International Journal of Sciences

Research Article Volume 13 – July 2024 (07)

Generalized Mass Additivity in Special Relativity

Amelia Carolina Sparavigna[1]

[1]Department of Applied Science and Technology, Polytechnic University of Turin

Abstract: Starting from literature about invariant mass, we will discuss the expression of the generalized additivity of masses in the special relativity for a system of particles. We will show the general formula of additivity in the framework of hyperbolic geometry too, showing it being invariant.

Keywords: Special Relativity, Hyperbolic Geometry

Introduction

In special relativity the mass is an invariant quantity and therefore is the same for all observers in all reference frames. Besides this invariant, we find in literature the 'relativistic mass', which is defined as dependent on velocity. We can find the relativistic mass given as:

$$m_{rel} = \frac{m}{\sqrt{1 - v^2/c^2}}$$

In our discussion, we will start from some literature about this 'relativistic' mass and continue considering the problem of mass additivity in special relativity. The hyperbolic geometric approach is given too. Before literature, it is fundamental to remember that energy and momentum are linked to the invariant mass by:

$$E^2 - (pc)^2 = (mc^2)^2$$

Einstein's thought

Hecht, 2009, discusses mass and energy in Einstein's thought. "Early on, Einstein embraced the idea of a speed-dependent mass but changed his mind in 1906 and thereafter carefully avoided that notion entirely. He shunned, and explicitly rejected, what later came to be known as 'relativistic mass'. Nonetheless many textbooks and articles credit him with the relation $E=mc^2$, where E is the total energy, m is the relativistic mass, and c is the vacuum speed of light. Einstein never derived this relation, at least not with that understanding of the meaning of its terms. He consistently related the *rest energy* of a system to its invariant inertial mass." This is what we can find in the abstract by Hecht, 2009. It is therefore required to read something more from the article.

"Einstein's first paper on relativity appeared when the concept of a speed-dependent electromagnetic mass had already become a topic of considerable interest. He accepted this idea but changed his mind after being confronted by a far more compelling insight" (Hecht, 2009). Hecht shows "that after reading Planck's 1906 article in which the concept of relativistic momentum was introduced, Einstein came to realize that it was the relativistic equations for energy and momentum that were primary. From that perspective it became clear that the inertial mass m was invariant, and he never again spoke of mass as being speed dependent. Over the next several years, no doubt unaware of Einstein's change of mind, a number of researchers continued to elaborate on the idea that inertial mass varied with relative speed v" (Hecht, 2009). For these researchers, the "Newtonian mass had to be replaced by the idea of *relativistic mass* $m_{rel}(v)$" (Hecht, 2009), where:

$$m_{rel} = m_0(1 - v^2/c^2)^{-1/2}$$

In this formula, m_0 is the rest mass, that is the inertial mass when $v=0$. Since it was usual to represent $(1 - v^2/c^2)^{-1/2}$ by γ, we find in literature $m_{rel} = m_0\gamma$ or just $m_{rel} = m\gamma$. "After 1908 there were two conflicting interpretations of relativistic dynamics: Einstein's invariant-mass perspective and the relativistic mass formulation. Meanwhile Einstein had shown that the energy of a system at rest was proportional to its inertial mass. Over the decades that followed, this extremely significant discovery took on the symbolic form $E = mc^2$, wherein E is the total energy and m is the relativistic mass. Surprisingly, Einstein never derived nor ever accepted this relation. As $E=mc^2$ was becoming the most widely recognized symbol of the Atomic Age, Einstein maintained that this general statement was formulated *somewhat inexactly*" (Hecht, 2009, Einstein, 1976). Among many others, Hecht is referring to Lev B. Okun, 1989, and his "concept of mass (mass, energy, relativity)".

Published at: http://www.ijsciences.com/pub/issue/2024-07/
DOI: 10.18483/ijSci.2777; Online ISSN: 2305-3925; Print ISSN: 2410-4477

Amelia Carolina Sparavigna (Correspondence)
+

The concept of mass

Okun hopes that he will "succeed to convince the reader that the term *rest mass* m_0 is superfluous, that instead of speaking of the *rest mass* m_0 one should speak of the mass *m* of a body which for ordinary bodies is the same, in the theory of relativity and in Newtonian mechanics, [and also] that in both theories the mass *m* does not depend on the reference frame, that the concept of mass dependent on velocity arose at the beginning of the twentieth century as a result of an unjustified extension of the Newtonian relation between momentum and velocity to the range of velocities comparable to the velocity of light in which it is invalid, and that at the end of the twentieth century one should bid a final farewell to the concept of mass dependent on velocity" (Okun, 1989).

Okun is stressing that "the fundamental relations of the theory of relativity for a freely moving particle (system of particle, body) are: $E^2 - (pc)^2 = (mc^2)^2$ (5.1) , $\vec{p} = \vec{v}\,E/c^2$ (5.2) , where *E* is the energy, ***p*** the momentum, *m* is the mass, and ***v*** the velocity of the particle (or system of particles, or body). It should be emphasized that the mass *m* and the velocity ***v*** for a particle or a body are the same quantities with which we deal in Newtonian mechanics. Like the four-dimensional coordinates *t* and ***r***, the energy *E* and the momentum ***p*** are the components of a four-dimensional vector. They change on the transition from one inertial system to another in accordance with the Lorentz transformation. The mass, however, is not changed - it is a Lorentz invariant" (Okun, 1989).

Moreover, as in the case of Newtonian mechanics, "the energy and momentum are additive — the total energy and total momentum of *n* free particles are, respectively, $E = \sum_{i=1}^{n} E_i$, $\vec{p} = \sum_{i=1}^{n} \vec{p_i}$ (5.3). With regard to the mass, in theory of relativity the mass of an isolated system is conserved (does not change with the time), but does not possess the property of additivity" (Okun, 1989). However, it is necessary "to include among the bodies not only *matter*, say atoms, but also *radiation* (photons)." (Okun, 1989).

"For massive particles (as we shall call all particles with nonzero mass, even if they are very light) the relations for the energy and momentum can be conveniently expressed in terms of the mass and velocity. For this we substitute (5.2) in (5.1): $E^2(1 - v^2/c^2) = m^2c^4$ (6.2) and, taking the square root, we obtain $E = mc^2(1 - v^2/c^2)^{-1/2}$. (6.3) Substituting (6.3) in (5.2), we obtain $\vec{p} = m\vec{v}(1 - v^2/c^2)^{-1/2}$. (6.4). It is obvious from (6.3) and (6.4) that a massive body (with $m \neq 0$) cannot move with the speed of light, since then the energy and momentum of the body would have to be infinite" (Okun, 1989).

"In the theory of relativity the mass of a system is not equal to the mass of the bodies that make up the system. This assertion can be illustrated by several examples" (Okun, 1989). Here a case: "Consider two photons moving in opposite directions with equal energies *E.* The total momentum of such a system is zero, and the total energy (it is the rest energy of the system of the two photons) is *2E*. Therefore, the *mass* of this system is $2E/c^2$. It is easy to show that a system of two photons will have zero mass only when they move in the same direction" (Okun. 1989).

Let us consider a system consisting of *n* particles. According to Okun, the mass of the system is determined by his formula (9.1):

$$m = \left[\left(\sum_{i=1}^{n} \frac{E_i}{c^2}\right)^2 - \left(\sum_{i=1}^{n} \frac{\vec{p_i}}{c}\right)^2\right]^{1/2}$$

In it, $\sum E_i$ is the sum of the energies of the bodies, and $\sum \vec{p_i}$ is the vector sum of momenta. Actually (9.1) tells us how to sum the masses.

Masses and particles

Further discussion is given by Landau and Lifshitz, 1971. Landau and Lifshits "emphasize that, although we speak of a *particle*, we have nowhere made use of the fact that it is *elementary*. Thus the formulas are equally applicable to any composite body consisting of many particles, where by *m* we mean the total mass of the body and by *v* the velocity of its motion as a whole". The formula (9.5) in Landau and Lifshitz, that of the *rest energy*, "is valid for any body which is at rest as a whole. We call attention to the fact that in relativistic mechanics the energy of a free body (i.e. the energy of any closed system) is a completely definite quantity which is always positive and is directly related to the mass of the body". Moreover, "in this connection we recall that in classical mechanics the energy of a body is defined only to within an arbitrary constant, and can be either positive or negative. The energy of a body at rest contains, in addition to the rest energies of its constituent particles, the kinetic energy of the particles and the energy of their interactions with one another. In other words, mc^2 is not equal to $\sum m_a c^2$ (where m_a are the masses of the particles), and so *m* is not equal to $\sum m_a$" (Landau & Lifshitz, 1971). It is clear that "the mass of a composite body is not equal to the sum of the masses of its parts. Instead only the law of conservation of energy, in which the rest energies of the particles are included, is valid" (Landau & Lifshitz, 1971).

In Landau and Lifshitz we can find also discussed the decay of particles. "Let us consider the spontaneous decay of a body of mass M into two parts with masses m_1 and m_2. The law of conservation of energy in the decay, applied in the system of reference in which the body is at rest, gives: $Mc^2 = E_{10} + E_{20}$ (11.1), where E_{10}, E_{20} are the energies of the emerging particles. Since $E_{10} > m_1c^2$ and $E_{20} > m_2c^2$ the equality (11.1) can be satisfied only if $M > m_1 + m_2$, i.e. a body can disintegrate spontaneously into parts the sum of whose masses is less than the mass of the body. On the other hand, if $M < m_1 + m_2$, the body is stable (with respect to the particular decay) and does not decay spontaneously. To cause the decay in this case, we would have to supply to the body from outside an amount of energy at least equal to its *binding energy*" (Landau & Lifshitz, 1971).

Landau and Lifshitz continue with discussing the conservation of momentum and energy in the decay process, arriving to determine uniquely the energies of the two emerging bodies. "In a certain sense the inverse of this problem is the calculation of the total energy [mass] M of two colliding particles in the system of reference in which their total momentum is zero. … The computation of this quantity gives a criterion for the possible occurrence of various inelastic collision processes, accompanied by a change in state of the colliding particles, or the *creation* of new particles. A process of this type can occur only if the sum of the masses of the *reaction products* does not exceed *M*" (Landau & Lifshitz, 1971).

An example of mass additivity

As an of mass additivity, let us pass to the Notes on Relativistic Dynamics, by Styer, 2021. "Two-particle system. Two particles move on the *x*-axis. Particle A has mass m_A and velocity (relative to frame F) v_A, particle B has mass m_B and velocity (relative to frame F) v_B. a) Show that the two-particle system has mass M, where (4.3):

$$M^2 = m_A^2 + m_B^2 + 2m_A m_B \frac{1 - v_A\, v_B/c^2}{\sqrt{(1-(v_A/c)^2)(1-(v_B/c)^2)}}$$

Frame F' moves relative to frame F at velocity V, so in this frame the two particles have velocities (4.4):

$$v'_A = \frac{v_A - V}{1 - v_A\, V/c^2} \quad \text{and} \quad v'_B = \frac{v_B - V}{1 - v_B\, V/c^2}$$

b) Show that in the frame F', the system has the same mass M given above" (Styer, 2021).

Note that, in (4.3), when the velocities are all equals, the additivity is the usual one.

Invariance

For point b) in Styer, 2021, let us use WolframAlpha software and calculate:

$$1 - \frac{(v_A - V)\,(v_B - V)/c^2}{(1 - v_A\, V/c^2)(1 - v_B\, V/c^2)}$$

It is equal to:

$$A = \frac{(V^2 - c^2)(v_A v_B - c^2)}{(v_A V - c^2)(v_B V - c^2)}$$

And

$$\sqrt{\left(1 - \frac{((v_A - V)/c)^2}{(1 - v_A\, V/c^2)^2}\right)\left(1 - \frac{((v_B - V)/c)^2}{(1 - v_B\, V/c^2)^2}\right)}$$

It is equal to

$$B = \sqrt{\frac{(v_A^2 - c^2)(v_B^2 - c^2)(c^2 - V^2)^2}{(v_A V - c^2)^2 (v_B V - c^2)^2}}$$

Therefore:

$$A/B = \frac{\dfrac{-(c^2 - V^2)(v_A v_B - c^2)}{(v_A V - c^2)(v_B V - c^2)}}{\sqrt{\dfrac{(v_A^2 - c^2)(v_B^2 - c^2)(c^2 - V^2)^2}{(v_A V - c^2)^2 (v_B V - c^2)^2}}} = \frac{-(v_A v_B - c^2)}{\sqrt{(-v_A^2 + c^2)(-v_B^2 + c^2)}} = \frac{1 - v_a\, v_b/c^2}{\sqrt{(1 - v_A^2/c^2)(1 - v_B^2/c^2)}}$$

Accordingly, we have invariance.

Generalized mass additivity

Let us pass to point a) in Styer, 2021, using (9.1) $M = \left[\left(\sum_{i=1}^{n} \frac{E_i}{c^2}\right)^2 - \left(\sum_{i=1}^{n} \frac{\vec{p_i}}{c}\right)^2\right]^{1/2}$.

Therefore:

$$M = \sqrt{(E_A/c^2 - E_B/c^2) - (p_A/c + p_B/c)^2}$$

$$M^2 = \left(m_A(1 - v_A^2/c^2)^{-1/2} + m_B(1 - v_B^2/c^2)^{-1/2}\right)^2 - \left(m_A(v_A/c)(1 - v_A^2/c^2)^{-1/2} + m_B(v_B/c)(1 - v_B^2/c^2)^{-1/2}\right)^2$$

$$M^2 = m_A^2(1-v_A^2/c^2)^{-1} + m_B^2(1-v_B^2/c^2)^{-1} + 2m_A m_B(1-v_A^2/c^2)^{-1/2}(1-v_B^2/c^2)^{-1/2} - m_A^2(v_A/c)^2(1-v_A^2/c^2)^{-1} - m_B^2(v_B/c)^2(1-v_B^2/c^2)^{-1} - 2m_A m_B(v_A v_B/c^2)(1-v_A^2/c^2)^{-1/2}(1-v_B^2/c^2)^{-1/2}$$

Then:

$$M^2 = m_A^2 + m_B^2 + 2m_A m_B \frac{1 - v_A v_B/c^2}{\sqrt{(1-(v_A/c)^2)(1-(v_B/c)^2)}}$$

This is a *generalized additivity* of masses in special relativity. The meaning is the following: this expression is making the *additivity* applicable in a wider manner. Generalization of additivity for integer numbers has been proposed in 2019 by Sparavigna. For what is regarding generalized entropies, see please Sparavigna, 2015. Note that the generalized additivity given in 2015 is quite different from the invariant expression given above.

In the case that we have three particles:

$$M^2 = m_A^2 + m_B^2 + m_C^2 + 2m_A m_B \frac{1 - v_A v_B/c^2}{\sqrt{(1-(v_A/c)^2)(1-(v_B/c)^2)}}$$
$$+2m_A m_C \frac{1 - v_A v_C/c^2}{\sqrt{(1-(v_A/c)^2)(1-(v_C/c)^2)}} + 2m_B m_C \frac{1 - v_B v_C/c^2}{\sqrt{(1-(v_B/c)^2)(1-(v_C/c)^2)}}$$

In general:

$$M^2 = \sum_{i=1}^n m_i^2 + \sum_{i=1}^n \sum_{j=1, j\neq i}^n m_i m_j \frac{1 - v_i v_j/c^2}{\sqrt{(1-(v_i/c)^2)\left(1-(v_j/c)^2\right)}}$$

That is:

$$M^2 = \sum_{i=1}^n \sum_{j=1}^n m_i m_j \frac{1 - v_i v_j/c^2}{\sqrt{(1-(v_i/c)^2)\left(1-(v_j/c)^2\right)}}$$

Hyperbolic geometry

Let us introduce $v/c = \tanh\beta$. Then $\gamma = 1/\sqrt{1-\tanh^2\beta} = \cosh\beta$ (Dray, 2012). We have:

$$M^2 = \sum_{i=1}^n \sum_{j=1}^n m_i m_j \cosh\beta_i \cosh\beta_j \left(1 - \tanh\beta_i \tanh\beta_j\right)$$
$$M^2 = \sum_{i=1}^n \sum_{j=1}^n m_i m_j \left(\cosh\beta_i \cosh\beta_j - \sinh\beta_i \sinh\beta_j\right)$$

Consequently, when $\beta_i = \beta_j = \beta$, we find immediately $M^2 = \sum_{i=1}^n \sum_{j=1}^n m_i m_j$.

About the invariant M^2 given above, let us consider frame F' moving relative to frame F at velocity V, so that in this frame:

$$M^2 = \sum_{i=1}^n \sum_{j=1}^n m_i m_j \left(\cosh\beta'_i \cosh\beta'_j - \sinh\beta'_i \sinh\beta'_j\right)$$

with velocities:

$$\frac{v'_i}{c} = \frac{(v_i - V)/c}{1 - v_i V/c^2} = \tanh\beta_i - \tanh\alpha \quad \text{and} \quad \frac{v'_j}{c} = \frac{(v_j - V)/c}{1 - v_j V/c^2} = \tanh\beta_j - \tanh\alpha$$

Using the properties: $\cosh^2\alpha - \sinh^2\alpha = 1$, $\sinh(\beta-\alpha) = \sinh\beta \cosh\alpha - \cosh\beta \sinh\alpha$, $\cosh(\beta-\alpha) = \cosh\beta \cosh\alpha - \sinh\beta \sinh\alpha$, it is easy to see that:

$$M^2 = \sum_{i=1}^n \sum_{j=1}^n m_i m_j \left(\cosh\beta_i \cosh\beta_j - \sinh\beta_i \sinh\beta_j\right)$$

We have again an invariant expression in the hyperbolic geometry.

References

1. Dray, T. (2012). The geometry of special relativity (Vol. 6). Boca Raton: CRC Press.
2. Einstein, A. (1976). $E=Mc^2$, Ideas and Opinions. Dell, New York, 1976, p. 330.
3. Hecht, E. (2009). Einstein on mass and energy. American Journal of Physics, 77(9), 799-806.
4. Landau, L. D., & Lifshits, E. M. (1976). Teoria dei campi. Editori Riuniti.
5. Landau, L. D., & Lifshitz, E. M. (1971). The classical theory of fields. Pergamon Press.
6. Okun, L.B. (1989). The concept of mass (mass, energy, relativity), Sov. Phys. Usp. 32, 629–638.
7. Planck, M. (1906).Das Prinzip der Relativität und die Grundgleichungen der Mechanik, Verh. Dtsch. Phys. Ges. 8, 136–141 1906.
8. Sparavigna, A. C. (2015). On the generalized additivity of Kaniadakis entropy. International Journal of Sciences, 4(02), 44-48.
9. Sparavigna, A.C. (0219). Composition Operations of Generalized Entropies Applied to the Study of Numbers. International Journal of Sciences, 8(04), 87-92.
10. Styer, D.F. (2021). Relativistic Dynamics. Oberlin College and Conservatory, https://www2.oberlin.edu/physics/dstyer/Modern/RelativisticDynamics.pdf