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On incentivizing innovation diffusion in a network of coordinating agents

Lorenzo Zino * and **Mengbin Ye** **

* *Department of Electronics and Telecommunications, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy (e-mail: lorenzo.zino@polito.it).*
** *Centre for Optimisation and Decision Science, Curtin University, Bentley 6102, WA, Australia (e-mail: mengbin.ye@curtin.edu.au).*

Abstract: Innovation diffusion is fundamental for societal growth and development, and understanding how to unlock it is key toward devising policies encouraging the adoption of new practices, e.g., sustainable innovations. Here, we propose a mathematical model to investigate such a problem. Specifically, we consider a coordination game —which is a standard game-theoretic model used to study innovation diffusion— and we embed it on an activity-driven network. Within this model, we integrate three policies to incentivize the adoption of the innovation: i) providing a direct advantage for adopting it, ii) making people sensitive to emerging trends at the population level, and iii) increasing the visibility of adopters of the innovation, respectively. We provide analytical insights to shed light on the effect of the joint use of these three policies on unlocking innovation diffusion, supported by numerical simulations.

Keywords: Dynamics on networks; Control over networks; Evolutionary game

1. INTRODUCTION

Innovation diffusion is the phenomenon in which a novel product, idea, or behavior is introduced in a population, successfully replacing the status quo (Rogers, 2003; Peyton Young, 2011; Acemoglu et al., 2011; Liu et al., 2012; Fagnani and Zino, 2017; Zhong et al., 2022). Hence, designing effective interventions to unlock such a phenomenon is key for its broad range of potential applications (Riehl et al., 2018), including assisting public authorities to incentivize sustainable practices (Otto et al., 2020).

Mathematical models have provided powerful tools to study innovation diffusion. In the past few decades, game-theoretic models based on the paradigm of network coordination games have increased in popularity, thanks to their ability to realistically capture human decision-making that occurs during innovation diffusion (Montanari and Saberi, 2010; Peyton Young, 2011). These models have subsequently been employed to study the key problem of controlling the network in order to favor innovation diffusion. Most of the literature focus on individual-level interventions, i.e., understanding where adopters of the innovation should be initially placed to incentivize diffusion (Como et al., 2021). However, interventions at such granularity cannot be implemented in many real-world scenarios, calling for the study of different policies.

Here, we fill in this gap by embedding a coordination game on a time-varying network, and incorporating three distinct control actions mirroring real-world intervention policies. First, we consider the intuitive use of incentives to increase the attractiveness of adopting the innovation over the status quo by increasing a certain entry of the payoff matrix of the coordination game. Second, we consider strategies based on making people aware of emerging trends, which is a recent intervention strategy examined in the empirical social psychology literature (Mortensen et al., 2019); this is modeled by adding an additional trend-seeking term to the dynamics, following Zino et al. (2022). Third, we consider interventions that make adopters of the innovation more visible, e.g., through the distribution of stickers (Hamann et al., 2015); this is modeled by a parameter that increases the probability that people interact with adopters of the innovation, inspired by Alessandretti et al. (2017). To the best of our knowledge, the first action has been studied in the literature (Montanari and Saberi, 2010; Peyton Young, 2011) and the second one has been recently addressed (Zino et al., 2022); the effect of the third action and its interplay with the other two is still to be investigated.

From the analysis of the stochastic model, we establish necessary and sufficient conditions to guarantee that innovation diffusion occurs with probability converging to 1 as the network size grows. In particular, we show a threshold behavior: if the fraction of initial adopters exceeds a threshold that depends on the control actions, innovation diffusion is guaranteed. Finally, we present a case study to discuss the interplay between the control actions.

Notation

We denote the set of nonnegative and strictly positive integer numbers by $\mathbb{N}$ and $\mathbb{N}_+$, respectively. A vector $\boldsymbol{x}$ is denoted with bold font, with ith entry x_i. The all-0 and all-1 (column) vectors are denoted by $\mathbf{0}$ and $\mathbf{1}$, respectively (with the appropriate dimension determined in the context). Given a stochastic event E, we denote its probability by $\mathbb{P}[E]$. Given a family of events E_n,

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parametrized by $n \in \mathbb{N}_+$, we say that E_n occurs with high probability (w.h.p.) if there exists a constant $K > 0$ such that $\mathbb{P}[E_n] \geq 1 - K/n$, for all $n \in \mathbb{N}_+$.

2. MODEL AND PROBLEM FORMULATION

2.1 Coordination game on time-varying networks

We consider a population of $n \in \mathbb{N}_+$ individuals, denoted by the set $\mathcal{V} = \{1, \dots, n\}$. At each discrete time-step $t \in \mathbb{N}$, each individual $i \in \mathcal{V}$ can choose whether to adopt the *status quo* (denoted by 0) or the *innovation* (denoted by 1). We denote by $x_i(t) \in \{0,1\}$ the *action* adopted by individual $i \in \mathcal{V}$ at time $t \in \mathbb{N}$, and each individual is able to revise this decision for the following time-step $t+1$. The actions of all the individuals in the population at time t are gathered in the n-dimensional *action state* vector $\boldsymbol{x}(t) \in \{0,1\}^n$. We further define the variable $z(t) := \frac{1}{n}\boldsymbol{x}(t)^\top \mathbf{1}$, which is the fraction of adopters of the innovation at time $t \in \mathbb{N}$.

At each time-step, each individual $i \in \mathcal{V}$ observes the actions of a fixed number of *social contacts* $k \in \mathbb{N}_+$, with $k \geq 2$, selected according to a mechanisms inspired by discrete-time activity-driven networks (Perra et al., 2012). Specifically, each social contact is selected uniformly at random from the entire population, independent of the others, i.e.,

$$a_{ij}^{\ell}(t) := \mathbb{P}[j \text{ is the } \ell\text{th contact of } i \text{ at time } t] = \frac{1}{n}. \quad (1)$$

We call this set of contacted individuals the neighbors of individual i at time t, denoted by $\mathcal{N}_i(t)$. Such a mechanism generates a directed time-varying (multi-)graph of interactions $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$, where $(i,j) \in \mathcal{E}(t) \iff j \in \mathcal{N}_i(t)$ (possibly, with multiple occurrences). Then, each individual $i \in \mathcal{V}$ makes their decision according to a 2-action *network coordination game* on $\mathcal{G}(t)$.

Specifically, $i \in \mathcal{V}$ engages in k symmetric 2-player coordination games with each of their k neighbors $j \in \mathcal{N}_i(t)$, where multiple occurrences of the same individuals are treated as distinct games. The *payoff* that individual i receives for choosing action $s \in \{0,1\}$ against $j \in \mathcal{N}_i(t)$ depends on s and on the action of j, $x_j(t) \in \{0,1\}$, and is represented thought the payoff matrix

$$\begin{array}{r c} & \begin{array}{cc} s=1 & s=0 \end{array} \\ \begin{array}{r} x_j(t)=1 \\ x_j(t)=0 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array}. \quad (2)$$

Briefly, i receives a unit reward if i and their peer coordinate on the same action, and no payoff if they fail at coordinating. Hence, the overall payoff that an agent i would receive for selecting action $s \in \{0,1\}$ when the system is in state $\boldsymbol{x}(t) = \boldsymbol{x}$ at time t can be expressed through the following payoff function:

$$f_i\big(s, \boldsymbol{x}, \mathcal{G}(t)\big) := \sum_{j \in \mathcal{N}_i(t)} \Big[(1-s)\big(1-x_j(t)\big) + s x_j(t)\Big]. \quad (3)$$

At each time step $t \in \mathbb{N}_+$, each agent revise their state according to a deterministic best-response dynamics for the payoff functions in Eq. (3). Specifically, we assume that agent $i \in \mathcal{V}$ revises their strategy to

$$x_i(t+1) = \begin{cases} 1 \text{ if } f_i\big(1, \boldsymbol{x}, \mathcal{G}(t)\big) > f_i\big(0, \boldsymbol{x}, \mathcal{G}(t)\big), \\ 0 \text{ if } f_i\big(1, \boldsymbol{x}, \mathcal{G}(t)\big) \leq f_i\big(0, \boldsymbol{x}, \mathcal{G}(t)\big), \end{cases} \quad (4)$$

i.e., each agent chooses to adopt the action that maximizes their payoff. If the two actions are equivalent, then the agent favors the status quo, consistent with the literature on status-quo bias.

2.2 Control actions to incentivize innovation diffusion

We present three control actions that can be implemented to incentivize innovation diffusion, each one captured by a scalar parameter, quantifying the effort placed in it.

Make the innovation more advantageous We introduce a parameter $u_a \geq 0$ that represents the *advantage* of the innovation with respect to the status quo. This adjusts the payoffs for the game, so that the payoff matrix for each coordination game becomes

$$\begin{array}{r c} & \begin{array}{cc} s=1 & s=0 \end{array} \\ \begin{array}{r} x_j(t)=1 \\ x_j(t)=0 \end{array} & \begin{bmatrix} 1+u_a & 0 \\ 0 & 1 \end{bmatrix} \end{array}, \quad (5)$$

yielding the following expression that replaces Eq. (3):

$$f_i\big(s, \boldsymbol{x}, \mathcal{G}(t)\big) := \sum_{j \in \mathcal{N}_i(t)} \Big[(1-s)\big(1-x_j(t)\big) + (1+u_a) s x_j(t)\Big]. \quad (6)$$

Note that $u_a = 0$ coincide with the baseline case discussed in the previous section, while $u_a > 0$ implies the innovation provides a superior payoff, capturing the implementation of policies that favor the adoption of the innovation, e.g., through tax reliefs.

Make people sensitive to emerging trends We introduce a parameter $u_t \in [0,1)$ that represents the sensitivity of individuals to emerging trends in the population, which has been identified in several recent experimental studies as an intervention method for incentivizing the adoption of innovation (Mortensen et al., 2019), ultimately ensuring its diffusion in social groups (Ye et al., 2021). Following (Zino et al., 2022), we assume that, at each time step, every individual, independently of the others, replaces with probability u_t the revision protocol in Eq. (4) with the following update rule:

$$x_i(t+1) = \begin{cases} 1 & \text{if } z(t) > z(t-1), \\ 0 & \text{if } z(t) < z(t-1), \\ x_i(t) & \text{if } z(t) = z(t-1). \end{cases} \quad (7)$$

In other words, individual $i \in \mathcal{V}$ follows the trend by updating their action to the one that has increased in adopters over the previous time step. We assume that the individual does not revise their action in the absence of any trend. If $u_t = 0$, individuals always use Eq. (4).

Increase visibility of adopters of the innovation We introduce a parameter $u_v \geq 0$ that represents the increase in visibility of the adopters of the innovation. Inspired by activity-driven networks with attractiveness (Alessandretti et al., 2017), we modify the network formation process, assuming that an individual has a higher probability of observing an adopter of the innovation. Specifically, we replace Eq. (1) with

$$a_{ij}^{\ell}(t) = \begin{cases} \frac{1+u_v}{n(1+z(t))} & \text{if } x_j(t) = 1, \\ \frac{1}{n(1+z(t))} & \text{if } x_j(t) = 0, \end{cases} \quad (8)$$

where $u_v = 0$ is baseline case discussed in the previous section. The parameter u_v thus captures the implementation of policies to increase the visibility of adopters of the innovation, such as using stickers to show others their commitment to the innovation (Hamann et al., 2015).

2.3 Problem Formulation

We initialize the system from $\boldsymbol{x}(0) = \mathbf{0}$. Then, a fraction $\zeta \in (0,1]$ of *initial adopters* of the innovation are introduced in the population. Hence, we randomly select ζn individuals, for which we set $x_i(1) = 1$, while the rest of the population has $x_j(1) = 0$.

We observe that the model proposed is parsimonious, as it is fully characterized by five parameters: i) the number of social contacts established by each individual $k \in \mathbb{N}_+$; ii) the fraction of initial adopters $\zeta \in (0,1)$; iii) the advantage of the innovation with respect to the status quo $u_a \geq 0$; iv) the sensitivity of individuals to trends $u_t \in [0,1)$; and v) the increase in visibility of adopters of the innovation with respect to the status quo $u_v \geq 0$. The first two parameters k and ζ are characteristics of the uncontrolled dynamics, while u_a, u_t, and u_v, instead, are each associated with one of the three control actions, and their magnitude quantifies the effort placed in that specific control action.

We consider large-scale populations ($n \to \infty$), and we introduce the following definition.

Definition 1. (Innovation diffusion). For any positive constant $\varepsilon > 0$, let us define two random times:

$$S_{n,\varepsilon} := \inf\{t \in \mathbb{N}_+ : z(t) \geq (1-\varepsilon)n\}\,, \tag{9a}$$

$$F_{n,\varepsilon} := \inf\{t \in \mathbb{N}_+ : z(t) \leq \varepsilon n\}\,, \tag{9b}$$

i.e., the time at which the fraction of adopters of the innovation becomes greater than $1-\varepsilon$ and less than ε, respectively. We say that innovation diffusion is guaranteed if $S_{n,\varepsilon} < F_{n,\varepsilon}$ w.h.p., for any $0 < \varepsilon < \min\{\zeta, 1-\zeta\}$.

Briefly, innovation diffusion is guaranteed if the fraction of adopters converges arbitrarily close to the all-1 consensus and does not converge arbitrarily close to the all-0 consensus. In the following, we will show that the system converges almost surely to one of these two configurations. Hence, the asymptotic behavior of the system is ultimately determined by understanding which model parameters, control actions, and initial conditions guarantee innovation diffusion in the sense of Definition 1.

In this paper, we cast the problem as determining the fraction of initial adopters ζ needed to guarantee innovation diffusion, assuming that the number of social contacts k and the three control actions u_a, u_t, u_v are given. Specifically, we will prove that the model is characterized by a threshold $\zeta_k^*(u_a, u_t, u_v)$ such that innovation diffusion is guaranteed if and only if (iff) $\zeta > \zeta_k^*(u_a, u_t, u_v)$, analytically characterizing such a threshold as a function of the control actions (i.e., u_a, u_t, and u_v). This will help build our understanding of how to optimally devise such interventions in order to reduce the fraction of initial adopters ζ needed to unlock innovation diffusion.

3. GENERAL PROPERTIES OF THE MODEL

Before analyzing the problem formulated in Section 2.3, we discuss some general properties of the game and the network. First, we characterize the Nash equilibria of the game, which given the stochastic nature of the network, should be understood with the following definition.

Definition 2. A (pure strategy) Nash equilibrium is a state vector $\boldsymbol{x} = [x_1, \dots, x_n]^\top \in \{0,1\}^n$ such that $f_i(x_i, \boldsymbol{x}, \mathcal{G}(t)) \geq f_i(1-x_i, \boldsymbol{x}, \mathcal{G}(t))$, for all $i \in \mathcal{V}$ and all possible realizations of $\mathcal{G}(t)$.

The game has two Nash equilibria, characterized in the following proposition.

Proposition 3. For any $u_a \geq 0$, the network game with payoff in Eq. (6) on the network $\mathcal{G}$ Eq. (8) has two (pure strategy) Nash equilibria: $\mathbf{0}$ and $\mathbf{1}$. That is, the pure configurations in which the entire population coordinate on action 0 or 1, respectively.

Proof. Observe that $\mathbf{1}$ is a Nash equilibrium. In fact, $f_i(1, \mathbf{1}, \mathcal{G}(t)) = k(1+u_a) > f_i(0, \mathbf{1}, \mathcal{G}(t)) = 0$, for any realization of $\mathcal{G}(t)$. A similar argument applies to $\mathbf{0}$. Finally, we show that any other state vector cannot be a Nash equilibrium. Assume that $\boldsymbol{x} \notin \{\mathbf{0}, \mathbf{1}\}$ is a Nash equilibrium. Then, there exists $i, j \in \mathcal{V}$ such that $x_i = 0$ and $x_j = 1$. For any $u_v \geq 0$, the network $\mathcal{G}(t)$ has at least $\lfloor k/2 \rfloor + 1$ instances of the link (i,j) with probability greater than $1/n^{\lfloor k/2 \rfloor + 1}$, which implies that $\boldsymbol{x}$ cannot be a Nash equilibrium, since $f_i(1, \boldsymbol{x}, \mathcal{G}(t)) \geq (\lfloor k/2 \rfloor + 1)(1+u_a) > f_i(0, \boldsymbol{x}, \mathcal{G}(t)) = (\lceil k/2 \rceil - 1)(1+u_a)$. □

The dynamics described in Section 2 induce a stochastic process $\boldsymbol{x}(t)$ on the state space $\{0,1\}^n$. In general, the process is non-Markovian, since Eq. (7) does not depend on just the state of the system at time t. Nonetheless, we can introduce an extended state space by defining an additional one-dimensional variable $y(t)$, taking values on $\{0, \frac{1}{n}, \dots, 1\}$, with the following update rule: $y(t+1) = \frac{1}{n}\boldsymbol{x}(t)^\top \mathbf{1}$. The process $\hat{\boldsymbol{x}}(t) := (\boldsymbol{x}(t), y(t))$ is a Markov chain. In fact, we can rewrite the condition $z(t) > z(t-1)$ in Eq. (7) as $\frac{1}{n}\boldsymbol{x}(t)^\top \mathbf{1} > y(t)$. This observation allows us to prove the following.

Proposition 4. The process $\boldsymbol{x}(t)$ converges almost surely in finite time to one of the two Nash equilibria $\mathbf{0}$ or $\mathbf{1}$.

Proof. Let us consider the process $\hat{\boldsymbol{x}}(t)$. First, we observe that if $\boldsymbol{x}$ is not a Nash equilibrium, then $(\boldsymbol{x}, y)$ cannot be a steady state of the process, independently of y, because of Eq. (4). Hence, steady states are in the form $(\mathbf{1}, y)$ and $(\mathbf{0}, y)$. At this stage, we observe that, from any configuration $(\mathbf{1}, y)$ with $y < 1$, one necessarily reaches $(\mathbf{1}, 1)$, while $(\mathbf{1}, 1)$ can be easily verified that is an absorbing state. A similar argument shows that $(\mathbf{0}, 0)$ is the only other absorbing state of the system. Finally, we observe that from any configuration there is a path with nonzero probability to reach at least one of the absorbing states. In fact, from $(\boldsymbol{x}(t), y(t)) = (\boldsymbol{x}, y)$ with $\boldsymbol{x} \notin \{\mathbf{0}, \mathbf{1}\}$, one can reach $(\boldsymbol{x}(t+1), y(t+1)) = (\mathbf{1}, \boldsymbol{x}^\top \mathbf{1})$ if all neighborhoods $\mathcal{N}_i(t)$, $i \in \mathcal{V}$, have a majority of agents playing 1 and all agents revise their action using Eq. (4). This occurs with nonzero probability (with an argument similar to the one used in the proof of Prop. 3). Then, from $(\mathbf{1}, \boldsymbol{x}^\top \mathbf{1})$, the absorbing state $(\mathbf{1}, 1)$ is reached in one step, as observed before. A similar argument can be applied concerning the absorbing state $(\mathbf{0}, 0)$. Hence, Markov chain theory yields the proof (Levin et al., 2006). □

We can explicitly write the transition probabilities of $\boldsymbol{x}(t)$, which are reported in the following.

Proposition 5. Let us define

$$\Pi_{k,u_a,u_v}(z) = \frac{1}{(1+u_v z)^k} \sum_{\ell=\lfloor \frac{k}{2+u_a} \rfloor +1}^{k} \binom{k}{\ell} (1+u_v)^\ell z^\ell (1-z)^{k-\ell}. \tag{10}$$

Then, for all $i \in \mathcal{V}$, the transition probabilities $p^i_{ab}(t) := \mathbb{P}[x_i(t+1) = b \,|\, x_i(t) = a]$ are equal to

$$p^i_{01}(t) = u_t \chi_{z(t)>z(t-1)} + (1-u_t)\Pi_{k,u_a,u_v}(z), \tag{11a}$$
$$p^i_{10}(t) = u_t \chi_{z(t)<z(t-1)} + (1-u_t)\big(1-\Pi_{k,u_a,u_v}(z)\big), \tag{11b}$$

where the indicator function $\chi_{z(t)>z(t-1)}$ assumes value 1 if the event "$z(t) > z(t-1)$" occurs and 0 otherwise.

Proof. First, observe that individual i with $x_i(t) = 0$ switches to action 1 iff one of the following two disjoint events occurs: i) i follows Eq. (7) (which occurs with probability u_t) and $z(t) > z(t-1)$ or ii) i follows Eq. (4) (which occurs with probability $1-u_t$) and $f_i\big(1, \boldsymbol{x}, \mathcal{G}(t)\big) > f_i\big(0, \boldsymbol{x}, \mathcal{G}(t)\big)$. For the latter event to occur, it is necessary and sufficient that at least $\lfloor k/(2+u_a) \rfloor + 1$ of the k neighbors of i at time t have state 1 (Zino et al., 2021). Each neighbor is sampled independently of the others and the probability that the generic ℓth neighbor of i has state 1 at time t is equal to

$$q^\ell_i(t) = \sum_{j:x_j(t)=1} a^\ell_{ij}(t) = \frac{(1+u_v)z(t)}{1+u_v z(t)}. \tag{12}$$

We observe that such a probability is independent of i and ℓ and depends on t only through $z(t)$, so we denote it as $q^\ell_i(t) = q(z(t))$. Hence, the number of 1 among the neighbors of i at time t follows a Binomial distribution with k trials with success probability $q(z(t))$, which yields the expression in Eq. (10). Finally, the sum of the probability of the two disjoint events yields Eq. (11a). The other transition probability in Eq. (11b) is obtained following a similar argument. □

We conclude this section with the generalization of a result from Zino et al. (2021), in which Hoeffding's inequality (Hoeffding, 1963) is used to establish a bound on $z(t+1)$ given $z(t)$. We sketch the proof, while details are omitted due to space constraints.

Lemma 6. The following hold true:

(i) if $z(t) > z(t-1)$ and $(1-u_t)\Pi_{k,u_a,u_v}(z) + u_t > z$, then there exists a positive constant $\delta > 0$ such that $\mathbb{P}[z(t+1) \leq z(t) | \mathcal{F}_t] \leq \exp\{-\delta n\}$, where $\mathcal{F}_t$ is the natural filtration of the process $\boldsymbol{x}(t)$ at time t;

(ii) if $z(t) < z(t-1)$ and $(1-u_t)\Pi_{k,u_a,u_v}(z) < z$, then there exists a positive constant $\delta > 0$ such that $\mathbb{P}[z(t+1) \geq z(t) | \mathcal{F}_t] \leq \exp\{-\delta n\}$.

Proof. First, we observe that the number of adopters of 1 at time $t+1$ is equal to the sum of the number of individuals who follow Eq. (7), which we denote as $N_t(t)$, and the number of the remaining $n - N_t(t)$ individuals who choose 1 according to Eq. (11). Hence, we write $z(t+1) = \frac{1}{n}N_t(t) + \frac{1}{n}\sum_{\ell=1}^{n-N_t(t)} b_\ell(t)$, where $N_t(t)$ is a binomial r.v. with n trials and success probability u_t; and $b_1(t), \dots, b_{N_t(t)}(t)$ is a sequence of $n - N_t(t)$ i.i.d. Bernoulli r.v.s, each one with the mean equal to $\Pi_{k,u_a,u_v}(z(t))$. We condition on $N_t(t)$ and use the law of total probability to compute $\mathbb{P}[z(t+1) \leq z(t)] = \sum_{m=0}^{n} \mathbb{P}[N_t(t) = m]\mathbb{P}[\frac{1}{n}m + \frac{1}{n}\sum_{\ell=1}^{n-m} b_\ell(t) < z] \leq \mathbb{P}[\frac{1}{n}N_t(t) \leq u_t - \sqrt{\delta}] + \mathbb{P}[\frac{1}{(1-u_t+\sqrt{\delta})n}\sum_{\ell=1}^{(1-u_t+\sqrt{\delta})n} b_\ell(t) < \Pi_{k,u_a,u_v}(z(t)) - \sqrt{\delta}]$, where the inequality is obtained by splitting the sum at $m = (u_t - \sqrt{\delta})n$ and simplifying the terms obtained by bounding some probabilities with 1 and re-arranging the terms (see Zino et al. (2022) for all the steps in a similar setting). Finally, using Hoeffding's inequality (Hoeffding, 1963), we bound the first term as $\mathbb{P}[\frac{1}{n}N_t(t) \leq u_t - \sqrt{\delta}] \leq \exp\{-2\delta n\}$, and similarly with the second term, yielding (i). The proof of (ii) follows a similar argument. □

4. MAIN RESULTS

We derive analytical insights into the effect of the intervention policies. While the impact of making the innovation more advantageous ($u_a > 0$) and making people sensitive to emerging trends ($u_t > 0$) has been extensively studied, both separately and combined (Zino et al., 2021; Zino et al., 2022), the effect of increasing the visibility of adopters of the innovation is yet to be analyzed. We start by considering its impact in the absence of other actions.

Theorem 7. For $u_a = u_t = 0$, there exists a threshold $\zeta^*_k(0,0,u_v) \in (0,1)$, which is the unique solution of $\Pi_{k,0,u_v}(z) = z$ in $(0,1)$. Innovation diffusion is guaranteed iff $\zeta > \zeta^*_k(0,0,u_v)$.

Proof. First, observe that $\Pi_{k,0,u_v}$ is the complementary cumulative distribution function (CCDF) of a binomial random variable (r.v.) with k trials and success probability $p(z,u_v) = \frac{(1+u_v)z}{1+u_v z}$, from Eq. (12). Since the CCDF of a binomial RV is a monotonically increasing function of its success probability, and the success probability is monotonically increasing in u_v, then we conclude that $\Pi_{k,0,u_v}$ is monotonically increasing in u_v. From explicitly computing the derivatives of Eq. (10), we verify that $\Pi_{k,0,u_v}(z)$ has stationary points at 0 and at 1 (both points in which $\Pi_{k,0,u_v}(z) = z$), and changes its convexity once at $\bar{z} = \lfloor k/2 \rfloor / (k-1+u_v(k-\lceil k/2 \rceil))$ from concave to convex. Hence, $\Pi_{k,0,u_v}(z) < z$ in an open neighborhood of $z = 0$ and $\Pi_{k,0,u_v}(z) > z$ in an open neighborhood of $z = 1$, which means that there necessarily exists a value z^* such that $\Pi_{k,0,u_v}(z^*) = z^*$. We prove now its uniqueness. Let z^* be the smallest nonzero solution of $\Pi_{k,0,u_v}(z) = z$ and let us assume that $z^* \leq \bar{z}$. Then, there cannot be another $z \in (0, \bar{z}]$ such that $\Pi_{k,0,u_v}(z) = z$ since the function is strictly convex up in $(0, \bar{z})$ and $\Pi_{k,0,u_v}(0) = 0$. Hence $\Pi_{k,0,u_v}(\bar{z}) \geq \bar{z}$. Finally, since $\Pi_{k,0,u_v}(\bar{z})$ is strictly convex down in $(\bar{z}, 1)$, then by definition of convexity we conclude that $\Pi_{k,0,u_v}(z) > z$, $\forall z \in (\bar{z}, 1)$. A similar argument is applied if $z^* \geq \bar{z}$, proving uniqueness of $z^* \in (0,1)$. Hence, $\Pi_{k,0,u_v}(z) > z$, $\forall z \in (z^*, 1)$

Now, we prove that if $z(0) > z^*$, then innovation diffusion is guaranteed. First, we introduce an additional random time capturing the first time-instant at which the adopters of the innovation do not increase: $Q_{n,\varepsilon} := \inf\{t \in \mathbb{N}_+ : z(t) \leq z(t-1)\}$. Clearly, $Q_{n,\varepsilon} \leq F_{n,\varepsilon}$. Hence, $S_{n,\varepsilon} < Q_{n,\varepsilon}$ w.h.p. is a sufficient condition to guarantee innovation diffusion. Moreover, we observe that $R_{n,\varepsilon} := \min\{Q_{n,\varepsilon}, S_{n,\varepsilon}\} \leq n$, since either $z(t+1) \geq z(t) + 1/n$ or

$z(t+1) \leq z(t)$. Observe that, for $t < R_{n,\varepsilon}$ and $z > z^*$ the two conditions of Lemma 6 item (i) are satisfied. Hence, by conditioning on $R_{n,\varepsilon}$, using the law of total probability, and item (i) of Lemma 6 in a recursive fashion, we write $\mathbb{P}[S_{n,\varepsilon} < F_{n,\varepsilon}] \geq \mathbb{P}[S_{n,\varepsilon} < Q_{n,\varepsilon}] = \sum_{r=0}^{\infty} \mathbb{P}[R_{n,\varepsilon} = r]\mathbb{P}[\nexists t < r : z(t+1) \leq z(t)|\mathcal{F}_r] = \sum_{r=0}^{n} \mathbb{P}[R_{n,\varepsilon} = r] \prod_{k=0}^{r-1} (1 - \mathbb{P}[z(k+1) \leq z(k)|\mathcal{F}_k]) \geq \sum_{r=0}^{n} \mathbb{P}[R_{n,\varepsilon} = r] (1 - \exp\{-\delta n\})^r \geq (1 - \exp\{-\delta n\})^n \geq 1 - \frac{K}{n}$, for some positive constant $K > 0$, which yields sufficiency.

Necessary condition is proved as follows. Assume that $z \leq z^*$. Then, we observe that necessarily $(1-u_t)\Pi_{k,u_a,u_v}(z) + u_t < z$, which implies that $\mathbb{P}[z(2) < z^*] \geq 1/2$. Moreover, if $z(2) < z^*$, then the conditions of item (ii) of Lemma 6 hold. Hence, we can use it recursively to bound $\mathbb{P}[F_{n,\varepsilon} < S_{n,\varepsilon}|z(2) < z^*] \geq 1 - K/n$, for some constant $K \geq 0$. Finally, we bound $\mathbb{P}[S_{n,\varepsilon} < F_{n,\varepsilon}] \leq 1 - \mathbb{P}[z(2) < z^*]\mathbb{P}[F_{n,\varepsilon} < S_{n,\varepsilon}|z(2) < z^*] \leq 1/2 + K/n$, which does not converge to 1, yielding the necessity claim. □

Now, we present our general result. The proof follows the same arguments of the one of Theorem 7, but it requires some additional technicalities due to the presence of $u_a, u_t \geq 0$, which are briefly discussed after the statement. A detailed proof is omitted due to space constraints.

Theorem 8. Define $u^*_{k,u_a} := \inf\{u_t \in [0,1] : (1 - u_t)\Pi_{k,u_a,u_v}(z) - z + u_t > 0, \forall z \in (0,1)\}$. The following hold:

(i) if $u_a > k-2$ or $u_t > u^*_{k,u_a}$, then innovation diffusion is guaranteed for any fraction of initial adopters $\zeta > 0$;
(ii) if $u_a < k-2$ and $u_t < \nu^*_{k,u_a}$, then there exists a threshold $\zeta^*_k(u_a, u_t, u_v) \in (0,1)$, which is the largest solution z of $(1-u_t)\Pi_{k,u_a,u_v}(z) + u_t - z = 0$ in $(0,1)$. Innovation diffusion is guaranteed iff $\zeta > \zeta^*_k(u_a, u_t, u_v)$.

Proof. First, for any $t < R_{n,\varepsilon}$, the condition $z > z^*$ guarantees that the conditions of Lemma 6 are satisfied, allowing its recursive use to guarantee diffusion, similar to Theorem 7. Finally, we observe that the two conditions in (i) guarantee that $u_t + (1-u_t)\Pi_{k,u_a,u_v}(z) > z$ for all $z \in (0,1)$, which concludes the proof for sufficiency. The proof for necessity is a bit more involved, since it is not necessarily true that $(1-u_t)\Pi_{k,u_a,u_v}(z) + u_t < z$ for all $z < z^*$. However, there exists necessarily an interval $[\tilde{z}, z^*)$ for some $0 < \tilde{z} < z^*$, in which the inequality above holds true. Using this observation, one can consider the different cases depending on whether ζ belongs to the interval or $z < \tilde{z}$. In the first case, the same argument used for proving necessity in Theorem 7 yields the claim. In the second case, one can bound the probability that $z(t)$ jumps from below $\tilde{z}$ to above z^* without passing into the interval (where the first argument could be used), and obtain the claim, similar to (Zino et al., 2022). □

Corollary 9. The threshold $\zeta^*_k(u_a, u_t, u_v)$ is monotonically nonincreasing in the control parameters u_a, u_t, and u_v.

Proof. By definition, Π_{k,u_a,u_v} is monotonically nondecreasing in u_a (since the higher u_a, the more positive terms are included into the summation in Eq. (12)). As a consequence, if $(1-u_t)\Pi_{k,u'_a,u_v} + u_t - z = 0$, then $(1-u_t)\Pi_{k,u_a,u_v} + u_t - z \geq 0$ for any $u_a > u'_a$, and thus its largest zero $\zeta^*_k(u_a, u_t, u_v) \leq \zeta^*_k(u'_a, u_t, u_v)$. Similarly, we observe that the function $(1-u_t)\Pi_{k,u_a,u_v} + u_t - z = 0$ is monotonically nondecreasing in u_t, being the probability $\Pi_{k,u_a,u_v} \leq 1$. Hence, the same argument used in the above proves monotonicity of the threshold with respect to u_t. Finally, as observed in the proof of Theorem 8, Π_{k,u_a,u_v} is monotonically increasing in u_v, hence $\zeta^*_k(u_a, u_t, u_v) \leq \zeta^*_k(u_a, u_t, u'_v)$ for all $u_v > u'_v$. □

4.1 Case study: $k = 3$

We consider the case in which each individual establishes $k = 3$ interactions, for which the analytical computation of the threshold is possible. First, using the monotonicity property observed in Corollary 9, we consider the effect of each control action independently, and we establish the following sufficient conditions.

Corollary 10. Innovation diffusion is guaranteed if either of the following conditions on a single control action holds:

(i) $u_a > 1$, for any initial $\zeta > 0$;
(ii) $u_t > \frac{1}{9}$ or $\zeta > \frac{1}{4} + \frac{1}{4}\sqrt{\frac{9u_t - 1}{u_t - 1}}$;
(iii) $\zeta > \frac{-2+(1+u_v)(\sqrt{4+4u_v+9u_v^2} - 3u_v)}{2u_v^3}$.

Proof. First, observe that $k - 2 = 1$. Hence, item (i) of Theorem 8, yields (i). Then, for $u_a \in [0,1]$,

$$\Pi_{k,u_a,u_v}(z) = \frac{(1+u_v)^2}{(1+u_v z)^3} 3z^2(1-z) + \frac{(1+u_v)^3}{(1+u_v z)^3} z^3, \quad (13)$$

For (ii), we refer to the computations in (Zino et al., 2022). Finally, for (iii), we write $\Pi_{k,u_a,u_t}(z) = z$ using Eq. (13), obtaining a fourth order equation with two trivial solutions ($z = 0$ and $z = 1$), and two other real solutions: one positive and one negative. The largest of the two is necessarily the unique solution in $(0,1)$, yielding (iii). □

From this result, we make several observations. First, we note that making the innovation more advantageous and making people sensitive to emerging trends will reduce the threshold to 0, guaranteeing innovation diffusion for any initial condition if $u_a > 1$ or $u_t > \frac{1}{9}$. On the other hand, increasing the visibility of adopters of the innovation reduces the fraction of initial adopters needed to unlock diffusion, but the threshold will never go to 0 under the sole effect of u_v. Second, we observe that the human tendency to interact with a limited number of other people to make decisions, captured by k, could hinder the effectiveness of making innovation more advantageous. In fact, for $k = 3$, an innovation would need to be twice as advantageous as the status quo to have any effect on its adoption. On the contrary, the other two control actions have a continuous impact on the threshold, i.e., any increase in the control yields a decrease in the threshold.

The sufficient conditions presented in Corollary 10 illustrate the impact of each single control action on the innovation diffusion process. However, the monotonicity properties highlighted in Corollary 9 suggest that a joint implementation of different control actions can be beneficial in further reducing the threshold. From item (ii) of Theorem 8 and the explicit expression in Eq. (13), we can derive the general expression for the threshold $\zeta^*_k(u_a, u_t, u_v)$, that is either 0 or the largest solution in $(0,1)$ of the following fourth order equation in z,

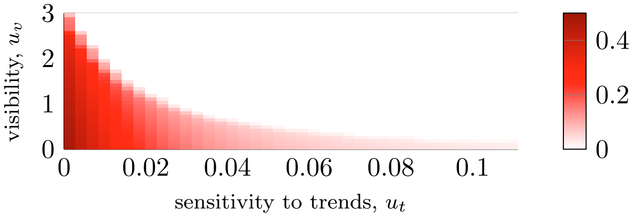

Fig. 1. Threshold $\zeta_3^*(u_a, u_t, u_v)$, computed analytically by solving Eq. (14), for different values of the control parameters u_t and u_v, and with $u_a \in [0, 1]$.

$$3(1-u_t)(1+u_v)^2 z^2(1-z) + (1-u_t)(1+u_v)^3 z^3 \\ +u_t(1+u_v z)^3 - z(1+u_v z)^3 = 0, \tag{14}$$

whose solutions can be computed and expressed in closed form, but are omitted due to their complexity. However, they can be used to shed light on the interplay between the two control actions u_t and u_v.

In Fig. 1, we illustrate the threshold, computed by solving Eq. (14). Note that $u_a > 1$ would guarantee diffusion, even in the absence of other control actions. The figure depicts a Pareto-front scenario, where the same value for the threshold can be obtained with different combinations of the two actions. Interestingly, an initial increase in the sensitivity to trends from $u_t = 0$ to $u_t = 0.04$ is sufficient to reduce the threshold by 80% from $\zeta_3(u_a, 0, 0) = 0.5$ to $\zeta_3(u_a, 0.4, 0) \approx 0.1$, while a further increase of the same amount would only halve the threshold to $\zeta_3(u_a, 0.7, 0) \approx 0.05$. On the contrary, in the absence of u_t, an increase in visibility would reduce the threshold to just $\zeta_3(u_a, 0, 0.6) = 0.0473$, but when $u_v > 0$ and $u_t > 0$ simultaneously, the threshold vanishes ($\zeta_3(u_a, 0.7, 0.6) = 0$). Optimal interventions could thus entail combining different actions.

5. CONCLUSION

We proposed a stochastic mathematical model for the diffusion of innovation. Building on the framework of network coordination games, we incorporated three realistic control actions —making the innovation more advantageous, making people sensitive to emerging trends, and increasing the visibility of adopters of the innovation— through the inclusion of additional mechanisms, each one regulated by a scalar parameter. Through the analysis of the stochastic dynamics obtained, we establish a threshold in the initial fraction of adopters of innovation: above this threshold, the innovation will spread through the entire network, replacing the status quo. The derivation of an analytical expression for such a threshold as a function of the three control actions allowed us to elucidate their impact.

Our findings pave the way for several promising lines of research. First, the inclusion of heterogeneity in the population and of constraints in the possible interactions that individuals may establish should be explored. Second, closed-form analytical expression for the threshold allows for studying the problem of designing an optimal control strategy by combining the different control actions to guarantee innovation diffusion. Efforts should be placed to analyze such optimization problems, toward gaining insight into the structure of optimal intervention policies. Finally, although each element of our model is supported by experimental studies and empirical evidence, real-world validation of the model is still missing.

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