Hardy and BMO spaces on Weyl chambers

Research Article

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https://doi.org/10.1515/forum-2023-0192
Received May 22, 2023; revised June 25, 2023

Abstract: Let W be a finite reflection group associated with a root system R in $\mathbb{R}^d$. Let C_+ denote a positive Weyl chamber distinguished by a choice of R_+, a set of positive roots. We define and investigate Hardy and BMO spaces on C_+ in the framework of boundary conditions given by a homomorphism $\eta \in \operatorname{Hom}(W, \widehat{\mathbb{Z}}_2)$ which attaches the $\pm$ signs to the facets of C_+. Specialized to orthogonal root systems, atomic decompositions in H^1_η and h^1_η are obtained and the duality problem is also treated.

Keywords: Root system, finite reflection group, Weyl chamber, Hardy space, bounded mean oscillation space

MSC 2020: Primary 46E30; secondary 42B30

Communicated by: Christopher D. Sogge

1 Introduction

Chang, Krantz and Stein [6] investigated Hardy spaces on smooth domains in $\mathbb{R}^d$. Later on, Chang, Dafni and Stein [5] substantially enhanced the theory by introducing new distribution spaces appropriate to the Dirichlet and Neumann problems on smooth bounded domains in $\mathbb{R}^d$. Smooth domains considered in [6] included bounded Lipschitz domains, bounded domains with C^∞ boundary, and (unbounded) special Lipschitz domains. For unbounded domains, an important step toward further development of the theory of Hardy spaces was done by Auscher and Russ [2], where strongly Lipschitz domains were considered (to be precise, this was done in a more general setting of elliptic second-order divergence operators); see also [9] for the special Lipschitz domain context and Section 3.3, where a brief account of the development of the theory of Hardy spaces on general subdomains of $\mathbb{R}^d$ is presented.

As an illustrative example, the theory of Hardy spaces on the upper half-spaces

$$\mathbb{R}^d_+ = \{(x', x_d) \colon x' \in \mathbb{R}^{d-1},\ x_d > 0\}, \quad d \geq 2,$$

was outlined in [6, Section 1]; see also [3] in the context of Hardy–Sobolev spaces. This open domain can be seen as a Weyl chamber in the framework of the root system

$$R = \{-e_d, e_d\}, \quad e_d = (0, \dots, 0, 1),$$

which is the simplest root system in $\mathbb{R}^d$. Consequently, the outlined theory was intimately connected with the relevant theory on $\mathbb{R}^d$ through the reflection $(x', x_d) \mapsto (-x', x_d)$, so that the reflection group staying behind was, up to an isomorphism, $\mathbb{Z}_2$.

The present paper includes, in the investigation, reflections coming from an arbitrary finite reflection group acting on $\mathbb{R}^d$. Speaking in a different way, apart from the groups of dilations and rotations which are naturally involved in the theory of Hardy spaces on $\mathbb{R}^d$, we also include finite reflection groups through the symmetries they generate.

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In fact, our study is devoted to the theory of Hardy and bounded mean oscillation spaces (BMO spaces for short) on Weyl chambers. Geometrically, for a given finite reflection group, Weyl chambers are open polyhedral cones in $\mathbb{R}^d$. They appear as connected open components of the set which emerges from $\mathbb{R}^d$ after removing all involved reflection hyperplanes; see Section 2 for details.

In our definition of Hardy and BMO spaces on a distinguished Weyl chamber C_+, we apply a procedure suggested earlier in [22, 23] by one of the authors. The procedure takes into account boundary conditions imposed on the facets (flat parts of the boundary) of C_+. This is encoded by a homomorphism η from W, the involved finite reflection group, into $\widehat{\mathbb{Z}}_2 = \{-1, 1\}$. In fact, our definition agrees with the commonly used definition of Hardy spaces associated with a given self-adjoint operator, for different classes of such operators. Namely, in the case of proposed definition, the resulting Hardy space $H^1_\eta(C_+)$ coincides with $H^1_{-\Delta^+_\eta}(C_+)$, where $-\Delta^+_\eta$ denotes the nonnegative self-adjoint extension of the (minus) Laplacian on C_+, related to the boundary conditions prescribed by η. Notably, for the distinguished homomorphisms triv $\equiv 1$ and sgn $=$ det, $-\Delta^+_{\mathrm{triv}}$ and $-\Delta^+_{\mathrm{sgn}}$ are the Neumann and Dirichlet Laplacians on C_+, respectively, and the η-Laplacians $-\Delta^+_\eta$ are in between; see [23] and Section 3 for details.

Our considerations are initially situated in the setting of a general root system in $\mathbb{R}^d$. Then we narrow the investigation to orthogonal root systems. In this framework, where the involved geometry is simplified and the corresponding Weyl chamber is

$$\mathbb{R}^d_{+,k} = \mathbb{R}^{d-k} \times (0,\infty)^k, \quad 1 \le k \le d,$$

we prove atomic decompositions for functions in $H^1_\eta(\mathbb{R}^d_{+,k})$ and $h^1_\eta(\mathbb{R}^d_{+,k})$. These are the contents of Theorems 4.6 and 4.8; in some sense, these results partly generalize those from [6]. It is worth noting that in these theorems we include characterizations in terms of extensions supported in specific regions depending on η. Accordingly, we continue investigation in this framework discussing η-BMO spaces and proving Theorems 5.5 and 5.6, where also intrinsic characterizations of η-BMO spaces are included. It is interesting to observe a natural duality between obtained results for η-Hardy and η-BMO spaces; see, for instance Theorems 4.6 and 5.5, Theorems 4.8 and 5.6, and Propositions 4.10 and 5.11. Finally, all these results are then braced together in Theorems 5.8 and 5.9, where relevant duality results are proved. We also verify that the introduced spaces are distinct for different η's. It is also worth mentioning that, for the two distinguished homomorphisms triv and sgn, the resulting spaces coincide with the well-known *extension by zero spaces* $H^1_z(\mathbb{R}^d_{+,k})$, $h^1_z(\mathbb{R}^d_{+,k})$, $\mathrm{BMO}_z(\mathbb{R}^d_{+,k})$ and $\mathrm{bmo}_z(\mathbb{R}^d_{+,k})$, and the *restriction spaces* $H^1_r(\mathbb{R}^d_{+,k})$, $h^1_r(\mathbb{R}^d_{+,k})$, $\mathrm{BMO}_r(\mathbb{R}^d_{+,k})$ and $\mathrm{bmo}_r(\mathbb{R}^d_{+,k})$, respectively; see Corollaries 4.9 and 5.7.

The paper is organized as follows. In Section 2, preliminaries on finite reflection groups and associated concepts are gathered. At the beginning of Section 3, a general procedure of defining η-function/distribution spaces on Weyl chambers is described and then η-Hardy spaces are discussed. In Section 4, we focus on orthogonal root systems and prove that the functions from the η-Hardy spaces $H^1_\eta(\mathbb{R}^d_{+,k})$ and $h^1_\eta(\mathbb{R}^d_{+,k})$ possess atomic decompositions. Section 5 is devoted to the introduction and investigation of the η-BMO spaces, $\mathrm{BMO}_\eta(\mathbb{R}^d_{+,k})$ and $\mathrm{bmo}_\eta(\mathbb{R}^d_{+,k})$, and then duality results are discussed.

Notation and terminology. The spaces $L^p(C_+)$, $0 < p \le \infty$, are considered with respect to the Lebesgue measure on C_+; all functions are Lebesgue measurable and complex-valued. By $\mathbb{1}_A$ we denote the characteristic function of a (measurable) subset $A \subset \mathbb{R}^d$. The symbol $\langle \cdot, \cdot \rangle$ means the inner product in $\mathbb{R}^d$, but it is also used for pairing functionals and testing functions; this should not lead to confusion. Cubes are always the Euclidean open cubes with sides parallel to the coordinate axes, and $l(Q)$ stands for the sidelength of a cube Q. Further, $e_1, \dots, e_d$ denote the unit vectors of coordinate axes in $\mathbb{R}^d$. Saying that a function F on $\mathbb{R}^d$ is supported in a subset $A \subset \mathbb{R}^d$ obviously means that $F = 0$ a.e. on the complement A^c. When writing estimates, for nonnegative X and Y we will frequently use the notation $X \lesssim Y$ to indicate that $X \le CY$ with a positive constant C independent of significant quantities. We shall write $X \simeq Y$ when simultaneously $X \lesssim Y$ and $Y \lesssim X$. If N, M are linear spaces with $N \subset M$, and both N and M are equipped with topologies (notably N and M are normed spaces), then writing $N \hookrightarrow M$ indicates that the identity map $\mathrm{Id}\colon N \to M$ is continuous, i.e. N is continuously embedded in M. In the normed/quasi-normed setting, this simply means that $\|x\|_M \lesssim \|x\|_N$ for $x \in N$, and the case of contraction (when the involved C satisfies $C = 1$) will be distinguished by writing $N \hookrightarrow_1 M$.

2 Preliminaries on finite reflection groups

Let R be a root system in the Euclidean space $\mathbb{R}^d$, that is, a finite set of nonzero vectors (called roots) such that for every $\alpha \in R$ we have $\sigma_\alpha(R) = R$, where

$$\sigma_\alpha(x) = x - \frac{2\langle \alpha, x\rangle}{\langle \alpha, \alpha\rangle}\alpha, \quad x \in \mathbb{R}^d,$$

is the orthogonal reflection in $\langle\alpha\rangle^\perp$, the reflection hyperplane orthogonal to α. The dimension of span(R) is called the *rank* of R and is denoted by rank(R). If, in addition, for every $\alpha \in R$ we have $R \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$, then R is called *reduced;* throughout, we assume root systems to be reduced without further mention.

The *finite reflection group* $W = W(R)$ *associated with* R (*reflection group* for short) is the subgroup of $O(\mathbb{R}^d)$ generated by the reflections σ_α, $\alpha \in R$. The set $\mathbb{R}^d \setminus \bigcup_{\alpha\in R}\langle\alpha\rangle^\perp$ splits into an even number (equal to $|W|$) of connected open components called the *Weyl chambers.* W acts (simply transitively) on the set of Weyl chambers, and hence they are mutually congruent. The action follows from the fact that W permutes the reflection hyperplanes (for every $g \in W$, g permutes the set of the reflection hyperplanes by the rule $g \cdot \langle\alpha\rangle^\perp = \langle g\alpha\rangle^\perp$), and so W also permutes the set of connected components of $\mathbb{R}^d \setminus \bigcup_{\alpha\in R}\langle\alpha\rangle^\perp$. A choice of $x_0 \in \mathbb{R}^d$ such that $\langle \alpha, x_0\rangle \neq 0$ for every $\alpha \in R$ gives the partition $R = R_+ \sqcup (-R_+)$, where $R_+ = \{\alpha \in R\colon \langle\alpha, x_0\rangle > 0\}$. Then R_+ is referred to as the *set of positive roots.* The partition distinguishes the chamber

$$C_+ = \{x \in \mathbb{R}^d\colon \langle x, \alpha\rangle > 0 \text{ for all } \alpha \in R_+\},$$

which is called the *positive Weyl chamber.*

Geometrically, as an intersection of a finite number of open half-spaces with supporting hyperplanes passing through the origin, C_+ is an *open polyhedral cone* in $\mathbb{R}^d$. Notably, when $d \geq 2$, C_+ is an example of a *special Lipschitz domain,* i.e., up to a rotation, the domain above the graph of a Lipschitz function defined on $\mathbb{R}^{d-1}$. A comment on rotational invariance of the definition of special Lipschitz domain is probably necessary. In most sources, e.g., [6, p. 304], the definition of special Lipschitz domain does not include a possible rotation; see, however, [20, Section 3.3]. Nevertheless, it is clear that rotation does not change essential properties of such domains. The class of special Lipschitz domains is a subclass of *strongly Lipschitz domains,* i.e. open connected proper subsets of $\mathbb{R}^d$ whose boundaries are covered by a finite union of rotated graphs of Lipschitz maps at most one of them being unbounded. It is also worth mentioning that, equipped with the Lebesgue measure, C_+ is a *space of homogeneous type* in the sense of Coifman and Weiss [7] (with the family of 'balls' being truncated cubes, i.e. sets of the form $Q \cap C_+$, where Q is a cube with center in C_+).

We now recall the concept of simple roots. The *system of simple roots* (*simple system* for short), called *fundamental system* in [11], is the unique subset $\Sigma \subset R_+$ which is a basis of lin$\{\alpha\colon \alpha \in R_+\}$ and each $\alpha \in R_+$ is a linear combination of vectors from this basis with nonnegative coefficients; see [14]. Consequently, $|\Sigma| = \operatorname{rank}(R)$,

$$C_+ = \{x \in \mathbb{R}^d\colon \langle x, \alpha\rangle > 0,\ \alpha \in \Sigma\},$$

and the closure $\overline{C_+}$ has exactly rank(R) *facets,* $\overline{C_+} \cap \langle\alpha\rangle^\perp$, $\alpha \in \Sigma$; further, $\mathcal{W}_\alpha := \langle\alpha\rangle^\perp$ will be called a *wall* of C_+. These facets are closed $(d-1)$-dimensional infinite cones (not necessarily congruent) in the hyperplanes $\langle\alpha\rangle^\perp$ (for $d = 1$ the facet is understood as the single point, the origin). If $|\Sigma| = 1$, i.e. $\Sigma = \{\alpha\}$, then the single facet coincides with $\langle\alpha\rangle^\perp$; otherwise, for $|\Sigma| \geq 2$, these facets are proper cones. If $|\Sigma| = d$, i.e. rank$(R) = d$, then C_+ is a *simplicial cone.* An overview of the variety of possible Weyl chambers in low dimensions can be found in [23, Appendix].

Throughout, Hom$(W, \widehat{\mathbb{Z}}_2)$ will stand for the group of homomorphisms from W to $\widehat{\mathbb{Z}}_2$, where $\widehat{\mathbb{Z}}_2 = \{1, -1\}$ with multiplication. The homomorphisms $\eta \equiv 1$ and $W \ni g \mapsto \det g$ will be denoted by triv and sgn, respectively. Heuristically, $\eta \in \operatorname{Hom}(W, \widehat{\mathbb{Z}}_2)$ serves for assigning signs to the walls of C_+ by setting $\operatorname{sign}_\eta(\mathcal{W}_\alpha) := \eta(\sigma_\alpha)$, $\alpha \in \Sigma$.

Perhaps the simplest example of a root system in $\mathbb{R}^d$ is furnished by an orthogonal system of vectors $E = \{v_1, \ldots, v_m\}$, $1 \leq m \leq d$. Then $R = E \cup (-E)$ is indeed a root system with E as a set of positive roots (it suffices to take $x_0 = v_1 + \cdots + v_m$, in fact, E is a simple system). It is worth noting that, unless $m = 1$, such a system is always *reducible,* i.e. splits into a disjoint union of nonempty mutually orthogonal sets (each such set is itself a root system). Reducible systems are equivalently called *decomposable.*

It is clear that, from the geometrical point of view, root systems in $\mathbb{R}^d$ leading to geometrically congruent positive Weyl chambers should be identified. It is equivalently clear that, for the geometry resulting from a given root system R, responsible is solely the configuration of the corresponding hyperplanes $\langle \alpha \rangle^\perp$, $\alpha \in R$. To avoid repetition in labeling the hyperplanes, in what follows we use positive roots for this labeling (a choice of a system of positive roots is immaterial for labeling). Thus, two root systems R_1 and R_2 in $\mathbb{R}^d$ are *isomorphic* provided there exists an orthogonal mapping in $\mathbb{R}^d$ that permutes the corresponding families of hyperplanes $\{\langle \alpha \rangle^\perp\}_{\alpha \in R_{1,+}}$ and $\{\langle \alpha \rangle^\perp\}_{\alpha \in R_{2,+}}$. Then $C_{1,+}$ and $C_{2,+}$ are congruent and the groups $W(R_1)$ and $W(R_2)$ are isomorphic.

Let R be an orthogonal root system in $\mathbb{R}^d$. Up to a rotation, R is isomorphic to the system $\{\pm e_{j_1}, \dots, \pm e_{j_k}\}$, where $1 \le j_1 < j_2 < \cdots < j_k = d$ and $1 \le k \le d$. Next, up to a permutation of the axes, the latter system is isomorphic to the root system $R_k := \{\pm e_{d-k+1}, \dots, \pm e_d\}$. With the choice

$$R_{k,+} := \{e_{d-k+1}, \dots, e_d\}$$

we denote the corresponding positive Weyl chamber by $\mathbb{R}^d_{+,k}$. Thus $\mathbb{R}^d_{+,k} = \mathbb{R}^{d-k} \times (0, \infty)^k$, and for $k = 1$ we write $\mathbb{R}^d_+$ rather than $\mathbb{R}^d_{+,1}$ to denote the upper half-space in $\mathbb{R}^d$; for $d = k = 1$ we simply write $\mathbb{R}_+$ to denote the half-line $(0, \infty)$.

Concluding, without loss of generality, the consideration of an orthogonal root system in $\mathbb{R}^d$ may be reduced to R_k for some $1 \le k \le d$, and to $\mathbb{R}^d_{+,k}$ as the corresponding positive Weyl chamber. Moreover, $W(R_k) \simeq \widehat{\mathbb{Z}}_2^k$ and the action of any $\varepsilon = (\varepsilon_j)_{j=1}^k \in \widehat{\mathbb{Z}}_2^k$ on $\mathbb{R}^d$ is through

$$x \longrightarrow \varepsilon x = (x_1, \dots, x_{d-k}, \varepsilon_1 x_{d-k+1}, \dots, \varepsilon_k x_d).$$

Consequently, we identify $\mathrm{Hom}(\widehat{\mathbb{Z}}_2^k, \widehat{\mathbb{Z}}_2)$ with $\mathbb{Z}_2^k$, where this time $\mathbb{Z}_2 = \{0, 1\}$ with addition modulo 2. In this identification, a homomorphism $\eta \in \mathbb{Z}_2^k$ of the reflection group represented by $\widehat{\mathbb{Z}}_2^k$ into $\widehat{\mathbb{Z}}_2$ acts through

$$\varepsilon \longrightarrow \varepsilon^\eta := \prod_{j=1}^k \varepsilon_j^{\eta_j} \quad \text{for } \varepsilon = (\varepsilon_j)_{j=1}^k.$$

The trivial homomorphism represented by $(0, \dots, 0)$, which in the general case is denoted by triv, in the case of orthogonal root systems will be denoted by $\mathbf{0}$. On the other hand, the homomorphism represented by $(1, \dots, 1)$, which in the general case is denoted by sgn, here will be denoted by $\mathbf{1}$.

For a comprehensive treatment of the general theory of finite reflection groups, we refer the reader to [14, 16] and [11, Chapter 4].

3 η-function and η-distribution spaces on $\overline{C_+}$

The essential aim of this section is to propose a general procedure of defining η-function/distribution spaces on $\overline{C_+}$. This is done in the following two subsections. Although the procedure is then applied only to Hardy and BMO spaces, we find it reasonable to present the procedure in broader generality due to possible subsequent investigations in other function space frameworks. Throughout, if not otherwise stated, R, W, R_+, C_+, and $\eta \in \mathrm{Hom}(W, \widehat{\mathbb{Z}}_2)$ are fixed.

Since W acts simply transitively on the set of Weyl chambers, the following notion, introduced in [22], makes sense. Namely, a function F on $\mathbb{R}^d$, identified up to a set of Lebesgue measure zero, is called *η-symmetric* provided

$$F(gx) = \eta(g)F(x), \quad g \in W,\ x \in \mathbb{R}^d.$$

Equivalently, F is η-symmetric if and only if $F = \mathcal{A}_\eta F$, where $\mathcal{A}_\eta$ stands for the *η-averaging operator*

$$\mathcal{A}_\eta F(y) = \frac{1}{|W|} \sum_{g \in W} \eta(g) F(gy), \quad y \in \mathbb{R}^d.$$

The *η-extension operator*

$$\mathcal{E}^\eta f(gx) = \eta(g) f(x), \quad x \in C_+,\ g \in W, \tag{3.1}$$

extends functions on C_+, possibly also identified up to a set of Lebesgue measure zero on C_+, to η-symmetric functions on $\mathbb{R}^d \setminus \bigcup_{\alpha\in R_+}\langle\alpha\rangle^\perp$, or on $\mathbb{R}^d$ if we identify functions on $\mathbb{R}^d$ up to a set of Lebesgue measure zero. Notice the useful equality

$$\int_{\mathbb{R}^d} \mathcal{E}^\eta f \cdot F = |W| \int_{C_+} f \cdot (\mathcal{A}_\eta F)|_{C_+} \tag{3.2}$$

for suitable functions f and F.

The concepts of η-averaging and η-extension operators were introduced in [22] and concerned functions identified up to a set of Lebesgue measure zero. When we wish to extend continuous functions on $\overline{C_+}$, i.e. if we want to replace C_+ by $\overline{C_+}$ in (3.1), then we use the well-known fact that the orbit of any $x \in \overline{C_+} \setminus C_+$ under the action of W meets $\overline{C_+} \setminus C_+$ only at x. So, by (3.1), $\mathcal{E}^\eta f$ is well defined on $\mathbb{R}^d$ for f given on $\overline{C_+}$; see Section 3.2, where this remark is applied.

It is clear that $\mathcal{A}_\eta$ maps the spaces $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{D}(\mathbb{R}^d)$, respectively, continuously into itself. Hence we can extend the action of $\mathcal{A}_\eta$ onto $\mathcal{S}'(\mathbb{R}^d)$ or $\mathcal{D}'(\mathbb{R}^d)$ by

$$\langle \mathcal{A}_\eta T, \varphi\rangle = \langle T, \mathcal{A}_\eta \varphi\rangle,$$

where $T \in \mathcal{S}'(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, and analogously in the other case. Obviously, this action is also continuous. Further, $T \in \mathcal{S}'(\mathbb{R}^d)$ or $T \in \mathcal{D}'(\mathbb{R}^d)$ are called *η-symmetric* provided $T = \mathcal{A}_\eta T$.

We are now ready to propose the following procedure. Let $\mathbb{X}$ be a Banach, quasi-Banach or topological vector space of functions, distributions or tempered distributions on $\mathbb{R}^d$. By $\mathbb{X}_\eta$ we shall denote the subspace of $\mathbb{X}$ consisting of all η-symmetric functions/distributions/tempered distributions in $\mathbb{X}$ with the norm/quasi-norm/family of semi-norms, or just the topology, inherited from that in $\mathbb{X}$. It frequently happens that $\mathbb{X}_\eta$ is closed in $\mathbb{X}$; this can always be verified by direct checking.

3.1 General procedure: Function spaces

If $\mathbb{X}$ is a vector space of functions on $\mathbb{R}^d$, then $\mathbb{X}_\eta$ can be also seen as the space of restrictions to $\overline{C_+}$ of η-symmetric functions from $\mathbb{X}$. Equivalently, a function f living on $\overline{C_+}$ belongs to $\mathbb{X}_\eta$ if and only if $\mathcal{E}^\eta f$ belongs to $\mathbb{X}$. Summarizing, we identify, as sets, $\mathbb{X}_\eta$ which is a subspace of $\mathbb{X}$ (of functions on $\mathbb{R}^d$), with the following space of functions on $\overline{C_+}$:

$$\{f : f \text{ is a function on } \overline{C_+}, \text{ and } \mathcal{E}^\eta f \in \mathbb{X}\},$$

and then we set $\|f\|_{\mathbb{X}_\eta} := \|\mathcal{E}^\eta f\|_{\mathbb{X}}$. If functions in $\mathbb{X}$ are identified up to sets of Lebesgue measure zero, we write C_+ rather than $\overline{C_+}$.

Of course, in some cases the procedure does not bring anything new in the following sense. Namely, assume that there is a well-established theory of a ‘category’ of spaces for any open subset of $\mathbb{R}^d$ (e.g., the category of L^p spaces). It may happen that, for any f on C_+ and any η, the η-extension $\mathcal{E}^\eta f$, as the function on $\mathbb{R}^d$, belongs to this category space. For instance, this happens for $\mathbb{X} = L^p(\mathbb{R}^d)$, $0 < p \le \infty$; then we have $\mathbb{X}_\eta = L^p(C_+)$ with norms that differ by the multiplicative constant $|W|$:

$$\|\cdot\|_{L^p(\mathbb{R}^d)_\eta} = |W| \|\cdot\|_{L^p(C_+)}.$$

On the scale of Sobolev spaces, for $\mathbb{X} = W^{1,p}(\mathbb{R}^d)$ or $\mathbb{X} = W^{1,p}_0(\mathbb{R}^d)$, $1 \le p < \infty$, the following assertions hold: $W^{1,p}(\mathbb{R}^d)_{\mathrm{triv}} = W^{1,p}(C_+)$ and $W^{1,p}_0(\mathbb{R}^d)_{\mathrm{sgn}} = W^{1,p}_0(C_+)$, with equivalence of norms. Both results are nontrivial; see [22].

One more comment is necessary in the case when $\mathbb{X}$ is a Banach space of functions on $\mathbb{R}^d$ identified modulo constants (e.g., BMO spaces). Since $\mathcal{E}^\eta \mathbb{1}_{C_+}$ is not a constant function unless $\eta = \mathrm{triv}$, for general η we remain with the definition of $\mathbb{X}_\eta$ as the subspace of $\mathbb{X}$. To be precise, $\mathbb{X}_\eta$ consists of these abstract classes $[f] \in \mathbb{X}$ such that there exists a representative $f_0 \in [f]$ which is an η-symmetric function. The norm in $\mathbb{X}_\eta$ is inherited from that in $\mathbb{X}$.

It is desirable to provide intrinsic characterizations of the spaces $\mathbb{X}_\eta$, identified as spaces of functions (or functions modulo constants) or distributions on $\overline{C_+}$. This means to impose necessary and sufficient conditions on f, a function or a distribution on $\overline{C_+}$, in terms of some objects, equivalent to the condition $\mathcal{E}^\eta f \in \mathbb{X}$. We do this in several occurrences; cf. Theorems 4.6 and 4.8.

3.2 General procedure: Distribution spaces

According to the observations made in the beginning of Section 3.1, we first identify $\mathcal{S}_\eta(\mathbb{R}^d)$, the (closed) subspace of $\mathcal{S}(\mathbb{R}^d)$ of η-symmetric Schwartz functions on $\mathbb{R}^d$, with

$$\mathcal{S}_\eta(\overline{C_+}) = \{f : \overline{C_+} \to \mathbb{C} \colon \mathcal{E}^\eta f \in \mathcal{S}(\mathbb{R}^d)\}.$$

This space can be seen as the space of *η-Schwartz functions* on $\overline{C_+}$. The topology in $\mathcal{S}_\eta(\overline{C_+})$, inherited from $\mathcal{S}(\mathbb{R}^d)$, has an intrinsic description as the topology generated by the family of seminorms

$$p_{\alpha,\beta}(f) = \sup_{x\in C_+} |x^\alpha \partial_\beta f(x)|, \qquad \alpha, \beta \in \mathbb{N}^d.$$

Indeed, $p_{\alpha,\beta}(f) \le p_{\alpha,\beta}(\mathcal{E}^\eta f)$, where the latter is understood as the relevant supremum taken over $\mathbb{R}^d$. In the opposite direction, it is easily seen that $p_{\alpha,\beta}(\mathcal{E}^\eta f)$ is dominated, up to a multiplicative constant, by a finite sum of $p_{\alpha_i,\beta_i}(f)$. With this topology, $\mathcal{S}_\eta(\overline{C_+})$ becomes a Fréchet space.

Obviously, $\mathcal{S}_\eta(\overline{C_+})$ is a subspace of $\mathcal{C}^\infty(\overline{C_+})$, the vector space of $\mathcal{C}^\infty$ functions on Ω such that, for any multi-index α, $\partial^\alpha f$ extends (uniquely) to a continuous function on $\overline{\Omega}$ (this is equivalent to the statement that, for any α, $\partial^\alpha f$ is uniformly continuous on bounded subsets of Ω). Moreover, the boundary behavior of functions from $\mathcal{S}_\eta(\overline{C_+})$ depends on η. Namely, with $\mathcal{W}_\alpha = \langle\alpha\rangle^\perp$, $\alpha \in \Sigma$, as the walls of C_+ with attached signs, i.e. $\operatorname{sign}_\eta(\mathcal{W}_\alpha) = \eta(\sigma_\alpha)$, it is easily seen that for $f \in \mathcal{S}_\eta(\overline{C_+})$ we have the following assertion: if $\operatorname{sign}_\eta(\mathcal{W}_\alpha) = -1$, then f vanishes on the facet $\overline{C_+} \cap \langle\alpha\rangle^\perp$, whereas, if $\operatorname{sign}_\eta(\mathcal{W}_\alpha) = 1$, then $\frac{\partial f}{\partial \vec{n}}$ vanishes on the $(d-1)$-dimensional interior of $\overline{C_+} \cap \langle\alpha\rangle^\perp$. The conclusion is that the spaces $\mathcal{S}_\eta(\overline{C_+})$ labeled by $\eta \in \operatorname{Hom}(W, \widehat{\mathbb{Z}}_2)$ are pairwise different.

Next we define $\mathcal{S}'_\eta(\overline{C_+})$ as the space of continuous linear functionals on $\mathcal{S}_\eta(\overline{C_+})$ with topology induced by the family of mappings $T \mapsto \langle T, \varphi\rangle$, $\varphi \in \mathcal{S}_\eta(\overline{C_+})$, and call it the space of *η-(tempered) distributions on $\overline{C_+}$*. With $\mathcal{S}_\eta(\overline{C_+})$ as a subspace of $\mathcal{S}(\mathbb{R}^d)$, it is obvious that $\mathcal{S}'(\mathbb{R}^d) \hookrightarrow \mathcal{S}'_\eta(\overline{C_+})$.

Following the procedure described in the lines preceding Section 3.1, if $\mathbb{X}$ is a subspace of the space of tempered distributions on $\mathbb{R}^d$ (the case of distributions can be treated analogously), i.e. $\mathbb{X} \subset \mathcal{S}'(\mathbb{R}^d)$, then $\mathbb{X}_\eta$ is defined to consist of η-symmetric members of $\mathbb{X}$. But $\mathbb{X}_\eta$ can be also identified with a subspace of $\mathcal{S}'_\eta(\overline{C_+})$, namely with

$$\{T \in \mathcal{S}'_\eta(\overline{C_+}) \colon \mathcal{E}^\eta T \in \mathbb{X}\},$$

where the extension operator on η-tempered distributions is $\langle \mathcal{E}^\eta T, \varphi\rangle := \langle T, \mathcal{A}_\eta \varphi\rangle$ for $T \in \mathcal{S}'_\eta(\overline{C_+})$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$ (and $\mathcal{A}_\eta\varphi$ is identified with its restriction to $\overline{C_+}$). Moreover, with this identification, the topology in $\mathbb{X}_\eta$ inherited from $\mathbb{X}$ agrees with that in

$$\{T \in \mathcal{S}'_\eta(\overline{C_+}) \colon \mathcal{E}^\eta T \in \mathbb{X}\}$$

inherited from $\mathcal{S}'_\eta(\overline{C_+})$. In what follows, we shall use this parallel description of distributional $\mathbb{X}_\eta$.

3.3 η-Hardy spaces on $\overline{C_+}$

We commence with presenting a brief account of the development of the theory of Hardy spaces on subdomains of $\mathbb{R}^d$.

Jonsson, Sjögren and Wallin [15] provided a construction of (global) Hardy spaces H^p and (local) Hardy spaces h^p for suitable closed subsets of $\mathbb{R}^d$. Additionally, the duals of these spaces were described as the homogeneous and inhomogeneous Lipschitz spaces, respectively. Miyachi [18] introduced H^p spaces on arbitrary open subsets of $\mathbb{R}^d$ together with identification of their duals. The initial definition of these spaces was given in terms of maximal functions. Then an atomic decomposition was established. Chang, Krantz and Stein [6] considered several versions of H^p and h^p spaces over open bounded subsets with smooth boundaries (mostly Lipschitz). Results of [6] were partially significantly improved in [5], where two versions of h^p spaces over open bounded subsets with smooth boundaries (mostly $\mathcal{C}^\infty$) were discussed. In one of these two versions, as the space of test functions, the subspace of $\mathcal{C}^\infty(\overline{\Omega})$ with functions vanishing on the boundary was chosen. Therefore, in some sense, this part of the theory concerned closed domains. Chang [4] established results on the dual spaces of local Hardy spaces discussed in [5]. See also the paper by Auscher and Russ [2] (or rather the first arXiv version of it [1]), where a theory of global and local Hardy spaces (as well as BMO spaces) on strongly Lipschitz domains of $\mathbb{R}^d$ is presented. We refer the reader to Stein [21, Chapter III] and Semmes [19] for a comprehensive treatment of the theory of Hardy spaces.

Following the procedure described above and comments made in Sections 3.1 and 3.2, we consider the spaces $H^p_\eta(\overline{C_+})$ and $h^p_\eta(\overline{C_+})$, $0 < p \le 1$. Here the 'initial spaces' $H^p(\mathbb{R}^d)$ and $h^p(\mathbb{R}^d)$ are equipped with (quasi-)norms $\|\cdot\|_{H^p(\mathbb{R}^d)}$ and $\|\cdot\|_{h^p(\mathbb{R}^d)}$, given by the grand maximal function and its truncated version, respectively, as described in [21, Chapter III] (this requires choosing an appropriate finite collection of seminorms on $\mathcal{S}(\mathbb{R}^d)$, but the resulting space is independent of this choice with equivalence of norms). It is routine to check that if $\mathbb{X}$ is one of the latter spaces, then $\mathbb{X}_\eta$ is a closed subspace of $\mathbb{X}$. In this way, for $p = 1$, $H^1_\eta(C_+)$ and $h^1_\eta(C_+)$ are Banach spaces of functions on C_+, while for $0 < p < 1$, $H^p_\eta(\overline{C_+})$ and $h^p_\eta(\overline{C_+})$, are quasi-Banach spaces of η-(tempered) distributions on $\overline{C_+}$.

It is worth pointing out that, in particular cases of a root system R and a homomorphism η, these spaces appeared in the literature. For example, for the half-space $\mathbb{R}^d_+$ as a Weyl chamber corresponding to the root system $R_1 = \{-e_d, e_d\}$ and $\eta = \mathrm{triv}$ or $\eta = \mathrm{sgn}$, for $p = 1$ the corresponding spaces were denoted in [6] by $H^1_e(\mathbb{R}^d_+)$ and $H^1_o(\mathbb{R}^d_+)$, respectively.

We mention that, in the case $p = 1$, the following two concepts of H^1 spaces, as well as h^1 spaces, on an arbitrary open subset Ω of $\mathbb{R}^d$ were fundamental; namely, the concepts of *restriction space* (to Ω) and *extension space* (by zero outside Ω).

- $H^1_r(\Omega)$ is understood as the space of restrictions to Ω of functions from $H^1(\mathbb{R}^d)$ with norm
$$\|f\|_{H^1_r(\Omega)} := \inf \|F\|_{H^1(\mathbb{R}^d)},$$
where $f = F|_\Omega$ and $F \in H^1(\mathbb{R}^d)$.
- $H^1_z(\Omega)$ is understood as the space of functions on Ω such that their extensions by zero outside Ω are in $H^1(\mathbb{R}^d)$, with norm
$$\|f\|_{H^1_z(\Omega)} := \|F\|_{H^1(\mathbb{R}^d)},$$
where $F \in H^1(\mathbb{R}^d)$ is the aforementioned extension: $F = f$ on Ω, and $F = 0$ on Ω^c.

Analogous definitions of $h^1_r(\Omega)$ and $h^1_z(\Omega)$ follow. Obviously,
$$H^1_z(\Omega) \hookrightarrow_1 H^1_r(\Omega) \quad \text{and} \quad h^1_z(\Omega) \hookrightarrow_1 h^1_r(\Omega).$$
Moreover, both inclusions are strict. For the first one, every function $f \in H^1_z(\Omega)$ satisfies $\int_\Omega f = 0$, whereas this may not happen for $f \in H^1_r(\Omega)$; for the second one, see [5, Proposition 6.4].

When it comes to a comparison of local and global Hardy spaces, then, clearly, $H^p(\mathbb{R}^d) \hookrightarrow_1 h^p(\mathbb{R}^d)$. Consequently, we have
$$H^1_z(C_+) \hookrightarrow_1 h^1_z(C_+), \quad H^1_r(C_+) \hookrightarrow_1 h^1_r(C_+), \quad H^p_\eta(C_+) \hookrightarrow_1 h^p_\eta(C_+).$$

Summing up, the theories of Hardy spaces on open subsets of $\mathbb{R}^d$ are well established. We mention that the differential operator staying behind these theories is the Laplacian; more precisely, including boundary conditions, the Neumann or the Dirichlet Laplacian.

An important line of generalizations of the classical theory of Hardy spaces (and bounded mean oscillation spaces) on subdomains of $\mathbb{R}^d$ is devoted to some classes of nonnegative self-adjoint operators. For instance, Auscher and Russ [2] investigated Hardy spaces for divergence operators on special Lipschitz domains of $\mathbb{R}^d$, and Duong and Yan [12] considered Hardy and BMO spaces associated with operators with heat kernel bounds. See the references in [2, 12] for further results.

Straightforward adaptations of these theories to sets which are Weyl chambers, do not include intimately associated symmetries. One of the aims of this paper is to bring into light the important role of these symmetries.

Let $\{p_t\}_{t>0}$ be the Gauss–Weierstrass kernel on $\mathbb{R}^d$. We consider the maximal operators

$$\mathcal{M}T(x) = \sup_{t>0} |\langle T, p_t(x - \cdot)\rangle|, \quad T \in \mathcal{S}'(\mathbb{R}^d),\ x \in \mathbb{R}^d,$$
$$mT(x) = \sup_{0<t<1} |\langle T, p_t(x - \cdot)\rangle|, \quad T \in \mathcal{S}'(\mathbb{R}^d),\ x \in \mathbb{R}^d.$$

In particular, for $T = F \in L^1(\mathbb{R}^d)$,

$$\mathcal{M}F(x) = \sup_{t>0} |p_t * F(x)|$$

is the *vertical maximal function*, and similarly for mF.

Analogously, let $\{P_t\}_{t>0}$ be the Poisson kernel on $\mathbb{R}^d$, i.e.

$$P_t(x) = c_n t/(t^2 + |x|^2)^{(n+1)/2},$$

and let $\mathcal{M}^*F$ and m^*F be the corresponding maximal operators with replacement of p_t by P_t, for $F \in L^1(\mathbb{R}^d)$.

It is a basic fact in the theory of Hardy spaces on $\mathbb{R}^d$ that, for $0 < p \leq 1$, $\mathcal{M}$ and m give rise to the norms/quasi-norms

$$\|T\|_{H^p,\max} := \|\mathcal{M}T\|_p \quad \text{and} \quad \|T\|_{h^p,\max} := \|mT\|_p$$

on $H^p(\mathbb{R}^d)$ and $h^p(\mathbb{R}^d)$, which are equivalent to the norms/quasi-norms $\|\cdot\|_{H^p}$ and $\|\cdot\|_{h^p}$, respectively.

In [22] (see also [17, 23]), one of the authors introduced a family $\{-\Delta_\eta^+\}$, $\eta \in \operatorname{Hom}(W, \widehat{\mathbb{Z}}_2)$, of (pairwise different) nonnegative self-adjoint extensions of the (minus) Laplacian $-\Delta_{C_+}$ considered with $C^\infty(C_+)$ as the initial domain. The imposed boundary conditions are given by setting the signs $\eta(\sigma_\alpha)$ on the facets $\overline{C}_+ \cap \mathcal{W}_\alpha$ of C_+, $\alpha \in \Sigma$.

It was proved in [22, Corollary 1.2] (see also [23, Proposition 2.1]) that the η-heat kernel on C_+, i.e. the integral kernels of the semigroup $\{e^{-t(-\Delta_\eta^+)}\}_{t>0}$ are

$$p_t^{\eta, C_+}(x, y) = \sum_{g \in W} \eta(g) p_t(gx - y), \quad x, y \in C_+,\ t > 0. \tag{3.3}$$

Similarly for the semigroup $\{e^{-t(-\Delta_\eta^+)^{1/2}}\}_{t>0}$, the integral kernels are (see [22, Theorem 1.1])

$$P_t^{\eta, C_+}(x, y) = \sum_{g \in W} \eta(g) P_t(gx - y), \quad x, y \in C_+,\ t > 0. \tag{3.4}$$

Obviously, (3.3) and (3.4) allow to extend p_t^{η, C_+} and P_t^{η, C_+} to $\mathbb{R}^d \times \mathbb{R}^d$, and frequently we shall consider p_t^{η, C_+} and P_t^{η, C_+} in this way. Observe that $p_t^{\eta, C_+}(x, y)$ is symmetric in x and y, and, for a given $x \in \mathbb{R}^d$,

$$p_t^{\eta, C_+}(x, \cdot) = |W| \mathcal{A}_\eta(p_t(x - \cdot)),$$

and analogously for P_t.

We define the corresponding maximal operators (note that $p_t^{\eta, C_+}(x, \cdot) \in \mathcal{S}_\eta(\mathbb{R}^d)$ for every $x \in \mathbb{R}^d$)

$$\mathcal{M}_\eta T(x) = \sup_{t>0} \big|\langle T, p_t^{\eta, C_+}(x, \cdot)\rangle\big|, \quad T \in \mathcal{S}_\eta'(\mathbb{R}^d),\ x \in C_+, \tag{3.5}$$

and

$$m_\eta T(x) = \sup_{0<t<1} \big|\langle T, p_t^{\eta, C_+}(x, \cdot)\rangle\big|, \quad T \in \mathcal{S}_\eta'(\mathbb{R}^d),\ x \in C_+. \tag{3.6}$$

It is now convenient to separate the cases $p = 1$ and $0 < p < 1$. For $p = 1$ let

$$H^1_{-\Delta_\eta^+}(C_+) = \{f \in L^1(C_+) \colon \|f\|_{H^1_\eta(C_+),\max} := \sup_{t>0} |e^{-t(-\Delta_\eta^+)} f| \in L^1(C_+)\}$$

and

$$H^1_{(-\Delta_\eta^+)^{1/2}}(C_+) = \{f \in L^1(C_+) \colon \|f\|_{H^1_\eta(C_+),\max^*} := \sup_{t>0} |e^{-t(-\Delta_\eta^+)^{1/2}} f| \in L^1(C_+)\},$$

with analogous definitions of $h^1_{-\Delta_\eta^+}(C_+)$ and $h^1_{(-\Delta_\eta^+)^{1/2}}(C_+)$, where taking the supremum reduces to $0 < t < 1$.

Proposition 3.1. *The spaces*

$$H^1_{-\Delta_\eta^+}(C_+) \quad \text{and} \quad H^1_{(-\Delta_\eta^+)^{1/2}}(C_+)$$

coincide with $H^1_\eta(C_+)$ with equivalence of norms. An analogous statement holds for the local spaces $h^1_{-\Delta_\eta^+}(C_+)$ and $h^1_{(-\Delta_\eta^+)^{1/2}}(C_+)$.

Proof. If $p = 1$ and $f \in L^1(C_+)$, then (3.5) takes the form

$$\mathcal{M}_\eta f(x) = \sup_{t>0}\Big|\int_{C_+} p_t^{\eta, C_+}(x,y)f(y)\,dy\Big| = \sup_{t>0}|e^{-t(-\Delta_\eta^+)}f(x)|, \quad x \in C_+.$$

Recall that

$$H^1_\eta(C_+) = \{f \in L^1(C_+) \colon \mathcal{E}^\eta f \in H^1(\mathbb{R}^d)\},$$

with norm $\|f\|_{H^1_\eta(C_+)} = \|\mathcal{E}^\eta f\|_{H^1(\mathbb{R}^d)}$, and similarly for $h^1_\eta(C_+)$.

It is easily seen (cf. (3.2)) that

$$\int_{C_+} p_t^{\eta, C_+}(x,y)f(y)\,dy = \int_{\mathbb{R}^d} p_t(x-y)\mathcal{E}^\eta f(y)\,dy, \quad x \in \mathbb{R}^d, \tag{3.7}$$

and hence

$$\mathcal{M}_\eta f(x) = \mathcal{M}(\mathcal{E}^\eta f)(x), \quad x \in \mathbb{R}^d.$$

Since $\mathcal{E}^\eta f$ is η-symmetric on $\mathbb{R}^d$, it follows that

$$\|\mathcal{M}_\eta f\|_{L^1(C_+)} = \frac{1}{|W|}\|\mathcal{M}(\mathcal{E}^\eta f)\|_{L^1(\mathbb{R}^d)},$$

and hence

$$H^1_\eta(C_+) = \{f \in L^1(C_+) \colon \mathcal{M}_\eta f \in L^1(C_+)\},$$

with norm

$$\|f\|_{H^1_\eta(C_+),\max} = \|\mathcal{M}_\eta f\|_{L^1(C_+)}$$

equivalent to $\|\cdot\|_{H^1_\eta(C_+)}$. The reasoning with replacement of p_t and $\|\cdot\|_{H^1_\eta(C_+),\max}$ by P_t and $\|\cdot\|_{H^1_\eta(C_+),\max^*}$, and using the maximal operator $\mathcal{M}^*_\eta$, is completely analogous.

Similarly, (3.6) takes the form

$$m_\eta f(x) = \sup_{0<t<1}\Big|\int_{C_+} p_t^{\eta, C_+}(x,y)f(y)\,dy\Big| = \sup_{0<t<1}|e^{-t(-\Delta_\eta^+)^{1/2}}f(x)|, \quad x \in C_+.$$

Arguments parallel to the ones just used lead to

$$h^1_\eta(C_+) = \{f \in L^1(C_+) \colon m_\eta f \in L^1(C_+)\} = \{f \in L^1(C_+) \colon m^*_\eta f \in L^1(C_+)\},$$

with norms

$$\|f\|_{h^1_\eta(C_+),\max} = \|m_\eta f\|_{L^1(C_+)} \quad \text{and} \quad \|f\|_{h^1_\eta(C_+),\max^*} = \|m^*_\eta f\|_{L^1(C_+)}$$

equivalent to $\|\cdot\|_{h^1_\eta(C_+)}$. □

Proposition 3.2. *For $0 < p < 1$, we have*

$$H^p_\eta(\overline{C}_+) = \{T \in \mathcal{S}'_\eta(\mathbb{R}^d) \colon \mathcal{M}_\eta T \in L^p(C_+)\},$$

with (quasi-)norm

$$\|f\|_{H^p_\eta(\overline{C}_+),\max} := \|\mathcal{M}_\eta T\|_{L^p(C_+)}$$

equivalent to $\|\cdot\|_{H^p_\eta(\overline{C}_+)}$. Analogous statements hold for the local spaces $h^p_\eta(\overline{C}_+)$, $0 < p < 1$.

Proof. Let $0 < p < 1$. Recall that, after suitable identification,

$$H^p_\eta(\overline{C}_+) = \{T \in \mathcal{S}'_\eta(\overline{C}_+) : \mathcal{E}^\eta T \in H^p(\mathbb{R}^d)\},$$

where $\langle \mathcal{E}^\eta T, \varphi\rangle := \langle T, \mathcal{A}_\eta \varphi|_{C_+}\rangle$ for $T \in \mathcal{S}'_\eta(\overline{C_+})$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$, with an analogous equality for $h^p_\eta(\overline{C}_+)$.

We shall use the following distributional version of (3.7):

$$\langle T, p_t^{\eta, C_+}(x,\cdot)\rangle = \langle \mathcal{E}^\eta T, p_t(x-\cdot)\rangle, \quad T \in \mathcal{S}'_\eta(\overline{C}_+),\ x \in \mathbb{R}^d;$$

notice that the pairings refer to two different distribution spaces. With this equality, we essentially copy the reasoning from the proof of Proposition 3.1 to obtain the claimed result. □

4 Atomic decomposition in H^1_η and h^1_η for orthogonal root systems

In this section, we consider the orthogonal root system R_k, $1 \le k \le d$, described in Section 2. Recall that, in this setting, $\mathbb{R}^d_{+,k} = \mathbb{R}^{d-k} \times (0,\infty)^k$ is the corresponding positive Weyl chamber. For any $e_j \in R_{k,+} = \{e_{d-k+1}, \dots, e_d\}$, we write

$$\langle e_j\rangle^\perp_+ = \{x \in \mathbb{R}^d : x_j > 0\} \quad \text{and} \quad \langle e_j\rangle^\perp_- = \{x \in \mathbb{R}^d : x_j < 0\}$$

for the 'upper' and 'lower' half-spaces with $\langle e_j\rangle^\perp$ as the supporting hyperplane. We also fix a homomorphism $\eta = (\eta_1, \dots, \eta_k) \in \mathbb{Z}_2^k$ and write

$$\mathbb{R}^{\eta,0}_+ = \bigcap_{i:\ \eta_i = 0} \langle e_{d-k+i}\rangle^\perp_+ = \{x \in \mathbb{R}^d : x_{d-k+i} > 0 \text{ when } \eta_i = 0,\ i = 1, \dots, k\},$$

$$\mathbb{R}^{\eta,1}_+ = \bigcap_{i:\ \eta_i = 1} \langle e_{d-k+i}\rangle^\perp_+ = \{x \in \mathbb{R}^d : x_{d-k+i} > 0 \text{ when } \eta_i = 1,\ i = 1, \dots, k\},$$

with the convention that $\mathbb{R}^{\mathbf{1},0}_+ = \mathbb{R}^{\mathbf{0},1}_+ = \mathbb{R}^d$ (then the relevant sets of indices are empty, recall that $\mathbf{0}$ represents the trivial homomorphism, while $\mathbf{1}$ represents the signum, i.e. the determinant homomorphism). Observe that

$$\mathbb{R}^{\mathbf{1},1}_+ = \mathbb{R}^{\mathbf{0},0}_+ = \mathbb{R}^d_{+,k} \quad \text{and} \quad \mathbb{R}^{\eta,0}_+ \cap \mathbb{R}^{\eta,1}_+ = \mathbb{R}^d_{+,k}.$$

The walls of $\mathbb{R}^d_{+,k}$ are

$$\mathcal{W}_i := \langle e_{d-k+i}\rangle^\perp, \quad i = 1, \dots, k,$$

and the signs attached by η to these walls are $\operatorname{sign}_\eta(\mathcal{W}_i) = (-1)^{\eta_i}$. In this way, the family of walls of $\mathbb{R}^{\eta,0}_+$ consists of the walls $\mathcal{W}_i$ (if any) labeled by the '+' sign, and the family of walls of $\mathbb{R}^{\eta,1}_+$ consists of the walls $\mathcal{W}_i$ (if any) which are labeled by the '–' sign.

The essential aims of this section are Theorems 4.6 and 4.8. To prove them, we shall need some auxiliary results: Lemmas 4.1 and 4.4. For brevity, we state and prove the lemmas only in the 'global' case, since the 'local' case is analogous and the proofs require only minor modifications; see Remarks 4.2 and 4.5.

In the statement of the first lemma, the (classical) atomic decomposition is applied. Recall that a function a on $\mathbb{R}^d$ is a (classical) atom if a is supported in a cube Q and satisfies the cancellation condition $\int_Q a = 0$ and the size condition $\|a\|_{L^2(\mathbb{R}^d)} \le |Q|^{-1/2}$. Then $f \in H^1(\mathbb{R}^d)$ if and only if $f = \sum_i \lambda_i a_i$, where a_i are atoms and $\{\lambda_i\} \in \ell^1$; the norm $\|f\|_{H^1(\mathbb{R}^d)}$ is comparable with the infimum of $\sum_i |\lambda_i|$ taken over all such decompositions. Notice that the first part of Lemma 4.1 is just a simple generalization of the known one-dimensional result; see [13, Lemma 7.40]. Until the end of this section, we write $\|\cdot\|_{L^2}$ rather than $\|\cdot\|_{L^2(\mathbb{R}^d)}$.

Lemma 4.1. *Let $F \in H^1(\mathbb{R}^d)$ and let e be a coordinate vector. The following two statements hold:*

(i) *If F is even in $\langle e\rangle^\perp$, i.e. $F = F \circ \sigma_e$, then $F\mathbb{1}_{\langle e\rangle^\perp_+}$ belongs to $H^1(\mathbb{R}^d)$ and admits an atomic decomposition with atoms supported in $\langle e\rangle^\perp_+$, and*

$$\|F\mathbb{1}_{\langle e\rangle^\perp_+}\|_{H^1(\mathbb{R}^d)} \lesssim \|F\|_{H^1(\mathbb{R}^d)}.$$

(ii) *The odd extension of $F|_{\langle e\rangle^\perp_+}$ to $\mathbb{R}^d$, namely $F\mathbb{1}_{\langle e\rangle^\perp_+} - (F\mathbb{1}_{\langle e\rangle^\perp_+}) \circ \sigma_e$, is in $H^1(\mathbb{R}^d)$, and*

$$\|F\mathbb{1}_{\langle e\rangle^\perp_+} - (F\mathbb{1}_{\langle e\rangle^\perp_+}) \circ \sigma_e\|_{H^1(\mathbb{R}^d)} \lesssim \|F\|_{H^1(\mathbb{R}^d)}.$$

Proof. For the first part, observe that if a is an atom, then also

$$\widetilde{a} = \frac{1}{2}(a + a \circ \sigma_e)\mathbb{1}_{\langle e\rangle_+^\perp}$$

is an atom. Indeed, let a be supported in Q. If $Q \subset \langle e\rangle_+^\perp$, then $\widetilde{a} = a/2$, and if $Q \subset \langle e\rangle_-^\perp$, then $\widetilde{a} = a \circ \sigma_e/2$. In the opposite situation, when $Q \cap \langle e\rangle^\perp \neq \emptyset$, then $\widetilde{a}$ is supported in a cube $\widetilde{Q}$ with property $|\widetilde{Q}| = |Q|$, and $\widetilde{Q} \subset \langle e\rangle_+^\perp$. Moreover, in all considered cases, $\|\widetilde{a}\|_{L^2} \le \|a\|_{L^2}$ and $\int_{\widetilde{Q}} \widetilde{a} = 0$. Hence, $\widetilde{a}$ is an atom supported in $\langle e\rangle_+^\perp$ regardless of the localization of Q. Let $F = \sum_{i\in\mathbb{N}} \lambda_i a_i$ be an atomic decomposition of F, even in $\langle e\rangle^\perp$. Hence

$$F\mathbb{1}_{\langle e\rangle_+^\perp} = \sum_{i\in\mathbb{N}} \lambda_i \widetilde{a}_i,$$

and the claim follows.

For the second statement, let $F = \sum_{i\in\mathbb{N}} \lambda_i a_i$ be an atomic decomposition of $F \in H^1(\mathbb{R}^d)$. Set

$$\widetilde{F} = F\mathbb{1}_{\langle e\rangle_+^\perp} - (F\mathbb{1}_{\langle e\rangle_+^\perp}) \circ \sigma_e.$$

Clearly,

$$\widetilde{F} = \sum_{i\in\mathbb{N}} \lambda_i\big(a_i\mathbb{1}_{\langle e\rangle_+^\perp} - (a_i\mathbb{1}_{\langle e\rangle_+^\perp}) \circ \sigma_e\big).$$

Let a, supported in a cube Q, be one of the atoms in the decomposition of F. Define

$$\widetilde{a} = (a\mathbb{1}_{\langle e\rangle_+^\perp} - (a\mathbb{1}_{\langle e\rangle_+^\perp}) \circ \sigma_e).$$

If $Q \subset \langle e\rangle_-^\perp$, then $\widetilde{a} = 0$. On the other hand, if $Q \subset \langle e\rangle_+^\perp$, then $\widetilde{a}$ is just the sum of two atoms supported in Q and $\sigma_e(Q)$, respectively. It remains to consider the case when Q is located on both sides of $\langle e\rangle^\perp$. Notice that then $\widetilde{a}$ is supported in a cube $\widetilde{Q}$ such that $Q \cup \sigma_e(Q) \subset \widetilde{Q}$ and $l(\widetilde{Q}) \le 2l(Q)$. This implies that $|Q| \le |\widetilde{Q}| \le 2^d|Q|$. Therefore,

$$\|\widetilde{a}\|_{L^2} \le 2\|a\|_{L^2} \le 2|Q|^{-1/2} \le 2^{1+d/2}|\widetilde{Q}|^{-1/2}.$$

Hence, $2^{-1-d/2}\widetilde{a}$ is an atom, $\widetilde{F}$ admits an atomic decomposition, and thus it is in $H^1(\mathbb{R}^d)$. Moreover,

$$\|\widetilde{F}\|_{H^1(\mathbb{R}^d)} \lesssim \|F\|_{H^1(\mathbb{R}^d)}.$$

This concludes the proof. □

Remark 4.2. Lemma 4.1 remains valid in the 'local' setting, i.e. when in the statement we replace $H^1(\mathbb{R}^d)$ by $h^1(\mathbb{R}^d)$, and the word 'atom' by the phrase 'local atom' (we present this notion later on in this section). The proof, mutatis mutandis, is a copy of that of Lemma 4.1.

Now we define two types of atoms, which will appear in the atomic decomposition of $H^1_\eta(\mathbb{R}^d_{+,k})$. This definition has its origins in [6, p. 294] (cf. also [1, Section 2.1]).

Definition 4.3. Let I_0, I_1 be two disjoint subsets of $\{1, \dots, d\}$. Fix a function $a(x)$ supported in a cube

$$Q \subset \bigcap_{i\in I_0\cup I_1} \langle e_i\rangle_+^\perp$$

and such that $\|a\|_{L^2} \le |Q|^{-1/2}$. We say that a is
- (I_0,I_1,A)-atom if $4Q \subset \bigcap_{i\in I_1}\langle e_i\rangle_+^\perp$ and $\int_Q a = 0$;
- (I_0,I_1,B)-atom if $2Q \subset \bigcap_{i\in I_1}\langle e_i\rangle_+^\perp$ and $4Q \not\subset \bigcap_{i\in I_1}\langle e_i\rangle_+^\perp$.

If any of the above intersections is taken over the empty set, then it is equal to $\mathbb{R}^d$ by convention.

Notice that if a is an atom of either type, then $\|a\|_{L^1(\mathbb{R}^d)} \le 1$. Notice also that in the case $I_1 = \emptyset$ there are no $(I_0,\emptyset,B)$-atoms, and the $(I_0,\emptyset,A)$-atoms are the classical atoms in $\mathbb{R}^d$ with supporting cubes contained in $\bigcap_{i\in I_0}\langle e_i\rangle_+^\perp$; notably, for $I_0 = \emptyset$ the $(\emptyset,\emptyset,A)$-atoms are the classical atoms in $\mathbb{R}^d$.

Usually, given η, we will consider

$$I_0 = I_{\eta,0} := d - k + \{i\colon \eta_i = 0\} \quad \text{and} \quad I_1 = I_{\eta,1} := d - k + \{i\colon \eta_i = 1\},$$

and then speak about (η, A)- and (η, B)-atoms, respectively. Obviously, for $\eta = \mathbf{0}$,

$$I_{\mathbf{0},0} = \{d-k+1, \dots, d\}, \quad I_{\mathbf{0},1} = \emptyset,$$

and then $(\mathbf{0}, A)$-atoms are just the classical atoms supported in $\mathbb{R}_+^{\mathbf{0},0} = \mathbb{R}_{+,k}^d$, and there are no $(\mathbf{0}, B)$-atoms. Geometrically, $(\eta, A/B)$-atoms are both supported in $\mathbb{R}_{+,k}^d$, but from the perspective of the larger $\mathbb{R}_+^{\eta,1}$, (η, A)-atoms are 'well inside' it, while (η, B)-atoms are 'boundary atoms', i.e. are relatively close to one of the walls of $\mathbb{R}_+^{\eta,1}$. In other words, from the perspective of $\mathbb{R}_{+,k}^d$, only the walls labeled by the '–' sign matter: (η, A)-atoms are supported relatively far from such walls, while (η, B)-atoms are supported relatively close to at least one of such walls (supports of atoms of either type may touch any wall of $\mathbb{R}_{+,k}^d$ labeled by the '+' sign).

In order to prove Theorem 4.6, we will consecutively apply Lemma 4.1 and the key Lemma 4.4 below.

Lemma 4.4. *Let I_0 and I_1 be disjoint subsets of $\{1, \dots, d\}$, $d \geq 1$, such that $|I_0 \cup I_1| < d$, and let*

$$n \in \{1, \dots, d\} \setminus (I_0 \cup I_1).$$

Let $F \in L^1(\mathbb{R}^d)$ be given by

$$F = \sum_{i\in\mathbb{N}} \lambda_i a_i + \mu_i b_i, \quad \text{with} \quad \sum_{i\in\mathbb{N}} |\lambda_i| + |\mu_i| < \infty, \tag{4.1}$$

where the a_i/b_i are (I_0,I_1,A/B)-atoms, respectively. Then we have the decomposition

$$F\mathbb{1}_{\langle e_n\rangle_+^\perp} = \sum_{i\in\mathbb{N}} \widetilde{\lambda_i}\widetilde{a_i} + \widetilde{\mu_i}\widetilde{b_i}, \quad \text{with} \quad \sum_{i\in\mathbb{N}} |\widetilde{\lambda_i}| + |\widetilde{\mu_i}| < \infty,$$

where the $\widetilde{a_i}/\widetilde{b_i}$ are (I_0,$I_1 \cup \{n\}$,A/B)-atoms, respectively. Moreover,

$$\sum_{i\in\mathbb{N}} |\widetilde{\lambda_i}| + |\widetilde{\mu_i}| \lesssim \sum_{i\in\mathbb{N}} |\lambda_i| + |\mu_i|. \tag{4.2}$$

Proof. Fix I_0, I_1, n, and F as in the statement so that F is supported in $U := \bigcap_{i\in I_0\cup I_1} \langle e_i\rangle_+^\perp$ (which is an open polyhedral cone with $|I_0 \cup I_1|$ mutually perpendicular walls when $|I_0 \cup I_1| > 0$, and is equal to $\mathbb{R}^d$ when $I_0 = I_1 = \emptyset$). By (4.1), we have

$$F\mathbb{1}_{\langle e_n\rangle_+^\perp} = \sum_{i\in\mathbb{N}} \lambda_i a_i \mathbb{1}_{\langle e_n\rangle_+^\perp} + \mu_i b_i \mathbb{1}_{\langle e_n\rangle_+^\perp}.$$

Firstly, we consider a_i. For $i \in \mathbb{N}$, let $Q_i \subset U$ be the cube corresponding to the (I_0, I_1, A)-atom a_i. We distinguish four cases depending on the location of Q_i with respect to $\langle e_n\rangle_+^\perp$:

(i) $4Q_i \subset \langle e_n\rangle_+^\perp$.
(ii) $2Q_i \subset \langle e_n\rangle_+^\perp$ and $4Q_i \not\subset \langle e_n\rangle_+^\perp$.
(iii) $Q_i \cap \langle e_n\rangle_+^\perp \neq \emptyset$ and $2Q_i \not\subset \langle e_n\rangle_+^\perp$.
(iv) $Q_i \cap \langle e_n\rangle_+^\perp = \emptyset$.

Notice that the fourth possibility will not be counted, because $a_i \mathbb{1}_{\langle e_n\rangle_+^\perp} \equiv 0$. Hence, we can write

$$\sum_{i\in\mathbb{N}} \lambda_i a_i \mathbb{1}_{\langle e_n\rangle_+^\perp} = \sum_{i\in\mathbb{N}} (\lambda_i^1 a_i^1 + \lambda_i^2 a_i^2 + \lambda_i^3 a_i^3 \mathbb{1}_{\langle e_n\rangle_+^\perp}),$$

where the a_i^m, $m \in \{1, 2, 3\}$, correspond to the m-th case above, and hence $\{\lambda_i^1\} \cup \{\lambda_i^2\} \cup \{\lambda_i^3\} \subset \{\lambda_i\}$.

We repeat these steps for b_i obtaining

$$F\mathbb{1}_{\langle e_n\rangle_+^\perp} = \sum_{i\in\mathbb{N}} (\lambda_i^1 a_i^1 + \lambda_i^2 a_i^2 + \lambda_i^3 a_i^3 \mathbb{1}_{\langle e_n\rangle_+^\perp} + \mu_i^1 b_i^1 + \mu_i^2 b_i^2 + \mu_i^3 b_i^3 \mathbb{1}_{\langle e_n\rangle_+^\perp}),$$

additionally with $\{\mu_i^1\} \cup \{\mu_i^2\} \cup \{\mu_i^3\} \subset \{\mu_i\}$. Letting $I_1' := I_1 \cup \{n\}$, observe that the a_i^1 are (I_0, I_1', A)-atoms and the a_i^2, b_i^1, b_i^2 are (I_0, I_1', B)-atoms. Thus, we are left with $a_i^3 \mathbb{1}_{\langle e_n\rangle_+^\perp}$ and $b_i^3 \mathbb{1}_{\langle e_n\rangle_+^\perp}$. We will show that the latter functions can be decomposed into a sum of (I_0, I_1', B)-atoms with coefficients that sum up to a uniformly majorized constant. For this, rather than to treat each of the two cases separately, we fix a function G supported in a cube $Q \subset U$, such that $Q \cap \langle e_n\rangle_+^\perp \neq \emptyset$, $2Q \not\subset \langle e_n\rangle_+^\perp$, and $\|G\|_{L^2} \leq |Q|^{-1/2}$ (note that $a_i^3 \mathbb{1}_{\langle e_n\rangle_+^\perp}$ and $b_i^3 \mathbb{1}_{\langle e_n\rangle_+^\perp}$ satisfy these conditions). We will verify that these conditions are sufficient to perform the decomposition.

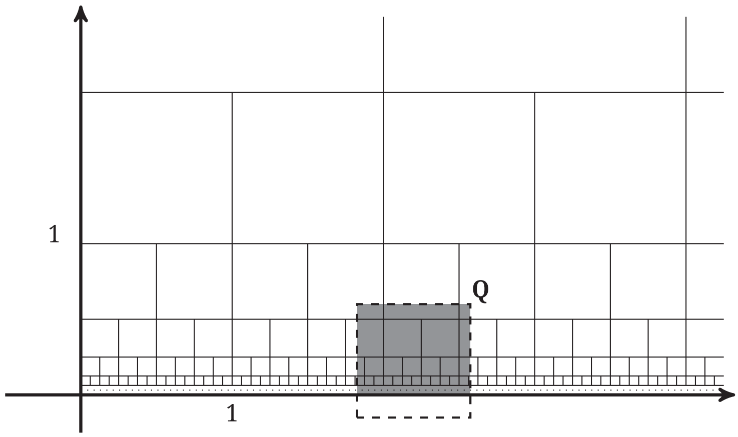

Figure 1: The Whitney decomposition in the case $d = 2$, $U = (0, \infty) \times \mathbb{R}$, and $n = 2$, with an example of the cube Q.

We introduce the Whitney partition of $U \cap \langle e_n \rangle_+^\perp$ (an open polyhedral cone with simultaneously perpendicular $|I_0 \cup I_1| + 1$ walls), relative to the hyperplane $\langle e_n \rangle^\perp$:

$$U \cap \langle e_n \rangle_+^\perp = \bigcup_{m \in \mathbb{Z}} \bigcup_{D \in \mathcal{D}_m} D,$$

where $\mathcal{D}_m$ consists of all dyadic cubes $D \subset U \cap \langle e_n \rangle_+^\perp$ with corners in $2^{-m} \cdot \mathbb{Z}^d$ such that

$$l(D) = \operatorname{dist}(D, \langle e_n \rangle^\perp) = 2^{-m};$$

see Figure 1 for a model situation. Notice that this means that $2D \subset U \cap \langle e_n \rangle_+^\perp$, but $4D \not\subset U \cap \langle e_n \rangle_+^\perp$.

Now we write

$$Q \cap \langle e_n \rangle_+^\perp = \bigcup_{m \in \mathbb{Z}} \bigcup_{j=1}^{j_m} Q \cap D_{mj}, \tag{4.3}$$

where $D_{mj} \in \mathcal{D}_m$ and $Q \cap D_{mj} \neq \emptyset$. In (4.3) and in similar places, equality is meant up to a set of Lebesgue measure zero.

We now establish a bound on j_m which is crucial for further development. Obviously, with given Q, j_m depends on the choice of I_0, I_1 and n, but it is geometrically clear that the 'worst' case is when $|I_0 \cup I_1| = d - 1$ (then the number of walls of $U \cap \langle e_n \rangle_+^\perp$ is d). Hence, estimating j_m, we can simply assume that $U = (0, \infty)^{d-1} \times \mathbb{R}$ and $n = d$, so that $U \cap \langle e_n \rangle_+^\perp = \mathbb{R}_+^d$. We claim that, for any admissible Q with $2^{-m_0-1} < l(Q) \leq 2^{-m_0}$, we have

$$j_m \leq C_d \times \begin{cases} 2^{(m-m_0)(d-1)}, & m \geq m_0, \\ 0, & m < m_0. \end{cases} \tag{4.4}$$

A dilation argument shows that it suffices to prove this estimate for $m_0 = 0$. Thus, let $Q' \subset (0, \infty)^{d-1} \times \mathbb{R}$ be such that

$$Q' \cap \langle e_d \rangle_+^\perp \neq \emptyset, \quad 2Q' \not\subset \langle e_d \rangle_+^\perp, \quad l(Q') = 1,$$

and (4.3) holds. By the geometry of the situation, it is clear that $j_m = 0$ for $m < 0$, and $j_m \leq C_d 2^{m(d-1)}$ for $m \geq 0$ ($C_d = 2^d - 1$ suffices), which proves the claim.

We now decompose $G\mathbb{1}_{\langle e_n \rangle_+^\perp}$ as

$$G\mathbb{1}_{\langle e_n \rangle_+^\perp} = \sum_{m=m_0}^{\infty} \sum_{j=1}^{j_m} \frac{1}{\alpha_{mj}} (\alpha_{mj} G\mathbb{1}_{Q \cap D_{mj}}),$$

where

$$\alpha_{mj} = |D_{mj}|^{-1/2} \|G\mathbb{1}_{Q \cap D_{mj}}\|_{L^2}^{-1}.$$

Observe that, with this choice of α_{mj}, the functions $G_{mj} := \alpha_{mj} G \mathbb{1}_{Q\cap D_{mj}}$ are (I_0, I_1', B)-atoms; this is because G_{mj} is supported in the cube D_{mj} and $\|G_{mj}\|_{L^2} = |D_{mj}|^{-1/2}$. Additionally, by using (4.4), we obtain

$$\begin{aligned}\sum_{m=m_0}^{\infty}\sum_{j=1}^{j_m}|\alpha_{mj}|^{-1} &\le \Big(\sum_{m=m_0}^{\infty}\sum_{j=1}^{j_m}|D_{mj}|\Big)^{1/2}\Big(\sum_{m=m_0}^{\infty}\sum_{j=1}^{j_m}\|G\mathbb{1}_{Q\cap D_{mj}}\|_{L^2}^2\Big)^{1/2}\\ &= \Big(\sum_{m=m_0}^{\infty}\sum_{j=1}^{j_m}|D_{mj}|\Big)^{1/2}\|G\mathbb{1}_{\langle e_n\rangle_+^\perp}\|_{L^2}\\ &\le \Big(\sum_{m=m_0}^{\infty}\sum_{j=1}^{j_m}2^{-dm}\Big)^{1/2}\|G\|_{L^2}\\ &\le (2^d C_d)^{1/2}.\end{aligned}$$

The latter inequality is a consequence of $\|G\|_{L^2} \le |Q|^{-1/2}$, and

$$\sum_{m=m_0}^{\infty}\sum_{j=1}^{j_m}2^{-dm} \le C_d 2^{-m_0(d-1)}\sum_{m=m_0}^{\infty}2^{-m} \le 2^d C_d|Q|.$$

Combining the above shows that $G = \sum_{k\in\mathbb{N}}\tau_k\hat{b}_k$, where the $\hat{b}_k$ are (I_0, I_1', B)-atoms and $\sum_k|\tau_k|$ is bounded by a universal constant not depending on G. Thus, for every $i \in \mathbb{N}$, $a_i^3\mathbb{1}_{\langle e_n\rangle_+^\perp}$ and $b_i^3\mathbb{1}_{\langle e_n\rangle_+^\perp}$ possess the analogous decompositions. This proves the existence of the required decomposition of $F\mathbb{1}_{\langle e_n\rangle_+^\perp}$ with coefficients satisfying (4.2) (which follows from observations made in between) and finishes the proof of the lemma. □

Remark 4.5. Lemma 4.4 remains valid in the 'local' setting, i.e. when in the statement we modify the phrases '$(I_0,I_1$,A/B)-atoms' and '$(I_0,I_1\cup\{n\}$,A/B)-atoms' by adding the word 'local' in front of them (we define these local analogues later on in this section, see Definition 4.7). The proof is a copy of that of Lemma 4.4, but we add the following comments.

Although local $(I_0, I_1, A/B)$-atoms differ from $(I_0, I_1, A/B)$-atoms by limiting first to the case $l(Q) \le 1$ and then admitting local (I_0, I_1, B)-atoms that satisfy merely the condition $l(Q) > 1$ (apart of the basic conditions on the supporting cube and the L^2-norm), the first step, the decomposition of $F\mathbb{1}_{\langle e_n\rangle_+^\perp}$ into a series of summands of six types, relies on distinguishing local atoms according to the location of their supporting cubes. Keeping the notation from the proof of Lemma 4.4, we arrive to the point where $a_i^3\mathbb{1}_{\langle e_n\rangle_+^\perp}$ and $b_i^3\mathbb{1}_{\langle e_n\rangle_+^\perp}$ should be decomposed into a series of local (I_0, I_1', B)-atoms with coefficients that sum up to a uniformly majorized constant. Again, the task reduces to considering a function G supported in a cube $Q \subset U$, such that

$$Q\cap\langle e_n\rangle_+^\perp \ne \emptyset,\quad 2Q \not\subset \langle e_n\rangle_+^\perp,\quad \|G\|_{L^2} \le |Q|^{-1/2},$$

and to decompose it into a series of local (I_0, I_1', B)-atoms with relevant control of coefficients.

We are now ready to state and prove one of the two main results of this section. Notice that implicitly Theorem 4.6 contains the statement that $(\eta, A/B)$-atoms belong to $H^1_\eta(\mathbb{R}^d_{+,k})$. Note, however, that for $\eta = \mathbf{0}$ there are no $(\mathbf{0}, B)$-atoms, and consequently the representation of f in (ii) has only $(\mathbf{0}, A)$-atoms, i.e. $\mu_i \equiv 0$. Also, for $\eta = \mathbf{1}$, the assumptions imposed on the support of F are void (recall that $\mathbb{R}_+^{\mathbf{1},0} = \mathbb{R}^d$ and $\mathbb{R}_+^{\mathbf{1},1} = \mathbb{R}^d_{+,k}$) and (iii) and (iv) are identical. Finally, observe that, since $\mathbb{R}_+^{\eta,0} \subset (\mathbb{R}_+^{\eta,1}\setminus\mathbb{R}^d_{+,k})^c$, for $\eta \ne \mathbf{1}$ condition (iv) looks a priori weaker than condition (iii), but a posteriori both are equivalent.

Theorem 4.6. *Let $f \in L^1(\mathbb{R}^d_{+,k})$ and $\eta \in \mathbb{Z}_2^k$. The following conditions are equivalent:*
(i) $f \in H^1_\eta(\mathbb{R}^d_{+,k})$.
(ii) $f = \sum_{i\in\mathbb{N}}\lambda_i a_i + \mu_i b_i$, *where a_i/b_i are (η,A/B)-atoms, respectively, and $\{\lambda_i\}, \{\mu_i\} \in \ell^1(\mathbb{N})$.*
(iii) *There exists $F \in H^1(\mathbb{R}^d)$ supported in $\mathbb{R}_+^{\eta,0}$ and such that $F|_{\mathbb{R}^d_{+,k}} = f$.*
(iv) *There exists $F \in H^1(\mathbb{R}^d)$ supported in $(\mathbb{R}_+^{\eta,1}\setminus\mathbb{R}^d_{+,k})^c$ and such that $F|_{\mathbb{R}^d_{+,k}} = f$.*
Moreover, the relevant quantities are comparable, namely

$$\|f\|_{H^1_\eta(\mathbb{R}^d_{+,k})} \simeq \inf\sum_{i\in\mathbb{N}}|\lambda_i| + |\mu_i| \simeq \inf_F\|F\|_{H^1(\mathbb{R}^d)},\tag{4.5}$$

where the first infimum is taken over all decompositions as in (ii) *and the second over all F as in* (iii) *or* (iv).

Proof. We fix $f \in L^1(\mathbb{R}^d_{+,k})$ throughout. Recall that the walls of $\mathbb{R}^d_{+,k}$ are $\mathcal{W}_i := \langle e_{d-k+i}\rangle^\perp$, $i = 1, \dots, k$, and $\operatorname{sign}_\eta(\mathcal{W}_i) = (-1)^{\eta_i}$.

"(ii) $\Longrightarrow$ (i)": Assume that f admits a decomposition as in (ii). To check that $f \in H^1_\eta(\mathbb{R}^d_{+,k})$ with

$$\|f\|_{H^1_\eta(\mathbb{R}^d_{+,k})} \lesssim \sum_{i\in\mathbb{N}} |\lambda_i| + |\mu_i|$$

uniformly on f, it suffices to verify that

$$\mathcal{E}^\eta f \in H^1(\mathbb{R}^d) \quad \text{and} \quad \|\mathcal{E}^\eta f\|_{H^1(\mathbb{R}^d)} \lesssim \sum_{i\in\mathbb{N}} |\lambda_i| + |\mu_i|,$$

again uniformly on f. By the linearity and boundedness of $\mathcal{E}^\eta$ from $L^1(\mathbb{R}^d_{+,k})$ to $L^1(\mathbb{R}^d)$, we have

$$\mathcal{E}^\eta f = \sum_{i\in\mathbb{N}} \lambda_i \mathcal{E}^\eta a_i + \mu_i \mathcal{E}^\eta b_i.$$

So, we are reduced to consider the action of $\mathcal{E}^\eta$ on $(\eta, A/B)$-atoms and to check that $\|\mathcal{E}^\eta a\|_{H^1(\mathbb{R}^d)} \lesssim 1$ for (η, A)-atoms, and analogously for (η, B)-atoms.

If a is an (η, A)-atom, then checking the above claim is immediate since $\mathcal{E}^\eta a$ is a sum of 2^k atoms and $\|\mathcal{E}^\eta a\|_{H^1(\mathbb{R}^d)} \lesssim 1$ follows. Now, let b be an (η, B)-atom supported in a cube $Q \subset \mathbb{R}^d_{+,k}$ (so from now on we assume $\eta \neq \mathbf{0}$). We will show that $\mathcal{E}^\eta b$ is a sum of a number of scaled atoms with scaling constant depending only on the dimension d, and hence $\|\mathcal{E}^\eta b\|_{H^1(\mathbb{R}^d)} \lesssim 1$ will follow. This will be done by representing the extension operator $\mathcal{E}^\eta$ as a composition of two extension operators related to contexts of two complementary subroot systems of R_k.

By the definition of (η, B)-atom, we have $4Q \cap \mathcal{W}_j \neq \emptyset$ for some $j \in \{1, \dots, k\}$ with $\eta_j = 1$. Denote by J the set of all such j's. Let $R_J = \{\pm e_{d-k+j} \colon j \in J\}$ with an analogous definition of R_{J^c}, where J^c stands for the complement of J in $\{1, \dots, k\}$; then R_J and R_{J^c} are complementary root (sub)systems of R_k, mutually orthogonal. Then

$$W(R_J) \simeq \prod_{j\in J} \widehat{\mathbb{Z}}_2$$

and similarly for $W(R_{J^c})$, and $W(R_k)$ naturally identifies with the direct product $W(R_J) \oplus W(R_{J^c})$. As a positive Weyl chamber associated with R_J we can choose

$$\mathbb{R}^d_{+,J} = \mathbb{R}^{d-k} \times \prod_{j\in J} Y_j,$$

where $Y_j = (0, \infty)$ for $j \in J$, and $Y_j = \mathbb{R}$ otherwise, and similarly for $\mathbb{R}^d_{+,J^c}$, a positive Weyl chamber associated with R_{J^c}. Clearly, $\mathbb{R}^d_{+,k} \subset \mathbb{R}^d_{+,J}$, the walls of $\mathbb{R}^d_{+,J}$ are $\mathcal{W}_j$, $j \in J$, and all of them are labeled by the '–' sign. Also, η can be viewed as the 'tensor product' of the corresponding homomorphisms on $W(R_J)$ and $W(R_{J^c})$, namely $\eta = \mathbf{1}_J \otimes \eta^c$, where $\mathbf{1}_J = (1, \dots, 1)$ ($|J|$ times) and η^c is obtained from η by removing all entries of η on the j's places, $j \in J$. Due to the product structure of $W(R_k)$ and the tensor product structure of η, it is easily seen that $\mathcal{E}^\eta f = \mathcal{E}^{\eta^c}(\mathcal{E}^{\mathbf{1}_J} f)$ for any f supported in $\mathbb{R}^d_{+,k}$; notice that if f is such a function, then $\mathcal{E}^{\mathbf{1}_J} f$ is supported in $\mathbb{R}^d_{+,J^c}$, so that the composition makes sense.

Let

$$b_J := \mathcal{E}^{\mathbf{1}_J} b = \sum_{g\in W(R_J)} \mathbf{1}_J(g) b \circ g.$$

Clearly, b_J has mean value zero. Set $Q_g = g(Q)$ for $g \in W(R_J)$ and choose a smallest cube Q_J (in the sense of sidelength) such that it contains each Q_g. Notice that the volumes of Q and Q_J are comparable. It follows that b_J is a scaled atom supported in Q_J, where the scaling constant depends only on the dimension d, and so $\|b_J\|_{H^1(\mathbb{R}^d)} \lesssim 1$. Finally, $\mathcal{E}^\eta b = \mathcal{E}^{\eta^c} b_J$ is a sum of $2^{k-|J|}$ scaled atoms, and hence $\|\mathcal{E}^\eta b\|_{H^1(\mathbb{R}^d)} \lesssim 1$.

"(i) $\Longrightarrow$ (iii)": If $f \in H^1_\eta(\mathbb{R}^d_{+,k})$, then $\mathcal{E}^\eta f \in H^1(\mathbb{R}^d)$ and it is even in each $\langle e_{d-k+i}\rangle^\perp$ such that $\eta_i = 0$. Hence, applying Lemma 4.1 (i) consecutively to each such e_{d-k+i} (if any) yields $(\mathcal{E}^\eta f)\mathbb{1}_{\mathbb{R}^{\eta,0}_+} \in H^1(\mathbb{R}^d)$, and this function has the required properties. Notably, we have

$$\|(\mathcal{E}^\eta f)\mathbb{1}_{\mathbb{R}^{\eta,0}_+}\|_{H^1(\mathbb{R}^d)} \lesssim \|\mathcal{E}^\eta f\|_{H^1(\mathbb{R}^d)} = \|f\|_{H^1_\eta(\mathbb{R}^d_{+,k})}.$$

"(iii) $\Longrightarrow$ (ii)": Let $F \in H^1(\mathbb{R}^d)$ be as in (iii). In particular, since F is supported in $\mathbb{R}_+^{\eta,0}$ and the latter region is a special Lipschitz domain, F admits a (classical) atomic decomposition whose atoms are supported in $\mathbb{R}_+^{\eta,0}$; see [6, Theorem 3.3] or the comments following [1, Theorem 1] and referring to the setting of strongly Lipschitz domains. In the notation of Lemma 4.4, this means a decomposition of the form (4.1) into $(I_{\eta,0},\emptyset,\mathrm{A})$-atoms (with $\mu_i \equiv 0$). Now, applying consecutively Lemma 4.4 to each $n \in I_{\eta,1}$ (if any), we obtain an atomic decomposition of $F\mathbb{1}_{\mathbb{R}^d_{+,k}}$ into (η, A)-atoms and (η, B)-atoms. Moreover, by (4.2), we have the relevant inequality in (4.5).

"(iv) $\Longrightarrow$ (iii)": Let $F \in H^1(\mathbb{R}^d)$ be as in (iv). We assume that $\eta \neq \mathbf{1}$. Let $1 \le \eta_{i_1} < \cdots < \eta_{i_r} \le d$, $1 \le r \le d$, be all entries of η which are equal to 1. We apply Lemma 4.1 (ii) to e_{i_1} and F to obtain $F_1 \in H^1(\mathbb{R}^d)$ being the odd extension of $F|_{\langle e_{i_1}\rangle_+^\perp}$. Then we continue this process to e_{i_2} and F_1 to get the odd extension of $F_1|_{\langle e_{i_2}\rangle_+^\perp}$, and so on. The final function F_r belongs to $H^1(\mathbb{R}^d)$ and

$$\|F_r\|_{H^1(\mathbb{R}^d)} \lesssim \|F\|_{H^1(\mathbb{R}^d)}.$$

Moreover, by the assumptions imposed on F and the way F_r was constructed, we have $F_r = (\mathcal{E}^\eta f)\mathbb{1}_{\mathbb{R}_+^{\eta,0}}$. Notably, F_r is supported in $\mathbb{R}_+^{\eta,0}$ and $F_r|_{\mathbb{R}^d_{+,k}} = f$.

"(iii) $\Longrightarrow$ (iv)": This implication is trivial.

Finally, notice that the comparability (4.5) follows by collecting the partial conclusions obtained in the proofs of the four implications. □

We now pass to the discussion of the local Hardy space h^1_η. Recall that a function a on $\mathbb{R}^d$ is a (classical) *local* atom if it is supported in a cube Q satisfying $\|a\|_{L^2} \le |Q|^{-1/2}$ and either $l(Q) > 1$, or $l(Q) \le 1$ and $\int_Q a = 0$. Then $f \in h^1(\mathbb{R}^d)$ if and only if $f = \sum_i \lambda_i a_i$, where the a_i are local atoms and $\{\lambda_i\} \in \ell^1$; the norm $\|f\|_{h^1(\mathbb{R}^d)}$ is comparable with the infimum of $\sum_i |\lambda_i|$ taken over all such decompositions.

We now introduce the local analogue of $(\eta, A/B)$-atoms. Again, if a is a local atom of either type, then $\|a\|_{L^1(\mathbb{R}^d)} \le 1$; also the case $I_0 = I_1 = \emptyset$ is admitted.

Definition 4.7. Let I_0, I_1 be two disjoint subsets of $\{1, \dots, d\}$. Fix a function $a(x)$ supported in a cube

$$Q \subset \bigcap_{i \in I_0 \cup I_1} \langle e_i\rangle_+^\perp$$

and such that $\|a\|_{L^2} \le |Q|^{-1/2}$. We say that a is
- local (I_0, I_1, A)-atom if $l(Q) \le 1$, $4Q \subset \bigcap_{i\in I_1} \langle e_i\rangle_+^\perp$, and $\int_Q a = 0$;
- local (I_0, I_1, B)-atom if either: (i) $l(Q) > 1$, or (ii) $l(Q) \le 1$, $2Q \subset \bigcap_{i\in I_1} \langle e_i\rangle_+^\perp$, and $4Q \not\subset \bigcap_{i\in I_1} \langle e_i\rangle_+^\perp$.

If $I_0 = I_{\eta,0}$ and $I_1 = I_{\eta,1}$ for some $\eta \in \mathbb{Z}_2^k$ (see the comments following Definition 4.3 where $I_{\eta,0}, I_{\eta,1}$ were defined), we refer to these atoms as to the local $(\eta, A/B)$-atoms.

Notice that for $\eta = \mathbf{0}$ there exist local $(\mathbf{0}, B)$-atoms (those with 'large supports'), unlike in the global case. In fact, we will avoid saying 'large' or 'small' supports in the future, because a function a supported in a 'small' cube Q, $l(Q) < 1$, which is placed anywhere in $\mathbb{R}^d_{+,k}$, is a local (η, B)-atom provided that $\|a\|_{L^2} \le 1$, since then it can be viewed as supported in a 'large' cube. Moreover, mind that, for such an a and a set $E \subset \mathbb{R}^d_{+,k}$, the restriction $a\mathbb{1}_E$ (if nontrivial) is still a local (η, B)-atom.

The local version of Theorem 4.6 is as follows.

Theorem 4.8. *Let $f \in L^1(\mathbb{R}^d_{+,k})$ and $\eta \in \mathbb{Z}_2^k$. The following conditions are equivalent:*
(i) $f \in h^1_\eta(\mathbb{R}^d_{+,k})$.
(ii) $f = \sum_{i\in\mathbb{N}} \lambda_i a_i + \mu_i b_i$, *where a_i/b_i are local $(\eta, A/B)$-atoms, and* $\{\lambda_i\}, \{\mu_i\} \in \ell^1(\mathbb{N})$.
(iii) *There exists $F \in h^1(\mathbb{R}^d)$ supported in $\mathbb{R}_+^{\eta,0}$ and such that $F|_{\mathbb{R}^d_{+,k}} = f$.*
(iv) *There exists $F \in h^1(\mathbb{R}^d)$ supported in $(\mathbb{R}_+^{\eta,1} \setminus \mathbb{R}^d_{+,k})^c$ and such that $F|_{\mathbb{R}^d_{+,k}} = f$.*
Moreover, the relevant quantities are comparable, namely

$$\|f\|_{h^1_\eta(\mathbb{R}^d_{+,k})} \simeq \inf \sum_{i\in\mathbb{N}} |\lambda_i| + |\mu_i| \simeq \inf_F \|F\|_{h^1(\mathbb{R}^d)},$$

where the first infimum is taken over all decompositions as in (ii) *and the second over all F as in* (iii) *or* (iv).

Proof. The proof is very similar to the proof of Theorem 4.6, that is why we only comment on the main differences; already at this point, we note that the implication "(iii) $\Longrightarrow$ (iv)" is trivial. We fix $f \in L^1(\mathbb{R}^d_{+,k})$ throughout.

"(ii) $\Longrightarrow$ (i)": It suffices to check that η-extensions of local atoms supported in cubes contained in $\mathbb{R}^d_{+,k}$ are in $h^1_\eta(\mathbb{R}^d_{+,k})$ with uniform norms. If a is a local (η, B)-atom supported in a cube $Q \subset \mathbb{R}^d_{+,k}$ such that $l(Q) > 1$, then $\mathcal{E}^\eta a$ is simply a sum of 2^k local (η, B)-atoms.

"(i) $\Longrightarrow$ (iii)": For this implication, we use a version of Lemma 4.1 (i) for $h^1(\mathbb{R}^d)$, see Remark 4.2, and copy the reasoning from the proof of the analogous implication in Theorem 4.6.

"(iii) $\Longrightarrow$ (ii)": With a local version of Lemma 4.4, see Remark 4.5, we again copy the reasoning from the proof of the analogous implication in Theorem 4.6 using, this time, [6, Theorem 3.10].

"(iv) $\Longrightarrow$ (iii)": We use a version of Lemma 4.1 (ii) for $h^1(\mathbb{R}^d)$, see Remark 4.2, and copy the reasoning from the proof of the analogous implication in Theorem 4.6. □

As a direct consequence of equivalences "(i) $\Longleftrightarrow$ (iii)" in Theorems 4.6 and 4.8, we obtain the following corollary.

Corollary 4.9. *For the Weyl chamber $\mathbb{R}^d_{+,k}$ corresponding to the root system R_k in $\mathbb{R}^d$, $1 \le k \le d$, and for $\eta = \mathbf{0}$ or $\eta = \mathbf{1}$, we have*

$$H^1_{\mathbf{0}}(\mathbb{R}^d_{+,k}) = H^1_z(\mathbb{R}^d_{+,k}) \quad \textit{and} \quad H^1_{\mathbf{1}}(\mathbb{R}^d_{+,k}) = H^1_r(\mathbb{R}^d_{+,k}),$$

with equivalence of norms. Consequently,

$$H^1_{\mathbf{0}}(\mathbb{R}^d_{+,k}) \hookrightarrow H^1_{\mathbf{1}}(\mathbb{R}^d_{+,k})$$

follows with strict inclusion. Analogous claims hold for the local Hardy spaces.

In the case of $k = 1$, i.e. for the root system $R_1 = \{-e_d, e_d\}$ and the half-space $\mathbb{R}^d_+$, the above result for global Hardy spaces is well known (see, e.g., [6, Corollaries 1.6 and 1.8] taken for $p = 1$).

It is also worth mentioning that, alternatively, the proof can be conducted by using results of [1]. Indeed, by [1, Theorem 1 (b2)] applied to $\Omega = \mathbb{R}^d_{+,k}$, we have (see [1] for the definition of $H^1_{z,a}(\mathbb{R}^d_{+,k})$)

$$H^1_{z,a}(\mathbb{R}^d_{+,k}) = H^1_z(\mathbb{R}^d_{+,k}).$$

Hence, by the equivalence "(i) $\Longleftrightarrow$ (ii)" in Theorem 4.6 applied for $\eta = \mathbf{0}$,

$$H^1_{z,a}(\mathbb{R}^d_{+,k}) = H^1_{\mathbf{0}}(\mathbb{R}^d_{+,k}),$$

and the first of the two claimed equalities follows. Similarly, by [1, Theorem 1 (a2)], one obtains (see [1] for the definition of $H^1_{r,a}(\mathbb{R}^d_{+,k})$)

$$H^1_{r,a}(\mathbb{R}^d_{+,k}) = H^1_r(\mathbb{R}^d_{+,k}).$$

Hence, by the equivalence "(i) $\Longleftrightarrow$ (ii)" in Theorem 4.6 applied for $\eta = \mathbf{1}$,

$$H^1_{r,a}(\mathbb{R}^d_{+,k}) = H^1_{\mathbf{1}}(\mathbb{R}^d_{+,k}),$$

and the second of the two claimed equalities follows. For the local Hardy spaces, we have

$$h^1_{\mathbf{0}}(\mathbb{R}^d_{+,k}) = h^1_z(\mathbb{R}^d_{+,k}) \quad \text{and} \quad h^1_{\mathbf{1}}(\mathbb{R}^d_{+}) = h^1_r(\mathbb{R}^d_{+,k}),$$

with equivalence of norms, and the proof of these equalities goes analogously. This time we use [1, Theorem 18 (b2)] and our Theorem 4.8 to achieve the first equality, and [1, Theorem 18 (a)] and Theorem 4.8 to obtain the second one.

The proposition concluding this section establishes relations between the η-Hardy spaces for different η's (cf. also Corollary 4.9).

Proposition 4.10. *If $\eta^{(1)} \le \eta^{(2)}$ (the lexicographical order), then*

$$H^1_{\eta^{(1)}}(\mathbb{R}^d_{+,k}) \hookrightarrow H^1_{\eta^{(2)}}(\mathbb{R}^d_{+,k}).$$

In particular, we have

$$H^1_{\mathbf{0}}(\mathbb{R}^d_{+,k}) \hookrightarrow H^1_\eta(\mathbb{R}^d_{+,k}) \hookrightarrow H^1_{\mathbf{1}}(\mathbb{R}^d_{+,k}),$$

with the relevant inclusions proper when $\eta \ne \mathbf{0}$ or $\eta \ne \mathbf{1}$, respectively. Analogous embeddings and statements hold for the local spaces $h^1_\eta(\mathbb{R}^d_{+,k})$.

Proof. For $\eta^{(1)} \leq \eta^{(2)}$, we have $\mathbb{R}_+^{\eta^{(1)},0} \subset \mathbb{R}_+^{\eta^{(2)},0}$. Hence the claimed continuous embedding is an immediate consequence of the equivalence "(i) $\Longleftrightarrow$ (iii)" in Theorem 4.6 and (4.5). For the local case, we repeat the same argument using Theorem 4.8.

When it comes to proper inclusions, this is obvious for the pair $\mathbf{0}$ and $\eta \neq \mathbf{0}$ (then $H^1_\eta(\mathbb{R}^d_{+,k})$ contains functions with nonvanishing integral over $\mathbb{R}^d_{+,k}$). Assume, a contrario, that the spaces coincide for the pair $\eta \neq \mathbf{1}$ and $\mathbf{1}$. Then, by the inverse mapping theorem, the corresponding norms are equivalent, and this implies an isomorphism (in the category of Banach spaces) of the dual spaces. But this is not the case; see Proposition 5.1 and Theorem 5.8. In the local case, we repeat the latter argument for both pairs, $\mathbf{0}$ and $\eta \neq \mathbf{0}$, and $\eta \neq \mathbf{1}$ and $\mathbf{1}$, and apply Proposition 5.1 and Theorem 5.9. □

In our final remark, we point out that distinguishing η-Hardy spaces (local or global, respectively) on the basis of the differences of the families of $(\eta, A/B)$-atoms (local or global) fails. To see this, given $\eta \in \mathbb{Z}_2^k$, we write $\mathcal{A}_\eta$ and $\mathcal{B}_\eta$ for the families of (global) (η, A)- and (η, B)-atoms, respectively (recall that $\mathcal{B}_\mathbf{0} = \emptyset$). Similarly, in the local case, we write $\mathcal{A}^{\text{loc}}_\eta$ and $\mathcal{B}^{\text{loc}}_\eta$. Although for $\eta \neq \eta'$ we have $\mathcal{A}_\eta \neq \mathcal{A}_{\eta'}$ and $\mathcal{B}_\eta \neq \mathcal{B}_{\eta'}$, and analogously in the local case, this is not sufficient for claiming that the η-Hardy spaces differ for different η's. Indeed, for every η, η' we have

$$\mathcal{A}_\eta \subset H^1_{\eta'}(\mathbb{R}^d_{+,k}),$$

and, when $\eta' \neq \mathbf{0}$,

$$\mathcal{B}_\eta \subset H^1_{\eta'}(\mathbb{R}^d_{+,k}).$$

This is because, for any (classical) atom a supported in $\mathbb{R}^d_{+,k}$, $\mathcal{E}^{\eta'}a$ is a scaled atom for any η', and $\mathcal{E}^{\eta'}a$ is a scaled atom when $\eta' \neq 0$ and $a \in L^2$ is supported in a cube $Q \subset \mathbb{R}^d_{+,k}$. In the local case, we have

$$\mathcal{A}^{\text{loc}}_\eta \cup \mathcal{B}^{\text{loc}}_\eta \subset h^1_{\eta'}(\mathbb{R}^d_{+,k})$$

for every η, η' (in case $\eta' = \mathbf{0}$, for $a \in \mathcal{B}_\eta$, $\mathcal{E}^{\eta'}a$ is a scaled local (η', A)-atom, and hence in $h^1_{\eta'}(\mathbb{R}^d_{+,k})$).

5 η-BMO spaces on Weyl chambers and duality

In the first part of this section, similarly to Section 3, let R, W, R_+, C_+, and $\eta \in \operatorname{Hom}(W, \widehat{\mathbb{Z}}_2)$ be fixed. Recall that $F \in L^1_{\text{loc}}(\mathbb{R}^d)$ belongs to $\operatorname{bmo}(\mathbb{R}^d)$ if

$$\|F\|_{\operatorname{bmo}(\mathbb{R}^d)} := \sup_{l(Q)<1} \frac{1}{|Q|} \int_Q |F - F_Q| + \sup_{l(Q)\geq 1} \frac{1}{|Q|} \int_Q |F| < \infty,$$

where F_Q stands for the mean value of F over Q. Analogously, $F \in L^1_{\text{loc}}(\mathbb{R}^d)$ is in $\operatorname{BMO}(\mathbb{R}^d)$ provided

$$\|F\|_{\operatorname{BMO}(\mathbb{R}^d)} := \sup_Q \frac{1}{|Q|} \int_Q |F - F_Q| < \infty;$$

here we identify functions modulo constants. It is well known that the norm $\|\cdot\|_{\operatorname{BMO}(\mathbb{R}^d)}$ is equivalent to $\|\cdot\|_{*,\operatorname{BMO}(\mathbb{R}^d)}$, where

$$\|F\|_{*,\operatorname{BMO}(\mathbb{R}^d)} := \sup_Q \inf_{c\in\mathbb{C}} \frac{1}{|Q|} \int_Q |F - c|.$$

More precisely,

$$\|\cdot\|_{*,\operatorname{BMO}(\mathbb{R}^d)} \leq \|\cdot\|_{\operatorname{BMO}(\mathbb{R}^d)} \leq 2\|\cdot\|_{*,\operatorname{BMO}(\mathbb{R}^d)}.$$

Similarly, for $F \in \operatorname{bmo}(\mathbb{R}^d)$,

$$\|F\|_{*,\operatorname{bmo}(\mathbb{R}^d)} \leq \|F\|_{\operatorname{bmo}(\mathbb{R}^d)} \leq 2\|F\|_{*,\operatorname{bmo}(\mathbb{R}^d)},$$

where

$$\|F\|_{*,\operatorname{bmo}(\mathbb{R}^d)} := \sup_{l(Q)<1} \inf_{c\in\mathbb{C}} \frac{1}{|Q|} \int_Q |F - c| + \sup_{l(Q)\geq 1} \frac{1}{|Q|} \int_Q |F|.$$

Following the procedure described in Section 3.1, we define

$$\mathrm{bmo}_\eta(C_+) = \{f \in L^1_{\mathrm{loc}}(C_+)\colon \mathcal{E}^\eta f \in \mathrm{bmo}(\mathbb{R}^d)\},$$

with norm $\|f\|_{\mathrm{bmo}_\eta(C_+)} = \|\mathcal{E}^\eta f\|_{\mathrm{bmo}(\mathbb{R}^d)}$. Analogously,

$$\mathrm{BMO}_\eta(C_+) = \{f \in L^1_{\mathrm{loc}}(C_+)\colon \mathcal{E}^\eta f \in \mathrm{BMO}(\mathbb{R}^d)\},$$

with norm $\|f\|_{\mathrm{BMO}_\eta(C_+)} = \|\mathcal{E}^\eta f\|_{\mathrm{BMO}(\mathbb{R}^d)}$, where for $\eta = \mathrm{triv}$ we identify functions on C_+ modulo constants, and for $\eta \neq \mathrm{triv}$ elements of $\mathrm{BMO}_\eta(C_+)$ are genuine functions, not equivalence classes modulo constants.

The above dichotomy is also reflected in the definition of the extension by zero of global BMO functions on open subdomains of $\mathbb{R}^d$. Namely, specified to C_+ as an open subdomain, $\mathrm{BMO}_z(C_+)$ is identified with the space of these (genuine) functions on C_+ such that f extended by zero outside C_+, and denoted by the same symbol, is in $\mathrm{BMO}(\mathbb{R}^d)$, with inherited norm

$$\|f\|_{\mathrm{BMO}_z(C_+)} = \|f\|_{\mathrm{BMO}(\mathbb{R}^d)};$$

see, e.g., [1, Section 2.3]. Unlikely, the concept of definition of $\mathrm{BMO}_r(C_+)$ is unchanged: $\mathrm{BMO}_r(C_+)$ (modulo constants) consists of restrictions to C_+ of BMO functions on $\mathbb{R}^d$ with norm

$$\|f\|_{\mathrm{BMO}(C_+)} := \inf_F \|F\|_{\mathrm{BMO}(\mathbb{R}^d)},$$

where the infimum is taken over all F such that $F|_{C_+} = f$. Obviously, $\mathrm{BMO}_z(C_+) \subset \mathrm{BMO}_r(C_+)$ with a continuous embedding in the sense that $[F] \in \mathrm{BMO}_r(\mathbb{R}^d)$ for $F \in \mathrm{BMO}_z(\mathbb{R}^d)$ and

$$\|[F]\|_{\mathrm{BMO}(\mathbb{R}^d)} \lesssim \|F\|_{\mathrm{BMO}(\mathbb{R}^d)}.$$

A similar convention is used in analogous situations when the space to which the embedding is made consists of abstract classes. Moreover, keeping this convention in mind, we shall use the symbols $\hookrightarrow$ and $\hookrightarrow_1$ in their previous sense.

Such a dichotomy in definitions does not occur in the case of local BMO spaces. We define $\mathrm{bmo}_z(C_+)$ and $\mathrm{bmo}_r(C_+)$ as the spaces of (genuine) functions on C_+ by the same principles as above, including the definitions of the norms. It is then obvious that $\mathrm{bmo}_z(C_+) \hookrightarrow_1 \mathrm{bmo}_r(C_+)$.

It is worth pointing out that, in particular cases of a root system R and a homomorphism η, these spaces appeared in the literature. For example, for the half-space $\mathbb{R}^d_+$ as a Weyl chamber corresponding to the root system $R_1 = \{-e_d, e_d\}$ and $\eta = \mathrm{triv} = \mathbf{0}$ or $\eta = \mathrm{sgn} = \mathbf{1}$, the corresponding spaces were denoted in [10], as $\mathrm{BMO}_e(\mathbb{R}^d_+)$ and $\mathrm{BMO}_o(\mathbb{R}^d_+)$, respectively. Unfortunately, in [10] it was not pointed out that $\mathrm{BMO}_o(\mathbb{R}^d_+)$ and $\mathrm{BMO}_z(\mathbb{R}^d_+)$ should be treated as spaces of genuine functions, not equivalence classes modulo constants.

When it comes to a comparison of local and global BMO spaces, then, clearly, $\mathrm{bmo}(\mathbb{R}^d) \hookrightarrow \mathrm{BMO}(\mathbb{R}^d)$ (we have $\hookrightarrow_1$ for the norms $\|\cdot\|_{*,\mathrm{bmo}}$ and $\|\cdot\|_{*,\mathrm{BMO}}$). Consequently,

$$\mathrm{bmo}_z(C_+) \hookrightarrow \mathrm{BMO}_z(C_+), \quad \mathrm{bmo}_r(C_+) \hookrightarrow \mathrm{BMO}_r(C_+), \quad \mathrm{bmo}_\eta(C_+) \hookrightarrow \mathrm{BMO}_\eta(C_+).$$

The spaces $\mathrm{bmo}_\eta(C_+)$, and thus also $\mathrm{BMO}_\eta(C_+)$, are nontrivial in the sense that they contain unbounded functions. To see this, recall that

$$F(x) = \log\frac{1}{|x|} \in \mathrm{BMO}(\mathbb{R}^d).$$

Moreover, also

$$\Psi(x) = \max\Big\{\log\frac{1}{|x|}, 0\Big\} \in \mathrm{BMO}(\mathbb{R}^d).$$

Since in addition $\Psi \in L^1(\mathbb{R}^d)$, it follows that $\Psi \in \mathrm{bmo}(\mathbb{R}^d)$. Now, choose $x_0 \in C_+$ such that $\mathrm{dist}(x_0, \partial C_+) > 1$. Then

$$\psi(x) = \mathbb{1}_{C_+}(x)\Psi(x - x_0), \quad x \in C_+,$$

is a function on C_+ such that $\mathcal{E}^\eta\psi \in \mathrm{bmo}(\mathbb{R}^d)$ ($\mathcal{E}^\eta\psi$ should be seen as a sum of $\mathrm{bmo}(\mathbb{R}^d)$ functions, which are reflections of ψ along a number of hyperplanes), and hence $\psi \in \mathrm{bmo}_\eta(C_+)$. To illustrate this general discussion, we present in dimension one an example of an odd unbounded function that belongs to bmo($\mathbb{R}$), and thus also to BMO($\mathbb{R}$) (surprisingly, we could not find such an example in the literature):

$$\Phi(x) = \operatorname{sgn} x \cdot \mathbb{1}_{[1,3]}(|x|) \cdot \log \frac{1}{||x| - 2|}, \quad x \in \mathbb{R}.$$

We now come to the discussion of the question if the η-BMO spaces are different for different η's. In dimension one, it is well known that, for F as above,

$$F_o(x) := \operatorname{sgn} x \cdot F(x) \notin \mathrm{BMO}(\mathbb{R}).$$

Hence, $\mathrm{BMO}_o(\mathbb{R}_+)$ and $\mathrm{BMO}_e(\mathbb{R}_+)$ differ. Replacing F_o by $\mathbb{1}_{[-1,1]}F_o$ gives the same conclusion in the local case, $\mathrm{bmo}_o(\mathbb{R}_+) \neq \mathrm{bmo}_e(\mathbb{R}_+)$ (note, however, that not only $F_o \notin \mathrm{bmo}(\mathbb{R})$ but also $F_o \notin \mathrm{BMO}(\mathbb{R})$). The same idea works in higher dimensions. To be precise, for $\eta' \neq \eta = \mathbf{0}$, $\mathrm{BMO}_{\eta'}(C_+)$ and $\mathrm{BMO}_{\mathbf{0}}(C_+)$ will differ in the sense that, for some f, $[f] \in \mathrm{BMO}_{\mathbf{0}}(C_+)$, but $f \notin \mathrm{BMO}_{\eta'}(C_+)$.

Proposition 5.1. *The spaces* $\mathrm{BMO}_\eta(C_+)$, $\eta \in \mathrm{Hom}(W, \widehat{\mathbb{Z}}_2)$, *are pairwise different. The same claim holds for the spaces* $\mathrm{bmo}_\eta(C_+)$.

Proof. We can assume that $d \geq 2$. Let $\eta^{(1)} \neq \eta^{(2)}$, say, $\eta^{(1)}(\sigma_\alpha) = 1$ and $\eta^{(2)}(\sigma_\alpha) = -1$ for some $\alpha \in \Sigma$. Take x_0, 'well inside' the $(d-1)$-dimensional interior of the facet $\mathcal{F}_\alpha = \overline{C_+} \cap \langle \alpha \rangle^\perp$, i.e. in the distance not smaller than 2 from the origin and from any of the $(d-2)$-dimensional faces (if any) of C_+, i.e. intersections of $\mathcal{F}_\alpha$ with any other different wall of C_+. Consider $\hat{\psi}$, the restriction to C_+ of $\psi(x) = \Psi(x - x_0)$, $x \in \mathbb{R}^d$, the latter function being in $\mathrm{bmo}(\mathbb{R}^d)$ (where Ψ is as defined earlier). Then $\mathcal{E}^{\eta^{(1)}}\hat{\psi} \in \mathrm{bmo}(\mathbb{R}^d)$. This is because $\mathcal{E}^{\eta^{(1)}}\hat{\psi}$ is a linear combination, with ± 1 coefficients, of shifts of Ψ, with ψ as one of the summands; here the radiality of Ψ plays a role. On the other hand, $\mathcal{E}^{\eta^{(2)}}\hat{\psi} \notin \mathrm{BMO}(\mathbb{R}^d)$. This is because $\mathcal{E}^{\eta^{(2)}}\hat{\psi}$ is a linear combination (with ± 1 coefficients) of shifts of the rotated function

$$\tilde{\psi}(x) = \psi(x)\mathbb{1}_{\langle \alpha \rangle_+^\perp} - \psi(x)\mathbb{1}_{\langle \alpha \rangle_-^\perp}, \quad x \in \mathbb{R}^d,$$

i.e. the function ψ broken along the hyperplane $\langle \alpha \rangle^\perp$; here $\langle \alpha \rangle_\pm^\perp$ stand for the 'upper' and 'lower' half-spaces with $\langle \alpha \rangle^\perp$ as the supporting hyperplane (C_+ is contained in $\langle \alpha \rangle_+^\perp$). The oscillations of $\mathcal{E}^{\eta^{(2)}}\hat{\psi}$, which agrees on $B(x_0, 1)$ with $\tilde{\psi}$, over small cubes centered at x_0 are unbounded.

This proves the two claims of the proposition. □

5.1 The case of orthogonal root systems

We now leave the general case and specify considerations to orthogonal root systems. Similarly to the case of Hardy spaces, we shall characterize $\mathrm{BMO}_\eta(\mathbb{R}^d_{+,k})$ and $\mathrm{bmo}_\eta(\mathbb{R}^d_{+,k})$ for the orthogonal root systems R_k; these are the contents of Theorems 5.5 and 5.6. Hence, in this and the next subsection, we assume $1 \leq k \leq d$ to be fixed. Also, we make use of the notation introduced in the previous section.

As mentioned before, we work simultaneously with function spaces and with function spaces modulo constants. In order to keep the notation transparent, we simply write $F \in \mathrm{BMO}(\mathbb{R}^d)$ even if F is a function (and not an abstract class modulo constants). This is a short form of saying that $F \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $[F] \in \mathrm{BMO}(\mathbb{R}^d)$. The same comment applies to $\mathrm{BMO}_{\mathbf{0}}(\mathbb{R}^d_{+,k})$ and $\mathrm{BMO}_z(\mathbb{R}^d_{+,k})$.

For the proof of the next theorem, we shall need an auxiliary lemma which, in some sense, is a dual result to Lemma 4.1. Notice that Lemma 5.2 (i) is true for abstract classes modulo constants as well.

Lemma 5.2. *Let e be a coordinate vector and let* $F \in \mathrm{BMO}(\mathbb{R}^d)$.

(i) *The even (with respect to $\langle e \rangle^\perp$) extension of $F|_{\langle e \rangle_+^\perp}$ to $\mathbb{R}^d$, namely $F\mathbb{1}_{\langle e \rangle_+^\perp} + (F\mathbb{1}_{\langle e \rangle_+^\perp}) \circ \sigma_e$, is in* $\mathrm{BMO}(\mathbb{R}^d)$, *and*

$$\|F\mathbb{1}_{\langle e \rangle_+^\perp} + (F\mathbb{1}_{\langle e \rangle_+^\perp}) \circ \sigma_e\|_{*,\mathrm{BMO}(\mathbb{R}^d)} \leq 2\|F\|_{*,\mathrm{BMO}(\mathbb{R}^d)}.$$

(ii) *If F is odd with respect to $\langle e \rangle^\perp$, then $F\mathbb{1}_{\langle e \rangle_+^\perp} \in$* $\mathrm{BMO}(\mathbb{R}^d)$ *and*

$$\|F\mathbb{1}_{\langle e \rangle_+^\perp}\|_{\mathrm{BMO}(\mathbb{R}^d)} \lesssim \|F\|_{\mathrm{BMO}(\mathbb{R}^d)}.$$

Proof. (i) Fix $F \in \mathrm{BMO}(\mathbb{R}^d)$ and let Q be a cube in $\mathbb{R}^d$. Our aim is to prove

$$\inf_{c\in\mathbb{C}} \frac{1}{|Q|}\int_Q |F\mathbb{1}_{\langle e\rangle_+^\perp} + (F\mathbb{1}_{\langle e\rangle_+^\perp})\circ\sigma_e - c| \le 2\|F\|_{*,BMO(\mathbb{R}^d)}.$$

The only nontrivial case is when $Q \cap \langle e\rangle^\perp \neq \emptyset$. We set $Q_\pm = Q \cap \langle e\rangle_\pm^\perp$. For any $c \in \mathbb{C}$, we obtain

$$\int_Q |F\mathbb{1}_{\langle e\rangle_+^\perp} + (F\mathbb{1}_{\langle e\rangle_+^\perp})\circ\sigma_e - c| = \int_{Q_+} |F - c| + \int_{\sigma_e(Q_-)} |F - c|.$$

Let $\widehat{Q} \subset \langle e\rangle_+^\perp$ be the cube such that $Q_+ \cup \sigma_e(Q_-) \subset \widehat{Q}$ and $|\widehat{Q}| = |Q|$. Thus,

$$\inf_{c\in\mathbb{C}} \frac{1}{|Q|}\int_Q |F\mathbb{1}_{\langle e\rangle_+^\perp} + (F\mathbb{1}_{\langle e\rangle_+^\perp})\circ\sigma_e - c| \le \inf_{c\in\mathbb{C}} \frac{2}{|\widehat{Q}|}\int_{\widehat{Q}} |F - c| \le 2\|F\|_{*,\mathrm{BMO}(\mathbb{R}^d)}.$$

(ii) Fix $F \in \mathrm{BMO}(\mathbb{R}^d)$ such that $F\circ\sigma_e = -F$. Let Q be a cube in $\mathbb{R}^d$. We shall check that

$$\inf_{c\in\mathbb{C}} \frac{1}{|Q|}\int_Q |F\mathbb{1}_{\langle e\rangle_+^\perp} - c| \lesssim \|F\|_{\mathrm{BMO}(\mathbb{R}^d)}.$$

As before, we assume $Q \cap \langle e\rangle^\perp \neq \emptyset$ and we use the same notation. Observe that

$$\inf_{c\in\mathbb{C}} \frac{1}{|Q|}\int_Q |F\mathbb{1}_{\langle e\rangle_+^\perp} - c| \le \inf_{c\in\mathbb{C}} \frac{1}{|Q|}\int_Q |F - c| + \frac{1}{|Q|}\int_{Q_-} |F| \le \|F\|_{*,\mathrm{BMO}(\mathbb{R}^d)} + \frac{1}{2|Q|}\int_{Q_-\cup\sigma_e(Q_-)} |F|.$$

Let $\widehat{Q} \subset \mathbb{R}^d$ be a cube symmetric in $\langle e\rangle^\perp$ such that $Q_- \cup \sigma_e(Q_-) \subset \widehat{Q}$ and $|\widehat{Q}| < 2^d|Q|$. Hence,

$$\frac{1}{2|Q|}\int_{Q_-\cup\sigma_e(Q_-)} |F| \le \frac{2^{d-1}}{|\widehat{Q}|}\int_{\widehat{Q}} |F| \le 2^{d-1}\|F\|_{\mathrm{BMO}(\mathbb{R}^d)},$$

where in the second inequality we used the fact that $F_{\widehat{Q}} = 0$. This concludes the proof. □

We shall need the fact (which is certainly a folklore result), that the definition of the $\mathrm{bmo}(\mathbb{R}^d)$-norm does not depend on choosing 1 as a breaking point, i.e. 1 can be replaced by an arbitrary positive number with equivalence of norms. For $a > 0$ and $F \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, we introduce the quantity

$$\|F\|_{*,\mathrm{bmo}_a(\mathbb{R}^d)} := \sup_{l(Q)<a} \inf_{c\in\mathbb{C}} \frac{1}{|Q|}\int_Q |F - c| + \sup_{l(Q)\ge a} \frac{1}{|Q|}\int_Q |F|$$

and define

$$\mathrm{bmo}_a(\mathbb{R}^d) = \{F\colon \|F\|_{*,\mathrm{bmo}(\mathbb{R}^d);a} < \infty\}$$

so that $\mathrm{bmo}_1(\mathbb{R}^d)$ is just $\mathrm{bmo}(\mathbb{R}^d)$.

The proof of the following technical result is simple, but we provide details for completeness.

Proposition 5.3. *Let $0 < a, b < \infty$. Then the spaces $\mathrm{bmo}_{*,a}(\mathbb{R}^d)$ and $\mathrm{bmo}_{*,b}(\mathbb{R}^d)$ coincide with equivalence of norms (and implicit constants depending on a and b).*

Proof. We explain the idea of the proof for $a = 1$ and $b = 2$; generalization to arbitrary $0 < a < b < \infty$ is straightforward. For notational simplicity, we let

$$|F|_Q = \frac{1}{|Q|}\int_Q |F| \quad \text{and} \quad |F - c|_Q = \frac{1}{|Q|}\int_Q |F - c|.$$

It is clear that

$$\|F\|_{*,\mathrm{bmo}_2(\mathbb{R}^d)} = \sup_{l(Q)<2} \inf_{c\in\mathbb{C}} |F - c|_Q + \sup_{l(Q)\ge 2} |F|_Q \le \sup_{l(Q)<1} \inf_{c\in\mathbb{C}} |F - c|_Q + \sup_{l(Q)\ge 1} |F|_Q + \sup_{1\le l(Q)<2} \inf_{c\in\mathbb{C}} |F - c|_Q.$$

Obviously,

$$\sup_{1\le l(Q)<2}\inf_{c\in\mathbb{C}}|F-c|_Q \le \sup_{1\le l(Q)<2}|F|_Q \le \sup_{l(Q)\ge 1}|F|_Q,$$

so that

$$\|F\|_{*,\mathrm{bmo}_2(\mathbb{R}^d)} \le 2\|F\|_{*,\mathrm{bmo}_1(\mathbb{R}^d)}$$

follows.

For the opposite inequality, we have

$$\|F\|_{*,\mathrm{bmo}_1(\mathbb{R}^d)} = \sup_{l(Q)<1}\inf_{c\in\mathbb{C}}|F-c|_Q + \sup_{l(Q)\ge 1}|F|_Q \le \sup_{l(Q)<2}\inf_{c\in\mathbb{C}}|F-c|_Q + \sup_{l(Q)\ge 2}|F|_Q + \sup_{1\le l(Q)<2}|F|_Q.$$

To estimate the latter summand, take a cube $\widehat{Q}$ with $1 \le l(\widehat{Q}) < 2$. It follows that $|F|_{\widehat{Q}} \le 2^d|F|_{2\widehat{Q}}$, and hence

$$\sup_{1\le l(Q)<2}|F|_Q \le 2^d \sup_{l(Q)\ge 2}|F|_Q.$$

This leads to

$$\|F\|_{*,\mathrm{bmo}_1(\mathbb{R}^d)} \le (2^d+1)\|F\|_{*,\mathrm{bmo}_2(\mathbb{R}^d)}.$$

□

Remark 5.4. Lemma 5.2 remains valid in the 'local' setting, i.e. when in the statement we replace $\mathrm{BMO}(\mathbb{R}^d)$ by $\mathrm{bmo}(\mathbb{R}^d)$. The proof follows that of Lemma 5.2 with the following changes. For the proof of the relevant statement replacing (i), for fixed $F \in \mathrm{bmo}(\mathbb{R}^d)$ and with the notation $\tilde{F} = F\mathbb{1}_{\langle e\rangle_+^\perp} + (F\mathbb{1}_{\langle e\rangle_+^\perp})\circ\sigma_e$, we wish to check that

$$\sup_{l(Q)<1}\inf_{c\in\mathbb{C}}\frac{1}{|Q|}\int_Q|\tilde{F}-c| + \sup_{l(Q)\ge 1}\frac{1}{|Q|}\int_Q|\tilde{F}| \le 2\|F\|_{*,\mathrm{bmo}(\mathbb{R}^d)}.$$

To achieve this, we take a cube Q in $\mathbb{R}^d$ and consider separately the cases $l(Q) < 1$ and $l(Q) \ge 1$ following the argument from the proof of (i) in Lemma 5.2.

For the proof of the relevant statement replacing (ii), for fixed $F \in \mathrm{bmo}(\mathbb{R}^d)$, we wish to check that

$$\sup_{l(Q)<1}\inf_{c\in\mathbb{C}}\frac{1}{|Q|}\int_Q|F\mathbb{1}_{\langle e\rangle_+^\perp}-c| + \sup_{l(Q)\ge 1}\frac{1}{|Q|}\int_Q|F\mathbb{1}_{\langle e\rangle_+^\perp}| \le 2\|F\|_{*,\mathrm{bmo}(\mathbb{R}^d)}.$$

Again, given a cube Q in $\mathbb{R}^d$, we consider separately the cases $l(Q) < 1$ and $l(Q) \ge 1$ following the argument from the proof of (ii) in Lemma 5.2. This time, considering the larger cube $\widehat{Q}$, we apply Proposition 5.3. With this, one can easily reach the desired claim.

For the proofs of the equivalence "(i) $\Longleftrightarrow$ (ii)" in Theorems 5.5 and 5.6, it is reasonable to introduce convenient notation. Given a suitable function f on $\mathbb{R}^d_{+,k}$ and a parameter $0 < a \le \infty$, we define

$$m_{\mathrm{osc},a}(f) = \sup_{l(Q')<a}\inf_{c\in\mathbb{C}}|f-c|_{Q'} \quad \text{and} \quad m_{\mathrm{mv},a}(f) = \max\{m^{(1)}_{\mathrm{mv},a}(f), m^{(2)}_{\mathrm{mv},a}(f)\},$$

where

$$m^{(1)}_{\mathrm{mv},a}(f) = \sup_{l(Q')\ge a}|f|_{Q'} \quad \text{and} \quad m^{(2)}_{\mathrm{mv},a}(f) = \sup_{\substack{l(Q'')<a\\ Q''\ \text{adjacent}}}|f|_{Q''}.$$

Here Q' and Q'' run over the collections of cubes in $\mathbb{R}^d_{+,k}$ satisfying the corresponding length restrictions, and, in case of $m^{(2)}_{\mathrm{mv},a}(f)$, additionally the Q'' are *adjacent* (in the sense as above the statement of Theorem 5.5); then $|f-c|_{Q'}$ and $|f|_{Q''}$ denote the *mean values* of $|f-c|$ and $|f|$ over corresponding cubes. Notice that only $m^{(2)}_{\mathrm{mv},a}(f)$ depends on η (which is not indicated) and for $\eta = \mathbf{0}$ (there are no adjacent cubes then) we set $m^{(2)}_{\mathrm{mv},a}(f) = 0$. For $a = 1$, we shall drop a in our notation writing simply $m_{\mathrm{loc}}(f)$ etc.

In the theorem that follows, mind that for $\eta = \mathbf{0}$ the statements concern abstract classes of functions, i.e. both f and F are considered modulo constants (it is clear what then $F|_{\mathbb{R}^d_{+,k}} = f$ means); otherwise, for $\eta \ne \mathbf{0}$, the statements concern genuine functions. Also, notice that for $\eta = \mathbf{0}$ the assumptions imposed on the support of F are void (note that $\mathbb{R}_+^{\mathbf{0},1} = \mathbb{R}^d$ and $\mathbb{R}_+^{\mathbf{0},0} = \mathbb{R}^d_{+,k}$) and (iii) and (iv) are identical. In addition, observe that, since

$$\mathbb{R}_+^{\eta,1} \subset (\mathbb{R}_+^{\eta,0} \setminus \mathbb{R}^d_{+,k})^{\mathrm{c}},$$

for $\eta \neq \mathbf{0}$ condition (iv) is seemingly weaker than condition (iii), but in fact these conditions occur to be equivalent. It is also worth pointing out a duality between conditions (iii) and (iv) in Theorem 4.6, and conditions (iii) and (iv) below. Finally, we define

$$M_1(f) = \sup_{Q'} \frac{1}{|Q'|} \int_{Q'} |f - f_{Q'}|, \quad M_2(f) = \sup_{Q''} \frac{1}{|Q''|} \int_{Q''} |f|,$$

where the first supremum is taken over all cubes $Q' \subset \mathbb{R}^d_{+,k}$ and the second supremum is taken over all cubes $Q'' \subset \mathbb{R}^d_{+,k}$ adjacent to at least one of the walls of $\mathbb{R}^d_{+,k}$ with the minus sign attached by η.

Theorem 5.5. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d_{+,k})$ and $\eta \in \mathbb{Z}_2^k$. The following conditions are equivalent:*
(i) $f \in \mathrm{BMO}_\eta(\mathbb{R}^d_{+,k})$.
(ii) $M_1(f) + M_2(f) < \infty$.
(iii) *There exists $F \in \mathrm{BMO}(\mathbb{R}^d)$ supported in $\mathbb{R}_+^{\eta,1}$ and such that $F|_{\mathbb{R}^d_{+,k}} = f$.*
(iv) *There exists $F \in \mathrm{BMO}(\mathbb{R}^d)$ supported in $(\mathbb{R}_+^{\eta,0} \setminus \mathbb{R}^d_{+,k})^{\mathrm{c}}$ and such that $F|_{\mathbb{R}^d_{+,k}} = f$.*
Moreover,

$$\|f\|_{\mathrm{BMO}_\eta(\mathbb{R}^d_{+,k})} \simeq M_1(f) + M_2(f) \simeq \inf_F \|F\|_{\mathrm{BMO}(\mathbb{R}^d)},$$

where the infimum is taken over all F as in (iii) *or in* (iv)*, respectively.*

Proof. "(i) $\Longleftrightarrow$ (ii)": Let $F = \mathcal{E}^\eta f$ for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d_{+,k})$. Proving the desired equivalence together with the comparability of the relevant quantities can be replaced by verifying that

$$\|F\|_{*,\mathrm{BMO}(\mathbb{R}^d)} \lesssim m_{\mathrm{osc},\infty}(f) + m^{(2)}_{\mathrm{mv},\infty}(f) \lesssim \|F\|_{*,\mathrm{BMO}(\mathbb{R}^d)}, \tag{5.1}$$

with implicit constants independent of f. Notice that $m_{\mathrm{osc},\infty}(f)$ measures the total oscillation of f over all cubes contained in $\mathbb{R}^d_{+,k}$, and $m^{(2)}_{\mathrm{mv},\infty}(f)$ gives the supremum of the mean values of $|f|$ over all adjacent cubes (if any) contained in $\mathbb{R}^d_{+,k}$. Therefore, proving the right-hand side in (5.1) reduces to checking that

$$m^{(2)}_{\mathrm{mv},\infty}(f) \lesssim \|F\|_{*,\mathrm{BMO}(\mathbb{R}^d)},$$

and additionally we can assume that $\eta \neq \mathbf{0}$. Let $Q'' \subset \mathbb{R}^d_{+,k}$ be adjacent. To verify that

$$|f|_{Q''} \lesssim \|F\|_{*,\mathrm{BMO}(\mathbb{R}^d)},$$

let $\widehat{Q''}$ be a smallest cube in $\mathbb{R}^d$ containing Q'' and symmetric with respect to any wall $\mathcal{W}_i$ of $\mathbb{R}^d_{+,k}$ with $\mathrm{sign}_\eta(\mathcal{W}_i) = -1$. Then $F_{\widehat{Q''}} = 0$, the volumes of Q'' and $\widehat{Q''}$ are comparable, and hence

$$\frac{1}{|Q''|} \int_{Q''} |f| \lesssim \frac{1}{|\widehat{Q''}|} \int_{\widehat{Q''}} |F| \le \|F\|_{\mathrm{BMO}(\mathbb{R}^d)} \lesssim \|F\|_{*,\mathrm{BMO}(\mathbb{R}^d)}.$$

To prove the left-hand side in (5.1), note that, due to symmetry reasons, estimating the mean values $|F - c|_Q$, the only cubes that matter are the ones crossed by at least one of the walls of $\mathbb{R}^d_{+,k}$ (for the remaining cubes these mean values are dominated by $m_{\mathrm{osc},\infty}(f)$). Let $Q = \prod_{j=1}^d (\alpha_j, \beta_j)$ be such a cube and let I denote the set of these $1 \le i \le k$ such that $\mathcal{W}_i$ crosses Q, i.e. $\alpha_{n-k+i} < 0 < \beta_{n-k+i}$. Again due to symmetry reasons, we can assume that $\beta_{n-k+i} \ge |\alpha_{n-k+i}|$ for every $i \in I$. Assume first that for every $i \in I$ we have $\mathrm{sign}_\eta(\mathcal{W}_i) = 1$ (this includes the case $\eta = \mathbf{0}$). Let Q' be a smallest cube in $\mathbb{R}^d_{+,k}$ that contains $Q^+ = Q \cap \mathbb{R}^d_{+,k}$. Then the volumes of Q and Q' are comparable, and Q' together with all reflections of Q' with respect to the walls $\mathcal{W}_i$, $i \in I$, covers Q. Consequently, we have

$$\frac{1}{|Q|} \int_Q |F - c| \lesssim \frac{1}{|Q'|} \int_{Q'} |f - c| \le m_{\mathrm{osc},\infty}(f). \tag{5.2}$$

In the complementary case, when $\operatorname{sign}_\eta(\mathcal{W}_i) = -1$ for some $i \in I$, set

$$I^- = \{i \in I \colon \operatorname{sign}_\eta(\mathcal{W}_i) = -1\}.$$

Let $Q'' \subset \mathbb{R}^d_{+,k}$ be a smallest cube containing $Q \cap \mathbb{R}^d_{+,k}$. Then Q'' is adjacent, the volumes of Q and Q'' are comparable, and Q'' together with all its reflections with respect to the walls $\mathcal{W}_i$, $i \in I^-$, covers Q. Consequently,

$$\inf_{c \in \mathbb{C}} |F - c|_Q \le \frac{1}{|Q|} \int_Q |F| \lesssim \frac{1}{|Q''|} \int_{Q''} |f| \lesssim m^{(2)}_{\mathrm{mv},\infty}(f). \tag{5.3}$$

"(iv) $\Longrightarrow$ (i)": Assuming that F is such as in (iv), it suffices to check that $\mathcal{E}^\eta f \in \mathrm{BMO}(\mathbb{R}^d)$ together with a corresponding bound. It is convenient to single out the case $\eta = \mathbf{1}$. Then we have to our disposal $F \in \mathrm{BMO}(\mathbb{R}^d)$ supported in $\mathbb{R}^d_{+,k}$ and such that $F|_{\mathbb{R}^d_{+,k}} = f$. It is clear that

$$\mathcal{E}^\eta f = \sum_{g \in W(R_k)} F \circ g,$$

and hence $\mathcal{E}^\eta f \in \mathrm{BMO}(\mathbb{R}^d)$ with

$$\|f\|_{\mathrm{BMO}_\eta(\mathbb{R}^d)} = \|\mathcal{E}^\eta f\|_{\mathrm{BMO}(\mathbb{R}^d)} \lesssim \|F\|_{\mathrm{BMO}(\mathbb{R}^d)}.$$

Assume now that $\eta \ne \mathbf{1}$. Let $1 \le \eta_{i_1} < \cdots < \eta_{i_s} \le d$, $1 \le s \le k$, be all entries of η which are equal to 0. We apply Lemma 5.2 (i) to e_{i_1} and F to obtain $F_1 \in \mathrm{BMO}(\mathbb{R}^d)$ being the even extension of $F|_{\langle e_{i_1}\rangle^\perp_+}$. Then we continue this process to e_{i_2} and F_1 to get the even extension of $F_1|_{\langle e_{i_2}\rangle^\perp_+}$, and so on. The final function F_s belongs to $\mathrm{BMO}(\mathbb{R}^d)$ and

$$\|F_s\|_{*,\mathrm{BMO}(\mathbb{R}^d)} \lesssim \|F\|_{*,\mathrm{BMO}(\mathbb{R}^d)}.$$

Moreover, by the assumptions imposed on F and the way F_s was constructed, we have $F_s = (\mathcal{E}^\eta f)\mathbb{1}_{\mathbb{R}^{\eta,1}_+}$. In particular, F_s is supported in $\mathbb{R}^{\eta,1}$ and $F_s|_{\mathbb{R}^d_{+,k}} = f$. Thus, for $\eta = \mathbf{0}$ we are done.

Notice that for $\eta = \mathbf{0}$ the above argument is valid modulo constants. Notably, this finishes the proof of the theorem in this case. Indeed, (i) clearly implies (iii) and (iv), which are identical. Thus, in the next parts of the proof we focus on the case $\eta \ne \mathbf{0}$.

To continue this part for $\mathbf{0} \ne \eta \ne \mathbf{1}$, we shall use an argument similar to that from the proof of "(iv) $\Longrightarrow$ (i)" in Theorem 4.6. Namely, consider the complementary root (sub)systems of R_k:

$$R_{k,0} = \{\pm e_{d-k+i} \colon \eta_i = 0\} \quad \text{and} \quad R_{k,1} = \{\pm e_{d-k+i} \colon \eta_i = 1\}.$$

As the positive Weyl chambers corresponding to these root systems we choose $\mathbb{R}^{\eta,0}_+$ and $\mathbb{R}^{\eta,1}_+$, respectively. We have the natural identification of $W(R_k)$ with the direct product $W(R_{k,1}) \oplus W(R_{k,0})$, and also η can be viewed as the 'tensor product' of the corresponding homomorphisms on $W(R_{k,1})$ and $W(R_{k,0})$, namely $\eta = \eta^{(1)} \otimes \eta^{(0)}$, where

$$\eta^{(1)} = (1, \dots, 1) \quad (\#\{i \colon \eta_i = 1\}\text{-times}) \qquad \text{and} \qquad \eta^{(0)} = (0, \dots, 0) \quad (\#\{i \colon \eta_i = 0\}\text{-times}).$$

Due to the product structure of $W(R_k)$ and the tensor product structure of η, it is easily seen that

$$\mathcal{E}^\eta f = \mathcal{E}^{\eta^{(1)}}(\mathcal{E}^{\eta^{(0)}} f)$$

for any f supported in $\mathbb{R}_{+,k}$; notice that if f is such a function, then $\mathcal{E}^{\eta^{(0)}} f$ is supported in $\mathbb{R}^{\eta,1}_+$, so that the composition makes sense.

To finish this part of the proof, we argue as follows. The equality $F_s = (\mathcal{E}^\eta f)\mathbb{1}_{\mathbb{R}^{\eta,1}_+}$ can be reinterpreted as $F_s = \mathcal{E}^{\eta^{(0)}} f$, and hence

$$\mathcal{E}^\eta f = \mathcal{E}^{\eta^{(1)}}(\mathcal{E}^{\eta^{(0)}} f) = \mathcal{E}^{\eta^{(1)}} F_s.$$

But F_s is supported in $\mathbb{R}^{\eta,1}$, so

$$\|f\|_{\mathrm{BMO}_\eta(\mathbb{R}^d)} = \|(\mathcal{E}^\eta f)\|_{\mathrm{BMO}(\mathbb{R}^d)} = \|(\mathcal{E}^{\eta^{(1)}} F_s)\|_{\mathrm{BMO}(\mathbb{R}^d)} \lesssim \|F_s\|_{\mathrm{BMO}(\mathbb{R}^d)} \lesssim \|F\|_{\mathrm{BMO}(\mathbb{R}^d)},$$

with the implicit constant independent of f.

"(i) $\Longrightarrow$ (iii)": Let $f \in \mathrm{BMO}_\eta(\mathbb{R}^d_{+,k})$. This means that $\mathcal{E}^\eta f \in \mathrm{BMO}(\mathbb{R}^d)$, and clearly $\mathcal{E}^\eta f$ is odd in each $\langle e_{d-k+i}\rangle^\perp$ such that $\eta_i = 1$ (if any). Hence, applying Lemma 5.2 (ii) consecutively to each such e_{d-k+i} (if any) yields $(\mathcal{E}^\eta f)\mathbb{1}_{\mathbb{R}^{\eta,1}_+} \in \mathrm{BMO}(\mathbb{R}^d)$. Consequently, $F := (\mathcal{E}^\eta f)\mathbb{1}_{\mathbb{R}^{\eta,1}_+}$ has the required properties and

$$\|F\|_{\mathrm{BMO}(\mathbb{R}^d)} \lesssim \|f\|_{\mathrm{BMO}_\eta(\mathbb{R}^d_{+,k})}$$

follows with the implicit constant independent of f.

"(iii) $\Longrightarrow$ (iv)": This is obvious; clearly, the former inequality remains in force.

Finally, observe that the claimed comparability of relevant quantities follows from the partial conclusions included in the proofs of the three implications. □

The local version of Theorem 5.5 is the following (note that for $\eta = \mathbf{0}$ the assumptions imposed on the support of F are void and (iii) and (iv) are identical). Again, observe a duality between conditions (iii) and (iv) in Theorem 4.8, and conditions (iii) and (iv) below. We define

$$M_1^{\mathrm{loc}}(f) = \sup_{Q'} \frac{1}{|Q'|}\int_{Q'} |f - f_{Q'}|, \quad M_2^{\mathrm{loc}}(f) = \sup_{Q''} \frac{1}{|Q''|}\int_{Q''} |f| < \infty,$$

where Q', Q'' are cubes included in $\mathbb{R}^d_{+,k}$ with the following restrictions: $l(Q') < 1$ in the first supremum, and in the second supremum either $l(Q'') \geq 1$, or Q'' is adjacent to at least one of the walls of $\mathbb{R}^d_{+,k}$ with the minus sign attached by η and $l(Q'') < 1$.

Theorem 5.6. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d_{+,k})$ and $\eta \in \mathbb{Z}_2^k$. The following conditions are equivalent:*
(i) $f \in \mathrm{bmo}_\eta(\mathbb{R}^d_{+,k})$.
(ii) $M_1^{\mathrm{loc}}(f) + M_2^{\mathrm{loc}}(f) < \infty$.
(iii) *There exists $F \in \mathrm{bmo}(\mathbb{R}^d)$ supported in $\mathbb{R}^{\eta,1}_+$ and such that $F|_{\mathbb{R}^d_{+,k}} = f$.*
(iv) *There exists $F \in \mathrm{bmo}(\mathbb{R}^d)$ supported in $(\mathbb{R}^{\eta,0}_+ \setminus \mathbb{R}^d_{+,k})^{\mathrm{c}}$ and such that $F|_{\mathbb{R}^d_{+,k}} = f$.*
Moreover,

$$\|f\|_{\mathrm{bmo}_\eta(\mathbb{R}^d_{+,k})} \simeq M_1^{\mathrm{loc}}(f) + M_2^{\mathrm{loc}}(f) \simeq \inf_F \|F\|_{\mathrm{bmo}(\mathbb{R}^d)},$$

where the infimum is taken over all F as in (iii) *or in* (iv), *respectively.*

Proof. The proof is similar to that of Theorem 5.5, so we only provide some comments (skipping the implication "(iii) $\Longrightarrow$ (iv)" which is obvious).

"(i) $\Longleftrightarrow$ (ii)": Similarly to the above notation, for a suitable function F on $\mathbb{R}^d$ define

$$M_{\mathrm{loc}}(F) = \sup_{l(Q)<1} \inf_{c\in\mathbb{C}} |F - c|_Q \quad \text{and} \quad M_{\mathrm{mv}}(F) = \sup_{l(Q)\geq 1} |F|_Q;$$

here Q runs over the collections of all cubes in $\mathbb{R}^d$ satisfying the corresponding length restrictions. By definition,

$$\|F\|_{*,\mathrm{bmo}(\mathbb{R}^d)} = M_{\mathrm{loc}}(F) + M_{\mathrm{mv}}(F).$$

Now, let $F = \mathcal{E}^\eta f$ for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d_{+,k})$. We are reduced to verifying that

$$M_{\mathrm{loc}}(F) + M_{\mathrm{mv}}(F) \lesssim m_{\mathrm{loc}}(f) + m_{\mathrm{mv}}(f) \lesssim M_{\mathrm{loc}}(F) + M_{\mathrm{mv}}(F). \tag{5.4}$$

We begin with comments on the right-hand side of (5.4). Obviously, $m_{\mathrm{loc}}(f) \leq M_{\mathrm{loc}}(F)$ and $m^{(1)}_{\mathrm{mv}}(f) \leq M_{\mathrm{mv}}(F)$. Hence, it suffices to check that

$$m^{(2)}_{\mathrm{mv}}(f) \lesssim M_{\mathrm{loc}}(F) + M_{\mathrm{mv}}(F),$$

additionally assuming that $\eta \neq \mathbf{0}$. Let $Q'' \subset \mathbb{R}^d_{+,k}$ be adjacent with $l(Q'') < 1$, and let $\widehat{Q''}$ be a smallest cube in $\mathbb{R}^d$ containing Q'' and symmetric with respect to any wall $\mathcal{W}_i$ of $\mathbb{R}^d_{+,k}$ with $\operatorname{sign}_\eta(\mathcal{W}_i) = -1$. Then the volumes of Q'' and $\widehat{Q''}$ are comparable and $F_{\widehat{Q''}} = 0$. Hence,

$$\frac{1}{|Q''|}\int_{Q''} |f| \lesssim \frac{1}{|\widehat{Q''}|}\int_{\widehat{Q''}} |F| \leq \|F\|_{\mathrm{bmo}(\mathbb{R}^d)} \lesssim \|F\|_{*,\mathrm{bmo}(\mathbb{R}^d)}.$$

To prove the left-hand side in (5.4), estimating the mean values $|F-c|_Q$ when $l(Q)<1$, and $|F|_Q$ when $l(Q)\geq 1$, again only the cubes crossed by at least one of the walls of $\mathbb{R}^d_{+,k}$ matter. We proceed as in the proof of the left-hand side in (5.1), keeping the same notation and applying the appropriate reducing observations, but additionally we split the reasoning to the cases $l(Q)<1$ and $l(Q)\geq 1$.

Assume first that $l(Q)<1$. Then, in the case when for every $i\in I$ we have $\operatorname{sign}_\eta(\mathcal{W}_i)=1$, reaching (5.2), the only change to be done is the replacement of $m_{\mathrm{osc},\infty}(f)$ by $m_{\mathrm{osc},2}(f)$ (this is since $l(Q')<2$). In the complementary case, when $\operatorname{sign}_\eta(\mathcal{W}_i)=-1$ for some $i\in I$, reaching (5.3), the change to be done is the replacement of $m^{(2)}_{\mathrm{mv},\infty}(f)$ by $m^{(2)}_{\mathrm{mv}}(f)$ (with $l(Q'')<l(Q)$). Concluding, we have

$$\mathcal{M}_{\mathrm{loc}}(F)\lesssim m_{\mathrm{osc},2}(f)+m^{(2)}_{\mathrm{mv}}(f)\lesssim m_{\mathrm{osc}}(f)+m_{\mathrm{mv}}(f),$$

where we used the fact that $m_{\mathrm{osc},2}(f)\lesssim m_{\mathrm{osc}}(f)+m^{(1)}_{\mathrm{mv}}(f)$ (see the proof of Proposition 5.3).

Assume now that $l(Q)\geq 1$. Taking $c=0$ in (5.2), we can replace $m_{\mathrm{osc},\infty}(f)$ by $m^{(1)}_{\mathrm{mv}}(f)$. Similarly, disregarding $\inf_{c\in\mathbb{C}}|F-c|_Q$ and replacing $m^{(2)}_{\mathrm{mv},\infty}(f)$ by $m^{(1)}_{\mathrm{mv}}(f)$ in (5.3), and then taking the supremum over all Q, $l(Q)\geq 1$, leads to

$$\mathcal{M}_{\mathrm{mv}}(F)\lesssim m^{(1)}_{\mathrm{mv}}(f).$$

The proof of the left-hand side in (5.4) is completed.

"(iv) $\Longrightarrow$ (i)": To check that $\mathcal{E}^\eta f\in \mathrm{bmo}(\mathbb{R}^d)$ together with a relevant bound, only cosmetic changes are needed in the case $\eta=\mathbf{1}$. Treating the case $\eta\neq\mathbf{1}$, we apply Remark 5.4 in place of Lemma 5.2 and follow the procedure described in "(iv) $\Longrightarrow$ (i)" in the proof of Theorem 5.5; this is sufficient to close the case $\eta=\mathbf{0}$. Finally, the case $\mathbf{0}\neq\eta\neq\mathbf{1}$ also needs only cosmetic changes.

"(i) $\Longrightarrow$ (iii)": The argument for this implication copies that of "(i) $\Longrightarrow$ (iii)" in the proof of Theorem 5.5 with obvious changes. □

As a direct consequence of Theorems 5.5 and 5.6, by using the equivalence "(i) $\Longleftrightarrow$ (iii)", we obtain the following corollary.

Corollary 5.7. *For the Weyl chamber $\mathbb{R}^d_{+,k}$ corresponding to the root system R_k in $\mathbb{R}^d$, we have*

$$\mathrm{BMO}_{\mathbf{0}}(\mathbb{R}^d_{+,k})=\mathrm{BMO}_r(\mathbb{R}^d_{+,k})\quad\text{and}\quad \mathrm{BMO}_{\mathbf{1}}(\mathbb{R}^d_{+,k})=\mathrm{BMO}_z(\mathbb{R}^d_{+,k}),$$

with equivalence of norms. In particular, it follows that

$$\mathrm{BMO}_{\mathbf{1}}(\mathbb{R}^d_{+,k})\hookrightarrow \mathrm{BMO}_{\mathbf{0}}(\mathbb{R}^d_{+,k})$$

with strict inclusion. Analogous statement holds for local bmo *spaces.*

5.2 Duality

We now state and prove duality results for η-Hardy spaces in the context of orthogonal root systems. Notice that, by Theorem 4.6, the vector space of (finite) linear combinations of (η,A)-atoms and (η,B)-atoms is dense in $H^1_\eta(\mathbb{R}^d_{+,k})$.

Theorem 5.8. *Let $\eta\in\mathbb{Z}^k_2$. The dual of $H^1_\eta(\mathbb{R}^d_{+,k})$ is $\mathrm{BMO}_\eta(\mathbb{R}^d_{+,k})$. More precisely, for any $b\in \mathrm{BMO}_\eta(\mathbb{R}^d_{+,k})$ the formula*

$$L_b(f)=\int\limits_{\mathbb{R}^d_{+,k}} bf \tag{5.5}$$

defined initially on the vector space of linear combinations of (η,A/B)-atoms, and then uniquely extended to the whole $H^1_\eta(\mathbb{R}^d_{+,k})$, gives a bounded linear functional on $H^1_\eta(\mathbb{R}^d_{+,k})$. Conversely, every element of $H^1_\eta(\mathbb{R}^d_{+,k})'$ is of this form. Moreover, $\|L_b\|\simeq\|b\|_{\mathrm{BMO}_\eta(\mathbb{R}^d_{+,k})}$.

Proof. First we note that (5.5) is a correct definition: if $f=\sum_{j=1}^n(\alpha_j a_j+\beta_j b_j)$, where each a_j/b_j is an $(\eta,A/B)$-atom supported in a cube in $\mathbb{R}^d_{+,k}$, then $\int_{\mathbb{R}^d_{+,k}}|bf|<\infty$. This is because a_j and b_j are square integrable and $b\in L^2_{\mathrm{loc}}(\mathbb{R}^d_{+,k})$

(since $\mathcal{E}^\eta b \in \mathrm{BMO}(\mathbb{R}^d)$, as a consequence of the John–Nirenberg inequality one has $\mathcal{E}^\eta b \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$, and hence $b \in L^2_{\mathrm{loc}}(\mathbb{R}^d_{+,k})$).

Now, fix $b \in \mathrm{BMO}_\eta(\mathbb{R}^d_{+,k})$. By Theorem 5.5, there exists $B \in \mathrm{BMO}(\mathbb{R}^d)$ supported in $\mathbb{R}^{\eta,1}_+$ and such that $B|_{\mathbb{R}^d_{+,k}} = b$. Let $f \in H^1_\eta(\mathbb{R}^d_{+,k})$ be a linear combination of $(\eta, A/B)$-atoms. By Theorem 4.6, there exists $F \in H^1(\mathbb{R}^d)$ supported in $\mathbb{R}^{\eta,0}_+$ and such that $F|_{\mathbb{R}^d_{+,k}} = f$. In fact, see the proof of Theorem 4.6, we can select $F = (\mathcal{E}^\eta f)\mathbb{1}_{\mathbb{R}^{\eta,0}_+}$ which has the required properties, and moreover satisfies

$$\|F\|_{H^1(\mathbb{R}^d)} \simeq \|f\|_{H^1_\eta(\mathbb{R}^d_{+,k})}.$$

Notably, F is a linear combination of (classical) atoms supported in $\mathbb{R}^d_{+,k}$. By the localization of the supports of B and F, and by the duality between $H^1(\mathbb{R}^d)$ and $\mathrm{BMO}(\mathbb{R}^d)$, we obtain

$$|L_b(f)| = \Big|\int_{\mathbb{R}^d_{+,k}} bf\Big| = \Big|\int_{\mathbb{R}^d} BF\Big| \le \|B\|_{\mathrm{BMO}(\mathbb{R}^d)}\|F\|_{H^1(\mathbb{R}^d)}.$$

Taking the infimum over B and F gives

$$|L_b(f)| \lesssim \|b\|_{\mathrm{BMO}_\eta(\mathbb{R}^d_{+,k})}\|f\|_{H^1_\eta(\mathbb{R}^d_{+,k})},$$

and this proves that

$$L_b \in H^1(\mathbb{R}^d_{+,k})' \quad \text{and} \quad \|L_b\| \lesssim \|b\|_{\mathrm{BMO}_\eta(\mathbb{R}^d_{+,k})}.$$

Conversely, fix $L \in H^1_\eta(\mathbb{R}^d_{+,k})'$. Let X be a subspace of $H^1(\mathbb{R}^d)$ composed of the functions vanishing in $\mathbb{R}^{\eta,1} \setminus \mathbb{R}^d_{+,k}$. Notice that, by Theorem 4.6, if $F \in X$, then

$$F|_{\mathbb{R}^d_{+,k}} \in H^1_\eta(\mathbb{R}^d_{+,k}) \quad \text{and} \quad \|F|_{\mathbb{R}^d_{+,k}}\|_{H^1_\eta(\mathbb{R}^d_{+,k})} \lesssim \|F\|_{H^1(\mathbb{R}^d)}.$$

The dual space of X is isometrically isomorphic to

$$Y := \mathrm{BMO}_r((\mathbb{R}^{\eta,1} \setminus \mathbb{R}^d_{+,k})^c),$$

namely the space of functions (modulo constants) from $\mathrm{BMO}(\mathbb{R}^d)$ restricted to $(\mathbb{R}^{\eta,1} \setminus \mathbb{R}^d_{+,k})^c$ with the usual norm $\|b\|_Y = \inf_B \|B\|_{\mathrm{BMO}(\mathbb{R}^d)}$, where the infimum is taken over all extensions $B \in \mathrm{BMO}(\mathbb{R}^d)$ of b to $\mathbb{R}^d$. The identification between the functionals from X' and the elements of Y is analogous to (5.5). This fact follows from the classical theory of Banach spaces (see [8, Theorem 10.1, p. 88]), because X is a closed subspace of $H^1(\mathbb{R}^d)$, but also from [3, 4], since $(\mathbb{R}^{\eta,1} \setminus \mathbb{R}^d_{+,k})^c$ is a strongly Lipschitz domain in $\mathbb{R}^d$ (to be precise, we should consider the interior of this set just to have the set open, but this is irrelevant).

Let us now define $\varphi : X \to \mathbb{C}$ by $\varphi(F) = L(F|_{\mathbb{R}^d_{+,k}})$. Notice that $\varphi \in X'$ and $\|\varphi\| \le \|L\|$. Indeed, this follows from the bound

$$|\varphi(F)| \le \|L\|\|F|_{\mathbb{R}^d_{+,k}}\|_{H^1_\eta(\mathbb{R}^d_{+,k})} \lesssim \|L\|\|F\|_{H^1(\mathbb{R}^d)}.$$

Hence, there exists (a function modulo constants) $B \in Y$ such that

$$\varphi(F) = \int_{\mathbb{R}^d} FB$$

for any F being a linear combination of atoms supported in $(\mathbb{R}^{\eta,1} \setminus \mathbb{R}^d_{+,k})^c$, and $\|\varphi\| = \|B\|_{\mathrm{BMO}(\mathbb{R}^d)}$.

At this point, it is appropriate to consider the case $\eta = \mathbf{0}$ separately. Notice that in this situation $(\mathbb{R}^{\eta,1}_+ \setminus \mathbb{R}^d_{+,k})^c$ is simply equal to $\mathbb{R}^d_{+,k}$ (to be precise, equal to $\overline{\mathbb{R}^d_{+,k}}$, but this is irrelevant). Thus, $Y = \mathrm{BMO}_r(\mathbb{R}^d_{+,k})$, and the proof of the duality is completed.

From now on, we assume that $\eta \neq \mathbf{0}$. By the definition of φ, if $F|_{\mathbb{R}^d_{+,k}} = 0$, then $\varphi(F) = 0$. Thus, B is constant in $(\mathbb{R}^{\eta,1}_+)^c$. We choose a representative $\widetilde{B}$ from the abstract class of B which vanishes in $(\mathbb{R}^{\eta,1}_+)^c$. Then, for any $f \in H^1_\eta(\mathbb{R}^d_{+,k})$ and for any extension $F \in X$ of it, we have

$$\varphi(F) = \int_{\mathbb{R}^d_{+,k}} f\widetilde{B}|_{\mathbb{R}^d_{+,k}}$$

and $\|\varphi\| = \inf_B \|B\|_{\mathrm{BMO}(\mathbb{R}^d)}$, where the infimum is taken over the functions $B \in \mathrm{BMO}(\mathbb{R}^d)$ such that $B|_{\mathbb{R}^d_{+,k}} = \widetilde{B}|_{\mathbb{R}^d_{+,k}}$ and B is supported in $\mathbb{R}^{\eta,1}_+$. By Theorem 5.5,

$$\widetilde{B}|_{\mathbb{R}^d_{+,k}} \in \mathrm{BMO}_\eta(\mathbb{R}^d_{+,k}) \quad \text{and} \quad \|\varphi\| \simeq \|\widetilde{B}|_{\mathbb{R}^d_{+,k}}\|_{\mathrm{BMO}_\eta(\mathbb{R}^d_{+,k})}.$$

This concludes the proof. □

The local version of Theorem 5.8 is the following theorem.

Theorem 5.9. *Let $\eta \in \mathbb{Z}_2^k$. The dual of $h^1_\eta(\mathbb{R}^d_{+,k})$ is $\mathrm{bmo}_\eta(\mathbb{R}^d_{+,k})$. More precisely, for any $b \in \mathrm{bmo}_\eta(\mathbb{R}^d_{+,k})$, the formula*

$$L_b(f) = \int_{\mathbb{R}^d_{+,k}} bf$$

defined initially on the vector space of linear combinations of local (η,A/B)-atoms, and then uniquely extended to the whole $h^1_\eta(\mathbb{R}^d_{+,k})$, gives a bounded linear functional on $h^1_\eta(\mathbb{R}^d_{+,k})$. Conversely, every element of $h^1_\eta(\mathbb{R}^d_{+,k})'$ is of this form. Moreover,

$$\|L_b\| \simeq \|b\|_{\mathrm{bmo}_\eta(\mathbb{R}^d_{+,k})}.$$

Proof. In the first part of the proof, we copy, mutatis mutandis, the argument from the proof of Theorem 5.8. Obviously, in place of Theorems 4.6 and 5.5, we now use their local versions, Theorems 4.8 and 5.6. The same local versions are used in the second part of the proof with necessary changes. □

Remark 5.10. In the general case of a strongly Lipschitz domain Ω, it was proved in [1, Theorem 5 (b), (d)], that the duals of $H^1_z(\Omega)$ and $H^1_r(\Omega)$ are $\mathrm{BMO}_r(\Omega)$ and $\mathrm{BMO}_z(\Omega)$, respectively. Therefore, in the general setting, since C_+ is a special Lipschitz domain, it follows that for $\eta = \mathrm{triv}$ or $\eta = \mathrm{sgn}$ we have that the dual of $H^1_{\mathrm{triv}}(C_+)$ is $\mathrm{BMO}_r(C_+)$, and the dual of $H^1_{\mathrm{sgn}}(C_+)$ is $\mathrm{BMO}_z(C_+)$.

The proposition concluding this section establishes relations between the spaces $\mathrm{BMO}^1_\eta(\mathbb{R}^d_{+,k})$ for different η's (cf. also Corollary 5.7) with an analogous statement in the local case, and is, in some sense, dual to Proposition 4.10.

Proposition 5.11. *If $\eta^{(1)} \le \eta^{(2)}$ (the lexicographical order), then*

$$\mathrm{BMO}_{\eta^{(2)}}(\mathbb{R}^d_{+,k}) \hookrightarrow \mathrm{BMO}_{\eta^{(1)}}(\mathbb{R}^d_{+,k}).$$

In particular, we have

$$\mathrm{BMO}_{\mathbf{1}}(\mathbb{R}^d_{+,k}) \hookrightarrow \mathrm{BMO}_\eta(\mathbb{R}^d_{+,k}) \hookrightarrow \mathrm{BMO}_{\mathbf{0}}(\mathbb{R}^d_{+,k}).$$

Analogous statements hold for the local spaces $\mathrm{bmo}_\eta(\mathbb{R}^d_{+,k})$.

Proof. If $\eta^{(1)} \le \eta^{(2)}$, then $\mathbb{R}^{\eta^{(2)},1}_+ \subset \mathbb{R}^{\eta^{(1)},1}_+$. Therefore, the claimed continuous embedding is an obvious consequence of the equivalence "(i) $\Longleftrightarrow$ (iii)" in Theorem 5.5. For the local case, we repeat the same argument using Theorem 5.6. □

Funding: P. Plewa acknowledges the financial support of Compagnia di San Paolo.

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