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Optimal Intervention in Non-Binary Super-Modular Games

Sebastiano Messina[1], Giacomo Como[1], Stephane Durand[2], and Fabio Fagnani[1]

***Abstract*— We study intervention design problems for general finite non-binary super-modular games. The considered interventions consist in constraining or incentivizing the players to play actions above designed lower bounds, with a cost for the system planner that is a separable increasing function of such bounds. We study the intervention of minimum cost for which a best response learning algorithm leads the system to its greatest Nash equilibrium. We show that, if the utility functions are unimodal, then the optimal intervention problem can be reformulated in terms of improvement paths, leading to a low complexity distributed iterative algorithm for its solution.**

***Index Terms*— Network games; super-modular games; network intervention; optimal targeting; equilibrium selection.**

I. Introduction

A fundamental problem in multi-player game theory is the design of optimal intervention policies capable of steering the system towards a desirable configuration. Examples are taxes or subsidies in economic models, prices and tolls in transportation systems, incentives in social activities [1].

An important family of games are super-modular games [2], modeling strategic complementarities [3]. Their applications include modeling of social and economic behaviors like adoption of a new technology, participation to an event, provision of a public good effort. Super-modular games are typically endowed with multiple Nash equilibria that admit a Pareto ordering and the problem of the minimal effort needed to push the system from a lower to a greater equilibrium is natural and relevant in all these applicative contexts.

For binary super-modular games, various intervention problems have been proposed and studied in the literature. The binary case includes the popular linear threshold model [4] for which optimal seeding problems have been studied. In particular, [5]–[7] study the problem of targeting a fixed number K of agents so that, if activated, they yield the maximal possible expansion of the contagion. In [8], [9] the complementary problem of the determination of the minimum number of agents that, if activated, will lead to a full cascade (target set selection) is instead considered together with variations of it where the intervention is instead modeled as a modification of the threshold. In [10], the target set selection problem is instead considered for general binary super-modular games. In the literature cited above, it is proven that all these problems are NP-complete and various algorithms are proposed for approximating solutions.

This paper studies intervention problems in finite non-binary super-modular games that drive the players to the highest Nash equilibrium and, in particular, it extends to such context a low complexity algorithm proposed in [10] to approximate optimal interventions for the binary case. We are not aware of previous similar analysis of the literature. A related contribution is [1] where the authors, for the case of quadratic games with continuous sets of actions, consider an intervention problem on the marginal individual utilities with the purpose of maximizing social welfare under a bounded cost. While the results proposed by the authors also apply for non super-modular games (e.g., public good games), the mechanism of their intervention is quite specific and their solution uses the explicit expression of the Nash equilibrium that depends linearly on the marginal utilities.

In this paper, we model the intervention on a finite non-binary super-modular game as the imposition of a lower bound on the action chosen by the various players. We consider the optimization problem of finding the minimum cost intervention (for general cost functions) for which in such restricted game an asynchronous best response dynamics leads to the maximum Nash equilibrium. This model is a natural generalization of the seeding problem considered in the binary case [8]–[10] and we show that it can be interpreted in economic applications as interventions with a fixed incentive (granted if a player maintains its effort above some prescribed level).

The main contribution of our paper is twofold. First, we individuate a crucial unimodality property of utility functions that, together with super-modularity, allows to establish an equivalence between the existence of best response paths and that of basic improvement paths where actions never decrease and modifications are of minimal size. This new property is always trivially verified in the binary case, while we show that this is not the case for larger action sets. Under super-modularity and unimodality we are then able to extend the algorithm proposed in [10] to this more general setting.

We conclude this section with a brief outline of the paper. In Section II, the intervention problem considered is formalized, all relevant concepts are introduced and a basic example is presented. Section III focuses on a fundamental property of the games considered, the equivalence between reachability of the maximum configuration through best response paths and the reachability through weak improvement paths of minimal size step. Section IV presents an iterative algorithm modeled as a reversible Markov chain that is proven to converge to the desired optimal intervention. Section V contains some simulation results. The paper ends with a conclusion section.

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II. Problem formulation

A. Finite supemodular unimodal games

We consider finite strategic form games with player set $\mathcal{V} = \{1, ..., n\}$, where $n \geq 2$. Every player i in $\mathcal{V}$ chooses her action x_i from a nonempty finite set $\mathcal{A}_i \subseteq \mathbb{R}$. We denote by $\mathcal{X} = \prod_{i \in \mathcal{V}} \mathcal{A}_i$ the strategy profile space. Given a strategy profile $\mathbf{x}$ in $\mathcal{X}$ and a player i, we indicate with $\mathbf{x}_{-i}$ the strategy profile of all players but i. Every player i is endowed with a utility function $u_i : \mathcal{X} \to \mathbb{R}$, so that

$$u_i(\mathbf{x}) = u_i(x_i, \mathbf{x}_{-i}) \tag{1}$$

denotes the utility of player i when she plays action x_i while the rest of the players play $\mathbf{x}_{-i}$. A game is formally identified by the triple $\Gamma = (\mathcal{V}, \{\mathcal{A}_i\}, \{u_i\})$. The best response correspondence for a player i is the set-valued map

$$\mathcal{B}_i(\mathbf{x}_{-i}) = \underset{a \in \mathcal{A}_i}{\operatorname{argmax}}\, u_i(a, \mathbf{x}_{-i}) \,, \tag{2}$$

while $\mathcal{B}_i^+(\mathbf{x}_{-i}) = \max \mathcal{B}_i(\mathbf{x}_{-i})$ denotes the maximal best response. The set of pure strategy Nash equilibria is

$$\mathcal{N} = \{\mathbf{x} \in \mathcal{X} |\ x_i \in \mathcal{B}_i(\mathbf{x}_{-i})\ \forall i \in \mathcal{V}\} \ .$$

For every player i in $\mathcal{V}$, we denote by $m_i = \min \mathcal{A}_i$ and $M_i = \max \mathcal{A}_i$, respectively, her minimal and maximal actions: for every non-maximal action $x_i < M_i$ (respectively, non-minimal action $x_i > m_i$), we define $x_i^+ = \min\{y \in \mathcal{A}_i : y \geq x_i\}$, ($x_i^- = \max\{y \in \mathcal{A}_i : y \leq x_i\}$). We equip the strategy profile space $\mathcal{X}$ with the standard partial order

$$\mathbf{x} \geq \mathbf{y} \iff x_i \geq y_i \qquad \forall i \in \mathcal{V} \,, \tag{3}$$

Then, $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ and $\mathbf{M} = (M_1, M_2, \ldots, M_n)$ are respectively the least and the greatest strategy profiles.

We consider super-modular games as per the following.

Definition 1: A game is *super-modular* if

$$u_i(x_i, \mathbf{x}_{-i}) - u_i(y_i, \mathbf{x}_{-i}) \geq u_i(x_i, \mathbf{y}_{-i}) - u_i(y_i, \mathbf{y}_{-i}) \,,$$

for every two strategy profiles $\mathbf{x} \geq \mathbf{y}$ and player i in $\mathcal{V}$.

It is well-known [2] that, for every finite super-modular game, the maximal best response of every player i is monotonically non-decreasing, i.e.,

$$\mathbf{x}_{-i} \geq \mathbf{y}_{-i} \Longrightarrow \mathcal{B}_i^+(\mathbf{x}_{-i}) \geq \mathcal{B}_i^+(\mathbf{y}_{-i}) \,, \tag{4}$$

and the set of Nash equilibria $\mathcal{N}$ is nonempty and it admits a least element and a greatest element.

In fact, we shall restrict to finite super-modular games that satisfy the following additional property.

Definition 2: A game is *unimodal* if the function $x_i \mapsto u_i(x_i, \mathbf{x}_{-i})$ is unimodal for every player i and strategy profile $\mathbf{x}_{-i}$, i.e., if there exists b_i in $\mathbb{R}$ such that $x_i \mapsto u_i(x_i, \mathbf{x}_{-i})$ is non-decreasing for $x_i \leq b_i$ and non-increasing for $x_i \geq b_i$.

Example 1: For a nonnegative square matrix $\mathbf{W} = (W_{ij})$ in $\mathbb{R}_+^{n \times n}$ and a vector $\mathbf{a} = (a_i)$ in $\mathbb{R}^n$, consider the game where every player $i = 1, \ldots, n$ has nonempty finite action set $\mathcal{A}_i \subseteq \mathbb{R}$ and utility

$$u_i(\mathbf{x}) = a_i x_i - \frac{1}{2} x_i^2 + x_i \sum_{j \neq i} W_{ij} x_j \,. \tag{5}$$

This game is super-modular, since, for every two actions x_i and y_i in $\mathcal{A}_i$ such that $z_i = x_i - y_i \geq 0$,

$$u_i(x_i, \mathbf{x}_{-i}) - u_i(y_i, \mathbf{x}_{-i}) = a_i z_i - \frac{x_i^2 - y_i^2}{2} + z_i \sum_{j \neq i} W_{ij} x_j$$

is a nondecreasing function of the strategy profile $\mathbf{x}_{-i}$. This game is also unimodal, since the map $x_i \mapsto u_i(x_i, \mathbf{x}_{-i}) = b_i x_i - x_i^2/2$, where $b_i = a_i + \sum_j W_{ij} x_i$, is non-decreasing for $x_i \leq b_i$ and non-increasing for $x_i \geq b_i$.

Such games can be interpreted as quantized versions of the quadratic games with strategic complements that are typically studied when the action sets are all (intervals of) the real line [11]–[14]. They are used, e.g., to model scenarios where $n \geq 2$ individuals in a social network (whose structure is captured by the matrix W) can engage in some common activity (e.g., study, partnership) at various levels of efforts, represented by her action set $\mathcal{A}_i$. If she chooses to engage at level x_i, she bears a cost $x_i^2/2$ and enjoys a direct (network independent) utility $a_i x_i$. Moreover, individuals benefit from engagement in the common activity by their neighbors in social network, as captured by the pairwise terms $W_{ij} x_i x_j$ in the utility (5).

B. Intervention optimization

We consider interventions on finite super-modular unimodal games as restrictions of the various action sets. Specifically, for a vector $\mathbf{h}$ in $\mathcal{X}$, to be referred as an intervention vector, we consider the game $\Gamma^{(\mathbf{h})}$ where the action set for player i is restricted to $\{x_i \in \mathcal{A}_i \,|\, x_i \geq h_i\}$. Hence, in particular, $\Gamma^{(\mathbf{m})} = \Gamma$ coincides with the original (unrestricted) game. We notice that the original game Γ is super-modular and unimodal if and only if so is the restricted game $\Gamma^{(\mathbf{h})}$ for every intervention vector $\mathbf{h}$ in $\mathcal{X}$. To every intervention we associate a separable cost

$$C(\mathbf{h}) = \sum_i c_i(h_i) \,, \tag{6}$$

where, for every player i in $\mathcal{V}$, $c_i : \mathcal{A}_i \to \mathbb{R}_+$ is a non-decreasing cost function such that $c_i(m_i) = 0$.

We then focus on the problem of determining a minimum cost intervention vector $\mathbf{h}$ such that the best response dynamics on the restricted game $\Gamma^{(\mathbf{h})}$ globally converges to the greatest strategy profile $\mathbf{M}$, as formalized below.

Definition 3: For an intervention vector $\mathbf{h}$ in $\mathcal{X}$:

(i) a tuple of strategy profiles $(\mathbf{x}^{(k)})_{0 \leq k \leq l}$ is a *BR-path* from $\mathbf{y}$ to $\mathbf{z}$ if $\mathbf{x}^{(0)} = \mathbf{y}$, $\mathbf{x}^{(l)} = \mathbf{z}$, and for every $0 \leq k < l$ there exists a player i_k in $\mathcal{V}$ such that

$$x_{i_k}^{(k+1)} \in \mathcal{B}_{i_k}(\mathbf{x}_{-i_k}^{(k)}) \,, \qquad \mathbf{x}_{-i_k}^{(k+1)} = \mathbf{x}_{-i_k}^{(k)} \,;$$

(ii) a *BR$^+$-path* is a BR-path such that

$$x_{i_k}^{(k+1)} \in \mathcal{B}_{i_k}^+(\mathbf{x}_{-i_k}^{(k)}) \,, \qquad 0 \leq k < l \,;$$

(iii) a strategy profile $\mathbf{x}$ in $\mathcal{X}$ is $\mathbf{M}$-*attracted* in $\Gamma^{(\mathbf{h})}$ if there exists a BR-path in $\Gamma^{(\mathbf{h})}$ from $\mathbf{x}$ to $\mathbf{M}$;

(iv) an intervention $\mathbf{h}$ is *sufficient* if every $\mathbf{x}$ in $\mathcal{X}^{(\mathbf{h})}$ is $\mathbf{M}$-attracted in $\Gamma^{(\mathbf{h})}$.

We then study the optimal intervention problem

$$\min_{\mathbf{h}\in\mathcal{O}} C(\mathbf{h}) \tag{7}$$

where $\mathcal{O}$ denotes the set of sufficient interventions and let

$$\mathcal{O}^* = \operatorname*{argmin}_{\mathbf{h}\in\mathcal{O}} C(\mathbf{h}) \tag{8}$$

denote the set of *minimum cost sufficient interventions*.

Remark 1: Given an intervention vector $\mathbf{h}$ in $\mathcal{X}$, consider a discrete-time asynchronous best response dynamics where at every instant a uniform random player updates her action to a best response in the restricted game $\Gamma^{(\mathbf{h})}$, choosing such best response uniformly at random in case of non-uniqueness. If the intervention $\mathbf{h}$ is sufficient, then from every profile $\mathbf{x}$ in $\mathcal{X}^{(\mathbf{h})}$ there exists a BR-path leading to $\mathbf{M}$, so that with probability one $\mathbf{M}$ will be reached in finite time.

Example 1 (continued): Consider the game with utilities (5) and assume that the action sets $\mathcal{A}_i$ are such that $m_i = 0$ and $a_i \in \mathcal{A}_i$ for all i. Assume that a feasible intervention consists in granting to each player i an extra utility $\lambda_i \geq 0$ whenever $x_a \geq h_i$. In other words, the players experiment a modified utility function $\tilde{u}_i(\mathbf{x}) = u_i(\mathbf{x}) + \lambda_i \mathbf{1}_{[h_i, M_i]}(x_i)$. The goal of the system planner is then to find a pair of vectors $(\lambda, \mathbf{h})$ for which every configuration $\mathbf{x}$ is $\mathbf{M}$-attracted and the cost for the planner $\sum_i \lambda_i$ is minimized.

We now show that this problem can be formulated as (7). First, a straightforward computation shows that

$$\lambda_i = c_i(h_i) = \frac{1}{2}[h_i - a_i]_+^2 \tag{9}$$

is the minimum value for which $u_i(h_i, \mathbf{x}_{-i}) \geq u_i(x_i, \mathbf{x}_{-i})$ for every $x_i \leq h_i$ and every $\mathbf{x}_{-i}$. We call $\tilde{\Gamma}^{(\mathbf{h})}$ the game equipped with utility functions $\tilde{u}_i$ with this specific value for λ_i and no restriction on the action sets and we compare it with the game $\Gamma^{(\mathbf{h})}$ with restricted action sets previously introduced. We notice that for configurations $\mathbf{x} \geq \mathbf{h}$, the two games are equivalent, in the sense that best response sets are always equal. This implies that if $\mathbf{h}$ is a sufficient intervention vector for $\Gamma^{(\mathbf{h})}$, then every configuration $\mathbf{x}$ is $\mathbf{M}$-attracted also for $\tilde{\Gamma}^{(\mathbf{h})}$. Indeed, by construction, starting from any initial configuration there exists a BR-path in $\tilde{\Gamma}^{(\mathbf{h})}$ leading to a configuration $\mathbf{x} \geq \mathbf{h}$ and from $\mathbf{x}$ the same BR-path that leads to $\mathbf{M}$ in $\Gamma^{(\mathbf{h})}$ also leads to $\mathbf{M}$ in $\tilde{\Gamma}^{(\mathbf{h})}$.

Conversely, suppose that $\mathbf{h}$ is such that every configuration $\mathbf{x}$ is $\mathbf{M}$-attracted in $\tilde{\Gamma}^{(\mathbf{h})}$. If we start from $\mathbf{x} \geq \mathbf{h}$, any BR-path in $\tilde{\Gamma}^{(\mathbf{h})}$ from $\mathbf{x}$ to $\mathbf{M}$ will pass through profiles $\mathbf{x}^{(k)} \geq \mathbf{h}$, hence it will also be a BR-path in $\Gamma^{(\mathbf{h})}$.

This shows that the optimal intervention problem for this example is equivalent to the minimization problem (7) with a cost function

$$C(\mathbf{h}) = \sum_{i\in\mathcal{V}} c_i(h_i)$$

III. Properties and equivalent formulation

In this section, we propose an equivalent description of the optimal intervention problem that will allow us to design a low complexity algorithm for its solution. Throughout, we assume we have fixed a finite super-modular unimodal game.

The following fact shows that in the description of the $\mathbf{M}$-attracted profiles we can always restrict to BR$^+$-paths.

Proposition 1: Consider a finite super-modular game. For a strategy profile $\mathbf{x}$, the following conditions are equivalent:

(i) there exists a BR-path from $\mathbf{x}$ to $\mathbf{M}$;
(ii) there exists a BR$^+$-path from $\mathbf{x}$ to $\mathbf{M}$.

Proof: Notice first that by definition (ii)⇒(i).

(i)⇒(ii): Consider a BR-path $(\mathbf{x}^{(k)})_{0\leq k\leq l}$ from $\mathbf{x}$ to $\mathbf{M}$ and construct a new path $(\mathbf{y}^{(k)})_{0\leq k\leq l}$ with $\mathbf{y}^{(0)} = \mathbf{x}$ and $y_{i_k}^{(k+1)} = \mathcal{B}_{i_k}^+(\mathbf{y}_{-i_k}^{(k)})$, $\mathbf{y}_{-i_k}^{(k+1)} = \mathbf{y}_{-i_k}^{(k)}$, for every $0 \leq k < l$. Clearly, $(\mathbf{y}^{(k)})_{0\leq k\leq l}$ is a BR$^+$-path, so that we are only left to show that $\mathbf{y}^{(l)} = \mathbf{M}$. We prove by induction that

$$\mathbf{y}^{(k)} \geq \mathbf{x}^{(k)}\,, \qquad \forall 0 \leq k \leq l\,. \tag{10}$$

Clearly, $\mathbf{y}^{(0)} = \mathbf{x} = \mathbf{x}^{(0)}$. Moreover, if $\mathbf{y}^{(k)} \geq \mathbf{x}^{(k)}$ for some $0 \leq k < l$, then, (4) and $x_{i_k}^{(k+1)} \in \mathcal{B}_{i_k}\big(\mathbf{x}_{-i_k}^{(k)}\big)$ imply that

$$y_{i_k}^{(k+1)} = \mathcal{B}_{i_k}^+\big(\mathbf{y}_{-i_k}^{(k)}\big) \geq \mathcal{B}_{i_k}^+\big(\mathbf{x}_{-i_k}^{(k)}\big) \geq x_{i_k}^{(k+1)}\,,$$

so that $\mathbf{y}^{(k+1)} \geq \mathbf{x}^{(k+1)}$. This proves (4). For $k = l$, we get $\mathbf{M} \leq \mathbf{x}^{(l)} \leq \mathbf{y}^{(l)} \leq \mathbf{M}$, thus completing the proof. ■

This yields an important consequence on the structure of the set of $\mathbf{M}$-attracted profiles, stated in the following result.

Corollary 1: If $\mathbf{x}$ is $\mathbf{M}$-attracted, then every $\mathbf{y} \geq \mathbf{x}$ is $\mathbf{M}$-attracted.

Proof: By virtue of Proposition 1, there exists a BR$^+$-path $(\mathbf{x}^{(k)})_{0\leq k\leq l}$ from $\mathbf{x}$ to $\mathbf{M}$. Consider now the BR$^+$-path

$$\mathbf{y}^{(0)} = \mathbf{y}\,, \qquad y_{i_k}^{(k+1)} = \mathcal{B}_{i_k}^+\big(\mathbf{y}_{-i_k}^{(k)}\big)\,, \quad \forall 0 \leq k < l\,.$$

Inductively, using the monotonicity of $\mathcal{B}_i^+$ as we did above, we prove that $\mathbf{y}^{(k)} \geq \mathbf{x}^{(k)}$ for every k. Hence $\mathbf{y}^{(l)} = \mathbf{M}$. ■

A different type of path is defined below where modifications are constrained to be monotonic and of minimum step and we only require that utilities do not decrease.

Definition 4 (weakly Improvement path): For a finite game Γ, a tuple of strategy profiles $(\mathbf{x}^{(k)})_{0\leq k\leq l}$ is a *weakly Improvement path (wI-path)* from the strategy profile $\mathbf{x}$ to the strategy profile $\mathbf{z}$ if $\mathbf{x}^{(0)} = \mathbf{x}$, $\mathbf{x}^{(l)} = \mathbf{z}$ and, for every $0 \leq k < l$ there exists i_k in $\mathcal{V}$ such that $\mathbf{x}_{-i_k}^{(k+1)} = \mathbf{x}_{-i_k}^{(k)}$ and

$$x_{i_k}^{(k+1)} = (x_{i_k}^{(k)})^+\,, \qquad u_{i_k}\big(\mathbf{x}^{(k+1)}\big) \geq u_{i_k}\big(\mathbf{x}^{(k)}\big)\,. \tag{11}$$

Observe that, in contrast to BR-paths, for every intervention vector $\mathbf{h}$, wI-paths from any $\mathbf{x} \geq \mathbf{h}$ remain the same in the restricted game $\Gamma^{(\mathbf{h})}$.

The following technical result builds on the unimodality assumption and will be instrumental to our future derivations.

Lemma 1: Consider a finite super-modular unimodal game and a strategy profile $\mathbf{x}$ in $\mathcal{X}$. If there exists a player i such that $x_i < M_i$ and $u_i(x_i, \mathbf{x}_{-i}) \leq u_i(x_i^+, \mathbf{x}_{-i})$, then

$$\mathcal{B}_i^+(\mathbf{x}_{-i}) \geq x_i^+\,.$$

Proof: Unimodality implies that $a \mapsto u_i(a, \mathbf{x}_{-i})$ is necessarily non-decreasing on the set $\{a \in \mathcal{A}_i \,|\, a \leq x_i^+\}$. This implies that the highest best response must necessarily belong to the set $\{a \in \mathcal{A}_i \,|\, a \geq x_i^+\}$. ■

The following result shows that $\mathbf{M}$-attractivity can be studied through weak Improvement paths.

Proposition 2: For a finite super-modular unimodal game and a strategy profile $\mathbf{x}$, the following are equivalent:

(i) there exists a BR-path from $\mathbf{x}$ to $\mathbf{M}$;

(ii) there exists a wI-path from $\mathbf{x}$ to $\mathbf{M}$.

Proof: (i)⇒(ii). Because of Proposition 1, we can assume that there exists a BR$^+$-path $(\mathbf{y}^{(k)})_{0\le k\le l}$ from $\mathbf{x}$ to $\mathbf{M}$. Define a new path $(\mathbf{x}^{(k)})_{0\le k\le l}$ with $\mathbf{x}^{(0)}=\mathbf{y}^{(0)}$ and

$$x_{i_k}^{(k+1)}=\max\left\{y_{i_k}^{(k+1)},\ x_{i_k}^{(k)}\right\}, \qquad \mathbf{x}_{-i_k}^{(k+1)}=\mathbf{x}_{-i_k}^{(k)}, \quad (12)$$

for $0\le k<l$, where i_k is such that $\mathbf{y}_{-i_k}^{(k+1)}=\mathbf{y}_{-i_k}^{(k)}$. Notice that $\mathbf{x}^{(k)}\ge\mathbf{y}^{(k)}$ for $0\le k\le l$. Now, observe that

$$u_{i_k}(x_{i_k}^{(k+1)},\mathbf{y}_{-i_k}^{(k)})\ge u_{i_k}(x_{i_k}^{(k)},\mathbf{y}_{-i_k}^{(k)}). \quad (13)$$

Indeed, when $x_{i_k}^{(k+1)}=x_{i_k}^{(k)}$, (13) trivially holds true as an equality, whereas when $x_{i_k}^{(k+1)}=y_{i_k}^{(k+1)}$, (13) holds true since $y_{i_k}^{(k+1)}=\mathcal{B}_{i_k}^+(\mathbf{y}_{-i_k}^{(k)})$ is a best response to $\mathbf{y}_{-i_k}^{(k)}$. Equations (13) and (12) and super-modularity yield

$$\begin{aligned} 0 &\le u_{i_k}(x_{i_k}^{(k+1)},\mathbf{y}_{-i_k}^{(k)})-u_{i_k}(x_{i_k}^{(k)},\mathbf{y}_{-i_k}^{(k)}) \\ &\le u_{i_k}(x_{i_k}^{(k+1)},\mathbf{x}_{-i_k}^{(k)})-u_{i_k}(x_{i_k}^{(k)},\mathbf{x}_{-i_k}^{(k)}), \end{aligned}$$

for $0\le k<l$. This shows that $(\mathbf{x}^{(k)})_{0\le k\le l}$ satisfies all the requirements for a weakly improvement path, except for possibly the first equation in (11), as it could still be the case that $x_{i_k}^{(k+1)}>(x_{i_k}^{(k)})^+$. If so, we consider the actions $a_1=(x_{i_k}^{(k)})^+<a_2=a_1^+<\cdots<a_r=x_{i_k}^{(k+1)}$, and interpolate the sequence of profiles from $\mathbf{x}^{(k)}$ to $\mathbf{x}^{(k+1)}$ with the subsequence $\mathbf{x}^{(k,s)}$ with $1\le s\le r$ such that $x_{i_k}^{(k,s)}=a_s$ and $\mathbf{x}_{-i_k}^{k,s}=\mathbf{x}_{-i_k}^{(k)}$ for all s. Notice that, by the unimodality assumption, necessarily $a\mapsto u_{i_k}(a,\mathbf{x}_{-i_k}^{(k)})$ is non decreasing in the interval $[x_{i_k}^{(k)},x_{i_k}^{(k+1)}]$, namely $u_{i_k}(\mathbf{x}^{k,s+1})\ge u_{i_k}(\mathbf{x}^{k,s})$ for every $1\le s<r$. If we carry on such an interpolation for every k for which $x_{i_k}^{(k+1)}>(x_{i_k}^{(k)})^+$, we finally obtain a wI-path leading from $\mathbf{x}$ to $\mathbf{M}$.

(ii)⇒(i). Consider a wI-path $(\mathbf{x}^{(k)})_{0\le k\le l}$ from $\mathbf{x}$ to $\mathbf{M}$ and define a new path $(\mathbf{y}^{(k)})_{0\le k\le l}$ with $\mathbf{y}^{(0)}=\mathbf{x}^{(0)}$ and

$$y_{i_k}^{(k+1)}=\mathcal{B}_{i_k}^+(\mathbf{y}_{-i_k}^{(k)}), \qquad \mathbf{y}_{-i_k}^{(k+1)}=\mathbf{y}_{-i_k}^{(k)}, \qquad \forall 0\le k<l.$$

By construction, $(\mathbf{y}^{(k)})_{0\le k\le l}$ is a BR$^+$-path and we only need to prove that $\mathbf{y}^{(l)}=\mathbf{M}$. This will directly follow from the fact that $\mathbf{y}^{(k)}\ge\mathbf{x}^{(k)}$ for every k. This last condition is proven by induction again, first noticing that is trivially true for $k=0$. Then, assume it to be true for some k, then monotonicity of $\mathcal{B}_{i_k}^+$ implies that

$$y_{i_k}^{(k+1)}=\mathcal{B}_{i_k}^+(\mathbf{y}_{-i_k}^{(k)})\ge\mathcal{B}_{i_k}^+(\mathbf{x}_{-i_k}^{(k)}) \quad (14)$$

Since $x_{i_k}^{(k+1)}=(x_{i_k}^{(k)})^+$ and $u_{i_k}(\mathbf{x}^{(k+1)})\ge u_{i_k}(\mathbf{x}^{(k)})$, Lemma 1 implies that $\mathcal{B}_{i_k}^+(\mathbf{x}_{-i_k}^{(k)})\ge x_{i_k}^{(k+1)}$. This and (14) imply that $\mathbf{y}^{(k+1)}\ge\mathbf{x}^{(k+1)}$, thus proving the claim. ■

The following conclusive result gives a simpler characterization of the set of sufficient interventions and constitutes the basis for the algorithm proposed in the next section.

Corollary 2: For a finite super-modular unimodal game Γ, the set of sufficient interventions can be characterized as

$$\mathcal{O}=\{\mathbf{h}\in\mathcal{X}\mid\exists\text{ a wI-path from }\mathbf{h}\text{ to }\mathbf{M}\} \quad (15)$$

Proof: ($\supseteq$) If there exists a wI-path from $\mathbf{h}$ to $\mathbf{M}$, then, by Proposition 2, there exists a BR-path from $\mathbf{h}$ to $\mathbf{M}$ in $\Gamma^{(\mathbf{h})}$. From Corollary 1, we deduce that all profiles $\mathbf{x}$ in $\mathcal{X}^{(\mathbf{h})}$ are $\mathbf{M}$-attracted in $\Gamma^{(\mathbf{h})}$ so that $\mathbf{h}\in\mathcal{O}$. ($\subseteq$) If $\mathbf{h}\in\mathcal{O}$, then there exists a BR-path from $\mathbf{h}$ to $\mathbf{M}$ in $\Gamma^{(\mathbf{h})}$. Then, by Proposition 2, there exists also a wI-path from $\mathbf{h}$ to $\mathbf{M}$. ■

The following example shows that unimodality is necessary for the above result to hold true.

Example 2: Consider the 2-player 3-action game with $\mathcal{A}_i=\{-1,0,1\}$ for $i=1,2$ and utilities

$$u_i(x_i,x_{-i})=-x_i+2x_i^2+x_ix_{-i}, \quad i=1,2$$

A direct check shows that the two possible BR-paths are $(0,0)\to(-1,0)\to(-1,-1)$ and $(0,0)\to(0,-1)\to(-1,-1)$, showing that $(0,0)$ is not $(1,1)$ attracted. However, $(0,0)\to(1,0)\to(1,1)$ is a wI-path leading to $(1,1)$.

IV. A DISTRIBUTED ALGORITHM FOR MINIMUM SUFFICIENT INTERVENTIONS

This section proposes a simple provably convergent iterative algorithm to compute sufficient interventions of minimum cost. It is an extension of an algorithm proposed in [10] in the special case of binary super-modular games and it is based on the characterization of $\mathcal{O}$ in Corollary 2. This suggests the possibility that minimum sufficient interventions may be searched for by starting from the greatest strategy profile $\mathbf{M}$ and iteratively following backwards a wI-path.

Assume we have fixed a finite super-modular unimodal game and non-decreasing cost functions $c_i:\mathcal{A}_i\to\mathbb{R}$. We put $d_i(x_i)=c_i(x_i^+)-c_i(x_i)$ for $x_i<M_i$ and $d_i(M_i)=0$.

We now introduce a family of discrete-time Markov chains $(Z_t^\varepsilon)_{t\ge0}$ on the strategy profile space $\mathcal{X}$, parameterized by a scalar ε in $[0,1]$. We will then prove that, for $0<\varepsilon\le1$, the Markov chain $(Z_t^\varepsilon)_{t\ge0}$ is time-reversible and that, as ε vanishes, its stationary distribution concentrates on the family of minimum cost sufficient interventions.

The dynamics of the Markov chain Z_t^ε are described as follows: at every time $t=0,1...$, given that $Z_t^\varepsilon=\mathbf{x}$, a player i is chosen uniformly at random from the whole player set $\mathcal{V}$ and she selects at random a direction to follow (up or down) with probability $1/2$. If she chooses to go down, then

- if $u_i(x_i^-,\mathbf{x}_{-i})>u_i(x_i,\mathbf{x}_{-i})$, then $Z_{t+1}^\varepsilon=Z_t^\varepsilon$;
- if $u_i(x_i^-,\mathbf{x}_{-i})\le u_i(x_i,\mathbf{x}_{-i})$, then

$$(Z_{t+1}^\varepsilon)_i=(Z_t^\varepsilon)_i^-, \qquad (Z_{t+1}^\varepsilon)_{-i}=(Z_t^\varepsilon)_{-i}.$$

On the other hand, if she chooses to go up, then

- if $u_i(x_i^+,\mathbf{x}_{-i})<u_i(x_i,\mathbf{x}_{-i})$, then $Z_{t+1}^\varepsilon=Z_t^\varepsilon$;
- if $u_i(x_i^+,\mathbf{x}_{-i})\ge u_i(x_i,\mathbf{x}_{-i})$, then $Z_{t+1}^\varepsilon=Z_t^\varepsilon$ with probability $1-\varepsilon^{d_i(x_i)}$, whereas

$$(Z_{t+1}^\varepsilon)_i=(Z_t^\varepsilon)_i^+, \qquad (Z_{t+1}^\varepsilon)_{-i}=(Z_t^\varepsilon)_{-i},$$

with probability $\varepsilon^{d_i(x_i)}$.

For a strategy profile $\mathbf{x} \in \mathcal{X}$, let

$$n^+(\mathbf{x}) = \{i \in \mathcal{V}: \; u_i(x_i^+, \mathbf{x}_{-i}) \geq u_i(x_i, \mathbf{x}_{-i})\}$$
$$n^-(\mathbf{x}) = \{i \in \mathcal{V}: \; u_i(x_i, \mathbf{x}_{-i}) \geq u_i(x_i^-, \mathbf{x}_{-i})\}$$

and

$$\alpha_\varepsilon(\mathbf{x}) = \frac{1}{2n}\left(\sum_{i \in n^+(\mathbf{x})} \varepsilon^{d_i(x_i)} + |n^-(\mathbf{x})|\right)$$

Then, the transition probabilities of this Markov chain are

$$P^{(\varepsilon)}_{\mathbf{x},\mathbf{y}} = \begin{cases} 1/2n & \text{if } \mathbf{y} = \left(x_i^-, \mathbf{x}_{-i}\right) \text{ and } u_i(\mathbf{y}) \leq u_i(\mathbf{x}) \\ \varepsilon^{d_i(x_i)}/2n & \text{if } \mathbf{y} = \left(x_i^+, \mathbf{x}_{-i}\right) \text{ and } u_i(\mathbf{y}) \geq u_i(\mathbf{x}) \\ 1 - \alpha_\varepsilon(\mathbf{x}) & \text{if } \mathbf{y} = \mathbf{x} \\ 0 & \text{otherwise}. \end{cases} \tag{16}$$

We first make some considerations in the special case $\varepsilon = 0$. Notice that, in this case, only transitions from higher to lower actions are allowed, hence the Markov chain $Z_t^{(0)}$ always converges to an absorbing state. We define

$$\mathcal{Z} = \left\{\mathbf{x} \in \mathcal{X} | \; \mathbb{P}\left(\exists t_0 \geq 0 : Z^{(0)}_{t_0} = \mathbf{x} | Z^{(0)}_0 = \mathbf{M}\right) \geq 0\right\} \tag{17}$$

to be the set of all states that are reachable by the Markov chain $Z_t^{(0)}$ when started from $\mathbf{M}$. We have the following.

Proposition 3: Consider a finite super-modular unimodal game and let $\mathcal{Z}$ be the set defined in (17). Then $\mathcal{O} = \mathcal{Z}$.

Proof: By definition, $\mathbf{x} \in \mathcal{Z}$ if and only if there exists a tuple of strategy profiles $(\mathbf{y}^{(k)})_{0 \leq k \leq l}$, such that $\mathbf{y}^{(0)} = \mathbf{M}, \mathbf{y}^{(l)} = \mathbf{x}$, and for every $0 \leq k < 1$ there exists i_k such that $\mathbf{y}^{(k+1)}_{-i_k} = \mathbf{y}^{(k)}_{-i_k}$, $y_i^{(k+1)} = (y_{i_k}^{(k)})^-$, and $u_{i_k}(\mathbf{y}^{(k+1)}) \leq u_{i_k}(\mathbf{y}^{(k)})$. Notice that this is equivalent to say that the reversed path $\left(\mathbf{x}^{(k)}\right)_{0 \leq k \leq l}$ with $\mathbf{x}^{(k)} = \mathbf{y}^{(l-k)}$ for $0 \leq k \leq l$ is a wI-path from $\mathbf{x}$ to $\mathbf{M}$. By Corollary 2 this is equivalent to $\mathbf{x} \in \mathcal{O}$. ■

Unfortunately, the absorbing states of the chain $Z_t^{(0)}$ are in general not in $\mathcal{O}^*$ so that running such Markov chain is not useful for our purposes. We now analyze the perturbed Markov chain Z_t^ε for $\varepsilon > 0$, starting with a technical lemma.

Lemma 2: Consider a discrete-time Markov Chain Z_t on $\mathcal{X}$ with transition matrix P such that:

(a) if $\mathbf{x} \neq \mathbf{y}$ differ in more than one entry or are such that $\mathbf{y}_{-i} = \mathbf{x}_{-i}$ and $y_i \notin \{x_i^-, x_i^+\}$, then $P_{\mathbf{x},\mathbf{y}} = 0$;

(b) if $\mathbf{x} \neq \mathbf{y}$ are such that $\mathbf{y}_{-i} = \mathbf{x}_{-i}$ and $y_i = x_i^+$, then

$$P_{\mathbf{x},\mathbf{y}} = P_{\mathbf{y},\mathbf{x}} \varepsilon^{d_i(x_i)}.$$

Then Z_t is reversible with respect to the invariant distribution

$$\mu_\mathbf{x} \propto \varepsilon^{\sum_i c_i(x_i)}, \qquad \mathbf{x} \in \mathcal{X}.$$

Proof: For $\mathbf{x} \neq \mathbf{y}$ such that $\mathbf{y}_{-i} = \mathbf{x}_{-i}$ and $y_i = x_i^+$,

$$\frac{\mu_\mathbf{y}}{\mu_\mathbf{x}} = \frac{\varepsilon^{c_i(x_i^+)}}{\varepsilon^{c_i(x_i)}} = \varepsilon^{d_i(x_i)} = \frac{P_{\mathbf{x},\mathbf{y}}}{P_{\mathbf{y},\mathbf{x}}}.$$

For all other $\mathbf{x} \neq \mathbf{y}$, we have $P_{\mathbf{x},\mathbf{y}} = P_{\mathbf{y},\mathbf{x}} = 0$. Hence, we have that $\mu_\mathbf{x} P_{\mathbf{xy}} = \mu_\mathbf{y} P_{\mathbf{yx}}$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{X}$. ■

Theorem 1: Consider a finite super-modular unimodal game. For $\varepsilon > 0$, the Markov chain Z_t^ε with transition probabilities (16):

(i) keeps the set $\mathcal{Z}$ invariant;

(ii) is time-reversible and ergodic on the set $\mathcal{Z}$;

(iii) has stationary probability distribution

$$\mu^{(\varepsilon)}_\mathbf{x} = \varepsilon^{\sum_i c_i(x_i)}/\mathcal{K}_\varepsilon, \qquad \mathbf{x} \in \mathcal{Z} \tag{18}$$

where $\mathcal{K}_\varepsilon = \sum_{\mathbf{x} \in \mathcal{Z}} \varepsilon^{\sum_i c_i(x_i)}$.

Proof: (i) Let $\mathbf{x}$ in $\mathcal{Z}$ be strategy profile that is reachable from the profile $\mathbf{M}$ by the Markov chain $Z_t^{(0)}$ and let $\mathbf{y}$ in $\mathcal{X}$ be a strategy profile such that $P^{(\varepsilon)}_{\mathbf{x},\mathbf{y}} > 0$. We need to prove that $\mathbf{y}$ belongs to $\mathcal{Z}$. If $\mathbf{y} = \left(x_i^-, \mathbf{x}_{-i}\right)$ for some player $i \in \mathcal{V}$ such that $u_i(\mathbf{y}) \leq u_i(\mathbf{x})$, then it follows that $0 \leq P^{(\varepsilon)}_{\mathbf{x},\mathbf{y}} = 1/2n$ and then $P^0_{\mathbf{x},\mathbf{y}} = 1/2n > 0$, thus implying that the strategy profile $\mathbf{y}$ belongs to $\mathcal{Z}$.

On the other hand if $\mathbf{y} = \left(x_i^+, \mathbf{x}_{-i}\right)$ for some player i in $\mathcal{V}$ we argue as follow. Since $\mathbf{x}$ in $\mathcal{Z}$ is a strategy profile reachable by the Markov chain $Z_t^{(0)}$ from the profile $\mathbf{M}$, we can find a sequence of profiles $\left(\mathbf{x}^{(k)}\right)_{k=0,\dots,l}$ such that $\mathbf{x}^{(0)} = \mathbf{M}$ and $\mathbf{x}^l = \mathbf{x}$ and $P^0_{\mathbf{x}^{(k-1)},\mathbf{x}^{(k)}} > 0$ fo $1 \leq k \leq l$. From (16), this is equivalent to

$$\mathbf{x}^{(k)} = ((x_{i_k}^{(k-1)})^-, \mathbf{x}^{(k-1)}_{-i_k}) \text{ and } u_{i_k}(\mathbf{x}^{(k)}) \leq u_{i_k}(\mathbf{x}^{(k-1)})$$

Let s in $\{1, ..., l\}$ be such that $i_s = i$ and consider $(\mathbf{z}^{(k)})_{0 \leq k < 1}$ such that $\mathbf{z}^{(k)} = \mathbf{x}^{(k)}$ for $k \leq s-1$ and $\mathbf{z}^{(k)} = (x_i^{(k+1)})^+, \mathbf{x}^{(k+1)}_{-i})$ for $k \geq s$. Notice that

$$\begin{aligned} \mathbf{z}^{(k)} &= \left((x_i^{(k+1)})^+, \mathbf{x}^{(k+1)}_{-i}\right) \\ &= \left((x_i^{(k+1)})^+, (x_{i_{k+1}}^{(k)})^-, \mathbf{x}^{(k)}_{-i,-i_{k+1}}\right) \\ &= \left((x_i^{(k)})^+, (x_{i_{k+1}}^{(k)})^-, \mathbf{x}^{(k)}_{-i,-i_{k+1}}\right) \\ &= \left((x_{i_{k+1}}^{(k)})^-, (x_i^{(k)})^+, \mathbf{x}^{(k)}_{-i,-i_{k+1}}\right) \\ &= \left((x_{i_{k+1}}^{(k)})^-, \mathbf{z}^{(k-1)}_{-i_{k+1}}\right) = \left((z_{i_{k+1}}^{(k-1)})^-, \mathbf{z}^{(k-1)}_{-i_{k+1}}\right) \end{aligned}$$

Using this relation and the super-modularity property,

$$0 \leq u_{i_{k+1}}(\mathbf{x}^{(k)}) - u_{i_{k+1}}(\mathbf{x}^{(k+1)}) \leq u_{i_{k+1}}(\mathbf{z}^{(k-1)}) - u_{i_{k+1}}(\mathbf{z}^{(k)})$$

for every $k \geq s-1$. This implies that $P^0_{\mathbf{z}^{(k-1)},\mathbf{z}^{(k)}} > 0$ for every $1 \leq k < l$. Since $\mathbf{z}^{(l-1)} = ((x_i^{(l)})^+, \mathbf{x}^{(l)}_{-i}) = \mathbf{y}$, this proves that the strategy profile $\mathbf{y}$ belongs to $\mathcal{Z}$.

(ii) It follows from Lemma 2 that

$$\varepsilon^{\sum_i c_i(x_i)} P^{(\varepsilon)}_{\mathbf{x},\mathbf{y}} = \varepsilon^{\sum_i c_i(y_i)} P^{(\varepsilon)}_{\mathbf{y},\mathbf{x}} \tag{19}$$

for every two strategy profiles $\mathbf{x}$ and $\mathbf{y}$ in $\mathcal{X}$. This implies that Z_t^ε is time-reversible with respect to the stationary distribution (18). Since the transitions that have positive probability for Z_t^0 have also positive probability for Z_t^ε, all profiles in $\mathcal{Z}$ can be reached from the profile $\mathbf{M}$ by the Markov chain Z_t^ε. Moreover, (19) implies that a transition probability $P^{(\varepsilon)}_{\mathbf{x},\mathbf{y}}$ is positive if and only if the reverse transition $P^{(\varepsilon)}_{\mathbf{y},\mathbf{x}}$ is positive. This implies that $\mathbf{M}$ is reachable from any other profile in $\mathcal{Z}$ and thus we conclude that Z_t^ε is ergodic on $\mathcal{Z}$.

(iii) Ergodicity and (19) imply that, for every $\varepsilon > 0$, the unique stationary distribution of Z_t^ε on $\mathcal{Z}$ is (18). ■

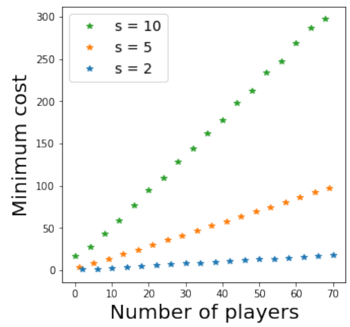

Fig. 1: Minimum cost of sufficient intervention, for Erds-Rnyi random graphs $E(n,p)$ with $p = 4n^{-1}\log n$.

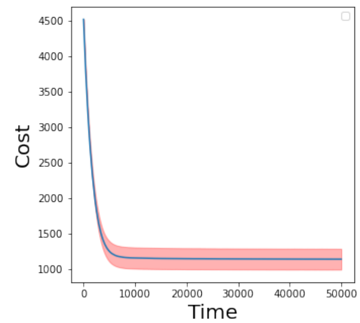

Fig. 2: Cost on the Facebook ego network, with $n = 334$ and action set $\mathcal{A} = \{0, 1, ..., 9\}$, averaged over 1000 runs.

Corollary 3: Consider a finite super-modular unimodal game and, for $\varepsilon > 0$, let $\mu^{(\varepsilon)}$ the Markov chain Z_t^ε with transition probabilities (16). Then,

$$\lim_{\varepsilon \to 0} \mu_{\mathbf{x}}^{(\varepsilon)} = \begin{cases} 1/|\mathcal{O}^*| & \text{if} \quad \mathbf{x} \in \mathcal{O}^* \\ 0 & \text{if} \quad \mathbf{x} \notin \mathcal{O}^* . \end{cases}$$

Proof: As ε vanishes, $\mu^{(\varepsilon)}$ converges to a uniform distribution on the set $\operatorname{argmin}_{\mathbf{x} \in \mathcal{Z}} \sum_i c_i(x_i)$, which coincides with $\mathcal{O}^*$, as we know from Proposition 3 that $\mathcal{Z} = \mathcal{O}$. ■

Theorem 1 and Corollary 3 suggest a simple iterative stochastic algorithm for (7): for small $\varepsilon > 0$, simulate (Z_t^ε) starting with $(Z_t^\varepsilon) = \mathbf{M}$ and keep track of the minimum cost intervention encountered thus far. In fact, Theorem 1(i) ensures that Z_t^ε is a sufficient intervention for all $t \geq 0$, while Theorem 1(iii) and Corollary 3 ensure that in the long run Z_t^ε will be close to $\mathcal{O}^*$ with high probability. Of course, the speed of convergence of Z_t^ε determines the efficiency of the proposed algorithm. However, anytime the algorithm is halted, it returns a sufficient intervention.

V. Numerical simulations

In this section, we present some numerical simulations of the algorithm proposed in Section IV. Specifically, we apply our algorithm to the game presented in the Example 1 of Subsection II-B with n players, action set $\mathcal{A}_i = \{0, 1, ..., s\}$.

First we consider the Erds-Rnyi random graph $E(n,p)$ with n nodes where undirected links between pairs of nodes are present with probability p, independently from one another. We consider the regime $p = 4\frac{\log n}{n}$, leading to a sparse graph that nevertheless remains connected with high probability as n grows large. The cost considered is (9).

We run Z_t^ε with $\varepsilon = 0.01$. Fig. 1 reports the minimum cost encountered in $T = 100n^2$ steps for $s = 2, 5, 10$.

Clearly, a fundamental parameter of our algorithm is the number of steps T. In Fig. 2 we report the evolution of the algorithm. The input to our model is an ego-network $\mathcal{G}$, that is a model of social network formed by an individual, the ego, and all the people with whom the ego has a social connection. The center node u of the ego-network (i.e., the "ego") is included in $\mathcal{G}$, so that $\mathcal{G}$ consists of node u's friends (the "alters"). We have run the simulation 1000 times and plotted the average cost of the profile found so far.

VI. Conclusion

We have formulated an optimal intervention problem for super-modular games with finite action set and proposed a low-complexity iterative algorithm for its solution. Two interesting research directions include the extension to super-modular games with continuous action sets and the study of optimal intervention problems for public good games where an external planner aims to maximize the total engagement.

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