

Quasihomomorphisms from the integers into Hamming metrics

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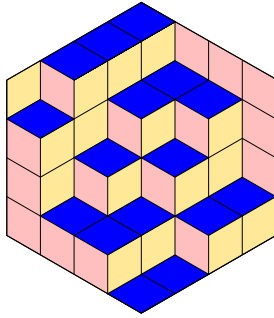
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# *ALGEBRAIC COMBINATORICS*

Jan Draisma, Rob H. Eggermont, Tim Seynnaeve, Nafie Tairi & Emanuele Ventura

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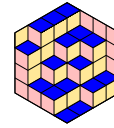


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# Quasihomomorphisms from the integers into Hamming metrics

Jan Draisma, Rob H. Eggermont, Tim Seynnaeve, Nafie Tairi  
& Emanuele Ventura

ABSTRACT A function  $f : \mathbb{Z} \rightarrow \mathbb{Q}^n$  is a  $c$ -quasihomomorphism if the Hamming distance between  $f(x+y)$  and  $f(x)+f(y)$  is at most  $c$  for all  $x, y \in \mathbb{Z}$ . We show that any  $c$ -quasihomomorphism has distance at most some constant  $C(c)$  to an actual group homomorphism; here  $C(c)$  depends only on  $c$  and not on  $n$  or  $f$ . This gives a positive answer to a special case of a question posed by Kazhdan and Ziegler.

## 1. INTRODUCTION

Let  $c$  be a nonnegative real number. A  $c$ -quasihomomorphism from a group  $G$  to a group  $H$  with a left-invariant metric  $d$  is a map  $f : G \rightarrow H$  such that  $d(f(xy), f(x)f(y)) \leq c$  for all  $x, y$  in  $G$ . A central question in geometric group theory, raised by Ulam in [17, Chapter 6], is whether there exists an actual homomorphism  $f' : G \rightarrow H$  such that  $d(f(x), f'(x))$  is at most some constant  $C$  for all  $x$ . (Related questions were studied before Ulam, e.g. by Turing in his work on approximability of groups [16].) Different versions of Ulam's question are of interest: for example,  $C$  may be allowed to depend on  $c, G, (H, d)$  but not on  $f$ ;  $G, (H, d)$  may be restricted to certain classes and  $C$  is only allowed to depend on  $c$ .

A well-known example where the answer to this question is negative is the case where  $G = H = \mathbb{Z}$  with the standard metric. Here, quasihomomorphisms modulo bounded maps are a model of the real numbers [15, 1], and the answer is yes only for those quasihomomorphisms that correspond to integers. In fact, this construction can be extended to construct completions of fields in general [11].

Much literature in this area focusses on *quasimorphisms*, which are quasihomomorphisms into the real numbers  $\mathbb{R}$  with the standard metric; we refer to [12] for a brief introduction. In particular, the concept of a quasimorphism features in bounded cohomology, see [13, 4, 6]. In another branch of the research on quasihomomorphisms  $H$  is assumed nonabelian, and one of the first positive results on the central question above is Kazhdan's theorem on  $\varepsilon$ -representations of amenable groups [9]. For more

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recent results on quasihomomorphisms into nonabelian groups we refer to [7, 8, 5, 2] and the references there.

The following instance of the central question was formulated by Kazhdan and Ziegler in their work on approximate cohomology [10].

QUESTION 1.1. *Let  $c \in \mathbb{N}$ . Does there exist a constant  $C = C(c)$  such that the following holds: For all  $n \in \mathbb{N}$  and all functions  $f : \mathbb{Z} \rightarrow \mathbb{C}^{n \times n}$  such that*

$$\text{for all } x, y \in \mathbb{Z}, \text{rk}(f(x + y) - f(x) - f(y)) \leq c,$$

*there exists a matrix  $g$  such that*

$$\text{for all } x \in \mathbb{Z}, \text{rk}(f(x) - xg) \leq C(c)?$$

Here,  $G$  equals  $\mathbb{Z}$  and  $H$  equals  $\mathbb{C}^{n \times n}$ , both with addition, and the metric on  $H$  is defined by  $d(A, B) := \text{rk}(A - B)$ . In [10, p1], the function  $R(\mathbb{Z}, c, \mathbb{C})$  denotes the minimal possible choice of  $C(c)$ . Our main result is an affirmative answer to Question 1.1 in the special case where all matrices  $f(x)$  are assumed to be *diagonal*.

DEFINITION 1.2. *Let  $(Q, +)$  be an abelian group. For an element  $v \in Q^n$ , the Hamming weight  $w_H(v)$  is the number of nonzero entries of  $v$ . For a pair of elements  $u, v \in Q^n$ , their Hamming distance is  $w_H(v - u)$ . This metric is clearly left-invariant, and indeed even bi-invariant.*

DEFINITION 1.3. *Let  $A$  be another abelian group. A function  $f : A \rightarrow Q^n$  is called a  $c$ -quasihomomorphism if*

$$(1) \quad \text{for all } x, y \in A, w_H(f(x + y) - f(x) - f(y)) \leq c.$$

REMARK 1.4. The map  $\text{diag} : \mathbb{C}^n \rightarrow \mathbb{C}^{n \times n}$  is an isometric embedding from  $\mathbb{C}^n$  with the Hamming metric to  $\mathbb{C}^{n \times n}$  with the rank metric. This connects Definition 1.3 to Question 1.1.

DEFINITION 1.5. *Let  $C \in \mathbb{N}$  and let  $f : A \rightarrow Q^n$  be a  $c$ -quasihomomorphism. A group homomorphism  $h : A \rightarrow Q^n$  is a  $C$ -approximation of  $f$  if the Hamming distance between  $f$  and  $h$  satisfies*

$$\text{for all } x \in A, w_H(f(x) - h(x)) \leq C.$$

We are ready to state our main result.

THEOREM 1.6 (Main Theorem). *Let  $c \in \mathbb{N}$ . Then there exists a constant  $C = C(c) \in \mathbb{N}$  such that for all  $n \in \mathbb{N}$  and  $c$ -quasihomomorphisms  $f : \mathbb{Z} \rightarrow \mathbb{Q}^n$ , we have:*

$$\text{for all } x \in \mathbb{Z}, w_H(f(x) - xf(1)) \leq C.$$

*Moreover, we can take  $C = 28c$ .*

REMARK 1.7. The coefficient 28 is probably not optimal. However, we certainly have that  $C(c) \geq c$ . Indeed, any map  $f : \mathbb{Z} \rightarrow \mathbb{Q}^n$  for which the only nonzero entries of  $f(x)$  are among the first  $c$ , is automatically a  $c$ -quasihomomorphism.

COROLLARY 1.8. *Theorem 1.6 also holds with  $\mathbb{Q}$  replaced by any torsion-free abelian group  $Q$ , with the same value of  $C = C(c)$ .*

*Proof.* Suppose, for a contradiction, that we have a  $c$ -quasihomomorphism  $f : \mathbb{Z} \rightarrow Q^n$  but  $w_H(f(y) - yf(1)) > C$  for some  $y \in \mathbb{Z}$ . Since  $Q$  is torsion-free, the natural map  $\iota$  from  $Q$  into the  $\mathbb{Q}$ -vector space  $V := \mathbb{Q} \otimes_{\mathbb{Z}} Q$  is injective. Consequently,  $g := \iota^n \circ f$  is a  $c$ -quasihomomorphism  $\mathbb{Z} \rightarrow V^n$  with  $w_H(g(y) - yg(1)) > C$ . Now choose any  $\mathbb{Q}$ -linear function  $\xi : V \rightarrow \mathbb{Q}$  that is nonzero on the nonzero entries of  $g(y) - yg(1)$ . Then  $h := \xi^n \circ g$  is a  $c$ -quasihomomorphism  $\mathbb{Z} \rightarrow \mathbb{Q}^n$  with  $w_H(h(y) - yh(1)) > C$ , a contradiction to Theorem 1.6.  $\square$

REMARK 1.9. As a referee kindly pointed out to us, our result fits in the broader context of  $\mathcal{G}$ -stability for a family  $\mathcal{G}$  of groups endowed with a bi-invariant metric; this was first introduced in [9] and further studied in [3] under the name of *Ulam stability*. Let  $\mathcal{G}$  be the family of groups  $\{\mathrm{GL}_n(\mathbb{C})\}_{n \geq 1}$  with the normalized rank metric, i.e.  $d(A, B) = \frac{1}{n} \mathrm{rk}(A - B)$ . Let  $\mathcal{G}_d$  be the subfamily of  $\mathcal{G}$  consisting of diagonal matrices. Theorem 1.6 shows that the abelian group  $\mathbb{Z}$  is uniformly  $\mathcal{G}_d$ -stable with a linear estimate.

Theorem 1.6 shows that for a  $c$ -quasihomomorphism  $f : \mathbb{Z} \rightarrow \mathbb{Q}^n$ , the group homomorphism  $\tilde{f} : \mathbb{Z} \rightarrow \mathbb{Q}^n$  defined by  $\tilde{f}(x) = xf(1)$  gives a  $C$ -approximation for some constant  $C \in \mathbb{N}$  independent on  $n$ . However,  $\tilde{f}$  need not be the homomorphism closest to  $f$ , as the next example shows.

EXAMPLE 1.10. Let  $c = 1$  and  $n \geq 3$ . Define  $f : \mathbb{Z} \rightarrow \mathbb{Q}^n$  to be

$$(2) \quad f(x) = \left( \left\lfloor \frac{2x}{5} \right\rfloor, \left\lfloor \frac{x}{5} \right\rfloor, \alpha_x, 0, \dots, 0 \right),$$

where  $\alpha_x \in \mathbb{Q}$  is arbitrary if  $5 \mid x$ , and  $\alpha_x = 0$  otherwise. Here  $\lfloor \cdot \rfloor$  denotes rounding to the nearest integer. To check that  $f$  is a 1-quasihomomorphism (1) we work mod 5. For simplicity, restrict to the case  $n = 3$ . Then, for  $k \in \mathbb{Z}$ ,

$$\begin{aligned} f(5k) &= (2k, k, \alpha_{5k}), & f(5k + 1) &= (2k, k, 0), \\ f(5k + 2) &= (2k + 1, k, 0), & f(5k + 3) &= (2k + 1, k + 1, 0), \\ f(5k + 4) &= (2k + 2, k + 1, 0). \end{aligned}$$

Let  $x = 5k + \ell_1$  and  $y = 5h + \ell_2$  with  $0 \leq \ell_1 \leq \ell_2 < 5$ . Then we can verify that

$$w_H(f(5(k + h) + (\ell_1 + \ell_2)) - f(5k + \ell_1) - f(5h + \ell_2)) \leq c = 1$$

in all cases. Roughly speaking, the check boils down to verifying that there are no cases where both  $\lfloor \frac{x+y}{5} \rfloor \neq \lfloor \frac{x}{5} \rfloor + \lfloor \frac{y}{5} \rfloor$  and  $\lfloor \frac{2(x+y)}{5} \rfloor \neq \lfloor \frac{2x}{5} \rfloor + \lfloor \frac{2y}{5} \rfloor$ , and moreover that if  $5 \mid x + y$ , then  $f(x + y) - f(x) - f(y) = (0, 0, \alpha_x)$  (because in this case,  $\frac{x}{5}$  is rounded down if and only if  $\frac{y}{5}$  is rounded up). Note that  $w_H(f(x) - xf(1)) \leq 3$  where equality is sometimes achieved (provided there is at least one  $x \neq 0$  for which we chose  $\alpha_x \neq 0$ ). However, there also exist 2-approximations of  $f$ . For instance, letting  $v = (\frac{2}{5}, \frac{1}{5}, 0, \dots, 0) \in \mathbb{Q}^n$ , one verifies that

$$w_H(f(x) - xv) \leq 2 \text{ for all } x \in \mathbb{Z}.$$

In [14], the authors show that for every 1-quasihomomorphism  $f : \mathbb{Z} \rightarrow \mathbb{Q}^n$ , and even for every 1-quasihomomorphism from  $\mathbb{Z}$  into the space of symmetric  $n \times n$ -matrices with the rank metric, there is a 2-approximation.

(This result is consistent with the second paragraph of [10], where a proof of the corresponding statement for general matrices is sketched. However, the above example shows that that proof is incomplete: viewing  $f$  as a map to the diagonal matrices, and assuming  $\alpha_0 = 0$  as is done in that paragraph, we obtain a counterexample to the statement in [10] that there exists either a subspace of codimension 1 living in the kernel of all matrices  $f(n + m) - f(n) - f(m)$  or else a subspace of dimension 1 containing all their images.)

On the other hand, the following shows that the best possible approximation of a given quasihomomorphism  $f$  is at most twice as close as the homomorphism  $x \mapsto xf(1)$ .

REMARK 1.11. Suppose that a map  $f : \mathbb{Z} \rightarrow \mathbb{Q}^n$  has a  $C'$ -approximation  $h$ . Then  $h(x) = xv$  for some  $v \in \mathbb{Q}^n$ , and

$$w_H(f(x) - xv) \leq C' \text{ for all } x \in \mathbb{N}.$$

Substituting  $x = 1$  yields  $w_H(f(1) - v) \leq C'$ . Thus

$$w_H(f(x) - xf(1)) \leq w_H(f(x) - xv) + w_H(xv - xf(1)) \leq 2C'.$$

REMARK 1.12. A result similar to Theorem 1.6 is easily proven in positive characteristic if we allow the constant  $C$  to depend on the characteristic. Let  $K$  be a field of characteristic  $p > 0$ , and let  $f : \mathbb{Z} \rightarrow K^n$  be a  $c$ -quasihomomorphism. Then there exists a constant  $C = C(p, c)$  such that  $w_H(f(x) - xf(1)) \leq C$ , for all  $x \in \mathbb{Z}$ .

To see this, we observe that for all  $u, v \in \mathbb{Z}$  with  $u \geq 1$ , we have

$$w_H(f(uv) - uf(v)) \leq (u - 1)c.$$

This follows by repeatedly applying the inequality  $w_H(f(uv) - f((u-1)v) - f(v)) \leq c$  if  $u > 1$ ; the case  $u = 1$  is trivial.

For  $x = kp + r$  with  $k \in \mathbb{Z}$  and  $0 \leq r \leq p - 1$ , we have

$$w_H(f(x) - xf(1)) = w_H(f(kp + r) - rf(1));$$

here we have used that  $pf(1) = 0$ . We rewrite the latter as

$$w_H(f(kp + r) - f(kp) - f(r) + f(kp) + f(r) - rf(1)).$$

We have  $w_H(f(kp+r) - f(kp) - f(r)) \leq c$ ;  $w_H(f(kp)) \leq (p-1)c$  using our observation with  $u = p, v = k$ ; and also  $w_H(f(r) - rf(1)) \leq (p-2)c$  (in the case  $r > 0$ ). In total, this gives  $w_H(f(x) - xf(1)) \leq 2(p-1)c$ , so we can take  $C = 2(p-1)c$ .

The remainder of this paper is organized as follows. In Section 2 we prove an auxiliary result of independent interest: maps from a finite abelian group into a torsion-free group that are almost a homomorphism, are in fact almost zero. Then, in Section 3, we apply this auxiliary result to the component functions of a  $c$ -quasihomomorphism  $\mathbb{Z} \rightarrow \mathbb{Q}^n$  to prove the Main Theorem.

## 2. ALMOST HOMOMORPHISMS ARE ALMOST ZERO

Let  $A$  be a finite abelian group and let  $H$  be a torsion-free abelian group. The only homomorphism  $A \rightarrow H$  is the zero map. The following proposition says that maps that are, in a suitable sense, close to being homomorphisms, are in fact also close to the zero map.

PROPOSITION 2.1. *Let  $a$  be a positive integer,  $A$  an abelian group of order  $a$ ,  $H$  a torsion-free abelian group,  $q \in [0, 1]$ , and  $f : A \rightarrow H$  a map. Suppose that the zero set*

$$Z(f) := \{b \in A \mid f(b) = 0\}$$

*has cardinality at most  $qa$ . Then the problem set*

$$P(f) := \{(b, c) \in A \times A \mid f(b+c) \neq f(b) + f(c)\}$$

*has cardinality at least  $\frac{(1-q)^2}{4}a^2 + \frac{(1-q)}{2}a$ .*

The contraposition of this statement says that if  $P(f)$  is a small fraction of  $a^2$ , so that  $f$  can be thought of as an (additive) ‘almost homomorphism’  $A \rightarrow H$ , then  $q$  must be close to 1 so that  $f$  is essentially zero.

*Proof.* Since  $H$  is torsion-free, it embeds into the  $\mathbb{Q}$ -vector space  $V := \mathbb{Q} \otimes_{\mathbb{Z}} H$ . By basic linear algebra, there exists a  $\mathbb{Q}$ -linear function  $\xi : V \rightarrow \mathbb{Q}$  such that  $\xi(f(b)) \neq 0$  for all  $b \notin Z(f)$ , so that  $Z(\xi \circ f) = Z(f)$ . Since  $P(\xi \circ f) \subseteq P(f)$ , it suffices to prove the proposition for  $\xi \circ f$  instead of  $f$ . In other words, we may assume from the beginning that  $H = \mathbb{Q}$ .

Set

$$B := \{b \in A \mid f(b) > 0\}.$$

Let  $\lambda_1 > \lambda_2 > \dots > \lambda_k > 0$  be the distinct values in  $f(B)$ , and for each  $i = 1, \dots, k$  set

$$B_i := \{b \in B \mid f(b) = \lambda_i\} \text{ and } n_i := |B_i|;$$

as well as  $n := n_1 + \dots + n_k = |B|$ .

Now for each  $c \in B_1$  and each  $b \in B$  we have

$$f(b) + f(c) = f(b) + \lambda_1 > \lambda_1$$

so that the left-hand side is not in  $f(B)$  and in particular not equal to  $f(b + c)$ . We have thus found  $n_1(n_1 + \dots + n_k)$  pairs  $(b, c) \in P(f)$  with  $c \in B_1$ .

Next, suppose  $(b, c)$  is a pair with  $c \in B_2$ ,  $b \in B$ , and  $(b, c) \notin P(f)$ . Then

$$f(b + c) = f(b) + f(c) > f(c) = \lambda_2$$

and hence  $b + c \in B_1$ . But given  $c$ , there are at most  $n_1$  values of  $b$  with  $b + c \in B_1$ . (Note that here we have used that  $A$  is a group.) Hence we have at least  $n_2(n_2 + \dots + n_k)$  pairs  $(b, c) \in P(f)$  with  $c \in B_2$ .

Similarly, we find at least  $n_i(n_i + \dots + n_k)$  pairs  $(b, c) \in P(f)$  with  $c \in B_i$ . In total, we have therefore found at least

$$(3) \quad \sum_{i=1}^k n_i(n_i + \dots + n_k) \geq \frac{n(n+1)}{2}$$

pairs in  $P(f)$ ; see Figure 1.

Let  $B' := \{b' \in A \mid f(b') < 0\}$  and  $n' := |B'|$ . Repeating the same argument above with  $B'$  and  $n'$ , we find at least  $n'(n'+1)/2$  further pairs in  $P(f)$ , disjoint from those found above. Since  $|Z(f)| \leq qa$ , we have  $n + n' \geq a(1 - q)$ . Therefore

$$|P(f)| \geq \frac{n(n+1)}{2} + \frac{n'(n'+1)}{2} = \frac{n^2 + n'^2}{2} + \frac{n + n'}{2} \geq \left(\frac{n + n'}{2}\right)^2 + \frac{n + n'}{2},$$

where the second inequality is the Cauchy-Schwarz inequality

$$(n^2 + n'^2) \left(\frac{1}{2^2} + \frac{1}{2^2}\right) \geq \left(\frac{n}{2} + \frac{n'}{2}\right)^2.$$

Since  $n + n' \geq a(1 - q)$ , we conclude that

$$|P(f)| \geq \left(\frac{a - qa}{2}\right) \left(\frac{a - qa}{2} + 1\right). \quad \square$$

REMARK 2.2. The lower bound in Proposition 2.1 is sharp. Let  $a = 2k + 1 \in \mathbb{Z}$ , consider  $A := \mathbb{Z}/a\mathbb{Z}$  and define  $f : A \rightarrow \mathbb{Z}$  as  $f(x) :=$ the representative of  $x + a\mathbb{Z}$  in  $\{-k, \dots, 0, \dots, k\}$ . We take  $q = \frac{Z(f)}{a} = \frac{1}{2k+1}$ . Then  $f(x + y) = f(x) + f(y)$  if and only if the right-hand side is still inside the interval  $\{-k, \dots, k\}$ , and a straightforward count shows that this is the case for  $3k^2 + 3k + 1$  pairs  $(x, y) \in A^2$ . Hence  $P(f)$  has size  $k(k + 1)$ , which equals  $\frac{(1-q)^2}{4}a^2 + \frac{(1-q)}{2}a$ .

A similar construction for  $a = 2k$  yields a problem set of size  $\frac{a^2}{4} = k^2$ , which equals the ceiling of the lower bound  $\frac{a^2}{4} - \frac{1}{4}$ .

Below, we will use the following strengthening of Proposition 2.1:

PROPOSITION 2.3. *Let  $a, A, H, q$  and  $f$  be as in Proposition 2.1. Furthermore, let  $p \in [0, \frac{1-q}{2})$  and let  $S \subseteq A$  be a subset of cardinality at most  $pa$ . Then the set*

$$P_S(f) := \{(b, c) \in A \times A \mid f(b+c) \neq f(b) + f(c) \text{ and } b+c \notin S\}.$$

has cardinality at least  $\frac{(1-q-2p)^2}{4}a^2 + \frac{(1-q-2p)}{2}a$ .

*Proof.* Keep the notation from the proof of Proposition 2.1. Recall  $n = |B|$  and  $n' = |B'|$ . Note that for a fixed  $b$ , there can be at most  $pa$  choices of  $c$  with  $b+c \in S$ . We then find at least  $n_i(n_i + \dots + n_k - pa)$  pairs  $(b, c) \in P_S(f)$  with  $b \in B_i$ . Letting  $k' \leq k$  be the largest index for which the second factor  $(n_{k'} + \dots + n_k - pa)$  is nonnegative, as in the proof of Proposition 2.1, we find that  $B$  contributes at least

$$\begin{aligned} \sum_{i=1}^{k'} n_i(n_i + \dots + n_k - pa) &= \sum_{i=1}^{k'} n_i(n - n_1 - \dots - n_{i-1} - pa) \\ (4) \qquad \qquad \qquad &\geq (n - pa)(n - pa + 1)/2 \end{aligned}$$

to  $P_S(f)$ ; see Figure 1. Similarly,  $B'$  contributes at least  $(n' - pa)(n' - pa + 1)/2$ , and these contributions are disjoint. The desired inequality follows as in the proof of Proposition 2.1 but with  $n, n'$  replaced by  $n - pa, n' - pa$ .  $\square$

The key ingredient for the proof of Theorem 1.6 is the following corollary of Proposition 2.3. Here, and in the rest of the paper, we write  $[a]$  for the set  $\{1, 2, \dots, a\}$ .

COROLLARY 2.4. *Let  $p, q \in [0, 1]$  such that  $p < \frac{1-q}{2}$ . Let  $f : [2a] \rightarrow \mathbb{Q}$  such that:*

- (1)  $|Z_a(f)| \leq qa$ , where  $Z_a(f) := \{x \in [a] \mid f(x) = 0\}$  is the zero set of  $f|_{[a]}$ .
- (2)  $|NP(f)| \leq pa$ , where  $NP(f) := \{x \in [a] \mid f(x+a) \neq f(x)\}$  is the nonperiodicity set.

Then

$$|P(f)| \geq \frac{(1-q-2p)^2}{4}a^2 + \frac{(1-q-2p)}{2}a,$$

where

$$P(f) = \{(x, y) \in [a] \times [a] \mid f(x+y) \neq f(x) + f(y)\}.$$

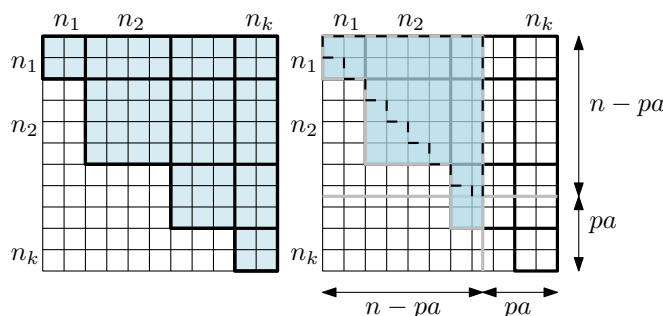


FIGURE 1. On the left, a graphical proof of the inequality (3): the left-hand side is the number of small squares in the shaded region, the right-hand side is the number of squares on or above the main diagonal. On the right, a proof of the inequality (4): the two expressions on top represent the area of the shaded region, while the bottom expression represents the area enclosed by the dashed line.

*Proof.* Let  $\tilde{f}$  be the restriction of  $f$  to the interval  $[a]$ , and identify  $\mathbb{Z}/a\mathbb{Z}$  with  $[a]$  with the group operation  $\star$  defined by  $x \star y := x + y \pmod{a}$ .

Let  $S = NP(f)$ , and apply Proposition 2.3 to  $\tilde{f}$ . We find that

$$P_S(\tilde{f}) = \{(b, c) \in \mathbb{Z}/a\mathbb{Z} \times \mathbb{Z}/a\mathbb{Z} \mid \tilde{f}(b \star c) \neq \tilde{f}(b) + \tilde{f}(c) \text{ and } b \star c \notin S\}$$

has cardinality at least  $\frac{(1-q-2p)^2}{4}a^2 + \frac{(1-q-2p)}{2}a$ . Since  $b \star c \notin S$  implies that  $\tilde{f}(b \star c) = f(b + c)$ , this set is contained in the problem set  $P(f)$ .  $\square$

### 3. PROOF OF THE MAIN THEOREM

The main goal of this section is to prove Theorem 1.6. We start with some definitions.

DEFINITION 3.1. Let  $1 < a$ , and  $f : [2a] \rightarrow \mathbb{Q}$ . We define the following problem sets of  $f$ :

$$P(f) := \{(x, y) \in [a] \times [a] \mid f(x + y) \neq f(x) + f(y)\},$$

and

$$P_1(f) := \{x \in [a] \mid f(x + 1) \neq f(x) + f(1)\},$$

and

$$P_a(f) := \{x \in [a] \mid f(x + a) \neq f(x) + f(a)\}.$$

Furthermore, we recall that  $Z_a(f)$  denotes the zero set of  $f|_{[a]}$ :

$$Z_a(f) := \{x \in [a] \mid f(x) = 0\}.$$

The following proposition says that  $P_1(f), P_a(f), P(f)$  cannot be simultaneously small.

PROPOSITION 3.2. Let  $p, q \in (0, 1)$  such that  $1 - q - 2p > 0$ ,  $a \in \mathbb{N}$  with  $1 < a$ , and let  $f : [2a] \rightarrow \mathbb{Q}$  such that  $f(a) \neq af(1)$ . Then at least one of the following holds:

- (i)  $|P_1(f)| > qa$ ,
- (ii)  $|P_a(f)| > pa$ ,
- (iii)  $|P(f)| \geq F(p, q)a^2$ ,

where

$$F(p, q) := \frac{(1 - q - 2p)^2}{4}.$$

*Proof.* Without loss of generality we can assume  $f(a) = 0$  and hence  $f(1) \neq 0$ . Indeed, suppose we have shown the statement for every  $\tilde{f}$  with  $\tilde{f}(a) = 0$ . Then for any  $f : [2a] \rightarrow \mathbb{Q}$  with  $f(a) \neq af(1)$ , we take  $\tilde{f} : [2a] \rightarrow \mathbb{Q}$  to be  $\tilde{f}(x) = af(x) - xf(a)$ . Now we observe that  $\tilde{f}(a) = 0 \neq a\tilde{f}(1)$ , and that  $P(f) = P(\tilde{f})$ ,  $P_1(f) = P_1(\tilde{f})$ ,  $P_a(f) = P_a(\tilde{f})$ .

To prove the proposition we will assume that ((i)) and ((ii)) are false, and prove that then ((iii)) must hold. Write  $Z_a(f) = \{x_1, \dots, x_m\}$ , where  $x_1 < \dots < x_m$ . Note that for  $1 \leq i < m$ , one of the elements  $x_i, x_i + 1, \dots, x_{i+1} - 1$  needs to be in  $P_1(f)$  since  $f(x_{i+1}) \neq f(x_i) + (x_{i+1} - x_i)f(1)$ . Likewise, at least one of the elements  $1, 2, \dots, x_1 - 1$  needs to be in  $P_1(f)$ . Thus we have

$$|Z_a(f)| \leq |P_1(f)| \leq qa,$$

and by assumption we have  $|NP(f)| = |P_a(f)| \leq pa$ . Now we can apply Corollary 2.4 to conclude.  $\square$

We now prove Theorem 1.6.

*Proof of the Main Theorem.* Consider a  $c$ -quasihomomorphism  $f = (f_1, \dots, f_n) : \mathbb{Z} \rightarrow \mathbb{Q}^n$ . Our goal is to show that for every  $a \in \mathbb{Z}$  we have  $w_H(f(a) - af(1)) \leq C$  for some constant  $C$  depending only on  $c$ . We start with the case  $a > 0$ .

Write  $I_a := \{i \in [a] \mid f_i(a) \neq af_i(1)\}$ , and note that  $|I_a| = w_H(f(a) - af(1))$ . We will show that  $|I_a| \leq C'$  for some constant  $C'$  depending on  $c$  only. To this end, fix small parameters  $p, q \in (0, 1)$  (to be optimized over later) and write  $f_i^a := f_i|_{[2a]}$  for the restriction of  $f_i$  to  $[2a]$ . By Proposition 3.2, for every  $i \in I_a$ , we have

- (i)  $|P_1(f_i^a)| > qa$ , or
- (ii)  $|P_a(f_i^a)| > pa$ , or
- (iii)  $|P(f_i^a)| \geq F(p, q)a^2$ .

Let  $m_0$  be the number of coordinates  $i \in I_a$  such that (iii) holds. We define  $m_1$  and  $m_2$  analogously, for (i) and (ii) respectively.

By counting the number of triples  $(i, x, y) \in [n] \times [a] \times [a]$  such that  $f_i(x + y) - f_i(x) - f_i(y) \neq 0$  in two ways, we see that

$$\sum_{x=1}^a \sum_{y=1}^a w_H(f(x + y) - f(x) - f(y)) = \sum_{i=1}^n |P(f_i^a)| \geq \sum_{i \in I_a} |P(f_i^a)|.$$

Because  $f$  is a  $c$ -quasihomomorphism, the very left-hand side is at most  $a^2c$ . On the other hand, the very right-hand side is at least  $m_0F(p, q)a^2$ , so

$$a^2c \geq \sum_{x=1}^a \sum_{y=1}^a w_H(f(x + y) - f(x) - f(y)) \geq \sum_{i \in I_a} |P(f_i^a)| \geq m_0F(p, q)a^2.$$

So we obtain  $m_0 \leq \frac{c}{F(p, q)}$ . Similarly we find

$$ac \geq \sum_{x=1}^a w_H(f(x + 1) - f(x) - f(1)) = \sum_{i=1}^n |P_1(f_i^a)| \geq \sum_{i \in I_a} |P_1(f_i^a)| > m_1qa,$$

so that  $m_1 < \frac{c}{q}$ . Finally,

$$ac \geq \sum_{x=1}^a w_H(f(x + a) - f(x) - f(a)) = \sum_{i=1}^n |P_a(f_i^a)| \geq \sum_{i \in I_a} |P_a(f_i^a)| > m_2pa.$$

So  $m_2 < \frac{c}{p}$ . But now  $|I_a| \leq m_0 + m_1 + m_2 < c(\frac{1}{F(p, q)} + \frac{1}{q} + \frac{1}{p}) =: C'$ .

The case  $a = 0$  is easy: we have

$$w_H(f(0)) = w_H(f(0) - f(0) - f(0)) \leq c.$$

Finally, let us consider the case  $a < 0$ . Then

$$\begin{aligned} w_H(f(a) - af(1)) &\leq w_H(f(a) + f(-a) - f(0)) + w_H(f(0)) \\ &\quad + w_H(f(-a) - (-a)f(1)) \leq 2c + C' =: C. \end{aligned}$$

This completes the proof of the qualitative part of the Main Theorem. To obtain the explicit bound  $28c$ , we minimize the function

$$2 + \frac{1}{q} + \frac{1}{p} + \frac{1}{F(p, q)} = 2 + \frac{1}{q} + \frac{1}{p} + \frac{4}{(1 - q - 2p)^2}.$$

This function is strictly convex for  $(p, q) \in \mathbb{R}_{>0}^2$ , so it has at most one minimum in the positive orthant. We find this by setting the partial derivatives to zero and solving for  $p, q$ . The minimum is  $\approx 27.6817$  and attained at  $(p, q) \approx (0.1167, 0.16500)$ .  $\square$

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