Analytical Relation Between Natural Frequency and Spectral Entropy in Information Theory of Single Degree of Freedom Systems

Analytical Relation Between Natural frequency and Spectral Entropy in Information Theory of Single Degree of Freedom Systems

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Abstract. The eighth (and last) axiom of Structural Health Monitoring (SHM) states that damage increases the complexity of a system. After its introduction, several methods to measure complexity have been explored in the scientific literature. Among these, entropy measures demonstrate to be very straightforward to apply, with a relatively simple concept behind them. Thus, based on this feature, researchers proposed damage identification methodology in several branches of research, such as mechanical engineering, civil engineering, etc. In this work, the authors present an analytical study on the dependency of spectral entropy measures from the modal characteristics of a structural system, showing that an analytical relation exists for what concerns a Single Degree of Freedom (SDoF) system. A comparison between spectral entropy of different physical quantities (displacements, velocity, accelerations, etc.) is discussed for their use in SHM. Finally, different contributions of variation in entropy (i.e., due to external sources and structural properties) are analysed to better understand the causes of these perturbations. The work is of particular interest for cultural heritage structures since many of these are characterised by the presence of protruding elements, most of the time assimilable to SDoF systems (e.g., lanterns, belfries, pinnacles, etc.), when their dynamic is analysed relatively to their base movement.

Keywords: Structural Health Monitoring, Damage Detection, Spectral Entropy, Information Theory, Cultural Heritage Structures, Protruding elements.

1 Introduction

Structural Health Monitoring (SHM) [1] is an active discipline in different fields of application and research. It can be defined as the "*process of implementing a damage detection strategy for engineering infrastructure*" [2]. The research effort on this topic led, in 2007, to the definition of several axioms [3].

In addition to the previous axioms, one has been emerging in recent years (i.e., since 2010). This conjecture would state that "*damage increases the complexity of a system",* leaving, in this definition, the term *complexity* not clearly defined [4]. Subsequently, the concept of complexity was better specified for applications in SHM [5], and some methods to measure it, for example, through the use of different definitions of entropy in information theory [6]. Among the various definitions, those based on frequency distributions (spectral entropies) emerge as particularly interesting for the analysis of damage to civil structures [7–11]. The spectral entropy can be estimated with different definitions provided by information theory.

In general, spectral entropy, such as Shannon Spectral Entropy (SSE) or Wiener Entropy (WE), can be used to determine how close a signal is to the Gaussian noise condition. When the signal derives from a structural system, however, the reading of the output from a monitoring sensing system intrinsically determines to consider two factors: (i) external perturbations and (ii) the system characteristics (filter effect operated on the output). In this work, the authors critically analyze these two effects in order to clarify the use of a measure of spectral entropy for SHM. The Continuous counterpart of WE (CWE), calculated on a scaled frequency domain, will be used in the paper. The study is of particular interest because it validates the potential of spectral entropy to define the level of damage on the basis of information obtained from the entire frequency range of the signal, rather than individual frequencies. The Single Degree of Freedom (SDoF) formulation has applications in cultural heritage such as e.g. for protruding elements [12].

Section 2 provides an analytical relation between CWE and modal parameters of structural systems. In Section 3, the effects of external perturbations are studied in detail. Section 4 summarises the knowledge gained in the previous sections to perform a critical analysis of the use of CWE for SHM. Conclusions are drawn in Section 5.

2 Analytic relation between CWE and modal parameters

Given a generic recorded signal $o(t)$, it can be expressed in the frequency domain $O(f)$ by applying the Fourier transform operator (where t and f denote time and frequency variables). If the signal comes from an observed system comparable to a Single Degree of Freedom (SDoF) (i.e., the main dominant mode is preponderant on the others), the signal can be defined by the combination of the system contribution $H(f)$ ($h(t)$ in the time domain) and a perturbation contribution $I(f)$ ($i(t)$ in the time domain), in the frequency domain:

$$O(f) = H(f) \cdot I(f) \tag{1}$$

where the square value of the modulus $|\cdot|$, applied to $O(f)$, $H(f)$, or $I(f)$ is referred here as the power spectrum of the respective quantity. It is known that if $i(t)$ is an impulsive function in time, $|O(f)|^2$ represents a rescaled version of $|H(f)|^2$, while if $i(t)$ is comparable to random Gaussian noise in time, $|O(f)|^2$ can be approximated to a noisy version of $|H(f)|^2$ (with a certain degree of approximation).

The WE, w_{P_z}, of a generic signal is defined as the ratio between the geometric and arithmetic mean of the power spectrum:

$$w_{P_z} = \frac{e^{\frac{1}{Z}\sum_{z=0}^{Z-1}\ln(P_z)}}{\frac{1}{Z}\sum_{z=0}^{Z-1} P_z} \tag{2}$$

In Eq. (2), Z is the total number of frequency bins, z, observed in the power spectrum P for a generic signal. Given that the arithmetic-geometric mean inequality, w_{P_z} is always lower than one, and because for real signals $P_z > 0$, the WE is always greater than zero. In the case of a continuous functions $P(f)$ in the frequency domain (e.g., the ratio between frequency resolution and sampling frequency tends to zero), the discrete operators in Eq. (2) would be replaced by their continuous counterparts, resulting in a potential loss of some properties of w_{P_z}. However, the arithmetic-geometric mean inequality would continue to be valid thanks to Jensen's inequality [13] applied to the concave function $\ln(\cdot)$. Thus, the Continuous WE (CWE), w, is here defined as:

$$w = \frac{e^{\frac{1}{y-x}\int_x^y \ln(P(r))dr}}{\frac{1}{y-x}\int_x^y P(r)dr} \tag{3}$$

where x and y are the extremes of integration, r is a generic integration variable, and w is still bounded between zero and one. Contrary to the discrete definition in Eq. (2), Eq. (3) can incorporate several definitions based on the domain of calculation of the integrals. In this study, a scaled domain β will be used instead of f (or in terms of circular frequency $\omega = 2\pi f$), in order to simplify the operations of integration. It is worth underlying that, in general, the CWE calculated over β will differ from that one calculated over f, which in turn will differ from that one calculated over ω.

Given the importance of $H(f)$ in the definition of $O(f)$, in the following pages, the CWE of $H(f)$ will be estimated starting from different physical quantities of $o(t)$ and $i(t)$, (i.e., displacement, velocity, and acceleration), supposing $i(t)$ is an acceleration rather than a force as more commonly done. In this sense, the following functions of β are reported for displacement, velocity and acceleration outputs, i.e., square modulus of compliance, mobility and accelerance, multiplied by the mass, respectively:

$$|H(\beta)|_C^2 = P_C(\beta) = \frac{1}{a\beta^2 + b\beta + c} \tag{4}$$

$$|H(\beta)|_M^2 = P_M(\beta) = \sqrt{a}\frac{\beta}{a\beta^2 + b\beta + c} \tag{5}$$

$$|H(\beta)|_A^2 = P_A(\beta) = a\frac{\beta^2}{a\beta^2 + b\beta + c} \tag{6}$$

with:

$$a = \left(\frac{\omega_s}{2}\right)^4,\ b = (4\zeta^2 - 2)\omega_n^2\left(\frac{\omega_s}{2}\right)^2,\ c = \omega_n^4 \tag{7}$$

$$\omega_n = 2\pi f_n \text{ and } \omega_s = 2\pi f_s \tag{8}$$

$$\beta = \left(\frac{\omega}{\omega_s/2}\right)^2 \text{ and } \omega = 2\pi f \tag{9}$$

where f_s, f_n, and ζ denote, respectively, the sampling frequency, the natural frequency, and the damping ratio, while Eq. (9) defines the scaled integration domain, ensuring that x=0 and y=1 in Eq. (3). Thus β is bounded between zero and one. Eq. (3) becomes:

$$w_\beta = \frac{e^{\int_0^1 \ln(P(\beta))d\beta}}{\int_0^1 P(\beta)d\beta} \tag{10}$$

Whatever the form of the power spectrum equation (see Eqs. (4)-(6)), if this is replaced in Eq. (10), all integration operations are performed, and the parameters of Eqs. (7)-(9) are replaced, it is then possible to obtain a closed formulation that relates the w_β to sampling frequency, natural frequency, and damping ratios. The results of the integrations are reported hereinafter. It is worth mentioning that the CWE equation that define w_β is valid just when $f \geq 0$, while the integral at the numerator is valid at the limit towards the lower extreme for what concern mobility and accelerance.

2.1 Displacement/acceleration

The CWE for output displacements, w_C, results to be:

$$\begin{gathered} w_C = -\frac{2\mu^{8\,\mu^2\zeta^2-4\mu^2+2}\cdot\zeta\cdot e^{4\mu^2\zeta\sqrt{1-\zeta^2}\sigma+2}\cdot\sqrt{1-\zeta^2}}{\sigma(\mu^4+4\mu^2\zeta^2-2\mu^2+1)^{2\mu^2\zeta^2-\mu^2+1}} \\ \sigma = \operatorname{atan}\left(\frac{2\zeta^2-1}{2\zeta\sqrt{1-\zeta^2}}\right) - \operatorname{atan}\left(\frac{2\mu^2\zeta^2-\mu^2+1}{2\mu^2\zeta\sqrt{1-\zeta^2}}\right) \\ \mu = \frac{f_n}{f_s/_2} \text{ and } f_n = \frac{\sqrt{K/_M}}{2\pi} \end{gathered} \tag{11}$$

where μ is the relative frequency, while K and M denote stiffness and mass of the system. Fig. 1 depicts Eq. (11).

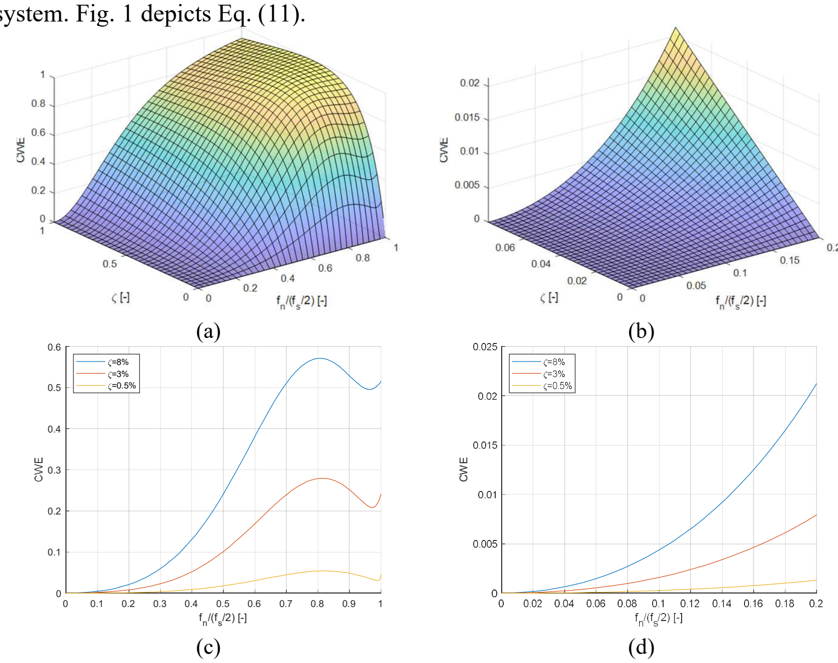


Fig. 1. Continuous Wiener Entropy (CWE) of displacement/acceleration signals: (a) CWE as function of damping ratio and normalized natural frequency; (b) zoom-in of (a); (c) CWE as function of normalized frequency for three common values of damping ratio for civil structures; (d) zoom-in of (c).

In Fig. 1, it is possible to appreciate how w_C globally diminishes at decreasing of damping and relative frequency values. Some exceptions are instead identified in correspondence of a relative frequency close to one and very high values of damping ratio (not plotted) that are out of the physical domain for civil structures (i.e., 0%-20%, or more likely 0.5%-8%). This result is quite interesting: for constant sampling frequency, it means that w_C decreases if natural frequency (and thus stiffness of the system) reduces. It is worth mentioning that this means that w_C would decrease if the mass of the system increases. On the other hand, a reduction of the modal damping characteristics of the system would result in a reduction of w_C as well.

2.2 Velocity/acceleration

As done for output displacements, the same can be obtained for the CWE of output velocities, w_M. However, the equation for this case is omitted for reasons of space (as it would be too long to write it). Fig. 2 depicts the obtained result for w_M.

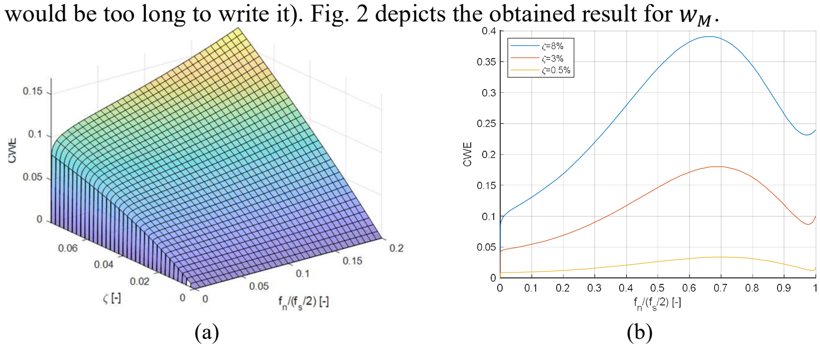
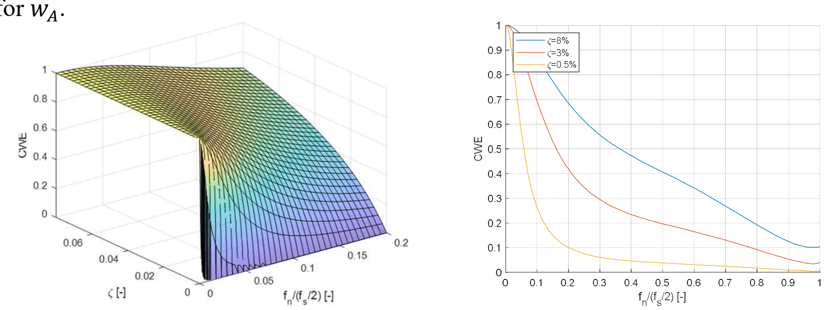


Fig. 2. Continuous Wiener Entropy (CWE) of velocity/acceleration signals: (a) CWE as a function of the damping ratio and normalised natural frequency (zoom-in); (b) CWE as a function of the normalised frequency for three common values of damping ratio for civil structures.

In Fig. 2, it is possible to appreciate how w_M globally reduces at decreasing values of both the damping and relative frequencys. Some exceptions are instead visible in correspondence of a relative frequency higher than 0.6-0.7. The result is quite interesting because, for constant sampling frequency, it means that w_V decreases if the natural frequency (and thus stiffness of the system) reduces but just when the main natural frequency of the system remains below the 65% of half-sampling frequency value, which for civil structures is quite a common finding. It is worth mentioning that this means that w_M would decrease if the mass of the system increases (always if the main natural frequency remains below 65% of the half-sampling frequency). Even in this case, a reduction in the modal damping characteristics of the system would result in a reduction of w_M as well, for any value of the relative frequency.

2.3 Acceleration/acceleration

As in the case of output displacements, the same can be obtained for the CWE of output accelerations, w_A. As for the velocity case, the equation of w_A is omitted for reasons of space (as it would be too long to write it). Fig. 3 depicts the obtained result for w_A.



(a) (b)

Fig. 3. Continuous Wiener Entropy (CWE) of acceleration/acceleration signals: (a) CWE as a function of the damping ratio and normalised natural frequency (zoom-in); (b) CWE as a function of the normalised frequency for three common values of damping ratio for civil structures.

In Fig. 3, it is possible to appreciate how w_A globally reduces at decreasing values of the damping ratio, but contrary to the other variables, it increases at decreasing values of the relative frequency. The result is quite interesting because, for constant sampling frequency, it means that w_A increases if natural frequency (and thus the stiffness of the system) reduces. It is worth mentioning that this means that w_A would increase if the mass of the system increases. Even in this case, a reduction in the modal damping characteristics of the system would result in a reduction of w_A for any value of relative frequency, except for relative frequency values very close to zero, for which $w_A = 1$ for any value of damping ratio (except for $\zeta = 0$ in Fig. 3a).

3 Causes of variation in CWE

In Section 2, the CWE of physical quantities was analysed as a function of some properties of the system. In this section, the effects of external perturbations on the CWE of $o(t)$ will be analyzed for different types of perturbations, i.e., impulse with different amplitudes, sinusoidal perturbation at varying frequency and amplitude, and Gaussian random noise with different values of standard deviations.

Recalling that the quantities in Eq. (1) are complex numbers, it is possible to write:

$$P_O = P_H \cdot P_I \tag{12}$$

where P_O, P_H, P_I are the power spectra of the output, transfer function, and input (in this order). Now, replacing P_O in Eq. (10) as function of β, and developing the calculations:

$$w_O = \frac{e^{\int_0^1 \ln(P_O(\beta))d\beta}}{\int_0^1 P_O(\beta)d\beta} = \frac{e^{\int_0^1 \ln(P_H(\beta))d\beta} \cdot e^{\int_0^1 \ln(P_I(\beta))d\beta}}{\int_0^1 P_H(\beta)P_I(\beta)d\beta} \tag{13}$$

where w_O is the CWE of the output. In general from Eq. (13) is clear as the CWE of the output power spectrum equals the ratio between the product of the geometric mean of transfer function and input power spectra, and the integral average of the product between the transfer function and input power spectra. Furthermore, recalling Eq. (10), it is possible to conclude that w_β is scale-invariant, an important property useful for the following discussion.

3.1 Impulse

Theoretically, if *i*(*t*) is a perfect impulse function, P_I is constant over β. Thus, thanks to the scale-invariant property of w_β Eq. (13) reads:

$$w_O = \frac{e^{\int_0^1 \ln(P_O(\beta))d\beta}}{\int_0^1 P_O(\beta)d\beta} = \frac{e^{\int_0^1 \ln(P_H(\beta))d\beta}}{\int_0^1 P_H(\beta)d\beta} = w_H \tag{14}$$

and thus, the CWE of the output equals the CWE of the transfer function w_H for any amplitude of the input (for any kind of quantity, i.e., displacement, velocity, and acceleration). Based on what has been said, the CWE of an input w_I tending to a single pulse will tend to one.

3.2 Sinusoid curve

The power spectrum of $i(t) = sin(\omega_0 t)$ is proportional to the product between π^2 and to the square value of the Dirac-delta function δ in $\omega = \omega_0$ (and $-\omega_0$), and zero elsewhere. Defining β_0 as the value of β related to ω_0, and recalling Eq. (13), it is possible to conclude that the result is undeterminate. However, if one think to real-world signals, the amplitude of the impulse located around β_0 would be finite, and the denominator of w_O in Eq. (13) would be proportional to the integral average of the power spectrum of the transfer function. The same cannot be said for the numerator part involving the external input, for which the presence of the $\ln(\cdot)$ function in the integral would bring the exponent of the exponential to tend towards $-\infty$, and consequently w_O would tend to zero, in accordance with the fact that *o*(*t*) will behave more deteministically due to the increse of information coming from the external sinusoidal input for any kind of output quantity, i.e., displacement, velocity, or acceleration. The conclusions are valid for any value of amplitude that multiply the sinusoidal function because the scale-invariant property of w_β. Based on what has been said, the CWE of an input w_I that is sinusoidal will tend to zero.

3.3 Gaussian random noise

Theoretically, a zero-input function will produce a zero-output observable function. In this sense, the Gaussian random noise condition is reached when the energy of a deterministic external perturbation tends to zero. Then, as demonstrated in Section 3.2, since the presence of a deterministic input will reduce the entropy value, its dissipation (and thus the tendency to reach random noise conditions) would inevitably lead to an increase in the entropy value, as can be appreciated by Eq. (13).

It is worth noting that the power spectrum of a continuous average of the Gaussian noise spectrum tends to be a constant (equal to the variance of the noise itself). In this case, the CWE evaluated on continuous averages of spectra obtained from ambient noise would tend, in principle, to the CWE of the transfer function, given the scale-invariance property of the CWE.

These results are quite important, because they demonstrate how spectral entropy: (i) will remain unchanged when external perturbations excite with the same magnitude all the frequency domain; (ii) will reduce if external deterministic signals are applied (because the energy of the input is narrowed in small locations of the power spectrum, even for the sum of sinusoidal functions); and (iii) will increase in the presence of Gaussian random noise (in other words, when the perturbation dissipates its effects, and the predominant frequencies in external perturbations are not recognised because their energy content becomes spread in frequency).

4 Critical discussion on the use of CWE for SHM

Being $H(f)$ a characteristic of the system, in the absence of internal variations, its shape remains constant over time. $I(f)$ instead, represents an observable quantity that describes the perturbations taking place on the system, which therefore determine variations in the form of $O(f)$. Remembering that spectral entropy can be used to measure how close a signal is to the Gaussian noise condition, it can be concluded that the more the Signal to Noise Ratio (SNR) of a signal is high the lower its entropy value. Since SNR is a function of the level of extractable information (i.e., pure signal), it can be said that, in the absence of changes in the structural state, the entropy of output signals observed by structural systems may vary for two causes, mainly,

- Changes in the amount of extractable information (pure signal level);
- Changes in the amount of noise level.

From this, it follows that the more an external perturbation assumes a deterministic behaviour, the lower the spectral entropy value of $O(f)$. The opposite case occurs when observing a system under *operational* conditions. Under these conditions, micro-tremors (ambient vibrations) are usually modeled with a Gaussian model (stochastic input), and the SNR value is very low. This consequently results in high spectral entropy values. Since a disturbance begins to act on a system, SNR of $I(f)$ increases due to the greater amount of information generated, and therefore the spectral entropy of $O(f)$ decreases. This means that the advent of an external perturbation allows observing a more deterministic behaviour of a system, thus leading to the decrease of entropy. The non-constant nature of $I(f)$ results in a continuous change in the spectral entropy of $O(f)$. Since the entropy of $O(f)$ is subject to variations due to external perturbations, a rigorous approach would be to monitor the entropy of $H(f)$ instead of $O(f)$. However, in order to carry out analyses of this kind, it would be necessary to know the form of $I(f)$ and this is not easy. In addition, we must consider that even $H(f)$ is, in principle, subject to variations due to Environmental and Operational Variations (EOVs) or to effects due to the occurrence of strong external perturbations (e.g., development of elastic nonlinearities of the structural behaviour).

To conclude, in the presence of an external perturbation, entropy analysis for identifying the variations in structural characteristics becomes more complicated, because $O(f)$ may differ significantly from $H(f)$. This would justify the research of models between spectral entropy and other characteristics able to synthesise/quantify the occurrence of a perturbation or the construction of derived measures of entropy that describe, in a simplified way, the only contribution made by $H(f)$ on the evaluation of entropy of $O(f)$. For example, the ratio of CWE of $O(f)$, e.g., CWE of acceleration / CWE of displacement, is a function proportional to the ratio of the integral average of displacement power spectrum and the integral average of acceleration power spectrum. Thus, the logarithmic terms of the input on the exponential in Eq. (13) are eliminated guaranteeing a more stability of the function in presence of external input.

5 Conclusions

In this paper, a critical analysis of the main causes of variation of a type of spectral entropy, the CWE, has been reported. This study aimed at discerning damage-related variations from unrelated ones, and it is of particular interest for cultural heritage structures that are mainly caractherized by the presence of protruding elements, which are intrinsically vulnerable. The conclusions of the study are summarised hereinafter:

- Given accelerations in input and displacements, velocities, or accelerations in output, the CWE always decreases for decreasing values of damping ratio, in a common range of variation of civil engineering interest (damping ratio from 0.5% to 8%, relative frequency between 0 and 0.2).
- Given constant sampling frequency, in a common range of variation of civil engineering interest, the CWE of displacements and velocities output responses decreases for decreasing values of natural frequency, while CWE of accelerations output responses increases for decreasing values of natural frequency.
- The CWE estimated from output responses of a system can be affected by the variations of external perturbations; thus, when the input is known, the CWE should be evaluated on the property of the system (e.g., frequency response functions). However, when the input is unknown, because the effects of the perturbations affect in the same way the CWE of displacements, velocities, and accelerations output responses (and given the paper results), a combination of these CWE may help to infer and reduce the unwanted variations in the spectral entropy due to external inputs.

It is worth mentioning that, in addition to the external input, other damage-unrelated perturbations can directly affect structural properties of monitored system (e.g., temperature affects natural frequencies) and thus indirectly the resulting entropy values. For this reason, a small unwanted variation in CWE can be still detected for real systems even in the case where the spectral entropy is evaluated on the property of the system (e.g., frequency response functions).

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