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***Abstract*—Mechanical energy harvesting is one of the most promising solutions to the renewable powering of dispersed Internet of Things devices. The design of such powering systems, however, is a challenging task, not only because a stochastic description is required to represent the very mechanical energy source which is random in nature, but also because a significant mismatch is often present between the electrical load and the equivalent circuit representing the harvester. In this contribution we propose a sophisticated solution technique allowing for the evaluation of the first two order moments of the output voltage for the stochastic differential equations model representing the device. The approach applies also to nonlinear harvesters exploiting a moment closure technique. We take into consideration also the presence of a reactive matching network aiming at the optimization of the energy flow between the harvesting device and the load. Results, besides validating the stochastic analysis technique, show an important improvement in the output power delivery and in the conversion efficiency.**

***Index Terms*—Energy harvesting, piezoelectric energy harvester, nonlinear dynamical systems, impedance matching, nonlinear resonance**

I. Introduction

Mechanical vibrations available in the environment are one of the most promising energy sources to feed energy harvesting devices exploited for the autonomous electrical powering of, e.g., individual elements of scattered wireless sensor networks [1]–[4]. Designing efficient harvesters is of course a major goal to make such technologies feasible for practical implementation, and an effective design relies on the availability of accurate and sound models [5]–[8]. As the first element of the modeling chain, the proper description of the energy source is at least as important as the description of the harvester collection and transduction mechanisms. Due to their inherent random nature, the natural environment for ambient vibration description is stochastic analysis [9], which mathematical intricacies need to be taken into account for reliable modeling. In particular, the forcing mechanical energy is described as a stochastic process that, in the simple case of negligible noise correlation time, can be represented as a white Gaussian noise process [10].

The harvester modeling is then completed by a mechanical oscillator excited by the random vibrations that embeds a transducer, typically realized with a piezoelectric material, to complete the energy conversion process. Recently, we showed that a convenient picture, also in view of the design stage, makes use of the electro-mechanical analogy to build an equivalent (electrical) circuit representing the behavior of the entire harvesting device [11]. Among others, this representation suggests the use of a matching network between the harvester itself and the electrical element representing the load, usually a resistance [12]–[14]. The goal of this matching network is to tame the sub-optimal operation of the harvester, so as to maximize the energy transfer to the load and, therefore, also the device conversion efficiency.

In this contribution, we describe the stochastic modeling of a piezoelectric energy harvester subject to random mechanical vibrations, such as the common example of cantilever covered by a piezoelectric material as shown in Fig. 1. The cantilever, besides being covered by the transduction material schematically connected to the load in the figure, is also completed by an inertial mass connected to the tip in order to enhance the mechanical swinging and, thus, the piezoelectric material excitation. This system can be well approximated by a simple linear model as far as the mass motion can be assumed one dimensional and of limited amplitude, or in other words as far as the vibration energy is small enough. Taking inspiration from our recent work [12], [14], a reactive LC matching network is interposed between the transducer and the resistive load.

We show novel analytical results from the solution of the Stochastic Differential Equation (SDE) system describing the entire device (including the matching network), making evident the large advantages in the use of such matching in terms of output (i.e., on the load) average voltage, output

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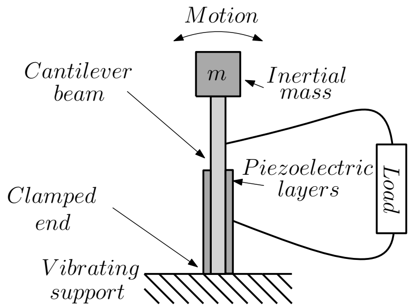

Fig. 1. Schematic representation of a piezoelectric cantilever beam energy harvester.

average power and conversion power efficiency.

The following section is devoted to the derivation of the SDE model and to the construction of a dimensionless version that makes calculations more convenient. After showing in Section III a power balance equation derived from the SDE model, we evaluate the output voltage and the conversion efficiency highlighting the advantages of the matching technique. Finally, conclusions are drawn in Section IV.

II. Piezoelectric harvester modeling

The first step in the modeling of a vibration energy harvester is the description of the energy source, i.e. random ambient vibrations. In terms of their spectral contents, energy of mechanical vibrations is typically concentrated at low frequencies. However, for a negligible noise correlation time and a realistic harvester characterized by a wide enough intrinsic frequency response, a white Gaussian noise process may represent a reasonable approximation, although nonphysical as a perfectly flat spectrum would correspond to an infinite signal energy (however limited by the finite bandwidth of the deterministic device). Also taking into account the well developed mathematical theory of deterministic systems excited by a white noise, the white process approximation is quite common in the literature [9], [10], [15]–[22]. According to such theory, a one dimensional white Gaussian noise is the time derivative of a Wiener process, the basic tool for stochastic analysis [9].

A. SDEs and variable transformations

In the following we adopt the standard notation used in probability: Capital letters denote random variables, while lower case letters denote their possible values. A one dimensional Wiener process $W_t = W(t)$ is characterized by an expectation value $\mathrm{E}[W_t] = 0$ (symbol $\mathrm{E}[X_t]$ denotes expectation, or average, of the stochastic process X_t), covariance $\mathrm{cov}(W_t, W_s) = \mathrm{E}[W_t\, W_s] = \min(t, s)$ and $W_t \sim \mathcal{N}(0, t)$, where symbol $\sim$ means "distributed as", and $\mathcal{N}(0, t)$ denotes the normal (Gaussian), zero average distribution.

In the general case, a system of d SDEs driven by a one dimensional Wiener process takes the form

$$\mathrm{d}\mathbf{Z}_t = \mathbf{a}(\mathbf{Z}_t)\mathrm{d}t + \epsilon\mathbf{B}(\mathbf{Z}_t)\mathrm{d}W_t \tag{1}$$

where $\mathbf{Z}_t$ is a vector (of size d) valued stochastic process, the d-dimensional vector $\mathbf{a}$ is called the drift function, while the d-dimensional vector $\mathbf{B}$ is the diffusion, and ϵ is a dimensionless parameter introduced to control the noise source amplitude. Noise is called additive (or not modulated) for a constant diffusion $\mathbf{B}$, while in the general case of a diffusion $\mathbf{B}(\mathbf{Z}_t)$ corresponds to a modulated, or multiplicative, noise. In the following, we consider a linear SDE system with linear drift and additive noise, i.e.

$$\mathrm{d}\mathbf{Z}_t = \mathbf{A}\mathbf{Z}_t\mathrm{d}t + \epsilon\mathbf{B}\mathrm{d}W_t \tag{2}$$

Practical manipulation of the equations is more convenient in case of dimensionless variables, and dimensionless time. To this aim, we consider a linear variable transformation $\mathbf{y} = \mathbf{P}\mathbf{z}$, where $\mathbf{P}$ is a constant, invertible real matrix. In practical cases, $\mathbf{P}$ is diagonal with entries represented by normalizing parameters of the original variables. Using Itô's lemma [9], [15], the following SDE system for the stochastic processes $\mathbf{Y}_t$ is obtained:

$$\mathrm{d}\mathbf{Y}_t = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}\mathbf{Y}_t\mathrm{d}t + \epsilon\mathbf{P}\mathbf{B}\,\mathrm{d}W_t \tag{3}$$

As a dimensionless time, we introduce the new time variable $\tau(t) = \omega\, t$, where $\omega > 0$ is a frequency. If $\mathbf{Y}_t$ solves (3), then $\mathbf{Y}_\tau$ solves the SDEs system

$$\mathrm{d}\mathbf{Y}_\tau = \frac{1}{\omega}\mathbf{P}\mathbf{A}\mathbf{P}^{-1}\mathbf{Y}_\tau\mathrm{d}\tau + \epsilon\mathbf{P}\mathbf{B}\,\mathrm{d}W_t \tag{4}$$

The time change theorem for Itô integrals [9, page 156] shows that

$$W_{\tau(t)} \sim \sqrt{\tau'(t)}\, W_t = \sqrt{\omega}\, W_t \tag{5}$$

Denoting as $\mathbf{X}_\tau$ the solution to the SDE system

$$\mathrm{d}\mathbf{X}_\tau = \frac{1}{\omega}\hat{\mathbf{A}}\mathbf{X}_\tau\mathrm{d}\tau + \frac{\epsilon}{\sqrt{\omega}}\hat{\mathbf{B}}\,\mathrm{d}W_\tau \tag{6}$$

where $\hat{\mathbf{A}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}$, and $\hat{\mathbf{B}} = \mathbf{P}\mathbf{B}$, it follows that $\mathbf{X}_\tau \sim \mathbf{Y}_\tau$, because they are solutions for the same SDEs system, for two different realizations of the Wiener process.

It is important to stress that $\mathbf{X}_\tau$ and $\mathbf{Y}_\tau$ only coincide in distribution. In practical application this information is the most relevant, because knowledge of the statistical properties, in particular the moments of the distribution, is more important that the knowledge of the detailed solution for a single specific realization of the noise process.

B. Harvester electro-mechanical model

Concerning the electro-mechanical system that forms the harvester, we can give an explicit form to the SDE system (1) starting from the laws of classical mechanics, from the characterization of piezoelectric materials, and from the circuit description of the electrical load [11], [12], [14].

An energy harvester for ambient mechanical vibrations is typically composed by an oscillating structure responsible for capturing mechanical kinetic energy. A schematic representation is shown in figure 1. The oscillating structure is represented by a cantilever beam, fixed at one end to a vibrating support and with an inertial mass at the opposite

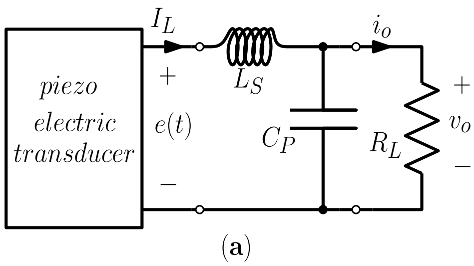

Fig. 2. Circuit representation of the LC matching network and of the elctrical load R_L.

end. Random vibrations of the support induce oscillations of the beam-mass system, that behaves as an inverted pendulum. The beam is covered with layers of piezoelectric materials that convert the mechanical stress and strain into electrical power

Coupling the mechanical oscillator to a lumped description of the piezoelectric material leads to

$$m\ddot{x} + \gamma\dot{x} + U'(x) + \alpha e = f_{\text{ext}}(t) \tag{7a}$$

$$C_{\text{pz}}\dot{e} + \alpha\dot{x} + I_{\text{L}} = 0 \tag{7b}$$

where the first equation (7a) describes the mechanical part, while (7b) represents the mechanical-electrical conversion. In system (7), m is the inertial mass, x is the state variable corresponding to the displacement of the mass from the rest position, γ is the internal friction constant, $U(x)$ is the elastic potential of the beam, α is the electro-mechanical coupling constant, e is the output voltage of the piezoelectric transducer, $f_{\text{ext}}(t)$ is the external force due to the vibrating support (a white Gaussian noise according to the previous discussion), C_{pz} is the electrical capacitance of the transducer, and I_{L} is the current through the electrical load. In the previous equations dots denote time derivative, while symbol $'$ refers to the derivation with respect to the function explicit variable.

For the sake of simplicity, it is assumed that the mechanical system is single-degree-of-freedom, that is oscillations occur along a single direction and the arc shaped motion can be approximated by a straight line. These approximations are justified if the beam has a rectangular cross-section, and if the oscillation amplitude is small enough with respect to the beam length. Moreover, the small oscillation assumption allows to use a linear pendulum model, assuming a quadratic form for the energy elastic potential of the beam, i.e. $U(x) = k\,x^2/2$.

C. Harvester and matching network model

The last stage of the model is the load. In the simplest (and most common) case, the electrical power must be provided to a device represented by an equivalent resistance R_L, which value is normally fixed by the nature of the load itself, and thus it cannot be tailored in the design procedure. However, as previously remarked, an LC matching network, ideally made of reactive elements only in order to avoid the introduction of losses, can be exploited to reduce the mismatch between the output of the transducer and R_L, as shown in Fig. 2 where L_S and C_P are the elements of the matching network. For the sake of simplicity, parasitic effects are assumed to be negligibly small, and the reactive components in the matching networks are assumed to be ideal elements. Notice that such matching network is characterized by a low-pass behavior, consistent with energy contents of mechanical vibrations that is usually concentrated at low frequencies. Notice also that perfect power transfer matching can be exactly obtained only at a single frequency, even if a wide body of literature suggests how to achieve a non ideal, however advantageous, matching over a wider bandwidth.

Kirchhoff laws and the characteristic relations of the reactive elements of the matching network yield

$$-I_L + C_P\dot{v}_o + G_L v_o = 0 \tag{8}$$

$$L_S\dot{I}_L + v_o - e = 0 \tag{9}$$

Combining (7), (8) and (9), and rewriting as an SDE system, finally provides the stochastic model

$$\mathrm{d}Z_1 = Z_2\,\mathrm{d}t \tag{10a}$$

$$\mathrm{d}Z_2 = \frac{1}{m}\left(-U'(Z_1) - \gamma\,Z_2 - \alpha\,Z_3\right)\mathrm{d}t + \frac{\epsilon}{m}\mathrm{d}W_t \tag{10b}$$

$$\mathrm{d}Z_3 = \frac{1}{C_{\text{pz}}}\left(\alpha Z_2 - Z_4\right)\mathrm{d}t \tag{10c}$$

$$\mathrm{d}Z_4 = \frac{1}{L_S}\left(Z_3 - Z_5\right)\mathrm{d}t \tag{10d}$$

$$\mathrm{d}Z_5 = \frac{1}{C_P}\left(Z_4 - G_L\,Z_5\right)\mathrm{d}t \tag{10e}$$

where $\mathbf{Z}_t = [Z_1, \ldots, Z_5]^T = [x, \dot{x}, e, I_L, v_o]^T$ is the state vector of electro-mechanical variables, and $\epsilon\,dW_t$ is the external forcing, that models ambient vibrations as a white Gaussian noise process with intensity ϵ.

The equations for the harvester in the absence of the matching network (i.e., the case of a purely resistive load directly connected to the harvester) can be easily recovered from (10). In particular, multiplying both sides of (10d) for L_S and setting $L_S = 0$ yields $Z_3 = Z_5$. Analogously, multiplying both sides of (10e) for C_P and setting $C_P = 0$ gives $Z_4 = G_L\,Z_5$. Substituting these results into (10c) we obtain the following equations for the energy harvester with resistive load:

$$\mathrm{d}Z_1 = Z_2\,\mathrm{d}t \tag{11a}$$

$$\mathrm{d}Z_2 = \frac{1}{m}\left(-U'(Z_1) - \gamma\,Z_2 - \alpha\,Z_3\right)\mathrm{d}t + \frac{\epsilon}{m}\mathrm{d}W_t \tag{11b}$$

$$\mathrm{d}Z_3 = \frac{1}{C_{\text{pz}}}\left(\alpha Z_2 - G_L\,Z_3\right)\mathrm{d}t \tag{11c}$$

The mathematical models for the energy harvester with resistive load has been extensively studied, and has been validated experimentally by several works [23]–[27].

III. Power balance, output voltage and conversion efficiency

Due to power balance, we can write the instantaneous power absorbed by the transducer as combination of the power transferred from the mechanical section to the transducer, and of the electrical power transferred from the transducer to the electrical load. Using the passive sign convention, considering the force $f_{\text{tr}}(t) = \alpha\, e = \alpha\, Z_3$ exerted by the mechanical part, we find

$$p_{\text{tr}}(t) = f_{\text{tr}}(t)\,\dot{x} - e\, I_L = C_{\text{pz}}\, e\, \dot{e} \tag{12}$$

where the last result follows from (7b). Integrating with respect to time and using an arbitrary constant K_E yields the energy stored in the transducer $E_{\text{tr}}(t) = C_{\text{pz}}\, Z_3^2/2 + K_E$.

The total energy stored in the harvester is the sum of the mass kinetic energy, of the elastic potential energy, and of the energy stored in the piezoelectric transducer and in the matching network elements. It takes the form

$$E(t) = \frac{1}{2} m\, Z_2^2 + U(Z_1) + \frac{1}{2} C_{\text{pz}}\, Z_3^2 + \frac{1}{2} L_S\, Z_4^2 + \frac{1}{2} C_P\, Z_5^2 + E_0 \tag{13}$$

where E_0 is an arbitrary constant.

Differentiating $E(t)$, exploiting (10) and applying Itô's lemma provides ($G_L = 1/R_L$)

$$\mathrm{d}E = \left(-\gamma Z_2^2 - G_L Z_5^2 + \frac{\epsilon^2}{2m} \right) \mathrm{d}t + \epsilon\, Z_2\, \mathrm{d}W_t \tag{14}$$

After taking stochastic expectation and using the martingale property of Itô stochastic integral, we find the power balance equation

$$\mathrm{E}\left[\frac{\mathrm{d}E(t)}{\mathrm{d}t} \right] = -\gamma\, \mathrm{E}\left[Z_2^2 \right] - G_L\, \mathrm{E}\left[Z_5^2 \right] + \frac{\epsilon^2}{2m} \tag{15}$$

At steady state, the harvester reaches an equilibrium where the average power injected by ambient vibrations: $P_{\text{in}} = \epsilon^2/(2m)$, equals the average power (internally dissipated by friction and absorbed by R_L) $P_{\text{out}} = \gamma\, \mathrm{E}\left[Z_2^2 \right] + G_L\, \mathrm{E}\left[Z_5^2 \right]$.

Finally, the power efficiency η is defined as the ratio between the average power absorbed by the load and injected by the ambient vibrations

$$\eta = \frac{2m\, G_L}{\epsilon^2}\, \mathrm{E}\left[Z_5^2 \right] \tag{16}$$

A. Estimation of the second-order moments

The previous equations are a function of the second order moments of the solution of the SDE system (10), that in the case of a linear harvester can be found as follows. Substituting the quadratic elastic potential for the beam in the SDE system and deriving the dimensionless version using as a normalization diagonal matrix

$$\mathbf{P} = \mathrm{diag}[l_0^{-1}, T l_0^{-1}, C_{\text{pz}} Q_0^{-1}, T Q_0^{-1}, C_{\text{pz}} Q_0^{-1}] \tag{17}$$

where $l_0 = 1$ and $Q_0 = 1$ are normalizing constants with dimension of a length and of a charge, respectively, and $T = 1/\omega = \sqrt{m/k}$ is a normalization time. The dimensionless SDE system is

$$\mathrm{d}\mathbf{X}_t = \hat{\mathbf{A}}\, \mathbf{X}_t\, \mathrm{d}t + \epsilon \hat{\mathbf{B}}\, \mathrm{d}W_t \tag{18}$$

where

$$\hat{\mathbf{A}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & -\rho & -\beta & 0 & 0 \\ 0 & \alpha & 0 & -1 & 0 \\ 0 & 0 & \mu & 0 & -\mu \\ 0 & 0 & 0 & \eta & -\delta \end{bmatrix} \qquad \hat{\mathbf{B}} = \begin{bmatrix} 0 \\ \sigma \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

having defined the positive parameters

$$\rho = \frac{\gamma}{\sqrt{mk}}, \quad \beta = \frac{\alpha}{C_{pz} k}, \quad \mu = \frac{m}{k\, C_{pz}\, L_S}$$

$$\eta = \frac{C_{pz}}{C_P}, \quad \delta = \frac{G_L}{C_P} \sqrt{\frac{m}{k}}, \quad \sigma = \frac{1}{m} \left(\frac{m}{k} \right)^{\frac{3}{4}}$$

Taking stochastic expectation in (18), we obtain the ordinary differential equation (ODE) system

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{E}[\mathbf{X}_t] = \hat{\mathbf{A}}\, \mathrm{E}[\mathbf{X}_t] \tag{20}$$

where, due to the positive parameters, matrix $\hat{\mathbf{A}}$ has negative eigenvalues. Thus, $\mathrm{E}[\mathbf{X}_t] \to 0$ for $t \to +\infty$.

Concerning the second order moments, we have

$$\begin{aligned} \mathrm{d}\left(\mathbf{X}_t \mathbf{X}_t^T \right) &= \mathrm{d}\mathbf{X}_t\, \mathbf{X}_t^T + \mathbf{X}_t\, \mathrm{d}\mathbf{X}_t^T + \mathrm{d}\mathbf{X}_t\, \mathrm{d}\mathbf{X}_t^T \\ &= \left(\hat{\mathbf{A}}\, \mathbf{X}_t\, \mathbf{X}_t^T + \mathbf{X}_t \mathbf{X}_t^T\, \hat{\mathbf{A}}^T + \epsilon^2 \hat{\mathbf{B}}\, \hat{\mathbf{B}}^T \right) \mathrm{d}t \\ &\quad + \epsilon \left(\hat{\mathbf{B}} \mathbf{X}_t^T + \mathbf{X}_t \hat{\mathbf{B}}^T \right) \mathrm{d}W_t \end{aligned} \tag{21}$$

where we used Itô's lemma, e.g. $\mathrm{d}t^2 = \mathrm{d}t\mathrm{d}W_t = 0$, and $\mathrm{d}W_t^2 = \mathrm{d}t$. Taking expectations on both sides, using the martingale property of Itô's integral and the fact that asymptotically $\mathrm{E}[\mathbf{X}_t] \to 0$, yields the Lyapunov equation

$$\frac{d\boldsymbol{\sigma}}{dt} = \hat{\mathbf{A}}\, \boldsymbol{\sigma} + \boldsymbol{\sigma}\, \hat{\mathbf{A}}^T + \epsilon^2 \hat{\mathbf{B}}\, \hat{\mathbf{B}}^T \tag{22}$$

where $\boldsymbol{\sigma} = \mathrm{E}[\mathbf{X}_t \mathbf{X}_t^T]$ is the symmetric covariance matrix. Since $\hat{\mathbf{A}}$ is a stable matrix, and $\hat{\mathbf{B}}\hat{\mathbf{B}}^T$ is symmetric, the solution of (22) is unique. The asymptotic value is obtained solving the stationary Lyapunov equation

$$\hat{\mathbf{A}}\, \boldsymbol{\sigma} + \boldsymbol{\sigma}\, \hat{\mathbf{A}}^T + \epsilon^2 \hat{\mathbf{B}}\, \hat{\mathbf{B}}^T = 0 \tag{23}$$

For an effective design of the matching network, we solved the stationary Lyapunov equation (23) for several values of the parameters L_S and C_P. Fig. 3 shows the root mean square value of the output voltage $v_{o(\text{rms})} = \sqrt{\mathrm{E}[v_o^2(t)]}$ as a function of the values of L_S and C_P. The other parameters are summarized in Table I, and they are well comparable to those used in several other recent works [28]–[31]. The output voltage shows a maximum for $L_S^{(\text{opt})} = 22.91$ H, and $C_P^{(\text{opt})} = 8.53$ nF, with $v_{o(\text{rms})}^{\max} = 20.59$ V.

It is worth mentioning that the relatively high optimum value of the inductance in the matching network is a consequence of the normalization assumed for the mass, set to the relatively high value of 1 g. It is well known that piezoelectric transducers require high inductance values for

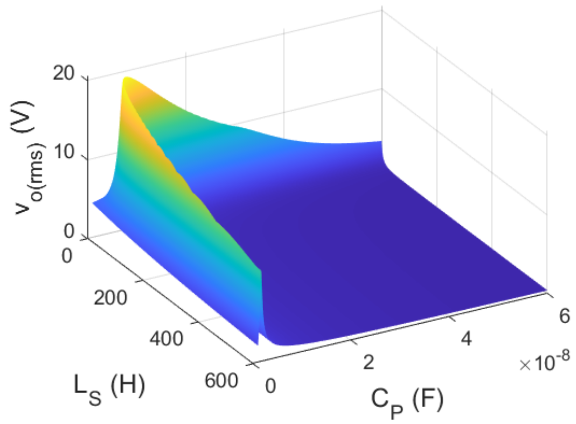

Fig. 3. Root mean square of the output voltage as a function of the matching network parameters L_S and C_P.

TABLE I
VALUES OF THE ENERGY HARVESTER PARAMETERS

Parameter	Value
m	1 g
γ	0.012 Ns/m
k	$5.4046 \cdot 10^3$ N/m
C_{pz}	80 nF
R_L	1 MΩ
α	0.0042 N/V (As/m)
ε	10^{-3} (dimensionless)

shunting and matching. Realization of high value inductances is an important research topic, and very promising results have been recently obtained, see for example [32].

For the sake of comparison, the energy harvester with resistive load (i.e., $C_p = L_S = 0$) shows an output voltage $v_{o(\mathrm{rms})}^{\max} = 4.43$ V. The average output voltage, average output power and power efficiency for the two harvesters are summarized in Table II.

IV. Conclusions

Energy harvesting from ambient mechanical vibrations is a promising technology to solve the problem of powering miniaturized, remote located and difficult to access electronic systems for the Internet of Things applications.

Energy harvesters should have very high efficiency, to cope with the limited energy density of ambient vibrations. Their design is particularly challenging, not only because the random nature of mechanical vibrations demands for a mathematical modelling in terms of stochastic processes, but also because they are electro-mechanical systems, with a significant impedance mismatch between the mechanical and the electrical domains.

In this work we have presented a detailed analysis of linear energy harvesters for ambient mechanical vibrations based on stochastic calculus. Parasitic vibrations have been modelled as a white Gaussian noise. A system of stochastic differential equations have been derived for the mechanical part of the harvester, the piezoelectric transducer and the electrical load. The system performance, namely the average electrical power provided to the load and the power conversion efficiency, have been traced back to the second order moments of the stochastic differential equations system model. We have shown that first and second order moments can be calculated using stochastic calculus, solving a stationary Lyapunov equation.

To cope with the impedance mismatch problem, we have designed a matching network that can be interposed between the energy harvester and the electrical load. The matching network parameters have been optimized in order to maximize the two performance indicators above, evaluating the covariance matrix expressed in the form of a stationary Lyapunov equation for a linear harvester. This approach permits to avoid Monte-Carlo analysis, based on long, resource demanding numerical simulations.

Our analysis shows that application of the matching network significantly boosts the performance, thus demonstrating the importance of this stage in the design of an optimized harvester.

TABLE II
ROOT MEAN SQUARE VALUE OF THE OUTPUT VOLTAGE, OUTPUT AVERAGE POWER AND POWER EFFICIENCY FOR THE ENERGY HARVESTER WITH THE TWO DIFFERENT LOAD SETUPS.

Configuration	Voltage (rms)	Maximum power	Efficiency
Resistive load	4.43 V	19.61 μW	3.9 %
Matched load	20.59 V	424.1 μW	84.8 %

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