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(Article begins on next page)

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Generalized Gini's mean difference through distortions and copulas, and related minimizing problems

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Abstract

Given a random variable X and considered a family of its possible distortions, we define two new measures of distance between X and each its distortion. For these distance measures, which are extensions of the Gini's mean difference, conditions are determined for the existence of a minimum, or a maximum, within specific families of distortions, generalizing some results presented in the recent literature.

Mathematics Subject Classification: 60E05, 60E15, 62C05

Keywords: Distance metrics; Distortion function; Copula; Proportional hazard model; Proportional reversed hazard model; Gini's mean difference.

1 Introduction and background

In the recent literature, minimizing problems of distance between random variables based on suitable measures have been examined (see, for instance, Ortega-Jiménez et al. [13], Delbaen and Majumdar [6], and references therein). This paper fits into this line of research, presenting new general measures of distance from a random variable X and its distortions, which can be possibly dependent on X, and describing conditions for such measures to present a minimum, or a maximum, within specific classes of distortions. For the definition of such distance measures (given in Eqs. (5) and (6) below), we provide in advance some assumptions on the variables considered, and some useful definitions.

Throughout the paper, we will assume that all random variables are absolutely continuous and that all distribution functions and copulas are continuously differentiable. In addition, the terms increasing and decreasing will be used in strict sense. Let X be a random variable with cumulative distribution function (c.d.f.) $F(x) = \mathsf{P}(X \leq x)$ and survival function (s.f.) $\overline{F}(x) = 1 - F(x)$, for $x \in \mathbb{R}$. Let

$$l = \inf\{x \in \mathbb{R} : F(x) > 0\}, \qquad r = \sup\{x \in \mathbb{R} : \overline{F}(x) > 0\}$$

denote respectively the lower and upper limits of the support of X, which may be finite or infinite. Moreover, given a c.d.f. $F(x)$, we will denote with $F^{-1}(u) = \sup\{x \colon F(x) \leq u\}$, $u \in [0, 1]$, the right-continuous version of its inverse, also named quantile function in statistical framework. We recall that, an

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increasing function $h : [0, 1] \to [0, 1]$, such that $h(0) = 0$ and $h(1) = 1$ is called distortion function. The quantities $h(F(x))$ and $h(\overline{F}(x))$ are called distorted c.d.f. and distorted s.f., respectively. For more details about distortion functions see for instance Section 2.4 in Navarro [11]. Let (X, Y) be a random vector with joint c.d.f. $F_{(X,Y)}(x, y) = \mathsf{P}(X \le x, Y \le y)$, joint s.f. $\overline{F}_{(X,Y)}(x, y) = \mathsf{P}(X > x, Y > y)$, marginal c.d.f.'s $F_X(x), F_Y(y)$ and marginal s.f.'s $\overline{F}_X(x)$, $\overline{F}_Y(y)$, for $x, y \in \mathbb{R}$. Thanks to Sklar theorem, one has

$$F_{(X,Y)}(x, y) = C(F_X(x), F_Y(y)), \qquad \overline{F}_{(X,Y)}(x, y) = \widehat{C}(\overline{F}_X(x), \overline{F}_Y(y)), \qquad x, y \in \mathbb{R},$$

where C and $\widehat{C}$ are copula distribution function and survival copula of (X, Y), respectively. For more details about copula functions see Nelsen [12]. We will consider only distorted s.f.'s and survival copulas. In addition, we recall that, if X and Y are nonnegative random variables with c.d.f.'s F and G, respectively, then X is said to be smaller than Y in the dispersive order, denoted as $X \le_d Y$, if and only if (see Section 3.B of Shaked and Shanthikumar [15])

$$F^{-1}(v) - F^{-1}(u) \ge G^{-1}(v) - G^{-1}(u) \qquad \text{whenever } 0 < u \le v < 1.$$

Moreover, if X and Y are absolutely continuous with density functions (d.f.'s) f and g, respectively, then

$$X \le_d Y \quad \text{if and only if} \quad f(F^{-1}(u)) \ge g(G^{-1}(u)) \quad \forall\, u \in (0, 1).$$

We recall that, the Gini's mean difference of a random variable X is defined as

$$\mathsf{GMD}(X) = \mathsf{E}|X - X'| = 2 \int_l^r F(x)\overline{F}(x)\mathrm{d}x, \tag{1}$$

where X' is an independent copy of X. It represents a special case of some information measures, such as the generalized cumulative entropies and extropies considered in Kattumannil et al. [9]. In addition we mention that the following quantities

$$\mathsf{RMD}(X) = \frac{\mathsf{GMD}(X)}{\mathsf{E}(X)}, \qquad \mathsf{G}(X) = \frac{\mathsf{RMD}(X)}{2}, \tag{2}$$

denote respectively the relative Gini's mean difference and the Gini's index of X, provided that the mean is finite and non-zero.

Let us now consider a family of distortion functions

$$\mathcal{H}_\alpha = \{h_\alpha : [0, 1] \to [0, 1],\ \alpha \in \mathcal{I} \subset \mathbb{R}\}. \tag{3}$$

We will denote with X_α the random variable with s.f. $\overline{F}_{X_\alpha}(x) = h_\alpha\left(\overline{F}(x)\right)$, for $x \in \mathbb{R}$, and with

$$\mathcal{F}_{X,h_\alpha} = \{X_\alpha : \overline{F}_{X_\alpha}(x) = h_\alpha\left(\overline{F}(x)\right),\ \alpha \in \mathcal{I},\ x \in \mathbb{R}\} \tag{4}$$

the family of all random variables with distorted s.f.'s from X through $h_\alpha \in \mathcal{H}_\alpha$.

We extend the Gini's mean difference in Eq. (1) in the cases in which the random variables are not necessarily independent or identically distributed. In particular, for X and X_α independent random variables, we define the *distorted Gini's mean difference of* X as

$$\eta_X(\alpha) \coloneqq \mathsf{E}|X - X_\alpha| = \int_l^r \left\{\overline{F}(x) + h_\alpha(\overline{F}(x))\left[1 - 2\overline{F}(x)\right]\right\} \mathrm{d}x. \tag{5}$$

For any $\alpha \in \mathcal{I}$, given the distortion function $h_\alpha \in \mathcal{H}_\alpha$, let (X, X_α) be a random vector with survival copula in the parametric family

$$\hat{\mathcal{C}}^\theta = \{\widehat{C}_\theta\,(u, v)\,;\, \theta \in \Theta,\ u, v \in [0, 1]\}.$$

By taking into account that $|x_1 - x_2| = x_1 + x_2 - 2\min\{x_1, x_2\}$, we define the *copula-distorted Gini's mean difference of* X as

$$\nu_X(\theta, \alpha) \coloneqq \mathsf{E}_{\widehat{C}_\theta}|X - X_\alpha| = \int_l^r \left\{\overline{F}(x) + h_\alpha(\overline{F}(x)) - 2\,\widehat{C}_\theta\left(\overline{F}(x), h_\alpha\left(\overline{F}(x)\right)\right)\right\} \mathrm{d}x. \tag{6}$$

We aim to study sufficient conditions on the distortion functions h_α in order to prove the existence of the minimum (or the maximum) for $\eta_X(\alpha)$. Along the same line, we obtain sufficient conditions on the

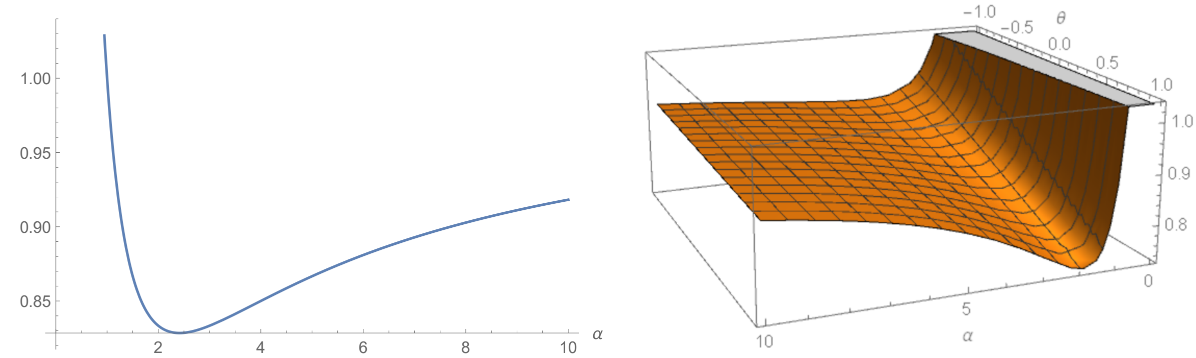

Figure 1: Plots of $\eta_X(\alpha)$ given in Eq. (7) (left) for $\alpha \in [0, 10]$, and $\nu_X(\theta, \alpha)$ given in Eq. (8) (right), for $\theta \in [-1, 1], \alpha \in [0, 10]$.

distortion functions h_α and the survival copulas $\widehat{C}_\theta$ in order to prove the existence of the minimum (or the maximum) for $\nu_X(\theta, \alpha)$.

At least when X and X_α have the same support, for a fixed survival copula $\widehat{C}_\theta$, a false intuition may suggest that the minimum is reached when $X =_{st} X_\alpha$, i.e. $h_\alpha(u) = u$, for all $u \in [0, 1]$. However, this is not necessarily true, as illustrated in the following example.

Example 1.1 *Suppose that the s.f. of X_α is obtained from X through the distortion $h_\alpha(u) = u^\alpha$, for $u \in [0, 1]$. If X has exponential distribution with $\mathsf{E}(X) = 1$, then*

$$\eta_X(\alpha) = \frac{1}{\alpha} + \frac{\alpha - 1}{\alpha + 1}, \qquad \alpha > 0. \tag{7}$$

This function reaches minimum in $\alpha = 1 + \sqrt{2}$, as shown in the left-hand-side of Figure 1. Moreover, consider the Farlie-Gumbel-Morgenstern family of survival copulas $\widehat{C}_\theta(u, v) = uv\,(1 + \theta(1 - u)(1 - v))$, for $u, v \in [0, 1]$ and $\theta \in [-1, 1]$. In these hypothesis, it follows that

$$\nu_X(\theta, \alpha) = \frac{1}{\alpha} + \frac{\alpha - 1}{\alpha + 1} - \frac{3\alpha\theta}{2 + 7\alpha + 7\alpha^2 + 2\alpha^3}, \qquad \theta \in [-1, 1],\, \alpha > 0, \tag{8}$$

which does not reach minimum in $\alpha = 1$, for all $\theta \in [-1, 1]$, as shown in right-hand-side of Figure 1.

Other examples will be illustrated in the rest of the paper.

We note that, if there exists a $\theta_I \in \Theta$ such that

$$\lim_{\theta \to \theta_I} \widehat{C}_\theta(u, v) = uv, \qquad \forall u, v \in [0, 1],$$

then

$$\nu_X(\theta_I, \alpha) = \eta_X(\alpha), \qquad \forall \alpha \in \mathcal{I}. \tag{9}$$

Remark 1.1 *The function defined in Eq. (6) can also be viewed as a variability measure in the sense of Bickel and Lehmann [4]. Indeed, it satisfies the following properties, for all $\theta \in \Theta$ and $\alpha \in \mathcal{I}$:*

1. *if X and Y have the same distribution, then $\nu_X(\theta, \alpha) = \nu_Y(\theta, \alpha)$ (law invariance);*
2. *$\nu_{X+\delta}(\theta, \alpha) = \nu_X(\theta, \alpha)$ for all X and for all constant $\delta \in \mathbb{R}$ (translation invariance);*
3. *$\nu_{\delta X}(\theta, \alpha) = \delta \nu_X(\theta, \alpha)$ for all X and for all $\delta \in \mathbb{R}^+$ (positive homogeneity);*
4. *$\nu_X(\theta, \alpha) \geq 0$ for any random variable X, with $\nu_X(\theta, \alpha) = 0$ for any degenerate random variable X (non-negativity);*
5. *$X \leq_d Y$ implies $\nu_X(\theta, \alpha) \leq \nu_Y(\theta, \alpha)$ (consistency with dispersive order).*

We remark that the copula-distorted Gini's mean difference in Eq. (6) represents an extension of the measure studied in Ortega-Jiménez et al. [14]. Through the paper we implicitly suppose that $\eta_X(\alpha)$ is continuous in $\alpha \in \mathcal{I}$ and $\nu_X(\theta, \alpha)$ is continuous in both $\theta \in \Theta$ and $\alpha \in \mathcal{I}$.

It is easy to see that, under suitable family of distortion functions, the right-hand-sides of Eqs. (5) and (6) can be expressed in terms of c.d.f.'s instead of s.f.'s, and copula distributions instead of survival copulas.

For different extended versions of Gini's mean difference see for instance Bassan et al. [1], Bernard and Müller [3], Capaldo et al. [5] and Vila et al. [17].

1.1 Plan of the paper

In Section 2 we define some useful notions for our aims and provide an example of application arising in reliability theory. In Section 3 we obtain sufficient conditions in order to prove the existence of the minimum or the maximum for the *distorted Gini's mean difference* $\eta_X(\alpha)$. In Section 4 we face the same problem for the *copula-distorted Gini's mean difference* $\nu_X(\theta, \alpha)$. Some final remarks are then given in Section 5.

2 Basic notions and results

The measures defined in Eqs. (5) and (6) can be applied also in contexts of reliability theory, where a nonnegative random variable X describes the lifetime of a device or a system. Therefore, a random lifetime will refer to a nonnegative random variable.

Definition 2.1 *A random lifetime X with s.f. $\overline{F}(x)$ is said to have the New Better than Used (NBU) property if $\overline{F}(x+t) \leq \overline{F}(x)\overline{F}(t)$ for all $x, t \geq 0$. If the inequality is reversed then X is said to have the New Worse than Used (NWU) property.*

Definition 2.2 *A random lifetime X with s.f. $\overline{F}(t)$ and d.f. $f(t)$ is said to have the increasing failure rate (IFR) property if its hazard rate (h.r.) $\lambda(t) := \frac{f(t)}{\overline{F}(t)}$ is increasing in t, for all t such that $\overline{F}(t) > 0$. If the h.r. is decreasing in t, then X is said to have the decreasing failure rate (DFR) property.*

For more details about aging notions see for instance Section 4.1 in [11].

In this paper, for the family of distortion functions $h_\alpha \in \mathcal{H}_\alpha$ involved in $\mathcal{F}_{X,h_\alpha}$ (see Eqs. (3) and (4)), we will consider the following assumptions, where $\mathrm{cl}(\mathcal{I})$ denotes the closure of $\mathcal{I}$:

Assumption 2.1 There exists a value $\alpha_I \in \mathrm{cl}(\mathcal{I})$ such that

$$\lim_{\alpha \to \alpha_I} h_\alpha(u) = u, \qquad \forall u \in [0,1].$$

Assumption 2.2 There exists a value $\alpha_0 \in \mathrm{cl}(\mathcal{I})$ such that

$$\lim_{\alpha \to \alpha_0} h_\alpha(u) = 0, \qquad \forall u \in [0,1).$$

Remark 2.1 *It easy to see that*

(a) *under the Assumption 2.1, one has* $\lim_{\alpha \to \alpha_I} \eta_X(\alpha) = \mathsf{GMD}(X)$*;*

(b) *under the Assumption 2.2, it follows that* $\lim_{\alpha \to \alpha_0} \nu_X(\theta, \alpha) = \mathsf{E}(X) - l$*, for all* $\theta \in \Theta$*.*

In various contexts of interest the quantile functions are used as an alternative to distribution functions. For a random variable X with c.d.f. $F(x)$ and d.f. $f(x)$, the quantile-density function of X is defined as

$$q(u) := \frac{1}{f\left(F^{-1}(u)\right)} = \frac{\mathrm{d}}{\mathrm{d}u} F^{-1}(u), \qquad u \in [0,1]$$

(see, for instance, Sunoj and Sankaran [16]). We recall that $f\left(F^{-1}(u)\right)$ is called density-quantile function, for $u \in [0,1]$. Referring to s.f.'s instead of c.d.f.'s, in the following we define a dual version of the quantile-density function.

Definition 2.3 *Let X be a random variable with s.f. $\overline{F}(x)$ and d.f. $f(x)$, with support in (l, r). The dual quantile-density function (d.q.d.f.) of X is defined as*

$$\tilde{q}(u) = \frac{1}{f\left(\overline{F}^{-1}(u)\right)} = -\frac{\mathrm{d}}{\mathrm{d}u} \overline{F}^{-1}(u), \qquad u \in [0,1]. \tag{10}$$

It is easy to see that, from Eq. (10), the d.q.d.f. $\tilde{q}(u)$ uniquely identifies the following location-family associated to $\overline{F}$:

$$\overline{\mathcal{F}} = \left\{\overline{G} : \overline{G}(x) = \overline{F}(x+k),\ \forall x \in \mathbb{R},\ k \in \mathbb{R}\right\}.$$

By means of the d.q.d.f. $\tilde{q}(u)$ we can derive a more tractable expression of the Eqs. (5) and (6).

Remark 2.2 *By setting $u = \overline{F}(x)$, Eqs. (5) and (6) can be expressed, respectively, as*

$$\eta_X(\alpha) = \int_0^1 \tilde{q}(u)\left\{u + h_\alpha(u)\,(1-2u)\right\} \mathrm{d}u, \qquad \alpha \in \mathcal{I}, \tag{11}$$

and

$$\nu_X(\theta,\alpha) = \int_0^1 \tilde{q}(u)\left\{u + h_\alpha(u) - 2\,\widehat{C}_\theta\left(u, h_\alpha(u)\right)\right\} \mathrm{d}u, \qquad \theta \in \Theta,\ \alpha \in \mathcal{I}. \tag{12}$$

Eqs. (11) and (12) allow to compare efficiently random variables with different supports, by making use of the following notions: the d.q.d.f. $\tilde{q}(u)$, the distortion functions in the family $\mathcal{H}_\alpha$, and also for the Eq. (12) the associated survival copulas in the family $\hat{\mathcal{C}}^\theta$.

In the main results of the paper, we often make use of the following assumptions regarding $\tilde{q}(u)$.

Assumption 2.3 One has

$$\tilde{q}(u) - \tilde{q}(1-u) \geq 0, \qquad \forall u \in \left[0, \frac{1}{2}\right].$$

Assumption 2.4 One has

$$\tilde{q}(u) - \tilde{q}(1-u) \leq 0, \qquad \forall u \in \left[0, \frac{1}{2}\right].$$

Proposition 2.1 *Let X be a random variable having d.q.d.f. $\tilde{q}(u)$, for $u \in [0,1]$.*
(i) If X is DFR, then Assumption 2.3 holds.
(ii) If Assumption 2.4 holds, then X is IFR.

Proof. The proof is an immediate consequence of Definitions 2.2 and 2.3. □

2.1 Hazard models and applications

Let us now consider some hazard models. We recall that the reversed h.r. of a random lifetime X is defined has $\tau(x) = f(x)/F(x)$, for $x \in (l,r)$.

Definition 2.4 *Let X be a random lifetime with support in (l,r), s.f. $\overline{F}(x)$ and h.r. $\lambda(x)$.*

(i) *The proportional hazard model is expressed by a random lifetime X_α having h.r.*

$$\lambda_\alpha(x) = \alpha\lambda(x), \qquad \alpha > 0,\ x \in (l,r).$$

(ii) *The proportional reversed hazard model is expressed by a random lifetime X_α having reversed h.r.*

$$\tau_\alpha(x) = \alpha\tau(x), \qquad \alpha > 0,\ x \in (l,r),$$

and having h.r.

$$\lambda_\alpha(x) = \alpha\lambda(x)g(x), \qquad \alpha > 0,\ x \in (l,r),$$

where

$$g_\alpha(x) = \frac{\left[1-\overline{F}(x)\right]^{\alpha-1} - \left[1-\overline{F}(x)\right]^{\alpha}}{1-\left[1-\overline{F}(x)\right]^{\alpha}}, \qquad \alpha > 0,\ x \in (l,r). \tag{13}$$

(iii) *The generalized additive hazard model is defined by a random lifetime X_α having h.r.*

$$\lambda_\alpha(x) = \lambda(x) + \alpha k(x), \qquad \alpha > 0,\ x \in (l,r),$$

where $k(x)$ is a suitable function such that $\lambda_\alpha(x) \geq 0$ for all $x \in (l,r)$.

(iv) *The power hazard model is defined by a random lifetime X_α having h.r.*

$$\lambda_\alpha(x) = \alpha\lambda(x^\alpha)x^{\alpha-1}, \qquad \alpha > 0,\ x \in (l,r).$$

Table 1: The hazard rates and the distortion functions for the hazard models introduced in Definition 2.4, with $g_\alpha(x)$ defined in Eq. (13), $x \in (l,r)$, $\alpha > 0$ and $u \in [0,1]$. In the case (iii) we set $K(t) = \int_0^t k(x)\mathrm{d}x$, for $t > 0$.

hazard models	hazard rates	distortion functions
(i) proportional hazard model	$\lambda_\alpha(x) = \alpha\lambda(x)$	$h_\alpha(u) = u^\alpha$
(ii) proportional reversed hazard model	$\lambda_\alpha(x) = \alpha\lambda(x)g_\alpha(x)$	$h_\alpha(u) = 1 - (1-u)^\alpha$
(iii) generalized additive hazard model	$\lambda_\alpha(x) = \lambda(x) + \alpha k(x)$	$h_\alpha(u) = u \exp\left\{-\alpha K\left(\overline{F}^{-1}(u)\right)\right\}$
(iv) power hazard model	$\lambda_\alpha(x) = \alpha\lambda(x^\alpha)x^{\alpha-1}$	$h_\alpha(u) = \overline{F}\left(\left(\overline{F}^{-1}(u)\right)^\alpha\right)$

For the proportional hazard model (i) see, for instance, Kumar and Klefsjö [10]. For the proportional reversed hazard model (ii) see Di Crescenzo [7], Gupta and Gupta [8]. We immediately note that, if $k(x) = 1$ for all x, the model (iii) corresponds to the additive hazard model (see Bebbington et al. [2]). In Table 1 we show the distortion functions corresponding to the hazard models considered in Definition 2.4. For such distortion functions one has that Assumptions 2.1 and 2.2 are both satisfied.

In reliability theory some interesting problems involve systems consisting of n components. Let $X_1, X_2, \dots, X_n$ describe the independent and identically distributed random lifetimes of each component, with s.f. $\overline{F}$. In the case of components connected in series, the system fails as soon as a component stops working. Therefore its lifetime is $X_{(1:n)} = \min\{X_1, X_2, \dots, X_n\}$, having s.f.

$$\overline{F}_{(1:n)}(x) = \left(\overline{F}(x)\right)^n. \tag{14}$$

When the components are connected in parallel, the system continues to work until the last component fails. So its lifetime is $X_{(n:n)} = \max\{X_1, X_2, \dots, X_n\}$, having s.f.

$$\overline{F}_{(n:n)}(x) = 1 - \left(1 - \overline{F}(x)\right)^n. \tag{15}$$

We remark that Eq. (14) represents the s.f. of X_α when $h_\alpha \in \mathcal{H}_\alpha$ comes from the proportional hazard model (i) in Table 1, for $\alpha = n$. On the other hand, Eq. (15) represents the s.f. of X_α when $h_\alpha \in \mathcal{H}_\alpha$ comes from the proportional reversed hazard model (ii) in Table 1, for $\alpha = n$.

In Example 1.1, under the model (i), we have shown that the minimum of Eq. (7) is not reached in $\alpha = 1$, for X exponentially distributed with $\mathsf{E}(X) = 1$. In these assumptions, it is easy to see that $\eta_X(2) < \eta_X(1)$. Therefore, if X describes the lifetime of a single component, in order to reduce the distance, in the sense of Eq. (5), between X and the lifetime of another independent item, it is better to consider a series systems with two independent copies of X instead of another single component distributed as X.

3 Results under independence

In this section we provide sufficient conditions for the existence of the minimum (or maximum) for the *distorted Gini's mean difference* defined in Eq. (5). A sketch about the main results of Section 3 is provided in Table 2.

Table 2: A sketch about the main results of Section 3, with indications of the involved quantities.

Theorems	Relevant topics	Results
Theorem 3.1	$\tilde{q}(u), h_\alpha(u)$	$\eta_X(\alpha) \le (\ge)\, \mathsf{GMD}(X)$
Theorem 3.2	$\tilde{q}(u), h_\alpha(u)$	$\eta_X(\alpha) \le (\ge)\, \mathsf{E}(X) - l$
Theorem 3.3	$\tilde{q}(u), \frac{\partial}{\partial\alpha}h_\alpha(u)$	$\lim\limits_{\alpha\to\alpha_I} \frac{\mathrm{d}}{\mathrm{d}\alpha}\eta_X(\alpha) \le (\ge)\, 0$
Theorem 3.4	$h_\alpha(u) = u\exp\left\{-\alpha K\left(\overline{F}^{-1}(u)\right)\right\}$ with $K(\cdot)$ increasing (decreasing), NBU (NWU) property, $\mathsf{G}(X)$	a local minimum (maximum) for $\eta_X(\alpha)$

In this section we assume that X is a random variable with s.f. $\overline{F}(x)$, d.f. $f(x)$, for $x \in \mathbb{R}$, and d.q.d.f. $\tilde{q}(u)$, for $u \in [0,1]$, having support in (l,r) and having finite expected value $\mathsf{E}(X)$. Moreover, we assume that $X_\alpha \in \mathcal{F}_{X,h_\alpha}$.

Theorem 3.1 *Let (X, X_α) be a random vector with independent components. For all $\alpha \in \mathcal{I}$ such that*

$$\tilde{q}(1-u)\,[1-u-h_\alpha(1-u)] \le (\ge)\, \tilde{q}(u)\,[u-h_\alpha(u)]\,, \qquad \forall u \in \left[0, \frac{1}{2}\right], \tag{16}$$

then

$$\eta_X(\alpha) \le (\ge)\, \mathsf{GMD}(X). \tag{17}$$

Proof. Making use of Eq. (11), with few calculations one has

$$\mathsf{GMD}(X) - \eta_X(\alpha) = \int_0^{\frac{1}{2}} (1-2u)\,\{\tilde{q}(u)\,[u-h_\alpha(u)] - \tilde{q}(1-u)\,[1-u-h_\alpha(1-u)]\}\,\mathrm{d}u.$$

Hence, Eq. (17) follows from Eq. (16).

□

The next result follows from Remark 2.1(a), recalling that α_I is defined as in Assumption 2.1.

Corollary 3.1 *Under the Assumption 2.1 and the hypothesis of Theorem 3.1, for all $\alpha \in \mathcal{I}$ such that Eq. (16) holds, one has*

$$\eta_X(\alpha) \le (\ge)\, \eta_X(\alpha_I). \tag{18}$$

Theorem 3.2 *Let (X, X_α) be a random vector with independent components. For all $\alpha \in \mathcal{I}$ such that*

$$\tilde{q}(u)h_\alpha(u) \le (\ge)\, \tilde{q}(1-u)h_\alpha(1-u), \qquad \forall u \in \left[0, \frac{1}{2}\right], \tag{19}$$

then

$$\eta_X(\alpha) \le (\ge)\, \mathsf{E}(X) - l. \tag{20}$$

Proof. Making use of Eq. (11), with few calculations one has

$$\mathsf{E}(X) - l - \eta_X(\alpha) = \int_0^{\frac{1}{2}} (1-2u)\,\{\tilde{q}(1-u)h_\alpha(1-u) - \tilde{q}(u)h_\alpha(u)\}\,\mathrm{d}u.$$

Therefore, the thesis follows from hypothesis (19).

□

The next result follows from Remark 2.1(b), recalling that α_0 is defined as in Assumption 2.2.

Corollary 3.2 *Under the Assumption 2.2 and the hypothesis of Theorem 3.2, for all $\alpha \in \mathcal{I}$ such that Eq. (19) holds, one has*

$$\eta_X(\alpha) \le (\ge)\, \eta_X(\alpha_0). \tag{21}$$

When both Eqs. (18) and (21) hold for $\alpha \in (\min\{\alpha_I, \alpha_0\}, \max\{\alpha_I, \alpha_0\}) \subseteq \mathrm{cl}(\mathcal{I})$, then there exists at least an $\alpha' \in (\min\{\alpha_I, \alpha_0\}, \max\{\alpha_I, \alpha_0\})$ such that $\eta_X(\alpha)$ attains a local minimum (maximum) in $\alpha = \alpha'$.

Remark 3.1 *Let X be uniformly distributed over $[l, r]$, with $l, r \in \mathbb{R}$. In this case Eq. (19) is valid for all $h_\alpha \in \mathcal{H}_\alpha$, so that from Theorem 3.2 one has*

$$\eta_X(\alpha) \le \frac{r-l}{2}, \qquad \forall \alpha \in \mathcal{I}.$$

In addition, we remark that under the models **(i)**, **(ii)** *and* **(iv)** *of Table 1, one has*

$$\eta_X(\alpha) = \frac{1}{2} + \frac{1}{1+\alpha} - \frac{2}{2+\alpha}, \qquad \alpha > 0.$$

Corollary 3.3 *Let (X, X_α) be a random vector with independent components. If Assumption 2.4 holds, then*

$$\mathsf{GMD}(X) \le \mathsf{E}(X) - l.$$

Proof. The thesis follows from Eq. (20) of Theorem 3.2, by noting that Assumption 2.4 represents a sufficient condition to Eq. (19) for $\alpha = \alpha_I$.

□

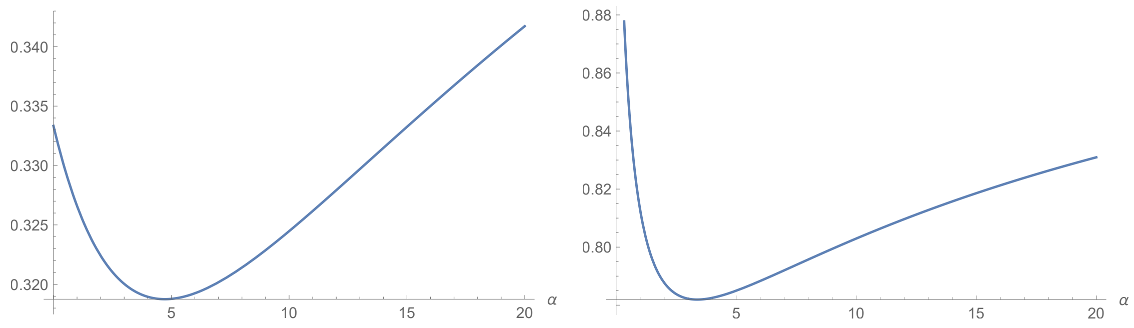

Figure 2: Plots of $\eta_X(\alpha)$ given in Eq. (25) (left) and given in Eq. (26) (right), for $\alpha \in [0, 20]$.

In the next theorem we aim to obtain sufficient conditions in order to specify the sign of $\lim_{\alpha \to \alpha_I} \frac{\mathrm{d}}{\mathrm{d}\alpha} \eta_X(\alpha)$, where due to Eq. (11) and by differentiation under the integral sign, one has

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} \eta_X(\alpha) = \int_0^1 \tilde{q}(u) \frac{\partial}{\partial \alpha} h_\alpha(u)\,(1-2u)\,\mathrm{d}u, \qquad \alpha \in \mathcal{I}. \tag{22}$$

Theorem 3.3 *Let (X, X_α) be a random vector with independent components. If*

$$\lim_{\alpha \to \alpha_I} \tilde{q}(u) \frac{\partial}{\partial \alpha} h_\alpha(u) \leq (\geq) \lim_{\alpha \to \alpha_I} \tilde{q}(1-u) \frac{\partial}{\partial \alpha} h_\alpha(1-u), \qquad \forall u \in \left[0, \frac{1}{2}\right], \tag{23}$$

then

$$\lim_{\alpha \to \alpha_I} \frac{\mathrm{d}}{\mathrm{d}\alpha} \eta_X(\alpha) \leq (\geq)\, 0. \tag{24}$$

Proof. Making use of Eq. (11), from Eq. (22), with some calculations one has

$$\lim_{\alpha \to \alpha_I} \frac{\mathrm{d}}{\mathrm{d}\alpha} \eta_X(\alpha) = \int_0^{\frac{1}{2}} (1-2u) \left\{ \lim_{\alpha \to \alpha_I} \left[\tilde{q}(u) \frac{\partial}{\partial \alpha} h_\alpha(u) - \tilde{q}(1-u) \frac{\partial}{\partial \alpha} h_\alpha(1-u) \right] \right\} \mathrm{d}u.$$

Therefore, Eq. (24) follows from Eq. (23)

□

Recalling Assumptions 2.1 and 2.2, if Eq. (20) holds for $\alpha = \alpha_I$ and Eq. (24) is valid, then there exists a local minimum (maximum) of $\eta_X(\alpha)$, due to the continuity of η_X.

Example 3.1 *Let X be uniformly distributed on $[0, 1]$. Let $X_\alpha \in \mathcal{F}_{X, h_\alpha}$, with distorted s.f. obtained through the generalized additive hazard model* (iii) *in Table 1, for $K(t) = \frac{t^2}{2}$. Since $\overline{F}^{-1}(u) = 1 - u$, with $u \in [0, 1]$, one has*

$$h_\alpha(u) = u \exp\left\{ -\alpha \frac{(1-u)^2}{2} \right\}, \qquad u \in [0, 1].$$

Therefore, Eq. (11) becomes

$$\eta_X(\alpha) = \frac{6 - 2e^{-\alpha/2} + \alpha}{2\alpha} - \frac{\sqrt{\frac{\pi}{2}}(2+\alpha)\mathsf{Erf}(\frac{\sqrt{\alpha}}{\sqrt{2}})}{\alpha^{\frac{3}{2}}}, \qquad \alpha > 0, \tag{25}$$

where $\mathsf{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp\{-t^2\}\, dt$, for $x > 0$, is the well known error function. Since h_α satisfies Eqs. (16) and (19), there exists the minimum for the $\eta_X(\alpha)$ given in Eq. (25). This is confirmed by the plot in the left-hand-side of Figure 2.

Example 3.2 *Let X be an exponential random variable having $\mathsf{E}(X) = 1$. Let $X_\alpha \in \mathcal{F}_{X, h_\alpha}$, with distorted s.f. obtained through the generalized additive hazard model* (iii) *in Table 1, for $K(t) = \frac{t^2}{2}$. Since $\overline{F}^{-1}(u) = -\log(u)$, with $u \in [0, 1]$, one has*

$$h_\alpha(u) = u \exp\left\{ -\alpha \frac{(-\log(u))^2}{2} \right\}, \qquad u \in [0, 1].$$

Therefore, Eq. (11) becomes

$$\eta_X(\alpha) = 1 + \frac{e^{\frac{1}{2\alpha}}\sqrt{\frac{\pi}{2}}\mathsf{Erfc}(\frac{1}{\sqrt{2\alpha}})}{\sqrt{\alpha}} - \frac{e^{\frac{2}{\alpha}}\sqrt{2\pi}\mathsf{Erfc}(\frac{\sqrt{2}}{\sqrt{\alpha}})}{\sqrt{\alpha}}, \qquad \alpha \geq 0. \tag{26}$$

where $\mathsf{Erfc}(x) = 1-\mathsf{Erf}(x)$ *is the complementary error function. Since* h_α *satisfies Eqs. (16) and (19), there exists the minimum for the* $\eta_X(\alpha)$ *given in Eq. (26). This is confirmed by the plot in the right-hand-side of Figure 2.*

Let us now consider the family of distortion functions $\mathcal{H}_\alpha$ under the generalized additive hazard model (iii) in Table 1. In this hypothesis, Eq. (5) becomes

$$\eta_X(\alpha) = \int_l^r \overline{F}(x)\left\{1 + e^{\{-\alpha K(x)\}}\left[1 - 2\overline{F}(x)\right]\right\}\mathrm{d}x, \tag{27}$$

and also

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\eta_X(\alpha) = \int_l^r K(x)e^{\{-\alpha K(x)\}}\overline{F}(x)\left[2\overline{F}(x) - 1\right]\mathrm{d}x. \tag{28}$$

Theorem 3.4 *Let* (X, X_α) *be a random vector with independent components having support* $(0, +\infty)$*. Let* $\eta_X(\alpha)$ *be as in Eq. (27) and let* $\mathsf{G}(X)$ *be the Gini's index of* X*. Suppose that* $K(x)$ *is increasing (decreasing) in* $x > 0$*. If* X *is NWU (NBU) and* $\mathsf{G}(X) \leq (\geq)\,\frac{1}{2}$*, then there exists the minimum (maximum) for* $\eta_X(\alpha)$*.*

Proof. If X is NWU (NBU), then

$$\int_0^{+\infty}\overline{F}(x)\left[2\overline{F}(x) - 1\right]\mathrm{d}x \leq (\geq)\, 2\int_0^{+\infty}\overline{F}(2x)\mathrm{d}x - \mathsf{E}(X) = 0.$$

Hence, from Eq. (28), if $K(x)$ is increasing (decreasing) in $x > 0$, then

$$\lim_{\alpha\to\alpha_I}\frac{\mathrm{d}}{\mathrm{d}\alpha}\eta_X(\alpha) = \int_0^{+\infty} K(x)\overline{F}(x)\left[2\overline{F}(x) - 1\right]\mathrm{d}x \leq (\geq)\, 0.$$

The thesis follows recalling that, from Eq. (2), one has $\mathsf{G}(X) \leq (\geq)\,\frac{1}{2}$ if and only if $\mathsf{GMD}(X) \leq (\geq)\,\mathsf{E}(X)$. $\square$

4 Results under dependence

In Section 3 we considered sufficient conditions for the existence of the minimum (or the maximum) for the distorted Gini's mean difference. In this section we face the same problem under the different hypothesis that X and X_α are dependent according to the survival copula $\widehat{C}_\theta \in \hat{\mathcal{C}}^\theta$, i.e. regarding the copula-distorted Gini's mean difference. A sketch about the main results of Section 4 is provided in Table 3.

Table 3: A sketch about the main results of Section 4, with indications of the involved quantities. In particular, $\partial_2\widehat{C}_\theta(u,u)$ is defined in Eq. (43).

Theorems	Relevant topics	Results
Theorem 4.1	$\widehat{C}_\theta\left(u, h_\alpha(u)\right), h_\alpha(u)$, Assumptions 2.1, 2.3, 2.4	$\nu_X(\theta,\alpha) \leq (\geq)\,\nu_X(\theta,\alpha_I)$
Theorem 4.2	$\widehat{C}_\theta\left(u, h_\alpha(u)\right), h_\alpha(u)$, Assumptions 2.3 and 2.4	$\nu_X(\theta,\alpha) \leq (\geq)\,\mathsf{E}(X) - l$
Theorem 4.3	$\partial_2\widehat{C}_\theta(u,u), \tilde{q}(u), \frac{\partial}{\partial\alpha}h_\alpha(u)$	$\lim_{\alpha\to\alpha_I}\frac{\partial}{\partial\alpha}\nu_X(\theta,\alpha) \geq (\leq)\, 0$
Theorem 4.4	$h_\alpha(u), \tilde{q}(u)$	$\nu_X(\theta,\alpha_I) \leq\, \mathsf{E}(X) - l$

In this section we assume that X is a random variable with s.f. $\overline{F}(x)$, d.f. $f(x)$, for $x \in \mathbb{R}$, and d.q.d.f. $\tilde{q}(u)$, for $u \in [0,1]$, having support in (l, r) and having finite expected value $\mathsf{E}(X)$. Moreover, we assume that $X_\alpha \in \mathcal{F}_{X,h_\alpha}$.

Theorem 4.1 *Let (X, X_α) be the random vector with survival copula $\widehat{C}_\theta \in \hat{\mathcal{C}}^\theta$. Under the Assumption 2.1, suppose that for all $\alpha \neq \alpha_I$ and for all $\theta \in \Theta$, one has for all $u \in \left[0, \frac{1}{2}\right]$*

$$\widehat{C}_\theta(u, h_\alpha(u)) + \widehat{C}_\theta(1-u, h_\alpha(1-u)) - \left[\widehat{C}_\theta(u,u) + \widehat{C}_\theta(1-u, 1-u)\right] - \frac{1}{2}[h_\alpha(u) + h_\alpha(1-u) - 1] = \delta_\theta(u) \quad (29)$$

and

$$\int_0^{\frac{1}{2}} \tilde{q}(1-u)\delta_\theta(u)\,du \geq (\leq)\, 0. \quad (30)$$

Let

$$\widehat{C}_\theta(u, h_\alpha(u)) - \widehat{C}_\theta(u,u) \geq (\leq)\, \frac{1}{2}\left[h_\alpha(u) - u\right], \qquad \forall u \in \left[0, \frac{1}{2}\right], \forall \theta \in \Theta. \quad (31)$$

If Assumption 2.3 (Assumption 2.4) holds, then

$$\nu_X(\theta, \alpha) \leq \nu_X(\theta, \alpha_I), \qquad \forall \theta \in \Theta, \forall \alpha \in \mathcal{I}. \quad (32)$$

If Assumption 2.4 (Assumption 2.3) holds, then

$$\nu_X(\theta, \alpha) \geq \nu_X(\theta, \alpha_I), \qquad \forall \theta \in \Theta, \forall \alpha \in \mathcal{I}. \quad (33)$$

Proof. From Assumption 2.1 and Eqs. (12) and (29), with few calculations one has

$$\nu_X(\theta, \alpha_I) - \nu_X(\theta, \alpha) = \int_0^{\frac{1}{2}} s(u)\left[\tilde{q}(u) - \tilde{q}(1-u)\right] \mathrm{d}u + \int_0^{\frac{1}{2}} \tilde{q}(1-u)\delta_\theta(u)\mathrm{d}u, \qquad \forall \theta \in \Theta,$$

where we have set $s(u) \coloneqq u - h_\alpha(u) + 2\left[\widehat{C}_\theta(u, h_\alpha(u)) - \widehat{C}_\theta(u,u)\right]$, for $u \in \left[0, \frac{1}{2}\right]$. Making use of Eqs. (30) and (31), Eq. (32) follows from Assumption 2.3 (Assumption 2.4). In the same way Eq. (33) can be obtained.

□

Theorem 4.2 *Let (X, X_α) be the random vector with survival copula $\widehat{C}_\theta \in \hat{\mathcal{C}}^\theta$. Suppose that for all $\theta \in \Theta$ and for all $\alpha \in \mathcal{I}$ one has*

$$\widehat{C}_\theta(u, h_\alpha(u)) + \widehat{C}_\theta(1-u, h_\alpha(1-u)) - \frac{1}{2}[h_\alpha(u) + h_\alpha(1-u)] = \delta_\theta(u), \qquad \forall u \in \left[0, \frac{1}{2}\right], \quad (34)$$

and

$$\int_0^{\frac{1}{2}} \tilde{q}(1-u)\delta_\theta(u)\,du \geq (\leq)\, 0. \quad (35)$$

Let

$$\widehat{C}_\theta(u, h_\alpha(u)) \geq (\leq)\, \frac{1}{2} h_\alpha(u), \qquad \forall u \in \left[0, \frac{1}{2}\right]. \quad (36)$$

If Assumption 2.3 (Assumption 2.4) holds, then

$$\nu_X(\theta, \alpha) \leq \mathsf{E}(X) - l, \qquad \forall \theta \in \Theta, \forall \alpha \in \mathcal{I}. \quad (37)$$

If Assumption 2.4 (Assumption 2.3) holds, then

$$\nu_X(\theta, \alpha) \geq \mathsf{E}(X) - l, \qquad \forall \theta \in \Theta, \forall \alpha \in \mathcal{I}. \quad (38)$$

Proof. From Eqs. (12) and (34), after few calculations we obtain

$$\mathsf{E}(X) - l - \nu_X(\theta, \alpha) = \int_0^{\frac{1}{2}} b(u)\left[\tilde{q}(u) - \tilde{q}(1-u)\right] \mathrm{d}u + \int_0^{\frac{1}{2}} \tilde{q}(1-u)\delta_\theta(u)\mathrm{d}u, \qquad \forall \theta \in \Theta,$$

where we have set $b(u) \coloneqq 2\widehat{C}_\theta(u, h_\alpha(u)) - h_\alpha(u)$, for $u \in \left[0, \frac{1}{2}\right]$. Making use of Eqs. (35) and (36), Eq. (37) follows from Assumption 2.3 (Assumption 2.4). In a similar way Eq. (38) can be obtained.

□

The next result follows from Remark 2.1(b), recalling that α_0 is defined as in Assumption 2.2.

Corollary 4.1 *Under the Assumption 2.2, Eqs. (37) and (38) become respectively*

$$\nu_X(\theta,\alpha) \le \nu_X(\theta,\alpha_0), \qquad \forall\theta\in\Theta, \forall\alpha\in\mathcal{I}, \tag{39}$$

$$\nu_X(\theta,\alpha) \ge \nu_X(\theta,\alpha_0), \qquad \forall\theta\in\Theta, \forall\alpha\in\mathcal{I}. \tag{40}$$

In addition, under the Assumption 2.1, one has

$$\nu_X(\theta,\alpha_I) \le \nu_X(\theta,\alpha_0), \qquad \forall\theta\in\Theta, \tag{41}$$

$$\nu_X(\theta,\alpha_I) \ge \nu_X(\theta,\alpha_0), \qquad \forall\theta\in\Theta. \tag{42}$$

As in the previous section, when both Eqs. (32) and (39) hold for $\alpha\in(\min\{\alpha_I,\alpha_0\},\max\{\alpha_I,\alpha_0\})$, it means that for any fixed $\theta\in\Theta$, there exists at least an $\alpha'\in\mathcal{I}$ such that $\nu_X(\theta,\alpha)$ attains a local minimum in $\alpha=\alpha'$. Similarly, when both Eqs. (33) and (40) hold for $\alpha\in(\min\{\alpha_I,\alpha_0\},\max\{\alpha_I,\alpha_0\})$, there exists a local maximum of $\nu_X(\theta,\alpha)$ in $\alpha=\alpha'$ for any fixed $\theta\in\Theta$.

Remark 4.1 *If X is uniformly distributed over $[l,r]$, $l,r\in\mathbb{R}$, the d.q.d.f. is given by $\tilde{q}(u)=r-l>0$, $u\in[0,1]$. Therefore, one can show that the thesis of Theorems 4.1 and 4.2 and Corollary 4.1 follow with a similar procedure without using Assumptions 2.3 or 2.4.*

For $u,v\in[0,1]$, we adopt the following notation

$$\partial_2\widehat{C}_\theta(u,v) := \frac{\partial}{\partial x_2}\widehat{C}_\theta(x_1,x_2)\Big|_{(x_1,x_2)=(u,v)}, \qquad \forall\theta\in\Theta. \tag{43}$$

As in Theorem 3.3, we now obtain sufficient conditions in order to specify the sign of $\lim_{\alpha\to\alpha_I}\frac{\partial}{\partial\alpha}\nu_X(\theta,\alpha)$, where due to Eq. (12) and by differentiation under the integral sign, one has

$$\frac{\partial}{\partial\alpha}\nu_X(\theta,\alpha) = \int_0^1 \tilde{q}(u)\frac{\partial}{\partial\alpha}h_\alpha(u)\left[1-2\partial_2\widehat{C}_\theta(u,h_\alpha(u))\right]\mathrm{d}u, \qquad \theta\in\Theta,\ \alpha\in\mathcal{I}. \tag{44}$$

Theorem 4.3 *Let (X,X_α) be the random vector with survival copula $\widehat{C}_\theta\in\hat{\mathcal{C}}^\theta$. Suppose that*

$$\partial_2\widehat{C}_\theta(1-u,1-u)+\partial_2\widehat{C}_\theta(u,u)=1, \qquad \forall u\in\left[0,\frac{1}{2}\right], \forall\theta\in\Theta, \tag{45}$$

and

$$\partial_2\widehat{C}_\theta(u,u) \le (\ge)\, \frac{1}{2}, \qquad \forall u\in\left[0,\frac{1}{2}\right], \forall\theta\in\Theta. \tag{46}$$

If

$$\lim_{\alpha\to\alpha_I}\tilde{q}(u)\frac{\partial}{\partial\alpha}h_\alpha(u) \le (\ge) \lim_{\alpha\to\alpha_I}\tilde{q}(1-u)\frac{\partial}{\partial\alpha}h_\alpha(1-u), \qquad \forall u\in\left[0,\frac{1}{2}\right] \tag{47}$$

then

$$\lim_{\alpha\to\alpha_I}\frac{\partial}{\partial\alpha}\nu_X(\theta,\alpha)\le 0, \qquad \forall\theta\in\Theta. \tag{48}$$

If

$$\lim_{\alpha\to\alpha_I}\tilde{q}(u)\frac{\partial}{\partial\alpha}h_\alpha(u) \ge (\le) \lim_{\alpha\to\alpha_I}\tilde{q}(1-u)\frac{\partial}{\partial\alpha}h_\alpha(1-u), \qquad \forall u\in\left[0,\frac{1}{2}\right] \tag{49}$$

then

$$\lim_{\alpha\to\alpha_I}\frac{\partial}{\partial\alpha}\nu_X(\theta,\alpha)\ge 0, \qquad \forall\theta\in\Theta. \tag{50}$$

Proof. In the assumption (45), making use of Eq. (44), with few calculations one has

$$\lim_{\alpha\to\alpha_I}\frac{\partial}{\partial\alpha}\nu_X(\theta,\alpha) = \int_0^{\frac{1}{2}}\left[1-2\partial_2\widehat{C}_\theta(u,u)\right]\left\{\lim_{\alpha\to\alpha_I}\left[\tilde{q}(u)\frac{\partial}{\partial\alpha}h_\alpha(u)-\tilde{q}(1-u)\frac{\partial}{\partial\alpha}h_\alpha(1-u)\right]\right\}\mathrm{d}u.$$

Therefore, Eq. (48) follows from Eqs. (46) and (47). In the same way Eq. (50) follows from Eqs. (46) and (49).

□

It easy to see that the independence copula satisfies Eqs. (45) and (46). Therefore Theorem 4.3 for $\frac{\partial}{\partial\alpha}\nu_X(\theta,\alpha)$ generalizes Theorem 3.3 for $\frac{\mathrm{d}}{\mathrm{d}\alpha}\eta_X(\alpha)$.

Recalling Assumptions 2.1 and 2.2, if Eqs. (41) and (48) hold, then there exists a local minimum of $\nu_X(\theta,\alpha)$, due to the continuity of ν_X, for any fixed $\theta \in \Theta$. For the same reasons, if Eqs. (42) and (50) hold, then there exists the local maximum of $\nu_X(\theta,\alpha)$.

Lemma 4.1 *For a copula function $C(u,v)$, for $u,v \in [0,1]$, one has*

$$C(u,u) \geq \max\{2u-1,0\}, \qquad \forall u \in [0,1]. \tag{51}$$

Proof. The proof immediately follows when $u = v$ in the left inequality of the following Fréchet-Hoeffding bounds

$$\max\{u+v-1,0\} \leq C(u,v) \leq \min\{u,v\},$$

that hold for every copula $C(u,v)$ and for all $u,v \in [0,1]$ (see [12]).

□

In the next theorem we illustrate other sufficient conditions leading to an inequality as given in Eq. (41).

Theorem 4.4 *Let (X,X_α) be the random vector with survival copula $\widehat{C}_\theta \in \hat{\mathcal{C}}^\theta$. Suppose that $h_\alpha(u)$ satisfies Assumptions 2.1 and 2.2 for all $u \in [0,1]$. If Assumption 2.4 holds and if*

$$\int_{\frac{1}{2}}^{1} \tilde{q}(u)(4u-3)\mathrm{d}u \geq 0, \tag{52}$$

then

$$\nu_X(\theta,\alpha_I) \leq \mathsf{E}(X) - l, \qquad \forall\theta \in \Theta.$$

Proof. From Assumptions 2.1 and 2.2, making use of Eq. (12), one has

$$\begin{aligned}
\mathsf{E}(X) - l - \nu_X(\theta,\alpha_I) &= \int_0^1 \tilde{q}(u)\left[2\widehat{C}_\theta(u,u) - u\right]\mathrm{d}u \\
&\geq \int_{\frac{1}{2}}^{1} \tilde{q}(u)(3u-2)\mathrm{d}u - \int_{\frac{1}{2}}^{1} \tilde{q}(1-u)(1-u)\mathrm{d}u \\
&\geq \int_{\frac{1}{2}}^{1} \tilde{q}(u)(4u-3)\mathrm{d}u,
\end{aligned}$$

where the inequalities come from Eq. (51) and Assumption 2.4, respectively. Finally, the thesis follows from Eq. (52).

□

Example 4.1 *Let X be a random variable with s.f. $\overline{F}(x) = 1 - x^2$, for $x \in [0,1]$ and d.q.d.f. $\tilde{q}(u) = 1/2(1-u)^{\frac{1}{2}}$, for $u \in [0,1]$. Let $X_\alpha \in \mathcal{F}_{X,h_\alpha}$, with $h_\alpha \in \mathcal{H}_\alpha$ satisfying the power hazard model* (iv) *of Table 1. Since $\overline{F}^{-1}(u) = (1-u)^{\frac{1}{2}}$, for $u \in [0,1]$, one has*

$$h_\alpha(u) = 1 - (1-u)^\alpha, \qquad u \in [0,1].$$

Consider the Farlie-Gumbel-Morgenstern (FGM) family of survival copulas, that is expressed by

$$\widehat{C}_\theta(u,v) = uv\left(1 + \theta(1-u)(1-v)\right), \qquad \theta \in [-1,1].$$

Therefore, Eq. (12) becomes

$$\nu_X(\theta,\alpha) = \frac{1}{3} + \frac{1}{1+2\alpha} - \frac{2}{3+2\alpha} - \frac{16\alpha(4+3\alpha)\theta}{(3+2\alpha)(5+2\alpha)(3+4\alpha)(5+4\alpha)}, \qquad \theta \in [-1,1],\ \alpha \in \mathcal{I}. \tag{53}$$

Since h_α satisfies Eq. (48) and since Eq. (52) holds, there exists the minimum for the $\nu_X(\theta,\alpha)$ given in Eq. (53). This is confirmed by the plot in Figure 3.

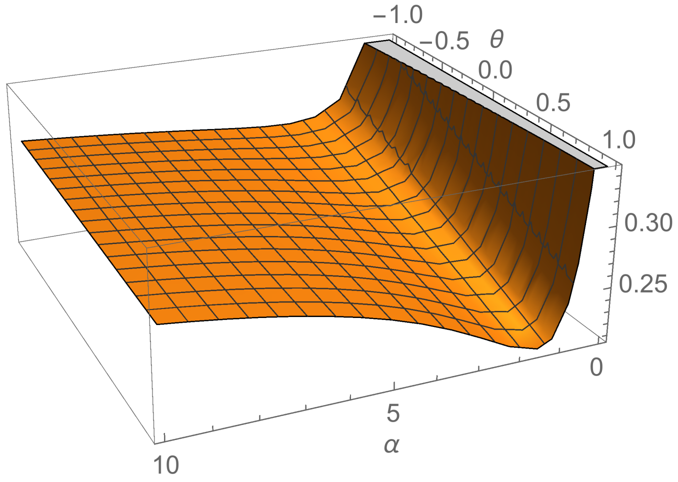

Figure 3: Plot of $\nu_X(\theta,\alpha)$ given in Eq. (53) for $\theta \in [-1,1]$ and $\alpha \in [0,10]$.

5 Conclusions and future purposes

In this paper we generalized the Gini's mean difference in both the directions of the distributions and of the dependences. We have studied sufficient conditions in order to determine the minimum, or the maximum, for the corresponding new measures of distance. We have also illustrated some examples in the context of hazard models.

Future developments can be oriented to study the measure $\nu_X(\theta,\alpha)$ for fixed distortions, in order to focus on the dependence expressed by the copula, with special attention to the case when $\alpha = \alpha_I$, i.e., when X and X_α are identically distributed.

It is worth noting that the Gini's index $\mathsf{G}(X)$, given in Eq. (2), can be suitably extended to the pair of (possibly dependent) random variables X and X_α studied so far. In this respect, under the same assumptions of Eq. (6), we define the *copula-distorted Gini's index of X* as

$$\mathcal{GI}_X(\theta,\alpha) = \frac{\nu_X(\theta,\alpha)}{\mathsf{E}(X)+\mathsf{E}(X_\alpha)}, \qquad \theta \in \Theta,\ \alpha \in \mathcal{I}. \tag{54}$$

Clearly, if X and X_α are independent, due to Eqs. (9), we refer to the "distorted Gini's index of X", which is obtained by replacing $\nu_X(\theta,\alpha)$ with $\eta_X(\alpha)$ in the right-hand-side of (54). The analysis and applications of the relative measure introduced in (54) may be the object of forthcoming investigations.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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