $F_e : \mathcal{C}_e \to \mathcal{D}$ and $R_e : \mathcal{D} \to \mathcal{C}_e$ become exact. □

Before proving the dual of Corollary 5.50, we observe that when $\mathcal{D}$ is a suspended category with suspension Σ, and $\perp$ is an additive stable comonad on it, then the Eilenberg-Moore category $\mathcal{D}^{\perp}$ of $\perp$-comodules inherits a structure of suspended category such that $U^{\perp} \dashv V^{\perp} : \mathcal{D} \to \mathcal{D}^{\perp}$ is an adjunction of additive functors commuting with the suspension. Explicitly, the suspension $\Sigma^{\perp} : \mathcal{D}^{\perp} \to \mathcal{D}^{\perp}$ is defined by

$$\begin{aligned} \Sigma^{\perp}(D, \rho_D) &:= (\Sigma D, \Sigma\rho_D : \Sigma D \to \Sigma{\perp}D = {\perp}\Sigma D), \\ \Sigma^{\perp}(f) &:= \Sigma f : (\Sigma D, \Sigma\rho_D) \to (\Sigma D', \Sigma\rho_{D'}), \end{aligned} \tag{5.6}$$

for every object (D, ρ_D) and every morphism $f : (D, \rho_D) \to (D', \rho_{D'})$ in $\mathcal{D}^{\perp}$.

Corollary 5.53. *Let $\mathcal{D}$ be a pre-triangulated category and let $\mathcal{C}$ be an idempotent complete suspended category. Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction of functors commuting with the suspension. Suppose that the stable comonad $FG : \mathcal{D} \to \mathcal{D}$ is an exact functor and that F is a stably semiseparable functor with associated idempotent natural transformation $e : \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}$. Then, the Eilenberg-Moore category $\mathcal{D}^{FG}$ is idempotent complete and pre-triangulated with distinguished triangles being exactly the ones whose image through the forgetful functor $U^{FG} : \mathcal{D}^{FG} \to \mathcal{D}$ is distinguished in $\mathcal{D}$. Moreover, with respect to this pre-triangulation, both the functor $U^{FG} : \mathcal{D}^{FG} \to \mathcal{D}$ and its right adjoint $V^{FG} : \mathcal{D} \to \mathcal{D}^{FG}$ become exact. Furthermore, there is a unique exact equivalence of categories $(K^{FG})_e : \mathcal{C}_e \to \mathcal{D}^{FG}$ such that $(K^{FG})_e \circ H = K^{FG}$ and $U^{FG} \circ (K^{FG})_e = F_e$.*

Proof. Assume that F is stably semiseparable. By Proposition 5.31 *ii*), there is a unique functor $(K^{FG})_e : \mathcal{C}_e \to \mathcal{D}^{FG}$ such that $(K^{FG})_e \circ H = K^{FG}$ and $U^{FG} \circ (K^{FG})_e = F_e$. Since $\mathcal{C}$ is idempotent complete, then the functor $(K^{FG})_e$ is an equivalence of categories. By Lemma 5.27 $\mathcal{C}_e$ is idempotent complete, so that also $\mathcal{D}^{FG}$ becomes idempotent complete. Note that $\mathcal{D}$ is suspended as it is pre-triangulated. Since $F \dashv G : \mathcal{D} \to \mathcal{C}$ is an adjunction of functors commuting with the suspension, the comonad $(FG, F\eta G, \epsilon)$ is stable. Moreover the functor $FG : \mathcal{D} \to \mathcal{D}$ is additive being an exact functor between pre-triangulated categories. Thus, the Eilenberg-Moore category $\mathcal{D}^{FG}$ inherits a structure of suspended category through the suspension Σ^{FG} (given as in (5.6)) such that $U^{FG} \dashv V^{FG} : \mathcal{D} \to \mathcal{D}^{FG}$ is an adjunction of additive functors commuting with the suspension. The cocomparison functor $K^{FG} : \mathcal{C} \to \mathcal{D}^{FG}$ commutes with the suspension as well. Indeed,

$$\begin{aligned}\left(\Sigma^{FG} \circ K^{FG}\right)(C) &= \Sigma^{FG}(FC, F\eta_C) = (\Sigma FC, \Sigma F\eta_C) = (F\Sigma C, F\Sigma\eta_C) \\ &= (F\Sigma C, F\eta_{\Sigma C}) = \left(K^{FG} \circ \Sigma\right)(C)\end{aligned}$$

and $\left(\Sigma^{FG} \circ K^{FG}\right)(f) = \Sigma^{FG}(Ff) = \Sigma Ff = F\Sigma f = \left(K^{FG} \circ \Sigma\right)(f)$. By Lemma 5.47 *ii)* the comonad $(FG, F\eta G, \epsilon)$ is stably separable and, by the dual of [12, Proposition 3.11], this means that $U^{FG} : \mathcal{D}^{FG} \to \mathcal{D}$ is a stably separable functor.

Then, Theorem 5.51, applied to the adjunction $U^{FG} \dashv V^{FG}$, yields a pre-triangulation on $\mathcal{D}^{FG}$ with distinguished triangles Δ being exactly the ones such that $U^{FG}(\Delta)$ is distinguished in $\mathcal{D}$. Moreover, with respect to this pre-triangulation, both functors U^{FG} and V^{FG} become exact. Note that $\Sigma^{FG} \circ (K^{FG})_e \circ H = \Sigma^{FG} \circ K^{FG} = K^{FG} \circ \Sigma = (K^{FG})_e \circ H \circ \Sigma = (K^{FG})_e \circ \Sigma_e \circ H$ and hence $\Sigma^{FG} \circ (K^{FG})_e = (K^{FG})_e \circ \Sigma_e$, i.e. the equivalence $(K^{FG})_e$ commutes with the suspension.

We check that $(K_{FG})_e$ preserves distinguished triangles. Let Δ be a distinguished triangle in $\mathcal{D}_e$. Then, by Theorem 5.52, $F_e(\Delta)$ is distinguished in $\mathcal{D}$. Since $U^{FG} \circ (K^{FG})_e = F_e$, we get that $U^{FG}\left((K^{FG})_e(\Delta)\right)$ is distinguished in $\mathcal{D}$. By definition of pre-triangulation on $\mathcal{D}^{FG}$ we obtain that $(K^{FG})_e(\Delta)$ is distinguished in $\mathcal{D}^{FG}$. Thus, $(K^{FG})_e$ is exact. □

Pre-triangulation on the Kleisli category

Let $\mathcal{C}$ be a suspended category with suspension Σ, and let $\top$ be an additive stable monad on it. The Kleisli category $\top\text{-Free}_{\mathcal{C}}$ inherits a structure of suspended category (cf. [12, Definition 2.4]).

Lemma 5.54. *Let $\mathcal{C}$ be a suspended category with the suspension Σ, and let $\top$ be an additive stable monad on it. Then, the Kleisli category $\top\text{-Free}_{\mathcal{C}}$ is a suspended category and the adjunction $V'_\top \dashv U'_\top : \top\text{-Free}_{\mathcal{C}} \to \mathcal{C}$ is an adjunction of additive functors commuting with the suspension. The suspension $\Sigma'_\top$ on $\top\text{-Free}_{\mathcal{C}}$ is given by*

$$\Sigma'_\top(C) := \Sigma C, \quad \Sigma'_\top(f) := \Sigma f. \tag{5.7}$$

Proof. The object ΣC belongs to $\top\text{-Free}_{\mathcal{C}}$, and for every $f : C \twoheadrightarrow D$, one has $\Sigma f : \Sigma C \to \Sigma\top D = \top\Sigma D$, i.e. $\Sigma f : \Sigma C \twoheadrightarrow \Sigma D$ is a morphism in $\top\text{-Free}_{\mathcal{C}}$; $\Sigma'_\top(\eta_C) = \Sigma\eta_C = \eta_{\Sigma C} = \eta_{\Sigma'_\top C}$ for every $C \in \mathcal{C}$, and for every $f : C \twoheadrightarrow D$, $g : D \twoheadrightarrow E$, one has $\Sigma'_\top(g \circ' f) = \Sigma(m_E \circ \top g \circ f) = \Sigma m_E \circ \Sigma\top g \circ \Sigma f = m_{\Sigma E} \circ \top\Sigma g \circ \Sigma f = \Sigma g \circ' \Sigma f = \Sigma'_\top(g) \circ' \Sigma'_\top(f)$. Similarly, one defines $(\Sigma'_\top)^{-1} : \top\text{-Free}_{\mathcal{C}} \to \top\text{-Free}_{\mathcal{C}}$, $C \mapsto \Sigma^{-1}(C)$, $f \mapsto \Sigma^{-1}(f)$ so that $\Sigma'_\top(\Sigma'_\top)^{-1} = \mathrm{Id}_{\top\text{-Free}_{\mathcal{C}}} = (\Sigma'_\top)^{-1}\Sigma'_\top$. We check that $U'_\top : \top\text{-Free}_{\mathcal{C}} \to \mathcal{C}$ and $V'_\top : \mathcal{C} \to \top\text{-Free}_{\mathcal{C}}$ commute with the suspension. In fact, we have

$$\begin{aligned} U'_\top\Sigma'_\top(C) &= U'_\top(\Sigma C) = \top\Sigma C = \Sigma\top C = \Sigma U'_\top(C), \\ U'_\top\Sigma'_\top\left(C \overset{f}{\twoheadrightarrow} D\right) &= U'_\top\left(\Sigma C \overset{\Sigma f}{\twoheadrightarrow} \Sigma D\right) = \left(\top\Sigma C \overset{m_{\Sigma D}\circ\top\Sigma f}{\to} \top\Sigma D\right) \\ &= \left(\Sigma\top C \overset{\Sigma m_D\circ\Sigma\top f}{\to} \Sigma\top D\right) = \Sigma\left(\top C \overset{m_D\circ\top f}{\to} \top D\right) = \Sigma U'_\top\left(C \overset{f}{\twoheadrightarrow} D\right)\end{aligned}$$

so that $U'_\top \Sigma'_\top = \Sigma U'_\top$, and

$$\begin{aligned} \Sigma'_\top V'_\top C &= \Sigma'_\top(C) = \Sigma(C) = V'_\top(\Sigma C), \\ \Sigma'_\top V'_\top \left(C \stackrel{f}{\to} D\right) &= \Sigma'_\top \left(C \stackrel{\eta_D \circ f}{\to} \top D\right) = \left(\Sigma C \stackrel{\Sigma\eta_D \circ \Sigma f}{\to} \Sigma\top D\right) = \left(\Sigma C \stackrel{\eta_{\Sigma D} \circ \Sigma f}{\to} \top\Sigma D\right) \\ &= V'_\top\left(\Sigma C \stackrel{\Sigma f}{\to} \Sigma D\right) = V'_\top \Sigma \left(C \stackrel{f}{\to} D\right) \end{aligned}$$

so that $\Sigma'_\top V'_\top = V'_\top \Sigma$. The unit of $(V'_\top, U'_\top)$ is η which commutes with the suspension. The counit $\beta' : V'_\top U'_\top \to \mathrm{Id}$ is given by $\beta'_Y = \mathrm{Id}_{\top Y} : \top Y \nrightarrow Y$, for every $Y \in \top\text{-Free}_{\mathcal{C}}$, and it commutes with the suspension. Indeed, since the monad is stable, we have $\Sigma\top = \top\Sigma$, so $\Sigma'_\top \beta_Y = \Sigma(\mathrm{Id}_{\top Y}) = \mathrm{Id}_{\Sigma\top Y} = \mathrm{Id}_{\top\Sigma Y} = \mathrm{Id}_{\top\Sigma'_\top Y} = \beta_{\Sigma'_\top Y}$, for every $Y \in \top\text{-Free}_{\mathcal{C}}$, thus $\Sigma'_\top \beta = \beta \Sigma'_\top$. Hence, $V'_\top \dashv U'_\top : \top\text{-Free}_{\mathcal{C}} \to \mathcal{C}$ is an adjunction of functors commuting with the suspension. □

As a consequence of Theorem 5.48 and Corollary 5.50, the pre-triangulation of $\mathcal{C}$ can be transferred also to the Kleisli category $GF\text{-Free}_{\mathcal{C}}$.

Corollary 5.55. *Let $\mathcal{C}$ be a pre-triangulated category and let $\mathcal{D}$ be an idempotent complete suspended category. Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction of functors commuting with the suspension. Suppose that the stable monad $GF : \mathcal{C} \to \mathcal{C}$ is an exact functor and that G is a stably semiseparable functor. Then, the Kleisli category $GF\text{-Free}_{\mathcal{C}}$ is pre-triangulated with distinguished triangles being exactly the ones whose image through the forgetful functor $U'_{GF} : GF\text{-Free}_{\mathcal{C}} \to \mathcal{C}$ is distinguished in $\mathcal{C}$. Moreover, with respect to this pre-triangulation, both the functor $U'_{GF} : GF\text{-Free}_{\mathcal{C}} \to \mathcal{C}$ and its left adjoint $V'_{GF} : \mathcal{C} \to GF\text{-Free}_{\mathcal{C}}$ become exact.*

Proof. Assume that G is stably semiseparable. From the proof of Corollary 5.50 we recall that $\mathcal{C}$ is suspended through Σ since it is pre-triangulated and, since $F \dashv G : \mathcal{D} \to \mathcal{C}$ is an adjunction of functors commuting with the suspension, the monad $(GF, G\epsilon F, \eta)$ is stable. Moreover, the functor $GF : \mathcal{C} \to \mathcal{C}$ is additive being an exact functor between pre-triangulated categories. Thus, by Lemma 5.54 the Kleisli category $GF\text{-Free}_{\mathcal{C}}$ inherits a structure of suspended category through the suspension Σ'_{GF}, given by $\Sigma'_{GF}(C) = \Sigma C$, $\Sigma'_{GF}(f) = \Sigma f$, such that $V'_{GF} \dashv U'_{GF} : GF\text{-Free}_{\mathcal{C}} \to \mathcal{C}$ is an adjunction of additive functors commuting with the suspension. Here we denote by Σ both the suspension of $\mathcal{C}$ and the one of $\mathcal{D}$, and by Σ' the suspension of $GF\text{-Free}_{\mathcal{C}}$. The Kleisli comparison functor $L_{GF} : GF\text{-Free}_{\mathcal{C}} \to \mathcal{D}$ commutes with the suspension as well. Indeed, for every $C \in GF\text{-Free}_{\mathcal{C}}$, we have

$$(L_{GF} \circ \Sigma')(C) = L_{GF}(\Sigma C) = F\Sigma C = \Sigma F C = (\Sigma \circ L_{GF})(C),$$

and, for every $f : C \nrightarrow D$ in $GF\text{-Free}_{\mathcal{C}}$, $(L_{GF} \circ \Sigma')(f) = L_{GF}(\Sigma f) = \epsilon_{F\Sigma D} \circ F\Sigma f = \Sigma\epsilon_{FD} \circ \Sigma F f = \Sigma(\epsilon_{FD} \circ Ff) = (\Sigma \circ L_{GF})(f)$. Furthermore, since L_{GF} is fully faithful (see Remark 1.74), then by Remark 5.44 L_{GF} is stably fully faithful.

As a consequence, by Lemma 5.45 *ii*) the functor $U'_{GF} : GF\text{-Free}_{\mathcal{C}} \to \mathcal{C}$ is stably separable since $U'_{GF} = G \circ L_{GF}$, where G is stably semiseparable. Thus, by applying Theorem 5.48 to the adjunction $V'_{GF} \dashv U'_{GF}$, we get a pre-triangulation on $GF\text{-Free}_{\mathcal{C}}$ with distinguished triangles Δ being exactly the ones such that $U'_{GF}(\Delta)$ is distinguished in $\mathcal{C}$. Moreover, with respect to this pre-triangulation, both functors U'_{GF} and V'_{GF} become exact. □

Remark 5.56. From the proof of Corollary 5.50 we know that the comparison functor K_{GF} commutes with the suspension, and so does also the functor $J_{GF} = K_{GF} \circ L_{GF}$. Indeed, $J_{GF}\Sigma'_{GF} = K_{GF}L_{GF}\Sigma'_{GF} = K_{GF}\Sigma L_{GF} = \Sigma_{GF}K_{GF}L_{GF} = \Sigma_{GF}J_{GF}$. Moreover, by Remark 5.44 J_{GF} is stably fully faithful. Thus, the stably separability of U'_{GF} in Corollary 5.55 follows also from the equality $U'_{GF} = U_{GF} \circ J_{GF}$ since U_{GF} is stably separable, as shown in Corollary 5.50.

Chapter 6

Further results on semiseparable functors

Related to the study of separable functors, several variations of the notion of separable functor have appeared in the literature. For instance, in [3] the *separability* of a functor $U : \mathcal{D} \to \mathcal{B}$, *relative* to functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{E} \to \mathcal{D}$, has been defined, and the standard results concerning separable functors have been extended to these functors. When both F and G are identity functors, one recovers the classical definition of separable functor. Since relative notions can be also formulated for faithfulness and (natural) fullness (see [3, Definition 2.4]), we here investigate a relative semiseparability, obtaining a "relative" version of Proposition 2.5.

As another instance, motivated by an example related to the tensor algebra, a stronger notion of separability has been recently introduced in [11] under the name of *heavily separable functor*. Explicitly, an heavily separable functor is a separable functor through a natural transformation $\mathcal{P}$ which is multiplicative. So, it is natural to wonder if the multiplicativity of the natural transformation associated to a semiseparable functor gives rise to a stronger notion of semiseparability. In Subsection 6.2.2 we will see that some of the examples considered in Chapter 3 are actually heavily semiseparable functors.

The aim of this chapter is to undertake the study of such variations of semiseparable functors, namely *relative semiseparable* functors and *heavily semiseparable* functors, planning to continue investigating them in future works.

6.1 Relative semiseparable functors

Consider functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{E} \to \mathcal{D}$, $U : \mathcal{D} \to \mathcal{B}$, that give rise to the functors

$$\mathrm{Hom}_{\mathcal{D}}(F-, G-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{E} \to \mathsf{Set}, \quad \mathrm{Hom}_{\mathcal{B}}(UF-, UG-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{E} \to \mathsf{Set},$$

and to a natural transformation

$$\left(\mathcal{F}^U_{F,G}\right)_{-,-} : \mathrm{Hom}_{\mathcal{D}}(F-, G-) \to \mathrm{Hom}_{\mathcal{B}}(UF-, UG-), \quad \left(\mathcal{F}^U_{F,G}\right)_{X,Y}(f) := U(f),$$

for every $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. On components we have $\left(\mathcal{F}^U_{F,G}\right)_{X,Y} = \mathcal{F}^U_{FX,GY}$, where $\mathcal{F}^U$ is the natural transformation (1.1) associated with U.

Then, see [3, Definition 2.4], the functor U is said to be

- (F, G)*-faithful* if $\left(\mathcal{F}^U_{F,G}\right)_{X,Y}$ is injective for every $X \in \mathcal{C}$, $Y \in \mathcal{E}$;

- (F, G)-*full* if $\left(\mathcal{F}^U_{F,G}\right)_{X,Y}$ is surjective for every $X \in \mathcal{C}$, $Y \in \mathcal{E}$;
- (F, G)-*fully faithful* if $\left(\mathcal{F}^U_{F,G}\right)_{X,Y}$ is bijective for every $X \in \mathcal{C}$, $Y \in \mathcal{E}$;
- (F, G)-*separable* if $\mathcal{F}^U_{F,G}$ is a split natural monomorphism;
- (F, G)-*naturally full* if $\mathcal{F}^U_{F,G}$ is a split natural epimorphism.

Remark 6.1. A functor U is (F, G)-fully faithful if, and only if, it is (F, G)-separable and (F, G)-naturally full. When both F and G are the identity functors, we recover the classical definitions of faithful, full, fully faithful, separable, naturally full functors.

Definition 6.2. We say that U is (F, G)-*semiseparable* if $\mathcal{F}^U_{F,G}$ is a regular natural transformation, i.e. there exists a natural transformation

$$\mathcal{P}^U_{F,G} : \mathrm{Hom}_{\mathcal{B}}(UF-, UG-) \to \mathrm{Hom}_{\mathcal{D}}(F-, G-)$$

such that $\mathcal{F}^U_{F,G}\mathcal{P}^U_{F,G}\mathcal{F}^U_{F,G} = \mathcal{F}^U_{F,G}$.

When both F and G are the identity functors, we recover Defintion 2.2.

The following is a relative version of Proposition 2.5.

Proposition 6.3. *Let* $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{E} \to \mathcal{D}$, $U : \mathcal{D} \to \mathcal{B}$ *be functors. Then,*

i) *U is (F, G)-separable if, and only if, U is (F, G)-semiseparable and (F, G)-faithful;*

ii) *U is (F, G)-naturally full if, and only if, U is (F, G)-semiseparable and (F, G)-full.*

Proof. It is similar to the proof of Proposition 2.5. □

As an application of the previous characterization, the next follows from [3, Proposition 3.6].

Example 6.4. Let $\mathcal{C}$ be an A-coring over an algebra A with a grouplike element g. Denote by $B := A^{\mathrm{co}\mathcal{C}} = \{b \in A \mid gb = bg\}$ the coinvariants of A with respect to g. Consider the adjunction $(- \otimes_B A, (-)^{\mathrm{co}\mathcal{C}})$ and the canonical map $\mathrm{can} : A \otimes_B A \to \mathcal{C}$, $a \otimes_B a' \mapsto aga'$. Then,

i) $(-)^{\mathrm{co}\mathcal{C}}$ is $(- \otimes_A \mathcal{C}, \mathrm{Id})$-separable if, and only if, $(-)^{\mathrm{co}\mathcal{C}}$ is $(- \otimes_A \mathcal{C}, \mathrm{Id})$-semiseparable and can is an epimorphism;

ii) $(-)^{\mathrm{co}\mathcal{C}}$ is $(- \otimes_A \mathcal{C}, \mathrm{Id})$-naturally full if, and only if, $(-)^{\mathrm{co}\mathcal{C}}$ is $(- \otimes_A \mathcal{C}, \mathrm{Id})$-semiseparable and can is a split monomorphism.

We show the following properties.

Lemma 6.5. (Cf. [3, Theorem 2.7]) *Consider the diagram of functors*

$$\begin{array}{ccccccccc} & & & & \mathcal{B} & & & & \\ & & & & \uparrow{\scriptstyle U} & & & & \\ \mathcal{C}' & \xrightarrow{F'} & \mathcal{C} & \xrightarrow{F} & \mathcal{D} & \xleftarrow{G} & \mathcal{E} & \xleftarrow{G'} & \mathcal{D}'. \end{array}$$

If U is (F, G)-semiseparable, then U is (FF', GG')-semiseparable.

Proof. If U is (F,G)-semiseparable, then there exists a natural transformation $\mathcal{P}^U_{F,G}$: $\mathrm{Hom}_{\mathcal{B}}(UF-,UG-) \to \mathrm{Hom}_{\mathcal{D}}(F-,G-)$ such that $\mathcal{F}^U_{F,G}\mathcal{P}^U_{F,G}\mathcal{F}^U_{F,G} = \mathcal{F}^U_{F,G}$. In particular, for any objects $X' \in \mathcal{C}'$, $Y' \in \mathcal{D}'$, the maps $\mathcal{P}^U_{FF'X',GG'Y'}$ such that

$$\mathcal{F}^U_{FF'X',GG'Y'}\mathcal{P}^U_{FF'X',GG'Y'}\mathcal{F}^U_{FF'X',GG'Y'} = \mathcal{F}^U_{FF'X',GG'Y'},$$

define a natural transformation $\mathcal{P}^U_{FF',GG'}$ such that $\mathcal{F}^U_{FF',GG'}\mathcal{P}^U_{FF',GG'}\mathcal{F}^U_{FF',GG'} = \mathcal{F}^U_{FF',GG'}$, hence U is (FF',GG')-semiseparable. □

Corollary 6.6. *Let $U : \mathcal{D} \to \mathcal{B}$ be a functor.*

i) Let $G : \mathcal{E} \to \mathcal{D}$ be a functor. Then, U is $(\mathrm{Id}_{\mathcal{D}}, G)$-semiseparable if, and only if, U is (F,G)-semiseparable, for every functor $F : \mathcal{C} \to \mathcal{D}$;

ii) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then, U is $(F, \mathrm{Id}_{\mathcal{D}})$-semiseparable if, and only if, U is (F,G)-semiseparable, for every functor $G : \mathcal{E} \to \mathcal{D}$.

Proof. *i)*. If U is (F,G)-semiseparable for every $F : \mathcal{C} \to \mathcal{D}$, then in particular it is so for $F = \mathrm{Id}_{\mathcal{D}}$. The "only if" part follows from Lemma 6.5.
ii). It follows by the same arguments. □

Corollary 6.7. *Let $U : \mathcal{D} \to \mathcal{B}$ be a functor. If U is semiseparable, then U is (F,G)-semiseparable for all functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{E} \to \mathcal{D}$.*

Proof. If U is semiseparable, then it is $(\mathrm{Id}_{\mathcal{D}}, \mathrm{Id}_{\mathcal{D}})$-semiseparable. By Lemma 6.5, it is (F,G)-semiseparable for all functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{E} \to \mathcal{D}$. □

Now, we look at the composition of relative semiseparable functors, extending Lemma 2.6 and Lemma 2.8 to this setting.

Lemma 6.8. *Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{E} \to \mathcal{D}$, $U : \mathcal{D} \to \mathcal{B}$, $U' : \mathcal{B} \to \mathcal{B}'$ be functors. Consider the composite $U' \circ U : \mathcal{D} \to \mathcal{B}'$.*

i) If U is (F,G)-semiseparable and U' is (UF,UG)-separable, then $U'U$ is (F,G)-semiseparable.

ii) If U is (F,G)-naturally full and U' is (UF,UG)-semiseparable, then $U'U$ is (F,G)-semiseparable.

iii) If $U'U$ is (F,G)-semiseparable and U' is (UF,UG)-faithful, then U is (F,G)-semiseparable.

iv) If $U'U$ is (F,G)-semiseparable and U is (F,G)-full, then U' is (UF,UG)-semiseparable.

Proof. *i)* and *ii)* follow as in Lemma 2.6 by defining $\mathcal{P}^{U'U}_{FX,GY} := \mathcal{P}^U_{FX,GY} \circ \mathcal{P}^{U'}_{UFX,UGY}$.
iii). It follows as in Lemma 2.8 by defining $\mathcal{P}^U_{FX,GY} := \mathcal{P}^{U'U}_{FX,GY} \circ \mathcal{F}^{U'}_{UFX,UGY}$.
iv). If $U'U$ is (F,G)-semiseparable through $\mathcal{P}^{U'U}_{F,G}$, then, for every $X \in \mathcal{C}$, $Y \in \mathcal{E}$, $\mathcal{F}^{U'U}_{FX,GY} \circ \mathcal{P}^{U'U}_{FX,GY} \circ \mathcal{F}^{U'U}_{FX,GY} = \mathcal{F}^{U'U}_{FX,GY}$, i.e.,

$$\mathcal{F}^{U'}_{UFX,UGY} \circ \mathcal{F}^U_{FX,GY} \circ \mathcal{P}^{U'U}_{FX,GY} \circ \mathcal{F}^{U'}_{UFX,UGY} \circ \mathcal{F}^U_{FX,GY} = \mathcal{F}^{U'}_{UFX,UGY} \circ \mathcal{F}^U_{FX,GY}.$$

Since $\left(\mathcal{F}^U_{F,G}\right)_{X,Y}$ is surjective for every $X \in \mathcal{C}$, $Y \in \mathcal{E}$, we have that $\mathcal{F}^{U'}_{UFX,UGY} \circ \mathcal{F}^U_{FX,GY} \circ \mathcal{P}^{U'U}_{FX,GY} \circ \mathcal{F}^{U'}_{UFX,UGY} = \mathcal{F}^{U'}_{UFX,UGY}$, so U' is (UF,UG)-semiseparable through $\mathcal{P}^{U'}_{UFX,UGY} := \mathcal{F}^U_{FX,GY} \circ \mathcal{P}^{U'U}_{FX,GY}$. □

We show an analogue of Proposition 2.11 for relative semiseparable functors.

Proposition 6.9. *Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{E} \to \mathcal{D}$, $U : \mathcal{D} \to \mathcal{B}$ be functors.*

i) If U is $(\mathrm{Id}_\mathcal{D}, G)$-semiseparable, then there is a unique natural transformation $e : G \to G$ such that $\mathcal{P}^U_{\mathrm{Id}_\mathcal{D},G} \circ \mathcal{F}^U_{\mathrm{Id}_\mathcal{D},G} = \mathrm{Hom}_\mathcal{D}(\mathrm{Id}, e)$. Moreover, $Ue = \mathrm{Id}_{UG}$, e is idempotent and e fulfills the following property: if $f, g : GX \to GY$ are morphisms in $\mathcal{D}$, then

$$Uf = Ug \quad \text{if, and only if,} \quad e_Y \circ f = e_Y \circ g, \tag{6.1}$$

which uniquely determines e in case G is full.

ii) If U is $(F, \mathrm{Id}_\mathcal{D})$-semiseparable, then there is a unique natural transformation $e : F \to F$ such that $\mathcal{P}^U_{F,\mathrm{Id}_\mathcal{D}} \circ \mathcal{F}^U_{F,\mathrm{Id}_\mathcal{D}} = \mathrm{Hom}_\mathcal{D}(e, \mathrm{Id})$. Moreover, $Ue = \mathrm{Id}_{UF}$, e is idempotent and e fulfills the following property: if $f, g : FX \to FY$ are morphisms in $\mathcal{D}$,

$$Uf = Ug \quad \text{if, and only if,} \quad e_Y \circ f = e_Y \circ g, \tag{6.2}$$

which uniquely determines e in case F is full.

Proof. i). Since U is $(\mathrm{Id}_\mathcal{D}, G)$-semiseparable, there is a natural transformation

$$\mathcal{P}^U_{\mathrm{Id}_\mathcal{D},G} : \mathrm{Hom}_\mathcal{B}(U-, UG-) \to \mathrm{Hom}_\mathcal{D}(-, G-)$$

such that $\mathcal{F}^U_{\mathrm{Id}_\mathcal{D},G}\mathcal{P}^U_{\mathrm{Id}_\mathcal{D},G}\mathcal{F}^U_{\mathrm{Id}_\mathcal{D},G} = \mathcal{F}^U_{\mathrm{Id}_\mathcal{D},G}$. By Yoneda Lemma (see e.g. [61, page 61]), a natural transformation

$$\mathcal{P}^U_{-,GX} \circ \mathcal{F}^U_{-,GX} : \mathrm{Hom}_\mathcal{D}(-, GX) \to \mathrm{Hom}_\mathcal{B}(U-, UGX) \to \mathrm{Hom}_\mathcal{D}(-, GX)$$

has the form $\mathrm{Hom}_\mathcal{D}(-, e_X)$ for a unique arrow $e_X : GX \to GX$ in $\mathcal{D}$. Set $e_X := \mathcal{P}^U_{GX,GX}(\mathrm{Id}_{UGX})$, for every $X \in \mathcal{E}$. For every $f : D \to GX$ in $\mathcal{D}$, we have $\mathcal{P}^U_{D,GX}\mathcal{F}^U_{D,GX}(f) = \mathcal{P}^U_{D,GX}(Uf) = \mathcal{P}^U_{GX,GX}(\mathrm{Id}_{UGX}) \circ f = e_X \circ f = \mathrm{Hom}_\mathcal{D}(\mathrm{Id}_D, e_X)(f)$. Moreover, for every morphism $f : X \to Y$ in $\mathcal{E}$, one has

$$\begin{aligned} Gf \circ e_X &= Gf \circ \mathcal{P}^U_{GX,GX}(\mathrm{Id}_{UGX}) = \mathcal{P}^U_{GX,GY}(UGf \circ \mathrm{Id}_{UGX}) \\ &= \mathcal{P}^U_{GX,GY}(\mathrm{Id}_{UGY} \circ UGf) = \mathcal{P}^U_{GY,GY}(\mathrm{Id}_{UGY}) \circ Gf = e_Y \circ Gf \end{aligned}$$

so that $Gf \circ e_X = e_Y \circ Gf$, i.e. $e = (e_X)_{X \in \mathcal{E}} : G \to G$ is a natural transformation. Note that

$$Ue_X = U\mathcal{P}^U_{GX,GX}(\mathrm{Id}_{UGX}) = \mathcal{F}^U_{GX,GX}\mathcal{P}^U_{GX,GX}\mathcal{F}^U_{GX,GX}(\mathrm{Id}_{GX}) = \mathcal{F}^U_{GX,GX}(\mathrm{Id}_{GX}) = \mathrm{Id}_{UGX},$$

and then

$$\begin{aligned} e_X \circ e_X &= \mathcal{P}^U_{GX,GX}(\mathrm{Id}_{UGX}) \circ e_X = \mathcal{P}^U_{GX,GX}(\mathrm{Id}_{UGX} \circ Ue_X) \\ &= \mathcal{P}^U_{GX,GX}(\mathrm{Id}_{UGX} \circ \mathrm{Id}_{UGX}) = \mathcal{P}^U_{GX,GX}(\mathrm{Id}_{UGX}) = e_X, \end{aligned}$$

for every $X \in \mathcal{E}$, hence e is an idempotent natural transformation such that $Ue = \mathrm{Id}_{UG}$. Now, consider morphisms $f, g : GX \to GY$ in $\mathcal{D}$. If $Uf = Ug$, then $\mathcal{P}^U_{GX,GY}(Uf) = \mathcal{P}^U_{GX,GY}(Ug)$, i.e. $\mathcal{P}^U_{GY,GY}(\mathrm{Id}_{UGY}) \circ f = \mathcal{P}^U_{GY,GY}(\mathrm{Id}_{UGY}) \circ g$, i.e. $e_Y \circ f = e_Y \circ g$. Conversely, from $e_Y \circ f = e_Y \circ g$ we get $Ue_Y \circ Uf = Ue_Y \circ Ug$, and hence $Uf = Ug$ as $Ue = \mathrm{Id}_{UG}$. Thus e fulfills (6.1). Finally, let $e' : G \to G$ be an idempotent natural transformation which fulfills (6.1), i.e. such that, if $f, g : X \to Y$ are morphisms in $\mathcal{E}$, then $Uf = Ug$ if, and only if, $e'_Y \circ f = e'_Y \circ g$. From $e'_X \circ e'_X = e'_X \circ \mathrm{Id}_{GX}$ we get $Ue'_X = U\mathrm{Id}_{GX}$ for every

$X \in \mathcal{E}$, hence $Ue' = \mathrm{Id}_{UG}$. From the property (6.1) of e we get $e_X \circ e'_X = e_X \circ \mathrm{Id}_{GX}$, i.e. $e_X \circ e'_X = e_X$. If we interchange the roles of e and e', in a similar way we get $e'_X \circ e_X = e'_X$. If we assume that G is full, then $e'_X = Gh$, for some $h : X \to X$ in $\mathcal{E}$. By naturality of e, we have $e_X = e_X \circ e'_X = e_X \circ Gh = Gh \circ e_X = e'_X \circ e_X = e'_X$, for every $X \in \mathcal{E}$, hence $e = e'$.
ii). It follows similarly. □

As a particular case we have the following result, similar to Corollary 2.12.

Corollary 6.10. *Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{E} \to \mathcal{D}$, $U : \mathcal{D} \to \mathcal{B}$ be functors. Then,*

i) U is $(\mathrm{Id}_\mathcal{D}, G)$-separable if, and only if, the associated idempotent natural transformation e coincides with Id_G;

ii) U is $(F, \mathrm{Id}_\mathcal{D})$-separable if, and only if, the associated idempotent natural transformation e coincides with Id_F.

Proof. We just prove *i*) as *ii*) follows similarly. If U is $(\mathrm{Id}_\mathcal{D}, G)$-separable, then $\mathcal{P}^U_{\mathrm{Id}_\mathcal{D},G} \circ \mathcal{F}^U_{\mathrm{Id}_\mathcal{D},G} = \mathrm{Id}$ and hence $e_X = \mathcal{P}^U_{GX,GX}(\mathrm{Id}_{UGX}) = \mathcal{P}^U_{GX,GX}\mathcal{F}^U_{GX,GX}(\mathrm{Id}_{GX}) = \mathrm{Id}_{GX}$, for every $X \in \mathcal{E}$. Conversely, suppose $e = \mathrm{Id}_G$. For every $f : D \to GY$ in $\mathcal{D}$, we have $\mathcal{P}^U_{D,GY}(Uf) = \mathcal{P}^U_{D,GY}(\mathrm{Id}_{UGY} \circ Uf) = \mathcal{P}^U_{GY,GY}(\mathrm{Id}_{UGY}) \circ f = e_Y \circ f = \mathrm{Id}_{GY} \circ f = f$, so that $\mathcal{P}^U_{\mathrm{Id}_\mathcal{D},G} \circ \mathcal{F}^U_{\mathrm{Id}_\mathcal{D},G} = \mathrm{Id}$ and U is $(\mathrm{Id}_\mathcal{D}, G)$-separable. □

Proposition 6.11. *Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{E} \to \mathcal{D}$, $U : \mathcal{D} \to \mathcal{B}$. Then,*

i) U is $(\mathrm{Id}_\mathcal{D}, G)$-separable if, and only if, U is $(\mathrm{Id}_\mathcal{D}, G)$-semiseparable and reflects split-monos $f : GX \to D$;

ii) U is $(F, \mathrm{Id}_\mathcal{D})$-separable if, and only if, U is $(F, \mathrm{Id}_\mathcal{D})$-semiseparable and reflects split-epis $f : D \to GX$.

Proof. i). If U is $(\mathrm{Id}_\mathcal{D}, G)$-separable, then by Proposition 6.3 *i*) it is $(\mathrm{Id}_\mathcal{D}, G)$-semiseparable and by [3, Theorem 2.8] it reflects split-monos $f : GX \to D$. On the other hand, if U is $(\mathrm{Id}_\mathcal{D}, G)$-semiseparable we can consider its associated idempotent natural transformation e such that $Ue_X = \mathrm{Id}_{UGX}$, for every object $X \in \mathcal{E}$. Thus, Ue_X is a split-mono, hence $e_X = \mathrm{Id}_{GX}$, so that by Corollary 6.10 U is $(\mathrm{Id}_\mathcal{D}, G)$-separable.
ii). It follows similarly. □

Furthermore, we show the following.

Theorem 6.12. *Consider the following functors*

$$\mathcal{A} \xrightarrow{V} \mathcal{D} \xrightarrow{U} \mathcal{B}.$$

Assume that V is full. If $U : \mathcal{D} \to \mathcal{B}$ is (V, V)-semiseparable, then there is a unique natural transformation $e : V \to V$ such that $\mathcal{P}^U_{V,V} \circ \mathcal{F}^U_{V,V} = \mathrm{Hom}_\mathcal{D}(e, \mathrm{Id})$. Moreover, $Ue = \mathrm{Id}_{UV}$, e is idempotent and it fulfills the following universal property: if $f, g : VX \to VY$ are morphisms in $\mathcal{D}$, then

$$Uf = Ug \quad \text{if, and only if,} \quad e_Y \circ f = e_Y \circ g. \tag{6.3}$$

Proof. If U is (V,V)-semiseparable, then there is a natural transformation

$$\mathcal{P}^U_{V,V} : \mathrm{Hom}_{\mathcal{B}}(UV-,UV-) \to \mathrm{Hom}_{\mathcal{D}}(V-,V-)$$

such that $\mathcal{F}^U_{V,V}\mathcal{P}^U_{V,V}\mathcal{F}^U_{V,V} = \mathcal{F}^U_{V,V}$. We show that

$$\mathcal{P}^U_{VX,V} \circ \mathcal{F}^U_{VX,V} : \mathrm{Hom}_{\mathcal{D}}(VX,V-) \to \mathrm{Hom}_{\mathcal{D}}(VX,V-)$$

has the form $\mathrm{Hom}_{\mathcal{D}}(e_X,\mathrm{Id})$, for a unique arrow $e_X : VX \to VX$ in $\mathcal{D}$. Set $e_X := \mathcal{P}^U_{VX,VX}(\mathrm{Id}_{UVX})$, for every $X \in \mathcal{A}$. Since V is full, for every $f : VX \to VY$ in $\mathcal{D}$ we have $f = V(g)$ for some $g : X \to Y$ in $\mathcal{A}$. Then, $\mathcal{P}^U_{VX,VY}\mathcal{F}^U_{VX,VY}(f) = \mathcal{P}^U_{VX,VY}(UVg) = Vg \circ \mathcal{P}^U_{VX,VX}(\mathrm{Id}_{UVX}) = f \circ e_X = \mathrm{Hom}_{\mathcal{D}}(e_X,\mathrm{Id}_{VY})(f)$, so $\mathcal{P}^U_{VX,VY}\mathcal{F}^U_{VX,VY} = \mathrm{Hom}_{\mathcal{D}}(e_X,\mathrm{Id}_{VY})$. Assume that there exists $e'_X : VX \to VX$ in $\mathcal{D}$ such that $\mathcal{P}^U_{VX,VY}\mathcal{F}^U_{VX,VY} = \mathrm{Hom}_{\mathcal{D}}(e'_X,\mathrm{Id}_{VY})$, for every $X,Y \in \mathcal{A}$. Then,

$$\begin{aligned} e'_X = \mathrm{Id}_{VX} \circ e'_X &= \mathrm{Hom}_{\mathcal{D}}(e'_X,\mathrm{Id}_{VX})(\mathrm{Id}_{VX}) = \mathcal{P}^U_{VX,VX}\mathcal{F}^U_{VX,VX}(\mathrm{Id}_{VX}) \\ &= \mathcal{P}^U_{VX,VX}(U\mathrm{Id}_{VX}) = \mathcal{P}^U_{VX,VX}(\mathrm{Id}_{UVX}) = e_X. \end{aligned}$$

Moreover, for every morphism $f : X \to Y$ in $\mathcal{A}$, we have $Vf \circ e_X = Vf \circ \mathcal{P}^U_{VX,VX}(\mathrm{Id}_{UVX}) = \mathcal{P}^U_{VX,VY}(UVf \circ \mathrm{Id}_{UVX}) = \mathcal{P}^U_{VX,VY}(\mathrm{Id}_{UVY} \circ UVf) = \mathcal{P}^U_{VY,VY}(\mathrm{Id}_{UVY}) \circ Vf = e_Y \circ Vf$, so that $Vf \circ e_X = e_Y \circ Vf$. Hence $e = (e_X)_{X\in\mathcal{A}} : V \to V$ is the unique natural transformation such that $\mathcal{P}^U_{V,V} \circ \mathcal{F}^U_{V,V} = \mathrm{Hom}_{\mathcal{D}}(e,\mathrm{Id})$.

Note that

$$Ue_X = U\mathcal{P}^U_{VX,VX}(\mathrm{Id}_{UVX}) = \mathcal{F}^U_{VX,VX}\mathcal{P}^U_{VX,VX}\mathcal{F}^U_{VX,VX}(\mathrm{Id}_{VX}) = \mathcal{F}^U_{VX,VX}(\mathrm{Id}_{VX}) = \mathrm{Id}_{UVX},$$

for every $X \in \mathcal{A}$. Since V is full, then $e_X = V(h)$, for some $h : X \to X$ in $\mathcal{A}$. We have that

$$\begin{aligned} e_X \circ e_X &= \mathcal{P}^U_{VX,VX}(\mathrm{Id}_{UVX}) \circ Vh = \mathcal{P}^U_{VX,VX}(\mathrm{Id}_{UVX} \circ UVh) = \mathcal{P}^U_{VX,VX}(UVh) \\ &= \mathcal{P}^U_{VX,VX}(Ue_X) = \mathcal{P}^U_{VX,VX}(\mathrm{Id}_{UVX}) = e_X, \end{aligned}$$

hence e_X is idempotent, for every $X \in \mathcal{A}$. Thus, e is an idempotent natural transformation such that $Ue = \mathrm{Id}_{UV}$.

Now, let $f,g : VX \to VY$ be morphisms in $\mathcal{D}$. Since V is full, there exist $h,k : X \to Y$ in $\mathcal{A}$ such that $f = V(h)$ and $g = V(k)$. If $Uf = Ug$, then $\mathcal{P}^U_{VX,VY}(Uf) = \mathcal{P}^U_{VX,VY}(Ug)$, i.e. $\mathcal{P}^U_{VX,VY}(UVh) = \mathcal{P}^U_{VX,VY}(UVk)$, i.e. $\mathcal{P}^U_{VY,VY}(\mathrm{Id}_{UVY}) \circ Vh = \mathcal{P}^U_{VY,VY}(\mathrm{Id}_{UVY}) \circ Vk$, i.e. $e_Y \circ f = e_Y \circ g$. Conversely, from $e_Y \circ f = e_Y \circ g$ we get $Ue_Y \circ Uf = Ue_Y \circ Ug$, and hence $Uf = Ug$ as $Ue = \mathrm{Id}_{UV}$. Thus, e fulfills (6.3).

Let $e' : V \to V$ be an idempotent natural transformation which fulfills (6.3), i.e. such that, if $f,g : VX \to VY$ are morphisms in $\mathcal{D}$, then $Uf = Ug$ if, and only if, $e'_Y \circ f = e'_Y \circ g$. From $e'_X \circ e'_X = e'_X \circ \mathrm{Id}_{VX}$ we get $Ue'_X = \mathrm{Id}_{UVX}$ for every $X \in \mathcal{A}$, so $Ue' = \mathrm{Id}_{UV}$. Thus, from the property (6.3) of e we have $e_X \circ e'_X = e_X$. If we interchange the roles of e and e', in a similar way we get $e'_X \circ e_X = e'_X$. Since V is full, then $e'_X = Vk$ for some $k : X \to X$ in $\mathcal{A}$. By naturality of e, for every $X \in \mathcal{A}$, we have $e_X = e_X \circ e'_X = e_X \circ Vk = Vk \circ e_X = e'_X \circ e_X = e'_X$, hence $e = e'$. □

Remark 6.13. Let $V : \mathcal{A} \to \mathcal{D}$ be a functor and assume that $U : \mathcal{D} \to \mathcal{B}$ is $(V,\mathrm{Id}_{\mathcal{D}})$-semiseparable. Then, by Corollary 6.6 U is (V,V)-semiseparable. In case V is full, by Theorem 6.12 we recover Proposition 6.9 *ii*).

6.1.1 Relative semiseparable functors and adjunctions

Let $L \dashv R : \mathcal{D}' \to \mathcal{D}$ be an adjunction with unit $\eta : \mathrm{Id} \to RL$ and counit $\epsilon : LR \to \mathrm{Id}$. Consider the following diagram of functors:

$$\mathcal{E} \xrightarrow{G} \mathcal{D} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}' \xleftarrow{F} \mathcal{C}. \qquad (L \dashv R)$$

We now provide a Rafael-type Theorem for relative semiseparable functors.

Theorem 6.14. *Let $L \dashv R : \mathcal{D}' \to \mathcal{D}$ be an adjunction with unit $\eta : \mathrm{Id}_{\mathcal{D}} \to RL$ and counit $\epsilon : LR \to \mathrm{Id}_{\mathcal{D}'}$. Consider functors $G : \mathcal{E} \to \mathcal{D}$, $F : \mathcal{C} \to \mathcal{D}'$. Then,*

i) L is $(\mathrm{Id}_{\mathcal{D}}, G)$-semiseparable if, and only if, ηG is a regular natural tranformation;

ii) R is $(F, \mathrm{Id}_{\mathcal{D}})$-semiseparable if, and only if, ϵF is a regular natural tranformation.

Proof. We show only $i)$ as $ii)$ follows similarly. Assume that L is $(\mathrm{Id}_{\mathcal{D}}, G)$-semiseparable through a natural transformation $\mathcal{P}^L_{\mathrm{Id}_{\mathcal{D}},G} : \mathrm{Hom}_{\mathcal{D}'}(L-, LG-) \to \mathrm{Hom}_{\mathcal{D}}(-, G-)$ such that $\mathcal{F}^L_{\mathrm{Id}_{\mathcal{D}},G}\mathcal{P}^L_{\mathrm{Id}_{\mathcal{D}},G}\mathcal{F}^L_{\mathrm{Id}_{\mathcal{D}},G} = \mathcal{F}^L_{\mathrm{Id}_{\mathcal{D}},G}$. We define $\nu : RLG \to G$ on components by setting

$$\nu_X := \mathcal{P}^L_{RLGX,GX}(\epsilon_{LGX}) : RLGX \to GX,$$

for any object X in $\mathcal{E}$. The naturality of ν follows from the one of $\mathcal{P}$. Indeed, for every $g : X \to Y$ in $\mathcal{E}$, we have

$$\begin{aligned} Gg \circ \nu_X = Gg \circ \mathcal{P}^L_{RLGX,GX}(\epsilon_{LGX}) &= \mathcal{P}^L_{RLGX,GY}(LGg \circ \epsilon_{LGX}) \\ = \mathcal{P}^L_{RLGX,GY}(\epsilon_{LGY} \circ LRLGg) &= \mathcal{P}^L_{RLGY,GY}(\epsilon_{LGY}) \circ RLGg = \nu_Y \circ RLGg. \end{aligned}$$

Moreover, by naturality of $\mathcal{P}^L$, for any X, Y in $\mathcal{E}$ and $g : LGX \to LGY$ in $\mathcal{D}'$, we have

$$\begin{aligned} \nu_Y \circ Rg \circ \eta_{GX} &= \mathcal{P}^L_{RLGY,GY}(\epsilon_{LGY}) \circ Rg \circ \eta_{GX} = \mathcal{P}^L_{GX,GY}(\epsilon_{LGY} \circ LRg \circ L\eta_{GX}) \\ &= \mathcal{P}^L_{GX,GY}(g \circ \epsilon_{LGX} \circ L\eta_{GX}) = \mathcal{P}^L_{GX,GY}(g \circ \mathrm{Id}_{LGX}) = \mathcal{P}^L_{GX,GY}(g). \end{aligned} \tag{6.4}$$

By Proposition 6.9 $i)$, the idempotent natural transformation $e : G \to G$ associated with L is defined by $e_X := \mathcal{P}^L_{GX,GX}(\mathrm{Id}_{LGX})$, for every $X \in \mathcal{E}$, hence by (6.4) $e_X = \nu_X \circ R\mathrm{Id}_{LGX} \circ \eta_{GX} = \nu_X \circ \eta_{GX}$, so that $e = \nu \circ \eta_G$. We compute $\eta_G \circ \nu \circ \eta_G = \eta_G \circ e = RLe \circ \eta_G = R\mathrm{Id}_{LG} \circ \eta_G = \eta_G$, as $Le = \mathrm{Id}_{LG}$. Thus, η_G is regular.

Conversely, assume η_G is regular, i.e. there exists a natural transformation $\nu : RLG \to G$ such that $\eta_G \circ \nu \circ \eta_G = \eta_G$. For any $f : LD \to LGY$ in $\mathcal{D}'$, define $\mathcal{P}^L_{D,GY}(f) := \nu_Y \circ Rf \circ \eta_D$.

By naturality of η_G and ν, for any $h : X \to Y$ in $\mathcal{D}$, $l : Z \to T$ in $\mathcal{E}$, and $k : LY \to LGZ$ in $\mathcal{D}'$, we have $\mathcal{P}^L_{X,GT}(LGl \circ k \circ Lh) = \nu_T \circ R(LGl \circ k \circ Lh) \circ \eta_X = (\nu_T \circ RLGl) \circ Rk \circ (RLh \circ \eta_X) = Gl \circ \nu_Z \circ Rk \circ \eta_Y \circ h = Gl \circ \mathcal{P}^L_{Y,GZ}(k) \circ h$, thus $\mathcal{P}^L_{-,G-} : \mathrm{Hom}_{\mathcal{D}'}(L-, LG-) \to \mathrm{Hom}_{\mathcal{D}}(-, G-)$ is a natural transformation. Note that $\mathcal{P}^L_{RLGX,GX}(\epsilon_{LGX}) = \nu_X \circ R\epsilon_{LGX} \circ \eta_{RLGX} = \nu_X \circ \mathrm{Id}_{RLGX} = \nu_X$, for every $X \in \mathcal{E}$, hence the correspondence between $\mathcal{P}^L$ and ν is bijective. For every $f : D \to GY$ in $\mathcal{D}$, we have that

$$\begin{aligned} (\mathcal{F}^L_{D,GY} \circ \mathcal{P}^L_{D,GY} \circ \mathcal{F}^L_{D,GY})(f) &= L(\mathcal{P}_{D,GY}(L(f))) = L(\nu_Y \circ RL(f) \circ \eta_D) = L(\nu_Y \circ \eta_{GY} \circ f) \\ &= \mathrm{Id}_{LY} \circ L\nu_Y \circ L\eta_{GY} \circ Lf = \epsilon_{LY} \circ L\eta_{GY} \circ L\nu_Y \circ L\eta_{GY} \circ Lf \\ &= \epsilon_{LGY} \circ L\eta_{GY} \circ Lf = \mathrm{Id}_{LGY} \circ Lf = Lf = \mathcal{F}^L_{D,GY}(f), \end{aligned}$$

hence L is $(\mathrm{Id}_{\mathcal{D}}, G)$-semiseparable. □

Remark 6.15. As in the separable and naturally full cases (see [3, Theorem 2.12, Theorem 2.13]), an alternative proof of Theorem 6.14 *i)* follows by observing that L is $(\mathrm{Id}_{\mathcal{D}}, G)$-semiseparable if, and only if, the natural transformation

$$\begin{aligned} &\Omega : \mathrm{Hom}_{\mathcal{D}}(-, G-) \to \mathrm{Hom}_{\mathcal{D}}(-, RLG-), \\ &\Omega(f) = RLf \circ \eta_X = \eta_{GY} \circ f, \end{aligned} \tag{6.5}$$

is regular. Note that Ω is the image of η_G under the fully faithful functor α given as in [3, Lemma 2.11].

6.2 Heavily semiseparable functors

In [11] a stronger notion of separable functor was introduced. This notion was motivated by the properties of the adjunction (T, P), where $\mathrm{P} : \mathsf{Bialg}_{\Bbbk} \to \mathfrak{M}$ is the functor that assigns to a $\Bbbk$-bialgebra B the $\Bbbk$-vector space of its primitive elements and acts on morphisms as the restriction on primitive elements, and $\mathrm{T} : \mathfrak{M} \to \mathsf{Bialg}_{\Bbbk}$ is the functor assigning to a vector space V the tensor algebra $\mathrm{T}V$ endowed with its canonical bialgebra structure such that the elements in V become primitive. Indeed, the unit η of the adjunction admits a natural retraction (i.e., $\nu \circ \eta = \mathrm{Id}_{\mathrm{Id}_{\mathcal{C}}}$) which satisfies the extra condition $\nu\nu = \nu \circ \mathrm{P}\epsilon\mathrm{T}$. The latter condition corresponds to a stronger notion of separability that can be formulated in terms of multiplicativity of the natural transformation $\mathcal{P}$ associated to a separable functor. Here we investigate the same extra condition on a semiseparable functor.

We recall that a *heavily separable* functor (*h-separable* for short, see [11, Definition 1.1]), is defined as a separable functor, through a natural transformation $\mathcal{P}$, such that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(FX, FY) \times \mathrm{Hom}_{\mathcal{D}}(FY, FZ) & \xrightarrow{\mathcal{P}_{X,Y} \times \mathcal{P}_{Y,Z}} & \mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \\ \circ \big\downarrow & & \big\downarrow \circ \\ \mathrm{Hom}_{\mathcal{D}}(FX, FZ) & \xrightarrow[\mathcal{P}_{X,Z}]{} & \mathrm{Hom}_{\mathcal{C}}(X, Z) \end{array} \tag{6.6}$$

is commutative for every X, Y, Z in $\mathcal{C}$, where the vertical arrows are the obvious compositions. On elements the above diagram means that $\mathcal{P}_{X,Z}(g \circ f) = \mathcal{P}_{Y,Z}(g) \circ \mathcal{P}_{X,Y}(f)$, for every $f : FX \to FY$, $g : FY \to FZ$ in $\mathcal{D}$.

Remark 6.16. We observe that in case $F : \mathcal{C} \to \mathcal{D}$ is a naturally full functor through a natural transformation $\mathcal{P}$, then the diagram (6.6) commutes. In fact, from $\mathcal{F} \circ \mathcal{P} = \mathrm{Id}$ it follows that $\mathcal{P}_{X,Z}(g \circ f) = \mathcal{P}_{X,Z}(\mathcal{F}_{Y,Z}\mathcal{P}_{Y,Z}(g) \circ f) = \mathcal{P}_{Y,Z}(g) \circ \mathcal{P}_{X,Y}(f)$, for every $f : FX \to FY$, $g : FY \to FZ$ in $\mathcal{D}$. Thus, any naturally full functor is "heavily naturally full", so such a property does not add further information on a naturally full functor.

As we will see in Example 6.30 and in Example 6.33, there are semiseparable functors that are not separable and whose associated natural transformation $\mathcal{P}$ is multiplicative. We propose here the notion of "heavily semiseparable" functor.

Definition 6.17. We say that a functor $F : \mathcal{C} \to \mathcal{D}$ is **heavily semiseparable** (**h-semiseparable** for short) if it is semiseparable through a natural transformation $\mathcal{P}$, such that the diagram 6.6 is commutative for every X, Y, Z in $\mathcal{C}$.

Lemma 6.18. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then, F is h-separable if, and only if, F is h-semiseparable and faithful.*

Proof. By Proposition 2.5, we have that F is h-separable if, and only if, F is semiseparable, faithful and the diagram (6.6) commutes, i.e. if, and only if, F is h-semiseparable and faithful. □

Remark 6.19. We observe that a semiseparable functor is not necessarily heavily semiseparable. In fact, for instance in [11, Example 3.13] it is shown that the extension $\mathbb{C}/\mathbb{R}$ is separable but not h-separable. As a consequence, the restriction of scalars functor $\varphi_* : \mathfrak{M}_\mathbb{C} \to \mathfrak{M}_\mathbb{R}$ is semiseparable but not h-semiseparable, otherwise, since it is faithful, by Lemma 6.18 it would be h-separable.

The next result describes the behavior of heavily semiseparable functors with respect to composition.

Lemma 6.20. *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ be functors and consider the composite $G \circ F : \mathcal{C} \to \mathcal{E}$.*

i) If F is h-semiseparable and G is h-separable, then $G \circ F$ is h-semiseparable.

ii) If F is naturally full and G is h-semiseparable, then $G \circ F$ is h-semiseparable.

iii) If $G \circ F$ is h-semiseparable and G is faithful, then F is h-semiseparable.

Proof. *i*). By Lemma 2.6 *i*), we know that $G \circ F$ is semiseparable with respect to $\mathcal{P}^{GF}_{X,Y} := \mathcal{P}^F_{X,Y} \circ \mathcal{P}^G_{FX,FY}$. Since F is h-semiseparable and G is h-separable, then the diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_\mathcal{E}(GFX,GFY)\times\mathrm{Hom}_\mathcal{E}(GFY,GFZ) & \xrightarrow{\mathcal{P}^G_{FX,FY}\times\mathcal{P}^G_{FY,FZ}} & \mathrm{Hom}_\mathcal{D}(FX,FY)\times\mathrm{Hom}_\mathcal{D}(FY,FZ) & \xrightarrow{\mathcal{P}^F_{X,Y}\times\mathcal{P}^F_{Y,Z}} & \mathrm{Hom}_\mathcal{C}(X,Y)\times\mathrm{Hom}_\mathcal{C}(Y,Z) \\
\downarrow{\circ} & & \downarrow{\circ} & & \downarrow{\circ} \\
\mathrm{Hom}_\mathcal{E}(GFX,GFZ) & \xrightarrow[\mathcal{P}^G_{FX,FZ}]{} & \mathrm{Hom}_\mathcal{D}(FX,FZ) & \xrightarrow[\mathcal{P}^F_{X,Z}]{} & \mathrm{Hom}_\mathcal{C}(X,Z)
\end{array}$$

commutes for every X, Y, Z in $\mathcal{C}$, so that $G \circ F$ is h-semiseparable.
ii). It follows similarly to *i*) by Lemma 2.6 *ii*) and Remark 6.16.
iii). By Lemma 2.8 we know that F is semiseparable through $\mathcal{P}^F_{X,Y} := \mathcal{P}^{GF}_{X,Y} \circ \mathcal{F}^G_{FX,FY}$. Since G is a functor and $G \circ F$ is h-semiseparable, then the diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_\mathcal{D}(FX,FY)\times\mathrm{Hom}_\mathcal{D}(FY,FZ) & \xrightarrow{\mathcal{F}^G_{FX,FY}\times\mathcal{F}^G_{FY,FZ}} & \mathrm{Hom}_\mathcal{E}(GFX,GFY)\times\mathrm{Hom}_\mathcal{E}(GFY,GFZ) & \xrightarrow{\mathcal{P}^{GF}_{X,Y}\times\mathcal{P}^{GF}_{Y,Z}} & \mathrm{Hom}_\mathcal{C}(X,Y)\times\mathrm{Hom}_\mathcal{C}(Y,Z) \\
\downarrow{\circ} & & \downarrow{\circ} & & \downarrow{\circ} \\
\mathrm{Hom}_\mathcal{D}(FX,FZ) & \xrightarrow[\mathcal{F}^G_{FX,FZ}]{} & \mathrm{Hom}_\mathcal{E}(GFX,GFZ) & \xrightarrow[\mathcal{P}^{GF}_{X,Z}]{} & \mathrm{Hom}_\mathcal{C}(X,Z)
\end{array}$$

commutes for every X, Y, Z in $\mathcal{C}$, so that F is h-semiseparable. □

Lemma 6.21. *A functor naturally isomorphic to a h-semiseparable functor is h-semiseparable.*

Proof. Cf. [11, Lemma 1.7] for the h-separable case. Let $\alpha : F \to G$ be a natural isomorphism of functors, where $G : \mathcal{C} \to \mathcal{D}$ is h-semiseparable with respect to $\mathcal{P}^G$. From the proof of Proposition 2.9 we know that F is semiseparable with respect to $\mathcal{P}^F_{X,Y} := \mathcal{P}^G_{X,Y} \circ \varsigma_{X,Y}$, where $\varsigma_{X,Y} : \mathrm{Hom}_\mathcal{D}(FX, FY) \to \mathrm{Hom}_\mathcal{D}(GX, GY)$ is defined by $\varsigma_{X,Y}(f) = \alpha_Y \circ f \circ \alpha_X^{-1}$. Since G is h-semiseparable we have that $\mathcal{P}^F_{X,Z}(g \circ f) = (\mathcal{P}^G_{X,Z} \circ \varsigma_{X,Z})(g \circ f) = \mathcal{P}^G_{X,Z}(\alpha_Z \circ g \circ f \circ \alpha_X^{-1}) = \mathcal{P}^G_{X,Z}(\alpha_Z \circ g \circ \alpha_Y^{-1} \circ \alpha_Y \circ f \circ \alpha_X^{-1}) = \mathcal{P}^G_{Y,Z}(\alpha_Z \circ g \circ \alpha_Y^{-1}) \circ \mathcal{P}^G_{X,Y}(\alpha_Y \circ f \circ \alpha_X^{-1}) = (\mathcal{P}^G_{Y,Z} \circ \varsigma_{Y,Z})(g) \circ (\mathcal{P}^G_{X,Y} \circ \varsigma_{X,Y})(f) = \mathcal{P}^F_{Y,Z}(g) \circ \mathcal{P}^F_{X,Y}(f)$, for every $f : FX \to FY$, $g : FY \to FZ$ in $\mathcal{D}$, thus F is h-semiseparable. □

It is known that a separable functor is both Maschke and dual Maschke, hence conservative, see Remark 1.12. Thus, we have the following characterization for h-separable functors in terms of Maschke and dual Maschke functors.

Proposition 6.22. *The following assertions are equivalent for a functor $F : \mathcal{C} \to \mathcal{D}$.*

(i) F is h-separable;

(ii) F is h-semiseparable and Maschke;

(iii) F is h-semiseparable and dual Maschke;

(iv) F is h-semiseparable and conservative.

Proof. It follows from Corollary 2.18 and Lemma 6.18. □

6.2.1 Heavily semiseparable adjoint functors

In this section we investigate the notion of h-semiseparability for functors which are part of an adjunction.

Theorem 6.23. (Rafael-type Theorem for h-semiseparability) *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction, with unit η and counit ϵ. Then,*

i) F is h-semiseparable if, and only if, there exists a natural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\eta \circ \nu \circ \eta = \eta$ (i.e., η is regular) and $\nu\nu = \nu \circ G\epsilon F$;

ii) G is h-semiseparable if, and only if, there exists a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\epsilon \circ \gamma \circ \epsilon = \epsilon$ (i.e., ϵ is regular) and $\gamma\gamma = F\eta G \circ \gamma$.

Proof. We only prove *i)* as *ii)* follows by duality. By Theorem 2.36 we know that F is semiseparable if, and only if, η is regular, i.e. there exists a natural transformation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $\eta \circ \nu \circ \eta = \eta$. Explicitly, if F is semiseparable through $\mathcal{P}$, then one defines $\nu_X := \mathcal{P}_{GFX,X}(\epsilon_{FX})$, for every $X \in \mathcal{C}$. On the other hand, given ν, one defines $\mathcal{P}_{X,Y} := \nu_Y \circ Gf \circ \eta_X$, for every $X, Y \in \mathcal{C}$. By Proposition 1.17 we know that $\mathcal{P} : \mathrm{Hom}_{\mathcal{D}}(F-, F-) \to \mathrm{Hom}_{\mathcal{C}}(-,-)$ is a natural transformation and the correspondence between $\mathcal{P}$ and ν is bijective. Assume that $\nu\nu = \nu \circ G\epsilon F$, i.e. $\nu \circ \nu GF = \nu \circ G\epsilon F$ holds. Then, for every $f : FX \to FY$, $g : FY \to FZ$ in $\mathcal{D}$, we have that

$$\begin{aligned}\mathcal{P}_{X,Z}(g \circ f) &= \nu_Z \circ G(g \circ f) \circ \eta_X = \nu_Z \circ Gg \circ Gf \circ \eta_X = \nu_Z \circ Gg \circ G\epsilon_{FY} \circ GF\eta_Y \circ Gf \circ \eta_X \\ &= \nu_Z \circ G\epsilon_{FZ} \circ GFGg \circ GF\eta_Y \circ Gf \circ \eta_X = \nu_Z \circ \nu_{GFZ} \circ GFGg \circ GF\eta_Y \circ Gf \circ \eta_X \\ &= \nu_Z \circ Gg \circ \eta_Y \circ \nu_Y \circ Gf \circ \eta_X = \mathcal{P}_{Y,Z}(g) \circ \mathcal{P}_{X,Y}(f).\end{aligned}$$

On the other hand, if $\mathcal{P}_{X,Z}(g \circ f) = \mathcal{P}_{Y,Z}(g) \circ \mathcal{P}_{X,Y}(f)$ holds true for every $f : FX \to FY$, $g : FY \to FZ$ in $\mathcal{D}$, then

$$\begin{aligned}\nu\nu_X &= \nu_X \circ \nu_{GFX} = \mathcal{P}_{GFX,X}(\epsilon_{FX}) \circ \mathcal{P}_{GFGFX,GFX}(\epsilon_{FGFX}) = \mathcal{P}_{GFGFX,X}(\epsilon_{FX} \circ \epsilon_{FGFX}) \\ &= \nu_X \circ G(\epsilon_{FX} \circ \epsilon_{FGFX}) \circ \eta_{GFGFX} = \nu_X \circ G\epsilon_{FX} \circ G\epsilon_{FGFX} \circ \eta_{GFGFX} = \nu_X \circ G\epsilon_{FX}.\end{aligned}$$ □

Remark 6.24. In the h-separable case, Theorem 6.23 recovers [11, Theorem 2.1].

In [11, Corollary 2.7] a characterization of h-separability of a right (resp., left) adjoint functor has been given in terms of the existence of a grouplike morphism (resp., an augmentation) of the associated comonad (resp., monad). Let us recall that the notion of a grouplike element in an R-coring $\mathcal{C}$, see Example 1.48, has been generalized in [65] to the notion of a grouplike natural transformation as follows.

Definition 6.25. [65, Definition 3.1] Let $\mathbb{C} := (\perp, \Delta, \varepsilon)$ be a comonad on a category $\mathcal{D}$. A natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to \perp$ is called a *grouplike morphism* provided $\varepsilon \circ \gamma = \mathrm{Id}$ and $\gamma\gamma = \Delta \circ \gamma$. Dually, see e.g. [59, Section 4], an *augmentation* of a monad $\mathbb{T} = (\top, m, \eta)$ on a category $\mathcal{D}$, is a natural transformation $\nu : \top \to \mathrm{Id}_{\mathcal{D}}$ such that $\nu \circ \eta = \mathrm{Id}_{\mathcal{D}}$ and $\nu\nu = \nu \circ m$.

In order to extend [11, Corollary 2.7] in the context of heavy semiseparability, we first introduce the following definitions.

Definition 6.26. Given a comonad $\mathbb{C} = (\perp, \Delta, \varepsilon)$ on a category $\mathcal{D}$, we say that a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to \perp$ is a **semi-grouplike morphism** if $\gamma\gamma = \Delta \circ \gamma$. Dually, given a monad $\mathbb{T} = (\top, m, \eta)$ on a category $\mathcal{D}$, we say that a natural transformation $\nu : \top \to \mathrm{Id}_{\mathcal{D}}$ is a **semi-augmentation** if $\nu\nu = \nu \circ m$.

Thus, as a consequence of Theorem 6.23 we get the next characterization.

Proposition 6.27. *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ. Then,*

i) G is h-semiseparable if, and only if, the comonad $(FG, F\eta G, \epsilon)$ has a semi-grouplike morphism $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $G(\epsilon \circ \gamma) = \mathrm{Id}_G$;

ii) F is h-semiseparable if, and only if, the monad $(GF, G\epsilon F, \eta)$ has a semi-augmentation $\nu : GF \to \mathrm{Id}_{\mathcal{C}}$ such that $F(\nu \circ \eta) = \mathrm{Id}_F$.

Proof. We only prove *i)* as *ii)* follows dually. By Theorem 6.23 *ii)* G is h-semiseparable if, and only if, there exists a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $\epsilon \circ \gamma \circ \epsilon = \epsilon$ (i.e., ϵ is regular) and $\gamma\gamma = F\eta G \circ \gamma$, i.e. if, and only if, there exists a natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $G(\epsilon \circ \gamma) = \mathrm{Id}_G$ holds true (by Lemma 2.38) and γ is a semi-grouplike morphism. □

By adding the faithfulness assumption on the right (resp., left) adjoint functor, which implies $\epsilon \circ \gamma = \mathrm{Id}$ (resp., $\nu \circ \eta = \mathrm{Id}$), we retrieve the announced result for the h-separable case.

Corollary 6.28. [11, Corollary 2.7] *Let $F \dashv G : \mathcal{D} \to \mathcal{C}$ be an adjunction with unit η and counit ϵ. Then,*

i) G is h-separable if, and only if, the comonad $(FG, F\eta G, \epsilon)$ has a grouplike morphism;

ii) F is h-separable if, and only if, the monad $(GF, G\epsilon F, \eta)$ has an augmentation.

6.2.2 Examples

In this section we provide some examples of heavily semiseparable functors.

Corings

As an application of the previous results we look at functors associated to an R-coring $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$, cf. Section 3.3. Since the forgetful functor $F : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_R$ is faithful, then it is h-semiseparable if, and only if, the coring $\mathcal{C}$ is h-coseparable (i.e., F is h-separable), i.e. by [11, Theorem 4.3] if, and only if, there exists an R-bimodule map $\gamma : \mathcal{C} \otimes_R \mathcal{C} \to R$, such that $\gamma \circ \Delta_{\mathcal{C}} = \varepsilon_{\mathcal{C}}$ and $\sum x_1\gamma(x_2 \otimes_R y) = \sum \gamma(x \otimes_R y_1)y_2$, $\sum \gamma(x \otimes_R y_1)\gamma(y_2 \otimes_R z) = \gamma(x\varepsilon_{\mathcal{C}}(y) \otimes_R z)$ for all x, y, z in $\mathcal{C}$. In [11, Theorem 4.5] it is shown that the right adjoint $G = (-) \otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}^{\mathcal{C}}$ of F is h-separable if, and only if, $\mathcal{C}$ has an invariant grouplike element. Here we prove a heavy semiseparable version of this result.

Theorem 6.29. *Given an R-coring* $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$*, the induction functor* $G = (-) \otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}^{\mathcal{C}}$ *is h-semiseparable if, and only if, the coring* $\mathcal{C}$ *has an invariant semi-grouplike element* $z \in \mathcal{C}$ *such that* $\varepsilon_{\mathcal{C}}(z)c = c$*, for every* $c \in \mathcal{C}$*.*

Proof. By Proposition 6.27 G is h-semiseparable if, and only if, the comonad $FG = (-)\otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}_R$ (cf. Example 1.63) has a semi-grouplike morphism $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ such that $G(\epsilon \circ \gamma) = \mathrm{Id}_G$. The latter condition is equivalent by Theorem 3.24 to the existence of an invariant $z \in \mathcal{C}^R = \{c \in \mathcal{C} \mid rc = cr, \text{ for all } r \in R\}$ such that $\varepsilon_{\mathcal{C}}(z)c = c$, for every $c \in \mathcal{C}$; moreover, from the proof of Theorem 3.24 we known that $z = l_{\mathcal{C}}\gamma_R(1_R)$, where $l_- : R \otimes_R - \to -$ is the canonical functorial isomorphism. Since γ is a semi-grouplike morphism, we have $(R \otimes_R \Delta_{\mathcal{C}}) \circ \gamma_R = \gamma_{R\otimes_R \mathcal{C}} \circ \gamma_R$, hence by applying $l_{\mathcal{C}\otimes_R\mathcal{C}}$ on both sides of the previous equality we get $\Delta_{\mathcal{C}}(z) = z \otimes_R z$, so z is a semi-grouplike element for the coring $\mathcal{C}$. Indeed, by naturality of γ we have that $\gamma_{\mathcal{C}} l_{\mathcal{C}}\gamma_R = (l_{\mathcal{C}}\gamma_R \otimes_R \mathcal{C})\gamma_R$, as $l_{\mathcal{C}}\gamma_R$ is in $\mathcal{M}_R$, so $\Delta_{\mathcal{C}}(z) = \Delta_{\mathcal{C}} l_{\mathcal{C}}\gamma_R(1_R) = l_{\mathcal{C}\otimes_R\mathcal{C}}(R \otimes_R \Delta_{\mathcal{C}})\gamma_R(1_R) = l_{\mathcal{C}\otimes_R\mathcal{C}}\gamma_{R\otimes_R\mathcal{C}}\gamma_R(1_R) = \gamma_{\mathcal{C}} l_{\mathcal{C}}\gamma_R(1_R) = (l_{\mathcal{C}}\gamma_R \otimes_R \mathcal{C})\gamma_R(1_R) = (l_{\mathcal{C}}\gamma_R \otimes_R \mathcal{C}) l_{\mathcal{C}}^{-1} l_{\mathcal{C}}\gamma_R(1_R) = (l_{\mathcal{C}}\gamma_R \otimes_R \mathcal{C}) l_{\mathcal{C}}^{-1}(z) = (l_{\mathcal{C}}\gamma_R \otimes_R \mathcal{C})(1_R \otimes_R z) = z \otimes_R z$. On the other hand, given a semi-grouplike element $z \in \mathcal{C}^R$, define the natural transformation $\gamma : \mathrm{Id}_{\mathcal{D}} \to FG$ given, for every $N \in \mathcal{M}_R$, by $\gamma_N : N \to N \otimes_R \mathcal{C}$, $n \mapsto n \otimes_R z$ (cf. the proof of *iii*) $\Rightarrow$ *i*) of Theorem 3.24). Then, since $\Delta_{\mathcal{C}}(z) = z \otimes_R z$, we get $\gamma\gamma_N = (N \otimes_R \Delta_{\mathcal{C}}) \circ \gamma_N$ for every $N \in \mathcal{M}_R$, as $\gamma\gamma_N(n) = (\gamma_{N\otimes_R\mathcal{C}} \circ \gamma_N)(n) = n \otimes_R z \otimes_R z = ((N \otimes_R \Delta_{\mathcal{C}}) \circ \gamma_N)(n)$, for all $n \in N$, so γ is a semi-grouplike morphism for the comonad $FG = (-) \otimes_R \mathcal{C} : \mathcal{M}_R \to \mathcal{M}_R$. □

Example 6.30. We recall from Example 3.26 that, given a morphism of rings $\varphi : R \to S$ such that the induction functor $\varphi^* = S \otimes_R (-)$ is naturally full, then (S, Δ, ε) is an R-coring, where $\Delta(s) = s \otimes_R 1_S = 1_S \otimes_R s$, for every $s \in S$, and $\varepsilon \in {}_R\mathrm{Hom}_R(S, R)$ is such that $\varphi \circ \varepsilon = \mathrm{Id}_S$. We have seen that $z := 1_S \in S^R$ fulfills the conditions of Theorem 3.24 guaranteeing that the functor $G = (-) \otimes_R S : \mathcal{M}_R \to \mathcal{M}^S$ is semiseparable and hence S is a semicosplit R-coring, but not cosplit in general (see Example 3.26 2)). We observe that $\Delta(z) = \Delta(1_S) = 1_S \otimes_R 1_S = z \otimes_R z$, hence z is semi-grouplike and G is heavily semiseparable, but not heavily separable in general.

Extension of scalars functor

In Proposition 3.1 we have seen that φ^* is semiseparable if, and only if, φ is a regular morphism of R-bimodules. Here we provide a characterization for the h-semiseparability of φ^*; cf. [11, Proposition 3.1] for the h-separable case.

Proposition 6.31. *Let* $\varphi : R \to S$ *be a morphism of rings. Then, the extension of scalars functor* $\varphi^* = S \otimes_R (-) : {}_R\mathcal{M} \to {}_S\mathcal{M}$ *is h-semiseparable if, and only if, there is a morphism* $E : S \to R$ *of R-bimodules which is multiplicative and such that* $\varphi \circ E \circ \varphi = \varphi$ *(i.e.,* $\varphi E(1_S) = 1_S$*).*

Proof. By Proposition 3.1 we know that φ^* is semiseparable if, and only if, φ is a regular morphism of R-bimodules, i.e. there is $E \in {}_R\mathrm{Hom}_R(S, R)$ such that $\varphi \circ E \circ \varphi = \varphi$, i.e., such that $\varphi E(1_S) = 1_S$. Given ν for φ^* as in Theorem 6.23, we define $E(s) = \nu_R(s \otimes_R 1_R)$. Conversely, given $E : S \to R$, we define $\nu_M = S \otimes_R M \to M$, $s \otimes_R m \mapsto E(s)m$, for every $M \in {}_R\mathcal{M}$ and it satisfies $\eta \circ \nu \circ \eta = \eta$, where the unit η is given by $\eta_M : M \to S \otimes_R M$, $m \mapsto 1_S \otimes_R m$. Moreover, the condition $\nu \circ \nu\varphi_*\varphi^* = \nu \circ \varphi_*\epsilon\varphi^*$ rewrites as $E(x)E(y)m = E(xy)m$, for every $x, y \in S$ and $m \in M$, for every $M \in {}_R\mathcal{M}$. Indeed, we have that

$$\begin{aligned} E(x)E(y)m &= E(x)\nu_M(y \otimes_R m) = \nu_M(E(x)(y \otimes_R m)) = (\nu_M \circ \nu_{\varphi_*(\varphi^*(M))})(x \otimes_R (y \otimes_R m)) \\ &= (\nu_M \circ \varphi_*\epsilon_{\varphi^*(M)})(x \otimes_R (y \otimes_R m)) = \nu_M(xy \otimes_R m) = E(xy)m, \end{aligned}$$

for every $x, y \in S$, $m \in M$. Thus, $\nu \circ \nu\varphi_*\varphi^* = \nu \circ \varphi_*\epsilon\varphi^*$ is equivalent to ask that E is multiplicative, so by Theorem 6.23 φ^* is h-semiseparable if, and only if, there is a morphism $E : S \to R$ of R-bimodules which is multiplicative and such that $\varphi \circ E \circ \varphi = \varphi$. □

As a consequence, we can restate [11, Proposition 3.1] as follows.

Corollary 6.32. *Let $\varphi : R \to S$ be a morphism of rings. Then, the extension of scalars functor $\varphi^* = S \otimes_R (-) : {}_R\mathcal{M} \to {}_S\mathcal{M}$ is h-separable if, and only if, φ^* is h-semiseparable through a morphism $E : S \to R$ of R-bimodules as in Proposition 6.31 such that $E(1_S) = 1_R$ holds true in addition, i.e. through a morphism of R-bimodules which is a ring homomorphism.*

The following is an example of a h-semiseparable functor, that is not h-separable.

Example 6.33. In Example 3.3 it is observed that, given morphisms of rings $\varphi : R \to S$ and $\psi : Q \to R$ whose induction functors φ^* and ψ^* are separable and naturally full, respectively, then the composition $\varphi^* \circ \psi^* \cong (\varphi \circ \psi)^*$ is semiseparable through $D \circ E \in {}_Q\mathrm{Hom}_Q(S, Q)$, where $E \in {}_R\mathrm{Hom}_R(S, R)$ is such that $E \circ \varphi = \mathrm{Id}$ (hence φ is injective) and $D \in {}_Q\mathrm{Hom}_Q(R, Q)$ is such that $\psi \circ D = \mathrm{Id}$. In case $\varphi \circ \psi$ is not injective, then $(\varphi \circ \psi)^*$ is not separable.

For instance, consider the canonical injection $\varphi : \mathbb{Q} \to \mathbb{Q}[X]$ of the field of rational numbers into the polynomial ring over it and let $\psi : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$ be given by the projection $\psi((q, q')) = q$ on the first component. Then, by defining $D : \mathbb{Q} \to \mathbb{Q} \times \mathbb{Q}$ as $D(q) = (q, 0)$ and $E : \mathbb{Q}[X] \to \mathbb{Q}$ to be the evaluation at 0 of the given polynomial, we know that $(\varphi \circ \psi)^*$ is semiseparable through $D \circ E$. Indeed, $D \circ E : \mathbb{Q}[X] \to \mathbb{Q} \times \mathbb{Q}$, $p(X) \mapsto (p(0), 0)$, is a morphism of $(\mathbb{Q} \times \mathbb{Q})$-bimodules as, for every $(u, v), (a, b) \in \mathbb{Q} \times \mathbb{Q}$ and $p(X) \in \mathbb{Q}[X]$, we have that $(D \circ E)((u, v) \cdot p(X) \cdot (a, b)) = D(E(\varphi(\psi((u, v)))p(X)\varphi(\psi((a, b))))) = D(E(up(X)a)) = D(up(0)a) = (up(0)a, 0) = (u, v)(p(0), 0)(a, b) = (u, v)D(p(0))(a, b) = (u, v)D(E(p(X)))(a, b)$. Moreover, it holds $\varphi \circ \psi \circ D \circ E \circ \varphi \circ \psi = \varphi \circ \mathrm{Id}_{\mathbb{Q}} \circ \mathrm{Id}_{\mathbb{Q}} \circ \psi = \varphi \circ \psi$. Since $D \circ E$ is multiplicative, as $(D \circ E)(p(X)q(X)) = (p(0)q(0), 0) = (p(0), 0)(q(0), 0) = (D \circ E)(p(X))(D \circ E)(q(X))$, then by Proposition 6.31 $(\varphi \circ \psi)^*$ is h-semiseparable. Note that $(\varphi \circ \psi)^*$ is not separable as $\varphi \circ \psi : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}[X]$, $(q, q') \mapsto q$, is not injective. Thus, it is not even h-separable.

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