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A Multi-period Location–Allocation Model for Nursing Home Network Planning Under Uncertainty

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Abstract This paper proposes a multi–period location–allocation problem arising in nursing home network planning. We present a strategic model in which the improvement of service accessibility through the planning horizon is appropriately addressed. Unlike previous research, the proposed model modifies the allocation pattern to prevent unacceptable deterioration of the accessibility criterion. In addition, the problem is formulated as a covering model in which the capacity of facilities as well as the demand elasticity are considered. The uncertainty in demands within each time period is captured by adopting a distributionally robust approach. The model is then applied to a real case study for nursing home planning network in Shiraz city, Iran.

Keywords Location–allocation problem, Health care, Strategic planning, Stochastic programming

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1 Introduction

An aging revolution is taking place world–wide. Increased longevity is one of the most important success of our era but it also raises a challenge for health systems that are put under the pressure to reform their care organization to be sustainable for an aging society. These increased expectations should be reconciled with the limited resources available. In 2012, people aged 60 years and more were 0.8 billion, 11% of the world population. By 2030 they will number 1.4 billion and will represent up 17% of the world population. By 2050, this number will rise up to 22% [37]. In addition, increased urbanization and migration will result in older people living alone. Each country will need a comprehensive approach to make the necessary transformation and to meet the challenge.

Improving services for older people entails to consider each component of care system as, for instance, specialized clinics, home care services, and nursing homes.

The impact of the aging society will fall predominantly on the long–term care sector requiring appropriate re–design of the long–term care services, including supported self–care, home–based and, especially, home nurses. This paper is motivated by the real problem of creating an efficient nursing home network to satisfy the elderly care demands in emerging countries. In particular, we shall focus on the Middle East considering Iran. In the last years, the Iranian society has experienced an exceptional increase of life expectancy: in 1960 this index was around 44 years against 73 years in 2012. It is supposed that even greater thresholds could be achieved with a widespread provision of public health care services in which all age groups are served equitably, including the elderly population (people aged 60 and over). The need of designing an efficient long–term care network is a quite new requirement in Iran, that for tradition, is a family–centered society. If in the past the idea of moving elderly family members to nursing home has always been disapproved, in recent years this trend has been inverted due to social changes, increased socio–economic difficulties and the opportunity to receive more professional care.

The problem of designing an efficient nursing home network has a strategic nature and involves decisions that have an impact over an extended planning horizon. The limited financial resources are typically spread over a time horizon making the adoption of a myopic policy, that ignores the inherent dynamic nature of the problem, highly inefficient. On the contrary, a long-sighted view, can be vital for the system financial survival.

To address this issue, in this paper, we propose a multi–period location model that incorporates the dynamic evolution of the system throughout the planning horizon; with the provision of new financial resources, new facilities can be located and additional demand can be gradually satisfied. To this aim, in consecutive periods, newly established facilities are located and the assignment pattern is improved assigning all previously covered demands to closer facilities, if possible. In general, by applying this incremental approach, the distance between covered demands and facilities sequentially reduces over time and the accessibility (perceived as an important service level) is improved over the entire planning horizon. Indeed, in presence of finite financial resources the system can fall, relatively easily and quickly, into very poor service levels, especially when the demand variability is high.

To the best of our knowledge, there is not any previous research on location literature addressing the strategic design of the nursing home network. In addition, although there is a vast literature on multi–period location models, only [1,3] address the important issue of satisfying the demands in an incremental fashion, whereas the variability of the demand of service, along the horizon, has been typically neglected [14,23]. This poses a challenge, since the determination of the optimal location of facilities at the beginning of the horizon should be made before the actual amount of demand is available.

In this paper, we deal with this issue by adopting a distributionally robust approach. We consider a general case in which only the mean and the deviation of the stochastic demands over each time period are known. Moreover, we opt for a risk–averse view point, considering the service levels as probabilistic constraints to be satisfied with a given reliability level [8,10,11]. We point out that there is a strong background on applying the chance constrained approach for strategic planning in the health care sector, supported by its risk–averse characteristic [7,16]. As a risk–averse approach, the chance constrained paradigm allows the decision maker to capture the demand uncertainty and to exclude the more risky situations depending on the aversion level. Compared with risk–neutral approach ([14,23]), in which only the expected values are considered, the risk–averse framework takes also the deviations of the uncertain parameters into account.

Another distinctive feature of the proposed model is the incorporation of the elasticity of the demand with respect to the distance traveled by the users (distance–elasticity of demand). There is a general consensus on the elasticity of demand in the health care sector [34,35]. The explicit consideration of the distance–elasticity of the demand enables the managers to gain valuable information about the participation level for care services, supporting in a more realistic fashion the decisions of upgrading or extending the facilities in the long–term.

To address the elasticity of demands, we define a user–specific distance threshold reflecting the preferences of users to access the service. This threshold is different from the manager–specific threshold which is related to the covering nature of the problem and represents the manager's preferences.

The main contributions of the proposed model are as follows:
(1) The model proposes a multi–period framework in which demand nodes are incrementally served.
(2) Unlike previous related research [1,3], the proposed model modifies the allocation pattern to prevent unacceptable deterioration of the accessibility criterion. In addition, the problem is formulated as a covering model in which the capacity of facilities is also considered.
(3) The uncertainty in demands within each time period is captured by adopting a distributionally robust chance constrained approach. In addition, the

model also incorporates the distance–elasticity of demands.

The rest of the paper is organized as follows: Section 2 presents a brief review on existing relevant literature. Section 3 describes the problem and presents a stochastic formulation along with its deterministic equivalent counterpart. Section 4 presents the real case study and shows the improvements achievable when implementing the recommendations provided by the model in terms of location–allocation configuration in the nursing home network in Shiraz city. Finally, conclusions and findings are reported in Section 5.

2 Literature Review

There is a vast literature on the development of single period location–allocation models in the health care context [6, 7, 19, 20]. Many existing researches have shown the advantage deriving from the adoption of a multi–period programming framework, when compared with a single period one, to deal with strategic location–allocation decisions. In particular, there is a wide literature on the application of multi–period location models for the public service sector [28, 30] and, especially, in the health care field [4].

Nevertheless, there are only a few papers in the literature addressing multi–period location–allocation models at the presence of stochasticity [21]. Among them, we refer to a recent work of Markovic et al. [21]. Recognizing the sparseness of literature, the authors presented a multi–period model to locate a set of flow–capturing facilities aimed at intercepting the stochastic traffic flows with evasive behavior. The proposed model allows the adjustment of facility locations over different time periods. A Lagrangian relaxation heuristic is proposed and tested on two road networks.

Adopting a risk–neutral approach, Albareda–Sambola et al. proposed a multi–period location–allocation model under cost uncertainty [2]. They considered two alternative strategies, including the scenario–dependent case in which the decision locations are made gradually with the evolution of randomness over the planning horizon and the scenario–independent case, where the locations decisions are made in an a priori fashion at the beginning of the horizon. For the scenario–dependent strategy, they presented a multi–stage stochastic location model while for the a priori case, a two–stage model is defined.

In another paper, Nickel et al. presented a multi–period model for the facility location problem in supply chain, where demands and interest rates are affected by uncertainty and represented by a set of scenarios [27]. The problem is formulated as a multi–stage stochastic model in which the objective is expressed as the maximization of the total benefit and the achieved service level.

Hernandez et al. ([18]) studied a multi–period mathematical model for the prison selection problem in which the demands are represented as stochastic parameters. The model determines the location and the size of new facilities for each time period and the capacity upgrade for both existing and new prisons over the planning horizon. The objective function minimizes the opening and expansion costs, the costs of transferring the convicted inmates from the court to their assigned prison, and the cost of overpopulation in prisons which is, indeed, a penalty term. In order to address the accessibility issue, the objective function accounts for the closeness of the inmate's prison to the court which facilitates the frequent visit of inmates and their families. To incorporate the uncertainty in demands, the authors applied a scenario tree generation approach and then solved the resulted model using a branch–and–cluster coordination method. As a case study, the Chilean prison system has been considered.

To get the reader familiar with the main issues arising in the health care sector, in the following we review those researches addressing the location–allocation planning of health care facilities.

In [33], a mathematical model for designing a network of long –term care facilities is presented. To reflect the patient's preference, the model imposes the closest assignment property in which patients are assigned to the nearest open facility. They also recognized the changes in the demand pattern and suggested developing a multi–period model in which the variation of demand through different periods is captured.

In [36], Zahiri et al. proposed a multi–period location model for an organ transplant problem in which the uncertainty in input data (cost) is handled by using a robust probabilistic programming approach. They also extended the model to a bi–objective one in which the minimization of total traveled time is considered, underlining the importance of dealing with distance–based measures even when the allocation is not directly done between pair of facilities and users, but between pair of facilities.

Benneyan et al. [5] proposed a location–allocation model, as well as its extended multi–period counterpart, to address the fluctuation of demands over time, for Veterans Health Administration facilities. They considered the objective function as a weighted sum of conflicting criteria, including travel time, unoccupied capacity, and uncovered demands.

In [26], Ndiaye and Alfares presented a multi–period location–allocation model for the establishment of seasonal health care facilities serving transient populations. The objective function minimizes the sum of opening and operating costs as well as the total traveled distance. The adoption of the multi–period framework enables the managers to handle the seasonal variability in operating costs and demands. Although the coverage issue has not been considered in the model, improvement of the accessibility is obtained by the incorporation of a distance threshold.

Rodriguez–Verjan et al. proposed in [29] a multi–period location–allocation model for home care services to minimize the total cost in a multiple resource system. What distinguishes the paper from other works is the modelization of some peculiarities of health care systems, like the authorization, different resources, pathologies and their evolution in time.

In [17], Ghaderi and Jabalameli presented a multi–period location model considering budget constraint on investment during each period. The objective function minimizes the total travel and operating costs. Both fixed and operating costs for located facilities and constructed links over each period time are considered. As a case study, they also presented an application of the model in the health sector.

Two different two–stage stochastic programming models for multi–period hospital network planning are presented in [14, 23]. The uncertainty in demand and supply is captured using different scenarios embedded in a two–stage stochastic framework. In the first model, the allocation decisions are postponed in the second stage, when the uncertainty realizes, whereas in the second model both location and allocation decisions can be taken in the first stage.

In [31], a multi–period location–allocation model for emergency blood supply scheduling problem was presented. A set of temporary blood facilities are located and assigned to the blood donors such that the total cost is minimized. The cost function is expressed as the total cost of transporting blood from blood facilities to the center as well as the cost of relocating blood facilities within consecutive periods. In addition, the total amount of unmet demands is penalized in the objective function. A coverage distance is imposed limiting the allocation of blood donors to blood facilities within the coverage radius. The number of available blood facilities over the planning horizon is fixed and a demand coverage constraint is imposed to guarantee the satisfaction of a specified percentage of demands. Enhanced with a Lagrangian relaxation solution approach, the model was implemented on a case study.

In a recent paper, Correia and Melo ([15]) proposed a multi–period location model in which the sensitivity of customers to delivery lead times has been incorporated. The novelty of the model comes from differentiating the customers who make the most contribution to the company's profit– and that should be responded on time–, from the others, for whom a maximum allowed delay is considered. Meanwhile, a subset of time periods over the planning horizon is specified in which strategical decisions such as the opening of new facilities, the closure of the existing ones, and the capacity acquisition decisions for new facilities are made. The tactical decisions about the distribution of services to customers can be made in any time period. Some additional assumptions, like considering different capacity levels for the facilities sited at potential locations or limiting the number of times that the customers are responded with delay over the planning horizon, are also considered.

The most significant contribution of the reviewed literature relies on modeling features of health care problems which had not been addressed before. Despite their undeniable novelty, there are still some potential gaps to be filled. For instance, none of the aforementioned models addresses the importance and the possibility of improving accessibility and service level through the planning horizon. Neglecting the modification of the allocation pattern, when there is such a possibility, may result in an overestimate of the system performance. To partially address this issue, Albareda–Sambola et al. ([1])

proposed a multi–period incremental location model to serve demands incrementally over a discrete planning horizon. Later on, Albareda–Sambola et al. introduced, in [3], three different multi–period incremental location models, differing in the definition of variables and presented some computational comparisons.

To the best of the authors' knowledge, the aforementioned papers are the only existing ones addressing incremental demand serving and budget limitation in a multi–period location problem. Although both contributions recognize that the allocation pattern might change through different time epochs, they do not address its negative results nor provide a solution for that. In addition, service levels were not considered.

Moreover, all the aforementioned studies, more or less, recognize the stochastic nature of problems in health care [5, 17, 29, 33], but only a few of them deal with uncertainty [14, 23, 36].

In addition, they share the same idea of minimizing the total cost or/and total traveled distance without considering the importance of the coverage concept in public health sector. It is notable that covering a particular demand node within the manager–specific distance threshold does not necessarily mean that all citizens of that zone (or at least a significant portion of them) will refer to the assigned facility, unless the preferences of the users in some way are incorporated. Hence, it might be impossible to come up with a single distance threshold in which both user and manager preferences are taken into consideration. In addition to the manager–specific distance threshold, which is related to the covering nature of the model, a user–specific distance threshold can be defined reflecting the user preferences. This also facilitates the injection of distance–elasticity of demands.

Our paper tries to address these important issues, only partially investigated in the scientific literature.

3 Problem description and mathematical formulation

The nursing homes problem can be modeled as a covering location–allocation model in which a finite number of demand nodes (population centers) should be served by a number of facilities (nursing homes). Allocation of demand nodes to facilities is carried out by taking into account a (manager–specific) distance threshold, D, which prevents the assignment of demands to distant facilities, representing the covering nature of the model. The location of facilities is chosen from a set of prespecified potential sites. The assumptions of the proposed model are as follows:

- A demand zone is satisfied provided that a nursing home is located within the manager–specific distance threshold.
- Whenever a demand zone is satisfied, its demand should be also fulfilled in the subsequent periods.
- Each demand node must be served by at most one facility during any time period (single assignment property).

- Due to budget restrictions, at any time period, a limited number of nursing homes can be established.
- Once a nursing home is opened in a time period, it should be kept open for all subsequent periods.
- Each nursing home can host only a limited number of people which is fixed over all periods.

Using the multi–period framework, with the establishment of new facilities, some demand nodes, which were not covered during previous periods, might be served. To investigate the possibility of enhancing the service levels, the reassignment to farther facilities is prevented for all subsequent periods, whereas previously covered demands can be only reassigned to closer facilities. Hence, throughout the planning horizon, the distance between demands and facilities is sequentially reduced. In general, this strategy helps to improve the accessibility criterion over consecutive periods.

In order to address the distance–elasticity of demands, we introduce another distance threshold, denoted by $\hat{D}$, which represents the preferences of users. Then, we define the "correction function", which is a function dependent on the user–specific threshold that estimates the expected portion of the demands from a population center that actually refers to the assigned facility. Obviously, all the people living in a covered population center will not necessarily refer to the facility assigned to. Hence, it is reasonable to have an estimation about the real value of referred demands.

In deterministic strategic planning, uncertainty is usually ignored and uncertain quantities are typically replaced by a single value forecast. While this approach could be accepted for single period problems, it is not realistic in multi–period problems where the horizon may span fifteen years. In these cases, a wrong decision may have serious consequences for many years, causing a deterioration of the system performance. Given the long–term nature of the problem, even forecasting the demand is difficult for the presence of unforeseen fluctuations in the population as well as inaccurate predictions of death rates. In our model, therefore, we explicitly account for uncertainty in the demand.

There are different approaches to deal with uncertainty: robust and worst–case methods often provide very conservative solutions. Chance constrained programming explicitly limits the probability of constraints violations. Since the main goal of the model is to ensure the provision of the service, we formulate the nursing homes problem as a probabilistic model with chance constraints.

In particular, our model includes, for each time period, a probabilistic constraint assuring that the stochastic demand can be covered by the opened nursing homes with a given probability value. This, in turn, enables the constraint to be violated with an acceptable violation probability, which is the risk the decision maker is willing to bear.

Since very often the assumption of full knowledge of the distribution of the random parameters fails, the uncertain demands are represented as random variables with unknown probability distribution function, but known expected value and variance. Under this assumptions, we formulate a distributionally robust problem, in which the nursing home network is designed to minimize the total number of uncovered demand points, while the chance constraints on the capacity of each nursing home are formulated considering any distribution with the given mean and variance. This approach is especially beneficial for cases in which scant information about nursing homes demand is available.

3.1 The multi–period probabilistic location-allocation model

The following notation is used in the model formulation:

Sets and indices:
I : set of demand nodes indexed by i
J : set of potential facility sites indexed by j
$H = \{0, 1, \ldots, T\}$: set of time periods indexed by t (time period 0 represents a dummy period)

Input Data and Parameters:
d_{ij} : shortest distance from demand node i to facility j
D : maximum acceptable service distance from the decision maker's point of view,
$\hat{D}$: maximum acceptable service distance from the user's point of view, ($\hat{D} \leq D$)
a_{ij} : element of the covering matrix equal to 1 if $d_{ij} \leq D$ and to 0 otherwise
$h_{it}(\omega)$: random demand generated at node i during period t
λ_{ij} : correction function, $\lambda_{ij} = (1 - \frac{\hat{d_{ij}}}{\hat{D}})\,\rho_0$, where ρ_0 is the participation probability when travel distance is negligible and $\hat{d_{ij}} = min(d_{ij}, \hat{D})$
Q_j : maximum amount of capacity for facility j
α : risk level
p_t : maximum number of facilities to be opened at period t

Decision Variables:

$$x_{ijt} = \begin{cases} 1 \text{ if demand node } i \text{ is allocated to facility } j \text{ during period } t \\ 0 \text{ otherwise} \end{cases}$$

$$y_{jt} = \begin{cases} 1 \text{ if a facility is located at site } j \text{ during period } t \\ 0 \text{ otherwise} \end{cases}$$

Considering the above notation, the mathematical formulation of the proposed multi–period probabilistic location–allocation model ($MPLM$) can be expressed as follows:

$$\min: \quad \sum_{t=1}^{T}\sum_{i\in I}(1-\sum_{j\in J}x_{ijt}) \tag{1}$$

s.t.

$$x_{ijt} \leq a_{ij}y_{jt} \quad \forall i \in I,\ \forall j \in J,\ t=1,\dots,T \tag{2}$$

$$\sum_{j\in J}x_{ijt} \leq \sum_{j\in J}x_{ij(t+1)} \quad \forall i \in I,\ t=1,\dots,T-1 \tag{3}$$

$$\sum_{j\in J}d_{ij}x_{ij(t+1)} \leq D(1-\sum_{j\in J}x_{ijt}) + \sum_{j\in J}d_{ij}x_{ijt} \quad \forall i \in I, t=1,\dots,T-1 \tag{4}$$

$$\sum_{j\in J}x_{ijt} \leq 1 \quad \forall i \in I,\ t=1,\dots,T \tag{5}$$

$$P\left(\sum_{i\in I}\lambda_{ij}h_{it}(\omega)x_{ijt} - Q_j y_{jt} \leq 0\right) \geq 1-\alpha \quad \forall j \in J,\ t=1,\dots,T \tag{6}$$

$$y_{jt} \leq y_{j(t+1)} \quad \forall j \in J,\ t=1,\dots,T-1 \tag{7}$$

$$\sum_{j\in J}(y_{j(t+1)} - y_{jt}) \leq p_{t+1} \quad t=0,\dots,T-1 \tag{8}$$

$$x_{ijt} \in \{0,1\} \quad \forall i \in I,\ \forall j \in J,\ t=1,\dots,T \tag{9}$$

$$y_{j0} = 0 \quad \forall j \in J \tag{10}$$

$$y_{jt} \in \{0,1\} \quad \forall j \in J,\ t=1,\dots,T \tag{11}$$

The objective function (1) minimizes the number of times that a demand node is not covered during the time horizon. Constraints (2) state that each demand node can be assigned only to open facilities which are within the distance threshold D. The elderly are not the only users of the system and the maximum acceptable service distance D was incorporated into the model not only for the sake of the residents of the nursing homes but their visitors and families too. In practice, the stay in nursing homes is more about years to days or months and families frequently travel to the nursing homes. Looking for possible ways to improve the accessibility of the systems, we came with the idea of imposing a maximum acceptable service distance, as a common idea in location–allocation literature. With the improvement of accessibility, the families are encouraged to go to the nursing homes more frequently which, in turn, has positive effects on the elderly's life as well. To be accessible, the daily nursing homes should not require traveling more than 5 or 6 km per day. This is a common idea which has been addressed in the public facility location context; for instance, in [18], the closeness of the prisons assigned to the prisoners with their families is taken into account.

Constraints (3) state that whenever a demand is covered, it should be covered for all upcoming periods. The next set of restrictions in (4) imply that, for each period, any previously covered demand is reallocated to a closer facility if possible; otherwise, the demand is covered by the same previous facility. Restrictions (5) state that each demand node, at any period, is served by at most one facility. The probabilistic capacity constraints in (6) ensure that the probability of not exceeding the capacity of each candidate facility, during each period, should be greater than or equal to a prespecified reliability level $1-\alpha$. It is worthwhile remarking that period-dependent capacities could be easily incorporated into the model by simply replacing Q_j with Q_{jt}. In our case study (see Section 4), based on the instructions imposed by the law to the governmental organization, the capacity of the nursing homes is fixed, and cannot be upgraded over the planning horizon.

Note that the term $\lambda_{ij} h_{it}(\omega)$ indicates the number of residents at demand node i who would, if assigned, effectively use the facility j during period t considering the distance–elasticity measure (see also [24,35]).

Constraints in (7) state that once a facility is located, it should remain open for all the subsequent periods.

Restrictions (8) impose a limit on the maximum number of newly established facilities at any period. Note that $\sum_{j=1}^{n}(y_{j(t+1)} - y_{jt})$ indicates the number of newly established facilities in period $t+1$ and y_{j0} represents the variable corresponding to the dummy period 0 which its value is set to zero. Finally, restrictions (9) - (11) define the nature of decision variables. We should mention that, based on the application at hand some modifications might be made on the model. For instance, the single assignment property can be mitigated allowing to split the demands of a node over different facilities reflecting the users' preferences. This can be easily incorporated by relaxing the binary variables x_{ijt}, leading to a more tractable problem. It is worthwhile mentioning that the users' preferences are incorporated into the model through the elasticity of the demands, and that, very often, the single assignment property is required by managers willing to allocate the cumulative demands of each demand zone to a single facility.

We should remark that different attitudes (egalitarian and utilitarian) may be considered in any health care model. The egalitarian approach considers an equal weight for different target demand points, whereas the utilitarian one focuses on high populated demand zones. The current assumptions of the model reflects an egalitarian approach, covering as much zones as possible, along the horizon, regardless of the variability of the demands in each zone. The utilitarian approach could be implemented incorporating into the objective function the expected demand, as usual in the maximal covering literature. Since the proposed model was motivated by a real case study, the objective function reflects the decision makers' preferences, and focuses on covering as many areas as possible. This was also motivated by the fact that there is not a significant difference among demand levels over different areas.

3.2 The deterministic equivalent formulation

We show that the problem $MPLM$ actually admits an explicit conic reformulation, which can then be conveniently solved using an outer approximation technique.

Let assume that the random demand value $h_{it}(\omega)$ follows an arbitrary distribution function but its mean (μ_{it}) and its variance (σ_{it}^2) are known. Specifically, we show that for any α value within $(0,1)$, the distributionally robust chance constraint

$$\inf_{h_{it}(\omega)\sim(\mu_{it},\sigma_{it}^2)} \mathrm{P}\left(\sum_{i\in I}\lambda_{ij}h_{it}(\omega)x_{ijt} - Q_j y_{jt} \le 0\right) \ge 1-\alpha \tag{12}$$

is equivalent to the convex second–order cone constraint [9,13]

$$\sqrt{\beta_\alpha \sum_{i\in I} x_{ijt}^2 \hat{\sigma}_{ijt}^2} - Q_j y_{jt} + \sum_{i\in I}\hat{\mu}_{ijt}\, x_{ijt} \le 0, \quad \forall j\in J,\ t=1,\dots,T \tag{13}$$

where $\hat{\mu}_{ijt} = \lambda_{ij}\mu_{it},\ \hat{\sigma}_{ijt} = \lambda_{ij}\sigma_{it}$, and $\beta_\alpha = \frac{1-\alpha}{\alpha}$.
In fact, we can rewrite (12) as

$$\inf_{\mathbf{h_t}\sim(\hat{\mathbf{h}},\mathbf{\Gamma})} \mathrm{P}\left(\mathbf{h}^T\,\tilde{\mathbf{x}} \le 0\right) \ge 1-\alpha \tag{14}$$

where $\mathbf{h_t}$ is the vector of $\left(h_{1t},\, h_{2t}\, \cdots,\, h_{|I|t},\, -Q_j y_{jt}\right)^T$ and $\tilde{\mathbf{x}}_{\mathbf{jt}}$ denotes the vector $\left(\lambda_{1j}\,x_{1jt},\, \lambda_{2j}\,x_{2jt},\, \cdots,\, \lambda_{|I|j}\,x_{|I|jt}, 1\right)$. In a similar way, $\hat{\mathbf{h}}$ and $\mathbf{\Gamma}$ represents the vector of expected values and the covariance matrix associated with $\mathbf{h_t}$. From now on, for the sake of simplicity, we denote $\mathbf{h_t}$ and $\tilde{\mathbf{x_{jt}}}$ by $\mathbf{h}$ and $\tilde{\mathbf{x}}$, respectively.
Let assume that $\mathbf{h} = \hat{\mathbf{h}} + \mathbf{\Gamma_f}\,\mathbf{z}$, where $\mu_{\mathbf{z}} = \mathbf{0}$, $\sigma_{\mathbf{z}}^2 = \mathbf{I}$, and $\mathbf{\Gamma_f}$ is the full-rank factorization matrix such that $\sigma^2(\mathbf{h}) = \mathbf{\Gamma_f}\,\mathbf{\Gamma_f}^T$. The following two cases are possible.

1. $\mathbf{\Gamma_f}^T\,\tilde{\mathbf{x}} \neq 0$. In this case, we have

$$\sup_{\mathbf{h}_t\sim(\hat{\mathbf{h}},\mathbf{\Gamma})} \mathrm{P}\left(\mathbf{h}^T\,\tilde{\mathbf{x}} \le 0\right) = \sup_{z\sim(\mathbf{0},\mathbf{I})} \mathrm{P}\left(\mathbf{z}^T\,\mathbf{\Gamma_f}^T\,\tilde{\mathbf{x}} \ge -\hat{\mathbf{h}}\,\tilde{\mathbf{x}}\right) = \frac{1}{1+q^2} \tag{15}$$

   where $q^2 = \inf_{\mathbf{z}^T\,\mathbf{\Gamma_f}^T\,\tilde{\mathbf{x}} > -\hat{\mathbf{h}}\,\tilde{\mathbf{x}}} \|z\|^2$ (The last equity holds based on Theorem 9 in [22]).

   (a) If $\hat{\mathbf{h}}\,\tilde{\mathbf{x}} > 0$, then by taking $\mathbf{z} = \mathbf{0}$, we can obtain the infimum $q^2 = 0$.

(b) If $\hat{\mathbf{h}}\,\tilde{\mathbf{x}} \leq 0$, then the problem is expressed as determining the squared distance from the origin of the hyperplane $\left\{\mathbf{z} \,|\, \mathbf{z}^T\,\Gamma_f^T\,\tilde{\mathbf{x}} = -\hat{\mathbf{h}}\,\tilde{\mathbf{x}}\right\}$ which is solved by taking $q^2 = \frac{(\hat{\mathbf{h}}\,\tilde{\mathbf{x}})^2}{\tilde{\mathbf{x}}^T\boldsymbol{\Gamma}^T\tilde{\mathbf{x}}}$.
So we have

$$q^2 = \begin{cases} 0 & \tilde{\mathbf{x}}^T\,\hat{\mathbf{h}} > 0 \\ \frac{\tilde{\mathbf{x}}^T\,\hat{\mathbf{h}}^2}{\sigma(\tilde{\mathbf{x}}^T\,\hat{\mathbf{h}})} & \tilde{\mathbf{x}}^T\,\hat{\mathbf{h}} \leq 0 \end{cases}$$

which represents a closed–form expression for q^2. Hence, the constraint in (14) holds iff $\frac{1}{1+q^2} \leq \alpha$. The last equation holds iff $\tilde{\mathbf{x}}^T\,\hat{\mathbf{h}} \leq 0$ and $(\tilde{\mathbf{x}}^T\,\hat{\mathbf{h}})^2 \geq \sigma(\tilde{\mathbf{x}}^T\,\hat{\mathbf{h}})\,\frac{1-\alpha}{\alpha}$ which is equivalent to (13).

2. $\boldsymbol{\Gamma_f}^T\,\tilde{\mathbf{x}} = 0$. In this case, we simply conclude that $\tilde{\mathbf{x}}^t\,\boldsymbol{\Gamma}\,\tilde{\mathbf{x}} = 0$ which results in

$$\inf_{\mathbf{h}\sim(\hat{\mathbf{h}},\boldsymbol{\Gamma})} \mathrm{P}\Bigg(\mathbf{h}^T\,\tilde{\mathbf{x}} \leq 0\Bigg) = 1,\ \text{if}\,\tilde{\mathbf{x}}^T\,\hat{\mathbf{h}} \leq 0 \tag{16}$$

Considering $\tilde{\mathbf{x}}^t\,\boldsymbol{\Gamma}\,\tilde{\mathbf{x}} = 0$, the equivalency of (12) and (13) is obtained.

By replacing restrictions (6) with (13) in $MPLM$, we come up with an integer non–linear deterministic equivalent formulation named as $(NDMPLM)$. To solve the model, a linearization of the model could be applied (as shown in Subsection 6.1 in Appendix) and then off–the shelf software such as CPLEX could be used.

Although, in theory, a linearized problem is computationally more attractive, in practice, the solution of the linearized MIP model, even for medium size problems, can be significantly time–consuming. In fact, the linearized model involves $3\ |I|\ |J|\ (|H|-1)\big(\,|I|+1\big)$ more constraints and $|I|\ |J|\ (|H|-1)\big(\,|I|+1\big)$ more binary variables than the nonlinear model.
Considering the special structure of the model, we can show that its continuous relaxation is convex.

Lemma 1 *The function* $F_{jt}(\mathbf{x},\mathbf{y}) = \sqrt{\beta_\alpha\,\sum_{i\in I} x_{ijt}^2\hat{\sigma}_{ijt}^2} - Q_j y_{jt} + \sum_{i\in I}\hat{\mu}_{ijt}\,x_{ijt}$ *is convex.*

Proof: We can rewrite F_{jt} as $Z(\mathbf{x}) - Q_j\,y_{jt} + \sum_{i\in I}\hat{\mu}_{ijt}\,x_{ijt}$ in which $Z(\mathbf{x}) = \sqrt{\beta_\alpha\,\sum_{i\in I} x_{ijt}^2\,\hat{\sigma}_{ijt}^2}$, or equivalently, $Z(\mathbf{x}) = \sqrt{\mathbf{x}\,\beta_\alpha\,\hat{\sigma}^2\,\mathbf{x}^T}$. As $Q_j y_{jt}$ and $\sum_{i\in I}\hat{\mu}_{ijt}\,x_{ijt}$ are linear terms, it suffices to show the convexity of $Z(\mathbf{x})$. Since $\beta_\alpha\,\hat{\sigma}^2$ defines a semidefinite positive matrix, there is a Cholesky decomposition for it as $\beta_\alpha\,\hat{\sigma}^2 = L\,L^T$. Hence, we can rewrite Z as $Z = \sqrt{\mathbf{x}\,L\,L^T\,\mathbf{x}^T} = \parallel \mathbf{x}\,L \parallel$, where $\parallel . \parallel$ is the Euclidean norm.
Assume $\mathbf{x_1},\ \mathbf{x_2}$ as two arbitrary vectors and $\lambda \in (0,1)$. We have

$$\parallel (\lambda\,\mathbf{x_1} + (1-\lambda)\mathbf{x_2})\,L \parallel \leq \parallel \lambda\,\mathbf{x_1}\,L \parallel + \parallel (1-\lambda)\mathbf{x_2}\,L \parallel$$

$$= \lambda \parallel \mathbf{x_1}\, L \parallel + (1-\lambda) \parallel \mathbf{x_2}\, L \parallel .$$

and the proof is complete.

Therefore, we may apply the outer approximation algorithm (AOA) on $NDMPLM$ to obtain the global optimal solution ([32]).

4 Case Study

4.1 Case study description and input data

In this section, we apply the proposed model on a real case study for the nursing home network design problem in Shiraz city, the sixth most populous city in Iran and the capital of Fars province. The model has been implemented in AIMMS 4.1 and solved by AOA [12]. The experiments were executed on a laptop Intel core i7 with a 2.7 GHz processor and 4 GB RAM. The average solution time, for all experiments, was less than 50 seconds.

For the current case study, an extended planning horizon including three periods from 2015 to 2025 has been considered.

Currently, seven nursing homes provide the residents with elderly care services such as rehabilitation, education, and welfare services [39]. Nursing homes are allowed to admit at most seventy recipients. The municipality of Shiraz, including nine municipal zones, is divided into 76 population centers, based on postal divisions [38]. Each population center represents a demand node in this study.

In addition to the location of the seven existing nursing homes, 17 more candidate locations have been considered. This enables the managers to site new facilities as well as relocating the existing ones, if necessary.

Tables 9, 10, and 11 in Appendix, show the location coordinates of all the population centers and the candidate facility locations. See also Fig. 1.

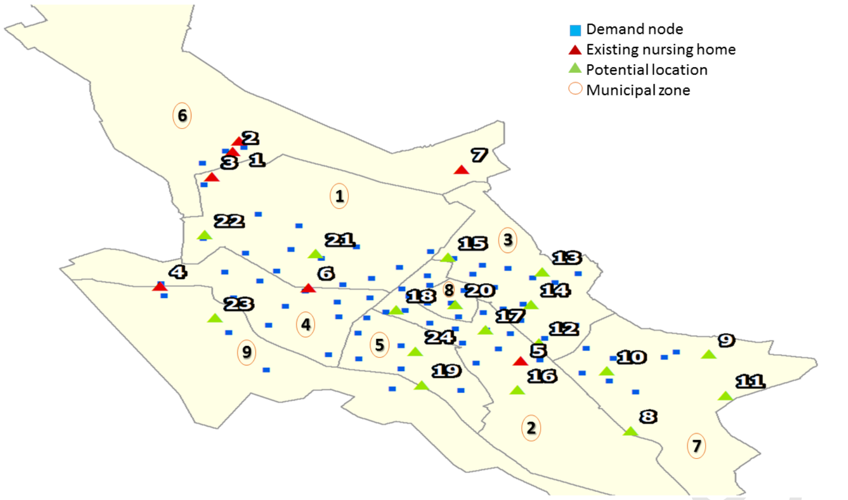

Fig. 1 Spatial distribution of demand nodes and the locations of nursing homes in Shiraz city

To estimate the Shiraz population, we applied the population projection data extracted from the UN population projection reports [40]. To estimate the elderly population during different years, we set the fertility rate based on the medium fertility rate scenario, introduced by the Population Division of the Department of Economic and Social Affairs of the United Nations [25].

Analyzing the statistical reports published by the Shiraz municipality [38], from 2006 to 2009, we found approximately an identical trend for population distribution of each municipal zone over different periods. The estimated demand at each demand zone was considered as its expected value and the variance was set to 0.2 of the expected value. We also assumed that the demand of each municipal zone is uniformly distributed among its nodes.
Table 1 shows the elderly population at any municipal zone for the next years.

Table 1 Estimated elderly population

Municipal zone	Elderly population during different years		
	2015	2020	2025
1	17096	21743	27840
2	17695	22505	28817
3	16188	20589	26363
4	17889	22751	29132
5	12778	16251	20809
6	13823	17581	22511
7	12772	16244	20799
8	5029	6396	8190
9	9843	12519	16029

The Euclidean distance was used to measure the travel distance between demand nodes and the nursing homes sites (see Subsection 6.2 in Appendix). Other characteristics of the problem are summarized in Table 2.

Table 2 Case study inputs

			p_t						
$\lvert I \rvert$	$\lvert J \rvert$	T	$t = 1$	$t = 2$	$t = 3$	α	D	$\hat{D}$	Q_j
76	24	3	2	3	5	0.05	5	5	70

4.2 Results and findings

Fig. 2 represents the optimal location of nursing homes, obtained by solving the model with $\alpha = 0.05$. The circles around the optimal locations $3, 13, 19$, and 23 represent the considered coverage radius.

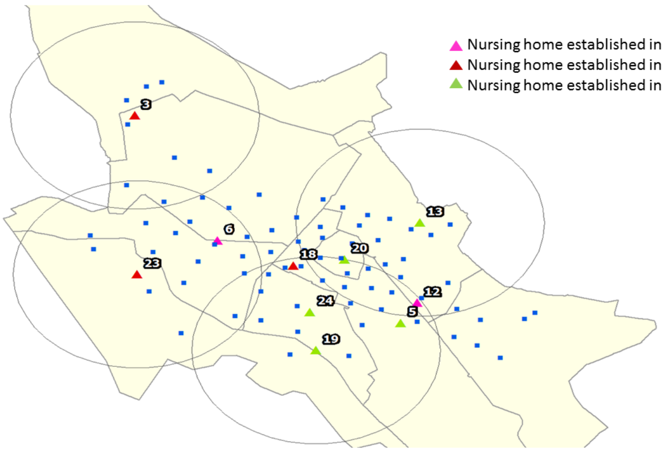

Fig. 2 Spatial distribution of optimal nursing homes' locations during each period

In addition, Table 3 shows the optimal location of active nursing homes for different periods.

Table 3 The optimal nursing homes' sites

Nursing home number	Time period $t = 1$	$t = 2$	$t = 3$
location3		*	*
location5			*
location6	*	*	*
location12	*	*	*
location13			*
location18		*	*
location19			*
location20			*
location23		*	*
location24			*

Based on the obtained results in Table 3, only three out of seven existing nursing homes are present in the optimal solutions. This shows that the current configuration of system can be improved by relocating nursing homes 1, 2, 4, and 7.
By calculating the distance between nodes and their assigned facility, we observed that the distance traveled by all covered demands is gradually decreasing through the planning horizon or at least is constant. The mean distance traveled by the residences is equal to 2.69, 2.61, and 2.02 kilometers with a deviation of 1.51, 1.27, and 1.12 kilometers, along the periods 1, 2, and 3, respectively (see Table 12 in Appendix). All demand nodes during periods 2 and 3 are covered and the number of uncovered nodes within the first period is limited to 9, including nodes 1, 2, 6, 27, 28, 29, 31, 57, and 58.
Table 4 classifies the nodes based on the relative improvement in the traveled distance within four categories. The relative improvement in the accessibility criterion, $s(i,t)$, for each covered node i over each period t, is calculated as follows.

$$s(i,t) = \frac{\sum_{j\in J} d_{ij}x_{ij(t-1)} - \sum_{j\in J} d_{ij}x_{ijt}}{\sum_{j\in J} d_{ij}x_{ij(t-1)}},\ t = 2,\ 3.$$

Table 4 Classifying covered nodes based on the relative improvement in accessibility criterion

Relative accessibility improvement	Time period $t=2$	$t=3$
$s(i,t) \leq 25\%$	$19, 30, 40, 41, 71, 73$	$13, 17, 25, 41, 43, 47, 48, 57, 59, 62, 66, 68$
$25\% < s(i,t) \leq 50\%$	$7, 9, 10, 32, 45, 67, 76$	$1, 7, 12, 14, 19, 44, 46, 56, 58, 70$
$50\% < s(i,t) \leq 75\%$	$8, 11, 23, 24, 25$	$3, 5, 6, 8, 18, 20, 26, 60, 61, 65$
$s(i,t) > 75\%$	$21, 22$	$45, 64$

The first row in Table 4 represents the index of demand nodes with up to 25% improvement in the accessibility criterion over periods 2 and 3. The second row shows the nodes with at least 25% and at most 50% improvement in $s(i,t)$. In a similar way, other rows present similar results for higher values of $s(i,t)$.
The maximum relative improvement in the accessibility criterion was, respectively, about 91% and 92% for demand nodes 21, 22 and 45, 64 over periods $1-2$ and $2-3$ (last row in Table 4). Obviously, the traveled distance of the demand nodes not reported in Table 4 does not change over different periods. Since we are addressing the multi–period service level based location problem in a coverage context, it can be possible that some demand nodes are not satisfied at the end of the planning horizon. Such demands can be assigned to the closest open facility provided that the facility capacity is increased or some external resources are available to serve unsatisfied demands.

Of course, the coverage radius strongly influences the system behavior (Fig. 3). By increasing the distance threshold values from 5 to 6 kilometers, it is possible to double the coverage, but nasty results are obtained for very narrow coverage radius values.

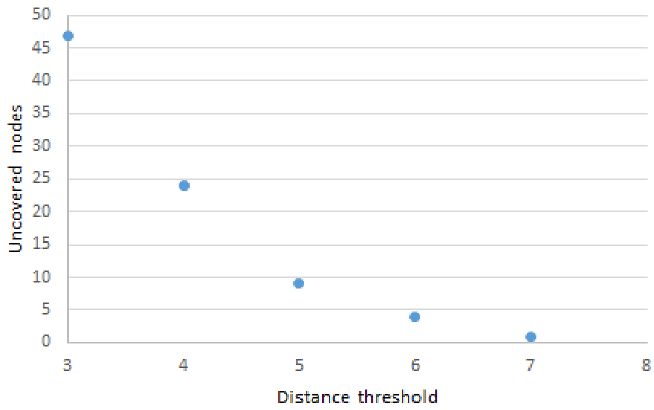

Fig. 3 Coverage versus distance threshold

We also investigated how the changes in the risk level α affects the coverage performance. Clearly, when the risk level is low, or equivalently, a high reliability level is required, the probabilistic capacity constraints (6) are tight, since the demands of covered nodes should be within the capacity of facilities with a high probability. In order to satisfy the capacity constraints, less demand nodes might be covered. On the contrary, when the decision–maker accepts higher risk levels, and lower reliability level are required, it is more likely that the capacity constraints (6) are violated so the number of uncovered nodes decreases and we are able to cover more demand points.
For the case with a distance threshold equal to 5 and a risk level of 0.01, the number of uncovered nodes over all periods is 13, but by increasing the risk level to 0.08, for the same distance threshold, the number of uncovered nodes decreases to 9 which shows about 31% improvement in the objective function.

Table 5 Sensitivity analysis with respect to α, ρ_0

α	ρ_0	D	Obj.	Expected satisfied demands
0.10	0.90	9	9	609
0.10	0.95	9	9	644
0.02	0.90	5	11	611
0.02	0.95	5	13	642

As another experiment, we have evaluated the impact of the participation probability ρ_0. For example as a result of advertising programs, this probability can increase, determining a higher participation level in the health program. Based on our observations, by a 5% increase in the participation probability (from 0.90 to 0.95), the expected amount of satisfied demands increased up to 6% (5%) for risk level equal to 0.10 (0.02) and distance coverage of 9 (5) kilometers. See Table 5.

As a final consideration, the role of constraints in (6) in improving the accessibility performance has been assessed by comparing the model solution with and without these set of constraints.

Although the objective function value in both cases is equal, about 27% of covered demands during periods 2 and 39% of covered demands in period 3 experience up to 93% increase in the traveled distance.

4.3 Current system evaluation

We have also carried out a set of experiments to provide some managerial insights about the current system performance. Specifically, we have considered the operational scenario that the managers are not strongly motivated to upgrade the system by adding more facilities, probably due to financial crisis, and instead, are interested to run the system with only the existing nursing homes. This requires keeping all the seven existing nursing homes active and banning the establishment of new facilities over the whole planning horizon. This in the mathematical model (1) - (11) can be expressed by imposing the additional set of constraints

$$y_{j1} - 1 = 0, \; j = 1, ..., 7$$

and solving the problem with $p_1 = 7$, $p_2 = p_3 = 0$.
Note that this new set of constraints in combination with (7) require the activation of existing nursing homes over all periods. The resulting model turned out to be infeasible due to the violation in the probabilistic capacity constraints (6). This supports the claim that the system should definitely be equipped with more facilities to address the increasing demand. Of course the capacity constraints are the most challenging constraints of the model and the managers might be willing to evaluate a capacity expansion, while keeping the current configuration of the facilities. To investigate this possibility, we removed the capacity constraints (6) from the aforementioned augmented model and evaluated the coverage performance of system. Although in this case the problem is feasible, 24 demand nodes will not be covered over different periods showing that even in the absence of capacity constraints, with the current configuration of nursing homes, not all the demand areas can be covered. Interestingly, the latter results in terms of coverage performance are still 54% worse than the results reported in Table 5 for the case with capacity constraints (6).
Finally, we investigated the multi–period behavior of the model assuming that the system upgrade is allowed from the second time period with $p_2 = 1$, $p_3 = 2$ and that the first period is run with all the currently existing facilities ($p_1 = 7$). Again, the latter assumption requires adding a set of constraints to the model. The optimal objective value is 12 in which 9, 1, and 2 zones are not covered over periods 1, 2, and 3, respectively, while the optimal objective value obtained by our model was 9 in which only 9 zones are not covered in the first period and all the demands are covered over the next periods. This again support our initial claim that the current configuration of facilities is not optimal and

even after upgrading the system, some demand nodes will never be covered. This will encourage the managers to modify the current system configuration and to relocate some facilities. Although the relocation of strategic facilities is costly and may involve unwanted consequences, it will improve the system performance.

4.4 Probabilistic versus deterministic and time-invariant model

In order to validate the probabilistic model, we have compared it with its deterministic counterpart, obtained by replacing the random variables with their expected value in constraints (6). Table 6 shows the resulting optimal sites. We observed that the assignment pattern associated with the deterministic model will result in the infeasibility of the problem at the presence of uncertainty. This shows the importance of incorporating the deviation of demands and adopting a probabilistic approach and provide evidence for the superiority of a risk–averse perspective over risk–neutral ones.

Table 6 The optimal nursing homes' sites for the deterministic model

Nursing home number	Time period $t = 1$	$t = 2$	$t = 3$
location2		*	*
location5			*
location6	*	*	*
location9			*
location12		*	*
location14			*
location17	*	*	*
location21			*
location23		*	*
location24			*

Apart from that, we also evaluated the left-hand side of the reliability constraints (6) obtained by the solution $(\hat{x}_{ijt}, \hat{y}_{jt})$ of the deterministic model expressed as

$$P\left(\sum_{i\in I} \lambda_{ij} h_{it}(\omega)\hat{x}_{ijt} \leq Q_j \hat{y}_{jt}\right) = F_{\Phi}(Q_j\, \hat{y}_{jt}), \ \ \forall j \in J,\ t = 1, ..., T$$

where $\Phi = \sum_{i\in I} \lambda_{ij} h_{it}(\omega)\hat{x}_{ijt}$ is a normally distributed random variable with the cumulative distribution function $F_{\Phi}(.)$ and $\hat{x}_{ijt}$ and $\hat{y}_{jt}$ are the optimal values of the deterministic model. As shown in Table 7, for some constraints, the probability of not exceeding the capacity is low and, considering a reliability level of 0.95 (corresponding to the risk level of α equal to 0.05), it is five times below the minimum required value. This cases are highlighted in bold in Table 7. This, again, supports our previous claim about the necessity of incorporating the stochasticity of uncertain parameters into the model.

We also investigated the importance of considering the temporal dependency of the stochastic demand.

Table 7 The reliability level of the deterministic model

Nursing home number	Time period $t = 1$	$t = 2$	$t = 3$
location2	-	0.99	**0.82**
location5	-	-	1
location6	0.99	**0.59**	**0.94**
location9	-	-	1
location12	-	1	1
location14	-	-	1
location17	**0.84**	**0.69**	0.99
location21	-	-	1
location23	-	1	1
location24	-	-	1

A comparison between the coverage performance resulting from the proposed model and the same model, where the mean and the variance of the demand are considered constant over time (stochastic time-invariant demand model), is presented in Fig. 4.

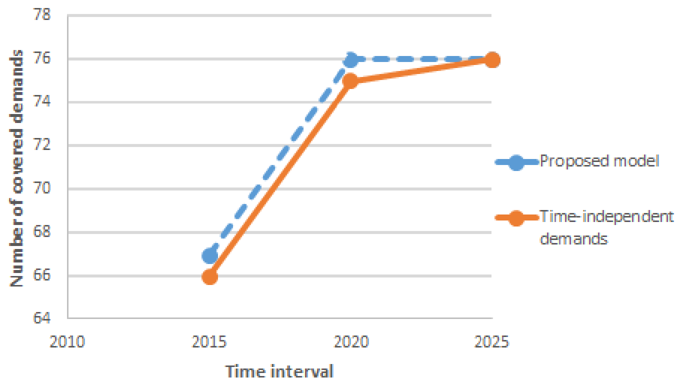

Fig. 4 Stochastic time-invariant demand model versus the proposed model

The proposed model outperforms the stochastic time-invariant demand model, in terms of coverage performance. In particular, in the first period, the latter model overestimates the coverage performance by 14%.

4.5 Monte Carlo simulation

The last part of this section is devoted to a Monte Carlo simulation investigating the validity of the proposed model with respect to the probabilistic chance constraints (13). The simulation results are expected to provide informative insights about the effectiveness of the proposed risk–averse approach, allowing to test how frequently the demand can be expected to exceed the capacity. To run the simulation, for each pair of candidate location j and period time

t in (13), and corresponding to each uncertain demand $h_{it}(\omega)$, $i \in I$, we generated 50000 different random values h_{its} drawn from the normal distribution $N(\mu_{it}, \sigma_{it})$ where each random value represents a scenario indexed by s. To validate the results of the stochastic model, the assignment and location variables x_{ijt} and y_{jt} were set to their optimal values, and the frequency of violation in constraints $\sum_{i \in I} \lambda_{ij}\, h_{its} - Q_j y_{jt} \leq 0,\ j \in J, t = 1, ..., T, s = 1, ..., 50000$ for different risk level values was calculated. Performing this procedure, we observed that for risk level values $\alpha \in \{0.01, 0.02, ..., 0.07\}$, all the constraints over all scenarios are satisfied, or equivalently, the frequency of violation is zero. These results are expected since lower risk levels are more conservative and it is more unlikely to experience any violation. The results of simulation for bigger values of α are reported in Table 8.

Table 8 The probability of violation in the Monte Carlo simulation

		j																							
α	t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
0.08	1	-	-	-	-	-	0	-	-	-	-	-	-	-	-	-	-	0	-	-	-	-	-	-	-
	2	-	-	0	-	-	0	-	-	-	-	0	-	-	-	-	-	0.009	-	-	-	-	-	0	-
	3	-	-	0	-	-	0	-	-	-	-	0	0	-	0	0	-	0	-	-	-	0	-	0	0
0.10	1	-	-	-	-	-	0	-	-	-	-	-	0	-	-	-	-	-	-	-	-	-	-	-	-
	2	0	-	-	-	-	0	-	-	-	-	-	0	-	-	-	-	-	0	-	-	-	-	0	-
	3	0	-	-	0	0	0	-	-	-	-	-	0	-	-	0	-	-	0.009	-	-	0	-	0	0
0.2	1	-	-	-	-	-	0	-	-	-	-	-	-	-	-	-	-	0	-	-	-	-	-	-	-
	2	-	-	0	-	-	0	-	-	-	-	-	0	-	-	-	-	0.009	-	-	-	-	-	0	-
	3	-	-	0	-	-	0	-	-	-	-	-	0	-	0	-	0	0	0	-	0	0	-	0	-
0.3	1	-	-	-	-	-	0	-	-	-	-	-	-	-	-	-	-	0	-	-	-	-	-	-	-
	2	-	-	0	-	-	0	-	-	-	-	-	0	-	-	-	-	0.009	-	-	-	-	-	0	-
	3	-	-	0	-	-	0	-	-	0	0	-	0	-	0	-	-	0	-	0	0	-	-	0	-

A few violations are experienced which are still less than the risk level. Since this violations are related to the candidate facilities 17 and 18, by adding other facilities to the network, possibly near the neighborhood of facility 17, the violation would be eliminated.

5 Conclusion

In this paper, we proposed a multi–period model for the nursing homes facility location problem. The multi–period perspective was adopted for handling the budget constraints as well as the fluctuation of demands over time.
The improvement of the accessibility performance was followed by the dynamic modification of the assignment pattern, if possible, while the deterioration of service level was strictly prohibited over the planning horizon.

We also discussed about the possibility of incorporating both the preferences of users and managers within a covering framework. This enabled us to address the elasticity of demands, based on the distance parameter, as well.
The imprecise nature of demands was tackled by applying a probabilistically constrained approach on the capacity constraints to satisfy with a given prob-

ability. Additionally, the deterministic equivalent formulation of the model as well as its linearized counterpart were introduced. The model was implemented on a real case study for nursing home location planning problem in Shiraz city, Iran. The analysis of the results provided us with important managerial insights about the current configuration of nursing home facilities and the possibility of improving the current performance.
It is mentionable that although we developed the model for nursing home planning network, it can also be applied for other strategic location decisions arising in the public sector. Extending the proposed model to address the issue of fairness will be interesting as a future research topic. This can be investigated through the division of demand nodes into different subsets (categories) based on their characteristics and special needs for health services. Hence, the incorporation of different constraints or the definition of other objectives specified for each demand category could be possible. For instance, an option could be the division of demand zones based on their geographical locations into marginal and non–marginal zones and the problem could be modeled as a bi–objective problem by adding another objective which minimizes the amount of uncovered demands associated with marginal zones over all periods. In addition, different coverage radii or types of facilities for zones in which the access to health care services is limited could be introduced.

6 Appendix

6.1 Linearization

Since the term under the square root in Eq. (13) is non–negative, if $\sum_{i\in I}\hat{\mu}_{ijt}\,x_{ijt} \leq Q_j y_{jt}$, we can rewrite it as follows:

$$\left(\sqrt{\beta_\alpha \sum_{i\in I} x_{ijt}^2\,\hat{\sigma}_{ijt}^2}\right)^2 \leq \left(Q_j y_{jt} - \sum_{i\in I}\hat{\mu}_{ijt}\,x_{ijt}\right)^2 \quad \forall j\in J,\ t=1,\ldots,T \tag{17}$$

which can be simplified in Eq. (18):

$$\begin{aligned}&\sum_{i\in I}\beta_\alpha\hat{\sigma}_{ijt}^2\,x_{ijt} - \ Q_j^2\,y_{jt} \ + 2\,Q_j \ \sum_{i\in I}\hat{\mu}_{ijt}x_{ijt}y_{jt}\\ &-\sum_{i\in I}\sum_{k\in I}\hat{\mu}_{ijt}\hat{\mu}_{kjt}\,x_{ijt}\,x_{kjt} \leq 0 \quad \forall j\in J, t=1,\ldots,T\end{aligned} \tag{18}$$

We introduce the auxiliary variables z_{ijt} and w_{ikjt} denoting the bilinear terms $x_{ijt}\,y_{jt}$ and $x_{ijt}\,x_{kjt}$ in (18) , respectively. The set of constraints (19) - (27) are also added to obtain a set of equivalent linear constraints for their non–linear counterparts in (18):

$$\sum_{i\in I}\beta_\alpha\hat{\sigma}_{ijt}^2\ x_{ijt} - Q_j^2\,y_{jt} + 2\,Q_j\ \sum_{i\in I}\hat{\mu}_{ijt}z_{ijt} - \sum_{i\in I}\sum_{k\in I}\hat{\mu}_{ijt}\,\hat{\mu}_{kjt}\,w_{ikjt} \leq 0$$

$$\forall j \in J,\ t = 1, \ldots, T \tag{19}$$

$$\sum_{i \in I} \hat{\mu}_{ijt}\, x_{ijt} \leq Q_j y_{jt} \qquad \forall i \in I,\ \forall j \in J \tag{20}$$

$$z_{ijt} \geq x_{ijt} + y_{jt} - 1 \qquad \forall i \in I,\ \forall j \in J,\ t = 1, \ldots, T \tag{21}$$

$$z_{ijt} \leq x_{ijt} \qquad \forall i \in I,\ \forall j \in J,\ t = 1, \ldots, T \tag{22}$$

$$z_{ijt} \leq y_{jt} \qquad \forall i \in I,\ \forall j \in J,\ t = 1, \ldots, T \tag{23}$$

$$w_{ikjt} \geq x_{ijt} + x_{kjt} - 1 \qquad \forall i, k \in I,\ \forall j \in J,\ t = 1, \ldots, T \tag{24}$$

$$w_{ikjt} \leq x_{ijt} \qquad \forall i, k \in I,\ \forall j \in J,\ t = 1, \ldots, T \tag{25}$$

$$w_{ikjt} \leq x_{kjt} \qquad \forall i, k \in I,\ \forall j \in J,\ t = 1, \ldots, T \tag{26}$$

$$z_{ijt},\ w_{ikjt} \in \{0, 1\} \qquad \forall i, k \in I,\ \forall j \in J,\ t = 1, \ldots, T \tag{27}$$

The mathematical model $NDMPLM$ amended with constraints (19) - (27), and the auxiliary variables, define the proposed model.

6.2 Coordinate transformation

The transformation in (28) was applied in order to convert the GPS coordinates of the demand nodes and facility locations, specified on the map, into the Cartesian coordinates, which is consistent with the Euclidean distance axiom.

$$\begin{aligned} x &= R \times cos(C \times 3.14/180) \times (s \times 3.14/180) \\ y &= R \times t \times 3.14/180 \end{aligned} \tag{28}$$

where (x, y) and (s, t) represent the Cartesian and the GPS coordinates, respectively, R is the approximate earth radius, and C shows the latitude of a hypothetical center point over the region.

Table 9 Location coordinates of population centers at zones 1-3

Zone	Population center	Location coordinates		Zone	Population center	Location coordinates	
		x	y			x	y
1	(1)	5069.3	3294.5	2	(9)	5077.1	3289.4
1	(2)	5069.3	3297.8	2	(10)	5076.3	3290.2
1	(3)	5071.2	3296	2	(11)	5075.6	3290.1
1	(4)	5072.6	3295.3	2	(12)	5076.4	3291.1
1	(5)	5070.8	3293.6	2	(13)	5076.2	3291.5
1	(6)	5072.3	3293.9	2	(14)	5077.1	3291.7
1	(7)	5071.8	3292.6	2	(15)	5077.1	3292.3
1	(8)	5074.1	3291.8	3	(1)	5085.7	3287.7
1	(9)	5075.1	3292.1	3	(2)	5081.1	3288.5
1	(10)	5073.4	3293.3	3	(3)	5080.3	3289.6
1	(11)	5076.1	3292.8	3	(4)	5079.7	3290.3
1	(12)	5077.2	3293.8	3	(5)	5078	3293.3
2	(1)	5078.7	3287	3	(6)	5079	3292.9
2	(2)	5078.3	3288.2	3	(7)	5079.8	3292.7
2	(3)	5080.9	3287.2	3	(8)	5079.4	3291.6
2	(4)	5079.5	3287.9	3	(9)	5078.6	3292.4
2	(5)	5079.9	3288.8	3	(10)	5080.4	3290.6
2	(6)	5079.1	3289	3	(11)	5082.2	3289.3
2	(7)	5079	3290.1	3	(12)	5081.5	3291.9
2	(8)	5078	3289.1	3	(13)	5082.3	3292.4

Table 10 Location coordinates of population centers at zones 4-9

Zone	Population center	Location x	coordinates y	Zone	Population center	Location x	coordinates y
4	(1)	5080.7	3292.2	6	(1)	5070.7	3300
4	(2)	5071.6	3289.3	6	(2)	5070.1	3299.8
4	(3)	5074	3289.8	6	(3)	5069.3	3299.1
4	(4)	5072.1	3290.5	7	(1)	5078.3	3291.4
4	(5)	5070.3	3291	7	(2)	5084.3	3285.3
4	(6)	5070.1	3292.5	7	(3)	5083.3	3286
4	(7)	5071.3	3292	7	(4)	5082.4	3286.4
4	(8)	5073.6	3287.5	7	(5)	5082.5	3287.9
4	(9)	5072.8	3291.4	7	(6)	5083.5	3287.3
4	(10)	5073.9	3290.7	7	(7)	5085.2	3287.4
4	(11)	5075.1	3290.9	8	(1)	5074.6	3294
5	(1)	5078.2	3285.4	8	(2)	5078.1	3289.8
5	(2)	5076.1	3286.7	8	(3)	5077.1	3290.6
5	(3)	5074.7	3287.3	8	(4)	5077.9	3290.6
5	(4)	5076.1	3288.1	9	(1)	5075.8	3285.5
5	(5)	5074.7	3288.8	9	(2)	5071.5	3286.6
5	(6)	5077.4	3288.5	9	(3)	5070.2	3288.9
5	(7)	5075	3289.8	9	(4)	5068	3291.1

Table 11 The coordinates of potential facility sites

Facility j	Location x	coordinates y	Facility j	Location x	coordinates y
1*	5071	3300	13	5081	3293
2*	5070	3300	14	5081	3291
3*	5070	3298	15	5078	3293
4*	5068	3292	16	5080	3285
5*	5080	3287	17	5079	3289
6*	5073	3292	18	5076	3290
7*	5078	3299	19	5077	3286
8	5084	3283	20	5078	3291
9	5087	3288	21	5073	3294
10	5083	3287	22	5069	3295
11	5087	3285	23	5070	3290
12	5081	3288	24	5077	3288

* Existing nursing homes

Table 12 Distance traveled by covered demands

Demand node	period			Demand node	period		
i	$t=1$	$t=2$	$t=3$	i	$t=1$	$t=2$	$t=3$
1	-	4.76	2.405	39	1.815	1.815	1.815
2	-	3.586	3.586	40	3.396	2.582	2.582
3	3.866	3.866	1.014	41	4.753	3.726	3.318
4	4.992	4.992	4.992	42	2.969	2.969	2.969
5	3.941	3.941	1.368	43	3.202	3.202	2.567
6	-	3.546	1.226	44	4.55	4.55	2.934
7	4.62	3.21	2.017	45	3.867	1.963	0.154
8	4.796	2.169	0.605	46	4.094	4.094	2.635
9	3.229	1.901	1.901	47	4.448	4.448	4.392
10	3.521	2.253	2.253	48	3.324	3.324	3.289
11	2.719	1.101	1.101	49	2.35	2.35	2.35
12	2.501	2.501	1.553	50	1.64	1.64	1.64
13	2.663	2.663	2.284	51	2.706	2.706	2.706
14	1.019	1.019	0.665	52	4.404	4.404	4.404
15	1.471	1.471	1.471	53	4.758	4.758	4.758
16	1.148	1.148	1.148	54	0.327	0.327	0.327
17	1.985	1.985	1.874	55	1.512	1.512	1.512
18	2.67	2.67	1.075	56	2.42	2.42	1.659
19	3.023	2.35	1.48	57	-	3.703	3.182
20	3.977	3.977	1.435	58	-	4.022	2.126
21	3.62	0.311	0.311	59	4.631	4.631	4.606
22	3.066	0.357	0.357	60	3.688	3.688	1.689
23	3.522	0.939	0.939	61	4.735	4.735	1.908
24	3.239	1.292	1.292	62	2.392	2.392	2.353
25	4.21	1.892	1.453	63	1.653	1.653	1.653
26	4.182	4.182	1.991	64	3.68	3.68	0.778
27	-	2.092	2.092	65	4.392	4.392	1.231
28	-	1.626	1.626	66	3.936	3.936	3.12
29	-	0.867	0.867	67	2.649	1.917	1.917
30	4.691	3.742	3.742	68	2.054	2.054	2.015
31	-	0.56	0.56	69	1.366	1.366	1.366
32	4.755	2.768	2.768	70	2.66	2.66	1.362
33	3.748	3.748	3.748	71	3.025	2.785	2.785
34	2.991	2.991	2.991	72	1.724	1.724	1.724
35	2.38	2.38	2.38	73	4.109	3.571	3.571
36	1.49	1.49	1.49	74	0.229	0.229	0.229
37	1.209	1.209	1.209	75	1.314	1.314	1.314
38	2.255	2.255	2.255	76	2.231	1.148	1.148

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