Convergent and oscillatory solutions in infinite-dimensional synchronization systems

CONVERGENT AND OSCILLATORY SOLUTIONS IN INFINITE-DIMENSIONAL SYNCHRONIZATION SYSTEMS

Alexandr P. Elsakov
Saint Petersburg State University of
Architecture and Civil Engineering
elsakov982@inbox.ru

Anton V. Proskurnikov
‘ Department of Electronics and Telecommunications,
Politecnico di Torino, Turin, Italy
anton.p.1982@ieee.org

Vera B. Smirnova
Saint Petersburg State University of
Architecture and Civil Engineering
and Saint Petersburg University
smirnova_vera_b@mail.ru

Article history:
Received 05.12.2023, Accepted 26.12.2023

Abstract

Control systems that arise in phase synchronization problems are featured by infinite sets of stable and unstable equilibria, caused by presence of periodic nonlinearities. For this reason, such systems are often called “pendulum-like”. Their dynamics are thus featured by multi-stability and cannot be examined by classical methods that have been developed to test the global stability of a unique equilibrium point. In general, only sufficient conditions for the solution convergence are known that are usually derived for pendulum-like systems of Lurie type, that is, interconnections of stable LTI blocks and periodic nonlinearities, which obey sector or slope restrictions. Most typically, these conditions are written as multi-parametric frequency-domain inequalities, which should be satisfied by the transfer function of the system’s linear part. Remarkably, if the frequency-domain inequalities hold outside some bounded range of frequencies, then the absence of periodic solutions with frequencies in this range is guaranteed, which can be considered as a weaker asymptotical property.

It should be noticed that validation of the frequency-domain stability condition for a given structure of the known linear part of the system is a self-standing non-trivial problem. In this paper, we demonstrate that a previously derived frequency-domain conditions for stability and absence of oscillations can be substantially simplified, which allows to tighten the estimates on the system’s parameters ensuring the corresponding asymptotic property. We demonstrate the efficiency of new criteria on specific synchronization systems.

Key words

Synchronization system, frequency-algebraic criteria, high-frequency oscillations,

1 Introduction

In this paper, we go on with developing multi-parametric frequency-algebraic criteria for the asymptotic behavior of infinite-dimension synchronization systems described by integral-differential equations (see [Smirnova and Proskurnikov, 2019] and bibliography there).

The simplest example of the synchronization system is the model of mathematical pendulum with viscous friction. So such systems are often called pendulum-like. Synchronization (or pendulum-like) control systems are featured by the properties like those of the mathematical pendulum. Their linear parts have zero eigenvalues, their nonlinear parts are periodic.

The class of synchronization systems is rather vast. It embraces phase-locked loops (PLL) [Best, 2003; Leonov and Kuznetsov, 2014; Leonov et al., 2015], communication systems [Solodov and Solodova, 1980], networks with periodic couplings [Dörfler and Bullo, 2014], electrical machines [Stoker, 1950; Halanay, 1975; Olmi et al., 2021]. vibrational units (rotors) [Blekhman, 2000; Sperling et al., 1997; Tomchina, 2020; Tomchina, 2022; Andrievsky and Boikov, 2021], biological systems [S.Somolinos, 1978; Burton, 1985].

The peculiarities of synchronization systems imply the specific character of their asymptotic behaviorThese systems have, as a rule, denumerable sets of equilibria, both

Lyapunov stable and unstable ones. The global stability of the synchronization system is usually treated as the convergence of every solution. Correspondingly, the methods of stability investigation devised for control systems with the unique equilibrium have turned out to be of no good for synchronization systems. These methods had to be adjusted for a new kind of asymptotic behavior. That is why the special tool has been developed [Gelig et al., 2004; Leonov et al., 1996] within the framework of traditional methods.

If a synchronization system is not stable, it may have periodic oscillations. Since high-frequency oscillations are undesired, the problem appeared to establish the estimates for the frequencies of possible oscillations.

The special procedures developed for stability investigation of synchronization systems turned out to be rather fruitful here. These procedures were combined with the tool of Fourier series and in such a way a number of frequency-algebraic criteria for the absence of high-frequency oscillations was obtained (see [Perkin et al., 2015] and bibliography there). The criteria were formulated in terms of the transfer function of the linear part of the system. They had the form of frequency-domain inequalities with varying parameters. The criteria give the opportunity to obtain estimates for domains with low-frequency oscillations. The more is a number of varying parameters the more "flexible" is the frequency-algebraic criteria and the more tight are the estimates for such domains in the space of parameters of the system.

In this paper, being based of results of [Proskurnikov and Smirnova, 2020] we choose the optimal algebraic conditions for varying parameters.

The optimal frequency-algebraic criterion for the absence of high-frequency oscillations is appllied to phase-locked loops with the proportional-integrating filter.

2 Problem setup

Consider a SISO synchronization system with distributed parameters which is described by an integro-differential Volterra equation

$$\begin{aligned}\frac{d\sigma}{dt} &= b(t) + \rho\varphi(\sigma(t-h)) - \\ &- \int_0^t \gamma(t-\tau)\varphi(\sigma(\tau))\,d\tau \quad (t>0).\end{aligned} \tag{1}$$

Here $\rho \in \mathbb{R}, h \geq 0; b, \gamma : [0, +\infty) \to \mathbb{R};\;\; \varphi : \mathbb{R} \to \mathbb{R}$. The solution of (1) is defined by specifying initial condition

$$\sigma(t)\Big|_{t\in[-h,0]} = \sigma^o(t). \tag{2}$$

The following assumptions are adopted.

A1) $b(t)$ is continuous, $\gamma(t)$ is piece-wise continuous,

$$b(t) \to 0 \text{ as } t \to +\infty. \tag{3}$$

A2) The linear part of (1) is stable;

$$b(t)e^{rt}, \gamma(t)e^{rt} \in L_2[0, +\infty) \quad (r > 0). \tag{4}$$

A3) $\varphi(\sigma)$ is $\mathbb{C}$–smooth, Δ– periodic and has a piece-wise continuous derivative; it has two roots $\sigma_1,\, \sigma_2 \in [0,\, \Delta)$ with

$$\varphi'(\sigma_1)\cdot\varphi'(\sigma_2) < 0.$$

A4) $\sigma_0(t) \in \mathbb{C}[-h,\, 0]$ and $\sigma(0+0) = \sigma^o(0)$.

A5)

$$\int_0^\infty \gamma(t)\,dt \neq \rho. \tag{5}$$

System (1) with assumptions A1)–A5) has a denumerable set of equilibria.

The main problem for synchronization system is to provide its global stability.

Definition 1. We say that equation (1) is globally stable (is gradient-like) if its any solution converges to a certain equilibrium.

Another significant problem is to guarantee the absence of high-frequency oscillations.

Definition 2. We say that equation (1) has a periodic solution $\sigma(t)$ if there exist a number $T > 0$ and an integer I such that

$$\sigma(t+T) = \sigma(t) + I\Delta, \quad \forall t \geq 0. \tag{6}$$

The number $\omega = \frac{2\pi}{T}$ is called the frequency of the periodic solution. In case $I = 0$ we fix a periodic solution of the first kind. If $I \neq 0$ we call $\sigma(t)$ a periodic solution of the second kind.

In [Smirnova and Proskurnikov, 2019] a number of frequency-algebraic theorems have been proved both for global stability and for the absence of periodic solutions. In this paper, we examine the problem of the absence of high-frequency periodic solutions. Our goal is to get the most accurate upper estimates for the frequencies of possible oscillations.

3 The absence of high frequency periodic solutions

Frequency-domain criteria are based on fundamental characteristics of linear and nonlinear parts of the control system.

The linear part of synchronization system (1) is entirely defined by its transfer function from the input φ to the output $(-\dot{\sigma})$:

$$K(p) = -\rho e^{-ph} + \int_0^\infty \gamma(t)e^{-pt}\,dt \quad (p \in \mathbb{C}).$$

The periodic function $\varphi(\sigma)$ is characterized by two constants

$$A_1 \triangleq \inf_{\sigma\in[0,\Delta)} \varphi'(\sigma), \quad A_2 \triangleq \sup_{\sigma\in[0,\Delta)} \varphi'(\sigma).$$

It is clear that

$$A_1 < 0 < A_2. \tag{7}$$

Suppose[1]

$$\alpha_1 \le A_1, \quad \alpha_2 \ge A_2. \tag{8}$$

Let $\alpha \triangleq (\alpha_1, \alpha_2)^T$.

We introduce then a non-negative accompanying function

$$\Phi(\sigma;\alpha) \triangleq \sqrt{(1-\alpha_1^{-1}\varphi'(\sigma))(1-\alpha_2^{-1}\varphi'(\sigma))} \tag{9}$$

where $\alpha_i^{-1} = 0$ if $\alpha_i = \pm\infty$ $(i = 1, 2)$, and constants

$$\nu \triangleq \frac{\int\limits_0^\Delta \varphi(\sigma)\,d\sigma}{\int\limits_0^\Delta |\varphi(\sigma)|\,d\sigma},$$

$$\nu_0(\alpha) \triangleq \frac{\int\limits_0^\Delta \varphi(\sigma)\,d\sigma}{\int\limits_0^\Delta |\varphi(\sigma)|\Phi(\sigma;\alpha)\,d\sigma},$$

$$\nu_1(\varepsilon,\tau,\alpha) \triangleq \frac{\int\limits_0^\Delta \varphi(\sigma)\,d\sigma}{\int\limits_0^\Delta |\varphi(\sigma)|\sqrt{\varepsilon+\tau\Phi^2(\sigma;\alpha)}\,d\sigma}.$$

Let $\varkappa \in \mathbb{R}$, $\varepsilon > 0$, $\tau > 0$, $\alpha_1 \le A_1$, $\alpha_2 \ge A_2$ be varying parameters. We shall need the functions:

$$\begin{gathered}\Pi(\omega;\varkappa,\varepsilon,\tau,\alpha) \triangleq \varkappa Re K(\imath\omega) - \\ -\tau(\alpha_2^{-1}+\alpha_1^{-1})\omega Im K(\imath\omega) - (\varepsilon+\tau)|K(\imath\omega)|^2 + \\ +|\alpha_1^{-1}\alpha_2^{-1}|\tau\omega^2 \ \ (\imath^2=-1), \quad \forall\omega\ge 0\end{gathered} \tag{10}$$

and constants

$$M(\varepsilon,\tau,\alpha) \triangleq \frac{\nu^2\nu_0^2(\alpha)}{4(\varepsilon\nu_0^2(\alpha)+\tau\nu^2)}. \tag{11}$$

From [Smirnova and Proskurnikov, 2019] we deduce, for a SISO system the following assertion.

Theorem 1. Suppose there exist an $\bar\omega \ge 0$ and numbers $\varkappa, \varepsilon > 0$, $\tau > 0$, $\delta > 0$, $\alpha_1 \le A_1$, $\alpha_2 \ge A_2$ such that the inequality

$$\Pi(\omega;\varkappa,\varepsilon,\tau,\alpha) \ge \delta \tag{12}$$

is valid for $\omega = 0$ and for $\omega \ge \bar\omega$, and one of the two inequalities is true: either

1)

$$4\varepsilon\delta\tau > \varkappa^2\nu_0^2(1-a)^2\varepsilon + \varkappa^2 a^2\nu^2\tau \tag{13}$$

for some $a \in [0, 1]$, or

2)

$$4\delta > \varkappa^2\nu_1^2. \tag{14}$$

Then

i) if $\bar\omega = 0$, the equation (1) is globaly stable;

ii) if $\bar\omega > 0$, the equation (1) has no periodic solutions with the frequency $\omega \ge \bar\omega$.

Let us analyze the conditions of Theorem 1.

Notice first that if $\varkappa = 0$ the inequality (12) is violated for $\omega = 0$ and if $\varkappa \ne 0$ one can scale the parameters $\varepsilon > 0$ and $\tau > 0$ by $|\varkappa|$. So we can put

$$\varkappa \triangleq \varkappa_0 = sign K(0) = sign\left(\int\limits_0^\infty \gamma(t)\,dt - \rho\right), \tag{15}$$

then $\varkappa^2 = 1$.

Consider algebraic condition 1). It is clear that one should choose the value of a in such a way that the value of δ is the minimal one. It follows from 1) that

$$\delta > \frac{1}{4}\left(\frac{a^2\nu^2}{\varepsilon} + \frac{(1-a)^2\nu_0^2(\alpha)}{\tau}\right). \tag{16}$$

It is easy to calculate that the minimal value of the right-hand part of (16) is achieved for

$$a = a_0 = \frac{\varepsilon\nu_0^2(\alpha)}{\varepsilon\nu_0^2(\alpha)+\tau\nu^2}, \tag{17}$$

and

$$\min_{a\in[0,\,1]}\left\{\frac{1}{4}\Big(\frac{a^2\nu^2}{\varepsilon} + \frac{(1-a)^2\nu_0^2(\alpha)}{\tau}\Big)\right\} = \\ = M(\varepsilon,\tau,\alpha).$$

The value a_0 is “the best” for the fixed set $\{\varkappa, \varepsilon, \tau, \alpha\}$ since it corresponds to the minimal possible value of δ. So inequality (16) should be substituted by

$$\delta > M(\varepsilon,\tau,\alpha). \tag{18}$$

Compare now the right-hand parts of inequalities (14) and (18). Notice that

$$M = \frac{\left(\int\limits_0^\Delta \varphi(\sigma)\,d\sigma\right)^2}{4\left(\varepsilon\left(\int\limits_0^\Delta |\varphi(\sigma)|\,d\sigma\right)^2 + \tau\left(\int\limits_0^\Delta \Phi(\sigma)|\varphi(\sigma)|\,d\sigma\right)^2\right)},$$

[1] α_i $(i = 1, 2)$ may be a number or $\pm\infty$.

$$\frac{\nu_1^2}{4} = \frac{\left(\int\limits_0^\Delta \varphi(\sigma)\, d\sigma\right)^2}{4\left(\int\limits_0^\Delta \sqrt{\varepsilon + \tau\Phi^2(\sigma)}|\varphi(\sigma)|\, d\sigma\right)^2}.$$

Proposition 1

$$M \geq \frac{\nu_1^2}{4}. \tag{19}$$

Proof. It is sufficient to prove that

$$\begin{gathered}\left(\int\limits_0^\Delta |\varphi|\, d\sigma\right)^2 + \frac{\tau}{\varepsilon}\left(\int\limits_0^\Delta \Phi|\varphi|\, d\sigma\right)^2 \leq \\ \leq \left(\int\limits_0^\Delta \sqrt{1 + \frac{\tau}{\varepsilon}\Phi^2}|\varphi|\, d\sigma\right)^2.\end{gathered} \tag{20}$$

Consider the function

$$\begin{gathered}W(y) \triangleq \left(\int\limits_0^\Delta |\varphi|\, d\sigma\right)^2 + y\left(\int\limits_0^\Delta \Phi|\varphi|\, d\sigma\right)^2 - \\ - \left(\int\limits_0^\Delta \sqrt{1 + y\Phi^2}|\varphi|\, d\sigma\right)^2 \quad (y \geq 0).\end{gathered} \tag{21}$$

Notice that

$$W(0) = 0 \tag{22}$$

and

$$\begin{gathered}W'(y) = \left(\int\limits_0^\Delta \Phi|\varphi|\, d\sigma\right)^2 - \\ - \left(\int\limits_0^\Delta \sqrt{1 + y\Phi^2}|\varphi|\, d\sigma\right) \cdot \int\limits_0^\Delta \frac{\Phi^2|\varphi|\, d\sigma}{\sqrt{1 + y\Phi^2}} = \\ = \left(\int\limits_0^\Delta \frac{\Phi(\sqrt{|\varphi|})^2\sqrt[4]{1 + y\Phi^2}\, d\sigma}{\sqrt[4]{1 + y\Phi^2}}\right)^2 - \\ - \int\limits_0^\Delta \sqrt{1 + y\Phi^2}|\varphi|\, d\sigma \int\limits_0^\Delta \frac{\Phi^2|\varphi|\, d\sigma}{\sqrt{1 + y\Phi^2}}.\end{gathered} \tag{23}$$

So by Cauchy-Bunyakovsky-Schwarz inequality

$$W(y) \leq 0, \quad \forall y > 0.$$

Thus (20) is true and consequently the Proposition is proved.

It follows that Theorem 1 can be substituted by the following assertion.

Theorem 2 Suppose there exist an $\bar{\omega} \geq 0$ and numbers $\varepsilon > 0$, $\tau > 0$, $\alpha_1 \leq A_1$, $\alpha_2 \geq A_2$ such that

$$\inf_{\{\omega=0\}\bigcup\{\omega\geq\bar{\omega}\}} \Pi(\omega; \varkappa_0, \varepsilon, \tau, \alpha) > \frac{\nu_1^2}{4}. \tag{24}$$

Then
i) if $\bar{\omega} = 0$, the equation (1) is globally stable;
ii) if $\bar{\omega} > 0$, the equation (1) has no periodic solutions with the frequency $\omega \geq \bar{\omega}$.

Theorem 2 implies that a triple $\{\varepsilon, \tau, \alpha\}$ of varying parameters is the element of the set

$$\begin{gathered}\Xi \triangleq \Big\{(\varepsilon, \tau, \alpha) : \varepsilon > 0, \tau > 0, \alpha_1 \leq A_1, \alpha_2 \geq A_2, \\ |K(0)| - (\varepsilon + \tau)K^2(0) > \frac{\nu_1^2(\varepsilon, \tau, \alpha)}{4}\Big\}.\end{gathered}$$

If Ξ is an empty set (i.e. $\Xi = \varnothing$) then Theorem 2 is inapplicable.

Suppose $\Xi \neq \varnothing$ and $(\varepsilon, \tau, \alpha) \in \Xi$. Let

$$\begin{gathered}\Omega(\varkappa_0, \varepsilon, \tau, \alpha) \triangleq \\ \triangleq \Big\{\omega > 0 : (\varepsilon, \tau, \alpha) \in \Xi \neq \varnothing, \\ \Pi(\omega; \varkappa_0, \varepsilon, \tau, \alpha) \leq \frac{\nu_1^2}{4}\Big\}.\end{gathered}$$

In virtue of (4) the set $\Omega(\varkappa_0, \varepsilon, \tau, \alpha)$ is bounded.

Next we determine

$$\tilde{\omega}(\varkappa_0, \varepsilon, \tau, \alpha) \triangleq \begin{cases} 0, \text{ if } \Omega = \varnothing, \\ \sup \Omega(\varkappa_0, \varepsilon, \tau, \alpha), \text{ if } \Omega \neq \varnothing. \end{cases}$$

Notice that every triple of varying parameters $(\varepsilon, \tau, \alpha) \in \Xi$ defines "its own" value of $\tilde{\omega}$.

It follows from Theorem 2 that if $\tilde{\omega}(\varkappa_0, \varepsilon, \tau, \alpha) = 0$ for a certain $(\varepsilon, \tau, \alpha)$, the equation (1) is globally stable. If $\tilde{\omega}(\varkappa_0, \varepsilon, \tau, \alpha) > 0$, the equation (1) has no periodic solutions with the frequency

$$\omega > \tilde{\omega}(\varkappa_0, \varepsilon, \tau, \alpha). \tag{25}$$

Let

$$\omega_0 \triangleq \inf_{(\varepsilon,\tau,\alpha)\in\Xi} \tilde{\omega}(\varkappa_0, \varepsilon, \tau, \alpha). \tag{26}$$

Theorem 3. Suppose $\Xi \neq \varnothing$. Then if $\omega_0 = 0$, the equation (1) is globally stable, and if $\omega_0 > 0$, the equation (1) has no periodic solutions with the frequency

$$\omega > \omega_0. \tag{27}$$

4 The estimates for domains with slow oscillations

In this section we apply Theorem 3 to the phase-locked loops (PLL) with the proportional integrating filter (PIF). The PLL can be described by the system

$$\begin{cases} \dot{z}(t) = -\frac{1}{T}z(t) - (1-s)\varphi(\sigma(t-h)), \\ \dot{\sigma}(t) = z(t) - sT\varphi(\sigma(t-h)) \\ \qquad (T > 0, s \in (0,1), h \geq 0), \end{cases} \tag{28}$$

where $\varphi(\sigma)$ is the characteristic of the phase detector (PD) and $h \geq 0$ is a time-delay.

The transfer function of (28) from φ to $(-\dot{\sigma})$ is as follows

$$K(p) = T\frac{sTp+1}{Tp+1}e^{-ph}. \tag{29}$$

System (28) can be easily reduced to the integral-differential equation (1) with

$$b(t) = z(0)e^{\frac{-t}{T}} - \\ - \begin{cases} \int\limits_{-h}^{t-h} e^{\frac{-(t-\lambda-h)}{T}}\varphi(\sigma^o(\lambda))\,d\lambda, & t < h, \\ \int\limits_{-h}^{0} e^{\frac{-(t-\lambda-h)}{T}}\varphi(\sigma^o(\lambda))\,d\lambda, & t \geq h; \end{cases} \tag{30}$$

$$\rho = -sT;$$

$$\gamma(t) = (1-s)\begin{cases} 0, & t < h. \\ e^{-\frac{(t-h)}{T}}, & t \geq h. \end{cases} \tag{31}$$

4.1 The undelayed PLL with a sine–shaped characteristic of phase detector (PD)

In this case

$$h = 0, \quad \varphi(\sigma) = \sin\sigma - \beta \quad (\beta \in (0,\,1)). \tag{32}$$

System (28), (32) has three parameters: $s,\ T,\ \beta$.
In our case

$$|\varphi'(\sigma)| \leq 1, \quad \alpha_1 \leq -1,\ \alpha_2 \geq 1. \tag{33}$$

Let $m_1 = -\alpha_1^{-1}$, $m_2 = \alpha_2^{-1}$. Evidently $m_i \in [0,\,1]$ $(i = 1, 2)$.
We have

$$|\nu_1| = \frac{2\pi\beta}{\int\limits_0^{2\pi} |\sin\sigma - \beta|\sqrt{\varepsilon + \tau\Phi^2(\sigma)}\,d\sigma} \tag{34}$$

where

$$\Phi(\sigma) = \sqrt{1 - m_1 m_2\cos^2\sigma + (m_1 - m_2)\cos\sigma}$$

Notice that $K(0) = T > 0$. Consequently $\varkappa = 1$.
Denoting $\bar{\varepsilon} = \varepsilon T$, $\bar{\tau} = \tau T$ one obtains:

$$\Pi(\omega;\,1,\varepsilon,\tau,\alpha) = \frac{\bar{\tau}}{T}m_1 m_2\omega^2 + T\frac{(a(T)\omega^2 + b)}{T^2\omega^2+1}, \tag{35}$$

where

$$a(T) = sT^2 - (\bar{\varepsilon} + \bar{\tau})s^2T^2 + \\ +\bar{\tau}(m_2 - m_1)(1-s), \\ b = 1 - \bar{\varepsilon} - \bar{\tau}.$$

Inequality (24) takes the form

$$\inf_{\{\omega=0\}\bigcup\{\omega\geq\omega_0\}} \left\{\bar{\tau}m_1m_2\omega^2 + \frac{T^2(a(T)\omega^2+b)}{T^2\omega^2+1}\right\} > \\ > \frac{T^2\nu_1^2(\bar{\varepsilon},\bar{\tau},\alpha)}{4} \tag{36}$$

and inequality

$$|K(0)| - (\varepsilon+\tau)|K(0)|^2 > \frac{\nu_1^2(\varepsilon,\tau,\alpha)}{4}$$

transforms into

$$1 - \bar{\varepsilon} - \bar{\tau} > \frac{\nu_1^2(\bar{\varepsilon},\bar{\tau},\alpha)}{4}$$

Changing with sufficiently small steps the values of $\bar{\varepsilon} \in (0;1)$, $\bar{\tau} \in (0;1-\bar{\varepsilon})$, $m_i \in (0;1]$ $(i = 1,2)$ we find for a pair $\{T^2, \beta\}$ the value of ω_0 from inequality (26). Thus we guarantee that for this pair of parameters the periodic solutions with $\omega > \omega_0$ are absent. In Fig. 1 we show for $s = 0.2$ the domains where ω_0 is equal to 3, 8, 10. The domains are situated under the curves. The leftmost curve corresponds to a gradient-like system $(\omega_0 = 0)$.

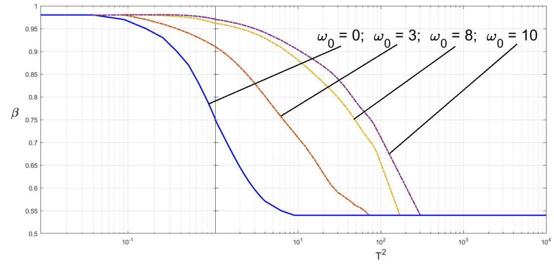

Figure 1. Domains where the oscillations with the frequency more than ω_0 are absent for undelayed PLL with PIF and a sine–shaped characteristic of PD

4.2 The case of sine–shaped characteristic of PD with a positive time-delay

In this case

$$h > 0, \;\; \varphi(\sigma) = \sin\sigma - \beta \quad (\beta \in (0,\,1)) \tag{37}$$

and formulas (33), (34) are valid. Also $\varkappa_0 = 1$. We suppose that $|\alpha_1| = \alpha_2 \geq 1$ and use the denotation $m \stackrel{\Delta}{=} |\alpha_i|^{-1}$.

Denoting $\bar{\varepsilon} \stackrel{\Delta}{=} \varepsilon T$, $\bar{\tau} \stackrel{\Delta}{=} \tau T$ we obtain

$$\begin{gathered}\Pi(\omega;\,1,\varepsilon,\tau,\alpha) = m^2\frac{\bar{\tau}}{T}\omega^2 + \\ +T\big((1+T^2 s\omega^2)\cos\omega h - \omega T(1-s)\sin\omega h - \\ -(\bar{\varepsilon}+\bar{\tau})(1+s^2T^2\omega^2)\big)(1+T^2\omega^2)^{-1}.\end{gathered} \tag{38}$$

Traditionally using $\hat{\omega} \stackrel{\Delta}{=} \omega T$ [Biswas et al., 1977; Belustina, 1992] we have

$$\begin{gathered}\Pi(\omega;\,1,\varepsilon,\tau,\alpha) = m^2\bar{\tau}\frac{\hat{\omega}^2}{T^3} + \\ +T\Big((1+\hat{\omega}^2 s)\cos\frac{\hat{\omega}h}{T} - \hat{\omega}(1-s)\sin\frac{\hat{\omega}h}{T} - \\ -(\bar{\varepsilon}+\bar{\tau})(1+s^2\hat{\omega}^2)\Big)(1+\hat{\omega}^2)^{-1}.\end{gathered} \tag{39}$$

Inequality (24) can be substituted by the inequality

$$\inf_{\hat{\omega}=0\bigcup\hat{\omega}\geq\bar{\omega}} P(\hat{\omega};\bar{\varepsilon},\bar{\tau},\alpha) > 0, \tag{40}$$

where

$$\begin{gathered}P(\hat{\omega};\bar{\varepsilon},\bar{\tau},\alpha) \stackrel{\Delta}{=} m^2\bar{\tau}\hat{\omega}^2 + \\ +T^4\Big\{\Big((1+\hat{\omega}^2 s)\cos\frac{\hat{\omega}h}{T} - \hat{\omega}(1-s)\sin\frac{\hat{\omega}h}{T} - \\ -(\bar{\varepsilon}+\bar{\tau})(1+\hat{\omega}^2 s^2)\Big)(1+\hat{\omega}^2)^{-1} - \frac{\nu_1^2(\bar{\varepsilon},\bar{\tau},\alpha)}{4}\Big\}.\end{gathered} \tag{41}$$

Consider the case of $s = 0.2$, $\frac{h}{T} = 0.1$. For each $\beta \in (0,1)$ and $T > 0$ the inequality (40) can be checked numerically. For any triple $\{\bar{\varepsilon},\bar{\tau},m\}$ it is easy to find a number $\Omega_0 = \Omega_0(\bar{\varepsilon},\bar{\tau},m)$ such that

$$P(\hat{\omega};\bar{\varepsilon},\bar{\tau},\alpha) > 0 \tag{42}$$

for $\hat{\omega} > \Omega_0$.

Then we fix the value of δ small enough and choose a step $h_\omega = h_\omega(\delta)$ in such a way that the inequalities

$$P(h_\omega k;\bar{\varepsilon},\bar{\tau},\alpha) > \delta \quad \Big(k \in \mathbb{N}\bigcup\{0\},\; h_\omega k \in [0,\Omega_0]\Big)$$

imply

$$P(\hat{\omega};\bar{\varepsilon},\bar{\tau},\alpha) > 0, \qquad \hat{\omega} \in [h_\omega k,\; h_\omega(k+1)].$$

Thus we find the value of $\bar{\omega}$ such that inequality (40) is true. Changing the values of $m \in (0;1]$, $\bar{\varepsilon} \in (0;1)$ and $\bar{\tau} \in (0; 1-\bar{\varepsilon})$ with small steps we define the value of ω_0 as the minimal of all possible $\bar{\omega}$.

In Fig. 2 the domains with $\frac{\omega_0}{T}$ equal to 2, 4, 7 are shown. They are situated under the curves. The leftmost one corresponds to the gradient-like system.

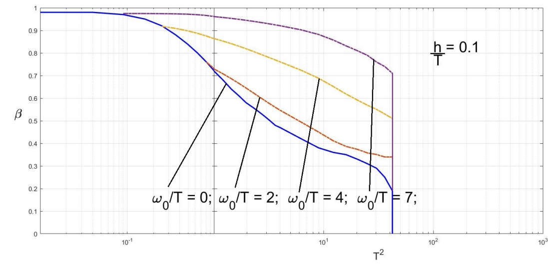

Figure 2. Domains of slow oscillations for PLL with PIF, a sine–shaped characteristic of PD, and delay $h = 0.1T$

4.3 The undelayed PLL with a triangle characteristic of PD

In this case

$$h = 0, \;\; \varphi(\sigma) = \begin{cases} \frac{2}{\pi}\sigma - \beta, & \sigma \in [-\frac{\pi}{2},\, \frac{\pi}{2}), \\ -\frac{2}{\pi}(\sigma-\pi) - \beta, & \sigma \in [\frac{\pi}{2},\, \frac{3\pi}{2}); \end{cases} \tag{43}$$

Notice that

$$|\varphi'(\sigma)| = \frac{2}{\pi}.$$

We confine ourselves to the variant of

$$-\alpha_1 = \alpha_2 > \frac{2}{\pi}.$$

It is easy to calculate that

$$\begin{gathered}|\nu| = \frac{2\beta}{1+\beta^2}, \\ |\nu_1| = \frac{|\nu|}{\sqrt{\varepsilon + \tau(1-\frac{4}{\pi^2}m^2)}} \quad (m = |\alpha_i|^{-1})\end{gathered} \tag{44}$$

Inequality (24) has the form (36).

In Fig. 3 for $s = 0.2$ the domains of slow oscillations with $\omega < \omega_0 = 3,\, 8,\, 10$ are shown. The leftmost curve corresponds to a globally stable system ($\omega_0 = 0$).

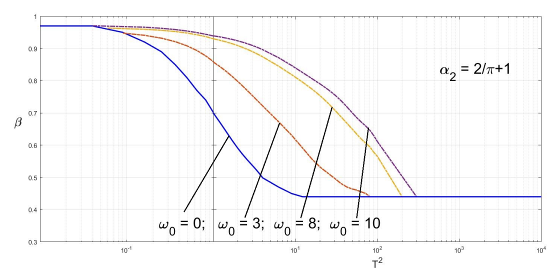

Figure 3. Domains where the oscillations with the frequency more than ω_0 are absent for undelayed PLL with PIF and a triangle characteristic of PD

5 Conclusion

This paper is devoted to the asymptotic behavior of infinite-dimensional synchronization (pendulum-like) systems. In particular, the problem of the absence of high-frequency oscillations is considered. Some frequency-algebraic criteria for the absence of oscillations of the prescribed frequency are analyzed, and the optimal one is applied to phase-locked loops with the proportionally-integrating filter.

References

Andrievsky, B. and Boikov, V. (2021). Bi directional controlled multiple synchronization of unbalanced rotors and its experimental evaluation. *Cybernetics and Physics*, **10** (2), pp. 63–64.

Belustina, L. (1992). Qualitative-numerical analysis for synchronization system with time-delay (in Russian). In *Proc. of International Seminar "Nonlinear circuits and systems"*, vol. 2, Moscow, pp. 50–59.

Best, R. (2003). *Phase-locked Loops: Design, Simulation and Applications*. McGraw-Hill.

Biswas, B. N., Banerjee, P., and Bhattaharia, A. K. (1977). Heterodyne phase locked loops-revisited. *IEEE Trans. on Commun.*, **25** (10), pp. 1164–1179.

Blekhman, I. (2000). *Vibrational mechanics*. World Scientific Publ. Co.

Burton, T. (1985). *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*. Acad. Press Inc., Orlando, FL.

Dörfler, F. and Bullo, F. (2014). Synchronization in complex networks of phase oscillators: A survey. *Automatica*, **50** (6), pp. 1539–1564.

Gelig, A., Leonov, G., and Yakubovich, V. (2004). *Stability of stationary sets in control systems with discontinuous nonlinearities*. World Scientific Publ. Co.

Halanay, A. (1975). Stability problem for synchronous machines. In *VII Intern. Konf. uber nichtlinear Schwingungen. Abh. der Acad. Wiss, DDR. Abt.-Naturwiss-Techn. Bd. II, N. 5*, Springer Int. Publ., pp. 407–421.

Leonov, G., Kuznetsov, N., Yuldashev, M., and Yuldashev, R. (2015). Hold-in, pull-in, and lock-in ranges of PLL circuits: Rigorous mathematical definitions and limitations of classical theory. *IEEE Trans. Circuits and Systems-I*, **62** (10), pp. 2454–2464.

Leonov, G. A. and Kuznetsov, N. V. (2014). *Nonlinear Mathematical Models Of Phase-Locked Loops. Stability and Oscillations.* Cambridge Univ. Press, Cambridge.

Leonov, G. A., Ponomarenko, D., and Smirnova, V. B. (1996). *Frequency-Domain Methods for Nonlinear Analysis. Theory and Applications*. World Scientific, Singapore–New Jersey–London–Hong Kong.

Olmi, S., Totz, C., and Schoell, E. (2021). Dynamik-control of intermittent renewable energy fluctuations in two layer power grids. *Cybernetics and Physics*, **10** (3), pp. 143–154,.

Perkin, A., Smirnova, V., Shepeljavyi, A., and Utina, N. (2015). Upper bounds for frequency of periodic regimes in many-dimensional and infinite dimensional phase synchronization systems. *Cybernetics and Physics*, **4** (2), pp. 41–48.

Proskurnikov, A. V. and Smirnova, V. B. (2020). Constructive estimates of the pull-in range for synchronization circuit described by integro-differential equations. In *Proc. of IEEE International Symposium on Circuts and Systems (ISCAS)*, Seville, Spain, p. 09180519 (5p.).

Smirnova, V. B. and Proskurnikov, A. V. (2019). Volterra equations with periodic nonlinearities:multistability, oscillations and cycle slipping. *Int. J. Bifurcation and Chaos*, **29** (5), pp. 1950068 (26p.).

Solodov, A. V. and Solodova, E. A. (1980). *Systems with variable delay*. Nauka (in Russian), Moscow.

Sperling, L., Merten, F., and Duckstein, H. (1997). Rotation und vibration in beispilen zur methode der direkten bewegungsteilung. *Technische Mechanik*, **17** (3), pp. 231–243.

S.Somolinos, A. (1978). Periodic solutions of the sunflower equation. *Quarterly of Appl.Math.*, (January), pp. 465–478.

Stoker, J. (1950). *Nonlinear vibrations in mechanical and electrical systems*. Interscience, New York.

Tomchina, O. (2022). Vibration field control of a two-rotor vibratory unit in the double synchronization mode. *Cybernetics and Physics*, **11** (4), pp. 246–252.

Tomchina, O. P. (2020). Control of oscillations in two-rotor cyberphysical vibration units with time-varying observer. *Cybernetics and Physics*, **9** (4), pp. 206–213.