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LCF-HCF strain–life model: Statistical distribution and design curves based on the maximum likelihood principle

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Abstract

In the paper, the statistical distribution of the total strain and the fatigue life in the low cycle fatigue (LCF)–high cycle fatigue (HCF) life range is analytically derived starting from the Coffin–Manson and Morrow model. The maximum likelihood principle is exploited for parameter estimation, thus allowing to consider both failures and runout specimens. A straightforward procedure for the estimation of the design curves based on the likelihood ratio confidence lower bound has been also developed. The proposed model has been validated with literature datasets obtained by testing steel and aluminum alloys, proving its effectiveness and representing a reliable alternative to the currently adopted literature models.

KEYWORDS

design curve, high cycle fatigue (HCF), low cycle fatigue (LCF), statistical distribution, strain–life

Highlights

- A method for the strain–life design curves is proposed.
- Likelihood ratio lower confidence bound is exploited for the design curves.
- The analytical formulation of the statistical distribution of total strain is derived.
- A procedure for parameter estimation based on maximum likelihood principle is proposed

1 | INTRODUCTION

The experimental assessment of the fatigue response of materials, together with an appropriate statistical modeling of the large experimental variability, is fundamental for the proper design of fatigue-resistant components. Experimental tests are carried out to assess the relationship between the applied load and the number of cycles to failure and to reliably design structures with safe-life methods. Two main experimental approaches are followed for assessing the fatigue response of specimens, that is, the stress-life and the strain–life approaches.[1–5] With the stress-life approach, the applied stress amplitude is kept constant during the fatigue test and the relationship between the applied stress amplitude and the number of cycles to failure is assessed. This testing methodology is mainly employed to assess the fatigue response in the high cycle fatigue (HCF) life region, with applied stress amplitudes significantly below the yield stress.[3] A power law equation (i.e., Basquin's equation[3,6]) is typically used to model the relationship between the applied stress amplitude and the number of cycles to failure, corresponding to a linear trend in a log–log S-N plot. Basquin's law, however, does not properly model the fatigue response for large applied stress amplitude and low fatigue life, that is, in the low cycle fatigue (LCF) life range.[3] Accordingly, when the LCF range is of interest, a strain–life approach is more appropriate. Fatigue tests are carried out by controlling and keeping the applied strain amplitude constant during the test.[7] Moreover, local strain–life methodologies are widely employed to predict the fatigue life of components with notches, welds or, in general, regions with local stress concentration[5,8,9] inducing large local deformations. The relationship between the applied strain amplitude and the number of cycles to failure is generally modeled with the Coffin–Manson and Morrow model,[3,7,9–14] which covers not only the LCF region but the whole LCF-HCF range. The strain life approach is employed for the design of components in service in many structural and critical applications, for example, in the automotive sector,[9] for nuclear applications,[15] for turbine,[16] and bridge components.[17] The interest among the scientific community mainly focuses on the assessment of the design curves, that is, the strain–life curves that are to be used for the design of components against the intrinsic variability of the fatigue phenomenon, thus ensuring a statistical safety margin with respect to experimental failures.

In the literature, different approaches have been proposed to assess the allowable strain amplitude, the "design strain amplitude" or in, general, "the design curves" to be considered when components are designed. For example, according to Keisler et al.[15] and to ASME Boiler and Pressure Vessel Code, the design curve is the most conservative curve among the best fit curve decreased by a factor of 2 considering the strain amplitude or the best fit curve decreased by a factor of 20 by considering the number of cycles to failure. In Williams et al.,[9] the design curves are assessed by separately estimating the plastic and the elastic design strain–life curves and then combining them, according to the Manson–Coffin and Morrow model. The elastic and plastic design curves are obtained with the approximate Owen tolerance limit.[9] More recent papers have also addressed this important subject. Harlow[18] focused on the confidence bounds for the median curve, suggesting that the mean square error (MSE) should be preferred in their estimation. Moreover, Harlow[18] proposed a "Generalized Lifetime approach," based on the Weibull cumulative distribution function, which ensures a better estimation of the confidence interval of the median curve with respect to the methodologies based on the Coffin–Manson and Morrow equation. In Beretta et al.,[16] a simplified log-normal format for calculating the failure probability of components subjected to loads in the LCF region is developed. The model is validated by assessing the design point, the one ensuring the target failure probability, of rotor steel and by computing the safety factor to be applied for a safe life design. Similarly, Zhu et al.[19] focus on the definition of safety factors for components subjected to LCF multiaxial stresses, based on the probabilistic assessment of the material and load variability together with the life scatter. In Castillo et al.,[20] a Weibull regression model, which considers directly the total strain amplitude rather than computing it as the sum of the elastic and the plastic strain amplitude, is proposed to estimate the strain–life curves. With the proposed model, runout specimens are also taken into account in the analysis. Finally, in Gu and Ma,[21] a method for the assessment of the strain–life design curves for datasets with a small number of data is proposed. The method is based on the Coffin–Manson and Morrow model and on repeated estimations of the unknown material parameters by considering datasets obtained by gradually removing data points from the original datasets. According to this literature analysis, several models addressing the assessment of the design strain–life curves have been developed, attesting that this research subject is of particular importance for industrial applications.

In the present paper, a methodology for the assessment of the strain–life design curves is proposed. The methodology is based on the application of the maximum likelihood principle and employs the Likelihood Ratio Confidence lower Bound (LRCB) for the estimation of the design curve. First, the statistical distribution of the total strain and of the fatigue life of parts subjected to a controlled strain amplitude is derived. Thereafter, the procedure for parameter estimation, which allows for taking into account failure and runout specimens and estimating the design curve with the LRCB is explained in detail. Finally, the proposed method is validated on literature datasets obtained through tests on steels and aluminum alloys.

2 | STRAIN LIFE APPROACH: METHODS

In this section, the model developed for the assessment of the strain–life curves is described. In Section 2.1, the cumulative distribution function (cdf) and the probability density function (pdf) of the total strain amplitude and the fatigue life are derived, whereas in Section 2.2, the procedure for parameter estimation is described. Finally, in Section 2.3, the methodology developed for the estimation of the design curve is reported.

In the following, the subscript "e" refers to the elastic deformation, whereas the subscript "p" refers to the plastic deformation. According to the standard notation in Halperin et al.,[22] the upper case is used for random variables, whereas the lower case is used for the realization of the random variable, that is, the random variable equal to a specific value (for example an experimental value).

2.1 | Statistical distribution of the strain amplitude and the fatigue life

The objective of the proposed methodology is to assess the cdf of the total strain as a function of the number of cycles to failure and, accordingly, the strain life curves at the required failure probability. Similarly, the fatigue life distribution as a function of the total strain amplitude will be obtained. The starting point is the Coffin–Manson and Morrow model,[3,7,9–14] which models the dependency between the total applied strain amplitude, ε_t, and the number of cycles to failure, n_f, with the expression reported in Equation (1):

$$\varepsilon_t = \varepsilon_e + \varepsilon_p = \frac{\sigma_f'}{E}\left(2n_f\right)^b + \varepsilon_f'\left(2n_f\right)^c, \quad (1)$$

being ε_e the elastic deformation, ε_p the plastic deformation, σ_f' the fatigue strength coefficient, b the fatigue strength exponent, E the material Young's modulus, ε_f' the fatigue ductility coefficient, and c the fatigue ductility exponent. By assuming that the rvs elastic strain, E_e, and total strain, E_t, are Lognormal (Beretta et al.[16] and references therein), then the plastic strain random variable E_p is approximately Lognormal too. This result originates from typical approximations for the sum of Lognormal random variables[23,24]: Given that $\mathrm{E}_t = \mathrm{E}_e + \mathrm{E}_p$ is Lognormal and E_e is Lognormal, then also E_p is approximately Lognormal.

According to Basquin's law, E_e is Lognormal with constant standard deviation, σ_e, and mean, μ_e, that depends on the fatigue life, $y = \ln(n_f)$:

$$\begin{cases} F_{\mathrm{E}_e}(\varepsilon_e;\mu_e,\sigma_e) = \phi_G\left(\dfrac{\ln(\varepsilon_e)-\mu_e}{\sigma_e}\right) \\ \mu_e = \ln\left(\dfrac{\sigma_f'}{E}\right) + b\ln(2n_f) = a_e + b_e\ln(n_f) = a_e + b_e y \end{cases}, \quad (2)$$

being $F_{\mathrm{E}_e}(\varepsilon_e;\mu_e,\sigma_e)$ the cdf of the elastic strain, a_e and b_e two constant coefficients to be estimated from the experimental data, and $\phi_G(\cdot)$ the standardized Normal cdf and $y = \ln(n_f)$.

Similarly, E_p is Lognormal with constant standard deviation, σ_p, and mean, μ_p, that depends on the fatigue life, according to the Coffin–Manson model:

$$\begin{cases} F_{\mathrm{E}_p}\left(\varepsilon_p;\mu_p,\sigma_p\right) = \phi_G\left(\dfrac{\ln(\varepsilon_p)-\mu_p}{\sigma_p}\right) \\ \mu_p = \ln\left(\varepsilon_f'\right) + c\ln(2n_f) = a_p + b_p\ln(n_f) = a_p + b_p y \end{cases}. \quad (3)$$

From the distributions of E_e and E_p, it is finally possible to compute the parameters of the Lognormal distribution followed by the total strain random variable E_t. In particular, according to Abu-Dayya and Beaulieu,[24] the mean, μ_t, and the standard deviation, σ_t, of the logarithm of E_t can be expressed according to Equation (4):

$$\begin{cases} \mu_t = 2\ln(u_1) - \ln(u_2)/2 \\ \sigma_t = \sqrt{\ln(u_2) - 2\ln(u_1)} \end{cases}, \quad (4)$$

where u_1 and u_2 are given by

$$\begin{cases} u_1 = e^{\mu_e+\sigma_e^2/2} + e^{\mu_p+\sigma_p^2/2} \\ u_2 = e^{2\mu_e+2\sigma_e^2} + e^{2\mu_p+2\sigma_p^2} + 2e^{\mu_e+\mu_p+\left(\sigma_e^2+\sigma_p^2+2\rho_{e,p}\sigma_e\sigma_p\right)/2} \end{cases}, \quad (5)$$

and $\rho_{e,p}$ is the correlation coefficient between E_e and E_p. The cdf followed by E_t for a given n_f is Lognormal with mean, μ_t, and standard deviation, σ_t:

$$F_{\mathrm{E}_t|n_f}(\varepsilon_t;\mu_t,\sigma_t) = \phi_G\left(\frac{\ln(\varepsilon_t)-\mu_t(n_f)}{\sigma_t(n_f)}\right). \quad (6)$$

Equation (6) can be also interpreted as the cdf of the fatigue life life n_f for a given total deformation ε_t:

$$F_{\mathrm{E}_t|n_f}\left(\varepsilon_t;\mu_t(n_f),\sigma_t(n_f)\right) = F_{N_f|\varepsilon_t}(n_f;\varepsilon_t) = \phi_G\left(\frac{\ln(\varepsilon_t)-\mu_t(n_f)}{\sigma_t(n_f)}\right). \quad (7)$$

An analytical expression for the pdf of the fatigue life can be assessed starting from Equation (7). The pdf is the derivative of Equation (7) with respect to the number of cycles to failure n_f:

$$f_{N_f|\varepsilon_t}(n_f;\varepsilon_t) = \frac{\partial F_{N_f|\varepsilon_t}(n_f;\varepsilon_t)}{\partial n_f}. \tag{8}$$

By rearranging Equation (8) and with easy passages, the pdf of the fatigue life becomes

$$f_{N_f|\varepsilon_t}(n_f;\varepsilon_t) = \frac{-\mu_t'\sigma_t + (\mu_t - \ln(\varepsilon_t))\sigma_t'}{\sigma_t} f_G(\ln(\varepsilon_t);\mu_t,\sigma_t), \tag{9}$$

where $f_G(\ln(\varepsilon_t);\mu_t,\sigma_t)$ is the Normal pdf of $\ln(E_t)$,

$$\begin{cases} \mu_t' = 2u_1'/u_1 - u_2'/(2u_2) \\ \sigma_t' = \dfrac{u_2'/u_2 - 2u_1'/u_1}{16\sigma_t} \end{cases}, \tag{10}$$

and

$$\begin{cases} u_1' = b_e/n_f e^{\mu_e+\sigma_e^2/2} + b_p/n_f e^{\mu_p+\sigma_p^2/2} \\ u_2' = 2\left(b_e/n_f e^{2\mu_e+2\sigma_e^2} + b_p/n_f e^{2\mu_p+2\sigma_p^2} + (b_e+b_p)/n_f e^{\mu_e+\mu_p+\left(\sigma_e^2+\sigma_p^2+2\rho_{e,p}\sigma_e\sigma_p\right)/2}\right) \end{cases}. \tag{11}$$

The cdfs of the fatigue life and of the total strain are therefore obtained starting from the Coffin–Manson and Morrow model, thus modeling the contribution of the elastic and the plastic strains in the total strain, in agreement with the experimental evidence, rather than only modeling the dependency of the total strain with the number of cycles with an assumed statistical distribution ensuring the best fitting for a specific dataset.[18] The proposed approach is statistical and phenomenological; that is, it is based on an assumed log-normal distribution for the elastic and the total stress amplitude. This choice of the initial distribution, a log-normal or a Weibull distribution, may affect the fitting capability and can have implications on the estimation of the lower bound strain life curve. However, the choice of a log-normal distribution has proven to properly work in the literature and will be confirmed in the following validations (Section 3). On the other hand, an approach based on physical assumptions and the compatibility condition between the probability distribution of the fatigue life and the applied stress has shown that the Weibull distribution should be employed for the fatigue analysis and the assessment of lower bound confidence levels.[25–27] Several literature models, however, assumed the log-normal distribution for describing the strain–life relationship and have been successfully validated on many datasets, proving that it can properly fit the experimental data, with limited differences and approximations with respect to a Weibull distribution.

Moreover the proposed approach cannot model an asymptotic trend, that is, a fatigue or endurance limit, in the HCF life region. Indeed, the strain–life relationship is generally assessed through experimental tests run in the LCF-HCF life region, with no data available in the VHCF life region to confirm an asymptotic trend. Literature models, like the one in Castillo et al.,[20] can model a potential endurance limit. However, it is worth noting that, thanks to its flexibility, the proposed model can also properly fit datasets showing an "asymptotic-like" trend in the HCF life range and with runout data occurring at number of cycles significantly larger than failures. In this case, the elastic strain curve would tend to a horizontal line, thus approaching an asymptotic trend. However, neglecting the potential fatigue limit means that failures at zero stress amplitude could occur, as pointed out in Fernández-Canteli et al.,[28] even if this can occur at number of cycles significantly beyond the VHCF life region and out of the range of interest for design.

2.2 | Parameter estimation and design curve

In this section, the procedure developed for the estimation of the material parameters involved in the proposed model is described. The set of material parameters to be estimated from the experimental data are a_e, b_e, σ_e, a_p, b_p, σ_p and the correlation coefficient, $\rho_{e,p}$. The parameters a_e, b_e are estimated through the application of the least square method between the logarithm of the applied elastic strain amplitude associated to each experimental failure and the corresponding logarithm of the number of cycles to failure. a_p, b_p are estimated with the same procedure, but by considering the plastic deformation. Plastic deformation can be obtained as the difference between the applied total deformation and the measured elastic deformation. The plastic strain amplitude computed from all the experimental data or from a subset not including small plastic deformations for fatigue lives in the HCF life range, especially if the excluded data deviate from linearity in a log–log plot, can be considered to estimate the material parameters a_p, b_p. For example, in Williams et al.,[9] a threshold for plastic strain amplitude is suggested for the interpolation of plastic strain amplitude data with respect to the number of cycles to failure.

Concurrently, the coefficient of correlation $\rho_{e,p}$ can be estimated as the correlation coefficient between the experimental elastic and plastic strain amplitudes. Accordingly, the $\rho_{e,p}$ value is not assumed but estimated from the experimental data for each investigated dataset.

The standard deviations σ_e and σ_p are finally estimated by applying the maximum likelihood principle, that is, by maximizing the Likelihood function $L[\boldsymbol{\theta}]$ defined in Equation (12):

$$L[\boldsymbol{\theta}] = \prod_{i_f=1}^{n} f_{N_f|\varepsilon_t}\left(n_{f,i};\varepsilon_{t,i}\right) \cdot \prod_{j=1}^{n_r}\left(1 - F_{N_f|\varepsilon_t}\left(n_f^{*};\varepsilon_{t,j}\right)\right), \tag{12}$$

being $\boldsymbol{\theta}$ the set of parameters to be estimated ($\boldsymbol{\theta} = \left(\sigma_e, \sigma_p\right)$), i_f the counter for the experimental failures, n the number of experimental failures ($i_f = 1 \ldots n$), j the counter for the experimental runout specimens, n_r the number of experimental runout specimens ($i_f = 1 \ldots n_r$), and n_f^{*} the runout number of cycles. With this method, runout specimens, neglected if the unknown parameters of the Coffin Manson and Morrow model (Equation 1) are estimated only by applying the least square method,[9] are considered, preventing the loss of the important information they contain. Indeed, even if the strain-based approaches are generally adopted for the analysis of strain-controlled LCF fatigue tests, they also properly work for experimental data covering the LCF-HCF life range[20,21,29] and thus with possible runout specimens. Figure 1 shows a flow chart of the described methodology. In Figure 1 and in the following, the notation $\tilde{}$ indicates the estimated parameters (e.g., $\tilde{a}_e, \tilde{b}_e$ are the estimates of the parameters a_e, b_e after the application of the least square method or $\boldsymbol{\theta}$ is the set of maximum likelihood estimates that satisfy Equation 12).

Once all the parameters have been estimated, the α-th quantile of the strain–life curve can be assessed by substituting $F_{N_f|\varepsilon_t}\left(n_f;\varepsilon_t\right)$ with α in Equation (7), and by solving numerically Equation (7) with respect to ε_t for the n_f of interest. Accordingly, by iteratively repeating this procedure for the life range of interest, the strain–life curve can be built point by point.

2.3 | Design curve with the LRCB

The design curves are here estimated as the LRCB of a high-reliability quantile strain–life curve. Differently from Williams et al.,[9] where the design curves for the elastic and the plastic strains are estimated separately with the approximate Owen tolerance limit and then combined, with the proposed procedure the design curve is estimated starting from the quantile of the total strain amplitude curve,[9] thus being capable of adapting to the experimental data and to the number of available data in each fatigue life range. On the other hand, the proposed procedure can be more complex to be applied, requiring an iterative procedure based on multiple iterations, but its implementation can be easily managed with the available numeric software. In the following, the definition of design curves provided in Williams et al.[9] and Tridello et al.[30] is considered, that is, the design curve corresponds to the lower bound, at a high confidence level, of a high-reliability quantile. According to a standard notation,[9,30] the design curve corresponds to the $R\alpha Cx_C$ curve, that is, the curve ensuring that, for each n_f, there is a $(1-\alpha)\%$ failure probability (α reliability) with the $x_C\%$ confidence level. Accordingly, the commonly adopted R90C90 design curve ensures that for each n_f in the range of interest, 90% of the components will not fail, with a 90% confidence level.

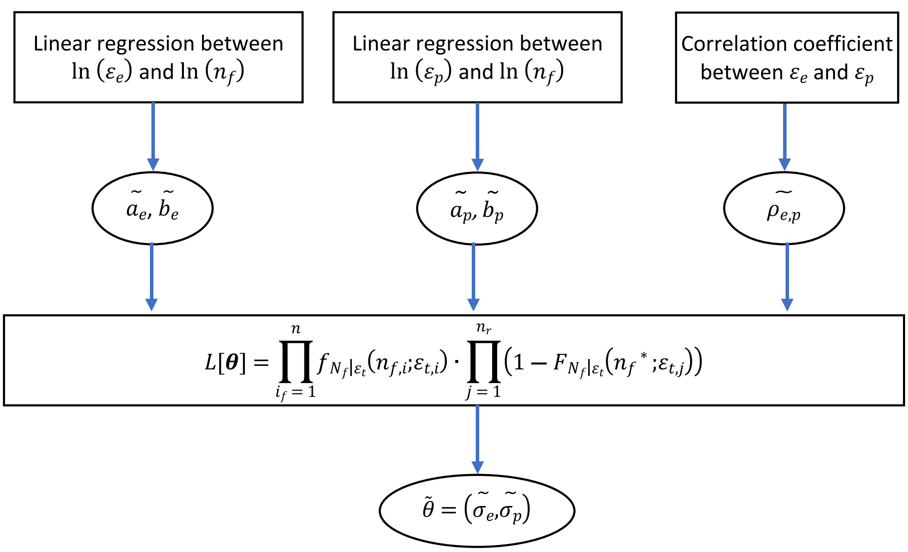

FIGURE 1 Procedure for the estimation of the material parameters in the proposed strain–life method. [Colour figure can be viewed at wileyonlinelibrary.com]

According to Tridello et al.,[30] the first step for the estimation of the design curve with the LRCB is the assessment of the profile likelihood function. In particular, for a specific n_f, the lower bound at the $x_C\%$ confidence level of the α quantile of the total strain, $\varepsilon_{t,\alpha}$, is obtained by solving Equation (13):

$$PL[\varepsilon_{t,\alpha}] = \frac{\max_{\theta_2}[L[\varepsilon_{t,\alpha},\theta_2]]}{L[\tilde{\boldsymbol{\theta}}]} \geq e^{-\frac{\chi^2(1;1-\beta_{th})}{2}}, \tag{13}$$

being $PL[\varepsilon_{t,\alpha}]$ the profile likelihood function, θ_2 one parameter between σ_e or σ_p, as detailed in the following, $\chi^2(1;1-\beta_{th})$ the $(1-\beta_{th})$-th quantile of a chi-square distribution with 1 degree of freedom and $L[\tilde{\boldsymbol{\theta}}]$ the Likelihood function computed for the best-fitting parameters (Section 2.2). According to Equation (13), the profile likelihood function must be a function of the α quantile of the total deformation, $\varepsilon_{t,\alpha}$. The numerator, $L[\varepsilon_{t,\alpha},\theta_2]$, must be thus rewritten in function of $\varepsilon_{t,\alpha}$; that is, the cdf of the fatigue life must be rewritten in function of $\varepsilon_{t,\alpha}$. This can be achieved according to the following steps:

1. From Equation (7), the α-th quantile of the total strain can be obtained by replacing $F_{N_f|\varepsilon_t}(n_f;\varepsilon_t)$ with α. By solving this equation with respect to σ_e or σ_p, an expression that correlates σ_e or σ_p with the α-th quantile of the total strain can be obtained (i.e., $\sigma_e(\varepsilon_{t,\alpha})$ or $\sigma_p(\varepsilon_{t,\alpha})$, respectively). Differently from Pyttel et al.,[26] an analytical closed-form solution for $\sigma_e(\varepsilon_{t,\alpha})$ or $\sigma_p(\varepsilon_{t,\alpha})$ does not exist, but a numerical solution can be easily obtained, according to Equation (14):

$$\alpha = \phi_G\left(\frac{\ln(\varepsilon_{t,\alpha}) - \mu_t(n_{f,\alpha})}{\sigma_t(n_{f,\alpha})}\right) \rightarrow \sigma_e(\varepsilon_{t,\alpha}) = f_e(\varepsilon_{t,\alpha}) \text{ or } \sigma_p(\varepsilon_{t,\alpha}) = f_p(\varepsilon_{t,\alpha}). \tag{14}$$

2. By substituting $\sigma_e(\varepsilon_{t,\alpha})$ or $\sigma_p(\varepsilon_{t,\alpha})$ in Equation (7), the cdf of the fatigue life in function of $\varepsilon_{t,\alpha}$ is obtained. In the first case ($\sigma_e(\varepsilon_{t,\alpha})$ in Equation 7), $\theta_2 = \sigma_p$. In the second case ($\sigma_p(\varepsilon_{t,\alpha})$ in Equation 7) $\theta_2 = \sigma_e$. By varying $\varepsilon_{t,\alpha}$ in a reasonable range for the specific n_f, the profile likelihood function can be built point by point. The $\varepsilon_{t,\alpha}$ solving Equation (13) provides the LRCB for the α quantile of the total strain of interest.

Practically, a procedure similar to that developed in Tridello et al.[30] has been implemented, briefly recalled in the following. The procedure involves iteratively varying $\varepsilon_{t,\alpha}$ in a reasonable range to build the $PL[\varepsilon_{t,\alpha}]$ function and find the $\varepsilon_{t,\alpha}$ value solving Equation (13):

1. First, $\sigma_e(\varepsilon_{t,\alpha}) = f_e(\varepsilon_{t,\alpha})$ is obtained by numerically solving Equation (14). The computed value is substituted in Equations (7) and (9), to express the profile likelihood as a function of $\varepsilon_{t,\alpha}$.
2. $PL[\varepsilon_{t,\alpha}]$ is thereafter built point by point. The first point is $\varepsilon_{t,\alpha} = \widetilde{\varepsilon_{t,\alpha}}$ (i.e., $\widetilde{\varepsilon_{t,\alpha}}$ computed for the selected n_f by solving Equation (8) for the α quantile of interest). For this value, $PL[\widetilde{\varepsilon_{t,\alpha}}] = 1$. Thereafter, $\varepsilon_{t,\alpha}$ is iteratively decreased with steps of 10^{-6}[mm/mm]. The steps can be adjusted, depending on the dataset. Accordingly, for each $\varepsilon_{t,\alpha}$, the $PL[\varepsilon_{t,\alpha}]$ function can be computed. The procedure is stopped when $PL[\varepsilon_{t,\alpha}]$ falls below a threshold set equal to $e^{-\frac{\chi^2(1;1-\beta_{th})}{2}}/2$.
3. $PL[\varepsilon_{t,\alpha}]$ points with respect to $\varepsilon_{t,\alpha}$ are interpolated with Piecewise Cubic Hermite Interpolating Polynomial (PCHIP). The PCHIP interpolating function, with which the data are interpolated with a cubic spline satisfying the Hermite interpolation condition, has been chosen since it ensures no overshoots and guarantees limited oscillation, being moreover computationally cheaper with respect to other interpolating approaches.
4. The $\varepsilon_{t,\alpha}$ value, $\varepsilon_{t,\alpha,x_r}$, at which the interpolating PCHIP function equals the $e^{-\frac{\chi^2(1;1-\beta_{th})}{2}}$ value corresponds to the LRCB of the investigated quantile of the total strain amplitude for the selected n_f. By iteratively repeating points 1 to 4 for different n_f, the design curve is built point by point.

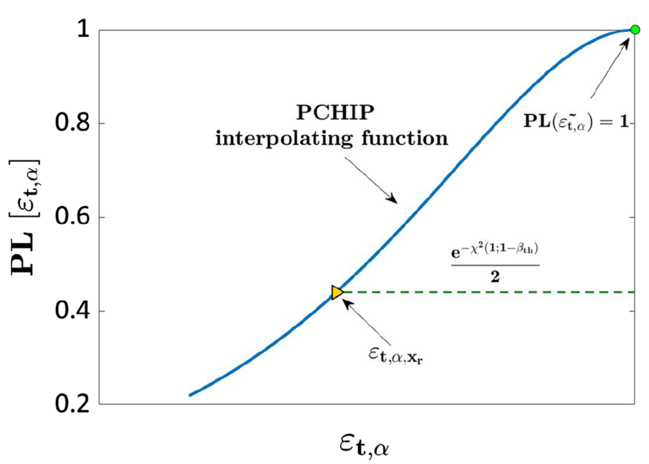

FIGURE 2 $PL[\varepsilon_{t,\alpha}]$ function with respect to $\varepsilon_{t,\alpha}$: Procedure for the estimation of the design curves. [Colour figure can be viewed at wileyonlinelibrary.com]

Figure 2 helps clarifying the procedure developed for the estimation of the design curve and shows the $PL[\varepsilon_{t,\alpha}]$ function with respect to $\varepsilon_{t,\alpha}$.

It must be noted that, according to the procedure shown in this section, the proposed design methodology requires a rather complex implementation, involving several iterations and optimizations. However, once it has been implemented, the design curves can be reliably estimated in a reasonable time. Other design methodologies, not discussed in the present paper, like the bootstrap method or the Bayesian approach implemented in the OpenBUGS software,[31] can be also adapted and employed to perform the parameter estimation process.

3 | LITERATURE VALIDATION

In this Section, the methodology described in Section 2 is validated with literature results. The procedure described in Section 2.3 has been implemented step by step in a Matlab script, which automatically estimates the design curve. The numerical solution to Equation (14) is obtained with the command *fzero*, whereas the maximization of the Likelihood function in Equations (12) and (13) is carried out with the *fminsearch* algorithm, based on the *Nelder–Mead* simplex algorithm.[32] In the following, the design curve corresponds to the R90C90 strain–life curve.

The effectiveness of the parameter estimation with the Maximum Likelihood Principle has been first investigated. The experimental datasets available in tabular form in Williams et al.[9] and obtained by testing the SAE 1137 carbon steel alloy and the ferric steel SAE 4512 have been considered. According to Williams et al.,[9] the SAE 1137 is a mild carbon steel, with an average Young modulus of 209 GPa. The SAE D4512 is a ferric steel used in industrial cast components, characterized by low ductility and high strength. The reader is kindly asked to refer to Williams et al.[9] for more details on these two investigated steels. Figure 3 shows the median curve and the 0.1-th quantile strain–life curves estimated with the proposed model, Figure 3A for the SAE 1137 carbon steel and Figure 3B for the ferric steel SAE D4512. The blue curves have been estimated by applying the least square method, with the standard deviations σ_e and σ_p estimated as the root mean square error (RMSE) of the plastic strain and the elastic strain, respectively. The red curves have been estimated by applying the maximum likelihood principle, as detailed in Section 2.

According to Figure 3, the two methodologies provide the same median curve, as expected. For the SAE 1137 carbon steel (Figure 3A), the 0.1-th quantile curve estimated with the maximum likelihood principle is closer to the experimental data and above the "least square" curve in the LCF life range, whereas an opposite trend is found in the HCF life range. One datapoint out of 14 is below the 0.1-th quantile, corresponding to about 10% of the data, as expected. For the SAE D4512 steel, the median and the 0.1-th quantile curves overlap, thus with the same estimated parameters. This analysis confirms that the procedure based on the Maximum Likelihood Principle provides reliable estimations of the standard deviations, which can be different from those obtained by simply applying the least square method to the elastic and plastic strain data.

The design curves have been then estimated for the same datasets[9] and compared in Figure 4, Figure 4A for the SAE 1137 carbon steel and Figure 4B for the SAE D4512 ferritic steel. In particular, the median and the design curves estimated with the methodology developed in Williams et al.,[9] based on the Owen tolerance interval, and those estimated with the methodology proposed in this paper are plotted. In Williams et al.,[9] the design curves have been estimated by considering a subset of data for the plastic strain amplitude and a subset for the elastic strain amplitude. For the SAE 1137, a transition life is "roughly estimated" as 22000 reversals. All the data below this transition life and the two data points immediately above have been considered for the LCF life range. Similarly, all the data with fatigue life above the estimated transition life and the two data immediately below are considered for the estimation of the elastic strain–life

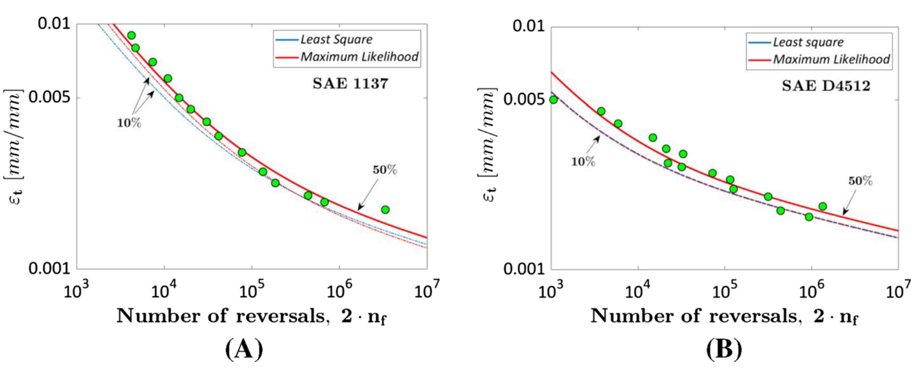

FIGURE 3 Median and 0.1-th quantile curves estimated with the least square method (blue curve) and with the maximum likelihood principle (red curve): (A) SAE 1137 carbon steel[9]; (B) SAE D4512 ferritic steel.[9] [Colour figure can be viewed at wileyonlinelibrary.com]

curve. For the SAE D4512 steel, only the data showing a plastic strain amplitude above 0.0005 mm/mm are considered for the plastic strain curve, whereas all the available data have been considered for elastic analysis. The elastic and plastic design curves are estimated by considering the approximate Owen tolerance limits. On the other hand, the design curves estimated with the approach developed in this work have been obtained by considering all the available elastic and plastic strain data. In the following figures, "ML" and "LRCB" refer to curves estimated with the proposed methodology, that is, the median curve estimated by applying the Maximum Likelihood Principle and the design curve estimated with the LRCB, respectively.

According to Figure 4, the approach in Williams et al.[9] and the one proposed in the present paper provide similar results. For the SAE 1137 steel, the median curves are close, with negligible differences. Similarly, the difference between the design curves is quite limited in the LCF range, with the curves being very close, and tends to increase in the HCF life range, with the LRCB curve being below that estimated in Williams et al.[9] On the other hand, a larger difference can be observed by considering the SAE D4512 steel (Figure 4B). Indeed the curve in Williams et al.[9] is more conservative for the whole LCF-HCF life range and below the LRCB design curve. However, even if less conservative, the LRCB design curve is below all the experimental failures too, in agreement with the definition of "design curve," being capable of properly modeling the experimental variability and adapting to the variability experimental data.

In the following, the influence of runout specimens on the design curves has been investigated by considering the same datasets in Williams et al.[9] Indeed, one of the strengths of the approach based on the LRCB is that it can take into account the influence of runout specimens by applying the maximum likelihood principle for the material parameter estimation. Indeed, if all the unknown constant coefficients, that is, also the standard deviations for the elastic and plastic life ranges, would have been estimated with linear regressions, the important information contained in runout specimens would have been neglected. On the other hand, by applying the maximum likelihood principle (Equation 12), runout data and the important information they contain can be reliably considered. In Williams et al.,[9] one runout specimen at $5 \cdot 10^6$ reversals has been experimentally found by testing the SAE 1137 steel. However, this experimental result has not been considered in the analysis carried out in Williams et al.,[9] since the least square method does not allow to take into account censored data ("If the sample did not fail, the data point should be excluded from this analysis because this analysis technique is only valid for failure occurrence and the analysis of an endurance limit involves a different approach," according to Williams et al.[9]). In Figure 5A, the experimental data for the SAE 1137 steel in Williams et al.[9] have been reanalyzed by considering also the runout specimen excluded in the previous analysis. The median and the design curves estimated with the LRCB and with the methodology in Williams et al.[9] are plotted. Moreover, to investigate also the influence of the number of runout specimens on the design curves, the dataset for the SAE 1137 steel has been modified by considering as runout number of failures the largest $2 \cdot n_f$ found experimentally ($3.3 \cdot 10^6$ cycles). Accordingly, the dataset for the SAE 1137 steel analyzed in Figure 5A has been modified, with one failure "transformed" into runout specimens, with a total of two runout specimens at different total strain amplitude. Figure 5B plots the median and the design curves estimated with the model based on the Owen tolerance limits[9] and the one developed in the present paper by considering the modified dataset which includes two runout specimens. It must be noted that the number of runout data should be limited. Indeed, as pointed out also in previous works,[28,30,33] low-stress amplitude runout data do not improve the design curve estimation, but, on the other hand, significantly increase the testing time. Accordingly, the proper testing strategy should be defined to avoid collecting large numbers of runout data. If available, however, runout data should be considered,

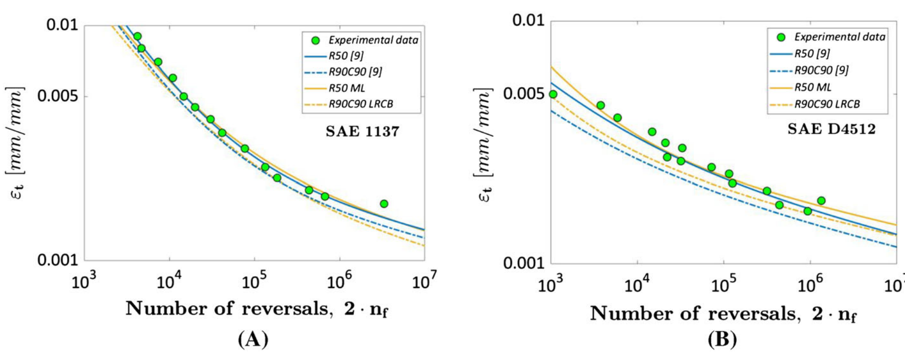

FIGURE 4 Median and design curves estimated through the methodology developed in Williams et al.[9] and in the present paper: (A) SAE 1137 carbon steel[9]; (B) SAE D4512 ferritic steel.[9] [Colour figure can be viewed at wileyonlinelibrary.com]

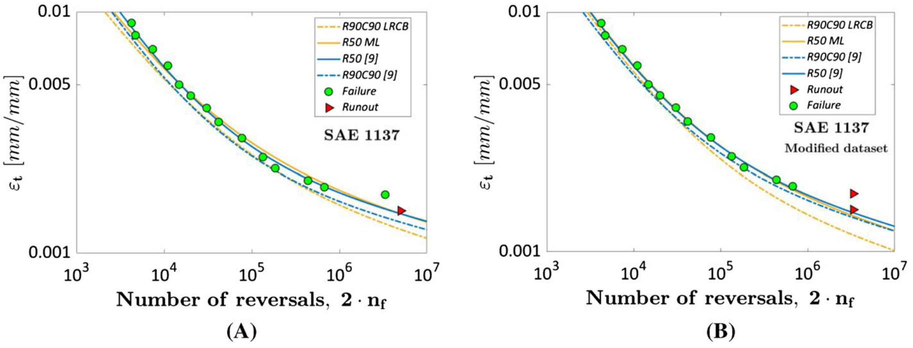

FIGURE 5 Median and design curves estimated through the methodology developed in Williams et al.[9] and in the present paper to highlight the influence of runout specimens: (A) SAE 1137 carbon steel with the runout data excluded in Williams et al.[9]; (B) modified SAE 1137 carbon steel dataset[9] with two runout specimens. [Colour figure can be viewed at wileyonlinelibrary.com]

and this is a strength of methodologies dealing with the strain–life curves.[20]

According to Figure 5, neglecting runout specimens can affect the estimated design curves, depending on the datasets. The design curves in Figure 5A for the SAE 1137 steel estimated with the Owen tolerance limits[9] and the LRCB are close, with limited differences. The standard deviation estimated with the procedure developed in the present paper varies by considering the runout specimen (from $7.79 \cdot 10^{-2}$ to $8.08 \cdot 10^{-2}$ for the elastic strain and from $4.12 \cdot 10^{-2}$ to $3.96 \cdot 10^{-2}$ for the plastic strain), but this variation is too small to be appreciable in Figure 5A. Indeed, one runout specimen with runout number of reversals close to the number of reversals of failed specimens has a limited influence on the estimated strain–life curves. The design curve estimated with the method in Williams et al.,[9] on the other hand, does not change.

The influence of runout specimens is more evident in Figure 5B, where two runout specimens have been considered. The variation of the design curve in Williams et al.[9] is due to the reduction of the number of failures for the elastic strain, thus increasing the multiplicative factors considered for assessing the elastic design curve. On the other hand, the LRCB design curve moves downward, since also the runout specimens are included within the analysis and with these two data having an influence on the estimated parameters and on the confidence bounds. The original dataset has been modified by "transforming" one failure into a runout. Accordingly, the number of failures in the dataset considered for estimating the design curves in Figure 5B is different from the number of failures in the original dataset. This justifies why the design curve counterintuitively moves downward. Indeed, without one failure in that region, the uncertainty associated to the strain–life curves increases and, depending on the failure and runout data available in that region, the design curve can move upward or downward. The estimated design curve, therefore, modifies to adapt to the dataset including the runout specimens and to model the experimental uncertainty, depending on the available failures and runout data in each life range. This analysis has proved that runout specimens can affect the shape of the design curve, depending on the dataset, and that they should be included in the analysis, especially if more than one runout has been experimentally found. It must be noted that runout data at low-stress amplitude or, in general, below a fatigue limit, if estimated, may not be considered in the analysis since not informative.[33] However, in the present paper, runout data occurred at strain amplitudes close to the strain amplitudes of failed specimens. Accordingly, in datasets like those considered in Figure 5A and in Figure 5B for which a fatigue limit has not been estimated, the removal of runout data would be arbitrary. For this reason, runout data have been considered in the analysis since they are informative and contain relevant information on the strain life relationship in the HCF life range, for which failures are generally not available.

The proposed methodology has been also validated with the datasets available in Borrego et al.[34] and in Harlow.[18] In Borrego et al.,[34] strain-controlled tests have been carried out on aluminum alloy 6082-T6 and 6060-T6 up to 10^6 number of reversals, thus investigating the LCF-HCF life range. Figure 6A,B plots the median and the design curves for the 6082-T6 and the 6060-T6 alloys, respectively. The design curves have been estimated with the models in Williams et al.[9] and the one proposed in the present paper, starting from data digitized with the *Engauge* software from the images reported in the original paper.[34] This further analysis allows verifying the effectiveness of the proposed methodology on datasets obtained through tests of aluminum alloys. Finally, the dataset in Harlow[18] obtained with tests on ASTM A969 hot dipped galvanized sheet steel has been considered since it is characterized by a large number of data for each tested total strain amplitude. The experimental data have been digitized from the ε_t-n_f plot in the original paper by using again the *Engauge* software. Due to the large number of data, many of which overlapping in the strain–life plot in the original paper, only the data for which the elastic strain amplitude and the corresponding

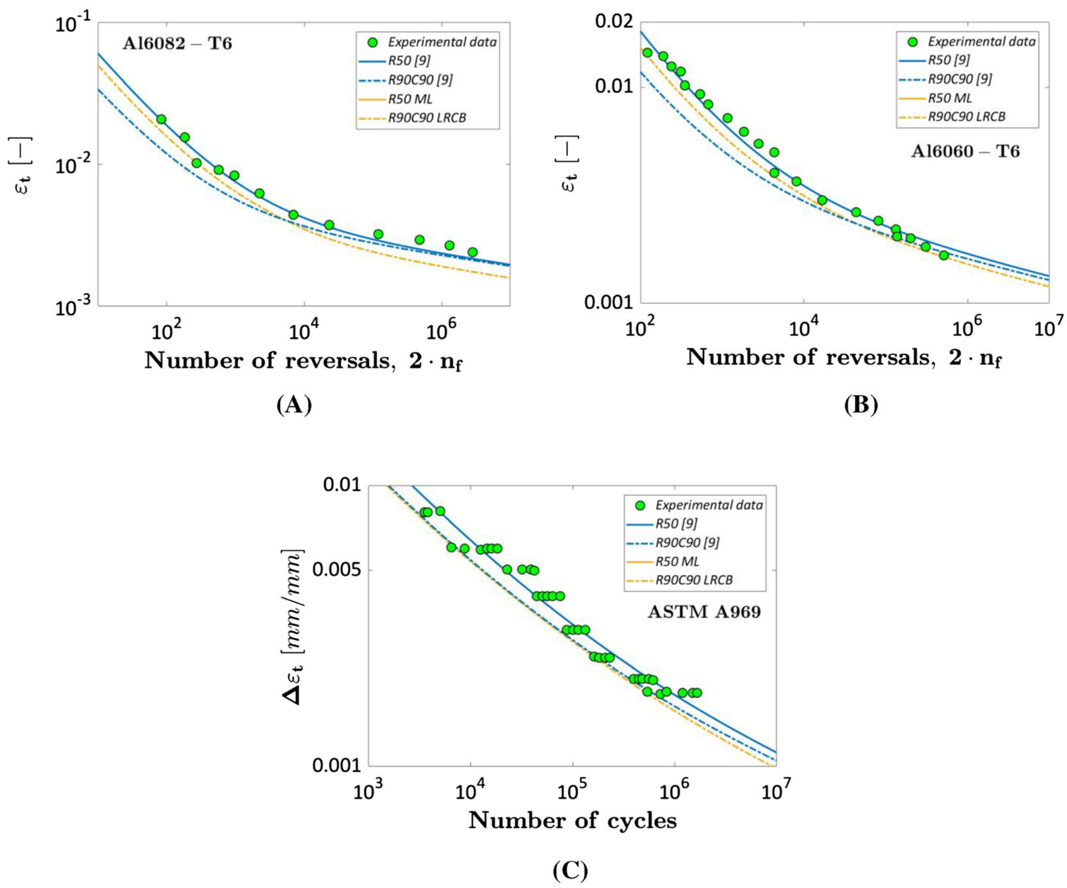

FIGURE 6 Median and design curves estimated through the methodology developed in Williams et al.[9] and in the present paper: (A) 6082-T6[34]; (B) 6060-T6 alloys[34]; (C) ASTM A969 hot dipped galvanized sheet steel.[18] [Colour figure can be viewed at wileyonlinelibrary.com]

plastic strain amplitude can be reliably extracted have been digitized. It is worth noting that the objective of this analysis is to validate the proposed model on datasets with different characteristics, and not to estimate the real strain life curves. In total, 37 data have been considered in the following analysis. Figure 6C plots the median and the design curves estimated with the LRCB and the methodology in Williams et al.[9] It must be noted that the objective of this further analysis is to prove the validity of the proposed method for different types of materials (steel and Al alloys) and for datasets with large number of available data.

According to Figure 6A,B, the developed model based on LRCB agrees even with the experimental data obtained by testing aluminum alloys. The median curve crosses the experimental failures and the design curve is below all the experimental failures, with a trend similar to that estimated with the model based on the approximate Owen tolerance limits.[9] In Figure 6A, one failure is below the LRCB design curve, whereas an opposite behavior is found for the Al6060 alloy, with the design curve estimated according to Williams et al.[9] being less conservative (one data below the design curve). A similar behavior can be found by analyzing Figure 6C, where a large number of experimental data is available. Two data and one data are below the design curve estimated according to Williams et al.[9] and with LRCB, respectively. The two design curves show the same trend, up to 10^5 cycles, with the LRCB design curve being more conservative in the HCF life range, but with limited differences.

It must be noted that, for the dataset reported in Figures 3 and 6B, the experimental data tend to be above the median curve in the LCF life region and below the median curve in the HCF life region. This criticality originates from the separate estimation of the parameters of the elastic and the plastic strain amplitude-life curves, without distinguishing between LCF and HCF life ranges. The authors employed the standard procedure commonly adopted in the literature[9] for the parameter estimation, whose main weakness is that the compatibility condition by the vertical and horizontal distributions is not fulfilled.[25] However, this criticality associated to this strategy for parameter estimation and the median curve is well-known in the literature, whereas the focus of the paper are the design curves and, in general, the development and the validation of an innovative procedure for the strain–life design curves, fundamental for the design of components. To overcome this weakness for the median curve, other methodologies, like the one reported in previous studies,[25,27,35] should be considered.

4 | CONCLUSIONS

In the present paper, a novel methodology for the estimation of the design strain–life curves is proposed and validated with literature datasets. The following conclusions can be drawn:

1. The statistical distribution of the total strain and of the fatigue life is analytically derived. Starting from the Manson–Coffin and Morrow model and by assuming a Lognormal distribution for the elastic and the total strain, the plastic strain has been considered approximately as lognormally distributed and the cumulative distribution function and the probability density function of the fatigue life is derived. The distribution of the fatigue life is therefore not based on an assumed statistical distribution for the total strain amplitude, but it embeds the well-known Manson–Coffin and Morrow model, which has proven to properly model the LCF-HCF range and the contribution of the elastic and the plastic strain on the fatigue life.
2. A straightforward procedure for parameter estimation based on the application of the maximum likelihood principle has been developed, allowing to consider both failures and runout data. The slope and the intercept of the linear trend of the elastic and the plastic strain are estimated by applying the least square methods, the coefficient of correlation is estimated as the correlation coefficient between the experimental elastic and plastic strain amplitudes, whereas the standard deviations of the elastic and plastic strains are estimated by maximizing the Likelihood function. The design curve is assessed by estimating the Likelihood Ratio Confidence bound (LRCB) of a high-reliability quantile strain–life curve and is built point by point through repeated optimizations to obtain the profile likelihood function. This procedure is more complex than those available in the literature, but it can be implemented step by step in numeric software.
3. The effectiveness and the fitting capability of the developed approach have been demonstrated through a validation on aluminium and steel alloys with different characteristics and sample sizes. The median curve has been found to be close to that estimated with literature models. The design curve, similarly, has proven to properly model the experimental variability, being below the experimental failures and close to literature models. The importance of taking into account also runout specimens in the estimation has been also demonstrated.

To conclude, the developed novel methodology for the estimation of the design strain–life curves represents a reliable alternative to widely adopted literature models. The drawback concerning its complexity is compensated by a straightforward implementation procedure and by its fitting capability and flexibility in modeling the fatigue response of datasets with different shapes and characteristics and containing also runout specimens.

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DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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