Politecnico
di Torino

ScuDo
Scuola di Dottorato ~ Doctoral School
WHAT YOU ARE, TAKES YOU FAR

Doctoral Dissertation
Doctoral Program in Pure and applied mathematics (36th cycle)

Curvature-dependent functionals: applications to membrane models and geometric flows

By

Anna Kubin

******

Supervisor(s):
Prof. M. Morandotti, Supervisor
Prof. L. Lussardi, Co-Supervisor

Doctoral Examination Committee:
Prof. G. Bellettini , Referee, Università di Siena
Prof. M. Röger, Referee, Technische Universität Dortmund
Prof. V. Chiadò Piat, Politecnico di Torino
Prof. M. Morini, Università di Parma
Prof. M. Ponsiglione, Sapienza Università di Roma

Politecnico di Torino
2024

Declaration

I hereby declare that the contents and organization of this dissertation constitute my own original work and do not compromise in any way the rights of third parties, including those relating to the security of personal data.

Anna Kubin
2024

* This dissertation is presented in partial fulfillment of the requirements for **Ph.D. degree** in the Graduate School of Politecnico di Torino (ScuDo).

Contents

Introduction

In this thesis, we study some curvature-dependent energies, with a specific emphasis on their applications to biological membranes and geometric flows. The contents of the thesis are based on the contributions of the research papers [32, 34, 33, 68] in collaboration with Daniele De Gennaro, Antonia Diana, Andrea Kubin, Luca Lussardi, and Marco Morandotti.

The modeling of biological membranes has become a dynamic topic in recent years, with numerous applications in biophysics, biology, and materials science. Biological membranes are modeled as regular surfaces in the three-dimensional space, and their equilibrium configurations are associated with the minima of the Canham–Helfrich functional. This functional takes into account the bending and stretching of the membrane and is formulated in terms of the curvatures of the membrane surface.

Geometric flows are evolution equations that describe the behavior of sets as they evolve over time based on their geometry. The velocity at which each set moves is determined by its geometric attributes, such as curvature. The equations that arise from this study are, in a suitable sense, parabolic differential equations that have broad applications across various fields, including materials science, computer vision, image processing, and physics. In particular, in this thesis we focus on two examples of geometric flows which depend on the mean curvature: the volume-preserving mean curvature flow and the surface diffusion flow, which describe the evolution of surfaces under the influence of surface tension and mass diffusion, respectively.

The thesis is organized into two main parts: biological membranes and geometric flows. Each part begins with a preliminary chapter that establishes the notation and reviews the auxiliary results required for the subsequent discussions.

In Chapter 2, which focuses on the Canham-Helfrich functional, we investigate the existence of minimizers for this functional on generalized Gauss graphs, a class of currents which include regular surfaces as a special case. As a first step, we extend the Canham–Helfrich energy, usually defined on regular surfaces, to generalized Gauss graphs, and then we prove lower semicontinuity and compactness, under a suitable condition on the bending constants ensuring coerciveness. Finally, we show the existence of a minimizer by applying the direct method of the Calculus of Variations.

In Chapter 4, we prove the stability of strictly stable critical sets of the perimeter, in the flat torus $\mathbb{T}^N$, for the volume-preserving mean curvature flow and surface diffusion flow. Chapter 5 and Chapter 6 address the time discretization of the volume-preserving mean curvature flow. In Chapter 5, we prove the stability of stable sets in $\mathbb{T}^N$ for the discrete flow, and we completely characterize the asymptotic behavior starting from any initial set in dimension $N = 2$. In Chapter 6, we focus on the time-discrete fractional mean curvature flow. We develop the long-time convergence analysis for this evolution, proving that the flow converges exponentially fast to a single ball under suitable hypotheses on the dimension N and on the fractional exponent s, namely: $N \leq 7$ and $s \approx 1$, or $N = 2$ and $s \in (0,1)$.

In the following we give a more detailed presentation of the contents of the thesis.

Biological membranes

The mathematical modeling of biological membranes is an active field of research that has received much attention in the last half century starting with the pioneering works of Canham [21] and Helfrich [59]. They modeled the membranes as regular surfaces in the space and associate the equilibrium configurations with the minimum of an energy functional depending on the curvatures.

If we denote by $M \subseteq \mathbb{R}^3$ a compact, oriented surface, by H and K its mean and Gaussian curvatures, respectively, and by H_0 a constant *spontaneous curvature*, the so-called *Canham–Helfrich energy functional* reads

$$\mathscr{E}(M) := \int_M \big(\alpha_H (H(p) - H_0)^2 - \alpha_K K(p)\big)\, \mathrm{d}\mathscr{H}^2(p),$$

where $\alpha_H, \alpha_K > 0$ are the physical, model-specific, *bending constants*.

The lipid bilayer that usually constitutes biological membranes is composed of amphiphiles, polar molecules featuring a hydrophilic head and a hydrophobic fatty tail, that are arranged in a fashion so that the tails are in the inner part of the bilayer, screened from the watery surrounding environment. Given the thickness of a few nanometers, one such bilayer can be effectively described as a surface M and the form of the energy depending on the mean curvature H of M responds to the need of explaining the bi-concave shape of red blood cells [21]. The competing contribution coming from the Gaussian curvature K was added by Helfrich [59], whereas the presence of the spontaneous mean curvature H_0 takes into account possibly preferred configurations: this is the case in which the asymmetry between the two layers, or the difference in the chemical potential across the membrane determine a natural bending of the membrane, even at rest.

If the surface M is without boundary, one can invoke the Gauss–Bonnet theorem and obtain that the term involving K gives a constant contribution (determined by the Euler characteristic $\chi(M)$ of M) to the energy, so that it can be neglected in view of the minimization of $\mathscr{E}$ among all surfaces with prescribed topology. In this case, and under the further constraint that the spontaneous curvature vanishes, the functional $\mathscr{E}$ reduces to the well-known *Willmore energy functional* [70, 89, 93, 96, 99]

$$\mathscr{W}(M) := \int_M H^2(p)\,\mathrm{d}\mathscr{H}^2(p).$$

Both functionals $\mathscr{E}$ and $\mathscr{W}$ are geometric in nature, since they depend on geometric features of the surface M, and can be studied in a number of different contexts, according to the regularity requests on M. Sobolev-type approaches to the minimization either of the Willmore functional (see [70] and the references therein) or of the Canham–Helfrich functional (see, *e.g.*, [25, 27, 60, 61, 80, 100]) assume that M has fixed topology, or even symmetry constraints. Aiming at considering more general surfaces, that include physically interesting configurations (see Figure 1), a successful approach is the one through varifolds [63, 101], see [17, 42, 43]. We point out that other frameworks are available in the study of geometric functionals: for instance, currents [47] have been used to tackle the minimization of the area functional. Despite not being suitable for the formulation of problems involving curvatures, due to their lack of an intrinsic notion of curvature, special classes of currents have been introduced to overcome this issue. Nonetheless, it is possible to apply the technical tools of the theory of currents to the class of the so-called *generalized Gauss graphs* [10], which are motivated by a generalization of the graph of the Gauss map on smooth surfaces M. Instead of generalizing M itself, this approach has the remarkable advantage to allow one to exploit the fact that the curvatures of M are coded in its Gauss map, see Section 1.3 for details.

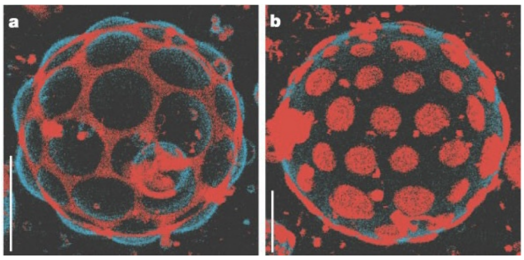

Fig. 1 Baumgart, Hess, Webb, Nature 2003 [13].

In order to ensure both the coercivity and the lower semicontinuity of the Canham–Helfrich energy we assume that the bending constants satisfy

$$4\alpha_H > \alpha_K > 0.$$

This condition is the same assumed in [27, Theorem 1] and [25, formula (1.9)] in the Sobolev setting, see also [60, 61], whereas the more restrictive condition $12\alpha_H > 5\alpha_K > 0$ is considered in [17] in the varifold setting. We note that the typical physical range of the parameters is $2\alpha_H \geq \alpha_K \geq 0$, see for example [12, 15, 97], the case $\alpha_K = 0$ essentially reducing to the Willmore functional $\mathscr{W}$.

In Chapter 2, we provide a suitable formulation of the Canham–Helfrich functional $\mathscr{E}$ in the class of generalized Gauss graphs and study some minimization problems. The main results are Theorems 2.3.5, 2.3.6, and 2.3.9 stating that, under the condition $4\alpha_H > \alpha_K > 0$, there exists a minimizer of the Canham–Helfrich functional in certain classes of generalized Gauss graphs, also enforcing area and enclosed volume constraints.

Geometric flows

We consider two smooth evolutions of sets $t \mapsto E_t$. The first one is the *forced mean curvature flow* which is defined by the equation

$$V_t(x) = -H_{E_t}(x) + f(x,t) \quad \text{for } x \in \partial E_t,$$

where V_t denotes the velocity in the normal direction, H_{E_t} denotes the scalar mean curvature of E_t, and f is a forcing term. In the case where f is zero, this flow is known as the classical *mean curvature flow*. When f is equal to the average of the mean curvature $\overline{H}_{E_t}$, we refer to it as *volume-preserving mean curvature flow*. In Chapter 6, we also consider the fractional counterpart of this evolution where the mean curvature H_{E_t} is substituted by the s–fractional mean curvature $H^s_{E_t}$.

The mean curvature flow is a fundamental concept with far-reaching geometric and physical applications, and it has a rich history dating back to its use in material science. One notable application is in physical systems involving multiple phases, such as the motion of grain boundaries in materials science, as first discussed by Mullins [84]. This model has then found numerous applications in image segmentation [22], and materials science [5]. Moreover, the fractional mean curvature flow has also been considered to model dislocation dynamics (see [64]).

The second evolution we consider is the *surface diffusion flow* which is a smooth flow of sets E_t evolving according to the law

$$V_t(x) = \Delta_{E_t} H_{E_t}(x) \quad \text{for } x \in \partial E_t,$$

where Δ_{E_t} denotes the Laplace-Beltrami operator on ∂E_t. Similar to the mean curvature flow, the surface diffusion flow has significant applications in materials science, particularly in physical systems with multiple phases. It was introduced by Mullins [85] as a model for surface dynamics in phase interfaces when the evolution is driven by mass diffusion in the interface.

The volume-preserving mean curvature flow can be viewed as a simplified second-order version of the surface diffusion flow, as both flows exhibit several common properties. Notably, both flows preserve the volume of the evolving sets while decreasing the perimeter (as discussed in Section 3.3). Furthermore, these evolutions can be considered, at least formally, as gradient flows of the perimeter functional with respect to suitable metrics. Specifically, the mean curvature flow can be regarded as the L^2-gradient flow of the perimeter, while the surface diffusion flow can be interpreted as its H^{-1}-gradient flow.

In both cases, the generated flows may present singularities of different kinds in a finite time-span even if the initial data is smooth. We can see merging or collision of near sets, pinch-offs or shrinking of connected components to points, and there exist examples of singular solutions even in the two dimensional case [77, 78]. Therefore, in general, only short-time existence results are available, see [45] for the mean curvature flow and [44] for the surface diffusion flow (see also [53] for the case of triple junction clusters), and it is an important question to identify sufficient conditions for global existence.

In Chapter 4 we focus on the stability of these two smooth evolutions in the flat torus $\mathbb{T}^N$, which is particularly interesting due to the great variety of possible limit points for the flows, namely periodic constant mean curvature hypersurfaces. In the Euclidean space only unions of balls have constant mean curvature, whereas the flat torus admits a much broader range of such sets. However, a full characterization of periodic constant mean curvature hypersurfaces is still not available in any dimension. For $N = 2$ the only sets with constant mean curvature are discs and stripes (also called lamellae), while for $N \geq 3$ there exist many nontrivial examples, as seen in Figure 2.

Given the (formal) gradient flow structure of the two evolutions, it is natural to expect that the flow starting close to “stable” sets for the perimeter exists for all times and asymptotically converges to those “stable” sets. We refer to this property as *dynamical stability.* We will properly define the notion of critical and stable set in

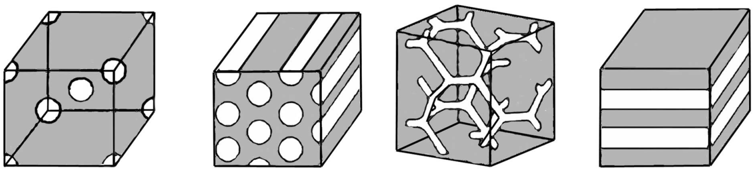

Fig. 2 The critical points in $\mathbb{T}^3$: balls, cylinders, gyroids and lamellae.

Definition 3.1.6, however we can summarize them as follows: critical sets are those with boundary with constant mean curvature, while stable sets are critical sets with positive definite second variation of the perimeter (i.e., they are “stable” for the perimeter functional).

It is a classical approach in the study of this type of flows to restrict the analysis to small enough neighborhoods of strictly stable sets, in order to show global existence and convergence of the flows. This method was employed in [3, 50, 51] (see also [35] for a complete survey), where the authors considered the surface diffusion and the Mullins-Sekerka flows in the $2,3$-dimensional flat torus. In these works, it was proved that strictly stable sets are dinamically stable for the aforementioned flows.

Regarding the volume-preserving mean curvature flow, recent progresses have been made in proving the dynamical stability of strictly stable sets in the 3-dimensional flat torus [86], while, in the Euclidean setting, the asymptotic convergence to a ball has been proven under various hypotheses on the initial set in [45, 52, 62, 72] (see also the approach based on weak solutions of [65] in $\mathbb{R}^2$ and the one in [14] in the anisotropic and crystalline setting). For the surface diffusion flow, some results have focused on the stability of balls [44, 98], infinite cylinders [71], triple junctions [54, 55], and also double bubbles [2].

Building upon recent developments in the study of geometric flows, in Chapter 4 we extend to all dimensions the aforementioned results on the dynamical stability of strictly stable sets in the flat torus, both for the surface diffusion flow and the volume-preserving mean curvature flow. Specifically, we will employ a quantitative Alexandrov-type estimate recently established in [33], based on prior results in the Euclidean setting [82] (see also [24, 34] for similar results in the nonlocal setting). More precisely, for initial sets close in the $C^{1,1}$-topology to a strictly stable set, we prove global existence and asymptotic convergence of both flows to a translated of such set exponentially fast. This is surprising for the surface diffusion flow, which is a fourth-order flow, but is made possible by applying estimates from [58], which provide a fourth-order counterpart to the results for mean curvature flows with rough initial data [67].

In Chapter 5, we focus on the volume-preserving mean curvature flow in the flat torus. As we already mentioned, singularities may appear in a finite time during the evolution even for smooth initial sets. Therefore, after the onset of singularities, the classical or smooth formulation of the flow ceases to hold and needs to be replaced by a weaker one. Due to the lack of a comparison principle, a natural approach is the minimizing movement approach proposed independently by Almgren, Taylor, and Wang in [6] and by Luckhaus and Sturzenhecker in [73] for the unconstrained case, and adapted to the volume-preserving setting in [83].

We briefly recall the scheme in the volume constrained setting. First of all we define a discrete-in-time approximation of the flow that will be called the *discrete (volume-preserving) flow*. Given any initial set E_0 and a time-step $h > 0$ we define iteratively $E_h^0 := E_0$, and for all $n \geq 0$ we set

$$E_h^{n+1} \in \operatorname{argmin}\left\{ P(F) + \frac{1}{h}\int_{F \triangle E_h^n} \operatorname{dist}_{\partial E_h^n}(x)\,\mathrm{d}x \ : \ F \subset \mathbb{T}^N,\, |F| = |E_0| \right\},$$

where $\operatorname{dist}_{\partial E_h^n}$ is the distance function from the set ∂E_h^n. We can define for every $t \geq 0$, the approximate flow by $E_h(t) := E_h^{[t/h]}$. Any limit point of this approximate flow as the time-step h converges to zero is called a *flat flow*. As for the classical mean curvature flow, this approach produces global-in-time solutions as shown in [83]. The existence of such global solutions then permits to analyse the equilibrium configurations reached in the long-time asymptotic.

In [82] the authors characterized, in the Euclidean space, the long-time behavior of the discrete flow: they proved that the evolution starting from an arbitrary bounded initial set converges exponentially fast to a finite union of disjoint balls with equal radii. Moreover, the same authors and collaborators were able to send the discretization parameter h to 0 in [65], in the case $N = 2$.

In Chapter 5, we develop the asymptotic analysis for the discrete flow in the flat torus $\mathbb{T}^N$. As we have already mentioned, in such a framework the class of possible long-time limits is much richer than in $\mathbb{R}^N$ as it includes not only union of balls with equal radii but also different type of critical sets for the perimeter. In particular, we prove the stability of strictly stable sets for the discrete flow, that is we show that the flow starting near a stable set asymptotically convergences to a translate of this set, and we completely characterize the long-time behavior starting from any initial set of finite perimeter when $N = 2$. Moreover, an estimate on the convergence speed is provided.

Finally, in Chapter 6 we consider the volume-preserving fractional mean-curvature flow and we characterize the asymptotic behaviour of its time-discretization. In particular, we prove that the discrete flow starting from any bounded set of finite fractional perimeter converges exponentially fast to a single ball if the dimension $N \leq 7$ and the fractional exponent $s \approx 1$, or for any $s \in (0,1)$ when $N = 2$.

Part I

Biological membranes

Chapter 1

Notation and preliminaries

In this chapter, we define the class of currents on which we will work in Chapter 2. These currents are known as *generalized Gauss graphs* and were introduced by Anzellotti, Serapioni, and Tamanini in [10]. We refer the reader to [40, 41] for the classical notions of differential geometry and to [46, 95] for comprehensive treatises on the theory of currents.

1.1 Exterior algebra and rectifiable currents

Let $k, N \in \mathbb{N}$ be such that $1 \leq k \leq N$. We say that $\alpha : (\mathbb{R}^N)^k \to \mathbb{R}$ is a *k-linear alternating form* if it is linear in each variable and it is alternating, i.e.

$$\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma)\alpha(v_1, \dots, v_k)$$

for every $v_1, \dots, v_k \in \mathbb{R}^N$ and every permutation σ of the indexes $\{1, \dots, k\}$, where $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation. We define $\bigwedge^0(\mathbb{R}^N) := \mathbb{R}$ and we denote by $\bigwedge^k(\mathbb{R}^N)$ the space of *k-covectors* in $\mathbb{R}^N$, that is the space of k-linear alternating forms on $\mathbb{R}^N$. We denote by $\bigwedge_k(\mathbb{R}^N)$ the dual space $(\bigwedge^k(\mathbb{R}^N))^* = \bigwedge^k((\mathbb{R}^N)^*)$, called the space of *k-vectors* in $\mathbb{R}^N$. We recall that, if $\{e_1, \dots, e_N\}$ is a basis of $\mathbb{R}^N$, then $\{e_{i_1} \wedge \ldots \wedge e_{i_k} : 1 \leq i_1 < \ldots < i_k \leq N\}$ is a basis of $\bigwedge_k(\mathbb{R}^N)$, where $\wedge$ denotes the *exterior (or wedge) product*. We recall that the exterior product $v \wedge w \in \bigwedge_{k+h}(\mathbb{R}^N)$ between $v \in \bigwedge_k(\mathbb{R}^N)$ and $w \in \bigwedge_h(\mathbb{R}^N)$ is characterized by the following properties: it is bilinear, and alternating, i.e. $e_i \wedge e_j = -e_j \wedge e_i$ for every $i \neq j$ and $e_i \wedge e_i = 0$ for every i. Similarly, one can define the exterior product on k-covectors. A k-vector v is called a *simple k-vector* if it can be written as $v = v_1 \wedge \ldots \wedge v_k$, for some $v_1, \dots, v_k \in \bigwedge_1(\mathbb{R}^N) \simeq \mathbb{R}^N$.

Let $\Omega \subset \mathbb{R}^N$ be an open set. A *(differential) k-form* ω on Ω is a (differentiable) map that to each $x \in \Omega$ associates $\omega(x) \in \bigwedge^k(\mathbb{R}^N)$. Given ω a 0-form on Ω (that is, a scalar function $\omega\colon \Omega \to \mathbb{R}$), we define $\mathrm{d}\omega$ as the 1-form on Ω given by the differential of ω; for $k > 0$, the definition of the *exterior differential operator* d is extended from k-forms to $(k+1)$-forms through the usual algebra of the exterior product. We denote by $\mathscr{D}^k(\Omega)$ the space of k-forms of class C^∞ with compact support in Ω.

Definition 1.1.1 (Current). *We define the space of k-currents* $\mathscr{D}_k(\Omega)$ *as the dual of* $\mathscr{D}^k(\Omega)$.

Given a sequence of currents $\{T_n\}_{n\in\mathbb{N}} \subset \mathscr{D}_k(\Omega)$ and a current $T \in \mathscr{D}_k(\Omega)$, we say that T_n converges weakly to T, and write $T_n \rightharpoonup T$, if and only if $\langle T_n, \omega\rangle \to \langle T, \omega\rangle$ for every $\omega \in \mathscr{D}^k(\Omega)$, where $\langle\cdot,\cdot\rangle$ denotes the dual product. We denote by $\partial T \in \mathscr{D}_{k-1}(\Omega)$ the *boundary* of the current $T \in \mathscr{D}_k(\Omega)$, defined as $\langle \partial T, \omega\rangle := \langle T, \mathrm{d}\omega\rangle$ for every $\omega \in \mathscr{D}^{k-1}(\Omega)$, while the boundary of a 0-current is set equal to 0. We notice that $\mathrm{d}\omega \in \mathscr{D}^k(\Omega)$ whenever $\omega \in \mathscr{D}^{k-1}(\Omega)$, that is, exterior differentiation preserves the compactness of the support, so that the duality $\langle \partial T, \omega\rangle$ is well defined. The *mass* of a current $T \in \mathscr{D}_k(\Omega)$ in the open set $W \subset \Omega$ is defined as

$$\mathbb{M}_W(T) := \sup\big\{\langle T, \omega\rangle : \omega \in \mathscr{D}^k(W),\ \|\omega(x)\| \le 1 \text{ for every } x \in W\big\},$$

for simplicity, we denote by $\mathbb{M}(T)$ the mass of T in Ω. Here, $\|\cdot\|$ denotes the *comass norm*, namely, for $\alpha \in \bigwedge^k(\mathbb{R}^N)$,

$$\|\alpha\| := \sup\{\langle \alpha, v\rangle : v \text{ is a simple } k\text{-vector with } |v| \le 1\},$$

where $|v| := |v_1 \wedge \ldots \wedge v_k|$ is the volume of the parallelepiped generated by $v_1, \ldots, v_k$.

We denote by $\mathscr{H}^k$ the k-dimensional Hausdorff measure. Given a set $M \subset \mathbb{R}^N$, we say that M is *k-rectifiable* if $M \subset \bigcup_{i=0}^{\infty} M_i$, for a certain $\mathscr{H}^k$-negligible subset $M_0 \subset \mathbb{R}^N$ and for certain k-dimensional C^1 surfaces $M_i \subset \mathbb{R}^N$, for $i > 0$. One can prove that, if M is a k-rectifiable set, for $\mathscr{H}^k$-almost every $p \in M$ there exists an approximate tangent space denoted by T_pM. We say that a map $\eta\colon M \to \bigwedge_k(\mathbb{R}^N)$ is an *orientation* of M if it is $\mathscr{H}^k$-measurable and if $\eta(p)$ is a unit simple k-vector that spans the approximate tangent space T_pM for $\mathscr{H}^k$-almost every $p \in M$. We say that a map $\beta\colon M \to \mathbb{R}$ is an *integer multiplicity* on M if it is $\mathscr{H}^k$-locally summable and with values in $\mathbb{N}$.

Definition 1.1.2 (Rectifiable current with integer multiplicity). *We say that a current* $T \in \mathscr{D}_k(\Omega)$ *is a k-rectifiable current with integer multiplicity if there exist a k-rectifiable set* $M \subset \mathbb{R}^N$*, an orientation* η *of* M*, and an integer multiplicity* β *on* M *such that for every* $\omega \in \mathscr{D}^k(\Omega)$ *we have*

$$\langle T, \omega\rangle = \int_M \langle \omega(p), \eta(p)\rangle \beta(p)\, \mathrm{d}\mathscr{H}^k(p).$$

We denote by $\mathscr{R}_k(\Omega)$ *the set of such currents and write* $T = [\![M, \eta, \beta]\!]$.

We remark that for k-rectifiable currents T we have

$$\mathbb{M}_W(T) = \int_{M\cap W} \beta(p)\,\mathrm{d}\mathscr{H}^k(p), \tag{1.1.1}$$

which simply returns $\mathscr{H}^k(M\cap W)$ if the multiplicity β is 1.

Oriented k-dimensional surfaces are a fundamental example of k-rectifiable currents with integer multiplicity. To each smooth oriented surface S is canonically associated the current T defined by $\langle T,\omega\rangle = \int_S \langle \omega(p),\tau(p)\rangle\,\mathrm{d}\mathscr{H}^k(p)$, where τ is the orientation of S. By Stokes Theorem, the boundary of T corresponds to the current associated with the boundary of S, and thus the notion of boundary for currents is compatible with the classical definition for oriented surfaces. Moreover, $\mathbb{M}(T) = \mathscr{H}^k(S)$, and therefore the mass naturally extend the concept of k-dimensional volume to k-currents.

We now state the celebrated Federer–Fleming theorem, which establishes the compactness and closeness for k-rectifiable currents with integer multiplicity.

Theorem 1.1.3 ([46, Theorem 4.2.17])**.** *Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathscr{R}_k(\Omega)$ such that $\partial T_n \in \mathscr{R}_{k-1}(\Omega)$ for any $n\in\mathbb{N}$. Assume that for any open set W with compact closure in Ω there exists a constant $c_W > 0$ such that*

$$\mathbb{M}_W(T_n) + \mathbb{M}_W(\partial T_n) < c_W.$$

Then there exist a subsequence $\{n_j\}_{j\in\mathbb{N}}$ and a current $T\in\mathscr{R}_k(\Omega)$ with $\partial T\in\mathscr{R}_{k-1}(\Omega)$ such that $T_{n_j} \rightharpoonup T$ as $j\to\infty$.

1.2 Gauss graphs

In the next chapter, our focus will be on two-dimensional objects in $\mathbb{R}^3$, thus we present all the concepts and results in this specific context. However, we note that everything can be readily adapted to a more general setting.

Let $M\subset\mathbb{R}^3$ be a compact two-dimensional surface of class C^2; we say that M is *orientable* if there exists a map ν from M to the unit sphere $\mathbb{S}^2$ of class C^1 on M such that, for every $p\in M$, the vector $\nu(p)$ is perpendicular to the tangent space T_pM. Once we fix a choice of such a map ν, we say that the surface M is *oriented* and we call ν the *Gauss map* of M. Since M is of class C^2, the Gauss map is differentiable at any $p\in M$ and, upon identifying $T_{\nu(p)}\mathbb{S}^2 \simeq T_pM$, its differential in p, $\mathrm{d}\nu_p\colon T_pM\to T_{\nu(p)}\mathbb{S}^2$, is a self-adjoint linear operator that has two real eigenvalues $\kappa_1(p)$ and $\kappa_2(p)$, called the *principal curvatures* of M at p. We define the mean and Gaussian curvatures of M at p by

$$H(p) := \kappa_1(p)+\kappa_2(p), \quad K(p) := \kappa_1(p)\kappa_2(p).$$

The map $\mathrm{d}\nu_p$ can be extended to a linear map $L_p\colon \mathbb{R}^3 \to \mathbb{R}^3$ by setting

$$L_p := \mathrm{d}\nu_p \circ \mathbf{P}_p, \tag{1.2.1}$$

where $\mathbf{P}_p\colon \mathbb{R}^3 \to T_pM$ denotes the orthogonal projection on the tangent space. With a little abuse of notation, we denote again by L_p the matrix associated with the linear map L_p. Observe that L_p has egeinvalues $\kappa_1(p)$, $\kappa_2(p)$, and 0; in particular, $\det L_p = 0$.

For convenience, we denote by $\mathbb{R}^3_x$ the space of points $p \in M$ and by $\mathbb{R}^3_y$ the space where $\nu(p)$ takes its values, and we consider the graph of the Gauss map

$$G := \left\{(p, \nu(p)) \in \mathbb{R}^3_x \times \mathbb{R}^3_y : p \in M\right\} \subset \mathbb{R}^3_x \times \mathbb{R}^3_y \simeq \mathbb{R}^6. \tag{1.2.2}$$

Since M is a two-dimensional surface of class C^2, G is a two-dimensional embedded manifold in $\mathbb{R}^3_x \times \mathbb{R}^3_y$ of class C^1. We remark that if M has a boundary then also G has a boundary which is given by $\partial G = \left\{(p, \nu(p)) : p \in \partial M\right\}$ and we notice that if $\partial M = \emptyset$ then $\partial G = \emptyset$.

We now define an orientation on G. We equip M with the orientation induced by ν and let $\tau(p) := *\nu(p)$, where $*\colon \bigwedge_1(\mathbb{R}^3) \to \bigwedge_2(\mathbb{R}^3)$ is the Hodge operator. In general, for every $0 \le k < N$, we recall that the Hodge operator

$$*\colon \textstyle\bigwedge_{N-k}(\mathbb{R}^N) \to \bigwedge_k(\mathbb{R}^N)$$

is the linear operator defined by $v \wedge *v = e_1 \wedge \ldots \wedge e_N$ for $v = e_1 \wedge \ldots \wedge e_{N-k}$, where $\{e_1, \ldots, e_N\}$ is the standard basis of $\mathbb{R}^N$. Notice that $\tau(p) \in \bigwedge_2(T_pM)$ for every $p \in M$, thus the field $p \mapsto \tau(p)$ is a tangent 2-vector field on M. Let $\Phi\colon M \to M \times \mathbb{S}^2 \subset \mathbb{R}^3_x \times \mathbb{R}^3_y$ be given by $\Phi(p) := (p, \nu(p))$ which is of class C^1 on M. Observe that $G = \Phi(M)$. For each $p \in M$ we have

$$\begin{aligned} \mathrm{d}\Phi_p\colon T_pM &\to T_pM \times T_{\nu(p)}\mathbb{S}^2 \subset \mathbb{R}^3_x \times \mathbb{R}^3_y \\ u &\mapsto (u, \mathrm{d}\nu_p(u)). \end{aligned}$$

Finally, we define $\xi : G \to \bigwedge_2(\mathbb{R}^3_x \times \mathbb{R}^3_y)$ as

$$\xi(p, \nu(p)) := \mathrm{d}\Phi_p(\tau_1(p)) \wedge \mathrm{d}\Phi_p(\tau_2(p)), \quad \text{for} \quad \tau = \tau_1 \wedge \tau_2. \tag{1.2.3}$$

Since $\mathrm{d}\Phi_p = I \oplus \mathrm{d}\nu_p$, where $I : \mathbb{R}^3_x \to \mathbb{R}^3_x$ denotes the identity map, we have that $|\xi| \ge 1$, hence we can normalize ξ obtaining

$$\eta := \frac{\xi}{|\xi|}, \tag{1.2.4}$$

which is an orientation of G. By combining [10, Proposition 1.1 and Example 1.2], we obtain the following formula for the area of G.

Proposition 1.2.1. *The area of the Gauss graph G of the smooth oriented surface $M \subset \mathbb{R}^3$ is given by*

$$\mathscr{H}^2(G) = \int_M |\xi(x,\nu(x))| \, \mathrm{d}\mathscr{H}^2(x),$$

and the norm of ξ is given by

$$\begin{aligned} |\xi(x,\nu(x))| &= \sqrt{1+\kappa_1(x)^2+\kappa_2(x)^2+(\kappa_1(x)\kappa_2(x))^2} \\ &= \sqrt{H(x)^2+(1-K(x))^2}. \end{aligned}$$

1.3 Generalized Gauss graphs

We now introduce the more general setting of generalized Gauss graphs. Let $\{e_1,e_2,e_3\}$ be the canonical basis of $\mathbb{R}^3_x$, and $\{\varepsilon_1,\varepsilon_2,\varepsilon_3\}$ be the one of $\mathbb{R}^3_y$. Moreover, we denote by $\{\mathrm{d}x_1,\mathrm{d}x_2,\mathrm{d}x_3\}$ and $\{\mathrm{d}y_1,\mathrm{d}y_2,\mathrm{d}y_3\}$ their dual basis, respectively.

Definition 1.3.1 (Stratification). *Given a 2-vector $\xi \in \bigwedge_2(\mathbb{R}^3_x \times \mathbb{R}^3_y)$, we define the stratification of ξ as the unique decomposition*

$$\xi = \xi_0+\xi_1+\xi_2, \quad \textit{where} \quad \xi_0 \in \textstyle\bigwedge_2(\mathbb{R}^3_x), \quad \xi_1 \in \bigwedge_1(\mathbb{R}^3_x)\wedge\bigwedge_1(\mathbb{R}^3_y), \quad \xi_2 \in \bigwedge_2(\mathbb{R}^3_y),$$

given by

$$\begin{aligned} \xi_0 &= \sum_{1\le i<j\le 3} \langle \mathrm{d}x_i \wedge \mathrm{d}x_j, \xi\rangle e_i \wedge e_j =: \sum_{1\le i<j\le 3} \xi_0^{ij} e_i \wedge e_j, \\ \xi_1 &= \sum_{1\le i,j\le 3} \langle \mathrm{d}x_i \wedge \mathrm{d}y_j, \xi\rangle e_i \wedge \varepsilon_j =: \sum_{1\le i,j\le 3} \xi_1^{ij} e_i \wedge \varepsilon_j, \\ \xi_2 &= \sum_{1\le i<j\le 3} \langle \mathrm{d}y_i \wedge \mathrm{d}y_j, \xi\rangle \varepsilon_i \wedge \varepsilon_j =: \sum_{1\le i<j\le 3} \xi_2^{ij} \varepsilon_i \wedge \varepsilon_j. \end{aligned}$$

Notice that the three equalities above serve as a definition of ξ_h^{ij}; ξ_0 and ξ_2 are represented by 3×3 skew-symmetric matrices while ξ_1 is represented by a 3×3 matrix.

From now on we consider $\Omega \subset \mathbb{R}^3_x$ an open set.

Definition 1.3.2 (Generalized Gauss graph). *We say that* $\Sigma = [\![G,\eta,\beta]\!]$ *is a generalized Gauss graph in* Ω*, and write* $\Sigma \in \mathrm{curv}_2(\Omega)$*, if*

$$\begin{cases} \Sigma \textit{ and } \partial\Sigma \textit{ are rectifiable currents supported on } \Omega \times \mathbb{S}^2, \\ \langle \Sigma, g\varphi^* \rangle = \displaystyle\int_G g(x,y)|\eta_0(x,y)|\beta(x,y)\,\mathrm{d}\mathscr{H}^2(x,y) \quad \textit{for all } g \in C_c(\Omega \times \mathbb{R}^3_y), \\ \langle \partial\Sigma, \varphi \wedge \omega \rangle = 0 \quad \textit{for all } \omega \in \mathscr{D}^0(\Omega \times \mathbb{R}^3_y), \end{cases} \tag{1.3.1}$$

where

$$\varphi(x,y) := \sum_{j=1}^{3} y_j \mathrm{d}x_j, \quad \varphi^*(x,y) := \sum_{j=1}^{3} (-1)^{j+1} y_j \mathrm{d}\hat{x}_j \tag{1.3.2}$$

and $\mathrm{d}\hat{x}_j = \mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2}$ *for* $1 \le j_1 < j_2 \le 3$*,* $j_1, j_2 \neq j$*.*

We can associate with the regular Gauss graph G the current $\Sigma_G \in \mathscr{R}_2(\Omega \times \mathbb{S}^2)$ given by $\Sigma_G := [\![G,\eta,1]\!]$, and this turns out to be an element of $\mathrm{curv}_2(\Omega)$, see [10, Section 2]. Moreover, one can show that the smallest weakly sequentially closed subset of

$$\{T \in \mathscr{D}_2(\Omega \times \mathbb{R}^3_y) : \mathbb{M}_W(T) + \mathbb{M}_W(\partial T) < \infty \text{ for every } W \Subset \Omega \times \mathbb{R}^3\}$$

containing the currents associated with the regular Gauss graphs is a subset of $\mathrm{curv}_2(\Omega)$ (see again [10, Sec. 2]). This is the motivation why the elements of $\mathrm{curv}_2(\Omega)$ are called generalized Gauss graphs, and it shows that $\mathrm{curv}_2(\Omega)$ contains, as a particular case, smooth oriented manifolds.

Remark 1.3.3. One can prove (see [10, Remark 2.3]) that the second condition in (1.3.1), i.e.

$$\langle \Sigma, g\varphi^* \rangle = \int_G g(x,y)|\eta_0(x,y)|\beta(x,y)\,\mathrm{d}\mathscr{H}^2(x,y),$$

is equivalent to

$$\langle \Sigma, \varphi \wedge \omega \rangle = 0 \quad \text{for every } \omega \in \mathscr{D}^1(\Omega \times \mathbb{R}^3_y) \tag{1.3.3}$$

together with

$$\langle \Sigma, g\varphi^* \rangle \ge 0 \quad \text{for all } g \in C_c(\Omega \times \mathbb{R}^3_y), \tag{1.3.4}$$

where φ and φ^* are defined in (1.3.2). One can easily note that (1.3.3), (1.3.4) and also the third condition in (1.3.1) are closed with respect to weak convergence of currents. From this and from Federer–Fleming Theorem 1.1.3, it follows that $\mathrm{curv}_2(\Omega)$ is a weakly sequentially closed subset of $\mathscr{R}_2(\Omega \times \mathbb{S}^2)$, that is, if $\Sigma_j \in \mathrm{curv}_2(\Omega)$ and $\Sigma_j \rightharpoonup \Sigma$, then $\Sigma \in \mathrm{curv}_2(\Omega)$.

The geometric meaning of the first condition in (1.3.1) is evident. The meaning of (1.3.3) and the third condition in (1.3.1) is given by the theorem below and can be summarised as follows: the variable y is orthogonal to the tangent space to p_1G, where we denote by $p_1 : \mathbb{R}^3_x \times \mathbb{R}^3_y \to \mathbb{R}^3_x$ the projection on the first component; a similar property holds for the support of the boundary $\partial\Sigma$. Finally, condition (1.3.4) fixes the orientation of the generalized Gauss graph.

Theorem 1.3.4 ([10, Theorem 2.9]). *Let* $\Sigma = [\![G,\eta,\beta]\!] \in \mathrm{curv}_2(\Omega)$. *Then*

$$\langle v,(y,0)\rangle = 0$$

for $\mathscr{H}^2$*-almost all* $(x,y) \in G$ *and for every* $v \in T_{(x,y)}G$. *In addition,*

$$(p_1|_G)^{-1}(x) \subset \{(x,\nu(x)),(x,-\nu(x))\}$$

for $\mathscr{H}^2$*-almost every* $x \in P := p_1G$, *where* $\nu\colon P \to \mathbb{S}^2$ *is* $\mathscr{H}^2$*-measurable, and* $\nu(x) \in (T_xP)^\perp$ *for* $\mathscr{H}^2$*-almost every* $x \in P$. *The analogous statements for* $\partial\Sigma = [\![G',\eta',\beta]\!]$ *are*

$$\langle v,(y,0)\rangle = 0$$

for $\mathscr{H}^1$*-almost all* $(x,y) \in G'$ *and for every* $v \in T_{(x,y)}G'$, *and*

$$(p_1|_{G'})^{-1}(x) \subset \big\{(x,y) : |y| = 1 \text{ and } y \text{ is orthogonal to } T_x(p_1G')\big\}$$

for $\mathscr{H}^1$*-almost every* $x \in P := p_1G'$.

In [10] the following structure theorem for generalized Gauss graphs $\Sigma = [\![G,\eta,\beta]\!] \in \mathrm{curv}_2(\Omega)$ was shown. It proves that the support G of Σ is the union of two parts: a "vertical" one and a (possibly double valued) graph over the 2-rectifiable set p_1G of a unit vector field ν normal to p_1G and approximately differentiable. A similar result for the boundary current $\partial\Sigma$ was proved in [10, Theorem 2.11].

Theorem 1.3.5 ([10, Theorem 2.10]). *Let* $\Sigma = [\![G,\eta,\beta]\!] \in \mathrm{curv}_2(\Omega)$, *then*

i) $G = G^{(0)} \cup G^{(1)} \cup \big(\bigcup_{j=1}^\infty G_j\big)$ *where* $G^{(0)}$, $G^{(1)}$ *and* G_j *are* $\mathscr{H}^2$*-measurable and*
  - $\mathscr{H}^2(G^{(0)}) = 0$,
  - $\forall (x,y) \in G \setminus G^{(0)}$, $T_{(x,y)}G$ *exists,*
  - $G^{(1)} = \{(x,y) \in G \setminus G^{(0)} : (\bigwedge^2 p_1)\eta(x,y) = 0\}$,
  - $\mathscr{H}^2(p_1G^{(1)}) = 0$,
  - $G_i \cap G_j = \emptyset$ *for* $i \neq j$.

ii) There are two-dimensional, oriented and embedded submanifolds $S_j \subset \Omega \times \mathbb{R}^3_y$, $j \in \mathbb{N}$ *of class* C^1 *such that:*

- *$G_j \subset S_j$,*
- *$p_1 S_j$ are class C^1 2-dimensional, oriented and embedded submanifolds of Ω,*
- *there are class C^1 functions $f_j : p_1 S_j \to \mathbb{R}^3_y$ such that*

$$S_j = \{(x, f_j(x)) : x \in p_1 S_j\}.$$

iii) $p_1 G \subset P^{(0)} \cup \big(\bigcup_{j=1}^{\infty} p_1 S_j\big)$ where $P^{(0)} = p_1(G^{(0)} \cup G^{(1)})$ and $\mathscr{H}^2(P^{(0)}) = 0$.

iv) It is possible to fix the orientation of each $p_1 S_j$, choosing a continuous normal vector field $\nu_j : p_1 S_j \to \mathbb{S}^2$ such that:

$$y = f_j(x) = \nu_j(x) \quad \textit{for } j \in \mathbb{N} \textit{ and } (x,y) \in R_j.$$

v) ν_j is approximately differentiable on $p_1 G_j$.

For a rectifiable current $\Sigma = [\![G, \eta, \beta]\!]$, according with the stratification of η, we define the strata Σ_i by

$$\Sigma_i(\omega) := \int_G \langle \omega(x,y), \eta_i(x,y)\rangle \beta(x,y)\, \mathrm{d}\mathscr{H}^2(x,y)$$

for every $\omega \in \mathscr{D}^2(\mathbb{R}^3_x \times \mathbb{R}^3_y)$, and we define the measure $|\Sigma_i|$ by

$$|\Sigma_i| = |\eta_i| \beta \mathscr{H}^2 \llcorner G.$$

Given $k \in \{1,2,3\}$, consider a multi-index $\lambda \in \{(\lambda_1, \ldots, \lambda_k) : 0 \le \lambda_1 < \ldots < \lambda_k \le 2\}$. A generalized Gauss graph $\Sigma \in \mathrm{curv}_2(\Omega)$ is said to be *λ-special* if

$$|\Sigma_{\lambda_i}| \ll |\Sigma_0| \quad \text{for} \quad i = 1, \ldots, k$$

and we write $\Sigma \in \mathrm{curv}_2^{\lambda}(\Omega)$. We set $\mathrm{curv}_2^*(\Omega) := \mathrm{curv}_2^{(0,1,2)}(\Omega)$ and we call its elements *special generalized Gauss graphs*; in the sequel, we will also make use of the space

$$\mathrm{curv}_2^{(0,1)}(\Omega) = \big\{\Sigma \in \mathrm{curv}_2(\Omega) : |\Sigma_1| \ll |\Sigma_0|\big\}.$$

Given $\Sigma = [\![G, \eta, \beta]\!] \in \mathrm{curv}_2(\Omega)$, we let $G^* := \{(x,y) \in G : \eta_0(x,y) \neq 0\}$ (notice that G^* is defined only $\mathscr{H}^2$-a.e.), and we remark that, if also $\Sigma \in \mathrm{curv}_2^*(\Omega)$, then $G = G^*$.

In order to consider functionals defined on generalized Gauss graphs, we introduce the following class of integrands.

Definition 1.3.6 (Standard integrand). *A function $f\colon \Omega \times \mathbb{R}^3_y \times \big(\bigwedge_1(\mathbb{R}^3_x) \wedge \bigwedge_1(\mathbb{R}^3_y)\big) \to \mathbb{R}$ is said to be a standard integrand in the setting of* $\mathrm{curv}_2(\Omega)$ *if* *(i)* f *is continuous;*

(ii) f *is convex in the last variable,* i.e.,

$$f(x,y,tp+(1-t)q) \leq tf(x,y,p)+(1-t)f(x,y,q),$$

for all $t \in (0,1)$*, for all* $(x,y) \in \Omega \times \mathbb{R}^3_y$*, and for all* $p,q \in \bigwedge_1(\mathbb{R}^3_x) \wedge \bigwedge_1(\mathbb{R}^3_y)$*;*

(iii) f *has superlinear growth in the last variable, i.e., there exists a continuous function* $\varphi\colon \Omega \times \mathbb{R}^3_y \times [0,+\infty) \to [0,+\infty)$*, non-decreasing in the last variable and such that* $\varphi(x,y,t) \to +\infty$ *locally uniformly in* (x,y) *as* $t \to +\infty$*, and with*

$$\varphi(x,y,|q|)|q| \leq f(x,y,q)$$

for all $(x,y,q) \in \Omega \times \mathbb{R}^3_y \times \left(\bigwedge_1(\mathbb{R}^3_x) \wedge \bigwedge_1(\mathbb{R}^3_y)\right)$.

Remark 1.3.7. A function f as in Definition 1.3.6 is called a (1)-standard integrand in [39, Definition 3.3].

Functions of this type are natural energy densities of functionals depending on curvatures, such as the Canham–Helfrich energy that we will study in Chapter 2.

The next theorem ensures that an integral functional with a standard integrand as density is lower semicontinuous.

Theorem 1.3.8 ([39, Theorem 3.2])**.** *Let* f *be a standard integrand in the setting of* $\mathrm{curv}_2(\Omega)$ *and, for every* $\Sigma = [\![G,\eta,\beta]\!] \in \mathrm{curv}_2(\Omega)$*, set*

$$I_f(\Sigma) := \int_{G^*} f\left(x,y,\frac{\eta_1(x,y)}{|\eta_0(x,y)|}\right)|\eta_0(x,y)|\beta(x,y)\,\mathrm{d}\mathscr{H}^2(x,y).$$

Consider a sequence $\{\Sigma_j\}_{j\in\mathbb{N}} \subset \mathrm{curv}_2^{(0,1)}(\Omega)$ *such that*

(i) $\Sigma_j \rightharpoonup \Sigma$*, where* $\Sigma \in \mathscr{R}_2(\Omega \times \mathbb{S}^2)$*;*

(ii) $\sup_{j\in\mathbb{N}} I_f(\Sigma_j) < +\infty$.

Then

$$\Sigma \in \mathrm{curv}_2^{(0,1)}(\Omega) \quad \textit{and} \quad I_f(\Sigma) \leq \liminf_{j\to\infty} I_f(\Sigma_j).$$

Finally, we recall a compactness result for special generalized Gauss graphs. Before stating it, we remark that, since $|\eta|^2 = |\eta_0|^2+|\eta_1|^2+|\eta_2|^2 = 1$, we can write $\frac{1}{|\eta_0|} = |\eta_0| + \frac{|\eta_1|^2}{|\eta_0|} + \frac{|\eta_2|^2}{|\eta_0|}$.

Corollary 1.3.9 ([38, Corollary 4.2]). *Consider a sequence* $\Sigma_j = [\![G_j, \eta_j, \beta_j]\!] \in \mathrm{curv}_2^*(\Omega)$ *such that*

$$\sup_{j\in\mathbb{N}}\left\{\int_{G_j^*}\left(|(\eta_j)_0(x,y)| + \frac{|(\eta_j)_1(x,y)|^2}{|(\eta_j)_0(x,y)|} + \frac{|(\eta_j)_2(x,y)|^2}{|(\eta_j)_0(x,y)|}\right)\beta_j(x,y)\,\mathrm{d}\mathscr{H}^2(x,y) + \mathbb{M}(\partial\Sigma_j)\right\} < +\infty.$$

Then there exist a subsequence $\{\Sigma_{j_k}\}_{k\in\mathbb{N}}$ *and* $\Sigma \in \mathrm{curv}_2^*(\Omega)$ *such that* $\Sigma_{j_k} \rightharpoonup \Sigma$ *as* $k \to \infty$.

Chapter 2

Minimization of the Canham-Helfrich energy

Contents

2.1 Introduction

In this chapter, which presents the results of [68], we investigate the existence of minimizers of the Canham–Helfrich functional among generalized Gauss graphs (see Section 1.3).

We recall that for any two-dimensional, compact, and oriented submanifold $M \subset \mathbb{R}^3$ (possibly with boundary), the *Canham–Helfrich functional* on M is defined by

$$\mathscr{E}(M) := \int_M \big(\alpha_H (H(p) - H_0)^2 - \alpha_K K(p)\big)\, \mathrm{d}\mathscr{H}^2(p), \tag{2.1.1}$$

where $H_0 \in \mathbb{R}$ is the *spontaneous curvature* and $\alpha_H, \alpha_K > 0$ are the *bending constants*.

We will provide a suitable formulation of the Canham–Helfrich functional in the class of generalized Gauss graphs and study three minimization problems. The main results of this chapter are Theorems 2.3.5, 2.3.6, and 2.3.9 stating that, under condition

$$4\alpha_H > \alpha_K > 0, \tag{2.1.2}$$

there exists a minimizer of the Canham–Helfrich functional (2.1.1) in certain classes of generalized Gauss graphs, also enforcing area and enclosed volume constraints, the latter being the physically relevant setup for biological applications. Their proof is a consequence of the direct method in the Calculus of Variations, once lower semicontinuity and compactness are proved.

The plan of the chapter is the following: in Section 2.2 we define the Canham–Helfrich energy of a generalized Gauss graph; Section 2.3 is devoted to the main results, and is complemented by a regularity result, Theorem 2.3.12.

2.2 The Canham–Helfrich energy of a Generalized Gauss graph

In this section, we are going to define the Canham–Helfrich energy of a generalized Gauss graph in a way that is the natural extension of the definition for smooth surfaces. Let $H_0 \in \mathbb{R}$. Here, $M \subset \mathbb{R}^3$ denotes a compact and oriented (with an understood choice of the normal ν) two-dimensional manifold of class C^2.

Lemma 2.2.1 ([74, Lemma 4.2]). *For $\xi \in \bigwedge_2(\mathbb{R}^3_x \times \mathbb{R}^3_y)$ as in* (1.2.3) *the following hold true*

$$\begin{cases} \xi_0 = \tau_1 \wedge \tau_2, \\ \xi_1 = \tau_1 \wedge \mathrm{d}\nu(\tau_2) - \tau_2 \wedge \mathrm{d}\nu(\tau_1), \\ \xi_1^{ij} = (\tau_1 \otimes \mathrm{d}\nu(\tau_2) - \tau_2 \otimes \mathrm{d}\nu(\tau_1))_{ij}, \\ \xi_2 = \mathrm{d}\nu(\tau_1) \wedge \mathrm{d}\nu(\tau_2) = \kappa_1\kappa_2\tau_1 \wedge \tau_2. \end{cases} \tag{2.2.1}$$

Proof. By the definition of ξ (see (1.2.3)) we have

$$\begin{aligned} \xi(p,\nu(p)) &= (\tau_1(p), \mathrm{d}\nu_p(\tau_1(p))) \wedge (\tau_2(p), \mathrm{d}\nu_p(\tau_2(p))) \\ &= \tau_1 \wedge \tau_2 + \tau_1 \wedge \mathrm{d}\nu_p(\tau_2) - \tau_2 \wedge \mathrm{d}\nu_p(\tau_1) + \mathrm{d}\nu_p(\tau_1) \wedge \mathrm{d}\nu_p(\tau_2) \\ &= \tau_1 \wedge \tau_2 + \kappa_2\tau_1 \wedge \tau_2 - \kappa_1\tau_2 \wedge \tau_1 + \kappa_1\kappa_2\tau_1 \wedge \tau_2. \end{aligned}$$

Then, recalling Definition 1.3.1, the equalities follow straightforwardly. □

Remark 2.2.2. If M is a two-dimensional oriented manifold of class C^2 with multiplicity $\bar{\beta}\colon M \to \mathbb{N}$, G is the Gauss graph associated with M via (1.2.2), and $\Sigma_G := [\![G,\eta,\beta]\!]$ with $\beta(x,y)=\bar{\beta}(x)$, then the equalities

$$\mathbb{M}(M) = \int_M \bar{\beta}(p)\,\mathrm{d}\mathscr{H}^2(p) = \int_G \frac{\beta(x,y)}{|\xi(x,y)|}\,\mathrm{d}\mathscr{H}^2(x,y) = \int_G |\eta_0(x,y)|\beta(x,y)\,\mathrm{d}\mathscr{H}^2(x,y) \qquad (2.2.2)$$

hold true by means of the area formula, (1.2.4), and the first identity in (2.2.1); here, by $\mathbb{M}(M)$ we mean the mass of the current $[\![M,*\nu,\bar{\beta}]\!]$, see (1.1.1) with $k=2$. In particular, if $\bar{\beta}\equiv 1$, we obtain

$$\mathscr{H}^2(M) = \int_G |\eta_0(x,y)|\,\mathrm{d}\mathscr{H}^2(x,y). \qquad (2.2.3)$$

The next two lemmas are proved in [74]. We provide the proof in our context for the sake of completeness.

Lemma 2.2.3 ([74, Lemma 4.5]). *Let* $\Sigma = [\![G,\eta,\beta]\!] \in \mathrm{curv}_2(\Omega)$ *be a generalized Gauss graph. Then*

- *for* $\mathscr{H}^2$*-almost every* $(x,y)\in G$

$$\sum_{i=1}^{3} \eta_1^{ij}(x,y)y_i = 0 \quad \textit{for all } 1\le j\le 3, \qquad (2.2.4)$$

- *for* $\mathscr{H}^2$*-almost every* $(x,y)\in G^*$

$$\sum_{j=1}^{3} \eta_1^{ij}(x,y)y_j = 0 \quad \textit{for all } 1\le i\le 3. \qquad (2.2.5)$$

Proof. As in the proof of [10, Proposition 2.4], we have that

$$\langle \eta(x,y),(y,0)\wedge(0,w)\rangle = 0 \quad \text{for all } w\in\mathbb{R}^3 \text{ and for } \mathscr{H}^2\text{-almost every } (x,y)\in G.$$

From this we deduce that $\sum_{ij}\eta_1^{ij}(x,y)y_iw_j = 0$ for all $w\in\mathbb{R}^3$, which implies (2.2.4).

By (ii) of Theorem 1.3.5, for $\mathscr{H}^2$-almost every $(x,y)\in G^*$, there are an embedded C^1 surface $S\subset\mathbb{R}^3$ and a map $\zeta\colon S\to\mathbb{S}^2$ of class C^1 such that

$$\zeta(x)=y, \quad \Lambda_2(\mathrm{I}\oplus\mathrm{d}\zeta_x)(*y) = \xi(x,y).$$

By Lemma 2.2.1, we obtain, for $i=1,2,3$ and $*y=\tau_1\wedge\tau_2$,

$$\sum_{j=1}^{3}\xi_1^{ij}y_j = e_i\cdot(\tau_1\otimes D\zeta(x)\tau_2 - \tau_2\otimes D\zeta(x)\tau_1)y = 0,$$

since $D\zeta(x)\tau_k \cdot y = D\zeta(x)\tau_k \cdot \zeta(x) = 0$ for $k = 1,2$ as ζ takes values in $\mathbb{S}^2$. Then (2.2.5) is proved recalling (1.2.4). □

We recall that the permutation symbols are given by

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (ijk) \text{ is an even permutation of } \{1,2,3\}, \\ -1 & \text{if } (ijk) \text{ is an odd permutation of } \{1,2,3\}, \\ 0 & \text{otherwise.} \end{cases}$$

For any $z \in \mathbb{R}^3$, we define

$$\Psi_z := \sum_{i,j,k=1}^{3} \varepsilon_{ijk} z_k \, \mathrm{d}x_i \wedge \mathrm{d}y_j. \tag{2.2.6}$$

Lemma 2.2.4 ([74, Lemma 4.6]). *For the linear operator L as in* (1.2.1)*, the following formulas hold*

$$\begin{aligned} H &= \operatorname{tr} L = \nu_1(\xi_1^{23} - \xi_1^{32}) - \nu_2(\xi_1^{13} - \xi_1^{31}) + \nu_3(\xi_1^{12} - \xi_1^{21}) = \langle \Psi_\nu, \xi_1 \rangle, \\ K &= \operatorname{tr}(\operatorname{cof} L) = \nu \cdot (\operatorname{cof} \xi_1)\nu, \end{aligned} \tag{2.2.7}$$

where L and ν are evaluated at $p \in M$ and ξ is evaluated at $(p, \nu(p))$.

Proof. Since $\{\tau_1, \tau_2, \nu\}$ is an orthonormal basis of $\mathbb{R}^3$, we observe that for any $r \in \mathbb{R}$

$$\begin{aligned} -r\operatorname{tr}(\operatorname{cof} L) + r^2 \operatorname{tr} L - r^3 &= \det(L - r\mathrm{I}) = \det(\tau_1|\tau_2|\nu)\det(L - r\mathrm{I}) \\ &= (L - r\mathrm{I})\nu \cdot [(L - r\mathrm{I})\tau_1 \times (L - r\mathrm{I})\tau_2] \\ &= -r(L\tau_1 \times L\tau_2) \cdot \nu + r^2(\tau_1 \times L\tau_2 - \tau_2 \times L\tau_1) \cdot \nu - r^3, \end{aligned} \tag{2.2.8}$$

where we used the fact that $L\nu = 0$. Therefore, from Lemma 2.2.1 we deduce that

$$\begin{aligned} \operatorname{tr} L = (\tau_1 \times L\tau_2 - \tau_2 \times L\tau_1) \cdot \nu &= \sum_{i,j,k=1}^{3} (\tau_{1,i} e_j \cdot L\tau_2 - \tau_{2,i} e_j \cdot L\tau_1)\nu_k \varepsilon_{ijk} \\ &= \sum_{i,j,k=1}^{3} \xi_1^{ij} \nu_k \varepsilon_{ijk} = \sum_{i<j} \sum_{k=1}^{3} (\xi_1^{ij} - \xi_1^{ji})\nu_k \varepsilon_{ijk} \\ &= \nu_1(\xi_1^{23} - \xi_1^{32}) - \nu_2(\xi_1^{13} - \xi_1^{31}) + \nu_3(\xi_1^{12} - \xi_1^{21}). \end{aligned}$$

Moreover, from (2.2.1) and (2.2.8) we also deduce that

$$\operatorname{tr}(\operatorname{cof} L) = (L\tau_1 \times L\tau_2) \cdot \nu = (L\tau_1 \wedge L\tau_2) \cdot \xi_0 = \kappa_1 \kappa_2.$$

Using (2.2.1) again and, since $\det(\xi_1) = 0$, by [94, Prop. 3.21], we have

$$\nu \cdot \operatorname{cof}(\xi_1)\nu = \nu \cdot \operatorname{cof}(\tau_1 \otimes L\tau_2 - \tau_2 \otimes L\tau_1)\nu = \det(\tau_1 \otimes L\tau_2 - \tau_2 \otimes L\tau_1 + \nu \otimes \nu) =: D.$$

We can represent the matrix $\tau_1 \otimes L\tau_2 - \tau_2 \otimes L\tau_1 + \nu \otimes \nu$ with respect to the basis $\{\tau_1, \tau_2, \nu\}$, obtaining

$$\begin{aligned} D = \det \begin{pmatrix} L\tau_2 \cdot \tau_1 & L\tau_2 \cdot \tau_2 & 0 \\ -L\tau_1 \cdot \tau_1 & -L\tau_1 \cdot \tau_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} &= \det \begin{pmatrix} L\tau_1 \cdot \tau_1 & L\tau_1 \cdot \tau_2 \\ L\tau_2 \cdot \tau_1 & L\tau_2 \cdot \tau_2 \end{pmatrix} \\ = \kappa_1 \kappa_2 \det \begin{pmatrix} \tau_1 \cdot \tau_1 & \tau_1 \cdot \tau_2 \\ \tau_2 \cdot \tau_1 & \tau_2 \cdot \tau_2 \end{pmatrix} &= \kappa_1 \kappa_2 = \operatorname{tr} \operatorname{cof} L, \end{aligned}$$

which concludes the proof. □

The next proposition provides the expression of the Canham–Helfrich functional defined on manifolds, seen as regular Gauss graphs. In turns, this suggests how to define the Canham–Helfrich functional for general elements in $\operatorname{curv}_2(\Omega)$.

Proposition 2.2.5. *Fix $y \in \mathbb{S}^2$ and let*

$$\mathscr{X}_y := \left\{ \zeta \in \textstyle\bigwedge_1(\mathbb{R}^3_x) \wedge \textstyle\bigwedge_1(\mathbb{R}^3_y) : \sum_{k=1}^{3} \zeta^{ki} y_k = \sum_{k=1}^{3} \zeta^{ik} y_k = \sum_{k=1}^{3} \zeta^{kk} = 0 \text{ for all } i = 1,2,3 \right\}. \tag{2.2.9}$$

Let $f_y \colon \mathscr{X}_y \to [0, +\infty)$ be defined by (recall (2.2.6)*)*

$$f_y(\zeta) := \alpha_H \langle \Psi_y, \zeta \rangle^2 - 2\alpha_H H_0 \langle \Psi_y, \zeta \rangle + \alpha_H H_0^2 - \alpha_K y \cdot (\operatorname{cof} \zeta) y. \tag{2.2.10}$$

Then, defining η as in (1.2.4)*, we have*

$$\mathscr{E}(M) = \int_{\Phi(M)^*} f_y\left(\frac{\eta_1(x,y)}{|\eta_0(x,y)|} \right) |\eta_0(x,y)| \, \mathrm{d}\mathscr{H}^2(x,y).$$

Proof. First observe that, by Lemma 2.2.3 and since by (2.2.1) the trace of ξ_1 is zero, $\eta_1(x,y)$ belongs to $\mathscr{X}_y$ for almost every $(x,y) \in \Phi(M)^*$. Moreover, by (2.1.1), Lemma 2.2.4, and the area formula, we have

$$\begin{aligned} \mathscr{E}(M) &= \int_M \big(\alpha_H (\operatorname{tr} L_p - H_0)^2 - \alpha_K \operatorname{tr}(\operatorname{cof} L_p) \big) \, \mathrm{d}\mathscr{H}^2(p) \\ &= \int_M \Big(\alpha_H \big(\langle \Psi_{\nu(p))}, \xi_1(p, \nu(p)) \rangle - H_0 \big)^2 - \alpha_K \nu(p) \cdot (\operatorname{cof} \xi_1(p, \nu(p))) \nu(p) \Big) \, \mathrm{d}\mathscr{H}^2(p) \\ &= \int_{\Phi(M)^*} \left(\alpha_H \left(\left\langle \Psi_y, \frac{\eta_1(x,y)}{|\eta_0(x,y)|} \right\rangle - H_0 \right)^2 - \alpha_K y \cdot \left(\operatorname{cof} \frac{\eta_1(x,y)}{|\eta_0(x,y)|} \right) y \right) |\eta_0(x,y)| \, \mathrm{d}\mathscr{H}^2(x,y), \end{aligned}$$

where we have used that $|\xi| = 1/|\eta_0| = |\det D\Phi|$. □

We are now ready to define the functional $\mathscr{E}$ on a generalized Gauss graph.

Definition 2.2.6. *The Canham–Helfrich functional defined on generalized Gauss graphs is the functional* $\mathscr{E}\colon \operatorname{curv}_2(\Omega) \to [-\infty, +\infty]$ *defined by*

$$\mathscr{E}(\Sigma) := \int_{G^*} f_y\left(\frac{\eta_1(x,y)}{|\eta_0(x,y)|}\right)|\eta_0(x,y)|\beta(x,y)\,\mathrm{d}\mathscr{H}^2(x,y), \tag{2.2.11}$$

for every $\Sigma = [\![G,\eta,\beta]\!] \in \operatorname{curv}_2(\Omega)$.

2.3 Existence and regularity of minimizers

2.3.1 Technical lemmas

For every $\zeta \in \bigwedge_1(\mathbb{R}^3_x) \wedge \bigwedge_1(\mathbb{R}^3_y)$ and for every $y \in \mathbb{S}^2$, let us define

$$g_y(\zeta) := \alpha_H \langle \Psi_y, \zeta\rangle^2 - \alpha_K y \cdot (\operatorname{cof}\zeta) y \qquad \text{and} \qquad h_y(\zeta) := 2\alpha_H H_0 \langle \Psi_y, \zeta\rangle \tag{2.3.1}$$

and let us identify ζ with a vector in $u = u[\zeta] \in \mathbb{R}^9$ by

$$u = u[\zeta] := (\zeta^{11}, \zeta^{12}, \zeta^{13}, \zeta^{21}, \zeta^{22}, \zeta^{23}, \zeta^{31}, \zeta^{32}, \zeta^{33}).$$

With these positions, we have (compare with the expression in (2.2.7))

$$\langle \Psi_y, \zeta\rangle = (0, y_3, -y_2, -y_3, 0, y_1, y_2, -y_1, 0) \cdot u = y_1(u_6 - u_8) - y_2(u_3 - u_7) + y_3(u_2 - u_4).$$

Lemma 2.3.1. *Let* (2.1.2) *holds. The function* $g_y\colon \bigwedge_1(\mathbb{R}^3_x) \wedge \bigwedge_1(\mathbb{R}^3_y) \to \mathbb{R}$ *defined in* (2.3.1) *is represented by a quadratic form* $u \mapsto u \cdot A_y u$ *on* $\mathbb{R}^9$*, that is* $g_y(\zeta) = u[\zeta] \cdot A_y u[\zeta]$*, where*

$$A_y = \begin{pmatrix}
0 & 0 & 0 & 0 & -\frac{\alpha_K}{2}y_3^2 & \frac{\alpha_K}{2}y_2y_3 & 0 & \frac{\alpha_K}{2}y_2y_3 & -\frac{\alpha_K}{2}y_2^2 \\
0 & \alpha_H y_3^2 & -\alpha_H y_2y_3 & -\gamma y_3^2 & 0 & \gamma y_1y_3 & \gamma y_2y_3 & -\alpha_H y_1y_3 & \frac{\alpha_K}{2}y_1y_2 \\
0 & -\alpha_H y_2y_3 & \alpha_H y_2^2 & \gamma y_2y_3 & \frac{\alpha_K}{2}y_1y_3 & -\alpha_H y_1y_2 & -\gamma y_2^2 & \gamma y_1y_2 & 0 \\
0 & -\gamma y_3^2 & \gamma y_2y_3 & \alpha_H y_3^2 & 0 & -\alpha_H y_1y_3 & -\alpha_H y_2y_3 & \gamma y_1y_3 & \frac{\alpha_K}{2}y_1y_2 \\
-\frac{\alpha_K}{2}y_3^2 & 0 & \frac{\alpha_K}{2}y_1y_3 & 0 & 0 & 0 & \frac{\alpha_K}{2}y_1y_3 & 0 & -\frac{\alpha_K}{2}y_1^2 \\
\frac{\alpha_K}{2}y_2y_3 & \gamma y_1y_3 & -\alpha_H y_1y_2 & -\alpha_H y_1y_3 & 0 & \alpha_H y_1^2 & \gamma y_1y_2 & -\gamma y_1^2 & 0 \\
0 & \gamma y_2y_3 & -\gamma y_2^2 & -\alpha_H y_2y_3 & \frac{\alpha_K}{2}y_1y_3 & \gamma y_1y_2 & \alpha_H y_2^2 & -\alpha_H y_1y_2 & 0 \\
\frac{\alpha_K}{2}y_2y_3 & -\alpha_H y_1y_3 & \gamma y_1y_2 & \gamma y_1y_3 & 0 & -\gamma y_1^2 & -\alpha_H y_1y_2 & \alpha_H y_1^2 & 0 \\
-\frac{\alpha_K}{2}y_2^2 & \frac{\alpha_K}{2}y_1y_2 & 0 & \frac{\alpha_K}{2}y_1y_2 & -\frac{\alpha_K}{2}y_1^2 & 0 & 0 & 0 & 0
\end{pmatrix}$$

for $\gamma := (\alpha_H - \frac{\alpha_K}{2})$*. Let*

$$v(-\alpha_K/2) := \begin{pmatrix} y_1^2-1 \\ y_1y_2 \\ y_1y_3 \\ y_1y_2 \\ y_2^2-1 \\ y_2y_3 \\ y_1y_3 \\ y_2y_3 \\ y_3^2-1 \end{pmatrix}, \qquad v(2\alpha_H-\alpha_K/2) := \begin{pmatrix} 0 \\ -y_3 \\ y_2 \\ y_3 \\ 0 \\ -y_1 \\ -y_2 \\ y_1 \\ 0 \end{pmatrix},$$

$$v_1(\alpha_K/2) := \begin{pmatrix} 2y_1y_2y_3 \\ y_3y_2^2-y_3y_1^2 \\ y_2y_3^2-y_2 \\ y_3y_2^2-y_3y_1^2 \\ -2y_1y_2y_3 \\ y_1-y_1y_3^2 \\ y_2y_3^2-y_2 \\ y_1-y_1y_3^2 \\ 0 \end{pmatrix}, \qquad v_2(\alpha_K/2) := \begin{pmatrix} y_1y_2^2-y_1y_3^2 \\ y_2^3-y_2 \\ y_3y_1^2+y_3y_2^2 \\ y_2^3-y_2 \\ y_1-y_1y_2^2 \\ 0 \\ y_3y_1^2+y_3y_2^2 \\ 0 \\ -y_1^3-y_1y_2^2 \end{pmatrix}.$$

Then these vectors are eigenvectors of the matrix A_y *with corresponding eigenvalues* $-\alpha_K/2$*,* $2\alpha_H-\alpha_K/2$*, and* $\alpha_K/2$ *with multiplicities* 1*,* 1*, and* 2*, respectively. The six vectors*

$$v_1(0) := \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix}, \; v_2(0) := \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}, \; v_3(0) := \begin{pmatrix} 0 \\ 0 \\ y \end{pmatrix},$$

$$v_4(0) := \begin{pmatrix} y_1e_1 \\ y_2e_1 \\ y_3e_1 \end{pmatrix}, \; v_5(0) := \begin{pmatrix} y_1e_2 \\ y_2e_2 \\ y_3e_2 \end{pmatrix}, \; v_6(0) := \begin{pmatrix} y_1e_3 \\ y_2e_3 \\ y_3e_3 \end{pmatrix}$$

generate the 5*-dimensional subspace associated with the eigenvector* 0*.*

The function $h_y\colon \bigwedge_1(\mathbb{R}^3_x)\wedge\bigwedge_1(\mathbb{R}^3_y)\to\mathbb{R}$ *defined in* (2.3.1) *is represented by a linear map* $u\mapsto u\cdot v_y$ *where* $v_y := -2\alpha_H H_0 v(2\alpha_H-\alpha_K/2)$*.*

Moreover, we have that

$$
\begin{aligned}
&\operatorname{span}\{v_1(0), v_2(0), v_3(0), v_4(0), v_5(0), v_6(0), v(-\alpha_K/2)\}\\
&= \operatorname{span}\left\{ \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ y \end{pmatrix}, \begin{pmatrix} y_1 e_1 \\ y_2 e_1 \\ y_3 e_1 \end{pmatrix}, \begin{pmatrix} y_1 e_2 \\ y_2 e_2 \\ y_3 e_2 \end{pmatrix}, \begin{pmatrix} y_1 e_3 \\ y_2 e_3 \\ y_3 e_3 \end{pmatrix}, \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \right\}
\end{aligned}
\tag{2.3.2}
$$

and by the isomorphism $\zeta \mapsto u[\zeta]$ *the space* $\mathscr{X}_y$ *introduced in* (2.2.9) *transforms to*

$$
\widetilde{\mathscr{X}_y} := \{u \in \mathbb{R}9 : u \perp \operatorname{span}\{v_1(0), v_2(0), v_3(0), v_4(0), v_5(0), v_6(0), v(-\alpha_K/2)\}\}. \tag{2.3.3}
$$

Proof. The claims follow by straightforward calculations. To prove (2.3.2), we observe that,

$$
v(-\alpha_K/2) = y_1 \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix} + y_2 \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} + y_3 \begin{pmatrix} 0 \\ 0 \\ y \end{pmatrix} - \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}
$$

and this concludes the proof. □

Lemma 2.3.1 shows that the quadratic form A_y (and therefore the function g_y) has both a negative eigenvalue and the zero eigenvalue, which prevent positive definiteness. Nonetheless, since the space $\mathscr{X}_y$ defined in (2.2.9) transforms to $\widetilde{\mathscr{X}_y}$ defined in (2.3.3), which is the orthogonal to the directions where there is loss of positive definiteness, we are able to prove, in the next Proposition, that it is possible to modify the integrand f_y defined in (2.2.10) to obtain the new function $\tilde{f}$ defined in (2.3.4) below, which is a standard integrand in the sense of Definition 1.3.6.

Proposition 2.3.2. *Let* (2.1.2) *hold. For* $y \in \mathbb{S}^2$, *define the map* $F_y \colon \mathbb{R}^9 \to \mathbb{R}$

$$
\begin{aligned}
F_y(u) &:= g_y(u) - h_y(u) + \alpha_H H_0^2 + \frac{\alpha_K}{2}|\pi_0 u|^2 + \alpha_K |\pi_{-\alpha_K/2} u|^2 \\
&= u \cdot A_y u - u \cdot v_y + \alpha_H H_0^2 + \frac{\alpha_K}{2}|\pi_0 u|^2 + \alpha_K |\pi_{-\alpha_K/2} u|^2,
\end{aligned}
$$

where g_y, h_y *are defined as in* (2.3.1), $\pi_0, \pi_{-\alpha_K/2} \colon \mathbb{R}^9 \to \mathbb{R}^9$ *are the orthogonal projections on* $\operatorname{span}\{v_1(0), \dots, v_5(0)\}$ *and* $\operatorname{span}\{v(-\alpha_K/2)\}$, *respectively. Moreover, let*

$$
\tilde{f} \colon \Omega \times \mathbb{S}^2 \times \big(\textstyle\bigwedge_1(\mathbb{R}^3_x) \wedge \bigwedge_1(\mathbb{R}^3_y)\big) \to \mathbb{R}, \qquad \tilde{f}(x, y, \zeta) := F_y(u[\zeta]). \tag{2.3.4}
$$

Then $\tilde f$ is continuous, convex in the third variable, and there exist two constants $c_1 > 0$ and $c_2 \geq 0$ such that

$$\tilde f(x,y,\zeta) \geq c_1|\zeta|^2 - c_2. \tag{2.3.5}$$

In particular, $\tilde f$ has uniform superlinear growth in the third variable.

Proof. Let $\pi_{2\alpha_H-\alpha_K/2}, \pi_{\alpha_K/2}\colon \mathbb{R}^9 \to \mathbb{R}^9$ be the orthogonal projections on $\mathrm{span}\{v(2\alpha_H - \alpha_K/2)\}$ and $\mathrm{span}\{v_1(\alpha_K/2), v_2(\alpha_K/2)\}$, respectively. For every $u \in \mathbb{R}^9$, by Lemma 2.3.1, we have

$$\begin{aligned} F_y(u) = & -\frac{\alpha_K}{2}|\pi_{-\alpha_K/2}u|^2 + \frac{\alpha_K}{2}|\pi_{\alpha_K/2}u|^2 + \Big(2\alpha_H - \frac{\alpha_K}{2}\Big)|\pi_{2\alpha_H-\alpha_K/2}u|^2 \\ & + 2\alpha_H H_0\, u\cdot v(2\alpha_H - \alpha_K/2) + \frac{\alpha_K}{2}|\pi_0 u|^2 + \alpha_K|\pi_{-\alpha_K/2}u|^2 + \alpha_H H_0^2 \\ = & \ \frac{\alpha_K}{2}|\pi_{-\alpha_K/2}u|^2 + \frac{\alpha_K}{2}|\pi_{\alpha_K/2}u|^2 + \Big(2\alpha_H - \frac{\alpha_K}{2}\Big)|\pi_{2\alpha_H-\alpha_K/2}u|^2 \\ & + \frac{\alpha_K}{2}|\pi_0 u|^2 + 2\alpha_H H_0\, u\cdot v(2\alpha_H - \alpha_K/2) + \alpha_H H_0^2. \end{aligned} \tag{2.3.6}$$

By (2.1.2), we deduce that F_y is convex (and therefore continuous) in u, so that $\tilde f$ is convex (and therefore continuous) in the third variable. Moreover, by reconstructing the norm $|u[\zeta]|^2 = |\zeta|^2$ from the projections $\pi_\bullet$ and by recalling that they are 1-Lipschitz functions, we have that

$$\tilde f(x,y,\zeta) = F_y(u[\zeta]) \geq \min\Big\{\frac{\alpha_K}{2}, 2\alpha_H - \frac{\alpha_K}{2}\Big\}|\zeta|^2 - 2\sqrt{2}\alpha_H|H_0||\zeta| + \alpha_H H_0^2$$

(the factor $\sqrt{2} = |v(2\alpha_H - \alpha_K/2)|$ comes from Schwarz inequality), from which we deduce the boundedness from below of $\tilde f$ and (2.3.5), with (a possible choice of)

$$c_1 = \frac{1}{4}\min\{\alpha_K, 4\alpha_H - \alpha_K\} \quad \text{and} \quad c_2 = \alpha_H H_0^2\Big(\frac{8\alpha_H}{\min\{\alpha_K, 4\alpha_H - \alpha_K\}} - 1\Big).$$

Finally, the continuity of $\tilde f$ with respect to y follows from the structure of the matrix A_y and of the vector v_y in Lemma 2.3.1. □

Proposition 2.3.3. *Let $\tilde f$ be the function defined in* (2.3.4). *The, for every $\Sigma = [\![G,\eta,\beta]\!] \in \mathrm{curv}_2(\Omega)$, it holds that*

$$\mathscr{E}(\Sigma) = \int_{G^*} \tilde f\Big(x,y,\frac{\eta_1(x,y)}{|\eta_0(x,y)|}\Big)|\eta_0(x,y)|\beta(x,y)\,\mathrm{d}\mathscr{H}^2(x,y). \tag{2.3.7}$$

Proof. Let $(x,y) \in G^*$. By Lemma 2.2.3 and Lemma 2.3.1 we have that $u[\xi_1(x,y)] \in \widetilde{\mathscr{X}_y}$, from which we obtain that $\pi_0 u[\xi_1(x,y)] = \pi_{-\alpha_K/2}u[\xi_1(x,y)] = 0$. Keeping (2.2.10), (2.3.1), and (2.3.4) into account, this implies that

$$\tilde{f}(x,y,\xi_1(x,y)) = F_y(u[\xi_1(x,y)]) = f_y(\xi_1(x,y)),$$

which, by (2.2.11), implies (2.3.7). □

Lemma 2.3.4. *Let* $A \Subset \Omega$ *and let* $\Sigma_j = [\![G_j, \eta_j, \beta_j]\!] \in \mathrm{curv}_2(\Omega)$ *be such that* $\mathrm{spt}\Sigma_j \subset A \times \mathbb{S}^2$ *for every* $j \in \mathbb{N}$ *and* $\Sigma_j \rightharpoonup \Sigma = [\![G, \eta, \beta]\!] \in \mathrm{curv}_2(\Omega)$ *as* $j \to \infty$*. Then* $\mathrm{spt}\Sigma \subset A \times \mathbb{S}^2$ *and*

$$\lim_{j\to\infty} \int_{G_j} |(\eta_j)_0(x,y)| \beta_j(x,y)\, \mathrm{d}\mathscr{H}^2(x,y) = \int_G |\eta_0(x,y)| \beta(x,y)\, \mathrm{d}\mathscr{H}^2(x,y). \tag{2.3.8}$$

In particular, if M_j, M *are two-dimensional oriented manifold of class* C^2 *contained in* A*, if* G_j, G *are the associated Gauss graphs by* (1.2.2)*, and* $\Sigma_j = \Sigma_{G_j} = [\![G_j, \eta_j, 1]\!]$*,* $\Sigma = \Sigma_G = [\![G, \eta, 1]\!]$ *are the associated currents, if* $\Sigma_j \rightharpoonup \Sigma$*, then* $\mathscr{H}^2(M_j) \to \mathscr{H}^2(M)$*.*

Proof. We first observe that the condition on the supports is closed, so that $\mathrm{spt}\Sigma \subset A \times \mathbb{S}^2$. Let $g \in C_c(\Omega \times \mathbb{R}^3_y)$ be such that $g = 1$ on $A \times \mathbb{S}^2$. Then the convergence

$$\int_{G_j} |(\eta_j)_0(x,y)| \beta_j(x,y)\, \mathrm{d}\mathscr{H}^2(x,y) = \Sigma_j(g\varphi^*) \to \Sigma(g\varphi^*) = \int_G |\eta_0(x,y)| \beta(x,y)\, \mathrm{d}\mathscr{H}^2(x,y)$$

follows immediately by (1.3.1). The proof of the last statement is obtained by combining (2.2.3) and (2.3.8):

$$\lim_{j\to\infty} \mathscr{H}^2(M_j) = \lim_{j\to\infty} \int_{G_j} |(\eta_j)_0(x,y)|\, \mathrm{d}\mathscr{H}^2(x,y) = \int_G |\eta_0(x,y)|\, \mathrm{d}\mathscr{H}^2(x,y) = \mathscr{H}^2(M).$$

This concludes the proof. □

2.3.2 Minimization problems

In this section we study various minimization problems for the energy $\mathscr{E}$ in (2.2.11). In the first two (see Theorems 2.3.5 and 2.3.6 below), reasonable sufficient conditions for unconstrained minimization are provided. In the third one (see Theorem 2.3.9 below), we tackle constrained minimization in terms of prescribed enclosed volume and surface area for a closed membrane.

For $A \Subset \Omega$ and $c > 0$, we define the class

$$\mathscr{X}^{(0,1)}_{A,c}(\Omega) := \left\{\Sigma = [\![G, \eta, \beta]\!] \in \mathrm{curv}^{(0,1)}_2(\Omega) : \mathrm{spt}\Sigma \subset A \times \mathbb{S}^2, \mathbb{M}(\partial\Sigma) + \mathbb{M}(\Sigma) \le c\right\} \tag{2.3.9}$$

of generalized Gauss graphs with compact support and equi-bounded masses. Our first existence result is the following.

Theorem 2.3.5. *Let* (2.1.2) *hold. The minimization problem*

$$\min\left\{\mathscr{E}(\Sigma):\Sigma\in\mathscr{X}^{(0,1)}_{A,c}(\Omega)\right\} \tag{2.3.10}$$

has a solution.

Proof. Let c_2 be the constant in (2.3.5) and, for every $\Sigma=[\![G,\eta,\beta]\!]\in\mathrm{curv}_2^{(0,1)}(\Omega)$, define the functional

$$\begin{aligned}\mathscr{E}^{(0,1)}(\Sigma):=&\int_{G^*}\left(\tilde{f}\left(x,y,\frac{\eta_1(x,y)}{|\eta_0(x,y)|}\right)+c_2\right)|\eta_0(x,y)|\beta(x,y)\,\mathrm{d}\mathscr{H}^2(x,y)\\ =&\mathscr{E}(\Sigma)+c_2\int_{G^*}|\eta_0(x,y)|\beta(x,y)\,\mathrm{d}\mathscr{H}^2(x,y),\end{aligned}$$

where the last equality follows from Proposition 2.3.3. Inequality (2.3.5) allows us to apply Theorem 1.3.8 and obtain that $\mathscr{E}^{(0,1)}$ is lower semicontinuous in $\mathrm{curv}_2^{(0,1)}(\Omega)$. By Lemma 2.3.4, it follows that also the functional $\mathscr{E}$ is lower semicontinuous in $\mathrm{curv}_2^{(0,1)}(\Omega)$. By Theorems 1.1.3 and 1.3.8, any minimizing sequence $\Sigma_j=[\![G_j,\eta_j,\beta_j]\!]\in\mathscr{X}^{(0,1)}_{A,c}(\Omega)$ for $\mathscr{E}$ is compact in $\mathscr{X}^{(0,1)}_{A,c}(\Omega)$. The thesis then follows from the direct method of the Calculus of Variations. □

Inequality (2.3.5) and Lemma 2.3.4 suggest that it is not necessary to bound the entire $\int_{G^*}\frac{\beta}{|\eta_0|}\mathrm{d}\mathscr{H}^2$ for $\Sigma=[\![G,\eta,\beta]\!]\in\mathrm{curv}_2^*(\Omega)$ in order to apply Theorem 1.3.9, so that we can consider the class

$$\begin{aligned}\mathscr{X}^*_{A,c}(\Omega):=\Bigg\{&\Sigma=[\![G,\eta,\beta]\!]\in\mathrm{curv}_2^*(\Omega):\mathrm{spt}\,\Sigma\subset A\times\mathbb{S}^2,\\ &\mathbb{M}(\partial\Sigma)+\int_{G^*}\left(|\eta_0(x,y)|+\frac{|\eta_2(x,y)|^2}{|\eta_0(x,y)|}\right)\beta(x,y)\,\mathrm{d}\mathscr{H}^2(x,y)\le c\Bigg\}.\end{aligned} \tag{2.3.11}$$

The bound on $\int_{G^*}\frac{|\eta_1(x,y)|^2}{|\eta_0(x,y)|^2}|\eta_0(x,y)|\beta(x,y)\,\mathrm{d}\mathscr{H}^2(x,y)$, together with the one on the second term in (2.3.11), imply the boundedness of the mass of Σ. Moreover, these bounds are needed in order to have closedness in the class $\mathrm{curv}_2^*(\Omega)$, which in general is not closed, contrary to $\mathrm{curv}_2(\Omega)$. In particular, for the regular Gauss graph G of a manifold M, they imply an L^4-bound on the curvatures of M, since

$$\int_{G^*}\left(|\eta_0|+\frac{|\eta_1|^2}{|\eta_0|}+\frac{|\eta_2|^2}{|\eta_0|}\right)\mathrm{d}\mathscr{H}^2=\int_M|\xi|^2\mathrm{d}\mathscr{H}^2=\int_M\left(H(x)^2+(1-K(x))^2\right)\mathrm{d}\mathscr{H}^2(x),$$

for the proof see [10, Proposition 1.1 and Example 1.2]. We present now our second existence result.

Theorem 2.3.6. *Let* (2.1.2) *hold. The minimization problem*

$$\min\left\{\mathscr{E}(\Sigma) : \Sigma \in \mathscr{X}^*_{A,c}(\Omega)\right\} \tag{2.3.12}$$

has a solution.

Proof. Let us consider a minimizing sequence $\Sigma_j = [\![G_j, \eta_j, \beta_j]\!] \in \mathscr{X}^*_{A,c}(\Omega)$ for the functional $\mathscr{E}$. By Proposition 2.3.3 and (2.3.5), we obtain

$$\begin{aligned}\mathscr{E}(\Sigma_j) &= \int_{G_j^*} \tilde{f}\left(x, y, \frac{(\eta_j)_1(x,y)}{|(\eta_j)_0(x,y)|}\right) |(\eta_j)_0(x,y)|\beta_j(x,y)\,\mathrm{d}\mathscr{H}^2(x,y)\\ &\geq c_1 \int_{G_j^*} \frac{|(\eta_j)_1(x,y)|^2}{|(\eta_j)_0(x,y)|^2} |(\eta_j)_0(x,y)|\beta_j(x,y)\,\mathrm{d}\mathscr{H}^2(x,y)\\ &\quad - c_2 \int_{G_j^*} |(\eta_j)_0(x,y)|\beta_j(x,y)\,\mathrm{d}\mathscr{H}^2(x,y).\end{aligned}$$

Now, by (2.3.11), the minimizing sequence satisfies the hypotheses of Corollary 1.3.9 and therefore there exist a subsequence $\{\Sigma_{j_k}\}_{k\in\mathbb{N}}$ and a special generalized Gauss graph $\Sigma_\infty \in \mathrm{curv}_2^*(\Omega)$ such that $\Sigma_{j_k} \rightharpoonup \Sigma_\infty$ as $k \to \infty$. The thesis follows from the direct method of the Calculus of Variations. □

Remark 2.3.7. We called the minimization problems (2.3.10) and (2.3.12) *unconstrained* because the classes $\mathscr{X}^{(0,1)}_{A,c}(\Omega)$ in (2.3.9) and $\mathscr{X}^*_{A,c}(\Omega)$ in (2.3.11) do not contain geometric constraints, namely, there are no generalized Gauss graphs excluded from these classes based on their geometry. In particular, this allows us to consider the zero current $\Sigma = 0$ as a competitor for both minimization problems, and it turns out to be an absolute minimizer if $H_0 = 0$. Indeed, in this case, (2.3.5) becomes $\tilde{f}(x,y,\zeta) \geq c_1|\zeta|^2$, so that $\mathscr{E} \geq 0$. Notice that also a generalized Gauss graph Π supported on a plane ($H = K = 0$) has zero energy, showing that both (2.3.10) and (2.3.12) have no unique solution.

On the other hand, if $H_0 \neq 0$, observe that a sphere Σ (or a portion of it, compatibly with A) with mean curvature $H = H_0$ makes the functional $\mathscr{E}$ negative. Indeed, since for spheres there holds $K = H^2/4$, we have $\mathscr{E}(\Sigma) = -\alpha_K H_0^2 \mathscr{H}^2(\Sigma)/4 < 0 = \mathscr{E}(0) < \alpha_H H_0^2 = \mathscr{E}(\Pi)$.

Given $\Sigma = [\![G, \eta, \beta]\!] \in \mathrm{curv}_2(\Omega)$, we define

$$\mathscr{A}(\Sigma) := \int_G |\eta_0(x,y)|\beta(x,y)\,\mathrm{d}\mathscr{H}^2(x,y). \tag{2.3.13}$$

In light of Remark 2.2.2, if Σ is a regular Gauss graph with multiplicity, the quantity $\mathscr{A}(\Sigma)$ has the geometric interpretation of mass of $p_1\Sigma$, see (2.2.2); in particular, if $\beta \equiv 1$, then $\mathscr{A}(\Sigma) = \mathscr{H}^2(M)$, the *area* of the manifold $M := p_1 G$, see (2.2.3).

We also define the quantity

$$\mathscr{V}(\Sigma) := \frac{1}{3}\int_G (x\cdot y)\,|\eta_0(x,y)|\beta(x,y)\,\mathrm{d}\mathscr{H}^2(x,y). \tag{2.3.14}$$

If Σ is a closed ($\partial\Sigma = 0$) regular Gauss graph with multiplicity $\beta \equiv 1$, by a simple application of the Divergence Theorem, the quantity $\mathscr{V}(\Sigma)$ has the geometric interpretation of the *enclosed volume* in $M := p_1\Sigma$. Indeed, if $M = \partial A$, then by means of the area formula we get

$$\frac{1}{3}\int_G (x\cdot y)\,|\eta_0(x,y)|\,\mathrm{d}\mathscr{H}^2(x,y) = \frac{1}{3}\int_M p\cdot \nu(p)\,\mathrm{d}\mathscr{H}^2(p) = \frac{1}{3}\int_A \mathrm{div}(p)\,\mathrm{d}p = \mathscr{L}^3(A).$$

Lemma 2.3.8. *Let* $A \Subset \Omega$ *and let* $\Sigma_j = [\![G_j,\eta_j,\beta_j]\!] \in \mathrm{curv}_2(\Omega)$ *be such that* $\mathrm{spt}\,\Sigma_j \subset A\times\mathbb{S}^2$ *and* $\partial\Sigma_j = 0$ *for every* $j\in\mathbb{N}$ *and* $\Sigma_j \rightharpoonup \Sigma = [\![G,\eta,\beta]\!] \in \mathrm{curv}_2(\Omega)$ *as* $j\to\infty$. *Then* $\mathrm{spt}\,\Sigma \subset A\times\mathbb{S}^2$, $\partial\Sigma = 0$, *and*

$$\lim_{j\to\infty}\int_{G_j}(x\cdot y)\,|(\eta_j)_0(x,y)|\beta_j(x,y)\,\mathrm{d}\mathscr{H}^2(x,y) = \int_G (x\cdot y)\,|\eta_0(x,y)|\beta(x,y)\,\mathrm{d}\mathscr{H}^2(x,y).$$

In particular, if $M_j = \partial E_j$ *and* $M = \partial E$ *for* E_j, E *sets of class* C^2 *contained in* A, *if* G_j, G *are the associated Gauss graphs by* (1.2.2), *and* $\Sigma_j = \Sigma_{G_j} = [\![G_j,\eta_j,1]\!]$, $\Sigma = \Sigma_G = [\![G,\eta,1]\!]$ *are the associated currents, if* $\Sigma_j \rightharpoonup \Sigma$, *then* $\mathscr{H}^3(E_j) \to \mathscr{H}^3(E)$.

Proof. The proof is the same as that of Lemma 2.3.4. □

Next we study constrained minimization problems, namely we prescribe the surface area and the enclosed volume. Given $a, v > 0$, we define the classes

$$\begin{aligned}\mathscr{X}^{(0,1)}_{A,c;a,v}(\Omega) := \Big\{ &\Sigma = [\![G,\eta,\beta]\!] \in \mathrm{curv}_2^{(0,1)}(\Omega) : \mathrm{spt}\,\Sigma \subseteq A\times \S 2, \partial\Sigma = 0,\\ &\mathbb{M}(\Sigma) \le c, \mathscr{A}(\Sigma) = a, \mathscr{V}(\Sigma) = v \Big\}.\end{aligned}$$

$$\begin{aligned}\mathscr{X}^{*}_{A,c;a,v}(\Omega) := \Big\{ &\Sigma = [\![G,\eta,\beta]\!] \in \mathrm{curv}_2^{*}(\Omega) : \mathrm{spt}\,\Sigma \subseteq A\times \S 2, \partial\Sigma = 0,\\ &\int_{G^*}\frac{|\eta_2(x,y)|^2}{|\eta_0(x,y)|}\beta(x,y)\,\mathrm{d}\mathscr{H}^2(x,y) \le c, \mathscr{A}(\Sigma) = a, \mathscr{V}(\Sigma) = v \Big\}.\end{aligned}$$

In order for two-dimensional closed oriented manifolds of class $\mathscr{C}^2$ to belong to these classes, we enforce the isoperimetric inequality

$$36\pi v^2 \leq a^3. \tag{2.3.15}$$

Theorem 2.3.9. *Let* (2.1.2) *hold and let* $a, v > 0$ *satisfy* (2.3.15). *The minimization problems*

$$\min\left\{\mathscr{E}(\Sigma) : \Sigma \in \mathscr{X}^{(0,1)}_{A,c;a,v}(\Omega)\right\}, \quad \min\left\{\mathscr{E}(\Sigma) : \Sigma \in \mathscr{X}^{*}_{A,c;a,v}(\Omega)\right\} \tag{2.3.16}$$

have a solution.

Proof. The proof is the same as that of Theorems 2.3.5 and 2.3.6, upon noting that Lemmas 2.3.4 and 2.3.8 provide the continuity for the area and enclosed volume constraints. □

We conclude this subsection with a remark on the necessity of assumption (2.1.2).

Remark 2.3.10 ($4\alpha_H \leq \alpha_K$)**.** In the case, then there exists a constant $r \geq 0$ such that $\alpha_K = 4\alpha_H + r$. For the Gauss graph G of a smooth surface M, we have

$$\mathscr{H}^2(G) = \int_M |\xi(x, \nu(x))| \,\mathrm{d}\mathscr{H}^2(x) = \int_M \sqrt{4H(x)^2 + (1-K(x))^2}\,\mathrm{d}\mathscr{H}^2(x),$$

where ξ is defined in (1.2.3) (see Proposition 1.2.1). We consider $M_j = \partial B_{1/j}$, where $B_{1/j}$ is the ball of radius $1/j$ centered in the origin, and we let $\Sigma_j := \Sigma_{G_j} = [\![G_j, \eta_j, 1]\!]$. Since the principal curvatures of M_j are both equal to j, we get from the above formula

$$\mathbb{M}(\Sigma_j) = \mathscr{H}^2(G_j) \leq \frac{4\pi}{j^2}\sqrt{j^4 + 14j^2 + 1},$$

which is uniformly bounded for every $j \in \mathbb{N} \setminus \{0\}$. Thus, for $\Omega = B_2$, we have that $\Sigma_j \in \mathrm{curv}_2^{(0,1)}(\Omega)$ for every $j \in \mathbb{N}$ and, since $\partial\Sigma_j = 0$, we also have that Σ_j belongs to $\mathscr{X}^{(0,1)}_{A,c}(\Omega)$, for every $j \in \mathbb{N} \setminus \{0\}$, for a suitable choice of A and c. Since Σ_j is a regular Gauss graph, $\mathscr{E}(\Sigma_j) = \mathscr{E}(M_j)$, so that, using the expression in (2.1.1), we obtain

$$\mathscr{E}(M_j) = 4\pi\left(\frac{\alpha_H H_0^2}{j^2} - \frac{4\alpha_H H_0}{j} - r\right), \tag{2.3.17}$$

using the fact that $H^2 = 4K$ for spheres. We now consider two cases.

(1) $r > 0$: the functional $\mathscr{E}$ is no longer lower semicontinuous, since $\Sigma_j \rightharpoonup 0$ and, by (2.3.17), $\liminf\limits_{j\to\infty} \mathscr{E}(M_j) = -4\pi r < 0 = \mathscr{E}(0)$.

(2) $r = 0$ and $H_0 = 0$: from (2.3.6) it is easy to see that $\mathscr{E} \geq 0$ and by (2.3.17) $\mathscr{E}(M_j) = 0$ for every $j \in \mathbb{N} \setminus \{0\}$, from which we obtain that $\mathscr{E}$ is minimized on spheres. We also notice that $\mathscr{E}$ is minimized on flat surfaces ($H = K = 0$).

The construction above can adapted to the constrained case by taking

$$M_j = \partial B_R \cup \partial B_{\rho/j}$$

for suitable $R, \rho > 0$, where all the spherical surfaces are oriented with the outward normal, such that $\mathscr{A}(M_j) = \mathscr{H}^2(M_j) = a$ and $\mathscr{V}(\Sigma_{M_j}) = v$. Then $\Sigma_{M_j} \in \mathscr{X}^{(0,1)}_{A,c}(\Omega)$, with area and volume constraints, and $\mathscr{E}(M_j)$ has an expression similar to that in (2.3.17), so that the same conclusions above hold.

The case $r = 0$ and $H_0 \neq 0$ is open and we do not have a counterexample at the moment.

Remark 2.3.11 ($\alpha_K = 0$)**.** In this case, the Canham–Helfrich functional $\mathscr{E}$ in (2.1.1) reduces to the functional

$$\mathscr{W}_0(M) := \alpha_H \int_M (H(p) - H_0)^2 \,\mathrm{d}\mathscr{H}^2(p),$$

which is non-negative and is minimized by a (portion of a) sphere with mean curvature $H = H_0$. Moreover, if $H_0 = 0$, this further reduces to the Willmore functional $\mathscr{W}$, which is again non-negative and minimized, for instance, on flat surfaces or on minimal surfaces. There is a vast literature on the Willmore functional both in the constrained and unconstrained case, see, *e.g.*, [69, 87, 88, 92, 91] in addition to those already mentioned in the Introduction.

Here we observe that Lemma 2.3.1 provides the eigenvalue $2\alpha_H$ with multiplicity 1 and the zero eigenvalue with multiplicity 8. Moreover, it is necessary for the coercivity of $\mathscr{E}$ that all the eigenvectors associated with the zero eigenvalue belong to $\widetilde{\mathscr{X}_y}^{\perp}$ and this is not the case. Therefore, we cannot prove the coercivity in (2.3.5) so that the direct method of the Calculus of Variations cannot be applied to show existence of minimizers. This suggests that the space of generalized Gauss graphs is not a good environment to study the Willmore functional $\mathscr{W}$.

2.3.3 Regularity of minimizers

We prove a regularity result for minimizers of $\mathscr{E}$.

Theorem 2.3.12. *Let* (2.1.2) *hold and let* $\Sigma \in \mathrm{curv}_2(\Omega)$ *be a solution either of problem* (2.3.10) *or of problem* (2.3.12) *with* $\partial\Sigma = 0$*, or of problem* (2.3.16)*. Then* $p_1\Sigma$ *is* C^2*-rectifiable, that is there exists a countable family* $\{S_j\}_{j\in\mathbb{N}}$ *of surfaces of class* C^2 *in* $\mathbb{R}^3$

such that

$$\mathscr{H}^2\left(p_1\Sigma\setminus\bigcup_{j\in\mathbb{N}} S_j\right)=0.$$

Proof. We start by observing that, by [37, Theorem 6.1], since $\partial\Sigma=0$ and $|\Sigma_1|\ll|\Sigma_0|$, we get that $p_1\Sigma$ is the support of a two-dimensional curvature varifold (see the proof of [37, Theorem 6.1] for the explicit construction). The regularity of Σ is now a consequence of [79, Theorem 1]. $\square$

Remark 2.3.13. We point out that Theorem 2.3.12 cannot be obtained using the Structure Theorem 1.3.5, which asserts that if Σ is a generalized Gauss graph then $p_1\Sigma$ is (only) C^1-rectifiable.

Part II

Geometric flows

Chapter 3

Notation and preliminaries

We start by recalling some definitions we will use in the following.

Let E be an open set with C^1 boundary and let $\nu = \nu_E(x)$ be its outer normal at $x \in \partial E$. Given a vector X, its tangential part on ∂E is $X_\tau = X - (X \cdot \nu)\nu$. In particular, we denote by ∇_τ the tangential gradient given by $\nabla_\tau \varphi = (\nabla \varphi)_\tau$, and similarly the tangential divergence div_τ; when no confusion arises we will drop the subscript τ. If E is also of class C^2, the second fundamental form B_E of ∂E is given by $D_\tau \nu$, its eigenvalues are called principal curvatures and its trace H_E is called mean curvature.

Let $\mathbb{T}^N := \mathbb{R}^N / \mathbb{Z}^N$ be the N-dimensional flat torus, that is the quotient space $\mathbb{R}^N / \sim$, where $\sim$ is the equivalence relation given by $x \sim y$ if and only if $x - y \in \mathbb{Z}^N$. The distance between two points $x, y \in \mathbb{T}^N$ is simply defined by

$$\mathrm{dist}_{\mathbb{T}^N}(x,y) = \min_{z \in \mathbb{Z}^N} |(x+z) - y|;$$

when no confusion arises we will drop the subscript $\mathbb{T}^N$. The definition of functional spaces on the torus is straightforward: for example, $L^p(\mathbb{T}^N)$ is identified as the subspace of $L^p_{loc}(\mathbb{R}^N)$ of functions that are one-periodic with respect to all coordinate directions.

3.1 The classical perimeter

In this section we recall some definitions and results about sets of finite perimeter in $\mathbb{R}^N$ and $\mathbb{T}^N$. For simplicity we focus on the ambient space $\mathbb{R}^N$, however, we note that, where not otherwise stated, all the definitions and results can be readily extended to the periodic setting.

We say that $u \in L^1(\mathbb{R}^N)$ is a *function of bounded variation* if its total variation is finite, that is

$$|Du|(\mathbb{R}^N) := \sup\left\{ \int_{\mathbb{R}^N} u(x) \operatorname{div}\varphi(x)\,\mathrm{d}x : \varphi \in C^1(\mathbb{R}^N;\mathbb{R}^N),\ \|\varphi\|_\infty \leq 1 \right\} < +\infty.$$

We denote the space of such functions by $BV(\mathbb{R}^N)$. Let $\mathrm{M}(\mathbb{R}^N)$ be the set of measurable subsets of $\mathbb{R}^N$. We say that $E \in \mathrm{M}(\mathbb{R}^N)$ is a *set of finite perimeter* if its characteristic function $\chi_E \in BV(\mathbb{R}^N)$. The *perimeter* $P(E)$ of E in $\mathbb{R}^N$ is simply the total variation $|D\chi_E|(\mathbb{R}^N)$, and the perimeter of E *relative to the open set* $\Omega \subset \mathbb{R}^N$ is $P(E;\Omega) := |D\chi_E|(\Omega)$. We refer to [8] and [75] for complete references about BV functions and sets of finite perimeter.

Definition 3.1.1 (Normal deformation). *Let $E \subset \mathbb{R}^N$ be an open set of class C^1. Given a function $f : \partial E \to \mathbb{R}$ such that $\|f\|_{L^\infty(\partial E)}$ is sufficiently small, we set*

$$\partial E_f := \{x + f(x)\nu_E(x) : x \in \partial E\}$$

and we call E_f the normal deformation of E induced by f.

Let $E \subset \mathbb{R}^N$ be an open set of class C^1. Let $X(\partial E)$ denote a functional space that can either be $L^p(\partial E)$, $W^{k,p}(\partial E)$, $C^{k,\alpha}(\partial E)$, for some $k \in \mathbb{N}$, $p \in [1,+\infty]$ and $\alpha \in [0,1]$. For any set $F = E_f$ with $f \in X(\partial E)$, with a slight abuse of notation, we set

$$\mathrm{dist}_X(F,E) := \|f\|_{X(\partial E)}.$$

Definition 3.1.2 (Convergence in $C^{1,\alpha}$). *Given $\alpha \in [0,1]$, a sequence $\{E_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ of open sets of class $C^{1,\alpha}$ is said to converge in $C^{1,\alpha}$ to a set $E \subset \mathbb{R}^N$ if: for any $x \in \partial E$, up to rotations and relabelling of the coordinates, there exist a cylinder $C = B' \times (-1,1)$, where $B' \subset \mathbb{R}^{N-1}$ is the unit ball centred at the origin, and functions $f, f_n \in C^{1,\beta}(B';(-1,1))$ such that, for n large enough, it holds*

$$\begin{gathered}(E-x)\cap C = \{(x',x_N) \in B' \times (-1,1) : x_N \leq f(x')\},\\ (E_n-x)\cap C = \{(x',x_N) \in B' \times (-1,1) : x_N \leq f_n(x')\},\\ f_n \to f \quad in \quad C^{1,\alpha}(B').\end{gathered}$$

A set of finite perimeter $E \subset \mathbb{R}^N$ is said to be a Λ-minimizer of the perimeter if there exists $\Lambda \geq 0$ such that

$$P(E) \leq P(F) + \Lambda|E \triangle F|.$$

We recall a classical result for Λ-minimizers, for the proof see [28, Lemma 3.6].

Theorem 3.1.3. *Let $\Lambda \geq 0$ and let $E \subset \mathbb{R}^N$ be an open bounded set of class C^2. Then for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, E) > 0$ with the following property: for every Λ-minimizer F such that $|E \triangle F| \leq \delta$, then, for every $\beta \in (0,1)$, F is of class $C^{1,\beta}$ and*

$$\operatorname{dist}_{C^{1,\beta}}(E,F) \leq \varepsilon.$$

Definition 3.1.4 (First and second variations)**.** *Let $E \in \mathrm{M}(\mathbb{R}^N)$ and $\mathscr{F} : \mathrm{M}(\mathbb{R}^N) \to \mathbb{R}$. For every map $X : \mathbb{R}^N \to \mathbb{R}^N$ of class C^2, we consider the associated flow $\Phi : \mathbb{R}^N \times (-1,1) \to \mathbb{R}^N$ defined by $\partial_t \Phi = X(\Phi)$, $\Phi(\cdot, 0) = \mathrm{I}$. We define the first and second variations of the functional $\mathscr{F}$ at E with respect to the flow Φ to be respectively the values*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathscr{F}(E_t), \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} \mathscr{F}(E_t)$$

where $E_t = \Phi(\cdot, t)(E)$.

Let $E \subset \mathbb{R}^N$ be a set of finite perimeter. It is a classical result that the the first variation of the perimeter has the following expression

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P(E_t) = \int_{\partial^* E} \operatorname{div}_\tau X \, \mathrm{d}\mathscr{H}^{N-1} = \int_{\partial^* E} H_E \nu_E \cdot X \, \mathrm{d}\mathscr{H}^{N-1}, \tag{3.1.1}$$

where H_E is the weak mean curvature of E.

We now consider the flat torus $\mathbb{T}^N$ and we recall some preliminary results from [4].

Theorem 3.1.5 ([4, Theorem 3.1])**.** *Let $E \subset \mathbb{T}^N$ be a set of class C^2, let X be as in Definition 3.1.4, then we have*

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} P(E_t) = & \int_{\partial E} \left(|D_\tau (X \cdot \nu_E)|^2 - |B_E|^2 (X \cdot \nu_E)^2 \right) \mathrm{d}\mathscr{H}^{N-1} \\ & - \int_{\partial E} H_E \operatorname{div}_\tau (X_\tau (X \cdot \nu_E)) \, \mathrm{d}\mathscr{H}^{N-1} + \int_{\partial E} H_E (\operatorname{div} X)(X \cdot \nu_E) \, \mathrm{d}\mathscr{H}^{N-1}. \end{aligned} \tag{3.1.2}$$

Since the expressions of the first and second variation only depend on the normal projection of X, we set

$$\partial P(E)[\varphi] := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P(E_t), \quad \partial^2 P(E)[\varphi] := \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} P(E_t),$$

where $\varphi = X \cdot \nu_E \in C^2(\partial E)$. We remark that due to the translation invariance of the perimeter functional, the second variation degenerates along flows of the form $\Phi(x,t) = x + t\eta$, where $\eta \in \mathbb{R}^N$. In view of this, it is convenient to introduce the subspace $T(\partial E)$ of $\tilde{H}^1(\partial E) := \left\{ \varphi \in H^1(\partial E) : \int_{\partial E} \varphi \, \mathrm{d}\mathscr{H}^{N-1} = 0 \right\}$ generated by the functions ν_i, $i = 1, \ldots, N$.

Its orthogonal subspace, in the L^2-sense, is denoted by $T^{\perp}(\partial E)$ and is given by

$$T^{\perp}(\partial E) = \left\{\varphi \in \tilde{H}^1(\partial E) : \int_{\partial E} \varphi \nu_i \, \mathrm{d}\mathscr{H}^{N-1} = 0,\ i = 1,\dots,N\right\}.$$

Definition 3.1.6 (Critical and strictly stable sets)**.** *We say that a set of finite perimeter E is a critical set of the perimeter functional if*

$$\partial P(E)[\varphi] = 0, \qquad \forall \varphi \in \tilde{H}^1(\partial E).$$

We say that E is a strictly stable set if it is a critical set of the perimeter of class C^2 and its second variation of the perimeter is positive in the sense that

$$\partial^2 P(E)[\varphi] > 0, \qquad \forall \varphi \in T^{\perp}(\partial E) \setminus \{0\}.$$

Remark 3.1.7. We remark that the last two integral in (3.1.2) vanish when E is a critical set for the perimeter and if $|\Phi(\cdot,t)(E)| = |E|$ for all $t \in [0,1]$. Indeed, if E is a C^2-regular critical set for the perimeter then its curvature is constant, therefore the second integral vanishes. Moreover, if the flow Φ is volume-preserving then it can be shown (see equation (2.30) in [26]) that

$$0 = \frac{\mathrm{d}^2}{\mathrm{d}t^2}|E_t| = \int_{\partial E} (\mathrm{div} X)(X \cdot \nu_E) \, \mathrm{d}\mathscr{H}^{N-1}.$$

Hence, if Φ is a volume-preserving variation of a regular critical set E we have

$$\partial^2 P(E)[X \cdot \nu_E] = \int_{\partial E} \left(|D_\tau (X \cdot \nu_E)|^2 - |B_E|^2 (X \cdot \nu_E)^2\right) \mathrm{d}\mathscr{H}^{N-1}. \tag{3.1.3}$$

The following result ensures that the second variation of a strictly stable set E is coercive on the subspace $T^{\perp}(\partial E)$.

Lemma 3.1.8 ([4, Lemma 3.6])**.** *Assume that E is a strictly stable set, then*

$$m_0 := \inf\left\{\partial^2 P(E)[\varphi] : \varphi \in T^{\perp}(\partial E),\ \|\varphi\|_{H^1(\partial E)} = 1\right\} > 0$$

and

$$\partial^2 P(E)[\varphi] \geq m_0 \|\varphi\|^2_{H^1(\partial E)} \quad \forall \varphi \in T^{\perp}(\partial E).$$

From Step 1 in the proof of [4, Theorem 3.9] one can prove the following result.

Lemma 3.1.9. *Assume that E is a strictly stable set, then there exists* $\bar{\delta} > 0$ *such that*

$$\inf\left\{\partial^2 P(E)[\varphi] : \varphi \in \tilde{H}^1(\partial E),\ \|\varphi\|_{H^1(\partial E)} = 1,\ \left|\int_{\partial E} \varphi \nu_E \, \mathrm{d}\mathscr{H}^{N-1}\right| \le \delta\right\} \ge \frac{m_0}{2},$$

for every $\delta \le \bar{\delta}$*, where* m_0 *is the constant in Lemma 3.1.8.*

We will also need the following lemma which shows that any set F sufficiently close to a smooth set E can be translated in such a way that the resulting set $\tilde{F}$ satisfies $\partial\tilde{F} = \{x + \varphi(x)\nu_E(x) : x \in \partial E\}$, with φ having a suitably small projection on $T(\partial E)$.

Lemma 3.1.10 ([4, Lemma 3.8])**.** *Let* $E \subset \mathbb{T}^N$ *be of class* C^3 *and let* $p > N-1$. *For every* $\delta > 0$ *there exist* $C > 0$ *and* $\eta_0 > 0$ *such that, if* $F \subset \mathbb{T}^N$ *satisfies* $\partial F = \{x + \psi(x)\nu_E(x) : x \in \partial E\}$ *for some* $\psi \in C^2(\partial E)$ *with* $\|\psi\|_{W^{2,p}(\partial E)} \le \eta_0$*, then there exist* $\sigma \in \mathbb{T}^N$ *and* $\varphi \in W^{2,p}(\partial E)$ *with the properties that*

$$|\sigma| \le C\|\psi\|_{W^{2,p}(\partial E)}, \quad \|\varphi\|_{W^{2,p}(\partial E)} \le C\|\psi\|_{W^{2,p}(\partial E)},$$

and

$$\partial F + \sigma = \{x + \varphi(x)\nu_E(x) : x \in \partial E\}, \quad \left|\int_{\partial E} \varphi \nu_E \, \mathrm{d}\mathscr{H}^{N-1}\right| \le \delta\|\varphi\|_{L^2(\partial E)}.$$

Let $E, F \subset \mathbb{T}^N$ be measurable sets, we define

$$\alpha(E,F) := \min_{x \in \mathbb{T}^N} |E \triangle (F + x)|.$$

Theorem 3.1.11 ([4, Corollary 1.2])**.** *Let* $E \subset \mathbb{T}^N$ *be a strictly stable set. Then, there exist* $\sigma = \sigma(E)$*,* $C = C(E) > 0$ *such that*

$$C\alpha^2(E,F) \le P(F) - P(E)$$

for all $F \subset \mathbb{T}^N$ *with* $|F| = |E|$ *and* $\alpha(E,F) < \sigma$.

3.2 The fractional perimeter

Let $s \in (0,1)$, we define the *s-fractional perimeter* as the following functional

$$P^s \colon \mathrm{M}(\mathbb{R}^N) \to [0, +\infty], \quad P^s(E) := \int_E \int_{E^c} \frac{1}{|x-y|^{N+s}} \, \mathrm{d}x \, \mathrm{d}y = \frac{1}{2}[\chi_E]^2_{H^{\frac{s}{2}}(\mathbb{R}^N)},$$

where we recall
$$[u]^p_{W^{s,p}(\mathbb{R}^N)} := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\,\mathrm{d}y$$

for every measurable function $u\colon \mathbb{R}^N \to \mathbb{R}$, $s \in (0,1)$ and $p \in [1,+\infty)$. More in general, for every $E,F \in \mathrm{M}(\mathbb{R}^N)$, we set

$$\mathscr{L}_s(E,F) := \int_E\int_F \frac{1}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y \tag{3.2.1}$$

and, for any open set $\Omega \subset \mathbb{R}^N$, we define the fractional perimeter of E relative to Ω as

$$P^s(E;\Omega) := \mathscr{L}_s(E\cap\Omega, E^c\cap\Omega) + \mathscr{L}_s(E\cap\Omega, E^c\setminus\Omega) + \mathscr{L}_s(E\setminus\Omega, E^c\cap\Omega).$$

Let $E \subset \mathbb{R}^N$ be a set of class C^2. Given $\varphi \in C^2(\partial E)$ with $\int_{\partial E}\varphi\,\mathrm{d}\mathscr{H}^{N-1} = 0$, the first variation of the s-fractional perimeter of E along φ (recall definition 3.1.4 and the subsequent discussion) is given by

$$\partial P^s(E)[\varphi] = \int_{\partial E} H^s_E(x) X(x)\cdot\nu_E(x)\,\mathrm{d}\mathscr{H}^{N-1}(x),$$

where $H^s_E(x)$ is the *s-fractional mean curvature of E* evaluated at $x \in \partial E$, that is

$$H^s_E(x) := \int_{\mathbb{R}^N} \frac{\chi^c_E(y)-\chi_E(y)}{|x-y|^{N+s}}\,\mathrm{d}y,$$

where the integral has to be intended in the principal value sense.

We recall two convergence theorems: the first one concerns the convergence of the fractional perimeter to the classical one (see [20, Theorem 1]); the second one regards the convergence of the fractional curvature (see for instance [1]).

Theorem 3.2.1. *Let E be a bounded set of class $C^{1,\alpha}$ for $\alpha \in (0,1)$. Then,*

$$\lim_{s\to 1^-}(1-s)P^s(E) = \omega_{N-1}P(E).$$

Theorem 3.2.2. *Let E be a bounded set of class C^2. Then,*

$$\lim_{s\to 1^-}(1-s)H^s_E = \omega_{N-1}H_E \quad \text{uniformly on } \partial E.$$

Finally, we recall the pointwise convergence of the fractional Gagliardo seminorms to the Sobolev one. The classical proof is contained in [16, Theorem 2], see also [57, Proposition 3.7] for the same result in a more general setting.

Theorem 3.2.3. *Assume $f \in H^s(\partial B)$. Then there exists a dimensional constant $C > 0$ such that*

$$\lim_{s\to 1^-}(1-s)[f]^2_{H^{\frac{1+s}{2}}(\partial B)} = C\|\nabla f\|^2_{L^2(\partial B)}.$$

3.3 Smooth flows

In this section we introduce two geometric evolutions and some of their properties.

Definition 3.3.1 (Smooth flow)**.** *Let $E_0, \{E_t\}_{t\in(0,T]} \subset \mathbb{R}^N$ be smooth open sets. We say that $\{E_t\}_{t\in[0,T]}$ is a smooth flow starting from E_0 if there exists $\Phi : \mathbb{R}^N \times [0,T] \to \mathbb{R}^N$ a smooth map such that $\Phi_0(\cdot) := \Phi(\cdot,0) = \mathrm{Id}$, $\Phi_t(\cdot) := \Phi(\cdot,t)$ are smooth diffeomorphisms and $E_t = \Phi_t(E_0)$ for every $t \in [0,T]$. The normal velocity of the flow on ∂E_t is defined by*

$$V_t(x) = \partial_t \Phi_t(y) \cdot \nu_{E_t}(x)$$

for every $t \in [0,T]$ and $x = \Phi_t^{-1}(y) \in \partial E_t$.

Definition 3.3.2 (Volume-preserving mean curvature flow and surface diffusion flow)**.** *We say that $\{E_t\}_{t\in[0,T]}$ is the volume-preserving mean curvature flow starting from E_0 if*

$$V_t(x) = -H_{\Sigma_t}(x) + \overline{H}_{\Sigma_t} \quad \textit{for } x \in \partial E_t. \tag{3.3.1}$$

Similarly, we define the surface diffusion flow starting from E_0 by

$$V_t(x) = \Delta_{E_t} H_{\Sigma_t}(x) \quad \textit{for } x \in \partial E_t. \tag{3.3.2}$$

If we consider H^s_Σ instead of H_Σ in (3.3.1) we call the evolution *fractional mean curvature flow*.

As already mentioned in the Introduction, these two evolutions share some similarity. Firstly, from the evolution laws (3.3.1) and (3.3.2), it follows that the volume of the evolving sets is preserved along the flows, indeed we have

$$\frac{\mathrm{d}}{\mathrm{d}t}|E_t| = \int_{\partial E_t} V_t(x)\,\mathrm{d}\mathscr{H}^{N-1}(x) \begin{cases} \overset{(3.3.1)}{=} \int_{\partial E_t}(-H_{E_t}(x) + \overline{H}_{E_t})\,\mathrm{d}\mathscr{H}^{N-1}(x) \\ \overset{(3.3.2)}{=} \int_{\partial E_t} \Delta_{E_t} H_{E_t}(x)\,\mathrm{d}\mathscr{H}^{N-1}(x) \end{cases} = 0.$$

Another important feature is that the perimeter does not increase during the evolutions. This property can be directly inferred from (3.3.1) for the mean curvature flow:

$$\frac{\mathrm{d}}{\mathrm{d}t}P(E_t) = \int_{\partial E_t} V_t H_{E_t}\,\mathrm{d}\mathscr{H}^{N-1} = -\int_{\partial E_t}(H_{E_t} - \bar{H}_{E_t})^2\,\mathrm{d}\mathscr{H}^{N-1} \le 0. \tag{3.3.3}$$

Similarly, for the surface diffusion flow given by (3.3.2), integration by parts yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}P(E_t) = \int_{\partial E_t} V_t H_{E_t}\,\mathrm{d}\mathscr{H}^{N-1} = -\int_{\partial E_t} |\nabla H_{E_t}|^2\,\mathrm{d}\mathscr{H}^{N-1} \leq 0. \tag{3.3.4}$$

3.4 Discrete flows

Let $F \neq \emptyset$ be a measurable subset of $\mathbb{T}^N$. In the following we will always assume that F coincides with its Lebesgue representative. Fixed $h > 0$, $m > 0$, we consider the minimum problem

$$\min\left\{P(E) + \frac{1}{h}\int_E \mathrm{sd}_F(x)\,\mathrm{d}x : E \subset \mathbb{T}^N,\, |E| = m\right\}, \tag{3.4.1}$$

where $\mathrm{sd}_F(x) := \mathrm{dist}_F(x) - \mathrm{dist}_{F^c}(x)$ is the signed distance from the set F. Observe that the minimum problem (3.4.1) is equivalent to

$$\min\left\{P(E) + \frac{1}{h}\int_{F\triangle E} \mathrm{dist}_{\partial F}(x)\,\mathrm{d}x : E \subset \mathbb{T}^N,\, |E| = m\right\}.$$

For every $E \subset \mathbb{T}^N$, we set

$$J_h(E,F) := P(E) + \frac{1}{h}\int_{F\triangle E} \mathrm{dist}_{\partial F}(x)\,\mathrm{d}x =: P(E) + \frac{1}{h}\mathscr{D}(E,F), \tag{3.4.2}$$

with a little abuse of notation we will sometimes denote by $J_h(\cdot,F)$ also the functional

$$E \mapsto P(E) + \frac{1}{h}\int_E \mathrm{sd}_F(x)\,\mathrm{d}x$$

and, when no ambiguity arises, we will write J_h instead of $J_h(\cdot,F)$.

Definition 3.4.1 (Discrete flow). *Let $E_0 \subset \mathbb{T}^N$ be a measurable set such that $|E_0| = m$, we define the discrete-in-time, volume-preserving mean curvature flow $\{E_h^n\}_{n\in\mathbb{N}}$ starting from E_0 (or for simplicity the discrete flow starting from E_0) in the following way: set $E_h^0 = E_0$, by induction assume that E_h^k is defined for $1 \leq k \leq n-1$, and let E_h^n to be a solution of* (3.4.1) *with F replaced by E_h^{n-1}, i.e.*

$$E_h^n \in \operatorname{argmin}\left\{P(E) + \frac{1}{h}\int_E \mathrm{sd}_{E_h^{n-1}}(x)\,\mathrm{d}x : E \subset \mathbb{T}^N,\, |E| = m\right\}.$$

Remark 3.4.2. We start by remarking that the sequence of the perimeters along the discrete flow is non-increasing. Indeed, by testing the minimality of E_h^n with E_h^{n-1}, we obtain

$$P(E_h^n) \le P(E_h^n) + \frac{1}{h}\int_{E_h^{n-1}\triangle E_h^n} \mathrm{dist}_{\partial E_h^{n-1}}(x)\,\mathrm{d}x \le P(E_h^{n-1}).$$

In particular, this proves that, even if the initial set E_0 is not of finite perimeter, the perimeters of the sets E_h^n, for $n \ge 1$, are uniformly bounded by a constant that only depends on the dimension N, the fixed volume m and h. To prove this, consider any set $E_0 \subset \mathbb{T}^N$ of volume m and let Q_m be the cube of the same volume. By testing the minimality of E_h^1 with Q_m as a competitor, we obtain

$$\begin{aligned} P(E_h^1) &\le P(Q_m) + \frac{1}{h}\int_{E_0\triangle Q_m} \mathrm{dist}_{\partial E_0}(x)\,\mathrm{d}x - \frac{1}{h}\int_{E_0\triangle E_h^1} \mathrm{dist}_{\partial E_0}(x)\,\mathrm{d}x \\ &\le P(Q_m) + \frac{1}{h}\int_{\mathbb{T}^N} \sqrt{N} = C(N,m,h), \end{aligned}$$

where we estimated $\mathrm{dist}_{\partial E_0} \le \mathrm{diam}(\mathbb{T}^N) = \sqrt{N}$.

We recall some preliminary results that can be found in [82]. First of all, we observe that the problem (3.4.1) admits a solution via the direct method of the calculus of variations. The regularity properties of the discrete flow are investigated in the following proposition. Some of the results are classical, others follow from [82, Proposition 2.3].

Proposition 3.4.3. *Let h, m, $M > 0$ and let $F \subset \mathbb{T}^N$ be a set with $|F| = m$ and $P(F) \le M$. Then, any solution $E \subset \mathbb{T}^N$ to* (3.4.1) *satisfies the following properties:*

i) There exist $c_0 = c_0(N) > 0$ and a radius $r_0 = r_0(m,h,N,M) > 0$ such that for every $x \in \partial^* E$ and $r \in (0, r_0]$ we have

$$|B_r(x) \cap E| \ge c_0 r^N \quad \text{and} \quad |B_r(x) \setminus E| \ge c_0 r^N.$$

In particular, E admits an open representative whose topological boundary coincides with the closure of its reduced boundary, i.e. $\partial E = \overline{\partial^* E}$.

ii) There exists $\Lambda = \Lambda(m,h,N,M) > 0$ such that E is a Λ-minimizer of the perimeter, that is

$$P(E) \le P(E') + \Lambda |E \triangle E'|$$

for all measurable set $E' \subset \mathbb{R}^N$ such that $\mathrm{diam}(E\triangle E') \le 1$.

iii) The following Euler-Lagrange equation holds: there exists $\lambda \in \mathbb{R}$ such that for all $X \in C_c^1(\mathbb{T}^N, \mathbb{R}^N)$ we have

$$\int_{\partial^* E} \frac{\mathrm{sd}_F}{h} X \cdot \nu_E \,\mathrm{d}\mathscr{H}^{N-1} + \int_{\partial^* E} \mathrm{div}_\tau X \,\mathrm{d}\mathscr{H}^{N-1} = \lambda \int_{\partial^* E} X \cdot \nu_E \,\mathrm{d}\mathscr{H}^{N-1}. \tag{3.4.3}$$

iv) There exists a closed set Σ, whose Hausdorff dimension is less than or equal to $N-8$, such that $\partial^* E = \partial E \setminus \Sigma$ is an $(N-1)$-submanifold of class $C^{2,\alpha}$ for all $\alpha \in (0,1)$ with

$$|H_E(x)| \leq \Lambda, \quad \textit{for all } x \in \partial E \setminus \Sigma.$$

v) There exist $k_0 = k_0(m,h,N,M) \in \mathbb{N}$, $d_0 = d_0(m,h,N,M) > 0$ and $a_0 = a_0(m,h,N,M) > 0$ such that E is made up of at most k_0 connected components with mutual Hausdorff distance at least a_0, and each one having diameter bounded from above by d_0.

The following result characterizes the stationary sets of the discrete scheme. The last assertion of the proposition is a technical result that will be employed in the proof of Lemma 5.3.1.

Proposition 3.4.4. *Every stationary set $E \subset \mathbb{T}^N$ for the discrete flow is a critical set of the perimeter.*
Viceversa, if $E \subset \mathbb{T}^N$ is a regular critical set of the perimeter, then there exists $h^* = h^*(E) > 0$ such that, for every $h < h^*$, the volume-preserving discrete flow starting from E is unique and given by $E_h^n = E$. Moreover, if E is a strictly stable set then it is also the unique volume-constrained minimizer of the functional

$$\tilde{J}_h(F) := P(F) + \frac{1}{h}\int_F \mathrm{dist}_E(x)\,\mathrm{d}x.$$

Proof. The first statement is an immediate consequence of (3.4.3). Since E is a stationary point for the discrete flow, it satisfies

$$\int_{\partial^* E} \mathrm{div}_\tau X \,\mathrm{d}\mathscr{H}^{N-1} = \lambda \int_{\partial^* E} X \cdot \nu_E \,\mathrm{d}\mathscr{H}^{N-1}$$

for all $X \in C^1_c(\mathbb{T}^N, \mathbb{T}^N)$, i.e. E is a critical point for the perimeter.

The second part follows using the same argument of the proof of [82, Proposition 3.2]. Indeed, recall that the second variation has the following expression

$$\partial^2 J_h(E)[\varphi] = \int_{\partial E} |\nabla \varphi|^2 + \left(\frac{1}{h} - |B_E|^2\right)\varphi^2 \,\mathrm{d}\mathscr{H}^{N-1},$$

which is positive if h is small enough. Then we proceed as in the proof of [82, Proposition 3.2].

Analogously, we prove that E is the unique volume-constrained minimizer of $\tilde{J}_h$. Firstly, observe that, by Theorem 3.1.11, E is a strict local L^1-minimizer of the perimeter and it is a global minimizer of the second term in $\tilde{J}_h$. Therefore, there exists $\varepsilon > 0$ such that

$$\tilde{J}_h(E) < \tilde{J}_h(F)$$

for all measurable set F such that $|F| = |E|$ and $|E \triangle F| \le \varepsilon$, i.e. E is an isolated local minimizer for $\tilde{J}_h$ in L^1 with the volume constraint, with minimality neighbourhood uniform with respect to h. Now, given any sequence $\{h_n\}_{n\in\mathbb{N}}$ going to zero, let F_n be a volume-constrained minimizer of $\tilde{J}_{h_n}$; we then easily deduce that $|E \triangle F_n| \to 0$ as $n \to \infty$, and therefore, for n large enough, $|E \triangle F_n| \le \varepsilon$. Finally, the strict minimality of E implies $F_n = E$.

□

In Chapter 6 we will consider a similar incremental minimum problem which defines the discrete-in-time approximation of the volume-preserving fractional mean curvature flow. More precisely, let $F \neq \emptyset$ be a bounded, measurable subset of $\mathbb{R}^N$ and fix $h, m > 0$, we consider the minimum problem

$$\min\left\{ P^s(E) + \frac{1}{h}\int_E \mathrm{sd}_F(x)\,\mathrm{d}x + \frac{1}{h^{\frac{s}{s+1}}}\big||E| - m\big| : E \subset \mathbb{R}^N \right\}, \tag{3.4.4}$$

where we recall $\mathrm{sd}_F(x) = \mathrm{dist}_F(x) - \mathrm{dist}_{F^c}(x)$. For the proof of the existence of minimizers of (3.4.4) see for example [23, Theorem 1.1]. We set $\mathscr{J}_h(F,\cdot) : \mathrm{M}(\mathbb{R}^N) \to (-\infty, +\infty]$ the functional

$$\mathscr{J}_h(F,E) := P^s(E) + \frac{1}{h}\int_E \mathrm{sd}_F(x)\,\mathrm{d}x + \frac{1}{h^{\frac{s}{s+1}}}\big||E| - m\big|. \tag{3.4.5}$$

Definition 3.4.5 (Fractional discrete flow)**.** *Let $E_0 \subset \mathbb{R}^N$ be a measurable set and fix $m > 0$, we define the discrete-in-time, volume-preserving fractional mean curvature flow $\{E_h^n\}_{n\in\mathbb{N}}$ starting from E_0 by*

$$E_h^n \in \mathit{argmin}\left\{ P^s(F) + \frac{1}{h}\int_F \mathrm{sd}_{E_h^{n-1}}(x)\,\mathrm{d}x + \frac{1}{h^{\frac{s}{1+s}}}\big||F| - m\big| : F \subset \mathbb{R}^N \right\}.$$

Remark 3.4.6. Proposition 3.4.3 holds also in the fractional setting for the problem (3.4.4). The only difference is that *iv*) becomes: the boundary ∂E is of class $C^{2,\alpha}$ for any $\alpha \in (0,s)$ outside of a closed set Σ of Hausdorff dimension at most $N-3$. Moreover, there exists $s_0 \in (0,1)$ such that, if $s \in (s_0,1)$, then ∂E is of class $C^{1,\alpha}$ for any $\alpha \in (0,1)$ outside a closed set Σ of Hausdorff dimension at most $N-8$. This follows from [90, Corollary 2] and [20, Theorem 5].

Chapter 4

Stability of the surface diffusion and volume-preserving mean curvature flows in the flat torus

Contents

4.1 Introduction

In this chapter, which contains the results of [32], we prove the stability of strictly stable sets (see Definition 3.1.6) for the volume-preserving mean curvature flow (3.3.1) and the surface diffusion flow (3.3.2) in the flat torus.

Theorem 4.1.1. *Let $E \subset \mathbb{T}^N$ be a strictly stable set and let $E_0 = E_{u_0} \subset \mathbb{T}^N$ be the normal deformation of E induced by $u_0 \in C^{1,1}(\partial E)$ (see Definition 3.1.1) with $|E_0| = |E|$. There exists $\delta = \delta(E) > 0$ such that if $\|u_0\|_{C^{1,1}(\partial E)} \leq \delta$, then*

- *(i) the volume-preserving mean curvature flow $\{E_t\}_{t\geq 0}$ starting from E_0 exists smooth for all times $t \geq 0$, and $E_t \to E + \tau$ as $t \to \infty$, for some $\tau \in \mathbb{T}^N$, in C^k for every $k \in \mathbb{N}$ exponentially fast;*
- *(ii) the surface diffusion flow $\{E_t\}_{t\geq 0}$ starting from E_0 exists smooth for all times $t \geq 0$, and $E_t \to E + \tau$ as $t \to \infty$, for some $\tau \in \mathbb{T}^N$, in C^k for every $k \in \mathbb{N}$ exponentially fast.*

Where with exponentially fast we mean that the sets E_t can be written as normal deformations of $E + \tau$ induced by functions $u(\cdot,t) \in C^\infty(\partial E + \tau)$ such that

$$\|u(\cdot,t)\|_{C^k(\partial E+\tau)} \leq C_k e^{-C_k t} \quad \text{for} \quad t > 0.$$

The plan of the chapter is the following: in Section 4.2 we show Schauder estimates for short-time solutions of the two flows; Section 4.3 is devoted to the proof of the main result which relays on a quantitative Alexandrov estimate for normal deformations of a strictly stable set (Theorem 5.2.1 in Chapter 5).

4.2 Preliminaries

In this section, we present improved short-time existence results for both the mean curvature and surface diffusion flows. Our goal is to establish the Schauder estimates, which play a crucial role in proving Theorem 4.1.1.

In the following, if not otherwise stated, we will always denote by E an open subset of $\mathbb{T}^N$. We start by recalling the definition of inner and outer ball conditions.

Definition 4.2.1 (Uniform ball condition)**.** *We say that a set E satisfies a uniform inner (respectively outer) ball condition with radius r if there exists $r > 0$ such that for every $x \in \partial E$ there exists a ball $B_r(y) \subset E$ (resp. $B_r(y) \subset E^c$) with $x \in \partial B_r(y)$.*

Note that all sets E of class $C^{1,1}$ satisfy a uniform inner and outer ball condition (see e.g. [31]). Arguing as in the proof of [4, Lemma 3.8], we can prove the following result.

Lemma 4.2.2. *Let $E \subset \mathbb{T}^N$ be of class C^∞ and let $m > 0$. There exists $\eta = \eta(m,E) > 0$ such that, for every $k \geq 2$, $u \in C^k(\partial E)$ with $\|u\|_{C^k(\partial E)} \leq m$, $\|u\|_{C^0(\partial E)} \leq \eta$ and for every $\sigma \in \mathbb{T}^N$ with $|\sigma| \leq \eta$, then $E_u + \sigma$ can be written as a normal deformation of E induced by a function $v : \partial E \to \mathbb{R}$ such that*

$$\|v\|_{C^0(\partial E)} \leq 2\eta, \quad \|v\|_{C^k(\partial E)} \leq C(E)(\|u\|_{C^k(\partial E)} + |\sigma|).$$

Proof. Being the set E smooth, it satisfies a uniform inner and outer ball condition, hence there exists a positive radius $r > 0$ such that the signed distance sd_E from the set E, defined by

$$\mathrm{sd}_E(x) = \begin{cases} \mathrm{dist}_{\partial E}(x) & \text{if } x \in E^c \\ -\mathrm{dist}_{\partial E}(x) & \text{if } x \in E, \end{cases}$$

is a function of class C^∞ (by the regularity of ∂E) in the r-tubular neighborhood $(\partial E)_r$, which we recall is $(\partial E)_r = \{x\colon \mathrm{dist}_{\partial E}(x) \leq r\}$ (for further properties of the distance function see [56, section 14.6]). Since, for some $k \geq 2$, u has C^k-norm bounded by m, we also have $\|u\|_{C^{1,1}(\partial E)} \leq m$. Then, there exists a radius $\rho = \rho(m,E)$ such that ∂E_u satisfies a uniform inner and outer ball condition of radius ρ. We can assume without loss of generality that $\rho < r$.

We now let $\eta \leq \rho/2$ to be chosen later, take any $|\sigma| < \eta$ and set $F = E_u + \sigma$. Clearly, F still satisfies a uniform inner and outer ball condition of radius ρ. Then, for every $y \in \partial F$ there exists $x \in \partial E_u$ such that $y = x + \sigma$, hence we have

$$\mathrm{dist}_{\partial E}(y) \leq |\sigma| + \mathrm{dist}_{\partial E}(x) < \eta + \|u\|_{C^0(\partial E)} \leq 2\eta,$$

and in particular $\partial F \subset (\partial E)_{2\eta} \subset (\partial E)_r$. We now define the map $T_u : \partial E \to \partial E$ as

$$T_u(x) := \pi_E(x + u(x)\nu_E(x) + \sigma) = y - \mathrm{sd}_E(y)\nabla \mathrm{sd}_E(y), \tag{4.2.1}$$

where π_E is the projection map on ∂E and $y = x + u(x)\nu_E(x) + \sigma \in \partial F$. By choosing η smaller, by interpolation, it holds $\|u\|_{C^1(\partial E)} + |\sigma| < \frac{1}{2}$, which implies that the function $x \mapsto x + u(x)\nu_E(x) + \sigma$ is a diffeomorphism (since it is a small perturbation of the identity). Moreover, since E is of class C^∞ (and possibly for η smaller), $\pi_E\big|_{\partial F} : \partial F \subset (\partial E)_{2\eta} \to \partial E$ is a diffeomorphism of class C^k, C^k-close to the identity. Therefore, $T_u \in C^k(\partial E)$ and, by (4.2.1), we get

$$\|T_u - I\|_{C^k(\partial E)} \leq C(\|u\|_{C^k(\partial E)} + |\sigma|). \tag{4.2.2}$$

Moreover, using again (4.2.1) and the invertibility of the map $x \mapsto x + u(x)\nu_E(x) + \sigma$, we obtain

$$\|T_u^{-1} - I\|_{C^k(\partial E)} \leq C(\|u\|_{C^k(\partial E)} + |\sigma|). \tag{4.2.3}$$

Using the fact that T_u is a diffeomorphism and (4.2.1), we can find a function $v : \partial E \to \mathbb{R}$ such that F is the normal deformation of E induced by v, more precisely for every $x \in \partial E$ it holds

$$x + u(x)\nu_E(x) + \sigma = T_u(x) + v(T_u(x))\nu_E(T_u(x)).$$

Finally, using the above expression and the bounds in (4.2.2) and (4.2.3), we conclude that there exists $C = C(E) > 0$ such that

$$\|v\|_{C^k(\partial E)} \leq \|T_u^{-1}\|_{C^k(\partial E)}(\|u\|_{C^k(\partial E)} + |\sigma| + \|T_u - I\|_{C^k(\partial E)}) \leq C(\|u\|_{C^k(\partial E)} + |\sigma|).$$

□

We now state a quantitative version of Alexandrov's theorem which we will prove in Section 5.2. It will be the main tool to prove the exponential stability of the flows.

Theorem 4.2.3. *Let $E \subset \mathbb{T}^N$ be a strictly stable set. There exist $\delta^* \in (0,1/2)$ and $C = C(E) > 0$ with the following property: for any $f \in C^1(\partial E) \cap H^2(\partial E)$ such that $\|f\|_{C^1(\partial E)} \leq \delta^*$ and satisfying*

$$|E_f| = |E|, \qquad \left| \int_{\partial E} f \nu_E \, \mathrm{d}\mathscr{H}^{N-1} \right| \leq \delta^* \|f\|_{L^2(\partial E)},$$

setting $\mathscr{H}_{E_f}(x) = H_{E_f}(x + f(x)\nu_E(x))$ for $x \in \partial E$, we have

$$\|f\|_{H^1(\partial E)} \leq C \|\mathscr{H}_{E_f} - \bar{\mathscr{H}}_{E_f}\|_{L^2(\partial E)}. \tag{4.2.4}$$

Remark 4.2.4. Note that equation (4.2.4) in particular implies that, under the hypotheses of Theorem 4.2.3, for any $\lambda \in \mathbb{R}$ it holds

$$\|f\|_{H^1(\partial E)} \leq C \|\mathscr{H}_{E_f} - \lambda\|_{L^2(\partial E)}.$$

We conclude by recalling the Poincaré and Gagliardo-Nieremberg inequalities on smooth hypersurfaces (see [11] for instance).

Lemma 4.2.5. *Let $\Sigma \subset \mathbb{T}^N$ be a smooth closed hypersurface and $f \in H^1(\Sigma)$. There exists $C = C(\Sigma) > 0$ such that*

$$\|f - \bar{f}\|_{L^2(\Sigma)} \leq C \|\nabla_\tau f\|_{H^1(\Sigma)}.$$

Theorem 4.2.6. *Let $\Sigma \subset \mathbb{T}^N$ be a smooth closed hypersurface. Let l, m, $k \in \mathbb{N}$ be such that $1 \leq l < m$, and let $1 \leq r \leq \infty$. There exists a constant C, depending on these constants and on Σ, with the following property: for every $u \in W^{l,p}(\partial\Sigma)$ we have*

$$\|\nabla^l u(\cdot,t)\|_{L^p(\Sigma)} \leq C \|u(\cdot,t)\|^{\theta}_{W^{m,r}(\Sigma)} \|u(\cdot,t)\|^{1-\theta}_{L^q(\Sigma)},$$

where

$$\frac{1}{p} = \frac{l}{N-1} + \theta\left(\frac{1}{r} - \frac{m}{N-1}\right) + (1-\theta)\frac{1}{q}$$

for all $\theta \in [l/m, 1)$ for which p is nonnegative.

4.2.1 Short-time existence for the mean curvature flow

Let $T > 0$ and let $E_0 \subset \mathbb{T}^N$ be a smooth open set. We recall that the volume-preserving mean curvature flow in $[0,T)$ starting from E_0 is the family of sets $\{E_t\}_{0\leq t<T}$ whose outer normal velocity is given by

$$V_t(x) = -H_{E_t}(x) + \bar{H}_{E_t}, \quad x \in \partial E_t,\ t \in (0,T),$$

see Section 3.3 for a rigorous definition of the flow. Assuming that the flow starting from E_0 exists, following classical computations (see for instance [76]) one can deduce that the evolution equation satisfied by u is

$$\partial_t u = \Delta_E u + \langle A(x,u,\nabla u), \nabla^2 u\rangle + J(x,u,\nabla u) - H_E + \bar{H}_{E_t},$$

where Δ_E is the Laplace-Beltrami operator on ∂E, A is a smooth tensor such that $A(\cdot,0,0) = 0$, and J is a smooth function.

In order to prove the stability of such flow, we need the following short-time existence result.

Theorem 4.2.7. *Let $\varepsilon > 0$, let $\beta \in (0,1)$ and let $E \subset \mathbb{T}^N$ be a smooth open set. There exists $\delta = \delta(\varepsilon,\beta,E) > 0$ with the following property: if E_0 is the normal deformation of E induced by $u_0 \in C^{1,1}(\partial E)$, $\|u_0\|_{C^{1,1}(\partial E)} \leq \delta$, and $|E_0| = |E|$, then there exists $T > 0$, which only depends on E, β and the bound on $\|u_0\|_{C^{1,1}(\partial E)}$, such that the volume-preserving mean curvature flow E_t starting from E_0 exists in $[0,T)$, the sets E_t are normal deformations of E induced by $u(\cdot,t) \in C^\infty(\partial E)$ for all $t \in (0,T)$, and*

$$\sup_{t\in(0,T)} \|u(\cdot,t)\|_{C^{1,\beta}(\partial E)} \leq \varepsilon. \tag{4.2.5}$$

Moreover, for every $k \in \mathbb{N}$, there exist two constants $c_k = c_k(N) > 0$ and $C_k = C_k(N,E) > 0$ such that

$$\sup_{t\in(0,T)} t^{c_k}\|\nabla^{k+2}u(\cdot,t)\|_{C^0(\partial E)} \leq C_k(\|u_0\|_{C^{1,1}(\partial E)} + 1). \tag{4.2.6}$$

We remark that the proof of this result is classical and can be derived from the Schauder estimates for quasi-linear parabolic equations, as u solves a lower-order, nonlinear perturbation of the heat equation. In the following subsection we will provide a brief outline of the proof for an analogous short-time existence result for the surface diffusion flow (see Theorem 4.2.17). Similar and simplified arguments would prove the previous result for the mean curvature flow, which is a second order flow.

For the sake of completeness, we provide here an alternative proof of Theorem 4.2.7 which follows from some results found in the literature. Even if these results are shown in the ambient space $\mathbb{R}^N$, the same arguments can be repeated in the flat torus. The first part of the theorem is the short-time existence result of [45].

Theorem 4.2.8 ([45, Main Theorem]). *Let $E \subset \mathbb{T}^N$ be a smooth open set and $\beta \in (0,1)$. There exists $\delta = \delta(E,\beta) > 0$ with the following property: if E_0 is the normal deformation of E induced by $u_0 \in C^{1,1}(\partial E)$, $\|u_0\|_{C^{1,1}(\partial E)} \leq \delta$, and $|E_0| = |E|$, then there exists $T > 0$, only depending on E, β and the bound on $\|u_0\|_{C^{1,1}(\partial E)}$, such that the volume-preserving mean curvature flow E_t starting from E_0 exists in $[0,T)$, and the sets E_t are normal deformations induced by $u(\cdot,t) \in C^\infty(\partial E)$ for all $t \in (0,T)$. Furthermore, the mapping $(t,E_0) \mapsto E_t$ is a local smooth semiflow on $C^{1,\beta}(E)$.*

We remark that the local smooth semiflow property in particular implies that $\|u(\cdot)\|_{C^{1,\beta}}$ depends continuously on $\|u_0\|_{C^{1,\beta}}$ (see for instance [7, pag. 66]). In particular, for every $\varepsilon > 0$ there exists $\delta(E,\varepsilon,\beta) > 0$ and $T(E,\varepsilon,\beta) > 0$ such that if $\|u_0\|_{C^{1,\beta}} \leq \delta$ then

$$\|u(\cdot,t)\|_{C^{1,\beta}} \leq \varepsilon \quad \text{for every } t \in (0,T). \tag{4.2.7}$$

In order to obtain the higher-order regularity inequalities, we apply some curvature estimates obtained recently in [66].

Theorem 4.2.9 ([66, Theorem 1.1]). *Assume that $E_0 \subset \mathbb{R}^N$ is an open bounded set satisfying a uniform inner and outer ball condition with radius r. Then, there exists a time $T = T(r,N) > 0$ such that the volume-preserving mean curvature flow E_t starting from E_0 exists in $[0,T)$ and it satisfies a uniform inner and outer ball condition of radius $r/2$. Moreover, it is smooth in $(0,T)$ and satisfies for every $k \in \mathbb{N}$*

$$\sup_{t\in(0,T)} \left(t^k \|H_{E_t}\|^2_{H^k(\partial E_t)}\right) \leq C_k, \tag{4.2.8}$$

where C_k depends on $k, |E_0|, r$.

Before proving the short time existence result, we remark a classical result concerning the uniform ball condition.

Remark 4.2.10. Let E be a smooth set satisfying a uniform ball condition of radius r_E. Then every small $C^{1,1}$-normal deformations of E satisfy a uniform ball condition of radius $r \approx r_E$. Indeed, it is easy to see that if E_f is the normal deformation of E induced by $f \in C^{1,1}(\partial E)$, then the Hausdorff distance between E and E_f is bounded by $\|f\|_{C^0(\partial E)}$. Furthermore, since $\nabla \mathrm{sd}_{E_f} = \nu_{E_f}$ and ν_{E_f} can be written as

$$\nu_{E_f} = \left(\nu_E - \sum_{i=1}^{N-1} \frac{\nabla f \cdot \nu_i}{1+\kappa_i f}\nu_i\right)\left(1 + \sum_{i=1}^{N-1} \frac{(\nabla f \cdot \nu_i)^2}{(1+\kappa_i f)^2}\right)^{-1/2},$$

where the family $\{v_i\}_{i=1,\dots,N-1}$ denotes an orthonormal frame of the tangent space on ∂E (see (5.2.5)), by expanding the above equation one can see that

$$\|\mathrm{sd}_{E_f} - \mathrm{sd}_E\|_{C^{1,1}(\partial E)} \le C_E \|f\|_{C^{1,1}(\partial E)},$$

which then implies that $E_f \to E$ in $C^{1,1}$ if $\|f\|_{C^{1,1}} \to 0$. Therefore, by [31, Theorem 2.6] and [31, Remark 2.7] one infers that the radius r of the uniform ball condition of the set E_f depends continuously on $\|f\|_{C^{1,1}}$ when it is small enough. In particular, for every $\varepsilon > 0$ there exists $\delta(r_E, \varepsilon) > 0$ such that, if $\|f\|_{C^{1,1}} \le \delta$ then

$$|r_E - r| \le \varepsilon.$$

Proof of Theorem 4.2.7. By Theorem 4.2.8 there exist a time $T' > 0$ and a family of evolving functions $u(\cdot, t)$, which are smooth in $(0, T')$ and satisfy the inequality (4.2.5). The second bound follows from classic elliptic regularity arguments that we now sketch. Fix $t \in (0, T')$, from the bound on $\sup_{t \in (0,T')} \|u\|_{C^{1,\beta}(\partial E)}$ and (up to rotations) for any given point $x = (x', x_N) \in \partial E$ we can parametrize in a cylinder $C = B'_r(x) \times (-L, L)$ both ∂E and ∂E_t as graphs of smooth functions g, g_t. From Theorem 4.2.9 there exists a time T'' (depending on E, δ by Remark (4.2.10)) such that the evolving sets E_t satisfy a uniform inner and outer ball condition of radius $r/2$ for any $t \in (0, T'')$. Let us set $T = \min\{T', T''\}$. From estimate (4.2.8) we get that

$$H_{E_t} = \operatorname{div}\left(\frac{\nabla g_t}{\sqrt{1 + |\nabla g_t|^2}}\right) = \frac{1}{\sqrt{1 + |\nabla g_t|}}\left(I - \frac{\nabla g_t \otimes \nabla g_t}{1 + |\nabla g_t|^2}\right) : \nabla^2 g_t$$

is bounded in $L^2(B'_r(x'))$ by a constant which depends on $|E_0|, T, r$. Then, by uniform geometric Calderon-Zygmund inequalities (see [36, Section 3] or [4, Lemma 7.2]) we deduce that, for some $\rho < r$, independent of x, in the ball $B'_\rho(x')$ the function g_t is bounded in $H^2(B'_\rho(x'))$ by a constant, depending only on the L^2-bound on H_{E_t}, the norm of the coefficients of the elliptic operator, which are in turn bounded by $\|u_0\|_{C^{1,1}}$ thanks to the previous step. Iterating this procedure, we bound the higher norms $H^k(B'_\rho(x'))$ of g_t, for every $k \in \mathbb{N}$. Then, we conclude by means of Sobolev embeddings and by a covering argument. □

4.2.2 Short-time existence for the surface diffusion flow

We now consider the evolution called *surface diffusion flow*, defined by

$$V_t(x) = \Delta_{E_t} H_{E_t}(x), \quad x \in \partial E_t,\ t \in (0, T). \tag{4.2.9}$$

As for the mean curvature flow, see Section 3.3, the equation above means that there exist a smooth open set $E \subset \mathbb{T}^N$ and a 1-parameter family of smooth diffeomorphism $\Phi_t : E \to \mathbb{T}^N$ such that $\Phi_t(x) = x + u(x,t)\nu_E(x)$, $\Phi_t(\partial E) = \partial E_t$ and

$$\partial_t u(x,t)\nu_E(x) \cdot \nu_{E_t}(\Phi_t(x)) = \Delta_{E_t} H_{E_t}(\Phi_t(x)).$$

Assuming that the diffeomorphisms above exist, arguing as in [76, pag. 21], one can deduce that the evolution equation satisfied by u is

$$\begin{aligned}\partial_t u =& -\Delta^2_{E_t} u - \frac{1}{\nu_E \cdot \nu_{E_t}} \Delta_{E_t}(\nu_E \cdot \nu_{E_t})\Delta_{E_t} u + \frac{1}{\nu_E \cdot \nu_{E_t}} \Delta_{E_t} P(x,u,\nabla u) \\ =& -\Delta^2_{E_t} u + \tilde{J}(x,u,\nabla u,\nabla^2 u,\nabla^3 u),\end{aligned} \tag{4.2.10}$$

where P is a smooth function (assuming that $\|u\|_{L^\infty}$ and $\|\nabla u\|_{L^\infty}$ are small), the function $\tilde{J}$ can be written as

$$\tilde{J}(x,u,\nabla u,\nabla^2 u,\nabla^3 u) = \langle \tilde{B}_1, \nabla^2 u\rangle + \langle \tilde{B}_2, \nabla^2 u \otimes \nabla^2 u\rangle + \langle \tilde{B}_3, \nabla^3 u\rangle + \tilde{b}_4$$

and $\tilde{B}_1, \tilde{B}_2, \tilde{B}_3$ and $\tilde{b}_4$ are tensor-valued, respectively scalar-valued functions depending on $(x,u,\nabla u)$ and smooth if their arguments are small enough. Here ∇ denote the covariant derivative on ∂E.

On the other hand, linearizing the Laplace-Beltrami operator yields the evolution equation (compare with [51, Section 3.1])

$$\partial_t u = -\Delta^2_E u + \langle A(x,u,\nabla u), \nabla^4 u\rangle + J(x,u,\nabla u,\nabla^2 u,\nabla^3 u), \tag{4.2.11}$$

where A is a smooth 4th-order tensor, vanishing when both u and ∇u vanish, and J is given by

$$\begin{aligned}J =& \langle B_1, \nabla^3 u \otimes \nabla^2 u\rangle + \langle B_2, \nabla^3 u\rangle + \langle B_3, \nabla^2 u \otimes \nabla^2 u \otimes \nabla^2 u\rangle \\ &+ \langle B_4, \nabla^2 u \otimes \nabla^2 u\rangle + \langle B_5, \nabla^2 u\rangle + b_6,\end{aligned} \tag{4.2.12}$$

where B_i, $i = 1, \dots 5$ and b_6 are smooth tensor-valued, respectively scalar-valued functions depending on $(x,u,\nabla u)$.

In this subsection we want to prove a short-time existence result for the surface diffusion flow, in particular we will obtain a priori estimates that will be used to prove the stability of the flow. We will follow the classical approach of linearization and fixed point to solve the nonlinear evolution problem, and then employ Schauder-type estimates to show higher order regularity of the flow. We will follow closely what has been done in [51], combining it with the results of [58].

To start we recall some classical results concerning the Cauchy problem for the biharmonic heat equation on a smooth Riemannian manifold Σ with metric g, which is the solution to the following problem

$$\begin{cases} \partial_t u = -\Delta_\Sigma^2 u + f(x,t) & \text{on } \Sigma \times [0,\infty) \\ u(\cdot,0) = u_0 & \text{on } \Sigma, \end{cases} \tag{4.2.13}$$

once the functions f, u_0 are assigned.

Theorem 4.2.11 ([49, Theorem 2]). *Given* (Σ, g) *a smooth Riemannian manifold, there exists a unique biharmonic heat kernel with respect to g denoted as* $b_g \in C^\infty(\Sigma \times \Sigma \times (0,\infty))$. *Moreover let* $T > 0$, *for any integers* $k, p, q \geq 0$ *and for any* $(x,y,t) \in \Sigma \times \Sigma \times (0,T)$ *we have*

$$|\partial_t^k \nabla_x^p \nabla_y^q b_g(x,y,t)|_g \leq Ct^{-\frac{n+4k+p+q}{4}} \exp\{-\delta\big(t^{-\frac{1}{4}} d_g(x,y)\big)^{\frac{4}{3}}\}, \tag{4.2.14}$$

where $|\cdot|_g = \sqrt{g(\cdot,\cdot)}$, ∇_x *and* ∇_y *are covariant derivatives with respect to g, and the constants* $C, \delta > 0$ *depend on T, g and* $p + q + 4k$.

Given the biharmonic heat kernel $b_g \in C^\infty(\Sigma \times \Sigma \times (0,\infty))$ on (Σ, g) and a function $u_0 \in C^0(\Sigma)$, we define for $(x,t) \in \Sigma \times (0,\infty)$

$$Su_0(x,t) = \int_\Sigma b_g(x,y,t) u_0(y) \mathrm{d}V_g(y) \tag{4.2.15}$$

where V_g is the Riemannian volume form. Hence, as usual, Su_0 is the solution to the homogeneous Cauchy problem

$$\begin{cases} \partial_t v + \Delta_\Sigma^2 v = 0 & \text{on } \Sigma \times (0,+\infty) \\ v(\cdot,0) = u_0(\cdot) & \text{on } \Sigma. \end{cases} \tag{4.2.16}$$

Moreover, since the biharmonic heat kernel is smooth for every $t > 0$, we get $Su_0 \in C^\infty(\Sigma \times (0,+\infty))$.

We now collect some results, which are shown in [58], about the solution of (4.2.13). The following Schauder-type estimates on the solution of the homogeneous problem (4.2.16) can then be proved, see [58, Theorem 3.8]. In particular, we modify slightly the formulation of the result, to fit our purposes.

Theorem 4.2.12 ([58, Theorem 3.8]). *Suppose* $u_0 \in C^{1,1}(\Sigma)$ *and fix* $T > 0$. *Then there exists* $C_1(\Sigma, T) > 0$ *such that*

$$\sup_{t \in (0,T)} \|Su_0\|_{C^{1,1}(\Sigma)} \leq C_1 \|u_0\|_{C^{1,1}(\Sigma)}, \tag{4.2.17}$$

Furthermore, for any $l, k \in \mathbb{N}$, *we have*

$$\sup_{t\in(0,T)} t^{l+\frac{k}{4}} \left\| (\partial_t)^l \nabla_g^{k+2} S u_0(t) \right\|_{C^0(\Sigma)} \le C_{l,k} \|u_0\|_{C^{1,1}(\Sigma)}, \tag{4.2.18}$$

for some constants $C_{l,k} > 0$ *depending on* l, k, Σ *and* T.

In order to study the evolution problem (4.2.11) we introduce the following two Banach spaces. Fix $0 < T < \infty$ and $0 < \beta < 1$. We define

$$Y_T := \left\{ u \in C^0\big(\Sigma \times (0,T)\big) : \|u\|_{Y_T} < \infty \right\}, \tag{4.2.19}$$

where

$$\begin{aligned} \|u\|_{Y_T} := & \sup_{t\in(0,T)} \left(t^{\frac{1}{2}} \|u(\cdot,t)\|_{C^0(\Sigma)} + t^{\frac{1}{2}+\frac{\beta}{4}} [u(\cdot,t)]_{C^\beta(\Sigma)} \right) \\ & + \sup_{(x,t)\in\Sigma\times(0,T)} \sup_{0<h<T-t} t^{\frac{1}{2}+\frac{\beta}{4}} \frac{|u(x,t+h)-u(x,t)|}{|h|^{\frac{\beta}{4}}} \end{aligned} \tag{4.2.20}$$

and $[\cdot]_{C^\beta}$ is the usual Hölder seminorm. Similarly, we introduce the space

$$X_T := \left\{ u \in C^0(\Sigma \times (0,T)) : u(\cdot,t) \in C^4(\Sigma),\ \|u\|_{X_T} < \infty \right\}, \tag{4.2.21}$$

where

$$\begin{aligned} \|u\|_{X_T} := & \sup_{t\in(0,T)} \Bigg(\sum_{k=0}^{4} t^{-\frac{1}{2}+\frac{k}{4}} \|\nabla^k u(\cdot,t)\|_{C^0(\Sigma)} + t^{\frac{1}{2}+\frac{\beta}{4}} [\nabla^4 u(\cdot,t)]_{C^\beta(\Sigma)} \\ & + t^{\frac{1}{2}} \|\partial_t u(\cdot,t)\|_{C^0(\Sigma)} + t^{\frac{1}{2}+\frac{\beta}{4}} [\partial_t u(\cdot,t)]_{C^\beta(\Sigma)} \Bigg) \\ & + \sup_{(x,t)\in\Sigma\times(0,T)} \sup_{0<h<T-t} t^{\frac{1}{2}+\frac{\beta}{4}} \frac{|\nabla^4 u(x,t+h)-\nabla^4 u(x,t)|_g}{|h|^{\frac{\beta}{4}}} \\ & + \sup_{(x,t)\in\Sigma\times(0,T)} \sup_{0<h<T-t} t^{\frac{1}{2}+\frac{\beta}{4}} \frac{|\partial_t u(x,t+h)-\partial_t u(x,t)|}{|h|^{\frac{\beta}{4}}}. \end{aligned} \tag{4.2.22}$$

Proposition 4.2.13. *The spaces* $(Y_T, \|\cdot\|_{Y_T})$ *and* $(X_T, \|\cdot\|_{X_T})$ *are Banach spaces.*

The proof of the completeness of the spaces Y_T and X_T is standard, indeed one can prove directly that all Cauchy sequence converge to a function in the space and the candidate limit is obtained using a diagonal argument.

Remark 4.2.14. Since the norm $\sum_{k=0}^4 \|\nabla^k u\|_{C^0}$ is equivalent to the norm $\|u\|_{C^0} + \|\nabla^4 u\|_{C^0}$ for $C^4(\Sigma)$, we have that the norm $\|\cdot\|_{X_T}$ defined in (4.2.22) is equivalent to the following norm

$$\|u\|'_{X_T} := \|u\|_{X_T} + \sum_{k=0}^{3} \sup_{(x,t)\in\Sigma\times(0,T)} \sup_{0<h<T-t} t^{-\frac{1}{2}+\frac{k}{4}+\frac{\beta}{4}} \frac{|\nabla^k u(x,t+h) - \nabla^k u(x,t)|_g}{|h|^{\frac{\beta}{4}}}.$$

Now we study the nonhomogeneous initial value problem

$$\begin{cases} \partial_t u + \Delta_\Sigma^2 u = f & \text{on } \Sigma\times(0,T) \\ u(\cdot,0) = 0 & \text{on } \Sigma, \end{cases} \tag{4.2.23}$$

where f is a function on $\Sigma\times(0,T)$. Given the biharmonic heat kernel $b_g \in C^\infty\big(\Sigma\times\Sigma\times(0,T)\big)$ on (Σ,g), the solution (if it exists) to the nonhomogeneous problem (4.2.23) should be given by Duhamel's principle

$$Vf(x,t) := \int_0^t \int_\Sigma b_g(x,y,t-s) f(y,s)\,\mathrm{d}V_g(y)\,\mathrm{d}s, \tag{4.2.24}$$

and, for every $\lambda > 0$, $Vf \in C^\infty(\Sigma\times(\frac{\lambda}{2},\lambda))$.

We then recall the following fundamental Schauder-type estimates proved in [58] on solutions of (4.2.23) (see [58, Remark 3.12] for the final comments on the constant C).

Theorem 4.2.15 ([58, Theorem 3.10]). *Fix $0<T<\infty$, if $f\in Y_T$, then $Vf\in X_T$ and there exists a constant $C>0$ depending on Σ, T such that*

$$\|Vf\|_{X_T} \le C\|f\|_{Y_T}. \tag{4.2.25}$$

Moreover, equation $(\partial_t + \Delta_\Sigma^2)Vf = f$ holds in the classical sense on $\Sigma\times(0,T)$ and thus $Vf\in C^\infty(\Sigma\times(0,T))$.

We now turn our attention to the evolution equation (4.2.11), and use the results above for the particular choice $\Sigma = \partial E$ with the Riemaniann metric induced by the Euclidean one. We consider the map

$$f[u](x) := \langle A(x,u,\nabla u), \nabla^4 u\rangle + J(x,u,\nabla u,\nabla^2 u,\nabla^3 u), \tag{4.2.26}$$

where A, J are the operators defined in (4.2.11). We now provide the fundamentals estimates on $f[u]$, which represents the nonlinear error generated linearizing (4.2.11).

Lemma 4.2.16. *For any ε, $m > 0$ there exist T, $\delta > 0$ depending on E, ε with the following properties. For every $u_0 \in C^{1,1}(\Sigma)$ and $\psi \in X_T$ satisfying $\|\psi\|_{X_T} \leq m$ it holds*

$$f[\psi + Su_0] \in Y_T. \tag{4.2.27}$$

Moreover, if $\|u_0\|_{C^{1,1}(\Sigma)} \leq \delta$ it holds

$$\|f[Su_0]\|_{Y_T} \leq \varepsilon(\|u_0\|_{C^{1,1}(\Sigma)} + 1). \tag{4.2.28}$$

Finally, $\psi_1, \psi_2 \in X_T$ satisfying $\|\psi_i\|_{X_T} \leq m$, it holds

$$\|f[\psi_1 + Su_0] - f[\psi_2 + Su_0]\|_{Y_T} \leq \varepsilon\|\psi_1 - \psi_2\|_{X_T}. \tag{4.2.29}$$

Proof. Let $T < 1$ to be chosen later and fic $\varepsilon, m > 0$. We prove only equation (4.2.28), giving a sketch of the proof for (4.2.29) and (4.2.27) as they are analogous; we also drop the dependence on the set E in the norms. For clarity of exposition, we prove the results for the simplified error term

$$\tilde{f}[u](x,t) := \langle A(x, u(x,t), \nabla u(x,t)), \nabla^4 u(x,t)\rangle + \langle B, \nabla^3 u(x,t) \otimes \nabla^2 u(x,t)\rangle, \tag{4.2.30}$$

where B is a (constant) tensor of the same dimension of $\nabla^3 u \otimes \nabla^2 u$ with $\|B\| < 1$. We will also write $A(x,t)$ and assume implicitly the dependence on $u, \nabla u$.

Firstly, we prove (4.2.28). In what follows we use the short-hand notation $u = Su_0$. From the definition of $\tilde{f}[\cdot]$ we have

$$\begin{aligned}
&\|\tilde{f}[u]\|_{C^0} \leq \|A\|_{C^0}\|\nabla^4 u\|_{C^0} + \|\nabla^3 u\|_{C^0}\|\nabla^2 u\|_{C^0},\\
&[\tilde{f}[u]]_{C^\beta} \leq \|\nabla^4 u\|_{C^0} \sup_{\tau \in \mathbb{T}^N}\Big(|\tau|^{-\beta}|A(x+\tau,t) - A(x,t)|\Big) + \|A\|_{C^0}[\nabla^4 u]_{C^\beta}\\
&\qquad + [\nabla^3 u]_{C^\beta}\|\nabla^2 u\|_{C^0} + \|\nabla^3 u\|_{C^0}[\nabla^2 u]_{C^\beta}.
\end{aligned} \tag{4.2.31}$$

Then, we multiply by $t^{\frac{1}{2}}$ the first equation in (4.2.31) to get

$$t^{\frac{1}{2}}\|\tilde{f}[u]\|_{C^0} \leq \|A\|_{C^0} t^{\frac{1}{2}}\|\nabla^4 u\|_{C^0} + t^{\frac{1}{4}} t^{\frac{1}{4}}\|\nabla^3 u\|_{C^0}\|\nabla^2 u\|_{C^0}.$$

By (4.2.18), with the choice of $l = 0$, $k = 0,1,2$, we have that all the terms $t^{\frac{1}{2}}\|\nabla^4 u\|_{C^0}$, $t^{\frac{1}{4}}\|\nabla^3 u\|_{C^0}$ and $\|\nabla^2 u\|_{C^0}$ are bounded by $\|u\|_{C^{1,1}}$ (times a constant that depends on E which we can suppose equal to one for simplicity). We now fix $\delta > 0$ sufficiently small, depending on ε and E, so that $\|A\|_{C^0}$ is bounded by ε, which can be done since A is a smooth tensor and $A(\cdot, 0, 0) = 0$. Finally, taking T small enough, depending on ε and E, we conclude

$$\sup_{t\in(0,T)} t^{\frac{1}{2}}\|\tilde{f}[u]\|_{C^0} \leq \varepsilon\|u_0\|_{C^{1,1}}.$$

Therefore, taking into account the full expression for the error term $f[u]$ given by (4.2.26), one can show that

$$\sup_{t\in(0,T)} t^{\frac{1}{2}}\|f[u]\|_{C^0} \leq C\varepsilon\left(\|u_0\|_{C^{1,1}}+1\right),$$

where the last constant comes from the term b_6.

Concerning the Hölder seminorm in space, we first remark that

$$\sup_{\tau\in\mathbb{T}^N}\frac{|A(x+\tau,t)-A(x,t)|}{|\tau|^{\beta}} \leq [A(\cdot,u,\nabla u)]_{C^\beta}+\|\partial_2 A\|_{C^0}[u]_{C^\beta}+\|\partial_3 A\|_{C^0}[\nabla u]_{C^\beta},$$

where $\partial_2 A$ and $\partial_3 A$ denote the derivative of $A(x,y,z)$ with respect to the second and third components. Therefore, employing again the bounds in (4.2.17) and (4.2.18) we can bound

$$t^{\frac{1}{2}}\|\nabla^4 u\|_{C^0}\sup_{\tau}\frac{|A(x+\tau,t)-A(x,t)|}{|\tau|^{\beta}} \leq \varepsilon\|u_0\|_{C^{1,1}},$$

where we took $\delta>0$ sufficiently small, depending on ε and E, such that

$$[A(\cdot,u,\nabla u)]_{C^\beta}+\|\partial_2 A\|_{C^0}[u]_{C^\beta}+\|\partial_3 A\|_{C^0}[\nabla u]_{C^\beta} \leq \varepsilon,$$

which is possible since A is smooth and $A(\cdot,0,0)=0$. Thus, multiplying by $t^{\frac{1}{2}+\frac{\beta}{4}}$ the second equation in (4.2.31) we obtain

$$\begin{aligned} t^{\frac{1}{2}+\frac{\beta}{4}}[\tilde{f}[u]]_{C^\beta} \leq{}& t^{\frac{\beta}{4}}\varepsilon\|u_0\|_{C^{1,1}}+\|A\|_{C^0}t^{\frac{1}{2}+\frac{\beta}{4}}[\nabla^4 u]_{C^\beta} \\ &+t^{\frac{1}{4}}t^{\frac{1}{4}+\frac{\beta}{4}}\|\nabla^3 u\|_{C^\beta}\|\nabla^2 u\|_{C^0}+t^{\frac{1}{4}}t^{\frac{1}{4}}\|\nabla^3 u\|_{C^0}t^{\frac{\beta}{4}}\|\nabla^2 u\|_{C^\beta}. \end{aligned} \tag{4.2.32}$$

Then, all the terms in (4.2.32) with the norms of u can be bounded employing (4.2.17) and (4.2.18), thus we can make the right-hand side above as small as needed taking T,δ small enough. Analogous calculations show a similar inequality for the complete error term $f[u]$.

Finally, we show how to bound the Hölder seminorm in time appearing in $\|\tilde{f}[u]\|_{Y_T}$. We fix $t\in(0,T), h\in(0,T-t)$. To ease notation, we omit to write the evaluation at x in the following. We have by the very definition of $\tilde{f}[u](t)$ that

$$\begin{aligned}&|\tilde{f}[u](t+h)-\tilde{f}[u](t)|\\&\quad\leq|\langle A(u(t+h),\nabla u(t+h)),\nabla^4u(t+h)\rangle-\langle A(u(t),\nabla u(t)),\nabla^4u(t)\rangle|\\&\quad+|\langle B,\big(\nabla^3u(t+h)\otimes\nabla^2u(t+h)\big)\rangle-\langle B,\big(\nabla^3u(t)\otimes\nabla^2u(t)\big)\rangle|.\end{aligned}$$

Now by the triangular inequality we obtain

$$\begin{aligned}&|\langle A(u(t+h),\nabla u(t+h)),\nabla^4u(t+h)\rangle-\langle A(u(t),\nabla u(t)),\nabla^4u(t)\rangle|\\&\leq\|A\|_{C^0}|\nabla^4u(t+h)-\nabla^4u(t)|+\|\partial_3A\|_{C^0}|\nabla u(t+h)-\nabla u(t)|\|\nabla^4u(t)\|_{C^0}\\&+\|\partial_2A\|_{C^0}|u(t+h)-u(t)|\|\nabla^4u\|_{C^0},\end{aligned}\tag{4.2.33}$$

and analogously

$$\begin{aligned}&|\langle B,\big(\nabla^3u(t+h)\otimes\nabla^2u(t+h)\big)\rangle-\langle B,\big(\nabla^3u(x,t)\otimes\nabla^2u(x,t)\big)\rangle|\\&\leq|\nabla^3u(t+h)-\nabla^3u(t)|\|\nabla^2u\|_{C^0}+\|\nabla^3u\|_{C^0}|\nabla^2u(t+h)-\nabla^2u(t)|.\end{aligned}\tag{4.2.34}$$

Therefore from formulas (4.2.33) and (4.2.34), we obtain

$$\begin{aligned}&|\tilde{f}[u](t+h)-\tilde{f}[u](t)|\\&\qquad\leq(\|\partial_2A\|_{C^0}|u(t+h)-u(t)|+\|\partial_3A\|_{C^0}|\nabla u(t+h)-\nabla u(t)|)\,\|\nabla^4u(t)\|_{C^0}\\&\qquad+\|A\|_{C^0}|\nabla^4u(t+h)-\nabla^4u(t)|+|\nabla^3u(t+h)-\nabla^3u(t)|\|\nabla^2u\|_{C^0}\\&\qquad+\|\nabla^3u\|_{C^0}|\nabla^2u(t+h)-\nabla^2u(t)|.\end{aligned}$$

Applying again (4.2.17), (4.2.18), and using the smallness of $\|A\|_{C^0}$, we conclude (4.2.28) by taking $T,\ \delta$ small enough.

Following the computations above one can easily prove that if $u_0\in C^{1,1}(\Sigma)$ and $\|\psi\|_{X_T}\leq m$, it holds

$$f[\psi+Su_0]\in Y_T.$$

The only difference is that, in addition to (4.2.17), (4.2.18) one can directly exploit the definition of $\|\cdot\|_{X_T}$ to obtain the required bounds. Also the proof for (4.2.29) is essentially the same, only much more tedious to write. We show the computations only for the term $\sup_{t\in(0,T)}t^{1/2}\|\cdot\|_{C^0}$ appearing in the norm of Y_T and for the simplified error term (4.2.30). For $u_i:=\psi_i+Su_0$ we can write

$$\begin{aligned}&|\tilde{f}[u_1]-\tilde{f}[u_2]|\\&=\big|\langle A(x,u_1,\nabla u_1),\nabla^4u_1\rangle-\langle A(x,u_2,\nabla u_2),\nabla^4u_2\rangle+\langle B,(\nabla^3u_1\otimes\nabla^2u_1-\nabla^3u_2\otimes\nabla^2u_2)\rangle\big|\\&\leq\|\nabla^4u_1\|_{C^0}(\|\partial_1A\|_{C^0}|\psi_1-\psi_2|+\|\partial_2A\|_{C^0}|\nabla\psi_1-\nabla\psi_2|)+\|A\|_{C^0}|\nabla^2\psi_1-\nabla^2\psi_2|\end{aligned}$$

$$+\|\nabla^3 u_1\|_{C^0}|\nabla^2\psi_1-\nabla^2\psi_2|+\|\nabla^2 u_2\|_{C^0}|\nabla^3\psi_1-\nabla^3\psi_2|.$$

Multiplying the inequality above by $t^{\frac{1}{2}}$ we have

$$\begin{aligned}
&t^{\frac{1}{2}}|\tilde{f}[u_1]-\tilde{f}[u_2]|\\
&\leq \Big(\|\nabla^4 u_1\|_{C^0}\Big(t\|\partial_1 A\|_{C^0}+t^{\frac{3}{4}}\|\partial_2 A\|_{C^0}\Big)+t^{\frac{1}{2}}\left(\|A\|_{C^0}+\|\nabla^3 u_1\|_{C^0}\right)\\
&\ +t^{\frac{1}{4}}\|\nabla^2 u_2\|_{C^0}\Big)\|\psi_1-\psi_2\|_{X_T}\\
&\leq t^{\frac{1}{4}}\Big(t^{\frac{1}{2}}\|\nabla^4 u_1\|_{C^0}\|A\|_{C^1}+\|A\|_{C^0}+t^{\frac{1}{4}}\|\nabla^3 u_1\|_{C^0}+|\nabla^2 u_2\|_{C^0}\Big)\|\psi_1-\psi_2\|_{X_T}.
\end{aligned}$$

Again, by definition of $\|\cdot\|_{X_T}$ and by (4.2.17),(4.2.18) we conclude taking $T,\ \delta$ small enough. $\square$

We are now able to prove a short-time existence result for the surface diffusion evolution. Thanks to the previous lemmas, we provide also higher order regularity estimates depending on the $C^{1,1}$-bound on the initial datum only. The proof follows closely the corresponding one in [58, 51].

Theorem 4.2.17. *Let $\varepsilon>0$ and let $E\subset\mathbb{T}^N$ be a smooth open set. There exist $\delta=\delta(\varepsilon,E)$, $T=T(\varepsilon,E)>0$ with the following property: if E_0 is the normal deformations of E induced by $u_0\in C^{1,1}(\partial E)$, $\|u_0\|_{C^{1,1}(\partial E)}\leq\delta$, and $|E_0|=|E|$, then the surface diffusion flow E_t starting from E_0 exists in $[0,T)$, the sets E_t are normal deformation of E induced by $u(\cdot,t)\in C^\infty(\partial E)$ for all $t\in(0,T)$, and*

$$\sup_{t\in(0,T)}\|u\|_{C^2(\partial E)}\leq\varepsilon. \tag{4.2.35}$$

Moreover, for every $k\in\mathbb{N}\setminus\{0\}$, there exist constants $C_k=C_k(\varepsilon,E)>0$ such that

$$\sup_{t\in[\frac{T}{2},T)}\|\nabla^{k+2}u\|_{C^0(\partial E)}\leq C_k(\|u_0\|_{C^{1,1}(\partial E)}+1). \tag{4.2.36}$$

Proof. In this proof we denote by $C>0$ a constant that depends on N and E and may change from line to line. Fix $\varepsilon>0$.

Step 1: We show existence for (4.2.11) via a fixed point argument. Let $T<1$, $\delta<1$ to be chosen later, and let $u_1\in C^\infty((0,T);C^\infty(\partial E))$ be the solution of

$$\begin{cases}\partial_t u_1=-\Delta_E^2 u_1 & \text{on } \partial E\times[0,T),\\ u_1(\cdot,0)=u_0 & \text{on } \partial E,\end{cases}$$

where $u_0\in C^{1,1}(\partial E)$ is such that $\|u_0\|_{C^{1,1}(\partial E)}\leq\delta$. The solution exists and it is given by (4.2.15), that is $u_1=0+Su_0=:\psi_1+Su_0$. Moreover (4.2.35) and (4.2.36) are satisfied by

u_1 thanks to Theorem 4.2.12, for δ small enough depending on ε. Let now u_2 be the solution of

$$\begin{cases} \partial_t u_2 = -\Delta_E^2 u_2 + f[u_1] & \text{on } \partial E \times [0,T), \\ u_2(\cdot,0) = u_0 & \text{on } \partial E, \end{cases}$$

where $f[u]$ is defined as in (4.2.26). By (4.2.15) and (4.2.24), the unique solution is given by $u_2 = Vf[u_1] + Su_0 = Vf[Su_0] + Su_0 =: \psi_2 + Su_0$. Moreover, by Theorem 4.2.15 and (4.2.28) we have the estimate

$$\|\psi_2\|_{X_T} \le C\|f[Su_0]\|_{Y_T} \le C\varepsilon(\|u_0\|_{C^{1,1}(\partial E)} + 1) \le m,$$

for m sufficiently large. We are then led to define an iterative scheme. We set u_1, u_2 as above and for $n \ge 3$ we let u_n be the solution to

$$\begin{cases} \partial_t u_n = -\Delta_E^2 u_n + f[u_{n-1}] & \text{on } \partial E \times [0,T), \\ u_n(\cdot,0) = u_0 & \text{on } \partial E, \end{cases} \tag{4.2.37}$$

and we split it as $u_n = Su_0 + Vf[u_{n-1}] =: \psi_n + Su_0$. We will show that the sequence ψ_n is converging in X_T. To do so, assume that $\psi_j \in X_T$ for $j = 1, \dots, n-1$ with

$$\|\psi_j\|_{X_T} \le m.$$

Then, by Theorem 4.2.15 and Lemma 4.2.16 we get $\psi_n \in X_T$ and

$$\begin{aligned} \|\psi_n\|_{X_T} = \|Vf[u_{n-1}]\|_{X_T} &\le C\|f[u_{n-1}]\|_{Y_T} = C\|f[\psi_{n-1} + Su_0]\|_{Y_T} \\ &\le C\sum_{j=2}^{n-1} \|f[\psi_j + Su_0] - f[\psi_{j-1} + Su_0]\|_{Y_T} + C\|f[Su_0]\|_{Y_T} \\ &\le C\Big(\sum_{j=1}^{n-1} \varepsilon^j\Big)(\|u_0\|_{C^{1,1}(\partial E)} + 1) \\ &\le C\varepsilon\Big(1 + \sum_{j=1}^{+\infty} \varepsilon^j\Big)(\|u_0\|_{C^{1,1}(\partial E)} + 1) \\ &\le C\varepsilon(\|u_0\|_{C^{1,1}(\partial E)} + 1) \le m. \end{aligned} \tag{4.2.38}$$

Moreover, Lemma 4.2.16 implies that, for $\delta(\varepsilon, E)$, $T(\varepsilon, E)$ small enough, it holds for all $n \ge 3$

$$\|\psi_{n+1} - \psi_n\|_{X_T} \le \varepsilon\|\psi_n - \psi_{n-1}\|_{X_T},$$

therefore ψ_n is a Cauchy sequence and admits a limit point ψ satisfying

$$\|\psi\|_{X_T} \leq C\varepsilon(\|u_0\|_{C^{1,1}(\partial E)} + 1). \tag{4.2.39}$$

We thus showed the existence of a fixed point $u = \psi + Su_0$ for the problem (4.2.37). Finally, by (4.2.17) and (4.2.39) it holds

$$\|u\|_{C^2(\partial E)} = \|\psi + Su_0\|_{C^2(\partial E)} \leq \|\psi\|_{X_T} + \|Su_0\|_{C^2(\partial E)} \leq C\varepsilon(\|u_0\|_{C^{1,1}(\partial E)} + 1). \tag{4.2.40}$$

Step 2: By (4.2.40) we get straightforwardly that (4.2.36) holds for $k = 0,1,2$. In order to prove (4.2.36) for $k \geq 3$, we consider $x \in \partial E$ and we work under local coordinate, $B'_r \cong U \subset \partial E$ such that the metric $(g^{ij})_{i,j=1,\dots,N-1}$ of ∂E satisfies $\frac{1}{2}\delta_{ij} \leq g_E^{ij} \leq 2\delta_{ij}$. Note in particular that the operator $-\Delta_E^2$ is uniformly elliptic in U. In the following we identify B'_r and $U \subset \partial E$. We also set g_t as the metric on ∂E_t (see [76, pag. 20] for details). Observe that u restricted to $B'_r \times [\frac{T}{2}, T)$ is of class C^∞ by the previous step. Recalling that $u = \psi + Su_0$, we have that the function ψ satisfies

$$\partial_t \psi = -\Delta_{g_t}^2 \psi + (\partial_t + \Delta_{g_t}^2)(Su_0) + f' =: -\Delta_{g_t}^2 \psi + \tilde{f}. \tag{4.2.41}$$

Taking ∇_g in (4.2.41) shows that the function $\nabla_g \psi$ satisfies the equation

$$\begin{aligned} \partial_t \nabla_g \psi &= -\Delta_{g_t}^2 \nabla_g \psi - (\nabla_g g_t^{ij}) g_t^{kl} (\psi)_{ijkl} - g_t^{ij} (\nabla_g g_t^{kl})(\psi)_{ijkl} + \nabla_g \tilde{f} \\ &=: -\Delta_{g_t}^2 \nabla_g \psi + F, \end{aligned} \tag{4.2.42}$$

where the error term F contains the derivative of ψ up to order four. To estimate $\|F\|_{C^{\beta/4}([\frac{T}{2},T];C^\beta(B'_r))}$ we first observe that, by (4.2.18), it follows

$$\|\nabla_g \left((\partial_t + \Delta_{g_t}^2)(Su_0)\right)\|_{C^{\beta/4}([\frac{T}{2},T);C^\beta(B'_1))} \leq C\varepsilon(\|u_0\|_{C^{1,1}(\partial E)} + 1).$$

Secondly, we remark that the other terms of F can be bounded analogously, recalling that they contain derivatives of ψ up to order four and using (4.2.39), to show that

$$\|F\|_{C^{\beta/4}([\frac{T}{2},T);C^\beta(B'_r))} \leq C\varepsilon(\|u_0\|_{C^{1,1}(\partial E)} + 1). \tag{4.2.43}$$

Note now that $\partial_t + \Delta_{g_t}^2$ is a uniformly parabolic operator, since the coefficients of $\Delta_{g_t}^2$ are close to the ones of Δ_E^2 depending on $\|u(\cdot,t)\|_{C^{1,1}(\partial E)}$ as $g_{E_u}^{ij} - g_E^{ij} = B(x,u,\nabla u)$ and B is a smooth function with $B(x,0,0) = 0$, see again [76, pag. 20]. Since $\nabla_g \psi$ solves (4.2.42), by the standard interior Schauder estimates and the bound (4.2.43), there exists $C > 0$, which depends on T and thus on ε and E, such that

$$\begin{aligned}\|\nabla_g \psi\|_{C^{1,\beta/4}([\frac{T}{2},T);C^{4,\beta}(B'_{r/2}))} &\le C\left(\|F\|_{C^{\beta/4}([\frac{T}{4},T);C^{\beta}(B'_r))} + \|\nabla_g \psi\|_{C^0(B'_r\times[\frac{T}{4},T))}\right)\\ &\le C\varepsilon(\|u_0\|_{C^{1,1}(\partial E)}+1),\end{aligned}$$

where we noted that $\|\psi\|_{C^1((B'_r\times[\frac{T}{4},T)))} \le \|\psi\|_{X_T}$ and employed again (4.2.39). Finally, we conclude

$$\sup_{t\in[\frac{T}{2},T)} \|\nabla^5 u\|_{C^0(\partial E)} \le C(\|u_0\|_{C^{1,1}(\partial E)}+1).$$

By induction, one can prove (4.2.36) for every $k \in \mathbb{N}$. □

4.3 Proof of the stability

4.3.1 Stability of the volume-preserving mean curvature flow

In this subsection, we study the evolution by mean curvature (3.3.1) of normal deformations of a strictly stable set, as defined in Definition 3.1.6. Suppose that E is a strictly stable set and that $E_0 = E_{u_0}$ is a smooth normal deformation of E. By Theorem 4.2.7, the volume-preserving mean curvature flow starting from E_0 exists in a short time interval, and the evolving sets E_t can be parametrized as normal deformations of the set E induced by functions $u(\cdot,t)$ satisfying

$$\begin{cases} u_t(x,t)\nu_{E_t}(p)\cdot\nu_E(x) = -\left(H_{E_t}(p) - \bar{H}_{E_t}\right) & x\in\partial E,\\ u(\cdot,0) = u_0 \end{cases}$$

where $p = x + u(x,t)\nu_E(x)$ and $\bar{H}_{E_t} = \fint_{\partial E_t} H_{E_t}$. The scalar product above (see (5.2.6)) can be written as

$$\nu_{E_t}(p)\cdot\nu_E(x) = \left(1+\sum_{j=1}^{N-1}\frac{(\partial_{\tau_j}u(x,t))^2}{(1+\kappa_j(x)u(x,t))^2}\right)^{-1/2},$$

where $\kappa_j(x)$ and $\tau_j(x)$ are, respectively, the principal curvatures and the principal directions of E at x. In particular, we remark that $\nu_{E_t}(p)\cdot\nu_E(x) = 1 + O(\|u(\cdot,t)\|_{H^1})$. We can prove the first part of Theorem 4.1.1 concerning the long-time behaviour of the volume-preserving mean curvature flow.

Proof of (i) *Theorem 4.1.1.* Let $\varepsilon,\ \delta(\varepsilon)\in(0,1)$ to be chosen later. In the following, if not otherwise stated, the constants depends on N, E and may change from line to line. Fix for instance $\beta = 1/2$ and suppose that δ is smaller than the constant given by

Theorem 4.2.7. We also use the short-hand notation $\pi_f := (\pi_E|_{E_f})^{-1}$.
Step 1. We start by proving that $P(E_t) - P(E) \le Ce^{-ct}$ as long as the flow exists.

Let $u_0 \in C^{1,1}(\partial E)$ with $\|u_0\|_{C^{1,1}} \le \delta < 1$. By Theorem 4.2.7 there exist a time $T > 0$, which depends on E and the bound on $\|u_0\|_{C^{1,1}} < 1$, and a smooth flow E_t starting from E_0 for $t \in [0,T)$. Moreover, $E_t = E_{u(\cdot,t)}$ and $u(\cdot,t)$ satisfies (4.2.5) and (4.2.6). Without loss of generality we can assume $T < \infty$.
We notice that, considering ε, δ smaller, the value of T does not change.

We recall the following well-known identities, holding along the smooth flow

$$\frac{\mathrm{d}}{\mathrm{d}t}|E_t| = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}P(E_t) = -\|H_{E_t} - \bar{H}_{E_t}\|^2_{L^2(\partial E_t)}. \tag{4.3.1}$$

Let δ^* be the constant given by Theorem 4.2.3, $p > N-1$ and $\eta = \eta(\delta^*, p)$ given by Lemma 3.1.10. By estimates (4.2.5), (4.2.6) and by interpolation we have that $\|u(\cdot,t)\|_{W^{2,p}(\partial E)} \le \eta$ for every $t \in [T/2, T)$, up to taking ε smaller and therefore δ smaller. Thus for any $t \in [T/2,T)$ we can apply Lemma 3.1.10 to find $\sigma_t \in \mathbb{T}^N$ and a function $\tilde{u}(\cdot,t)$ such that $E_t + \sigma_t = E_{\tilde{u}(\cdot,t)}$ and

$$|\sigma_t| \le C\|u(\cdot,t)\|_{W^{2,p}(\partial E)}, \ \|\tilde{u}(\cdot,t)\|_{W^{2,p}(\partial E)} \le C\|u(\cdot,t)\|_{W^{2,p}(\partial E)},$$
$$\left|\int_{\partial E_t} \tilde{u}(\cdot,t)\nu_{E_t}\right| \le \delta^*\|\tilde{u}(\cdot,t)\|_{L^2(\partial E)}.$$

Furthermore, Lemma 4.2.2 (taking δ smaller if needed) implies that $\|\tilde{u}(\cdot,t)\|_{C^1(\partial E)} \le \delta^*$. We then apply Theorem 4.2.3 to the set $E_t + \sigma_t$ to obtain

$$\|\tilde{u}(\cdot,t)\|_{H^1(\partial E)} \le C\|\mathscr{H}_{E_t+\sigma_t} - \lambda\|_{L^2(\partial E)} \tag{4.3.2}$$

for any $\lambda \in \mathbb{R}$, where we recall $\mathscr{H}_{E_t+\sigma_t}(x) = H_{E_t}(x + \tilde{u}(x)\nu_E(x))$. From the previous equation, first by the change of variable $y = x + \tilde{u}(x,t)\nu_E(x)$ (estimating the Jacobian with the bounds on $\tilde{u}$ and Lemma 4.2.2), and then by translation invariance, we arrive at

$$\|\tilde{u}(\cdot,t)\|_{H^1(\partial E)} \le C\|H_{E_t+\sigma_t} - \lambda\|_{L^2(\partial E_t+\sigma_t)} = C\|H_{E_t} - \lambda\|_{L^2(\partial E_t)}. \tag{4.3.3}$$

We now claim that

$$P(E_t + \sigma_t) - P(E) = P(E_{\tilde{u}(\cdot,t)}) - P(E) \le C\|\tilde{u}(\cdot,t)\|^2_{H^1(\partial E)}, \tag{4.3.4}$$

which is a classical result but we provide a proof for the sake of completeness.

Let us define, for every $x \in \partial E$, the function

$$Q(x) := \left(1 + \sum_{j=1}^{N-1} \frac{(\partial_{\tau_j} \tilde{u}(x,t))^2}{(1+\kappa_j(x)\tilde{u}(x,t))^2}\right)^{\frac{1}{2}}$$

where $\tau_1(x), \dots, \tau_{N-1}(x)$ and $\kappa_1(x), \dots, \kappa_{N-1}(x)$ are, respectively, the principal directions and curvatures of ∂E at x. Then by Lemma 5.2.3 we have

$$\begin{aligned}
P(E_t + \sigma_t) = P(E_{\tilde{u}(\cdot,t)}) &= \int_{\partial E} Q(x) \prod_{i=1}^{N-1} (1 + \kappa_i(x)\tilde{u}(t,x))\, d\mathscr{H}^{N-1}(x) \\
&= P(E) + \int_{\partial E} \left(H_E \tilde{u}(\cdot,t) + O(\tilde{u}(\cdot,t)^2) + O(|D\tilde{u}(\cdot,t)|^2)\right) d\mathscr{H}^{N-1} \\
&\le P(E) + C\|\tilde{u}(\cdot,t)\|^2_{H^1(\partial E)},
\end{aligned}$$

where we have used that $H_E = \sum_{i=1}^{N-1} \kappa_i$ and the inequality

$$\left| \int_{\partial E} \tilde{u}(\cdot,t)\, d\mathscr{H}^{N-1} \right| \le C \int_{\partial E} \tilde{u}(\cdot,t)^2\, d\mathscr{H}^{N-1},$$

which follows from the fact that $|E_t| = |E_0|$ (see Remark 5.2.4). Hence, we have proved (4.3.4).

We now define the functional $\mathscr{E}(t) = P(E_t) - P(E)$, which is non increasing by (4.3.1). Moreover, by translation invariance, from (4.3.3), (4.3.4) and for any $\lambda \in \mathbb{R}$ we have

$$P(E_t) - P(E) = P(E_t + \sigma_t) - P(E) \le C\|H_{E_t} - \lambda\|^2_{L^2(\partial E_t)}. \tag{4.3.5}$$

Since for any $t \in (0,T)$ equation (4.3.5) for the particular choice of $\lambda = \bar{H}_{E_t}$ implies

$$\mathscr{E}'(t) = -\|H_{E_t} - \bar{H}_{E_t}\|^2_{L^2(\partial E_t)} \le -C\mathscr{E}(t),$$

by Gronwall's inequality we conclude (recalling $\mathscr{E}(0) \ge \mathscr{E}(T/2)$)

$$\mathscr{E}(t) \le \mathscr{E}(0) e^{-C(t-T/2)}, \qquad \forall t \in [T/2, T). \tag{4.3.6}$$

Step 2. We now show that the flow exists for every $t \ge 0$ and it converges exponentially fast to E up to translations.

Up to taking δ smaller, we can use the quantitative isoperimetric inequality in Theorem 3.1.11 to find the existence of translations τ_t such that

$$C|E \triangle (E_t + \tau_t)|^2 \le P(E_t) - P(E) \le P(E_0) - P(E).$$

Furthermore, since all the evolving sets $\{E_t\}_{t\in[T/2,T)}$ satisfy a uniform inner and outer ball condition by Remark 4.2.10, by classical convergence results (see e.g. [31, Theorem 3.2]) we have that $E_t + \tau_t$ is C^1-close to E. In particular, there exist smooth (by the implicit map theorem) functions $v(\cdot,t) : \partial E \to \mathbb{R}$ such that $E_t + \tau_t = E_{v(\cdot,t)}$ and

$$|\tau_t| \le \max_{x\in\partial E_t+\sigma_t} \mathrm{dist}_{\partial E_t}(x) \le \|u(\cdot,t)\|_{C^0(\partial E)} + \|v(\cdot,t)\|_{C^0(\partial E)} \le 2\varepsilon,$$

up to taking δ smaller. Therefore, recalling (4.3.6), we have

$$\|v(\cdot,t)\|^2_{L^1(\partial E)} \le C(P(E_0) - P(E))e^{-C(t-T/2)}. \tag{4.3.7}$$

By Lemma 4.2.2, we also have $\|v(\cdot,t)\|_{C^k(\partial E)} \le C(\|u(\cdot,t)\|_{C^k(\partial E)} + |\tau_t|)$ for every $k \ge 2$. For every $t \in [T/2,T)$, by combining the previous estimate with (4.2.6), (4.3.7) and interpolation inequalities, for any $l \in \mathbb{N}$ there exist $k(l) \in \mathbb{N}, \theta(l) \in (0,1)$ and $C = C(E,l) > 0$ such that

$$\|\nabla^l v(\cdot,t)\|_{C^0} \le C\|v(\cdot,t)\|^{\theta}_{L^1}\|v(\cdot,t)\|^{1-\theta}_{C^k} \le CT^{-\frac{k}{4}(1-\theta)}(P(E_0) - P(E))^{\frac{\theta}{2}} e^{-C(t-T/2)}. \tag{4.3.8}$$

Choosing $\mathscr{E}(0) = P(E_0) - P(E)$ small (hence choosing δ small) we can then apply again Theorem 4.2.7 with the new initial set $E_{v(\cdot,T/2)} = E_{T/2} + \tau_{T/2}$ to get existence of the translated flow up to the time $3T/2$. We remark that, by uniqueness, the flow above is well defined since it coincides in $[T/2,T)$ with the flow E_t translated by τ_t and estimate (4.3.6) now holds for all $t \in [T/2,3T/2)$. By induction, choosing at every step the times $t = nT/2$, we can iterate the procedure above to prove that the flow exists for all times $t \in [0,\infty)$. Moreover, for every $t \in (0,\infty)$ there exists a translation τ_t such that $E_t + \tau_t = E_{v(\cdot,t)}$ with v satisfying (4.3.8). In particular, we have that $v \to 0$ exponentially in C^k for any k, as $t \to \infty$ and thus $E_t + \tau_t \to E$ in C^k for every k. This also implies (reasoning as in (4.3.3)) that $\|H_{E_t} - \bar{H}_{E_t}\|_{L^2(\partial E)} \to 0$ exponentially fast.

Step 3. We conclude by showing the convergence of the whole flow to a translate of E.

Let us prove the convergence of the translations $\{\tau_t\}_{t\ge 0}$. By compactness we can find a sequence $t_n \to \infty$ such that $\tau_{t_n} \to \tau$. Defining

$$\mathscr{D}(F,G) := \int_{F\triangle G} \mathrm{dist}_{\partial G}(x)\,\mathrm{d}x, \tag{4.3.9}$$

following the computations of [3, pag. 21] we

$$
\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathscr{D}(E_t,E-\tau)\right| &= \left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{E_t\triangle(E-\tau)} \mathrm{dist}_{\partial E\tau_t}(x)\,\mathrm{d}x\right| \\
&= \left|\int_{E_t} \mathrm{div}(\mathrm{sd}_{E-\tau}(x)V_t(x)\nu_{E_t}(x))\,\mathrm{d}x\right| \\
&= \left|-\int_{\partial E_t} \mathrm{sd}_{E-\tau}(x)(H_{E_t}(x)-\bar{H}_{E_t}(x))\,\mathrm{d}\mathscr{H}^{N-1}(x)\right| \\
&\leq P(E_0)\|H_{E_t}-\bar{H}_{E_t}\|_{L^2(\partial E)}\left(\sup_{x\in\partial E_t}\mathrm{dist}_{\partial E-\tau}(x)\right) \\
&\leq Ce^{-Ct}\left(\sup_{x\in\mathbb{T}^N}\mathrm{dist}_{\partial E-\tau}(x)\right) \leq Ce^{-Ct},
\end{aligned}
\tag{4.3.10}
$$

where we recall that V_t is the velocity of the flow in the normal direction (see (3.3.1)). Clearly, condition (4.3.10) implies that $\mathscr{D}(E_t,E-\tau)$ admits a limit as $t\to+\infty$. By the previous step and since $\tau_{t_n}\to\tau$, we deduce that

$$
\mathscr{D}(E_t,E-\tau)\to 0 \quad \text{as } t\to+\infty.
$$

Assume now that $\sigma\in\mathbb{T}^N$ is the limit of τ_{s_n} along a subsequence $s_n\to\infty$ as $n\to+\infty$. By the previous step, $E_{s_n}\to E-\sigma$, therefore

$$
0=\lim_{n\to+\infty}\mathscr{D}(E_{s_n},E-\tau)=\mathscr{D}(E-\sigma,E-\tau),
$$

which implies $\sigma=\tau$ by definition (4.3.9). This concludes the proof as the exponential convergence follows from Step 2. □

4.3.2 Stability of the surface diffusion flow

We now focus on surface diffusion flow, which we defined in (4.2.9). As in the previous subsection, we consider E a strictly stable set and $E_0=E_{u_0}$ a smooth normal deformation of E. By Theorem 4.2.17, the surface diffusion flow starting from E_0 exists smooth in an interval $[0,T)$, moreover the evolving sets E_t can be written as normal deformations of E induced by functions $u(\cdot,t)$ satisfying

$$
\begin{cases}
u_t(x,t)\nu_{E_t}(p)\cdot\nu_E(x)=\Delta_{E_t}H_{E_t}(p) & \forall x\in\partial E,\\
u(x,0)=u_0(x)
\end{cases}
$$

where $p=x+u(x,t)\nu_E(x)$.

Now, we aim to show the stability result (ii) of Theorem 4.1.1 for the surface diffusion flow. Due to the similarity of the arguments needed with those employed to prove item (i) of Theorem 4.1.1, we will only highlight the main differencies between the two.

Proof of (ii) *Theorem 4.1.1.* Firstly, Theorem 4.2.17 ensures the existence of a smooth flow E_t for $t \in (0,T)$ of normal deformations of E induced by functions $u(\cdot,t) \in C^\infty(\partial E)$ and satisfying (4.2.35) and (4.2.36). We recall the following identities, holding along the flow E_t as long as it exists smooth,

$$\frac{\mathrm{d}}{\mathrm{d}t}|E_t| = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t}P(E_t) = \int_{\partial E} H_{E_t}(x)\Delta_{E_t}H_{E_t}(x)\,dx = -\|\nabla H_{E_t}\|^2_{L^2(\partial E_t)} \leq 0. \tag{4.3.11}$$

Denoting by C_{E_t} the constant in the Poincaré inequality of Lemma 4.2.5, we get

$$\|H_{E_t} - \bar{H}_{E_t}\|_{L^2(\partial E_t)} \leq C_{E_t}\|\nabla H_{E_t}\|_{L^2(\partial E_t)}.$$

Combining the previous inequality with (4.3.11), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}P(E_t) \leq -C_{E_t}\|H_{E_t} - \bar{H}_{E_t}\|_{L^2(\partial E_t)}.$$

Since $\|u(\cdot,t)\|_{C^{1,1}(\partial E)} \leq c$ for every $t \in (0,T)$, the Poincaré constants C_{E_t} are uniformly bounded in the same time interval and the bound depends on $E, \|u\|_{C^{1,1}(\partial E)}$ (see e.g. the results in [36]). Thus, we obtain the estimate $\frac{\mathrm{d}}{\mathrm{d}t}P(E_t) \leq -C\|H_{E_t} - \bar{H}_{E_t}\|_{L^2(\partial E_t)}$ uniformly in $(0,T)$. We then conclude by following the same arguments of part (i). □

Chapter 5

Stability of the discrete flow in the flat torus

Contents

5.1 Introduction

Within this chapter, which contains the results of [33], we analyse the asymptotic of the discrete mean curvature flow (see Definition 3.4.1) in the flat torus $\mathbb{T}^N$. We recall that in $\mathbb{T}^N$ the class of possible long-time limits is much richer than in $\mathbb{R}^N$ as it includes not only union of balls with equal radii but also different types of critical sets for the perimeter, for example cylinders and stripes. The first main result of the chapter is the theorem below. It provides a complete characterization of the long-time behaviour of the discrete flow in $\mathbb{T}^N$ starting near a strictly stable set (see Definition 3.1.6). Moreover, an estimate on the convergence speed is provided.

Theorem 5.1.1. *Let E be a strictly stable set in the flat torus. Then there exist $\delta^* = \delta^*(E) > 0$ and $h^* = h^*(E) > 0$ with the following property: if $h < h^*$ and $E_0 \subset \mathbb{T}^N$ is a set of finite perimeter satisfying*

$$|E_0| = |E|, \qquad \overline{E}_0 \subset (E)_{\delta^*},$$

then every discrete volume preserving mean curvature flow $\{E_h^n\}_{n\in\mathbb{N}}$ starting from E_0 converges to $E+\tau$ in C^k for every $k \in \mathbb{N}$, for some translation $\tau \in \mathbb{T}^N$. Moreover, the convergence is exponentially fast in the following sense: the sets E_h^n are normal deformations of $E+\tau$ induced by functions $u_h^n \in C^\infty(\partial E+\tau)$ such that

$$\|u_h^n\|_{C^k(\partial E+\tau)} \le c_k e^{-c_k n}, \quad \text{for every } k \in \mathbb{N},$$

where c_k are constants that only depend on k and E.

The second result of the chapter gives a complete characterization of the asymptotic of the discrete flow starting from any initial set in dimension $N = 2$. In order to state the precise result we first introduce the following notation: we call *lamella* any connected set in $\mathbb{T}^2$ whose 1-periodic extension in $\mathbb{R}^2$ is a stripe bounded by two parallel lines.

Theorem 5.1.2. *Fix h, $m > 0$ and an initial set $E_0 \subset \mathbb{T}^2$ with finite perimeter and such that $|E_0| = m$. Let $\{E_h^n\}_{n\in\mathbb{N}}$ be a discrete flow starting from E_0 and let P_∞ be the limit of the non-increasing sequence $P(E_h^n)$. Then one of the following holds:*

i) $\{E_h^n\}_{n\in\mathbb{N}}$ converges to a disjoint union of l discs of equal radii and total area m, where $l = \pi^{-1}(4m)^{-1}P_\infty^2 \in \mathbb{N}$;

ii) $\left\{(E_h^n)^c\right\}_{n\in\mathbb{N}}$ converges to a disjoint union of l discs of equal radii and total area $1-m$, where $l = \pi^{-1}(4-4m)^{-1}P_\infty^2 \in \mathbb{N}$;

iii) $\{E_h^n\}_{n\in\mathbb{N}}$ converges to a disjoint union of l lamellae of total area m, with the same slope and $l \le P_\infty/2$. Moreover, the equality $l = P_\infty/2 \in \mathbb{N}$ holds if and only if the limit is given by vertical or horizontal lamellae.

In all cases the convergence is exponentially fast in C^k for every $k \in N$.

The plan of the chapter is the following: in Section 5.2 we establish the Alexandrov-type estimate (Theorem 5.2.1) for strictly stable sets in the flat torus, which is a crucial step for the proof of the main result; Section 5.3 is devoted to prove an L^1-estimate for initial sets of the discrete flow sufficiently "close" to a strictly stable set; Section 5.4 is devoted to the stability result (Theorem 5.1.1); finally in Section 5.5 we completely characterize the long-time behaviour of the flow in dimension two (Theorem 5.1.2).

5.2 A quantitative Alexandrov Theorem in the flat torus

In this section, we will prove that, in the flat torus, strictly stable sets satisfy a quantitative Alexandrov-type estimate.

Theorem 5.2.1. *Let $E \subset \mathbb{T}^N$ be a strictly stable set. There exist $\delta \in (0,1/2)$ and $C > 0$, depending only on N and E, with the following property: for any $f \in C^1(\partial E) \cap H^2(\partial E)$ such that $\|f\|_{C^1(\partial E)} \leq \delta$ and satisfying*

$$\left| \int_{\partial E} f \, \mathrm{d}\mathscr{H}^{N-1} \right| \leq \delta \|f\|_{L^2(\partial E)}, \qquad \left| \int_{\partial E} f \nu_E \, \mathrm{d}\mathscr{H}^{N-1} \right| \leq \delta \|f\|_{L^2(\partial E)},$$

we have

$$\|f\|_{H^1(\partial E)} \leq C \|H_{E_f} - \overline{H}_{E_f}\|_{L^2(\partial E)} =: C \left(\int_{\partial E} (H_{E_f}(x + \nu_E(x) f(x)) - \overline{H}_{E_f})^2 \, \mathrm{d}\mathscr{H}^{N-1} \right)^{\frac{1}{2}}, \quad (5.2.1)$$

where we recall E_f is the normal deformation of E induced by f (see Definition 3.1.1), H_{E_f} is the mean curvature of E_f, and $\overline{H}_{E_f} = \fint_{\partial E} H_{E_f}(x + \nu_E(x) f(x)) \, \mathrm{d}\mathscr{H}^{N-1}$.

To prove this theorem we reproduce some arguments similar to the ones used in the proof of [82, Theorem 1.3]. In this section, we consider $E \subset \mathbb{T}^N$ a strictly stable set. Thanks to some classical results for sets of finite perimeter (see for example [75, Theorem 27.4]), the previous hypothesis implies that E is connected and it is of class C^∞.

Remark 5.2.2. For some particular choices of the set E, a geometric explanation of the condition

$$\left| \int_{\partial E} f \nu_E \, \mathrm{d}\mathscr{H}^{N-1} \right| \leq \delta \|f\|_{L^2(\partial E)} \qquad (5.2.2)$$

can be found. It is the case for the ball, the cylinder or the lamella.

First consider the case of the ball, for instance let E be the ball B with radius one and centered at the origin. We show that in this case, by assuming that $|E_f| = |B|$ and enforcing the following condition on the barycenters

$$\mathrm{bar}(E_f) = \mathrm{bar}(B) = 0$$

inequality (5.2.2) follows. Indeed, the barycenter in polar coordinates is given by

$$0=\frac{1}{(N+1)\omega_N}\int_{\partial B}(1+f(x))^{N+1}x\,\mathrm{d}\mathscr{H}^{N-1}(x)$$

and thus, by a simple Taylor expansion, we obtain

$$\begin{aligned}0&=\int_{\partial B}\left(1+(N+1)f(x)+\frac{1}{2}Rf(x)^2\right)x\,\mathrm{d}\mathscr{H}^{N-1}(x)\\&=(N+1)\int_{\partial B}f(x)x\,\mathrm{d}\mathscr{H}^{N-1}(x)+\frac{1}{2}\int_{\partial B}xRf(x)^2\,\mathrm{d}\mathscr{H}^{N-1}(x)\end{aligned}$$

where $|R(x)|\le C(N)$ for every $x\in\partial B$. We can then estimate

$$\left|\int_{\partial B}f(x)x\,\mathrm{d}\mathscr{H}^{N-1}(x)\right|\le C\|f\|^2_{L^2(\partial E)},$$

and, provided $\|f\|_{C^1(\partial E)}\le\delta$, the conclusion follows recalling $\nu_B(x)=x$.

Now, we take (for example) E equal to the cylinder $C=(-1/2,1/2)\times B^{N-1}$, then from $|E_F|=|C|$ and the condition

$$\mathrm{bar}(E_f)=\mathrm{bar}(C)=0,$$

we obtain for the last $N-1$ components of the barycenter

$$0=\frac{1}{N\omega_{N-1}}\int_{\partial B^{N-1}}(1+f(x))^N x\,\mathrm{d}\mathscr{H}^{N-2}(x),$$

and the conclusion follows as before.

The case of a lamella L follows immediately by assuming that $|E_f|=|L|$, recalling that in this case ν_L is a constant vector.

Let $f\in C^1(\partial E)$ with L^∞ norm sufficiently small. First of all, we compute the $(N-1)$-Jacobian of the map

$$\Phi:\partial E\to\partial E_f\subset\mathbb{R}^N,\qquad x\mapsto x+f(x)\nu_E(x).$$

Given $x\in\partial E$, we choose an orthonormal basis

$$\mathscr{B}':=\{v_1(x),\dots,v_{N-1}(x)\}$$

of the tangent T_xE of ∂E at x such that, in this basis, the second fundamental form of E, $B_E(x) : T_xE \to T_xE \subset \mathbb{R}^N$, has the following expression

$$B_E(x) = \begin{pmatrix} \kappa_1(x) & & \\ & \ddots & \\ & & \kappa_{N-1}(x) \\ 0 & \dots & 0 \end{pmatrix},$$

where $\kappa_1(x), \dots, \kappa_{N-1}(x)$ are the principal curvatures of E in x. We then complete $\mathscr{B}'$ to a basis $\mathscr{B}$ of the whole $\mathbb{R}^N$ with the normal vector $v_N(x) := \nu_E(x)$. In the following, to simplify the notation, we will drop the dependence on x. The tangential differential of Φ with respect to the basis $\mathscr{B}$ is given by

$$\mathrm{d}\Phi = I + \nu_E \otimes \nabla f + f\,\mathrm{d}\nu_E,$$

where I is the immersion $T_xE \hookrightarrow \mathbb{R}^N$, ∇f is the tangential gradient of f and $\mathrm{d}\nu_E$ is the tangential differential of ν_E. Given the regularity of ∂E, we recall that $\mathrm{d}\nu_E$ is equal to B_E. Moreover, by definition of $\mathscr{B}$, we have that

$$(\nu_E \otimes \nabla f)(v_i, v_j) = \delta_{N,i} \nabla f \cdot v_j, \quad i = 1, \dots, N,\ j = 1, \dots, N-1.$$

Thanks to the previous observations we obtain

$$\mathrm{d}\Phi = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & \dots & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & \ddots & \\ & & 0 \\ \partial_{v_1} f & \dots & \partial_{v_{N-1}} f \end{pmatrix} + \begin{pmatrix} \kappa_1 f & & \\ & \ddots & \\ & & \kappa_{N-1} f \\ 0 & \dots & 0 \end{pmatrix},$$

thus we find the following expression

$$\mathrm{d}\Phi = \begin{pmatrix} 1 + \kappa_1 f & & \\ & \ddots & \\ & & 1 + \kappa_{N-1} f \\ \partial_{v_1} f & \dots & \partial_{v_{N-1}} f \end{pmatrix}. \tag{5.2.3}$$

By Binet formula, the Jacobian $J\Phi$ can be explicitly computed as

$$J\Phi = \left(\prod_{i=1}^{N-1} (1 + \kappa_i f)^2 + \sum_{j=1}^{N-1} (\partial_{v_j} f)^2 \prod_{i \neq j} (1 + \kappa_i f)^2 \right)^{1/2}$$

$$= \prod_{i=1}^{N-1}(1+\kappa_i f)\left(1+\sum_{j=1}^{N-1}\frac{(\partial_{v_j} f)^2}{(1+\kappa_j f)^2}\right)^{1/2}. \tag{5.2.4}$$

To show the previous formula, we characterize the minors of $\mathrm{d}\Phi$. If we omit the N-th row of $\mathrm{d}\Phi$, we obtain the minor

$$M_N = \begin{pmatrix} 1+\kappa_1 f & & \\ & \ddots & \\ & & 1+\kappa_{N-1} f \end{pmatrix},$$

if we omit the i-th row of $D\Phi$ for $1 \le i \le N-1$, we obtain the minor

$$M_i = \begin{pmatrix} 1+\kappa_1 f & & & & & \\ & \ddots & & & & \\ & & 1+\kappa_{i-1} f & & & \\ & & & 1+\kappa_{i+1} f & & \\ & & & & \ddots & \\ & & & & & 1+\kappa_{N-1} f \\ \partial_{v_1} f & \dots & \partial_{v_{i-1}} f & \partial_{v_{i+1}} f & \dots & \partial_{v_{N-1}} f \end{pmatrix}.$$

We then deduce (5.2.4) by explicitly computing

$$\det(M_N)^2 = \prod_{i=1}^{N-1}(1+\kappa_i f)^2, \quad \det(M_i)^2 = (\partial_{v_i} f)^2 \prod_{j\neq i}(1+\kappa_j f)^2.$$

The previous formula for $J\Phi$ allows us to calculate some quantities that will be useful later on. Observe that, if $\|f\|_{C^1(\partial E)}$ is small enough, the map Φ is a diffeomorphism from ∂E to $\Phi(\partial E) = \partial E_f$, and thus the tangential differential $\mathrm{d}\Phi : T_x E \to T_{\Phi(x)} E_f$ is a surjective map. In particular, this allows us to calculate the normal vector ν_{E_f} in $\Phi(x)$. We remark that a vector v orthogonal to every column of (5.2.3) is a normal vector to the whole tangent space $T_{\Phi(x)} E_f$, therefore a possible v is given by

$$v = -\sum_{i=1}^{N-1}\frac{\partial_{v_i} f}{1+\kappa_i f} v_i + \nu_E,$$

where the sign of the component along ν_E is taken positive so that the case $f = 0$ is consistent with the orientation of ν_E. Since $|v| \ge 1$, by normalizing v we obtain the normal vector

$$\nu_{E_f} = \left(\nu_E - \sum_{i=1}^{N-1} \frac{\partial_{\nu_i} f}{1+\kappa_i f}\nu_i\right)\left(1+\sum_{j=1}^{N-1}\frac{(\partial_{\nu_j} f)^2}{(1+\kappa_j f)^2}\right)^{-1/2}, \tag{5.2.5}$$

moreover, we remark that

$$\nu_E \cdot \nu_{E_f} = \left(1+\sum_{j=1}^{N-1}\frac{(\partial_{\nu_j} f)^2}{(1+\kappa_j f)^2}\right)^{-1/2}. \tag{5.2.6}$$

We can now compute explicitly the formula for the first variation of the perimeter.

Lemma 5.2.3. *Setting* $Q := \left(1+\sum_{j=1}^{N-1}\frac{(\partial_{\nu_j} f)^2}{(1+\kappa_j f)^2}\right)^{1/2}$*, the following formulas hold true:*

1. *If* $f \in C^1(\partial E)$ *with* $\|f\|_{L^\infty(\partial E)}$ *sufficiently small, then*

$$P(E_f) = \int_{\partial E} Q \prod_{i=1}^{N-1}(1+\kappa_i f)\,\mathrm{d}\mathscr{H}^{N-1}.$$

2. *If* $f \in C^1(\partial E)$ *with* $\|f\|_{L^\infty}$ *sufficiently small, then the first variation* $\partial P(E_f)[\varphi]$ *exists for all* $\varphi \in C^1(\partial E)$ *and is given by*

$$\begin{aligned}\partial P(E_f)[\varphi] = &\int_{\partial E} \varphi Q \sum_{i=1}^{N-1} \kappa_i \prod_{j\neq i}(1+\kappa_j f)\,\mathrm{d}\mathscr{H}^{N-1}\\ &+ \int_{\partial E} \frac{1}{Q}\prod_{i=1}^{N-1}(1+\kappa_i f)\left(\sum_{j=1}^{N-1}\frac{\partial_{\nu_j}\varphi\,\partial_{\nu_j} f}{(1+k_j f)^2} - \varphi\sum_{j=1}^{N-1}\frac{k_j\,(\partial_{\nu_j} f)^2}{(1+k_j f)^3}\right)\mathrm{d}\mathscr{H}^{N-1}.\end{aligned} \tag{5.2.7}$$

Proof. The first formula is a straightforward consequence of the area formula

$$P(E_f) = \int_{\partial E_f} \mathrm{d}\mathscr{H}^{N-1} = \int_{\partial E} J\Phi\,\mathrm{d}\mathscr{H}^{N-1}$$

and of the expression of the Jacobian $J\Phi$ in (5.2.4). Now, (5.2.7) easily follows by taking the derivatives

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} P(E_{f+\varepsilon\varphi})$$

in the first formula. □

In the following, with C we will refer to a positive constant, possibly changing from line to line, and we will specify its explicit dependence when needed.

Remark 5.2.4. We observe that, if $\|f\|_{L^\infty(\partial E)}$ is small enough and $|E_f| = |E|$, then there exists a constant $C > 0$, only depending on E, such that

$$\left|\int_{\partial E} f(x)\,\mathrm{d}\mathscr{H}^{N-1}(x)\right| \leq C\int_{\partial E} f(x)^2\,\mathrm{d}\mathscr{H}^{N-1}(x). \tag{5.2.8}$$

Firstly, since ∂E is regular, for every $\varepsilon > 0$ sufficiently small there exists a tubular neighborhood N of ∂E such that N is diffeomorphic to $\partial E \times (-\varepsilon,\varepsilon)$ via the diffeomeorphism $\Psi(x,t) = x + \nu_E(x)t$. The Jacobian of Ψ is given by

$$J\Psi(x,t) = \prod_{i=1}^{N-1}(1+\kappa_i(x)t). \tag{5.2.9}$$

Secondly, if $\|f\|_{L^\infty(\partial E)}$ is small enough, we remark that the condition $|E_f| = |E|$ is equivalent to

$$0 = |E_f| - |E| = \int_{\partial E}\int_0^{f(x)} J\Psi(x,t)\,\mathrm{d}t\,\mathrm{d}\mathscr{H}^{N-1}(x).$$

Then, we can conclude that

$$\begin{aligned}
0 &= \int_{\partial E}\int_0^{f(x)} J\Psi(x,t)\,\mathrm{d}t\,\mathrm{d}\mathscr{H}^{N-1}(x)\\
&= \int_{\partial E} f(x)\,\mathrm{d}\mathscr{H}^{N-1}(x) + \int_{\partial E}\int_0^{f(x)} (J\Psi(x,t)-1)\,\mathrm{d}t\,\mathrm{d}\mathscr{H}^{N-1}(x)\\
&= \int_{\partial E} f(x)\,\mathrm{d}\mathscr{H}^{N-1}(x) + \int_{\partial E}\int_0^{f(x)} (H_E(x)t + o(t))\,\mathrm{d}t\,\mathrm{d}\mathscr{H}^{N-1}(x),
\end{aligned}$$

that implies (5.2.8) for a constant depending only on N and the principal curvatures of E.

We are now able to prove the following stability result; it ensures that the second variation of the perimeter (3.1.3) remains strictly positive for small normal deformations of a strictly stable set E.

Lemma 5.2.5. *Fix $N \geq 2$. There exists $\delta = \delta(E) > 0$ small such that, if $f \in L^\infty(\partial E) \cap H^1(\partial E)$ with $\|f\|_{L^\infty(\partial E)} \leq \delta$,*

$$\left|\int_{\partial E} f(x)\,\mathrm{d}\mathscr{H}^{N-1}(x)\right| \leq \delta\|f\|_{L^2(\partial E)} \quad \text{and} \quad \left|\int_{\partial E} f(x)\nu_E(x)\,\mathrm{d}\mathscr{H}^{N-1}(x)\right| \leq \delta\|f\|_{L^2(\partial E)}, \tag{5.2.10}$$

then we have

$$\partial^2 P(E)[f] = \int_{\partial E}(|\nabla f(x)|^2 - |B_E(x)|^2 f(x)^2)\,\mathrm{d}\mathscr{H}^{N-1}(x) \geq \frac{m_0}{8}\|f\|^2_{H^1(\partial E)},$$

where m_0 is the constant given by Lemma 3.1.8.

Proof. Set $g = f - \bar{f}$, where $\bar{f} = \fint_{\partial E} f \, \mathrm{d}\mathscr{H}^{N-1}$, then g has zero average and, by the first inequality in (5.2.10), we have

$$\bar{f}^2 = \frac{1}{P(E)^2}\left(\int_{\partial E} f \, \mathrm{d}\mathscr{H}^{N-1}\right)^2 \leq C\delta^2 \|f\|^2_{L^2(\partial E)}. \tag{5.2.11}$$

If δ is sufficiently small, from (5.2.11) we obtain

$$\|g\|^2_{L^2(\partial E)} = \|f - \bar{f}\|^2_{L^2(\partial E)} = \|f\|^2_{L^2(\partial E)} - \bar{f}^2 P(E) \geq \|f\|^2_{L^2(\partial E)}\left(1 - C\delta^2\right) \geq \frac{1}{2}\|f\|^2_{L^2(\partial E)}.$$

Using the previous inequality, (5.2.11) again and the second inequality in (5.2.10) we infer that the function g satisfies

$$\left|\int_{\partial E} g\nu_E \, \mathrm{d}\mathscr{H}^{N-1}\right| \leq \left|\int_{\partial E} f\nu_E \, \mathrm{d}\mathscr{H}^{N-1}\right| + \left|\int_{\partial E} \bar{f}\nu_E \, \mathrm{d}\mathscr{H}^{N-1}\right| \leq C\delta\|g\|_{L^2(\partial E)}.$$

Then, we can apply Lemma 3.1.9 to obtain

$$\partial^2 P(E)[g] \geq \frac{m_0}{2}\|g\|^2_{H^1(\partial E)},$$

provided δ small enough. We conclude

$$\begin{aligned}
\partial^2 P(E)[f] &= \partial^2 P(E)[g] - \partial^2 P(E)[g] + \partial^2 P(E)[f] \\
&= \partial^2 P(E)[g] - 2\bar{f}\int_{\partial E} |B_E(x)|^2 f(x) \, \mathrm{d}\mathscr{H}^{N-1}(x) + \bar{f}^2 \int_{\partial E} |B_E(x)|^2 \, \mathrm{d}\mathscr{H}^{N-1}(x) \\
&\geq \frac{m_0}{2}\|g\|^2_{H^1(\partial E)} - C|\bar{f}|\|f\|_{L^2(\partial E)} \geq \frac{m_0}{2}(\|g\|^2_{L^2(\partial E)} + \|\nabla g\|^2_{L^2(\partial E)}) - C\delta\|f\|^2_{L^2(\partial E)} \\
&\geq \frac{m_0}{4}(\|f\|^2_{L^2(\partial E)} + \|\nabla f\|^2_{L^2(\partial E)}) - C\delta\|f\|^2_{L^2(\partial E)} \geq \frac{m_0}{8}\|f\|^2_{H^1(\partial E)},
\end{aligned}$$

up to taking δ smaller if needed, and where the constant $C > 0$ only depends on E. □

Remark 5.2.6. Remark 5.2.4 ensures that the conclusion of the previous lemma also holds if we replace the hypothesis $|\int_{\partial E} f \, \mathrm{d}\mathscr{H}^{N-1}| \leq \delta\|f\|_{L^2(\partial E)}$ with $\|f\|_{L^\infty(\partial E)}$ small enough and $|E_f| = |E|$.

We are now able to prove the generalized version of the quantitative Alexandrov's inequality in the periodic setting, Theorem 5.2.1.

Proof of Theorem 5.2.1. First of all we notice that, if we take the constant C in (5.2.1) to be bigger than $\sqrt{P(E)/2}$, then it is enough to consider only the case $\|H_{E_f} - \overline{H}_{E_f}\|_{L^2(\partial E)} \leq 1$.

Set $p = x + f(x)\nu_E(x)$ and let $\varphi \in C^1(\partial E)$, by the definition of scalar mean curvature H_{E_f} and a change of coordinates we obtain

$$\partial P(E_f)[\varphi] = \int_{\partial E} (H_{E_f}\nu_{E_f})(p) \cdot \nu_E \, \varphi \, J\Phi \, \mathrm{d}\mathscr{H}^{N-1}. \tag{5.2.12}$$

Combining (5.2.12), (5.2.4) and (5.2.6) we obtain

$$\begin{aligned}\partial P(E_f)[\varphi] &= \int_{\partial E} H_{E_f} \varphi J\Phi \left(1 + \sum_{j=1}^{N-1} \frac{(\partial_{\nu_j} f)^2}{(1+\kappa_j f)^2}\right)^{-1/2} \mathrm{d}\mathscr{H}^{N-1} \\ &= \int_{\partial E} H_{E_f} \varphi \prod_{i=1}^{N-1} (1+\kappa_i f) \, \mathrm{d}\mathscr{H}^{N-1}.\end{aligned}$$

In the following, with a slight abuse of notation, with the symbol $O(g)$ we will mean any function h of the form $h(x) = r(x)g(x)$, where $|r(x)| \leq C$ for all $x \in \partial E$ and C is a constant depending only on N and E.

By a simple Taylor expansion we have

$$\partial P(E_f)[\varphi] = \int_{\partial E} H_{E_f} \varphi \left(1 + H_E f + O(f^2)\right) \mathrm{d}\mathscr{H}^{N-1}. \tag{5.2.13}$$

From (5.2.7) and again by Taylor expansion, we obtain

$$\begin{aligned}\partial P(E_f)[\varphi] &= \int_{\partial E} \left(H_E + f \sum_{i=1}^{N-1} \kappa_i \sum_{s \neq i} \kappa_s + O(f^2) + O(|\nabla f|^2)\right) \varphi \, \mathrm{d}\mathscr{H}^{N-1} \\ &\quad + \int_{\partial E} (\nabla f + h) \cdot \nabla \varphi \, \mathrm{d}\mathscr{H}^{N-1} \\ &= \int_{\partial E} \left(H_E + f H_E^2 - |B_E|^2 f + O(f^2) + O(|\nabla f|^2)\right) \varphi \, \mathrm{d}\mathscr{H}^{N-1} \\ &\quad + \int_{\partial E} (\nabla f + h) \cdot \nabla \varphi \, \mathrm{d}\mathscr{H}^{N-1}\end{aligned} \tag{5.2.14}$$

where h is a vector field satisfying $|h| \leq C(|f| + |\nabla f|^2))|\nabla f|$. Set $R = O(f^2) + O(|\nabla f|^2)$, by comparing (5.2.13) and (5.2.14) we infer that

$$\begin{aligned}\int_{\partial E} (\nabla f \cdot \nabla \varphi - |B_E|^2 f \varphi) \, \mathrm{d}\mathscr{H}^{N-1} &= \int_{\partial E} (H_{E_f} - H_E)(1 + H_E f + R) \, \varphi \, \mathrm{d}\mathscr{H}^{N-1} \\ &\quad - \int_{\partial E} (h \cdot \nabla \varphi + (O(f^2) + O(|\nabla f|^2))\varphi) \, \mathrm{d}\mathscr{H}^{N-1}.\end{aligned} \tag{5.2.15}$$

Testing (5.2.15) with $\varphi = 1$, we get

$$\int_{\partial E} (H_{E_f} - H_E)\,(1 + H_E f + R)\,\mathrm{d}\mathscr{H}^{N-1} = \int_{\partial E} (O(|f|) + O(|\nabla f|^2))\,\mathrm{d}\mathscr{H}^{N-1},$$

then, for δ sufficiently small, using Hölder inequality we obtain

$$\begin{aligned}
\left|\overline{H}_{E_f} - H_E\right| &= \left| -\fint_{\partial E} (H_{E_f} - H_E)(H_E f + R)\,\mathrm{d}\mathscr{H}^{N-1} + \fint_{\partial E} (O(|f|) + O(|\nabla f|^2))\,\mathrm{d}\mathscr{H}^{N-1} \right| \\
&\leq \left| \fint_{\partial E} (H_{E_f} - \overline{H}_{E_f})(H_E f + R)\,\mathrm{d}\mathscr{H}^{N-1} \right| + \left| \fint_{\partial E} (\overline{H}_{E_f} - H_E)(H_E f + R)\,\mathrm{d}\mathscr{H}^{N-1} \right| \\
&\quad + \int_{\partial E} (O(|f|) + O(|\nabla f|^2))\,\mathrm{d}\mathscr{H}^{N-1} \\
&\leq \delta\,\frac{|H_E| + C\delta}{P(E)} \|H_{E_f} - \overline{H}_{E_f}\|_{L^2(\partial E)} + \delta\,(|H_E| + C\delta)\,|\overline{H}_{E_f} - H_E| \\
&\quad + \int_{\partial E} (O(|f|) + O(|\nabla f|^2))\,\mathrm{d}\mathscr{H}^{N-1},
\end{aligned}$$

with $C = C(N, E)$ since $\delta \leq 1$. For δ small enough, recalling that $\|H_{E_f} - \overline{H}_{E_f}\|_{L^2(\partial E)} \leq 1$, the previous inequality implies

$$\frac{1}{2}|\overline{H}_{E_f} - H_E| \leq C\delta \|H_{E_f} - \overline{H}_{E_f}\|_{L^2(\partial E)} + \int_{\partial E} (O(|f|) + O(|\nabla f|^2))\,\mathrm{d}\mathscr{H}^{N-1} \leq C\delta. \tag{5.2.16}$$

Using the bound $\|f\|_{C^1(\partial E)} \leq \delta$ and the definition of h we easily see that

$$h \cdot \nabla f = \delta\, O(|\nabla f|^2).$$

Testing (5.2.15) with $\varphi = f$, using Hölder's inequality and by the previous remark, we get

$$\begin{aligned}
\int_{\partial E} (|\nabla f|^2 - |B_E|^2 f^2)\,\mathrm{d}\mathscr{H}^{N-1} &= \int_{\partial E} \left(H_{E_f} - H_E\right)(1 + H_E f + R) f\,\mathrm{d}\mathscr{H}^{N-1} \\
&\quad + \delta \int_{\partial E} (O(f^2) + O(|\nabla f|^2))\,\mathrm{d}\mathscr{H}^{N-1} \\
&= \int (H_{E_f} - \overline{H}_{E_f})(1 + H_E f + R) f\,\mathrm{d}\mathscr{H}^{N-1} + \int (\overline{H}_{E_f} - H_E)(1 + H_E f + R) f\,\mathrm{d}\mathscr{H}^{N-1} \\
&\quad + \delta \int_{\partial E} (O(f^2) + O(|\nabla f|^2))\,\mathrm{d}\mathscr{H}^{N-1} \\
&\leq C\|H_{E_f} - \overline{H}_{E_f}\|_{L^2(\partial E)} \|f\|_{L^2(\partial E)} + |\overline{H}_{E_f} - H_E| \int (1 + H_E f + R) f\,\mathrm{d}\mathscr{H}^{N-1} \\
&\quad + \delta \int_{\partial E} (O(f^2) + O(|\nabla f|^2))\,\mathrm{d}\mathscr{H}^{N-1}
\end{aligned}$$

$$
\begin{aligned}
&= C\|H_{E_f}-\overline{H}_{E_f}\|_{L^2(\partial E)}\|f\|_{L^2(\partial E)}+|\overline{H}_{E_f}-H_E|\int(f+O(f^2)+fO(|\nabla f|^2))\,\mathrm{d}\mathscr{H}^{N-1}\\
&\quad+\delta\int_{\partial E}(O(f^2)+O(|\nabla f|^2))\,\mathrm{d}\mathscr{H}^{N-1}.
\end{aligned}
\tag{5.2.17}
$$

By (5.2.8), (5.2.16) and by Hölder inequality, we obtain

$$
|\overline{H}_{E_f}-H_E|\int(f+O(f^2)+fO(|\nabla f|^2))\,\mathrm{d}\mathscr{H}^{N-1}\le\delta\int_{\partial E}(O(f^2)+O(|\nabla f|^2)).
$$

Finally, by the above inequality, (5.2.8) again and by combining (5.2.17) with (5.2.16) we deduce that, for any $\eta>0$, it holds

$$
\begin{aligned}
\int_{\partial E}(|\nabla f|^2-|B_E|^2f^2)\,\mathrm{d}\mathscr{H}^{N-1}&\le C\|H_{E_f}-\overline{H}_{E_f}\|_{L^2(\partial E)}\|f\|_{H^1(\partial E)}+\delta\int_{\partial E}(O(f^2+|\nabla f|^2))\,\mathrm{d}\mathscr{H}^{N-1}\\
&\le\frac{1}{\eta}C^2\|H_{E_f}-\overline{H}_{E_f}\|^2_{L^2(\partial E)}+\eta\|f\|^2_{H^1(\partial E)}+C\delta\|f\|^2_{H^1(\partial E)}.
\end{aligned}
\tag{5.2.18}
$$

The conclusion then follows combining (5.2.18) with Lemma 5.2.5 and taking δ and η sufficiently small. □

5.3 Uniform L^1-estimate on the discrete flow

In this section, we prove Proposition 5.3.3, which plays a crucial role in the proof of the main result of the chapter. Additionally, we show that the hypothesis of closeness of the initial set to the stable set E in Theorem 5.1.1 cannot be weakened to closeness only in L^1.

5.3.1 Uniform L^1 estimate

In this subsection we prove a uniform L^1-estimate on the discrete flow starting from an initial set E_0 sufficiently “close” to a strictly stable set of the perimeter. Before we recall the definition of Hausdorff distance and some of its properties, for a complete reference see e.g. [9, Section 4.4], [81, Section 10.1].

Let C_1, $C_2\subset\mathbb{T}^N$ be closed sets, we define the *Hausdorff distance* between C_1 and C_2 as

$$
\mathrm{d}_H(C_1,C_2):=\inf\{\rho>0:C_1\subset(C_2)_\rho,C_2\subset(C_1)_\rho\},
$$

where we recall that by C_ρ we denote the closed δ-neighbourhood of C, that is $C_\rho=\{x\in\mathbb{T}^N:\mathrm{dist}_C(x)\le\rho\}$. Given C_n, C closed sets in $\mathbb{T}^N$, we say that $\{C_n\}_{n\in\mathbb{N}}$ converges to

C in the Hausdorff distance and we write $C_n \xrightarrow{H} C$, if $\mathrm{d}_H(C_n, C) \to 0$ as $n \to \infty$. We recall that the space of closed subsets of a compact set equipped with the Hausdorff metric is compact (see e.g [9, Theorem 4.4.15] or [81, Proposition 10.1]) and also that the convergence in the Hausdorff distance is equivalent to the uniform convergence of the respective distance functions, i.e.

$$C_n \xrightarrow{H} C \quad \Longleftrightarrow \quad \mathrm{dist}_{C_n} \to \mathrm{dist}_C \quad \text{uniformly}.$$

In the following, given two open smooth sets E_1, E_2, we will denote by $\mathrm{d}_H(E_1, E_2)$ the Hausdorff distance between their closures.

Lemma 5.3.1. *Let $E \subset \mathbb{T}^N$ be a strictly stable set and let $\varepsilon > 0$. Then, there exist $\delta = \delta(\varepsilon, E) > 0$ and $h^* = h^*(E) > 0$ such that, for every $h < h^*$ and for every set E_0 satisfying*

$$|E_0| = |E|, \qquad \mathrm{d}_H(E_0, E) \leq \delta,$$

we have

$$|E \triangle F| \leq \varepsilon,$$

where F is a solution of (3.4.1) *with E_0 replacing E.*

Proof. Let $h^* = h^*(E)$ be the constant given by Proposition 3.4.4 so that, for every $h < h^*$, E is the unique volume-constrained global minimizer of the functional

$$\tilde{J}_h(G) := P(G) + \frac{1}{h} \int_G \mathrm{dist}_E(x)\,\mathrm{d}x. \tag{5.3.1}$$

Fix $h < h^*$ and let $\{E_n\}_{n \in \mathbb{N}}$ be a sequence of sets satisfying

$$|E_n| = |E|, \qquad \overline{E}_n \xrightarrow{H} \overline{E}. \tag{5.3.2}$$

Consider F_n a solution of (3.4.1) with E_n replacing E. We claim that

$$F_n \xrightarrow{L^1} E.$$

If we prove the claim, the conclusion easily follows.

First, Remark 3.4.2 ensures that $\{F_n\}_{n \in \mathbb{N}}$ is a sequence of sets with uniformly bounded perimeters, with the bound depending only on N, m, h. Therefore, there exist F a set of finite perimeter such that $|F| = m$ and a (unrelabelled) subsequence of $\{F_n\}_{n \in \mathbb{N}}$ such that

$$F_n \xrightarrow{L^1} F.$$

Now, let K be a compact subset of $\mathbb{T}^N$ such that, up to a subsequence, we have

$$\overline{E_n^c} \xrightarrow{H} K.$$

From the second property in (5.3.2) we easily deduce that $(\overline{E})^c \subset K$, and therefore $K^c \subset \overline{E}$. In particular, this inclusion implies that

$$\int_{K^c} \mathrm{dist}_K(x)\,\mathrm{d}x = \int_E \mathrm{dist}_K(x)\,\mathrm{d}x \geq \int_G \mathrm{dist}_K(x)\,\mathrm{d}x$$

for every $G \subset \mathbb{T}^N$. Setting

$$\bar{J}_h(G) := P(G) + \frac{1}{h}\int_G (\mathrm{dist}_E(x) - \mathrm{dist}_K(x))\,\mathrm{d}x,$$

from the previous remark and from the fact that E is the unique minimizer of (5.3.1), we have

$$\begin{aligned}
\bar{J}_h(G) &= \tilde{J}_h(G) - \frac{1}{h}\int_G \mathrm{dist}_K(x)\,\mathrm{d}x \\
&> \tilde{J}_h(E) - \frac{1}{h}\int_G \mathrm{dist}_K(x)\,\mathrm{d}x \\
&\geq \tilde{J}_h(E) - \frac{1}{h}\int_E \mathrm{dist}_K(x)\,\mathrm{d}x = \bar{J}_h(E),
\end{aligned}$$

for any measurable set $G \subset \mathbb{T}^N$ with $|G| = |E|$. Finally, we obtain

$$\begin{aligned}
\bar{J}_h(F) &= P(F) + \frac{1}{h}\int_F (\mathrm{dist}_E(x) - \mathrm{dist}_K(x))\,\mathrm{d}x \\
&\leq \liminf_{n\to\infty} P(F_n) + \frac{1}{h}\int_F (\mathrm{dist}_E(x) - \mathrm{dist}_K(x))\,\mathrm{d}x \\
&= \liminf_{n\to\infty}\left(P(F_n) + \frac{1}{h}\int_{F_n}\left(\mathrm{dist}_{E_n}(x) - \mathrm{dist}_{E_n^c}(x)\right)\,\mathrm{d}x\right) \\
&\leq \liminf_{n\to\infty}\left(P(E) + \frac{1}{h}\int_{E}\left(\mathrm{dist}_{E_n}(x) - \mathrm{dist}_{E_n^c}(x)\right)\,\mathrm{d}x\right) \\
&= P(E) - \frac{1}{h}\int_E \mathrm{dist}_K(x)\,\mathrm{d}x = \bar{J}_h(E)
\end{aligned}$$

where we exploited the lower-semicontinuity of the perimeter and the minimality of F_n. Since E is the unique volume-constrained minimizer of $\bar{J}_h$, the set F must coincide with E and this concludes the proof. □

Remark 5.3.2. We remark that under the hypotheses of Lemma 5.3.1 we could have just assumed the one-sided inclusion

$$\overline{E}_0 \subset (E)_{\delta^*}$$

instead of

$$\mathrm{d}_H(E_0,E) \leq \delta$$

for a suitable $\delta^* \leq \delta$. Indeed, let $\delta_n \to 0$ and $E_n \subset (E)_{\delta_n}$ such that $|E_n| = |E|$. We prove that $\overline{E}_n$ converges to $\overline{E}$ in the sense of Kuratowski, and thus in the Hausdorff distance (see [9] for the definition of Kuratowski convergence and its properties). Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence such that $x_n \in \overline{E}_n$ and $x_n \to y$. For every $n \in \mathbb{N}$, there exists $y_n \in E$ such that $|x_n - y_n| \leq \delta_n$. Therefore, for any $\varepsilon > 0$ there exists n_0 such that, for $n \geq n_0$, we have

$$|y_n - y| \leq |y_n - x_n| + |x_n - y| \leq \delta_n + \varepsilon,$$

that is $y_n \to y$. Since $\{y_n\}_{n\in\mathbb{N}} \subset E$, we have $y \in \overline{E}$.

Fix now $y \in \overline{E}$. Assume by contradiction that there exists $\delta > 0$ such that $\mathrm{dist}_{E_n}(y) > \delta$, i.e. it doesn't exist a sequence of elements in $\overline{E}_n$ converging to y. From this (and up to subsequences) it follows

$$E_n \subset (E)_{\delta_n} \setminus B_\delta(y) \qquad \forall n \in \mathbb{N}.$$

Thus we have

$$\begin{aligned} m = \lim_{n\to\infty} |E_n| &\leq \lim_{n\to\infty} |(E)_{\delta_n} \setminus B_\delta(y)| \\ &\leq \lim_{n\to\infty} |(E)_{\delta_n} \setminus (B_\delta(y) \cap E)\,| \\ &= \lim_{n\to\infty} |(E)_{\delta_n}| - |B_\delta(y) \cap E| = m - |B_\delta(y) \cap E| \end{aligned}$$

which is a contradiction.

We are now able to prove the main estimate that will be used in the proof of Proposition 5.4.2.

Proposition 5.3.3 (Uniform L^1-estimate)**.** *Let $E \subset \mathbb{T}^N$ be a strictly stable set. Then, for every $\varepsilon > 0$ there exist $\delta^* = \delta^*(\varepsilon, E) > 0$ and $h^* = h^*(E) > 0$ with the following property: for every $h < h^*$, if E_0 is a measurable set such that*

$$|E_0| = |E|, \qquad \overline{E}_0 \subset (E)_{\delta^*},$$

then the discrete flow $\{E_h^n\}_{n\in\mathbb{N}}$ *starting from* E_0 *satisfies*

$$\alpha(E,E_h^n) \leq \varepsilon$$

for every $n \in \mathbb{N}$*, where we recall* $\alpha(G,F) = \min_{x\in\mathbb{T}^N} |G\triangle(F+x)|$ *for every measurable sets* $G,F \subset \mathbb{T}^N$*.*

Proof. Fix $h < h^*$, where $h^* = h^*(E)$ is the constant given by Lemma 5.3.1 and let $\sigma = \sigma(E)$, $C = C(E)$ be the constants of Theorem 3.1.11. Moreover, let $\delta := \delta(\sigma,E)$ be the constant given by Lemma 5.3.1 with σ replacing ε. Set $\delta^* \leq \delta$ to be chosen later and consider E_0 such that

$$|E_0| = |E|, \qquad \overline{E}_0 \subset (E)_{\delta^*}.$$

Recall that, from Remark 5.3.2 and from the hypothesis $\overline{E}_0 \subset (E)_{\delta^*}$, without loss of generality, we can assume $\mathrm{d}_H(E_0,E) \leq \delta^*$. Moreover, by the regularity of E, we can also suppose $\alpha(E_0,E) \leq \tilde{C}\delta^*$, for a suitable constant $\tilde{C} > 0$ that only depends on E. From Lemma 5.3.1 we have that

$$|E_h^1\triangle E| \leq \sigma. \tag{5.3.3}$$

Let x_0 be such that $\alpha(E_0,E) = |E_0\triangle(E+x_0)|$. By choosing $E+x_0$ as a competitor for the minimality of E_h^1 and estimating $\mathrm{dist}_{\partial E_0} \leq \mathrm{diam}(\mathbb{T}^N) = \sqrt{N}$, we find

$$P(E_h^1) - P(E) \leq \frac{1}{h}\int_{E_0\triangle(E+x_0)} \mathrm{dist}_{\partial E_0}(x)\,\mathrm{d}x \leq \frac{\sqrt{N}}{h}\alpha(E_0,E) \leq \frac{\sqrt{N}}{h}\tilde{C}\delta^*.$$

By (5.3.3), we can apply Theorem 3.1.11 and the previous estimate to obtain

$$\alpha(E_h^1,E) \leq \frac{1}{\sqrt{C}}\sqrt{P(E_h^1)-P(E)} \leq \frac{1}{\sqrt{C}}\sqrt{\frac{\sqrt{N}}{h}\alpha(E,E_0)} \leq \frac{1}{\sqrt{C}}\sqrt{\frac{\sqrt{N}}{h}\tilde{C}\delta^*} \leq \min\{\sigma,\delta,\varepsilon\},$$

where we have chosen δ^* such that $\delta^* \leq Ch(\min\{\sigma,\delta,\varepsilon\})^2/(\tilde{C}\sqrt{N})$. Since E_h^1 is a Λ-minimizer and E is regular, up to taking δ^* smaller, the classical regularity theory for Λ-minimizers (see Theorem 3.1.3) implies

$$\mathrm{d}_H(\partial E_h^1, \partial E + x_1) \leq \delta,$$

where x_1 is such that $\alpha(E_h^1,E) = |E_h^1\triangle(E+x_1)|$.

Now we iterate the procedure: by induction, suppose that

$$\alpha(E_h^{n-1},E) \leq \min\{\sigma,\delta,\varepsilon\}, \qquad \mathrm{d}_H(\partial E_h^{n-1}, \partial E + x_{n-1}) \leq \delta \tag{5.3.4}$$

where x_{n-1} is such that $|E_h^{n-1}\triangle(E+x_{n-1})| = \alpha(E_h^{n-1},E)$. Observe that the second inequality in (5.3.4) implies that $\mathrm{d}_H(E_h^{n-1},E+x_{n-1}) \le \delta$, therefore E_h^{n-1} and $E+x_{n-1}$ satisfy the hypotheses of Lemma 5.3.1 and thus

$$|E_h^n\triangle(E+x_{n-1})| \le \sigma.$$

Observe that by definition $\alpha(E_h^n,E+x_{n-1}) = \alpha(E_h^n,E)$. Now, by Theorem 3.1.11 and the monotonicity of the perimeters along the discrete flow we obtain

$$\begin{aligned}\alpha(E_h^n,E) &\le \frac{1}{\sqrt{C}}\sqrt{P(E_h^n)-P(E)}\\ &\le \frac{1}{\sqrt{C}}\sqrt{P(E_h^1)-P(E)}\\ &\le \frac{1}{\sqrt{C}}\sqrt{\frac{\sqrt{N}}{h}\tilde{C}\delta^*} \le \min\{\sigma,\delta,\varepsilon\}.\end{aligned}$$

Again, thanks to the choice of δ^*, the hypotheses of Theorem 3.1.3 are satisfied and thus

$$\mathrm{d}_H(\partial E_h^n,\partial E+x_n) \le \delta,$$

where x_n is such that $\alpha(E_h^n,E) = |E_h^n\triangle(E+x_n)|$. This concludes the proof. □

5.3.2 Some remarks on the hypothesis of the L^1-estimate

In this subsection we show that Proposition 5.3.3 does not hold if we weaken the hypothesis of closeness in the Hausdorff distance between the starting set E_0 and the strictly stable set E. In particular, we prove that the sole hypothesis of closeness in L^1 and in perimeter is not enough. We remark that a modification of this example yields the same result in $\mathbb{R}^N$.

Fix $h>0$ and $G\subset\mathbb{T}^N$. Recall that, for any set $F\subset\mathbb{T}^N$ such that $|F|=|G|$, we have set

$$J_h(F,G) := P(F) + \frac{1}{h}\int_{F\triangle G}\mathrm{dist}_{\partial G}(x)\,\mathrm{d}x. \tag{5.3.5}$$

Proposition 5.3.4. *There exist $m>0$ and a sequence $\{E_n\}_{n\in\mathbb{N}}\subset\mathbb{T}^N$ with the following properties: $|E_n|=m$ for every $n\in\mathbb{N}$, $P(E_n)$ is uniformly bounded and, letting F_n be any volume-constrained minimizer of* (5.3.5) *with E_n instead of G, we have*

$$E_n \xrightarrow{L^1} E, \quad P(E_n)\to P(E) \quad \textit{but} \quad F_n \xrightarrow{L^1} F,$$

where E is a lamella and F is such that $|E\triangle F|>0$.

Proof. Let $m>0$ such that the ball of volume m has perimeter strictly less than the one of the lamella of the same volume; we remark that for every smaller volume $m' \le m$ the same property holds. Let E be a lamella of measure m, that is a set whose 1-periodic extension in $\mathbb{R}^N$ is stripe bounded by two parallel hyperplanes. Recall that E is a strictly stable set of the perimeter in $\mathbb{T}^N$. From the assumption on m it follows that E is only a local minimizer of the perimeter and not a global one.

Step 1. Firstly, we construct a sequence $\{E_n\}_{n\in\mathbb{N}}$ such that $E_n \to E$ in L^1 and $\partial E_n \to \mathbb{T}^N$ in the Hausdorff distance. We define E_n by adding to E some balls contained in $\mathbb{T}^N\setminus E$ and of overall small volume, and by subtracting to E balls contained in E with the same overall volume.

Recall that $\mathbb{T}^N=[0,1]^N/\mathbb{Z}^N$. In the following, with a little abuse of notation, we will identify $\mathbb{T}^N$ and $[0,1)^N$. We define

$$I_n := \left\{\underline{k}=(k_1,\dots,k_N)\in\mathbb{Z}^N : 0\le k_i\le 2^n-1 \quad \forall i=1,\dots N\right\},$$
$$\mathscr{P}_n := \left\{Q_{n,\underline{k}} := \left[0,\frac{1}{2^n}\right)^N+\frac{\underline{k}}{2^n} : \underline{k}\in I_n\right\},$$

for every $n\in\mathbb{N}$. Up to choosing m smaller, we can assume that $m=1/2^s$ for some $s\in\mathbb{N}$. Moreover, we can suppose, up to translations, that $E=[0,1)^{N-1}\times(0,1/2^s)$, thus for $n\ge s$ we have

$$E=\mathrm{Int}\left(\bigcup_{\underline{k}\in I_n,\, 0\le k_N\le 2^{n-s}-1} Q_{n,\underline{k}}\right),$$

where $\mathrm{Int}(\cdot)$ denotes the interior of a set in $\mathbb{T}^N$. For every $n\ge s$ and $\underline{k}\in I_n$, we consider the balls $B_{n,\underline{k}}\subset Q_{n,\underline{k}}$ centered in the center of the cube $Q_{n,\underline{k}}$ and of radius $r_{n,\underline{k}}$ chosen in such a way that

$$\left|\bigcup_{\underline{k}\in I_n,\, 0\le k_N\le 2^{n-s}-1} B_{n,\underline{k}}\right| = \left|\bigcup_{\underline{k}\in I_n,\, 2^{n-s}\le k_N\le 2^n-1} B_{n,\underline{k}}\right|. \tag{5.3.6}$$

Moreover, we can also take the radii $r_{n,\underline{k}}$ sufficiently small so that

$$\lim_{n\to\infty}\left|\bigcup_{\underline{k}\in I_n} B_{n,\underline{k}}\right|=0, \qquad \lim_{n\to\infty} P\left(\bigcup_{\underline{k}\in I_n} B_{n,\underline{k}}\right)=0. \tag{5.3.7}$$

Set now

$$A_n := \bigcup_{\underline{k}\in I_n,\, 0\le k_N\le 2^{n-s}-1} B_{n,\underline{k}} \subset \mathrm{Int}\left(\bigcup_{\underline{k}\in I_n,\, 0\le k_N\le 2^{n-s}-1} Q_{n,\underline{k}}\right)=E,$$

$$C_n := \bigcup_{\underline{k}\in I_n,\, 2^{n-s}\le k_N\le 2^n-1} B_{n,\underline{k}} \subset \bigcup_{\underline{k}\in I_n,\, 2^{n-s}\le k_N\le 2^n-1} Q_{n,\underline{k}} \subset \mathbb{T}^N \setminus E.$$

Define $E_n = (E\cup C_n)\setminus A_n$ and observe that, by (5.3.6), we have $|E_n| = |E|$. Now, by (5.3.7), we also obtain

$$E_n \overset{L^1}{\to} E \quad \text{and} \quad P(E_n)\to P(E).$$

Observe that, from the definition of A_n and C_n, we have that

$$(\partial A_n)_{\sqrt{N}/2^n} \cup (\partial C_n)_{\sqrt{N}/2^n} = \mathbb{T}^N$$

and therefore the set $\partial E_n = \partial E \cup \partial C_n \cup \partial A_n$ converges in the Hausdorff metric to the whole $\mathbb{T}^N$ as $n\to+\infty$. Therefore we have constructed a sequence $\{E_n\}_{n\in\mathbb{N}}$ that satisfies

$$E_n \overset{L^1}{\to} E, \qquad P(E_n)\to P(E), \qquad \partial E_n \overset{H}{\to} \mathbb{T}^N. \tag{5.3.8}$$

Step 2. Let E_n be the sets previously defined. We consider the space $X = \{F \subset \mathbb{T}^N : F \text{ is measurable}\}$ endowed with the L^1-distance, i.e. $\mathrm{dist}_{L^1}(F,G) = |F\triangle G|$ for every $F,G\in X$. We extend our functional in the following way

$$\tilde{J}_h(F,E) := \begin{cases} J_h(F,E) & \text{if } P(F)<\infty,\ |F|=m,\\ +\infty & \text{otherwise} \end{cases}$$

and we set $J_n := \tilde{J}_n(\cdot,E_n)$. We then prove the Γ-convergence of the functionals J_n to the perimeter functional in X (see for istance [30] for the definition of Γ-convergence and its properties), that is

$$\Gamma(X) - \lim_{n\to\infty} J_n = P. \tag{5.3.9}$$

We can clearly restrict ourselves to consider sets of finite perimeter and volume m, otherwise the result is trivial. For any given set F of measure m and finite perimeter we choose the sequence constantly equal to F as a recovery sequence for F. Indeed, by (5.3.8) we have

$$J_n(F) = P(F) + \frac{1}{h}\int_{F\triangle E_n} \mathrm{dist}_{\partial E_n} \to P(F).$$

We now prove the $\liminf$ inequality. Given a sequence F_n that converges to F in L^1, by the L^1-semicontinuity of the perimeter, we have

$$P(F) \le \liminf_{n\to\infty} P(F_n) \le \liminf_{n\to\infty}\left(P(F_n) + \frac{1}{h}\int_{F_n\triangle E_n} \mathrm{dist}_{\partial E_n}\right)$$

and thus (5.3.9) is proved. Therefore, thanks to the equi-coercivity of the functionals J_n, any sequence of volume-constrained global minimizers of J_n converges in L^1, up to a subsequence, to a volume-constrained global minimizer of the perimeter in the torus. Let $\{F_n\}_{n\in\mathbb{N}}$ be a sequence of global minimizers of the functional J_n and let F be such that $F_n \to F$ in L^1. We know that F is a global minimizer of the perimeter and that by the choice of m the lamella is not a global minimizer. Therefore it must hold $|E\triangle F| > 0$. ☐

5.4 Convergence of the flow

In this section, we will prove the first main result of the chapter concerning the stability of the discrete flow.

5.4.1 Convergence of the flow up to translations

In this subsection we characterize the long-time behaviour up to translations of the discrete mean curvature flow in the flat torus starting near a regular strictly stable set. We start by recalling a Lemma.

Lemma 5.4.1 ([82, Lemma 3.6]). *Let $\{E_h^n\}_{n\in\mathbb{N}}$ be a volume-preserving discrete flow starting from E_0 and let $E_h^{k_n}$ be a subsequence such that $E_h^{k_n} + \tau_n \to F$ in $L^1(\mathbb{T}^N)$ for some set F and a suitable sequence $\{\tau_n\}_{n\in\mathbb{N}} \subset \mathbb{T}^N$. Then $\mathrm{dist}_{\partial E_h^{k_n-1}}(\cdot + \tau_n) \to \mathrm{dist}_{\partial F}$ uniformly.*

Proposition 5.4.2. *Let $E \subset \mathbb{T}^N$ be a strictly stable set. Then there exist $\delta^* = \delta^*(E) > 0$ and $h^* = h^*(E) > 0$ with the following property: if $h < h^*$ and $E_0 \subset \mathbb{T}^N$ is a set of finite perimeter satisfying*

$$|E_0| = |E|, \qquad \overline{E}_0 \subset (E)_{\delta^*},$$

then, for every discrete flow $\{E_h^n\}_{n\in\mathbb{N}}$ starting from E_0, there exists a sequence of translations $\tau_n \in \mathbb{T}^N$ such that

$$E_h^n + \tau_n \to E \quad \text{in } C^k \text{ for every } k \in \mathbb{N}.$$

Proof. Let $\varepsilon > 0$ be sufficiently small and let $\delta^* = \delta^*(\varepsilon, E)$, $h^* = h^*(E)$ be the constants given by Proposition 5.3.3. Fix E_0 an initial set satisfying $|E| = |E_0|$ and $\overline{E}_0 \subset (E)_{\delta^*}$. It is enough to show that any (unrelabelled) subsequence of the discrete flow starting from E_0 admits a further subsequence converging in C^k and up to translations to E. We divide the proof into three steps.

Step 1. (Existence and regularity of a limit point) From Proposition 3.4.3 we remark that, for $n \geq 1$, the sets E_h^n are uniform Λ-minimizers with uniformly bounded, non-increasing perimeters. Therefore, by the compactness of (uniform) Λ-minimizers, we can conclude that there exists a subsequence $\{E_h^{k_n}\}_{n\in\mathbb{N}}$ and a Λ-minimizer E_h^∞ such that

$$E_h^{k_n} \xrightarrow{L^1} E_h^\infty, \quad P(E_h^{k_n}) \to P(E_h^\infty), \quad \mathrm{sd}_{E_h^{k_n-1}} \to \mathrm{sd}_{E_h^\infty} \text{ uniformly.}$$

Let G be a set of finite perimeter such that $|G| = m$. By the minimality of $E_h^{k_n}$ we have

$$P(E_h^{k_n}) + \frac{1}{h}\int_{E_h^{k_n}} \mathrm{sd}_{E_h^{k_n-1}}(x)\,\mathrm{d}x \leq P(G) + \frac{1}{h}\int_G \mathrm{sd}_{E_h^{k_n-1}}(x)\,\mathrm{d}x$$

and, taking the limit as $n \to \infty$, we obtain

$$P(E_h^\infty) + \frac{1}{h}\int_{E_h^\infty} \mathrm{sd}_{E_h^\infty}(x)\,\mathrm{d}x \leq P(G) + \frac{1}{h}\int_G \mathrm{sd}_{E_h^\infty}(x)\,\mathrm{d}x.$$

We have thus proved that E_h^∞ is a fixed point for the discrete flow and thus, by Proposition 3.4.4, it is a critical point for the perimeter.

Let $\tau_\infty \in \mathrm{argmin}_x |(E_h^\infty + x)\triangle E|$. By Proposition 5.3.3 we have $\alpha(E, E_h^{k_n}) \leq \varepsilon$ for every $n \in \mathbb{N}$. Now, up to taking ε smaller, Theorem 3.1.3 and the smoothness of E, yields both the $C^{1,\beta}$-closeness between $E_h^\infty + \tau_\infty$ and E, and the $C^{1,\beta}$ regularity of $E_h^\infty + \tau_\infty$ (and thus of E_h^∞), for every $\beta \in (0,1)$. From Proposition 3.4.3 *(iv)* it follows that E_h^∞ is of class $C^{2,\beta}$, therefore we conclude that E_h^∞ has constant classical mean curvature and thus it is of class C^∞. Finally, the smoothness of E_h^∞ allows us to use Theorem 3.1.3 to to ensure that the sets $E_h^{k_n}$ are of class $C^{1,\beta}$ for n large enough and to improve the convergence of the subsequence to

$$E_h^{k_n} \to E_h^\infty \quad \text{in} \quad C^{1,\beta}. \tag{5.4.1}$$

Step 2. (Convergence in $C^{2,\beta}$ of the flow and $C^{2,\beta}$-closeness to E) In this step we will prove that E_h^∞ is $C^{2,\beta}$-close to E and that the convergence of $E_h^{k_n}$ to E_h^∞ is in $C^{2,\beta}$. Without loss of generality, we assume that $\alpha(E, E_h^\infty) = |E \triangle E_h^\infty|$ so that the translations introduced by the previous step do not appear.

First of all we remark that, owing to the compactness of ∂E_h^∞, it suffices to show that the result holds locally. By a compactness argument and the definition of convergence of sets in $C^{1,\beta}$ (Definition 3.1.2), up to rotations and relabelling the coordinates, we can find a cylinder $C = B' \times (-L, L)$, where $B' \subset \mathbb{R}^{N-1}$ is a ball centred at the origin, and functions $f_\infty, f_n \in C^{1,\beta}(B'; (-L, L))$ describing locally $\partial E_h^\infty \cap C$ and $\partial E_h^{k_n} \cap C$ respectively.

We remark that the convergence (5.4.1) now reads as

$$f_{k_n} \to f_\infty \quad \text{in} \quad C^{1,\beta}(B'). \tag{5.4.2}$$

We now prove that the curvatures $H_{E_h^{k_n}}$ of the sequence $E_h^{k_n}$ are converging in $C^{0,\beta}$ to the curvature of E_h^∞ in the following sense

$$H_{E_h^{k_n}}(\cdot, f_{k_n}(\cdot)) \to H_{E_h^\infty}(\cdot, f_\infty(\cdot)) \quad \text{in} \quad C^{0,\beta}(B'). \tag{5.4.3}$$

We will follow an argument used in Step 3 of the proof of [4, Theorem 4.3].

Since we described $\partial E_h^{k_n} \cap C$ as a graph, the following formula for the curvature of $\partial E_h^{k_n}$ holds

$$\operatorname{div}\left(\frac{\nabla f_{k_n}(\cdot)}{\sqrt{1+|\nabla f_{k_n}(\cdot)|^2}}\right) = H_{E_h^{k_n}}(\cdot, f_{k_n}(\cdot)) \quad \text{on} \quad B' \tag{5.4.4}$$

and an analogous formula holds for ∂E_h^∞. From (5.4.4) and the Euler-Lagrange equation (3.4.3), by integrating on B', we then obtain

$$\begin{aligned} \lambda_{k_n}\mathscr{H}^{N-1}(B') - \frac{1}{h}\int_{B'} \mathrm{sd}_{E_h^{k_n-1}}(x', f_{k_n}(x'))\, \mathrm{d}\mathscr{H}^{N-1}(x') \qquad\qquad & \qquad (5.4.5)\\ = \int_{B'} H_{E_h^{k_n}}(x', f_{k_n}(x'))\, \mathrm{d}\mathscr{H}^{N-1}(x') & \\ = \int_{B'} \operatorname{div}\left(\frac{\nabla f_{k_n}(x')}{\sqrt{1+|\nabla f_{k_n}(x')|^2}}\right) \mathrm{d}\mathscr{H}^{N-1}(x') & \\ = \int_{\partial B'} \frac{\nabla f_{k_n}(y)}{\sqrt{1+|\nabla f_{k_n}(y)|^2}} \cdot y \, \mathrm{d}\mathscr{H}^{N-2}(y), & \end{aligned}$$

where we set $y = x'/|x'|$ and integrated by parts in the last line. We can then exploit the convergence (5.4.2) and the formula (5.4.4) for the curvature of E_h^∞ to prove

$$\begin{aligned} \int_{\partial B'} \frac{\nabla f_{k_n}(y)}{\sqrt{1+|\nabla f_{k_n}(y)|^2}} \cdot y \, \mathrm{d}\mathscr{H}^{N-2}(y) &\to \int_{\partial B'} \frac{\nabla f_\infty(y)}{\sqrt{1+|\nabla f_\infty|^2(y)}} \cdot y \, \mathrm{d}\mathscr{H}^{N-2}(y) \\ &= \int_{B'} \operatorname{div}\left(\frac{\nabla f_\infty(x')}{\sqrt{1+|\nabla f_\infty(x')|^2}}\right) \mathrm{d}\mathscr{H}^{N-1}(x') \\ &= H_{E_h^\infty}\mathscr{H}^{N-1}(B'). \end{aligned}$$

Now, Lemma 5.4.1 ensures that $\mathrm{sd}_{E_h^{k_n-1}} \to \mathrm{sd}_{E_h^\infty}$ uniformly and we can use the convergence (5.4.2) to obtain $\mathrm{sd}_{E_h^{k_n-1}}((\cdot, f_{k_n}(\cdot))) \to \mathrm{sd}_{E_h^\infty}((\cdot, f_\infty(\cdot))) = 0$ uniformly on B', since

$\partial E_h^\infty \cap C = \{(x', f_\infty(x')) : x' \in B')\}$ by definition. Therefore we find

$$\int_{B'} \mathrm{sd}_{E_h^{k_n-1}}((x', f_{k_n}(x')))\, \mathrm{d}\mathscr{H}^{N-1}(x') \to \int_{B'} \mathrm{sd}_{E_h^\infty}((x', f_\infty(x')))\, \mathrm{d}\mathscr{H}^{N-1}(x') = 0.$$

We then conclude that (5.4.5) converges to $H_{E_h^\infty}\mathscr{H}^{N-1}(B')$ and thus it must hold

$$\lambda_{k_n} \to H_{E_h^\infty}.$$

From (3.4.3), the previous result and the fact that the signed distance functions are all equi-lipschitz, we conclude that for any $\beta \in (0,1)$, the sequence $(H_{E_h^{k_n}}(\cdot, f_{k_n}(\cdot)))$ is bounded in $C^{0,\beta}(B')$ and thus it converges uniformly to $H_{E_h^\infty}(\cdot, f_\infty(\cdot))$. This proves the convergence (5.4.3).

We remark that the previous result also hold if we describe the sets of the flow $E_h^{k_n}$ as normal deformations of E_h^∞, that is there exist functions $\varphi_{k_n} : \partial E_h^\infty \to \mathbb{R}$ such that $E_h^{k_n} = (E_h^\infty)_{\varphi_{k_n}}$. In this case the convergence (5.4.1) reads as

$$\varphi_{k_n} \to 0 \quad \text{in} \quad C^{1,\beta}(\partial E_h^\infty),$$

and this and Lemma 5.4.1 ensure that

$$\mathrm{sd}_{E_h^{k_n-1}}(\cdot + \varphi_{k_n}(\cdot)\nu_{E_h^\infty}(\cdot)) \to \mathrm{sd}_{E_h^\infty}(\cdot) = 0 \quad \text{uniformly on } \partial E_h^\infty.$$

Now, the convergence of the curvatures reads as

$$H_{E_h^{k_n}}(\cdot + \varphi_{k_n}(\cdot)\nu_{E_h^\infty}(\cdot)) \to H_{E_h^\infty}(\cdot) \quad \text{in} \quad C^{0,\beta}(\partial E_h^\infty).$$

We can then apply directly [4, Lemma 7.2] to obtain that the subsequence $E_h^{k_n}$ is converging to E_h^∞ in $C^{2,\beta}$.

To prove the $C^{2,\beta}$-closeness of the limit point we argue by contradiction. Assume that a sequence of limit points $\{E_h^{\infty,l}\}_{l\in\mathbb{N}}$ is converging in $C^{1,\beta}$ to E but there exists $\sigma > 0$ such that

$$\mathrm{dist}_{C^{2,\beta}}(E, E_h^{\infty,l}) > \sigma$$

for every l large enough. Again, we describe locally $\partial E_h^{\infty,l}$ and ∂E as graphs of suitable functions $f_{\infty,l}, f : B' \to (-L, L)$ and we can repeat the same argument previously employed to prove that

$$H_{E_h^{\infty,l}}((\cdot, f_{\infty,l}(\cdot))) \to H_E((\cdot, f(\cdot))) \qquad \text{in } C^{0,\beta}(B').$$

This time the argument is simpler, since the limit points are stationary sets for the perimeter and thus their Euler-Lagrange equation is

$$H_{E_h^{\infty,l}} = \lambda_{E_h^{\infty,l}} \in \mathbb{R} \quad \text{on} \quad \partial E_h^{\infty,l}.$$

Again, Lemma 7.2 in [4] yields the desired contradiction.

Step 3. (Uniqueness up to translations and C^k convergence) By the previous step we can find a suitable function $\varphi_\infty \in C^{2,\beta}(\partial E)$ such that $E_h^\infty = E_{\varphi_\infty}$. Up to introducing a further translation given by Lemma 3.1.10, the hypotheses of Theorem 5.2.1 are satisfied and thus

$$\|\varphi_\infty\|_{H^1(\partial E)} \leq C\|H_{E_h^\infty} - \overline{H}_{E_h^\infty}\|_{L^2(\partial E)} = 0,$$

since the set E_h^∞ is a stationary set for the perimeter. Therefore E_h^∞ is a translated of the set E.

A standard bootstrap method based on the elliptic regularity theory combined with the Euler-Lagrange equation (3.4.3) yields the convergence in C^k for every $k \in \mathbb{N}$. □

5.4.2 Exponential convergence of the whole flow

In this subsection we will prove that the translations introduced in Proposition 5.4.2 decay to zero exponentially fast. In order to prove this result we will estimate the decay of the dissipations via a dissipation-dissipation inequality, which in turn relies on the quantitative Alexandrov type estimate established in Theorem 5.2.1. We start by recalling some preliminary results from [82].

For every measurable sets E, F, we recall the notation

$$\mathscr{D}(E,F) := \int_{E\Delta F} \mathrm{dist}_{\partial E}(x)\,\mathrm{d}x.$$

The following lemma is an adaptation to our case of [82, Lemma 3.8]. We report its proof for the reader's convenience.

Lemma 5.4.3 (*A priori* estimates)**.** *Let $\eta > 0$ and let $E \subset \mathbb{T}^N$ be a strictly stable set. There exists $\delta > 0$ with the following property: if $f_1, f_2 \in C^1(\partial E)$ with $\|f_i\|_{C^1(\partial E)} \leq \delta$ and $|E_{f_i}| = |E|$ for $i = 1,2$ we have*

$$C_1(1-\eta)\|f_1 - f_2\|_{L^2}^2 \leq \mathscr{D}(E_{f_1}, E_{f_2}) \leq C_1(1+\eta)\|f_1 - f_2\|_{L^2}^2 \tag{5.4.6}$$

$$\frac{1-\eta}{2}\int_{\partial E_{f_1}} \mathrm{sd}_{E_{f_2}}^2 \,\mathrm{d}\mathscr{H}^{N-1} \leq \mathscr{D}(E_{f_1}, E_{f_2}) \leq \frac{1+\eta}{2}\int_{\partial E_{f_1}} \mathrm{sd}_{E_{f_2}}^2 \,\mathrm{d}\mathscr{H}^{N-1} \tag{5.4.7}$$

$$|\mathrm{bar}(E_{f_1}) - \mathrm{bar}(E_{f_2})|^2 \leq C_2\|f_1 - f_2\|^2_{L^2} \leq \frac{C_2}{C_1(1-\eta)}\mathscr{D}(E_{f_1}, E_{f_2}) \tag{5.4.8}$$

for suitable constants $C_1, C_2 > 0$.

Proof of Lemma 5.4.3. The proof of equations (5.4.6) and (5.4.7) are quite analogous to the corresponding ones in [82]. We recall it for the sake of completeness and to highlight the minor differences between the two versions.

We start by observing that for any $\eta' > 0$, if δ is sufficiently small, then for every $p_0 \in \partial E_{f_2}$ the boundary of E_{f_2} in a small disc must be contained in a cone

$$\partial E_{f_2} \cap B_{4\bar{\delta}}(p_0) \subset G := \left\{ y \in \mathbb{R}^N \ : |(y - p_0) \cdot \nu_{E_{f_2}}(p_0)|^2 \leq \frac{\eta'^2}{1+\eta'^2}|y - p_0|^2 \right\}. \tag{5.4.9}$$

We then divide the rest of the proof into two steps.
Step 1. If δ is small enough , for every point $p = \lambda p_0 \in B_{2\delta}(p_0)$ $(\lambda > 0)$, we have that

$$\frac{1}{1+\eta'}|p - p_0| \leq \mathrm{dist}(p, \partial E_{f_2}) \leq |p - p_0|.$$

Indeed the second inequality is trivial by definition, since $p_0 \in \partial E_{f_2}$. Concerning the first one, set $q \in \partial E_{f_2}$ such that $\mathrm{dist}(p, \partial E_{f_2}) = |p - q|$, in particular $|p - q| \leq |p - p_0| \leq 2\delta$. From (5.4.9) we infer that $q \in G$ and thus we have

$$\mathrm{dist}(p, \partial E_{f_2}) \geq \mathrm{dist}(p, G) = \frac{1}{\sqrt{1+\eta'^2}}|p - p_0| \geq \frac{1}{1+\eta'}|p - p_0|$$

where we used the explicit formula for the projection of a point on a cone. If $p_0 := s + f_2(s)\nu_E(s) \in \partial E_{f_2}$ with $s \in \partial E$, we set

$$p_t := p_0 + t\frac{f_1(s) - f_2(s)}{|f_1(s) - f_2(s)|}\nu_E(s) \quad \text{for all } t \in [0, c|f_1(s) - f_2(s)|]$$

for an appropriate constant c such that the quantities defined are regular. We deduce that

$$\frac{1}{1+\eta'}t \leq \mathrm{dist}(p_t, \partial E_{f_2}) \leq t. \tag{5.4.10}$$

Keeping the same notation and using the coarea formula (also recall (5.2.9)), we infer that

$$\begin{aligned}\mathscr{D}(E_{f_1}, E_{f_2}) &= \int_{E_{f_1} \triangle E_{f_2}} \mathrm{dist}(x, \partial E_{f_2})\,\mathrm{d}x \\ &= \int_{\partial E} \mathrm{d}\mathscr{H}^{N-1}(s) \int_0^{c|f_1(s)-f_2(s)|} \mathrm{dist}(p_t, \partial E_{f_2}) J\Phi(s,t)\,\mathrm{d}t\end{aligned}$$

$$= \int_{\partial E} \mathrm{d}\mathscr{H}^{N-1}(s) \int_0^{c|f_1(s)-f_2(s)|} \mathrm{dist}(p_t, \partial E_{f_2})\, \mathrm{d}t$$
$$+ \int_{\partial E} \mathrm{d}\mathscr{H}^{N-1}(s) \int_0^{c|f_1(s)-f_2(s)|} \mathrm{dist}(p_t, \partial E_{f_2})(J\Phi(s,t)-1)\, \mathrm{d}t. \tag{5.4.11}$$

Recalling that for every $s \in \partial E$ we have that $J\Phi(s,\cdot)-1$ is Lipschitz continuous with constant H_E, for δ small enough and using (5.4.10), we get

$$\mathscr{D}(E_{f_1}, E_{f_2}) \leq (1+\delta H_E) \int_{\partial E} \mathrm{d}\mathscr{H}^{N-1}(s) \int_0^{c|f_1(s)-f_2(s)|} t\, \mathrm{d}t \tag{5.4.12}$$
$$= \frac{1+\delta H_E}{2} C^2 \int_{\partial E} |f_1(s)-f_2(s)|^2\, \mathrm{d}\mathscr{H}^{N-1}(s), \tag{5.4.13}$$

from which the second inequality in (5.4.6) follows by taking δ small enough. On the other hand, by (5.4.10) we also have

$$\mathscr{D}(E_{f_1}, E_{f_2}) \geq \frac{1-\delta H_E}{1+\eta'} \int_{\partial E} \mathrm{d}\mathscr{H}^{N-1}(s) \int_0^{c|f_1(s)-f_2(s)|} t\, \mathrm{d}t$$
$$= \frac{1-\delta H_E}{1+\eta'} C^2 \int_{\partial E} |f_1(s)-f_2(s)|^2\, \mathrm{d}\mathscr{H}^{N-1}(s), \tag{5.4.14}$$

from which the first inequality in (5.4.6) follows by taking η' and δ small enough.
Step 2. The inequalities (5.4.7) and (5.4.8) are now easy consequences. Indeed, by (5.4.10) we have that, for every $p_1 = (1+f_1(s))\nu_E(s) \in \partial E_{f_1}$, it holds

$$\frac{c}{1+\eta'} |f_1(s)-f_2(s)| \leq \mathrm{dist}(p_1, E_{f_2}) \leq c|f_1(s)-f_2(s)|.$$

Therefore (5.4.7) follows from (5.4.13) and (5.4.14), by taking η' and δ smaller if needed and using the same change of coordinates used previously (recall that $J\Phi$ and its inverse are estimated from above by $1+C\delta$ for a suitable constant $C>0$).

Finally, we prove (5.4.8). For δ small enough, we can bound the Jacobian by 2 and therefore we obtain

$$|\mathrm{bar}(E_{f_1}) - \mathrm{bar}(E_{f_2})|\,|E| = \left| \int_{E_{f_1}\setminus E_{f_2}} x\, \mathrm{d}x - \int_{E_{f_2}\setminus E_{f_1}} x\, \mathrm{d}x \right|$$
$$= \left| \int_{\partial E \cap \{f_1 > f_2\}} \mathrm{d}\mathscr{H}^{N-1}(s) \int_{f_2(s)}^{f_1(s)} (s+t\nu_E(s)) J\Phi(s)\, \mathrm{d}t \right.$$
$$\left. - \int_{\partial E \cap \{f_1 < f_2\}} \mathrm{d}\mathscr{H}^{N-1}(s) \int_{f_1(s)}^{f_2(s)} (s+t\nu_E(s)) J\Phi(s)\, \mathrm{d}t \right|$$
$$\leq 2 \left| \int_{\partial E} \big(2s + (|f_1(s)|+|f_2(s)|)\nu_E(s)\big)\, |f_1(s)-f_2(s)|\, \mathrm{d}\mathscr{H}^{N-1}(s) \right|$$

$$\leq C\|f_1 - f_2\|_{L^2}$$

and the conclusion follows from (5.4.6). □

The following lemma proves the crucial dissipation-dissipation inequality (5.4.16) (see [82, Lemma 3.9]). This result will play a central role in the proof of Theorem 5.1.1. Its proof is based on the Alexandrov-type estimate contained in Theorem 5.2.1.

Lemma 5.4.4. *Let $h > 0$ and let $E \subset \mathbb{T}^N$ be a strictly stable set. There exist constants C, $\delta > 0$ with the following property: for any pair of normal deformations E_{f_1}, E_{f_2} with $f_i \in C^2(\partial E)$, $\|f_i\|_{C^1(\partial E)} \leq \delta$, and such that $|E_{f_2}| = |E|$, $|\int_{\partial E} \nu_E f_2 \, \mathrm{d}\mathscr{H}^{N-1}| \leq \delta \|f_2\|_{L^2(\partial E)}$ and*

$$H_{E_{f_2}} + \frac{\mathrm{sd}_{E_{f_1}}}{h} = \lambda \quad on \quad \partial E_{f_2} \tag{5.4.15}$$

for some $\lambda \in \mathbb{R}$, we have

$$\mathscr{D}(E, E_{f_2}) \leq C\mathscr{D}(E_{f_2}, E_{f_1}). \tag{5.4.16}$$

Proof. By Theorem 5.2.1, for δ sufficiently small, we get

$$\begin{aligned} \|f_2\|^2_{L^2(\partial E)} &\leq C\|H_{E_{f_2}} - \overline{H}_{E_{f_2}}\|^2_{L^2(\partial E)} \leq C\|H_{E_{f_2}} - \lambda\|^2_{L^2(\partial E)} \\ &\leq 2C\|H_{E_{f_2}} - \lambda\|^2_{L^2(\partial E_{f_2})} = \frac{2C}{h^2}\int_{\partial E_{f_2}} \mathrm{sd}^2_{E_{f_1}} \, \mathrm{d}\mathscr{H}^{N-1}, \end{aligned}$$

where the third inequality follows by bounding the Jacobian of the change of variables by 2 (see (5.2.4)). By combining the previous inequalities with (5.4.6) and (5.4.7), we obtain the thesis. □

We are now able to prove our main result. The proof relies on our previous result Proposition 5.4.2, however this time we have to show that the translations introduced converge to an appropriate translation ξ. To achieve this result, we will obtain in Step 1 some estimates on the dissipations along the flow by comparing the energy with a suitable competitor. Once the (exponential) decay of the dissipations is proved, the convergence of the translations follows (see Step 2). The last step is devoted to prove the exponential convergence of the sets.

Proof of Theorem 5.1.1. Let $h^* > 0$, $\delta^* > 0$ and $\{\tau_n\}_{n\in\mathbb{N}}$ be given by Proposition 5.4.2. Fix $h < h^*$ and set $E_n := E^n_h$. We split the proof in three steps.

Step 1. (Exponential convergence of dissipations) Testing the minimality of E_n with E_{n-1} we obtain

$$P(E_n) + \frac{1}{h}\mathscr{D}(E_n, E_{n-1}) \leq P(E_{n-1}).$$

Recalling that $P(E_n) \to P(E)$ and summing the previous inequality from $n+1$ to $+\infty$ we get

$$\sum_{k=n+1}^{+\infty} \frac{1}{h} \mathscr{D}(E_k, E_{k-1}) \le P(E_n) - P(E). \tag{5.4.17}$$

We will now construct a suitable competitor to estimate the dissipation at the step $n-1$ with the difference of perimeters. Since, by Proposition 5.4.2, we have

$$E_n + \tau_n \to E \quad \text{in} \quad C^k \quad \forall k \in \mathbb{N}, \tag{5.4.18}$$

the translated sets of the flow, for n large enough, can be written as normal deformations of the set E, that is there exists $g_n : \partial E \to \mathbb{R}$ such that

$$E_n + \tau_n = E_{g_n},$$

where E_{g_n} was defined in Definition 3.1.1. The convergence (5.4.18) then reads as $g_n \to 0$ in C^k as $n \to \infty$. Let σ_n be the translations introduced by Lemma 3.1.10 with $E_n + \tau_n$ instead of F. From the convergence in C^k of $E_n + \tau_n$ to E, we deduce that $\sigma_n \to 0$ as $n \to \infty$. Therefore, setting

$$F_n := E_n + \tau_n + \sigma_n,$$

we have that $F_n \to E$ in C^k and $F_n = E_{f_n}$ with $f_n : \partial E \to \mathbb{R}$ satisfying

$$\left| \int_{\partial E} f_n \nu_E \, \mathrm{d}\mathscr{H}^{N-1} \right| \le \delta \|f_n\|_{L^2(\partial E)} \quad \text{and} \quad \|f_n\|_{W^{2,p}(\partial E)} \le C \|g_n\|_{W^{2,p}(\partial E)}$$

for $p > N-1$. Consider now the competitor

$$\mathscr{E}_n := E - \tau_{n-1} - \sigma_{n-1}.$$

From the minimality of E_n we easily deduce

$$P(E_n) + \frac{1}{h}\mathscr{D}(E_n, E_{n-1}) \le P(\mathscr{E}_n) + \frac{1}{h}\mathscr{D}(\mathscr{E}_n, E_{n-1}) = P(E) + \frac{1}{h}\mathscr{D}(E, E_{n-1} + \tau_{n-1} + \sigma_{n-1}) \tag{5.4.19}$$

where we used the translational invariance of the dissipations. From Lemma 5.4.1 we obtain that the sequence $E_{n-2} + \tau_{n-1} + \sigma_{n-1}$ converges in C^k to the same limit of $E_{n-1} + \tau_{n-1} + \sigma_{n-1}$, that is to E. In particular, for n large enough we can write $E_{n-2} + \tau_{n-1} + \sigma_{n-1} = E_\psi$ for a suitable function $\psi : \partial E \to \mathbb{R}$ (depending on n) and with $\|\psi\|_{C^1(\partial E)}$ small. From Lemma 5.4.4 we can then estimate the right hand side of (5.4.19) with

$$\mathscr{D}(E, E_{n-1} + \tau_{n-1} + \sigma_{n-1}) = \mathscr{D}(E, F_{n-1}) = \mathscr{D}(E, E_{f_{n-1}}) \le C\mathscr{D}(E_{f_{n-1}}, E_\psi)$$

$$
\begin{aligned}
&= C\mathscr{D}(E_{n-1} + \tau_{n-1} + \sigma_{n-1}, E_{n-2} + \tau_{n-1} + \sigma_{n-1}) \\
&= C\mathscr{D}(E_{n-1}, E_{n-2}).
\end{aligned}
$$

From the previous inequality and (5.4.19) we obtain

$$
P(E_n) - P(E) = P(E_n) - P(\mathscr{E}_n) \leq \frac{C}{h}\mathscr{D}(E_{n-1}, E_{n-2}). \tag{5.4.20}
$$

Now, (5.4.17) and (5.4.20) yield

$$
\begin{aligned}
\sum_{k=n-1}^{\infty} \frac{1}{h}\mathscr{D}(E_k, E_{k-1}) &= \sum_{k=n+1}^{\infty} \frac{1}{h}\mathscr{D}(E_k, E_{k-1}) + \frac{1}{h}\mathscr{D}(E_n, E_{n-1}) + \frac{1}{h}\mathscr{D}(E_{n-1}, E_{n-2}) \\
&\leq \frac{C+1}{h}\mathscr{D}(E_{n-1}, E_{n-2}) + \frac{1}{h}\mathscr{D}(E_n, E_{n-1}) \\
&\leq \frac{C+1}{h}\left(\mathscr{D}(E_{n-1}, E_{n-2}) + \mathscr{D}(E_n, E_{n-1})\right).
\end{aligned}
$$

We then apply Lemma 5.4.5 below (with $l = 2$) to conclude

$$
\mathscr{D}(E_n, E_{n-1}) \leq \left(1 - \frac{1}{C+1}\right)^{n/2} \left(P(E_0) - P(E)\right). \tag{5.4.21}
$$

Step 2. (Exponential convergence of barycenters) Set

$$
b = \left(1 - \frac{1}{C+1}\right)^{\frac{1}{4}} \in (0,1). \tag{5.4.22}
$$

From (5.4.18) and Lemma 5.4.1 both the sequences $\{E_n + \tau_n\}_{n\in\mathbb{N}}$ and $\{E_{n-1} + \tau_n\}_{n\in\mathbb{N}}$ converge in C^k to E. Therefore, for n large enough, there exist some functions $f_{1,n}$, $f_{2,n} \in C^k(\partial E)$ such that

$$
E_n + \tau_n = E_{f_{1,n}}, \quad E_{n-1} + \tau_n = E_{f_{2,n}}
$$

and $\|f_{i,n}\|_{C^k(\partial E)} \to 0$ as $n \to \infty$ for $i = 1, 2$. From (5.4.8) and (5.4.21) we can estimate for n sufficiently large

$$
\begin{aligned}
|\mathrm{bar}(E_n) - \mathrm{bar}(E_{n-1})| &= |\mathrm{bar}(E_n + \tau_n) - \mathrm{bar}(E_{n-1} + \tau_n)| \\
&= |\mathrm{bar}(E_{f_{1,n}}) - \mathrm{bar}(E_{f_{2,n}})| \\
&\leq C\sqrt{\mathscr{D}(E_{f_{1,n}}, E_{f_{2,n}})} = \sqrt{\mathscr{D}(E_n, E_{n-1})} \\
&\leq C\left(P(E_0) - P(E)\right)^{1/2} b^n.
\end{aligned}
$$

In turn, the above estimate implies that $\{\mathrm{bar}(E_n)\}_{n\in\mathbb{N}}$ satisfies the Cauchy condition, thus the whole sequence admits a limit $\bar{\xi}\in\mathbb{T}^N$. Moreover, the convergence is exponentially fast in the sense that

$$|\mathrm{bar}(E_{f_{1,n}})-\bar{\xi}|\leq\sum_{k=n+1}^{\infty}|\mathrm{bar}(E_{f_{1,n}})-\mathrm{bar}(E_{f_{2,n}})|\leq C\left(P(E_0)-P(E)\right)^{1/2}\frac{b^n}{1-b}$$

for n large enough. Recalling (5.4.18) we thus conclude that there exists a suitable translation $\xi\in\mathbb{T}^N$ such that for every $k\in\mathbb{N}$

$$E_n\to E-\xi\quad\text{in } C^k\quad\text{as } n\to\infty.$$

Step 3. (Exponential convergence of the sets) By the previous step we can write, for n large, the boundaries of the evolving sets as radial graphs of the limit set $E-\xi$. Precisely, for n large enough there exist functions f_n such that

$$E_n+\xi=E_{f_n}\quad\text{and}\quad\|f_n\|_{C^k(\partial E)}\to 0\quad\text{as } n\to\infty. \tag{5.4.23}$$

From (5.4.6) and for n large enough we have $\|f_n-f_{n-1}\|_{L^2(\partial E)}\leq 2\sqrt{\mathscr{D}(E_n,E_{n-1})}$ and thus, recalling (5.4.21) and arguing as in Step 2, we get

$$\|f_n\|_{L^2(\partial E)}\leq\sum_{k=n+1}^{\infty}\|f_n-f_{n-1}\|_{L^2(\partial E)}\leq\left(P(E_0)-P(E)\right)^{1/2}\frac{b^n}{1-b} \tag{5.4.24}$$

where b is as in (5.4.22). The above estimate yields the exponential decay of the L^2-norms of the radial graphs. We recall the well-known Gagliardo-Nieremberg inequality: for every $j\in\mathbb{N}$ there exists $C>0$ such that, if g is smooth enough on the boundary of a smooth set E, then

$$\|D^kg\|_{L^2(\partial E)}\leq C\|D^{2k}g\|^{1/2}_{L^2(\partial E)}\|g\|^{1/2}_{L^2(\partial E)} \tag{5.4.25}$$

where D^k stands for the collection of all the k-th order derivatives of g, see e.g. [11, Theorem 3.70]. Now, by (5.4.23) for every k there exists n_k such that $\sup_{n\geq n_k}\|D^{2k}f_n\|_{L^2(\partial E)}\leq 1$, therefore we may apply (5.4.25) to f_n to deduce from (5.4.24) that also $\|D^kf_n\|_{L^2(\partial E)}$ decays exponentially fast for all $k\in\mathbb{N}$. The Sobolev immersion theorem then yields the exponential decay in C^k for every k thus completing the proof of the result. □

Lemma 5.4.5. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of non-negative numbers. Assume furthermore that there exist $c>1$, $l\in\mathbb{N}$ such that $\sum_{n=k}^{\infty}a_n\leq c\sum_{j=k}^{k+l-1}a_j$ for every $k\in\mathbb{N}$. Then*

$$a_k\leq\left(1-\frac{1}{c}\right)^{\frac{k}{l}}S$$

for every $k \in \mathbb{N}$*, where* $S = \sum_{n=1}^{\infty} a_n$.

The proof of the previous lemma can be found in [82, Lemma 3.11].

5.5 Two-dimensional case

In this section, we completely characterize the long-time behaviour of the discrete flow in dimension two. This particular choice for the dimension is purely technical and can be justified as follows. In the two-dimensional flat torus we have a complete characterization of the critical points of the perimeter: they consist in unions of disjoint discs (having the same area) or in unions of disjoint lamellae (possibly having different areas), or their complements. It turns out that these sets are all strictly stable. This allows us to conclude that either the connected components of any limit point of the discrete flow or the ones of their complements are strictly stable sets. We remark that in higher dimension this could not be true anymore.

Fix h, $m > 0$ and let $\{E_h^n\}_{n\in\mathbb{N}}$ be a flow with initial set $E_0 \subset \mathbb{T}^2$ such that $|E_0| = m$. We recall that, by Proposition 3.4.3, there exists $a_0 > 0$ such that the distance between the connected components of the set E_h^n is at least a_0. Moreover, the proposition also provides a bound from below on the diameter of the connected components. Set

$$P_\infty := \lim_{n\to\infty} P(E_h^n)$$

as the limit of the monotone sequence of the perimeters along the discrete flow. Let F be any possible limit point of the sequence $\{E_n^h\}_{n\in\mathbb{N}}$. We observe that if F is a union of discs then its number of connected components must be $\pi^{-1}P_\infty^2/(4m)$ and therefore the form of the limit point is uniquely determinated up to translations. Analogously, if F is the complement of a union of discs, F^c is made of $\pi^{-1}P_\infty^2/(4-4m)$ connected components and thus it is uniquely determinated up to translations of its complement. In the case when F is a union of lamellae the number of connected components is, in general, less than or equal to $P_\infty/2$, and we have no information on the area of the single components.

Since we will consider h as a fixed parameter, from now on we will denote by E_n the set E_h^n.

Remark 5.5.1 (Remarks on the uniform $C^{1,\alpha}$-closeness to limit points)**.** We remark that for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that for every $n \geq n_0$ it holds

$$|E_n \triangle \bigcup_{i=1}^{l_n} F_{i,n}| \leq \varepsilon \quad \text{or} \quad |E_n^c \triangle \bigcup_{i=1}^{L_n} F_{i,n}| \leq \varepsilon, \tag{5.5.1}$$

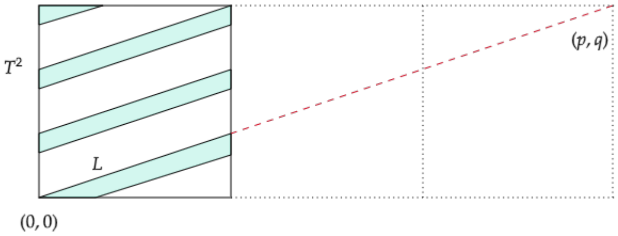

Fig. 5.1 The lamella L in light blue, the line a dashed in red.

where, in the first case, $\bigcup_{i=1}^{l_n} F_{i,n}$ is a union of disjoint lamellae or a union of disjoint discs, with $F_{i,n}$ having the same mass of the i-th connected component of E_n; l_n is either less than or equal to $P_\infty/2$ if $F_{i,n},\ i=1,\dots,l_n$, are lamellae or $l_n = \pi^{-1}P_\infty^2/(4m)$ if they are discs; in the second case, $\bigcup_{i=1}^{L_n} F_{i,n}$ is a union of disjoint discs, with $F_{i,n}$ having the same mass of the i-th connected component of E_n^c and $L_n = \pi^{-1}P_\infty^2/(4-4m)$. This can be easily proved recalling that any subsequence of the flow admits a further subsequence converging in L^1 to a set of the aforementioned form.

Moreover, the classical regularity theory of Λ-minimizers implies that the previous result can be improved. Consider, for the sake of simplicity, that E_n satisfies the first inequality in (5.5.1) (the other case being analogous). Then one can prove that for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that for every $n \geq n_0$ it holds

$$E_n = \bigcup_{i=1}^{l_n} (F_{i,n})_{f_{i,n}} \quad \text{where} \quad f_{i,n} \in C^{1,\alpha}(\partial F_{i,n}),\ \|f_{i,n}\|_{C^{1,\alpha}(\partial F_{i,n})} \leq \varepsilon. \tag{5.5.2}$$

Remark 5.5.2. In this remark, we identify $\mathbb{T}^2$ with the unit square $[0,1)^2$. We prove that for a fixed $M > 0$ there exists a finite number of slopes such that, for any lamella L having one of those slopes, we have $P(L) \leq M$.

Fix a lamella L. Let $a \subset \mathbb{T}^2$ be one of the two components of the boundary of L, and suppose that $(0,0) \in a$. Since a is a closed curve in $\mathbb{T}^2$, by periodicity, the line in $\mathbb{R}^2$ passing through the origin and with the same slope of a must also pass through a point of the form $(p,q) \in \mathbb{N} \times \mathbb{N}$ with p,q coprime or equal to $(0,1)$ or $(1,0)$. We then remark that the length in $\mathbb{T}^2$ of a is equal to the one of the segment between the origin and (p,q), that is $\text{length}(a) = |(p,q)|$. Since $P(L) = 2\,\text{length}(a)$, in order to have $P(L) \leq M$, the point (p,q) must be contained in the disc of radius $M/2$. Our claim follows since in the disc of radius $M/2$ there is a finite number of points belonging to $\mathbb{N} \times \mathbb{N}$.

In the following lemma we characterize the geometric form of any limit point of the discrete flow.

Lemma 5.5.3 (Uniqueness of the form of the limit). *Fix h, $m > 0$ and an initial set $E_0 \subset \mathbb{T}^2$ with mass m. Let $\{E_n\}_{n\in\mathbb{N}}$ be a discrete flow starting from E_0. Then either one of the following holds:*

i) the limit points of the flow are disjoint unions of l discs of total area m, where $l = \pi^{-1}(4m)^{-1}P_\infty^2$ belongs to $\mathbb{N}$,

ii) the limit points of the flow are the complement of disjoint unions of l discs of total area $1-m$, where $l = \pi^{-1}(4-4m)^{-1}P_\infty^2$ belongs to $\mathbb{N}$.

iii) the limit points of the flow are disjoint unions of l lamellae of total area m, with the same slope and $l \le P_\infty/2$. Moreover, the equality $l = P_\infty/2 \in \mathbb{N}$ holds if and only if the limit is given by vertical or horizontal lamellae.

Proof. We first employ a compactness argument and then use Lemma 5.4.1 to conclude. We start by fixing some notation. We denote by

$$\mathscr{E}_B := \bigcup_{i=1}^{l_B} B_i \tag{5.5.3}$$

any disjoint union of $l_B = 4^{-1}\pi m^{-1}P_\infty^2$ discs each having radius $2m/P_\infty$; we denote by

$$\mathscr{E}_{B^c} := \left(\bigcup_{i=1}^{l_{B^c}} B_i\right)^c \tag{5.5.4}$$

the complement of any disjoint union of $l_{B^c} = 4^{-1}\pi(1-m)^{-1}P_\infty^2$ discs, each of radius $2(1-m)/P_\infty$; we denote by

$$\mathscr{E}_L := \bigcup_{i=1}^{l_L} L_i \tag{5.5.5}$$

any disjoint union of $l_L \le P_\infty/2$ lamellae having the same slope (and possibly having different masses). We remark that, for every fixed P_∞ and m, the following holds

$$i := \inf\{ \mathrm{d}_H(\mathscr{E}_B,\mathscr{E}_L) \wedge \mathrm{d}_H(\mathscr{E}_{B^c},\mathscr{E}_L) \wedge \mathrm{d}_H(\mathscr{E}_{B^c},\mathscr{E}_B) \ : \ \mathscr{E}_L,\mathscr{E}_B,\mathscr{E}_{B^c} \text{ as above}\} > 0, \tag{5.5.6}$$

This is clear if we compare the families $\mathscr{E}_B,\mathscr{E}_{B^c}$ and a union of lamellae having the same slope. Since, by Remark 5.5.2, there is a finite number of possible slopes for the lamellae, we conclude (5.5.6). From Remark 5.5.1 the discrete flow is eventually C^1-close to a limit point of the form $\mathscr{E}_L,\mathscr{E}_B$ or $\mathscr{E}_{B^c}$. Assume now by contradiction that the flow does not converge to a fixed configuration. Then, without loss of generality, we can assume that for every $0 < \varepsilon < i/3$ there exist infinitely many indexes such that

$$\mathrm{d}_H(E_{n-1}, \mathscr{E}_B) \le \varepsilon \quad \text{and} \quad \mathrm{d}_H(E_n, \mathscr{E}_L) \le \varepsilon.$$

Therefore we get

$$\mathrm{d}_H(\mathscr{E}_B, \mathscr{E}_L) \le \mathrm{d}_H(\mathscr{E}_B, E_{n-1}) + \mathrm{d}_H(\mathscr{E}_L, E_n) + \mathrm{d}_H(E_n, E_{n-1}) \le 2\varepsilon + \mathrm{d}_H(E_n, E_{n-1}).$$

To reach the contradiction (compare (5.5.6)), it is enough to show that for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that for every $n \ge n_0$ it holds

$$\mathrm{d}_H(E_{n-1}, E_n) \le \varepsilon. \tag{5.5.7}$$

Assume by contradiction the existence of a subsequence n_k along which the flow satisfies

$$\mathrm{d}_H(E_{n_k-1}, E_{n_k}) > \varepsilon.$$

Up to a further subsequence, $E_{n_k} \to F$, with F being a set of the form $\mathscr{E}_B, \mathscr{E}_L$ or $\mathscr{E}_{B^c}$. But then Lemma 5.4.1 implies $\mathrm{sd}_{E_{n_k-1}} \to \mathrm{sd}_F$ uniformly, which is clearly a contradiction.

Finally, we observe that in case *iii*) the number of connected component is given by $\frac{P_\infty}{2|(p,q)|}$, where we used the same notation of Remark 5.5.2. Thus, $l = P_\infty/2$ if and only if (p,q) is equal to $(0,1)$ or to $(1,0)$ that means that the lamella is either vertical or horizontal. □

Thanks to the previous lemma we can then conclude the proof of Theorem 5.1.2, the main result of this section. While the proofs of assertions *i*) and *ii*) of Theorem 5.1.2 are similar to the one of [82, Theorem 3.4], the third one is slightly different, the main issue being that we can not fix the mass of the connected components of the limiting configuration. We will prove nonetheless the exponential convergence of the dissipations that, in turn, yields the convergence of the mass of the connected components of the flow. We start by a simple remark.

Remark 5.5.4 ($C^{1,\alpha}$**-closeness to lamellae).** Let $\varepsilon > 0$. Consider two lamellae L_1, L_2 having the same slope, possibly having different area and two $C^{1,\alpha}$-deformations E_1, E_2, respectively, of L_1 and L_2. Suppose also that

$$\mathrm{dist}_{C^{1,\alpha}}(E_i, L_i) \le \varepsilon, \quad i = 1, 2.$$

Then the closeness in L^∞ of E_1 and E_2 implies that E_2 and L_1 are close in $C^{1,\alpha}$. Indeed, we first remark that

$$\mathrm{dist}_{C^{1,\alpha}}(L_2, L_1) = \mathrm{dist}_{L^\infty}(L_2, L_1)$$

since the components of the boundaries of L_1 and L_2 differ only by a translation. Moreover, the hypothesis $\mathrm{dist}_{L^\infty}(E_1,E_2) \leq \varepsilon$ implies $\mathrm{dist}_{L^\infty}(L_2,L_1) \leq 2\varepsilon$. Now, let f_2 be a suitable function such that $E_2 = (L_2)_{f_2}$, then $\|f_2\|_{C^{1,\alpha}(\partial L_2)} \leq \varepsilon$ and there exists a constant $|c| \leq \mathrm{dist}_{L^\infty}(L_1,L_2) \leq 2\varepsilon$ such that $E_2 = (L_1)_{f_2+c}$. Therefore we obtain

$$\mathrm{dist}_{C^{1,\alpha}}(E_2,L_1) = \|f_2+c\|_{C^{1,\alpha}(\partial L_1)} \leq \|f_2\|_{C^{1,\alpha}(\partial L_2)} + |c| \leq \varepsilon + 2\varepsilon = 3\varepsilon.$$

Proof of Theorem 5.1.2. By Lemma 5.5.3, we can assume that all the limit points of the flow are sets either of the form $\mathscr{E}_B$, $\mathscr{E}_{B^c}$ or $\mathscr{E}_L$ (see (5.5.3), (5.5.4), (5.5.5)). To conclude we need to prove that the whole sequence converges in C^k and exponentially fast to a unique configuration.

In the case when the limit points are of the form $\mathscr{E}_B$, the proof follows the same spirit of [82, Theorem 3.4], but it is easier since we work in a compact space. The case when the limit points are of the form $\mathscr{E}_{B^c}$ is at all analogous: we simply remark that, if F is a minimizer of 3.4.2, then its complement is a minimizer of the same problem with E^c instead of E and with $1-m$ instead of m. By studying the evolution of the complement of the discrete flow, we can conclude as before.

Now, suppose that the limit points are of the form $\mathscr{E}_L$. We begin by observing that any subsequence of the flow admits a further subsequence converging in L^1 to a union of disjoint lamellae. Firstly, we prove the exponential decay of the dissipations. Testing the minimality of E_s with E_{s-1} we obtain

$$P(E_s) + \frac{1}{h}\mathscr{D}(E_s,E_{s-1}) \leq P(E_{s-1}).$$

Summing for $s \geq n+1$ we have

$$\sum_{s=n+1}^{+\infty} \frac{1}{h}\mathscr{D}(E_s,E_{s-1}) \leq P(E_n) - P_\infty. \tag{5.5.8}$$

With the notation previously introduced, for every ε we can choose n large enough such that (5.5.2) holds. Let $F_{i,n}$ be the sets given by (5.5.2): by Lemma 5.5.3, we know that $F_{i,n}$, $i=1,\dots,l_n$, are eventually lamellae and $l_n = l \geq P_\infty/2$.

We will now construct a suitable competitor to estimate the dissipation at the step $n-1$ with the difference of the perimeters. For n large enough consider the competitor $\mathscr{L}_n = \bigcup_{i=1}^{l} F_{i,n-1}$. We remark that, by definition and for n large enough, this competitor has perimeter $P(\mathscr{L}_n) = P_\infty$. By Proposition 3.4.3, there exists $s_0 = s_0(m,h,N,E_0) > 0$ such that the connected components $E_{i,n}$ of E_n satisfy $\mathrm{dist}\left(E_{i,n},E_{j,n}\right) \geq s_0$ for every $i \neq j$, moreover Remark 5.5.1 ensures that $\mathrm{dist}\left(F_{i,n-1},F_{j,n-1}\right) \geq s_0/2$ holds for n large enough and $i \neq j$. Thus, we can localize the dissipations

$$\begin{aligned}\mathscr{D}(E_n,E_{n-1}) &= \sum_{i=1}^{l}\mathscr{D}(E_{i,n},E_{i,n-1}), \\ \mathscr{D}(\mathscr{L}_n,E_{n-1}) &= \sum_{i=1}^{l}\mathscr{D}(F_{i,n-1},E_{i,n-1}).\end{aligned} \tag{5.5.9}$$

Testing the minimality of E_n with $\mathscr{L}_n$ and using the previous equality we have

$$P(E_n)+\frac{1}{h}\mathscr{D}(E_n,E_{n-1})\leq P(\mathscr{L}_n)+\frac{1}{h}\sum_{i=1}^{l}\mathscr{D}(F_{i,n-1},E_{i,n-1}). \tag{5.5.10}$$

Recalling Remark 5.5.4 and equations (5.5.2) and (5.5.7), we then obtain that the connected components of both E_{n-1} and E_{n-2} are small normal $C^{1,\alpha}$-deformations of the connected components of $\mathscr{L}_{n-1}$. Thus we can assume that both $E_{i,n-1}$ and $E_{i,n-2}$ can be described as normal deformation of $F_{i,n-1}$ for $i=1,\dots,k$. Let $f_{i,n-1}$ and $f_{i,n-2}$ be the functions (having small $C^{1,\alpha}$-norms) that describe respectively these deformations. Now, recalling Lemma 5.4.4, we can estimate

$$\begin{aligned}\mathscr{D}(F_{i,n-1},E_{i,n-1}) =&\mathscr{D}(F_{i,n-1},(F_{i,n-1})_{f_{i,n-1}})\leq C\mathscr{D}((F_{i,n-1})_{f_{i,n-1}},(F_{i,n-1})_{f_{i,n-2}}) \\ =&C\mathscr{D}(E_{i,n-1},E_{i,n-2}).\end{aligned}$$

Thus, from equations (5.5.9) and (5.5.10) we get

$$P(E_n)-P_\infty=P(E_n)-P(\mathscr{L}_n)\leq\frac{C}{h}\sum_{i=1}^{l}\mathscr{D}(E_{i,n-1},E_{i,n-2})=\frac{C}{h}\mathscr{D}(E_{n-1},E_{n-2})$$

and then (5.5.8) clearly yields

$$\begin{aligned}\sum_{s=n-1}^{\infty}\frac{1}{h}\mathscr{D}(E_s,E_{s-1}) &= \sum_{s=n+1}^{\infty}\frac{1}{h}\mathscr{D}(E_s,E_{s-1})+\frac{1}{h}\mathscr{D}(E_{n-1},E_{n-2})+\frac{1}{h}\mathscr{D}(E_n,E_{n-1}) \\ &\leq P(E_n)-P_\infty+\frac{1}{h}\mathscr{D}(E_{n-1},E_{n-2})+\frac{1}{h}\mathscr{D}(E_n,E_{n-1}) \\ &\leq\frac{C+1}{h}\mathscr{D}(E_{n-1},E_{n-2})+\frac{1}{h}\mathscr{D}(E_n,E_{n-1}) \\ &\leq\left(\frac{C+1}{h}\mathscr{D}(E_{n-1},E_{n-2})+\frac{1}{h}\mathscr{D}(E_n,E_{n-1})\right).\end{aligned}$$

We can then conclude using the same arguments of [82, Theorem 3.4]. □

Chapter 6

Asymptotic of the fractional discrete flow

Contents

6.1 Introduction

In this chapter, which is based on the results of [34], we develop the asymptotic analysis of the discrete flow in the fractional setting (see (3.4.4)). We consider the ambient space $\mathbb{R}^N$ and assume that the dimension N is such that any Λ-minimizer of the fractional perimeter is a smooth set. Namely, we will suppose that either:

- $N = 2$ and $s \in (0,1)$;
- $N \leq 7$ and $s \in (s_0, 1)$, where $s_0 > 0$ is the constant of [20, Theorem 3].

This is a technical hypothesis that could be dropped if, for instance, we knew that the evolving sets were smooth. In particular, it is essential to characterize the possible long-time limit points of the discrete flow. More precisely, the main theorem of the chapter is the following.

Theorem 6.1.1. *Let m, $M > 0$ and let $E_0 \subset \mathbb{R}^N$ be a bounded set with $P^s(E_0) \le M$, $|E_0| = m$. Then, for $h = h(s,M,m) > 0$ small enough the following holds: for any discrete flow E_h^n starting from E_0, there exists $\xi \in \mathbb{R}^N$ such that*

$$E_h^n - \xi \to B^{(m)} \quad \text{as} \quad n \to \infty \quad \text{in } C^k \text{ for every } k \in \mathbb{N},$$

where $B^{(m)}$ denotes the ball centered at the origin with volume equal to m. Moreover, the convergence is exponentially fast, meaning that the sets $E_h^n - \xi$ can be written as normal deformations of $B^{(m)}$ (recall Definition 3.1.1) induced by functions $f_n \in C^\infty(B^{(m)})$ such that $\|f_n\|_{C^k(\partial B^{(m)})} \le c_k e^{-c_k n}$, for some constants $c_k > 0$ possibly depending on k, m and M.

The proof of the main result follows the approach outlined in [82] regarding the asymptotic convergence of the discrete flow in the local setting. In Section 6.2, we establish a fractional quantitative Alexandrov theorem for normal deformations of a ball. This result is a crucial step in proving the asymptotic convergence of the discrete flow to a single ball, which is proved in Section 6.3. It is important to emphasize a significant distinction between the classical setting and the fractional one. In the classical setting, the possible limit points of the flow consist of unions of disjoint balls, each with the same radius. However, in the fractional case, the limit points reduce to a single ball. This is a peculiar feature of the nonlocal perimeter, that penalizes non-connected components.

6.2 A fractional quantitative Alexandrov Theorem

In this section, we are going to prove a quantitative Alexandrov-type inequality. We denote by $B \subset \mathbb{R}^N$ the unit ball centered at the origin and by ω_N its volume.

Theorem 6.2.1. *There exists $\delta = \delta(N) > 0$ with the following property: for any $f \in C^2(\partial B)$ such that $\|f\|_{C^1(\partial B)} \le \delta$, $|B_f| = \omega_N$ and $\mathrm{bar}(B_f) = \int_{B_f} x \,\mathrm{d}x = 0$, and for any $s \in (0,1)$, there exists $C = C(N,s) > 0$ such that*

$$\|f\|_{H^{\frac{1+s}{2}}(\partial B)} \le C \|H^s_{B_f} - \bar{H}^s_{B_f}\|_{L^2(\partial B)},$$

where we have set $\bar{H}^s_{B_f} := \fint_{\partial B} H^s_{B_f}(x + f(x)x) \,\mathrm{d}\mathscr{H}^{N-1}(x)$.

We will use the notation $\mathrm{d}\mathscr{H}^{N-1}_x = \mathrm{d}\mathscr{H}^{N-1}(x)$ and

$$[f]^2_{\frac{1+s}{2}} = [f]^2_{H^{\frac{1+s}{2}}(\partial B)} = \int_{\partial B}\int_{\partial B} \frac{|f(x)-f(y)|^2}{|x-y|^{N+s}} \,\mathrm{d}\mathscr{H}^{N-1}_x \,\mathrm{d}\mathscr{H}^{N-1}_y.$$

We start by finding representation formulas for the s-fractional perimeter and its first variation.

Lemma 6.2.2. *The following equalities hold true:*

1. *If $f \in C^2(\partial B)$ with $\|f\|_\infty$ sufficiently small, then*

$$P^s(B_f) = \frac{P^s(B)}{P(B)} \int_{\partial B} (1+f)^{N-s} \,\mathrm{d}\mathscr{H}^{N-1} + \\ + \frac{1}{2} \int_{\partial B} \int_{\partial B} \int_{1+f(y)}^{1+f(x)} \int_{1+f(y)}^{1+f(x)} F_{|x-y|}(r,\rho)\,\mathrm{d}r\,\mathrm{d}\rho\,\mathrm{d}\mathscr{H}^{N-1}_x\,\mathrm{d}\mathscr{H}^{N-1}_y, \tag{6.2.1}$$

where, for every $\theta, r, \rho \in (0,+\infty)$, we have set

$$F_\theta(r,\rho) := \frac{r^{N-1}\rho^{N-1}}{((r-\rho)^2 + r\rho\theta^2)^{\frac{N+s}{2}}}.$$

2. *If $f \in C^2(\partial B)$ with $\|f\|_\infty$ sufficiently small, then, for every $\psi \in C^1(\partial B)$, we have*

$$\partial P^s(B_f)[\psi] = (N-s)\frac{P^s(B)}{P(B)} \int_{\partial B} (1+f)^{N-s-1} \psi \,\mathrm{d}\mathscr{H}^{N-1} \\ + 2 \int_{\partial B} \int_{\partial B} \int_{f(y)}^{f(x)} \psi(x) F_{|x-y|}(1+f(x), 1+\rho)\,\mathrm{d}\rho\,\mathrm{d}\mathscr{H}^{N-1}_x\,\mathrm{d}\mathscr{H}^{N-1}_y. \tag{6.2.2}$$

Proof. To obtain (6.2.1) we use the calculations in the proof of [48, Theorem 2.1]. Using polar coordinates, we rewrite

$$P^s(B_f) = \int_{\partial B} \int_{\partial B} \int_0^{1+f(x)} \int_{1+f(y)}^{+\infty} F_{|x-y|}(r,\rho)\,\mathrm{d}r\,\mathrm{d}\rho\,\mathrm{d}\mathscr{H}^{N-1}_x\,\mathrm{d}\mathscr{H}^{N-1}_y.$$

Then by simmetry we get

$$P^s(B_f) = \frac{1}{2} \int_{\partial B} \int_{\partial B} \Big(\int_0^{1+f(x)} \int_{1+f(y)}^{+\infty} F_{|x-y|}(r,\rho)\,\mathrm{d}r\,\mathrm{d}\rho \\ + \int_0^{1+f(y)} \int_{1+f(x)}^{+\infty} F_{|x-y|}(r,\rho)\,\mathrm{d}r\,\mathrm{d}\rho \Big) \,\mathrm{d}\mathscr{H}^{N-1}_x\,\mathrm{d}\mathscr{H}^{N-1}_y.$$

Using the convention $\int_a^b = -\int_b^a$, we formally have

$$\int_0^b \int_a^{+\infty} + \int_0^a \int_b^{+\infty} = \int_a^b \int_a^b + \int_0^a \int_a^{+\infty} + \int_0^b \int_b^{+\infty},$$

which implies

$$\begin{aligned} P^s(B_f) = \int_{\partial B}\int_{\partial B}\Big(\frac{1}{2}\int_{1+f(y)}^{1+f(x)}\int_{1+f(y)}^{1+f(x)} F_{|x-y|}(r,\rho)\,\mathrm{d}r\,\mathrm{d}\rho \\ + \int_0^{1+f(x)}\int_{1+f(x)}^{+\infty} F_{|x-y|}(r,\rho)\,\mathrm{d}r\,\mathrm{d}\rho\Big)\,\mathrm{d}\mathscr{H}^{N-1}_x\,\mathrm{d}\mathscr{H}^{N-1}_y. \end{aligned} \tag{6.2.3}$$

By rescaling variables, for every $x \in \partial B$, we obtain that

$$\begin{aligned} &\int_{\partial B}\int_0^{1+f(x)}\int_{1+f(x)}^{+\infty} F_{|x-y|}(r,\rho)\,\mathrm{d}r\,\mathrm{d}\rho\,\mathrm{d}\mathscr{H}^{N-1}_y \\ &\qquad = (1+f(x))^{N-s}\int_{\partial B}\int_0^1\int_1^{+\infty} F_{|x-y|}(r,\rho)\,\mathrm{d}r\,\mathrm{d}\rho\,\mathrm{d}\mathscr{H}^{N-1}_y. \end{aligned}$$

By symmetry, we observe that the triple integral on the right hand side does not depend on $x \in \partial B$, then, by taking $f = 0$ in (6.2.3), we can calculate its value and get

$$P^s(B) = P(B)\int_{\partial B}\int_0^1\int_1^{+\infty} F_{|x-y|}(r,\rho)\,\mathrm{d}r\,\mathrm{d}\rho\,\mathrm{d}\mathscr{H}^{N-1}_y.$$

Now, equation (6.2.1) follows from the previous observations.

To prove (6.2.2), we take the derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} P^s(B_{f+t\psi})$$

in formula (6.2.1) and, recalling that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\alpha(t)}^{\beta(t)}\int_{\alpha(t)}^{\beta(t)} F(r,\rho)\,\mathrm{d}\rho\,\mathrm{d}r\right] &= \int_{\alpha(t)}^{\beta(t)}(F(\beta(t),\rho)\beta'(t) - F(\alpha(t),\rho)\alpha'(t))\,\mathrm{d}\rho \\ &\quad + \int_{\alpha(t)}^{\beta(t)}(F(r,\beta(t))\beta'(t) - F(r,\alpha(t))\alpha'(t))\,\mathrm{d}r \end{aligned}$$

for every α, $\beta : \mathbb{R} \to \mathbb{R}$ of class C^1 and $F \in C^0(\mathbb{R}\times\mathbb{R})$, we conclude

$$\begin{aligned} \partial P^s(B_f)[\psi] &= \int_{\partial B}\int_{\partial B}\int_{1+f(y)}^{1+f(x)}\big(\psi(x)F_{|x-y|}(1+f(x),\rho) - \psi(y)F_{|x-y|}(1+f(y),\rho)\big)\,\mathrm{d}\rho \\ &\quad + (N-s)\frac{P^s(B)}{P(B)}\int_{\partial B}(1+f)^{N-s-1}\psi\,\mathrm{d}\mathscr{H}^{N-1}. \end{aligned}$$

By symmetry and by a simple change of coordinates we obtain (6.2.2). □

Lemma 6.2.3. *If $f \in C^2(\partial B)$ with $\|f\|_{C^1(\partial B)} \leq \delta$ sufficiently small, then we have*

$$\partial P^s(B_f)[1] = (N-s)\frac{P^s(B)}{P(B)}\int_{\partial B}\left(1+(N-s-1)f+O(f^2)\right)\mathrm{d}\mathscr{H}^{N-1} + O([f]^2_{\frac{1+s}{2}}), \tag{6.2.4}$$

$$\begin{aligned}\partial P^s(B_f)[f] =&(N-s)\frac{P^s(B)}{P(B)}\int_{\partial B}\left(1+(N-s-1)f+O(f^2)\right)f\,\mathrm{d}\mathscr{H}^{N-1}\\ &+\int_{\partial B}\int_{\partial B}\frac{(f(x)-f(y))^2}{|x-y|^{N+s}}\,\mathrm{d}\mathscr{H}^{N-1}_x\,\mathrm{d}\mathscr{H}^{N-1}_y + O([f]^2_{\frac{1+s}{2}})\|f\|_{C^1(\partial B)}.\end{aligned} \tag{6.2.5}$$

With a slight abuse of notation, we have denoted with $O(f)$ a function $g(x) = r(x)f(x)$ for every $x \in \partial B$, where $\|r\|_{L^\infty(\partial B)} \leq C$, for some constant C depending only on the apriori bound $\|f\|_{C^1(\partial B)} \leq 1$.

Proof. Let $\psi \in C^1(\partial B)$, by expanding the first term in (6.2.2), we obtain

$$\begin{aligned}\partial P^s(B_f)[\psi] =&(N-s)\frac{P^s(B)}{P(B)}\int_{\partial B}(1+(N-s-1)f+O(f^2))\psi\,\mathrm{d}\mathscr{H}^{N-1}\\ &+2\int_{\partial B}\int_{\partial B}\int_{f(y)}^{f(x)}\psi(x)F_{|x-y|}(1+f(x),1+\rho)\,\mathrm{d}\rho\,\mathrm{d}\mathscr{H}^{N-1}_x\,\mathrm{d}\mathscr{H}^{N-1}_y\,\mathrm{d}\rho.\end{aligned} \tag{6.2.6}$$

We remark that, fixed $x, y \in \partial B$, for every ρ that varies between the values $f(y)$ and $f(x)$, we have $|f(x)-\rho| \leq \|\nabla f\|_\infty |x-y| \leq \delta|x-y|$. From this observation we can expand the denominator of $F_{|x-y|}(1+f(x),1+\rho)$, when $x \neq y$, and get

$$\begin{aligned}&|(f(x)-\rho)^2+(1+f(x))(1+\rho)|x-y|^2|^{-\frac{N+s}{2}}\\ &=\frac{1}{|x-y|^{N+s}}\left(\frac{(f(x)-\rho)^2}{|x-y|^2}+f(x)+\rho+f(x)\rho+1\right)^{-\frac{N+s}{2}}\\ &=\frac{1}{|x-y|^{N+s}}\left(1+O(\|f\|_{C^1})\right).\end{aligned} \tag{6.2.7}$$

Plugging formula (6.2.7) into the second addend of (6.2.6) and by symmetry again, we obtain

$$\begin{aligned}&2\int_{\partial B}\int_{\partial B}\int_{f(y)}^{f(x)}\psi(x)F_{|x-y|}(1+f(x),1+\rho)\,\mathrm{d}\rho\,\mathrm{d}\mathscr{H}^{N-1}_x\,\mathrm{d}\mathscr{H}^{N-1}_y\\ &=2\int_{\partial B\times\partial B}\frac{\psi(x)}{N}\frac{(1+f(x))^{N-1}}{|x-y|^{N+s}}\left((1+f(x))^N-(1+f(y))^N\right)(1+O(\|f\|_{C^1}))\,\mathrm{d}\mathscr{H}^{N-1}_x\,\mathrm{d}\mathscr{H}^{N-1}_y\\ &=\int\frac{\left(\psi(x)(1+f(x))^{N-1}-\psi(y)(1+f(y))^{N-1}\right)\left((1+f(x))^N-(1+f(y))^N\right)}{N|x-y|^{N+s}}(1+O(\|f\|_{C^1})).\end{aligned}$$

Now, if $\psi = 1$ by a simple Taylor expansion we conclude

$$2\int_{\partial B\times\partial B}\int_{f(y)}^{f(x)} F_{|x-y|}(1+f(x),1+\rho) = (N-1)\int_{\partial B\times\partial B}\frac{(f(x)-f(y))^2}{|x-y|^{N+s}}(1+O(\|f\|_{C^1})) = O([f]^2_{\frac{1+s}{2}}),$$

while the choice $\psi = f$ yields

$$2\int_{\partial B\times\partial B}\int_{f(y)}^{f(x)} f(x)F_{|x-y|}(1+f(x),1+\rho) = \int_{\partial B\times\partial B}\frac{(f(x)-f(y))^2}{|x-y|^{N+s}}(1+O(\|f\|_{C^1})).$$

□

In order to prove Theorem 6.2.1, we need the following lemma, which states the coercivity of the second variation of the fractional perimeter of a ball with respect to normal deformations. Its proof is contained in [48, Theorem 8.1]. We start by defining

$$\lambda_1^s := s(N-s)\frac{P^s(B)}{P(B)}. \tag{6.2.8}$$

Lemma 6.2.4. *There exists $\delta > 0$ small such that, if $f \in C^2(\partial B)$ with $\|f\|_{C^1(\partial B)} \le \delta$, $|B_f| = \omega_N$ and $\mathrm{bar}(B_f) = 0$, then we have*

$$\begin{aligned}\partial^2 P^s(B)[f] &= \int_{\partial B}\int_{\partial B}\frac{(f(x)-f(y))^2}{|x-y|^{N+s}}\,\mathrm{d}\mathcal{H}_x^{N-1}\,\mathrm{d}\mathcal{H}_y^{N-1} - \lambda_1^s\int_{\partial B}|f|^2\,\mathrm{d}\mathcal{H}^{N-1}\\ &\ge \frac{1}{4}\Big([f]^2_{\frac{1+s}{2}} + \lambda_1^s\|f\|^2_{L^2(\partial B)}\Big).\end{aligned}$$

We are now in position to prove Theorem 6.2.1.

Proof of Theorem 6.2.1. Without loss of generality, we assume that $\|H^s_{B_f} - \bar{H}^s_{B_f}\|_{L^2} \le 1$. Let $\Phi : \partial B \to \partial B_f \subset \mathbb{R}^N$ be the map defined by $\Phi(x) = (1+\ f(x))x$, by direct computations one can prove that

$$J\Phi(x) = (1+f(x))^{N-1}\sqrt{1+(1+f(x))^{-2}|\nabla f(x)|}.$$

For every $\psi \in C^1(\partial B)$, let

$$X : \mathbb{R}^N \to \mathbb{R}^N, \quad X(x) := \frac{x}{|x|}\psi\left(\frac{x}{|x|}\right).$$

Employing the area formula we get

$$\begin{aligned}\partial P^s(B_f)[\psi] &= \int_{\partial B_f} H^s_{B_f} \nu_{B_f} \cdot X \,\mathrm{d}\mathscr{H}^{N-1} \\ &= \int_{\partial B} H^s_{B_f}(p) \nu_{B_f}(p) \cdot x \psi(x) J\Phi(x) \,\mathrm{d}\mathscr{H}^{N-1}_x \\ &= \int_{\partial B} H^s_{B_f}(p)\, \psi(x)\, (1+f(x))^{N-1} \,\mathrm{d}\mathscr{H}^{N-1}_x,\end{aligned}$$

where we have set $p=(1+f(x))x$ (for more details see Section 5.2 and [82, Section 1]). Now, by a simple Taylor expansion we obtain

$$\partial P^s(B_f)[\psi] = \int_{\partial B} H^s_{B_f}(p)\, \psi(x)\, (1+(N-1)f(x)+O(f^2)) \,\mathrm{d}\mathscr{H}^{N-1}_x. \tag{6.2.9}$$

We recall that

$$H^s_B(x) = (N-s)\frac{P^s(B)}{P(B)} \quad \text{for all } x \in \partial B.$$

If $\psi=1$, by combining formulas (6.2.4) and (6.2.9), we infer

$$\int_{\partial B} (H^s_{B_f}(p) - H^s_B)(1+(N-1)f(x)+O(f^2)) \,\mathrm{d}\mathscr{H}^{N-1}_x = \int_{\partial B} O(f) \,\mathrm{d}\mathscr{H}^{N-1} + O([f]^2_{\frac{1+s}{2}}) \tag{6.2.10}$$

and if $\psi=f$, by combining equations (6.2.5) and (6.2.9), we get

$$\begin{aligned}&\int_{\partial B}\int_{\partial B} \frac{(f(x)-f(y))^2}{|x-y|^{N+s}} \,\mathrm{d}\mathscr{H}^{N-1}_x \,\mathrm{d}\mathscr{H}^{N-1}_y - s(N-s)\frac{P^s(B)}{P(B)} \int_{\partial B} f^2 \,\mathrm{d}\mathscr{H}^{N-1} \\ &= \int_{\partial B} \left(H^s_{B_f}(p) - H^s_B\right) (1+(N-1)f(x)+O(f^2)) f(x) \,\mathrm{d}\mathscr{H}^{N-1}_x + O([f]^2_{\frac{1+s}{2}})\|f\|_{C^1}.\end{aligned} \tag{6.2.11}$$

Using the same arguments of the proof of Theorem 5.2.1 (see also [82, Theorem 1.3]) we can conclude. For the sake of completeness we present a sketch of the proof.

By (6.2.10), for δ sufficiently small, using Hölder's inequality we obtain

$$\begin{aligned}\left|\bar{H}^s_{B_f} - H^s_B\right| &\le \left| - \fint_{\partial B} (H^s_{B_f} - H^s_B)((N-1)f + O(f^2)) \,\mathrm{d}\mathscr{H}^{N-1}\right| \\ &\quad + \int_{\partial B} O(|f|) \,\mathrm{d}\mathscr{H}^{N-1} + O([f]^2_{\frac{1+s}{2}}) \\ &\le \left| \fint_{\partial B} (H^s_{B_f} - \bar{H}^s_{B_f})((N-1)f + O(f^2)) \,\mathrm{d}\mathscr{H}^{N-1}\right| \\ &\quad + \left| \fint_{\partial B} (\bar{H}^s_{B_f} - H^s_B)((N-1)f + O(f^2)) \,\mathrm{d}\mathscr{H}^{N-1}\right| \\ &\quad + \int_{\partial B} O(|f|) \,\mathrm{d}\mathscr{H}^{N-1} + O([f]^2_{\frac{1+s}{2}})\end{aligned}$$

$$
\begin{aligned}
&\leq \delta \frac{N-1+C\delta}{P(B)} \|H^s_{B_f} - \bar{H}^s_{B_f}\|_{L^2} + \delta\,(N-1+C\delta)\,|\bar{H}^s_{B_f} - H^s_B| \\
&\quad + \int_{\partial B} O(|f|)\,\mathrm{d}\mathscr{H}^{N-1} + O([f]^2_{\frac{1+s}{2}}),
\end{aligned}
$$

with $C = C(N)$. Recalling that $\|H_{B_f} - \bar{H}_{B_f}\|_{L^2} \leq 1$, For δ small enough, the previous inequality implies

$$
\frac{1}{2}|\bar{H}^s_{B_f} - H^s_B| \leq C\delta \|H^s_{B_f} - \bar{H}^s_{B_f}\|_{L^2} + \int_{\partial B} O(|f|)\,\mathrm{d}\mathscr{H}^{N-1} + O([f]^2_{\frac{1+s}{2}}) \leq C\delta. \tag{6.2.12}
$$

By (6.2.11) and using again Hölder's inequality, we get

$$
\begin{aligned}
&\int_{\partial B}\int_{\partial B} \frac{(f(x)-f(y))^2}{|x-y|^{N+s}}\,\mathrm{d}\mathscr{H}^{N-1}_x\,\mathrm{d}\mathscr{H}^{N-1}_y - s(N-s)\frac{P^s(B)}{P(B)}\int_{\partial B} f^2\,\mathrm{d}\mathscr{H}^{N-1} \\
&= \int_{\partial B}\left(H^s_{B_f}(p) - H^s_B\right)(1+(N-1)f+O(f^2))f\,\mathrm{d}\mathscr{H}^{N-1} + O([f]^2_{\frac{1+s}{2}})\|f\|_{C^1} \\
&= \int_{\partial B}(H^s_{B_f}(p) - \bar{H}^s_{B_f})(1+(N-1)f+O(f^2))f\,\mathrm{d}\mathscr{H}^{N-1} \\
&\quad + \int_{\partial B}(\bar{H}^s_{B_f} - H^s_B)(1+(N-1)f+O(f^2))f\,\mathrm{d}\mathscr{H}^{N-1} + O([f]^2_{\frac{1+s}{2}})\|f\|_{C^1} \\
&\leq C\|H^s_{B_f} - \bar{H}^s_{B_f}\|_{L^2}\|f\|_{L^2} + |\bar{H}^s_{B_f} - H^s_B|\int_{\partial B}(1+(N-1)f+O(f^2))f\,\mathrm{d}\mathscr{H}^{N-1} \\
&\quad + O([f]^2_{\frac{1+s}{2}})\|f\|_{C^1}.
\end{aligned}
\tag{6.2.13}
$$

Since $|B_f| = \omega_N$, we have

$$
\left|\int_{\partial B} f\,\mathrm{d}\mathscr{H}^{N-1}\right| = \int_{\partial B} O(f^2)\,\mathrm{d}\mathscr{H}^{N-1}. \tag{6.2.14}
$$

By (6.2.12) and (6.2.14), we obtain

$$
|\bar{H}^s_{B_f} - H^s_B|\int_{\partial B}(f+O(f^2))\,\mathrm{d}\mathscr{H}^{N-1} \leq \delta\int_{\partial B} O(f^2).
$$

Finally, by combining the above inequality, (6.2.12), (6.2.13) and (6.2.14), we deduce that, for any $\eta > 0$, it holds

$$
\begin{aligned}
&\int_{\partial B}\int_{\partial B} \frac{(f(x)-f(y))^2}{|x-y|^{N+s}}\,\mathrm{d}\mathscr{H}^{N-1}_x\,\mathrm{d}\mathscr{H}^{N-1}_y - s(N-s)\frac{P^s(B)}{P(B)}\int_{\partial B} f^2\,\mathrm{d}\mathscr{H}^{N-1} \\
&\leq C\|H^s_{B_f} - \bar{H}^s_{B_f}\|_{L^2}\|f\|_{L^2} + C\delta(\|f\|^2_{L^2} + [f]^2_{\frac{1+s}{2}}) \qquad (6.2.15) \\
&\leq \frac{1}{\eta}C^2\|H^s_{B_f} - \bar{H}^s_{B_f}\|^2_{L^2} + \eta\|f\|^2_{L^2} + C\delta(\|f\|^2_{L^2} + [f]^2_{\frac{1+s}{2}}). \qquad (6.2.16)
\end{aligned}
$$

The conclusion then follows combining (6.2.16) with Lemma 6.2.4 and taking δ and η sufficiently small. □

Remark 6.2.5. By slightly changing the last step in the previous proof we can prove the quantitative Alexandrov result in the classical case, [82, Theorem 1.1]. First, we remark that (6.2.15) can be read as

$$\begin{aligned}(1-s)\left(\int_{\partial B}\int_{\partial B}\frac{(f(x)-f(y))^2}{|x-y|^{N+s}}\,\mathrm{d}\mathscr{H}^{N-1}_x\,\mathrm{d}\mathscr{H}^{N-1}_y - s(N-s)\frac{P^s(B)}{P(B)}\int_{\partial B} f^2\,\mathrm{d}\mathscr{H}^{N-1}\right)\\ \le C\left\|(1-s)\left(H^s_{B_f}-\bar{H}^s_{B_f}\right)\right\|_{L^2}\|f\|_{L^2}+C\delta(1-s)(\|f\|^2_{L^2}+[f]^2_{\frac{1+s}{2}}),\end{aligned}$$

by Lemma 6.2.4, we obtain

$$\begin{aligned}\frac{1-s}{4}\left(\lambda^s_1\|f\|^2_{L^2}+[f]^2_{\frac{1+s}{2}}\right)\le\frac{C^2}{\eta}\left\|(1-s)\left(H^s_{B_f}-\bar{H}^s_{B_f}\right)\right\|^2_{L^2}+\eta\|f\|^2_{L^2}\\ +C\delta(1-s)(\|f\|^2_{L^2}+[f]^2_{\frac{1+s}{2}}).\end{aligned}\tag{6.2.17}$$

By recalling the definition of λ^s_1 (see (6.2.8)), and by Theorem 3.2.1 we obtain

$$\lim_{s\to 1}(1-s)\lambda^s_1=(N-1)\omega_{N-1}.$$

Finally, using Theorems 3.2.2, 3.2.3, we can take the limit as $s\to 1^-$ in the inequality 6.2.17 and get

$$\frac{1}{4}\left((N-1)\omega_{N-1}\|f\|^2_{L^2}+C\|\nabla f\|^2_{L^2}\right)\le\frac{C^2}{\eta}\left\|\omega_{N-1}\left(H_{B_f}-\bar{H}_{B_f}\right)\right\|^2_{L^2}+\eta\|f\|^2_{L^2}+C\delta\|\nabla f\|^2_{L^2},$$

where $C=C(N)$ and we also used that, by uniform convergence, $(1-s)\bar{H}^s_{B_f}\to\omega_{N-1}\bar{H}_{B_f}$. We then conclude by taking η and δ sufficiently small. Finally, the hypothesis $f\in C^2(\partial B)$ can be weakened to $f\in C^1(\partial B)\cap H^2(\partial B)$ by approximation.

6.3 The asymptotic of the fractional discrete flow

We recall the following density estimate holding for one-sided minimizers of the fractional perimeter, which can be found in [19, Theorem 4.1].

Proposition 6.3.1. *There exists a constant $C=C(N,s)>0$ with the following property: given $E\subset\mathbb{R}^N$, R, $\mu>0$ and $x_0\in\partial E$ such that*

$$P^s(E)\le P^s(E\setminus B_r(x_0))+\mu|E\cap B_r(x_0)|\quad\text{for all } 0<r<R,$$

then

$$Cr^N \le |E \cap B_r(x_0)| \quad \text{for all } 0 < r < \min\{R, \mu^{-1/s}\}.$$

We employ the density estimate above to bound the distance function between two consecutive sets of the discrete flow (see Definition 3.4.5). The proof follows the line of [83, Proposition 3.2] where it is proved in the local case, see also [73].

Proposition 6.3.2. *There exists a constant* $\gamma = \gamma(N,s) > 0$ *with the following property: let* $F \subset \mathbb{R}^N$ *be a bounded set of finite fractional perimeter and let* E *be a minimizer of* $\mathscr{J}_h(F,\cdot)$*, see* (3.4.5)*, then*

$$\sup_{E \triangle F} \operatorname{dist}_{\partial F} \le \gamma h^{1/1+s}.$$

Proof. Let $\gamma = \max\left\{3, 2^{s+1/s} P^s(B)^{1/s} C^{-1/s}\right\}$, where $C = C(N,s)$ is the constant given by the Proposition 6.3.1. Let $c > \gamma$ and $x_0 \in E \triangle F$. Suppose by contradiction that $\operatorname{dist}_{\partial F}(x_0) > ch^{1/1+s}$. Since the other case is analogous, we assume $x_0 \in E \setminus F$. We then have

$$\mathrm{sd}_F(x_0) > ch^{1/1+s} \tag{6.3.1}$$

and thus any ball $B_r(x_0)$ of radius $r \le ch^{1/1+s}/2$ is contained in F^c. By the minimality of E, we have $\mathscr{J}_h(F,E) \le \mathscr{J}_h(F, E \setminus B_r(x_0))$, therefore

$$P^s(E) \le P^s(E \setminus B_r(x_0)) - \frac{1}{h} \int_{E \cap B_r(x_0)} \mathrm{sd}_F \, \mathrm{d}x + \frac{1}{h^{s/1+s}} |E \cap B_r(x_0)|.$$

We use (6.3.1) and $r \le ch^{1/1+s}/2$ to infer that

$$-\frac{1}{h} \int_{E \cap B_r(x_0)} \mathrm{sd}_F \, \mathrm{d}x < -\frac{c}{2h^{s/1+s}} |E \cap B_r(x_0)|.$$

Then we have

$$P^s(E) \le P^s(E \setminus B_r(x_0)) - \frac{1}{h^{s/1+s}} \left(\frac{c}{2} - 1\right) |E \cap B_r(x_0)|. \tag{6.3.2}$$

By assumption $c > 3$ and we can apply Proposition 6.3.1 with $\mu = 0$ and obtain

$$Cr^N \le |E \cap B_r(x_0)| \qquad \forall 0 < r < \frac{c}{2} h^{1/1+s}. \tag{6.3.3}$$

On the other hand, from (6.3.2) we deduce, for every $0 < r < ch^{1/1+s}/2$, that

$$\frac{1}{h^{s/1+s}} \left(\frac{c}{2} - 1\right) |E \cap B_r(x_0)| \le P^s(E \setminus B_r(x_0)) - P^s(E) \le P^s(B_r^c) = P^s(B) r^{N-s} \tag{6.3.4}$$

(where the last inequality follows from the subadditivity of the perimeter on E and B_r^c). Combining (6.3.3) and (6.3.4), we get that

$$Cr^N \leq |E \cap B_r(x_0)| \leq P^s(B)\left(\frac{c}{2}-1\right)^{-1} h^{s/1+s} r^{N-s} \leq 2P^s(B) h^{s/1+s} r^{N-s}$$

for all $0 < r < ch^{1/1+s}/2$, which gives the desired contradiction to the choice of c as soon as $r \to ch^{1/1+s}/2$. □

As a corollary of the previous result we obtain the following density estimates, their proof is an adaptation of the one of [83, Corollary 3.3].

Corollary 6.3.3. *Let $F \subset \mathbb{R}^N$ be a bounded set of finite fractional perimeter and let E be a minimizer of $\mathscr{J}_h(F,\cdot)$, see* (3.4.5)*. Then for every $r \in (0, \gamma h^{1/1+s})$ and for every $x_0 \in \partial^* E$, it holds*

$$\min\{|B_r(x_0) \setminus E|, |E \cap B_r(x_0)|\} \geq cr^N \tag{6.3.5}$$

$$cr^{N-s} \leq P^s(E, B_r(x_0)) \leq Cr^{N-s}, \tag{6.3.6}$$

where γ is the constant given by Proposition 6.3.2 and the constants c, C only depend on N and s.

Proof. Since E is a minimizer of $\mathscr{J}_h(F,\cdot)$, for any $x_0 \in \partial E$, it holds that $\mathscr{J}_h(F,E) \leq \mathscr{J}_h(F, E \cup B_r(x_0))$, which implies

$$\begin{aligned} P^s(E) &\leq P^s(E \cup B_r(x_0)) + \frac{1}{h}\int_{B_r(x_0)\setminus E} \mathrm{sd}_F \, \mathrm{d}x + \frac{1}{h^{s/1+s}}|B_r(x_0) \setminus E| \\ &\leq P^s(E \cup B_r(x_0)) + \frac{C}{h^{s/1+s}}|B_r(x_0) \setminus E|, \end{aligned}$$

where we used that $r < \gamma h^{1/1+s}$ and bounded $\mathrm{sd}_F \leq \gamma h^{1/1+s}$ by Proposition 6.3.2. Analogously, one can show that

$$\begin{aligned} P^s(E) &\leq P^s(E \setminus B_r(x_0)) + \frac{C}{h^{s/1+s}}|E \cap B_r(x_0)| \\ &= \mathscr{L}_s(E \setminus B_r(x_0), E^c \setminus B_r(x_0)) + \mathscr{L}_s(E \setminus B_r(x_0), B_r(x_0)) + \frac{C}{h^{s/1+s}}|E \cap B_r(x_0)|, \end{aligned} \tag{6.3.7}$$

where $\mathscr{L}_s(\cdot,\cdot)$ was defined in (3.2.1). Therefore, by Proposition 6.3.1, we deduce

$$\min\{|E \cap B_r(x_0)|, |B_r(x_0) \setminus E|\} \geq cr^N \qquad \forall 0 < r < \gamma h^{1/1+s}.$$

The first inequality in (6.3.6) is now an immediate consequence of the relative isoperimetric inequality. To prove the second inequality, by (6.3.7) we get

$$\begin{aligned} P^s(E,B_r(x_0)) &= \mathscr{L}_s(E\cap B_r(x_0),E^c)+\mathscr{L}_s(E\setminus B_r(x_0),E^c\cap B_r(x_0)) \\ &= P^s(E)-\mathscr{L}_s(E\setminus B_r(x_0),E^c\setminus B_r(x_0)) \\ &\le \mathscr{L}_s(E\setminus B_r(x_0),B_r(x_0))+\frac{C}{h^{s/1+s}}|B_r(x_0)\setminus E| \\ &\le P^s(B_r(x_0))+\frac{C\gamma^s}{r^s}\omega_N r^N \le C(N,s)r^{N-s}, \end{aligned}$$

where we used that $r<\gamma h^{1/1+s}$. □

Remark 6.3.4. From the monotonicity of the energy $P^s(\cdot)+h^{-\frac{s}{1+s}}||\cdot|-m|$ along the discrete flow starting from E_0 (recall Defnition 3.4.5) with $|E_0|=m$, $P^s(E_0)\le M$, one can observe that $|E_n^{(h)}|\in(m/2,3m/2)$ for all $n\in\mathbb{N}$ and for $h=h(m,M)$ small.

We now characterize the stationary sets E for the discrete flow. We say that E is a *stationary set* for the discrete flow if it is a fixed set for the functional (3.4.5), that is, $E=E_h^n$ for every $n\in\mathbb{N}$. In the following, we will always assume that either:

- $N=2$ and $s\in(0,1)$;
- $N\le 7$ and $s\in(s_0,1)$, where s_0 is the constant of [20, Theorem 3].

This hypothesis is essential for the proof of the following result.

Proposition 6.3.5. *Every stationary set E for the discrete flow is a critical set of the s-perimeter, that is, a single ball.*

Proof. It is an immediate consequence of the Euler-Lagrange equation (3.4.3). Since E is a stationary point for the discrete flow, it satisfies

$$\int_{\partial E} H_E^s\,\mathrm{d}\mathscr{H}^{N-1}=\lambda\int_{\partial E} X\cdot\nu_E\,\mathrm{d}\mathscr{H}^{N-1}$$

for all $X\in C_c^1(\mathbb{R}^N,\mathbb{R}^N)$, i.e. E is a critical point for the s-perimeter. By [18, Theorem 1.1] (or [29, Theorem 1.1]), we conclude that E is a single ball having constant fractional mean curvature $H_E^s=\lambda$. □

Before proving the convergence of the flow up to translations, we recall a uniform convergence result contained in [82, Lemma 3.5] that will be used in the proof of the next proposition. The proof in the fractional setting is analogous and will be omitted.

Lemma 6.3.6. *Let* $\{E_h^n\}_{n\in\mathbb{N}}$ *be a discrete flow starting from* E_0 *and let* $E_h^{k_n}$ *be a subsequence such that* $E_h^{k_n}+\tau_n \to F$ *in* L^1 *for some set* F *and a suitable sequence* $\{\tau_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^N$*. Then* $\mathrm{dist}_{\partial E_h^{k_n-1}}(\cdot+\tau_n)\to \mathrm{dist}_{\partial F}$ *uniformly.*

The following result proves the convergence of the discrete flow, up to translations, to a union of disjointed balls, all having the same radius. The proof follows closely the one of [82, Proposition 3.6]. Moreover, we prove that the flow eventually has fixed volume. We remark that, at this point, we cannot rule out that the flow is converging to a union of balls (each at infinite distance from the others), and that the translations introduced could be different along different subsequences. We will provide a sharper result in the final theorem.

Proposition 6.3.7. *Let* m*,* $M>0$ *and* E_0 *be an initial bounded set with* $P^s(E_0)\le M$*,* $|E_0|=m$*. Then there exists* $h^*=h^*(s,M,m)>0$ *such that, for any* $h<h^*$ *and for any discrete flow* E_h^n *starting from* E_0*, the following properties hold:*

i) for n *sufficiently large* $|E_h^n|=m$*;*

ii) there exists

$$P^s_\infty=\lim_{n\to\infty}P^s(E_h^n);$$

iii) E_h^n *is made of* $K=(P^s_\infty/\omega^s_N)^{\frac{N}{s}}(\omega_N/m)^{\frac{N}{s}-1}$ *distinct connected components* $E_h^{n,i}$*, and* $E_h^{n,i}-\mathrm{bar}(E_h^{n,i})$ *converges in* C^k*, for every* $k\in\mathbb{N}$*, to the ball centered at the origin with mass* m/K*.*

Proof. Let $\{E_h^{k_n}\}_{n\in\mathbb{N}}$ be any subsequence of $\{E_h^n\}_{n\in\mathbb{N}}$. By Proposition 3.4.3 (and Remark 3.4.6), each set $E_h^{k_n}$ is made up of $l_n\le k_0$ connected components having diameter uniformly bounded by d_0. Therefore, there exist l_n balls $B_{d_0}(\xi_n^i)$, each containing a different component of $E_h^{k_n}$ and such that $E_h^{k_n}\subset\cup_{i=1}^{l_n}B_{d_0}(\xi_n^i)$. Up to subsequences, we can assume that $l_n=\tilde{l}$, and for all $1\le i<j\le\tilde{l}$ the following limits exist

$$\limsup_{n\to\infty}|\xi_n^i-\xi_n^j| =: d^{i,j}\in[0,+\infty].$$

Now we define the following equivalence classes: we say that $i\equiv j$ if and only if $d^{i,j}<+\infty$. Denote by $l\le\tilde{l}$ the number of such equivalence classes, let $j(i)$ be a representative for each class $i\in\{1,\dots,l\}$, and set $\sigma_n^i:=\xi_n^{j(i)}$ for $i=1,\dots,l$. We have constructed a subsequence $E_h^{k_n}$ satisfying $E_h^{k_n}\subset\cup_{i=1}^{l}B_R(\sigma_n^i)$, where $R=d_0+\max\{d^{i,j}:d^{i,j}<+\infty\}+1$, and for all $i\not\equiv j$ it holds $|\sigma_n^i-\sigma_n^j|\to+\infty$ as $n\to+\infty$.

Now, fix $1\le i\le l$, and set

$$F_n^i:=E_h^{k_n}-\sigma_n^i,\quad \tilde{F}_n^i:=(E_h^{k_n}-\sigma_n^i)\cap B_R,\quad m_n^i:=|\tilde{F}_n^i|.$$

Up to a subsequence, we have $m_n^i \to m^i > 0$. Moreover, by Lemma 6.3.6 and by the compactness of sets of equi-bounded fractional perimeters, there exist $\tilde{F}^i \subset B_R$ such that, up to a subsequence,

$$\tilde{F}_n^i \to \tilde{F}^i \text{ in } L^1, \quad \mathrm{sd}_{E_h^{k_n-1}}(\cdot + \sigma_n^i) \to \mathrm{sd}_{\tilde{F}^i}(\cdot) \text{ locally uniformly.} \tag{6.3.8}$$

Let $\tilde{G}^i$ be any bounded set with $|\tilde{G}^i| = m^i$ and let $\tilde{G}_n^i := \left(\frac{m_n^i}{m^i}\right)^{\frac{1}{N}} \tilde{G}^i$. We set now $G_n^i := (F_n^i \setminus \tilde{F}_n^i) \cup \tilde{G}_n^i$ so that, for n sufficiently large, $|F_n^i| = |G_n^i|$ (since the components of $F_n^i \setminus \tilde{F}_n^i$ diverge). By the minimality of $E_h^{k_n}$ we have

$$P^s(F_n^i) + \frac{1}{h}\int_{F_n^i} \mathrm{sd}_{E_h^{k_n-1}}(x + \sigma_n^i)\,\mathrm{d}x \le P^s(G_n^i) + \frac{1}{h}\int_{G_n^i} \mathrm{sd}_{E_h^{k_n-1}}(x + \sigma_n^i)\,\mathrm{d}x.$$

Using that for any disjoint measurable sets A, B we have

$$P^s(A \cup B) = P^s(A) + P^s(B) - 2\int_A \int_B \frac{1}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y,$$

for n sufficiently large, we obtain

$$\begin{aligned} &P^s(\tilde{F}_n^i) - 2\int_{\tilde{F}_n^i} \int_{F_n^i \setminus \tilde{F}_n^i} \frac{1}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{h}\int_{\tilde{F}_n^i} \mathrm{sd}_{E_h^{k_n-1}}(x + \sigma_n^i)\,\mathrm{d}x \\ &\le P^s(\tilde{G}_n^i) - 2\int_{\tilde{G}_n^i} \int_{F_n^i \setminus \tilde{F}_n^i} \frac{1}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{h}\int_{\tilde{G}_n^i} \mathrm{sd}_{E_h^{k_n-1}}(x + \sigma_n^i)\,\mathrm{d}x. \end{aligned}$$

Passing to the limit as $n \to \infty$, using (6.3.8) and the uniform boundedness of $\tilde{F}_n^i$ and $\tilde{G}_n^i$, we deduce that

$$P^s(\tilde{F}^i) + \frac{1}{h}\int_{\tilde{F}^i} \mathrm{sd}_{\tilde{F}^i}(x)\,\mathrm{d}x \le P^s(G^i) + \frac{1}{h}\int_{G^i} \mathrm{sd}_{\tilde{F}^i}(x)\,\mathrm{d}x.$$

This minimality property holds for every G^i with finite perimeter and volume m^i, therefore we deduce that $\tilde{F}^i$ is a fixed point for the discrete scheme with prescribed volume m^i, and, thus by Proposition 6.3.5, it is a ball. Moreover, since the sets $\tilde{F}^i$ are uniform Λ-minimizers by Proposition 3.4.3 (and Remark 3.4.6), we also deduce, by Theorem 3.1.3, that $\tilde{F}_n^i$ converges to $\tilde{F}^i$ in $C^{1,\alpha}$ for every $\alpha \in (0,1)$. In particular, for n large enough, $\tilde{F}_n^i$ has only one connected component.

We have shown that, for n large enough, $E_h^{k_n}$ is made up by a fixed number K of connected components $(E_h^{k_n})^i$, $i = 1, \dots, K$, and $(E_h^{k_n})^i - \mathrm{bar}((E_h^{k_n})^i) \to B_{R_i}$ where $|B_{R_i}| = m^i$. Now, we show that all the radii R_i are equal to R. To this aim, we consider the Euler-Lagrange equation (3.4.3)

$$\frac{1}{h}\mathrm{sd}_{E_h^{k_n-1}} + H^s_{E_h^{k_n}} = \lambda_n \quad \text{on } \partial E_h^{k_n}.$$

By Proposition 6.3.2, we deduce that

$$|\lambda_n| \le h^{-1}\|\mathrm{sd}_{E_h^{k_n-1}}\|_{L^\infty(\partial E_h^{k_n})} + \|H^s_{E_h^{k_n}}\|_{L^\infty(\partial E_h^{k_n})} \le c + \|H^s_{E_h^{k_n}}\|_{L^\infty(\partial E_h^{k_n})}.$$

To bound the right hand side, we use the Λ-minimality of $E_h^{k_n}$ to obtain

$$\|H^s_{E_h^{k_n}}\|_{L^\infty(\partial E_h^{k_n})} \le \Lambda.$$

Therefore, by passing to a further subsequence, we can assume $\lambda_n \to \lambda \in \mathbb{R}$. Arguing as before, we can localize the Euler-Lagrange equation to each single $\tilde{F}^i_n$ and obtain

$$\frac{1}{h}\mathrm{sd}_{\tilde{F}^i_n}(x+\sigma^i_n) + H^s_{\tilde{F}^i_n}(x) = \lambda_n \quad x \in \partial \tilde{F}^i_n.$$

We can then pass to the limit as $n \to \infty$ thanks to Lemma 6.3.6 and the continuity of the fractional mean curvature, and obtain

$$H^s_{\tilde{F}^i} = \lambda \quad \text{on } \partial \tilde{F}^i.$$

In particular, this shows that, for every i, $\tilde{F}^i$ is a ball of radius $R_i = c\lambda^{-s}$, for a suitable constant c depending only on s and N. In order to prove that eventually $|E_h^n| = m$, we proceed as follows. Set $|B_{R_i}| = c_1\lambda^{-sN}$ and $P^s(B_{R_i}) = c_2\lambda^{-s(N-s)}$, for suitable constants c_1, c_2 depending on N, s. From Remark 6.3.4, we take $h = h(s, M)$ small enough such that

$$|E_h^{k_n}| \in \left[\frac{m}{2}, \frac{3m}{2}\right], \quad P^s(E_h^{k_n}) \le P^s(E_0) \le M$$

and, for n large enough, this implies

$$\sum_{i=1}^K m^i_n \in \left[\frac{m}{2}, \frac{3m}{2}\right], \quad \sum_{i=1}^K P^s(\tilde{F}^i_n) = P^s(E_h^{k_n}) + 2\sum_{i<j}\int_{\tilde{F}^i_n}\int_{\tilde{F}^j_n} \frac{1}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y \le 2M.$$

Passing to the limit as $n \to \infty$ we obtain

$$Kc_1\lambda^{-sN} \in \left[\frac{m}{2}, \frac{3m}{2}\right], \quad Kc_2\lambda^{-s(N-s)} \le M,$$

which implies

$$\lambda^{s^2} \le \frac{2c_1M}{mc_2}. \tag{6.3.9}$$

If we suppose that $|E_h^{k_n}| \neq m$ for infinitely many indexes, then, from the Euler-Lagrange equation (3.4.3), we know that $\lambda = \mathrm{sgn}(m - |E_h^{k_n}|)h^{-\frac{s}{1+s}}$ which is a contradiction to (6.3.9) if h is sufficiently small. We have thus proved item *i)*. Since, for n large enough, $|E_h^n| = m$, the sequence $\{P^s(E_h^n)\}_{n\in\mathbb{N}}$ is eventually non-increasing, from which item *ii)* follows. Knowing the exact values of the volume and s-perimeter of any limit point, we are able to compute K and obtain the convergence in L^1 of the whole sequence. Moreover, arguing as in [23] we conclude the convergence in C^k for every $k \in \mathbb{N}$ via a bootstrap argument. □

In order to prove the main theorem, we need to collect some results of [82]. Firstly, recall $\mathscr{D}(E,F) := \int_{F\triangle E} \mathrm{dist}_{\partial F}(x)\,\mathrm{d}x$.

Lemma 6.3.8 ([82, Lemma 3.8])**.** *Let $\eta > 0$. There exists $\delta > 0$ with the following property: if $f_1, f_2 \in C^1(\partial B)$ with $\|f_i\|_{C^1(\partial B)} \leq \delta$ and $|B_{f_i}| = |B|$ for $i = 1,2$ we have*

$$C_1(1-\eta)\|f_1 - f_2\|^2_{L^2(B)} \leq \mathscr{D}(B_{f_1}, B_{f_2}) \leq C_1(1+\eta)\|f_1 - f_2\|^2_{L^2(B)}, \tag{6.3.10}$$

$$\frac{1-\eta}{2}\int_{\partial B_{f_2}} \mathrm{sd}^2_{B_{f_1}}\,\mathrm{d}\mathscr{H}^{N-1} \leq \mathscr{D}(B_{f_1}, B_{f_2}) \leq \frac{1+\eta}{2}\int_{\partial B_{f-2}} \mathrm{sd}^2_{B_{f_1}}\,\mathrm{d}\mathscr{H}^{N-1}, \tag{6.3.11}$$

$$|\mathrm{bar}(B_{f_1}) - \mathrm{bar}(B_{f_2})|^2 \leq C_2\|f_1 - f_2\|^2_{L^2(B)} \leq \frac{C_2}{C_1(1-\eta)}\mathscr{D}(B_{f_1}, B_{f_2}),$$

for suitable constants $C_1, C_2 > 0$.

The following lemma proves a crucial dissipation-dissipation inequality, which plays a central role in the proof of Theorem 6.1.1. Its proof is based on the Alexandrov-type estimate contained in Theorem 6.2.1, and follows the lines of [82, Lemma 3.9].

Lemma 6.3.9. *Let $h > 0$. There exist constants $C = C(h,m,s)$, $\delta > 0$ with the following property: given two normal deformations $B^{(m)}_{f_1}$, $B^{(m)}_{f_2}$ of the ball $B^{(m)}$ of volume m centered at the origin with $f_i \in C^2(\partial B^{(m)})$, $\|f_i\|_{C^1(\partial B^{(m)})} \leq \delta$, and such that $|B^{(m)}_{f_2}| = m$, $\mathrm{bar}(B^{(m)}_{f_2}) = 0$ and*

$$H^s_{B^{(m)}_{f_2}} + \frac{\mathrm{sd}_{B^{(m)}_{f_1}}}{h} = \lambda \quad \text{on} \quad \partial B^{(m)}_{f_2} \tag{6.3.12}$$

for some $\lambda \in \mathbb{R}$, we have

$$\mathscr{D}(B^{(m)}, B^{(m)}_{f_2}) \leq C\mathscr{D}(B^{(m)}_{f_2}, B^{(m)}_{f_1}).$$

Proof. By Theorem 6.2.1, for δ sufficiently small, we get by using (6.3.12)

$$\begin{aligned}\|f_2\|^2_{L^2(\partial B^{(m)})} &\le C\|H^s_{B^{(m)}_{f_2}} - \overline{H}^s_{B^{(m)}_{f_2}}\|^2_{L^2(\partial B^{(m)})} \le C\|H^s_{B^{(m)}_{f_2}} - \lambda\|^2_{L^2(\partial B^{(m)})}\\ &\le C\|H^s_{B^{(m)}_{f_2}} - \lambda\|^2_{L^2(\partial B^{(m)}_{f_2})} = \frac{C}{h^2}\int_{\partial B^{(m)}_{f_2}} \mathrm{sd}^2_{B^{(m)}_{f_1}}\,\mathrm{d}\mathscr{H}^{N-1},\end{aligned}$$

where the third inequality follows by bounding the Jacobian of the change of variables by 1 (up to taking δ sufficiently small). By combining the previous inequalities with (6.3.10) and (6.3.11), we obtain the thesis. □

We now prove Theorem 6.1.1. We will follow closely the proofs of Theorem 5.1.1 and [82, Theorem 3.3]. The main difference is that we use the fractional perimeter framework instead of the classical one. We present only a sketch of the proof.

Proof of Theorem 6.1.1. We start by outlining the proof of the exponential decay of the dissipations following Step 1 in [82, Theorem 3.3].

From Proposition 6.3.7 we know that any limit point of the discrete flow is given by the union of K disjoint balls, all having volume m/K. We then use two competitors to obtain a discrete Gronwall-type inequality. Firstly, testing the minimality of E^k_h with E^{k-1}_h and summing from $n+1$ to infinity, we obtain

$$\sum_{k\ge n+1} \mathscr{D}(E^k_h, E^{k-1}_h) \le P^s(E^n_h) - P^s_\infty = P^s(E^n_h) - KP^s(B^{(m/K)}),$$

where we used that for n sufficiently large $|E^n_h| = m$. On the other hand, recalling Proposition 6.3.7, the sets $(E^n_h)^i - \mathrm{bar}((E^n_h)^i) =: (E^n_h)^i - \xi^i_n$ are eventually $C^{1,\alpha}$-deformations of $B^{(m/K)}$, having volume $|(E^n_h)^i| = m^i_n$. We consider the admissible competitor for E^n_h given by

$$\mathscr{B}_n = \bigcup_{i=1}^{K}\left(B^{(m^i_{n-1})} + \xi^i_{n-1}\right).$$

Testing the minimality of E^n_h against $\mathscr{B}_n$, one obtains, by employing Lemma 6.3.9, that

$$P^s(E^n_h) - P^s(\mathscr{B}_n) \le C\mathscr{D}(E^{n-1}_h, E^{n-2}_h).$$

Recalling that, if a measurable set F has L disjointed connected components F^i, $i = 1,\dots,L$, then

$$P^s(F) = \sum_{i=1}^{L} P^s(F^i) - 2\sum_{i<j}\int_{F^i}\int_{F^j}\frac{1}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y,$$

by concavity, we estimate

$$P^s(\mathscr{B}_n) \leq \sum_{i=1}^{K} P^s(B^{(m^i_{n-1})}) \leq KP^s(B^{(m/K)}).$$

Thus, combining the previous estimates, we obtain the discrete Gronwall-type estimate

$$\sum_{k \geq n+1} \mathscr{D}(E^k_h, E^{k-1}_h) \leq C\mathscr{D}(E^{n-1}_h, E^{n-2}_h).$$

Finally, employing [82, Lemma 3.10] we conclude the exponential convergence of the dissipations

$$\mathscr{D}(E^n_h, E^{n-1}_h) \leq \left(1 - \frac{1}{C+1}\right)^{\frac{n}{2}} (P^s(E_0) - KP^s(B^{(m/K)})).$$

From now on, one can follow directly the proof of [82, Theorem 3.3] employing Lemma 6.3.8 to conclude that the discrete flow E^n_h is eventually contained in a compact set and converges in C^k to a union of K disjoint balls. Finally, by Proposition 6.3.5 we deduce that the limit point is indeed a single ball, having volume equal to m, thus reaching the conclusion of the proof. ☐

References

[1] Abatangelo, N. and Valdinoci, E. (2014). A notion of nonlocal curvature. *Numer. Funct. Anal. Optim.*, 35(7-9):793–815.

[2] Abels, H., Arab, N., and Garcke, H. (2021). Standard planar double bubbles are dynamically stable under surface diffusion flow. *Comm. Anal. Geom.*, 29(5):1007–1060.

[3] Acerbi, E., Fusco, N., Julin, V., and Morini, M. (2019). Nonlinear stability results for the modified Mullins-Sekerka and the surface diffusion flow. *J. Differential Geom.*, 113(1):1–53.

[4] Acerbi, E., Fusco, N., and Morini, M. (2013). Minimality via second variation for a nonlocal isoperimetric problem. *Comm. Math. Phys.*, 322(2):515–557.

[5] Allen, S. M. and Cahn, J. W. (1979). A microscopic theory for antiphase boundary motion and its application to antiphase domain coarsening. *Acta Metallurgica*, 27(6):1085–1095.

[6] Almgren, F., Taylor, J. E., and Wang, L. (1993). Curvature-driven flows: a variational approach. *SIAM J. Control Optim.*, 31(2):387–438.

[7] Amann, H. (1993). Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems. In *Function spaces, differential operators and nonlinear analysis (Friedrichroda, 1992)*, volume 133 of *Teubner-Texte Math.*, pages 9–126. Teubner, Stuttgart.

[8] Ambrosio, L., Fusco, N., and Pallara, D. (2000). *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York.

[9] Ambrosio, L. and Tilli, P. (2004). *Topics on analysis in metric spaces*, volume 25 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford.

[10] Anzellotti, G., Serapioni, R., and Tamanini, I. (1990). Curvatures, functionals, currents. *Indiana Univ. Math. J.*, 39(3):617–669.

[11] Aubin, T. (1998). *Some nonlinear problems in Riemannian geometry*. Springer Monographs in Mathematics. Springer-Verlag, Berlin.

[12] Baumgart, T., Das, S., Webb, W., and Jenkins, J. (2005). Membrane elasticity in giant vesicles with fluid phase coexistence. *Biophysical Journal*, 89(2):1067–1080.

[13] Baumgart, T., Hess, S., and Webb, W. (2003). Imaging coexisting fluid domains in biomembrane models coupling curvature and line tension. *Nature*, 425:821–824.

[14] Bellettini, G., Caselles, V., Chambolle, A., and Novaga, M. (2006). Crystalline mean curvature flow of convex sets. *Arch. Ration. Mech. Anal.*, 179(1):109–152.

[15] Bellettini, G. and Mugnai, L. (2010). Approximation of Helfrich's functional via diffuse interfaces. *SIAM J. Math. Anal.*, 42(6):2402–2433.

[16] Bourgain, J., Brezis, H., and Mironescu, P. (2001). *Another look at Sobolev spaces*. IOS, Amsterdam.

[17] Brazda, K., Lussardi, L., and Stefanelli, U. (2020). Existence of varifold minimizers for the multiphase Canham-Helfrich functional. *Calc. Var. Partial Differential Equations*, 59(3):Paper No. 93, 26.

[18] Cabré, X., Fall, M. M., Solà-Morales, J., and Weth, T. (2018). Curves and surfaces with constant nonlocal mean curvature: meeting Alexandrov and Delaunay. *J. Reine Angew. Math.*, 745:253–280.

[19] Caffarelli, L., Roquejoffre, J.-M., and Savin, O. (2010). Nonlocal minimal surfaces. *Comm. Pure Appl. Math.*, 63:1111–1144.

[20] Caffarelli, L. and Valdinoci, E. (2013). Regularity properties of nonlocal minimal surfaces via limiting arguments. *Adv. Math.*, 248:843–871.

[21] Canham, P. B. (1970). The minimum energy of bending as a possible explanation of the biconcave shape of the human red blood cell. *J. Theor. Biol.*, 26(1):61–81.

[22] Capuzzo Dolcetta, I., Finzi Vita, S., and March, R. (2002). Area-preserving curve-shortening flows: from phase separation to image processing. *Interfaces Free Bound.*, 4(4):325–343.

[23] Cesaroni, A. and Novaga, M. (2016). Volume constrained minimizers of the fractional perimeter with a potential energy. *preprint*.

[24] Cesaroni, A. and Novaga, M. (2022). Stability of the ball under volume preserving fractional mean curvature flow. *Advances in Calculus of Variations*.

[25] Choksi, R., Morandotti, M., and Veneroni, M. (2013). Global minimizers for axisymmetric multiphase membranes. *ESAIM Control Optim. Calc. Var.*, 19(4):1014–1029.

[26] Choksi, R. and Sternberg, P. (2007). On the first and second variations of a nonlocal isoperimetric problem. *J. Reine Angew. Math.*, 611:75–108.

[27] Choksi, R. and Veneroni, M. (2013). Global minimizers for the doubly-constrained Helfrich energy: the axisymmetric case. *Calc. Var. Partial Differential Equations*, 48(3-4):337–366.

[28] Cicalese, M. and Leonardi, G. P. (2012). A selection principle for the sharp quantitative isoperimetric inequality. *Arch. Ration. Mech. Anal.*, 206(2):617–643.

[29] Ciraolo, G., Figalli, A., Maggi, F., and Novaga, M. (2018). Rigidity and sharp stability estimates for hypersurfaces with constant and almost-constant nonlocal mean curvature. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 2018(741):275–294.

[30] Dal Maso, G. (1993). *An introduction to* Γ*-convergence*, volume 8 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA.

[31] Dalphin, J. (2018). Uniform ball property and existence of optimal shapes for a wide class of geometric functionals. *Interfaces Free Bound.*, 20(2):211–260.

[32] De Gennaro, D., Diana, D., Kubin, A., and Kubin, A. (2023a). Stability of the surface diffusion flow and volume-preserving mean curvature flow in the flat torus. *preprint*.

[33] De Gennaro, D. and Kubin, A. (2023). Long time behaviour of the discrete volume preserving mean curvature flow in the flat torus. *Calc. Var. Partial Differential Equations*, 62(3):Paper No. 103, 39.

[34] De Gennaro, D., Kubin, A., and Kubin, A. (2023b). Asymptotic of the discrete volume-preserving fractional mean curvature flow via a nonlocal quantitative Alexandrov theorem. *Nonlinear Analysis*, 228:113200.

[35] Della Corte, S., Diana, A., and Mantegazza, C. (2022). Global existence and stability for the modified Mullins-Sekerka and surface diffusion flow. *Math. Eng.*, 4(6):Paper No. 054, 104.

[36] Della Corte, S., Diana, A., and Mantegazza, C. (2023). Uniform Sobolev, interpolation and Calderón–Zygmund inequalities for graph hypersurfaces. arXiv:2107.12234v4.

[37] Delladio, S. (1996). Do generalized Gauss graphs induce curvature varifolds? *Boll. UMI B*, 10(4):991–1017.

[38] Delladio, S. (1997). Special generalized Gauss graphs and their application to minimization of functionals involving curvatures. *J. Reine Angew. Math.*, 486:17–43.

[39] Delladio, S. (2001). On hypersurfaces in $\mathbb{R}^{n+1}$ with integral bounds on curvature. *J. Geom. Anal.*, 11:17–42.

[40] do Carmo, M. P. (1994). *Differential forms and applications*. Universitext. Springer-Verlag, Berlin. Translated from the 1971 Portuguese original.

[41] do Carmo, M. P. (2016). *Differential geometry of curves & surfaces*. Dover Publications, Inc., Mineola, NY, second edition.

[42] Eichmann, S. (2019). The Helfrich boundary value problem. *Calc. Var. Partial Differential Equations*, 58(1):Paper No. 34, 26.

[43] Eichmann, S. (2020). Lower semicontinuity for the Helfrich problem. *Ann. Global Anal. Geom.*, 58(2):147–175.

[44] Escher, J., Mayer, U. F., and Simonett, G. (1998). The surface diffusion flow for immersed hypersurfaces. *SIAM J. Math. Anal.*, 29(6):1419–1433.

[45] Escher, J. and Simonett, G. (1998). The volume preserving mean curvature flow near spheres. *Proc. Amer. Math. Soc.*, 126(9):2789–2796.

[46] Federer, H. (1969). *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York.

[47] Federer, H. and Fleming, W. H. (1960). Normal and integral currents. *Ann. of Math. (2)*, 72:458–520.

[48] Figalli, A., Fusco, N., Maggi, F., Millot, V., and Morini, M. (2015). Isoperimetry and stability properties of balls with respect to nonlocal energies. *Comm. Math. Phys.*, 336(1):441–507.

[49] Friedman, A. (1964). *Partial differential equations of parabolic type*. Prentice-Hall, Inc., Englewood Cliffs, N.J.

[50] Fusco, N., Julin, V., and Morini, M. (2018). The surface diffusion flow with elasticity in the plane. *Comm. Math. Phys.*, 362(2):571–607.

[51] Fusco, N., Julin, V., and Morini, M. (2020). The surface diffusion flow with elasticity in three dimensions. *Arch. Ration. Mech. Anal.*, 237(3):1325–1382.

[52] Gage, M. (1986). On an area-preserving evolution equation for plane curves. In *Nonlinear problems in geometry (Mobile, Ala., 1985)*, volume 51 of *Contemp. Math.*, pages 51–62. Amer. Math. Soc., Providence, RI.

[53] Garcke, H. and Gößwein, M. (2019). On the surface diffusion flow with triple junctions in higher space dimensions. *Geometric Flows*, 5:1 – 39.

[54] Garcke, H. and Gößwein, M. (2021). Non-linear stability of double bubbles under surface diffusion. *Journal of Differential Equations*, 302:617–661.

[55] Garcke, H., Ito, K., and Kohsaka, Y. (2008). Nonlinear stability of stationary solutions for surface diffusion with boundary conditions. *SIAM J. Math. Anal.*, 40(2):491–515.

[56] Gilbarg, D. and Trudinger, N. (1977). *Elliptic partial differential equations of second order*, volume 224. Springer.

[57] Han, B. and Pinamonti, A. (2021). On the asymptotic beahaviour of the fractional sobolev seminorms in metric measure spaces: Bourgain-brezis-mironescu's theorem revisited. *preprint*.

[58] He, W. and Zeng, Y. (2021). The Calabi flow with rough initial data. *Int. Math. Res. Not. IMRN*, (10):7470–7551.

[59] Helfrich, W. (1973). Elastic properties of lipid bilayers: Theories and possible experiments. *Z. Naturforsch. C*, 28(11-12):693–703.

[60] Helmers, M. (2013). Kinks in two-phase lipid bilayer membranes. *Calc. Var. Partial Differential Equations*, 48(1-2):211–242.

[61] Helmers, M. (2015). Convergence of an approximation for rotationally symmetric two-phase lipid bilayer membranes. *Q. J. Math.*, 66(1):143–170.

[62] Huisken, G. (1987). The volume preserving mean curvature flow. *J. Reine Angew. Math.*, 382:35–48.

[63] Hutchinson, J. E. (1986). Second fundamental form for varifolds and the existence of surfaces minimising curvature. *Indiana Univ. Math. J.*, 35(1):45–71.

[64] Imbert, C. (2009). Level set approach for fractional mean curvature flows. *Interfaces Free Bound.*, 11(1):153–176.

[65] Julin, V., Morini, M., Ponsiglione, M., and Spadaro, E. (2022). The asymptotics of the area-preserving mean curvature and the mullins–sekerka flow in two dimensions. *Math. Ann.*

[66] Julin, V. and Niinikoski, J. (2022). Consistency of the flat flow solution to the volume preserving mean curvature flow. arXiv:2206.05002.

[67] Koch, H. and Lamm, T. (2012). Geometric flows with rough initial data. *Asian J. Math.*, 16(2):209–235.

[68] Kubin, A., Lussardi, L., and Morandotti, M. (2022). Direct minimization of the Canham–Helfrich energy on generalized Gauss graphs. *preprint*.

[69] Kuwert, E., Mondino, A., and Schygulla, J. (2014). Existence of immersed spheres minimizing curvature functionals in compact 3-manifolds. *Math. Ann.*, 359(1-2):379–425.

[70] Kuwert, E. and Schätzle, R. (2012). *The Willmore functional*, volume 13 of *CRM Series*. Ed. Norm., Pisa.

[71] LeCrone, J. and Simonett, G. (2013). On well-posedness, stability, and bifurcation for the axisymmetric surface diffusion flow. *SIAM J. Math. Anal.*, 45(5):2834–2869.

[72] Li, H. (2009). The volume-preserving mean curvature flow in Euclidean space. *Pacific J. Math.*, 243(2):331–355.

[73] Luckhaus, S. and Sturzenhecker, T. (1995). Implicit time discretization for the mean curvature flow equation. *Calc. Var. Partial Differential Equations*, 3(2):253–271.

[74] Lussardi, L. and Röger, M. (2016). Gamma convergence of a family of surface-director bending energies with small tilt. *Arch. Ration. Mech. Anal.*, 219(3):985–1016.

[75] Maggi, F. (2012). *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge. An introduction to geometric measure theory.

[76] Mantegazza, C. (2011). *Lecture notes on mean curvature flow*, volume 290 of *Progress in Mathematics*. Birkhäuser/Springer Basel AG, Basel.

[77] Mayer, U. F. (2001). A singular example for the averaged mean curvature flow. *Experiment. Math.*, 10(1):103–107.

[78] Mayer, U. F. and Simonett, G. (2000). Self-intersections for the surface diffusion and the volume-preserving mean curvature flow. *Differential Integral Equations*, 13(7-9):1189–1199.

[79] Menne, U. (2013). Second order rectifiablity of integral varifolds of locally bounded first variation. *J. Geom. Anal.*, 23(2):709–763.

[80] Mondino, A. and Scharrer, C. (2020). Existence and regularity of spheres minimising the Canham-Helfrich energy. *Arch. Ration. Mech. Anal.*, 236(3):1455–1485.

[81] Morel, J.-M. and Solimini, S. (1995). *Variational methods in image segmentation*, volume 14 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA. With seven image processing experiments.

[82] Morini, M., Ponsiglione, M., and Spadaro, E. (2022). Long time behavior of discrete volume preserving mean curvature flows. *J. Reine Angew. Math.*, 784:27–51.

[83] Mugnai, L., Seis, C., and Spadaro, E. (2016). Global solutions to the volume-preserving mean-curvature flow. *Calc. Var. Partial Differential Equations*, 55(1):Art. 18, 23.

[84] Mullins, W. W. (1956). Two-dimensional motion of idealized grain boundaries. *Journal of Applied Physics*, 27:900.

[85] Mullins, W. W. (1957). Theory of thermal grooving. *Journal of Applied Physics*, 28:333–339.

[86] Niinikoski, J. (2021). Volume preserving mean curvature flows near strictly stable sets in flat torus. *J. Differential Equations*, 276:149–186.

[87] Novaga, M. and Pozzetta, M. (2020). Connected surfaces with boundary minimizing the Willmore energy. *Math. Eng.*, 2(3):527–556.

[88] Pozzetta, M. (2021). On the Plateau-Douglas problem for the Willmore energy of surfaces with planar boundary curves. *ESAIM Control Optim. Calc. Var.*, 27(suppl.):Paper No. S2, 35.

[89] Rivière, T. (2008). Analysis aspects of Willmore surfaces. *Invent. Math.*, 174(1):1–45.

[90] Savin, O. and Valdinoci, E. (2013). Regularity of nonlocal minimal cones in dimension 2. *Calc. Var. Partial Differential Equations*, 48(1-2):33–39.

[91] Scharrer, C. (2022). Some geometric inequalities for varifolds on Riemannian manifolds based on monotonicity identities. *Ann. Global Anal. Geom.*, 61(4):691–719.

[92] Schätzle, R. (2010). The Willmore boundary problem. *Calc. Var. Partial Differential Equations*, 37(3-4):275–302.

[93] Schygulla, J. (2012). Willmore minimizers with prescribed isoperimetric ratio. *Arch. Ration. Mech. Anal.*, 203(3):901–941.

[94] Serre, D. (2010). *Matrices*, volume 216 of *Graduate Texts in Mathematics*. Springer, New York, second edition. Theory and applications.

[95] Simon, L. (1983). *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University*. Australian National University, Centre for Mathematical Analysis, Canberra.

[96] Simon, L. (1993). Existence of surfaces minimizing the Willmore functional. *Comm. Anal. Geom.*, 1(2):281–326.

[97] Templer, R. H., Khoo, B. J., and Seddon, J. M. (1998). Gaussian curvature modulus of an amphiphilic monolayer. *Langmuir*, 14(26):7427–7434.

[98] Wheeler, G. (2012). Surface diffusion flow near spheres. *Calc. Var. Partial Differential Equations*, 44(1-2):131–151.

[99] Willmore, T. J. (1965). Note on embedded surfaces. *An. Şti. Univ. "Al. I. Cuza" Iaşi Secţ. I a Mat. (N.S.)*, 11B:493–496.

[100] Wojtowytsch, S. (2017). Helfrich's energy and constrained minimisation. *Commun. Math. Sci.*, 15(8):2373–2386.

[101] Young, L. C. (1951). Surfaces paramétriques généralisées. *Bull. Soc. Math. France*, 79:59–84.