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Addendum to “A formulation of volumetric growth as a mechanical problem subjected to non-holonomic and rheonomic constraint”

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Abstract

In this work, we critically review some aspects of our own article “*A formulation of volumetric growth as a mechanical problem subjected to non-holonomic and rheonomic constraint*” [Mathematics and Mechanics of Solids, DOI: 10.1177/10812865231152228], which has been recently published in this Journal. The reason for undertaking this critique is that, after exploring some fundamental literature, which was not included in our original paper, we have noticed that, if the “canonical doctrine” on non-holonomic and rheonomic constraints is followed, some of our conclusions should be partially rephrased, and the procedure adopted to obtain them can be shortened. In fact, some of the main results of our article, although remaining unaltered, can be retrieved in a more straightforward way, while some other results should be reconsidered, and some statements should be corrected. On the basis of these considerations, the scope of this work is to present the necessary amendments to our previous paper, and to recast the core messages of our article, which remain valid, in an alternative form that is more concise and consistent with the standard theory of non-holonomic constraints.

Keywords

Growth mechanics; Bilby-Kröner-Lee multiplicative decomposition; Non-holonomic constraints; Virtual displacements; Lagrange multipliers; Principle of Virtual Work; Dissipation; Cahn-Hilliard model.

1 Introduction

After a further inspection into the fundamental literature on Analytical Mechanics (see e.g. [9, 4]), we have reached the conclusion that some of the results presented in our recent article “*A formulation of volumetric growth as a mechanical problem subjected to non-holonomic and rheonomic constraint*” [5], although remaining valid, can/should be re-obtained by rephrasing them consistently with the classical approach to non-holonomic and rheonomic constraints, and with the classical definition of virtual displacements, contextualized to systems subjected to such constraints [9, 4, 1, 3, 2].

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28 1.1 Backstory

29 As for the original article [5], the point of departure of this work is the fact that, in several bio-
 30 mechanical problems dealing with the mechanics of volumetric growth, the mass balance law of a
 31 growing body can be put in the form of a *non-holonomic and rheonomic constraint*¹ on the so-called
 32 *growth tensor* \mathbf{K} , i.e.,

$$\check{\mathcal{C}}_{\mathbf{K}} \circ (\mathbf{F}, \mathbf{K}, \dot{\mathbf{K}}, \mathcal{X}, \mathcal{T}) := \mathbf{K}^{-\mathbf{T}} : \dot{\mathbf{K}} - \check{R}_{\gamma(\text{ph})} \circ (\mathbf{F}, \mathbf{K}, \mathcal{X}, \mathcal{T}) = 0, \quad \text{in } \mathcal{B} \times \mathcal{I} \quad (1)$$

33 (see Equation (8) of [5]). Here and in the sequel, the notation is the same as in [5]. However, we
 34 recall that: \mathcal{B} is the reference placement of the body under consideration; \mathcal{I} is the time line; \mathbf{F}
 35 is the deformation gradient tensor; the auxiliary maps $\mathcal{X} : \mathcal{B} \times \mathcal{I} \rightarrow \mathcal{B}$ and $\mathcal{T} : \mathcal{B} \times \mathcal{I} \rightarrow \mathcal{I}$
 36 are defined by $\mathcal{X}(X, t) = X$ and $\mathcal{T}(X, t) = t$, for each pair $(X, t) \in \mathcal{B} \times \mathcal{I}$; $R_{\gamma(\text{ph})} \equiv \check{R}_{\gamma(\text{ph})} \circ$
 37 $(\mathbf{F}, \mathbf{K}, \mathcal{X}, \mathcal{T})$ is referred to as “*growth law*” [5] and is assumed to be given from the outset, e.g.
 38 phenomenologically, in such a way that it can be expressed as a function of \mathbf{F} and \mathbf{K} , as well as of
 39 \mathcal{X} and \mathcal{T} , in order to account for its explicit dependence on the points of \mathcal{B} and time. The growth
 40 tensor is introduced through the Bilby-Kröner-Lee (BKL) decomposition of \mathbf{F} , i.e., $\mathbf{F} = \mathbf{F}_e \mathbf{K}$, into
 41 an elastic and a growth part, represented by the tensors \mathbf{F}_e and \mathbf{K} , respectively.

42 In [5], we have studied the constraint (1) by following an approach that we have developed
 43 by taking inspiration from a paper by Nadile [8], and from a consideration on non-holonomic and
 44 rheonomic constraints given in [6]. In particular, Lanczos [6] writes:

45 *“Non-holonomic auxiliary conditions which are rheonomic [...] require particular care.*
 46 *Here it is necessary to know what conditions exist between the δq_k if the variation is not*
 47 *performed instantaneously but during the infinitesimal time δt . The auxiliary conditions*
 48 *now take the form $A_{i1}\delta q_1 + \dots + A_{in}\delta q_n + B_i\delta t = 0$ [...]”* (see page 66 of [6])

49 although, few lines afterwards the text quoted above, he adds that the virtual displacements are
 50 taken “without *varying the time*” [6]. Accordingly, δt should be set equal to zero, and only the
 51 sum $\sum_{k=1}^n A_{ik}\delta q_k$ contributes to the equations of motion of the considered system. On the other
 52 hand, Nadile’s study [8] treats time as a fictitious, additional Lagrangian parameter of his theory,
 53 and determines an Euler-Lagrange equation associated with it that is similar to Equation (29e) or
 54 (69b) of [5].

55 In the sequel, however, we show how the crucial point of our previous work [5] can be obtained
 56 also without considering time as a fictitious Lagrangian parameter and, for this purpose, we adhere
 57 to the classical formulation of non-holonomic and rheonomic constraints [9, 4, 1, 3, 2]. Therefore,
 58 the term $B_i\delta t$ reported above, which reads $[\check{R}_{\gamma(\text{ph})} \circ (\mathbf{F}, \mathbf{K}, \mathcal{X}, \mathcal{T})]\delta\mathcal{T}$ in our framework [5], disap-
 59 pears from the formulation of the Principle of Virtual Work (PVW). Indeed, consistently with the
 60 standard definition of virtual displacements [9], given also for the constraints of the type shown in
 61 Equation (1), the virtual displacement $\delta\mathbf{K}$ is accompanied by the condition $\delta\mathcal{T} = 0$.

62 1.2 Main changes with respect to the original article [5]

63 In the remainder of this work, we review the most important results of [5], and we reformulate
 64 them in an alternative and more straightforward manner in light of the “canonical doctrine” on
 65 non-holonomic and rheonomic constraints [9, 4, 1, 2, 3], which *does not* require viewing time as

¹Here, we are adopting the terminology of [6], in the same manner as we did in our work [5].

66 a “*fictitious Lagrangian parameter*” [5]. Moreover, to facilitate the comparison with the origi-
 67 nal article [5], we highlight the sentence(s) and/or mathematical expression(s) that must/can be
 68 rephrased, and we specify the section(s) of [5] in which they are to be found. Note that, in the
 69 blocks of reformulated text reported below, the references that feature in quotation marks refer to
 70 [5].

71 2 Review of Abstract and Sections 1, 2, 3, and 4 of [5]

72 If time is not viewed as a fictitious, additional Lagrangian parameter, then the following remarks
 73 apply:

74 1. At the fifth line of the abstract of [5], the sentence

75 “*For our purposes, [...] unitary.*”

76 should be rephrased as

77 “*For our purposes, we put the constraint in Pfaffian form.*”

78 Moreover, at the ninth line of the Abstract, the wording “*Lagrange multipliers*” should be
 79 replaced with “*Lagrange multiplier*”.

80 2. The core messages reported in Section 1 (Introduction) of [5] remain identical to the ones
 81 announced in the original article.

82 3. If the standard approach to the study of non-holonomic and rheonomic constraints is followed
 83 (see e.g. [9, 4]), Sections 2 and 3 of [5] remain unaltered, while Section 4 of [5] is no longer
 84 necessary to obtain the boundary value (sub-)problem expressed by Equations (29a)–(29d)
 85 and (29g) of [5], which is indeed one of the crucial points of our work. Accordingly, in the
 86 sequel we introduce neither the fictitious Lagrangian parameter \mathfrak{T} , nor the virtual variations
 87 $\delta\mathfrak{T}$ and $\delta\mathcal{T}$, and we study the constraint on \mathbf{K} either as assigned in Equation (1) or in the
 88 form

$$\hat{\mathcal{C}}_{\mathbf{K}} \circ (\mathbf{F}, \mathbf{K}, \dot{\mathbf{K}}, \omega) := \mathbf{K}^{-\text{T}} : \dot{\mathbf{K}} - \hat{R}_{\gamma(\text{ph})} \circ (\mathbf{F}, \mathbf{K}, \omega) = 0, \quad \text{in } \mathcal{B} \times \mathcal{I}. \quad (2)$$

89 Equation (2) is privileged if the dependence of the phenomenological growth law $\hat{R}_{\gamma(\text{ph})}$ on
 90 the mass fraction of the nutrients is highlighted (see the discussion on this topic reported in
 91 [5]).

92 Note also that Appendix A2 of [5] fits in the context developed therein, but it is not necessary
 93 in the present framework.

94 4. Since the constraints (1) and (2) are affine in the generalized velocity $\dot{\mathbf{K}}$, they comply with
 95 Chetaev’s conditions (see [4, 3, 2, 7]), which, in our case, read

$$\left[\frac{\partial \hat{\mathcal{C}}_{\mathbf{K}}}{\partial \dot{\mathbf{K}}} \circ (\mathbf{F}, \mathbf{K}, \dot{\mathbf{K}}, \mathcal{X}, \mathcal{T}) \right] : \delta \mathbf{K} = 0 \quad \Rightarrow \quad \mathbf{K}^{-\text{T}} : \delta \mathbf{K} = 0, \quad (3a)$$

$$\left[\frac{\partial \hat{\mathcal{C}}_{\mathbf{K}}}{\partial \dot{\mathbf{K}}} \circ (\mathbf{F}, \mathbf{K}, \dot{\mathbf{K}}, \omega) \right] : \delta \mathbf{K} = 0 \quad \Rightarrow \quad \mathbf{K}^{-\text{T}} : \delta \mathbf{K} = 0, \quad (3b)$$

96 where the tensor $\delta\mathbf{K}$ is the generalized virtual displacement associated with \mathbf{K} . In fact,
 97 with respect to the framework presented in [5], Equations (3a) and (3b) replace Equations
 98 (19) and (20b) of [5], and constitute the forms of the restriction that the components of $\delta\mathbf{K}$
 99 must fulfill in order for $\delta\mathbf{K}$ to be compatible with the associated constraint consistently with
 100 the “canonical doctrine” on non-holonomic and rheonomic constraints [9, 4]. Here, thus, in
 101 accordance with Pars [9] and Gantmacher [4], $\delta\mathbf{K}$ is taken at *fixed time*, i.e., with $\delta\mathcal{T} = 0$
 102 (see, in this respect, also the way in which Gantmacher [4], at pages 13 and 14 of his book,
 103 introduces the virtual displacements, and the motivation he gives for regarding them as the
 104 “*displacements in the case of ‘frozen’ constraints*” [4]).

105 In particular, by employing Equation (3a) or (3b) in the constrained version of the PVW,
 106 the “*technical difficulties*” mentioned at the beginning of Section 4 of [5] disappear. We also
 107 remark that the form of Chetaev’s condition given in Equation (3a), or (3b), substitutes, in
 108 the present framework, the one supplied in the footnote 5 of [5].

109 In addition, it is worth to clarify the following points pertaining to the Introduction of [5]:

- 110 • In the Introduction (Section 1) of [5], the sentence six lines after the beginning of the section

111 “*However, [...] Kozlov [12–15].*”

112 is incomplete, since some fundamental literature on these constraints was not cited. Therefore,
 113 by including some references on the topic, the sentence quoted above should read

114 “*However, [...], the formulation of the PVW becomes less obvious when the con-*
 115 *sidered constraints are non-holonomic and rheonomic, although [...] due to Kozlov*
 116 *[12–15]”, and although there does exist classical literature on the topic (see e.g.*
 117 *[9, 4, 1, 3, 2]).*

- 118 • Although in our opinion the conceptual novelty of [5] is preserved, the sentence in the second
 119 paragraph, third page of Section 1, i.e.,

120 “*Compared [...] Lagrange multiplier technique.*”

121 could be made clearer by reformulating it as follows:

122 “*Compared with the formulation summarized above, [...] the approach that we*
 123 *are proposing is novel because it [...] provides a constrained version of the PVW,*
 124 *relying on the Lagrange multiplier technique”, which may lead to deeper insights*
 125 *on the mechanics of inelastic processes, such as growth, remodeling, and aging. The*
 126 *principal advantage of our point of view is that it grants the ability to take into*
 127 *account a priori both experimentally observable growth laws and growth-conjugated*
 128 *generalized forces that could resolve other possible biological features specifically*
 129 *associated with growth itself.*

- 130 • Moreover, the sentence four lines after the one quoted previously

131 “*To the best of our knowledge, [...] in completely different frameworks.*”

132 should be clarified as follows

133 “To the best of our knowledge, this procedure is not standard for the case of
 134 non-holonomic and rheonomic constraints”, although it applies also to systems
 135 subjected to such constraints with some clarifications about the way in which virtual
 136 displacements comply with the given constraints [3, 2]. In our work, we propose an
 137 extension of the standard procedure, thereby generalizing some results put forward
 138 by Nadile [8] in a completely different framework (in fact, for discrete systems) to
 139 the context of Continuum Mechanics.

140 3 Review of Section 5 of [5]

141 With respect to Section 5 of [5], we discuss the following modifications, which apply if time is not
 142 viewed as a fictitious, additional Lagrangian parameter.

- 143 1. Consistently with the present framework, the sentence starting three lines after the beginning
 144 of Section 5:

145 “First, [...] as follows:”

146 and ending with Equation (23) of [5] should be reformulated as (from here on, for the sake
 147 of readability, in the sentences taken from [5] that feature in the blocks of reformulated text,
 148 we do not report the mathematical symbols and words that are related to viewing time as a
 149 fictitious Lagrangian parameter):

150 “First, we recall that the kinematic descriptors of the present theory, which is of
 151 grade one in χ , and of grade zero in \mathbf{K} [54], are given as follows:

$$(\chi, \mathbf{F}, \mathbf{K}, \delta\chi, \text{Grad}\delta\chi, \delta\mathbf{K}).” \quad (4)$$

152 This amounts to avoiding the introduction of $\delta\mathfrak{T}$ and $\delta\mathcal{T}$, or, equivalently, to setting $\delta\mathfrak{T} =$
 153 $\delta\mathcal{T} = 0$, which means that the virtual displacements $\delta\chi$ and $\delta\mathbf{K}$ are taken here at fixed time.

- 154 2. The duality pairings in Equations (24a) and (24b) should be re-considered in light of the fact
 155 that the Lagrange multiplier $\mu_{\mathfrak{T}}$ and its virtual variation $\delta\mu_{\mathfrak{T}}$ are not present in the “classical
 156 doctrine”. Accordingly, the sentence immediately after Equation (23) of [5]

157 “Then, [...] duality:”

158 should be reformulated as (the emphasized text highlights the modifications of the original
 159 text)

160 “Then, since we are going to append the *constraint*, both in the rescaled *form*”
 161 $t_c[\hat{\mathcal{C}}_{\mathbf{K}} \circ (\mathbf{F}, \mathbf{K}, \dot{\mathbf{K}}, \omega)]$ and in the Chetaev form $\mathbf{K}^{-\text{T}} : \delta\mathbf{K} = 0$ [7], “to the expression
 162 of the PVW that one would have in the absence of constraints, we introduce the
 163 Lagrange *multiplier* $\mu_{\mathbf{K}}$, along with *its* virtual *variation* $\delta\mu_{\mathbf{K}}$, so that the following
 164 duality pairings apply:”

$$\mu_{\mathbf{K}} \div \mathbf{K}^{-\text{T}} : \delta\mathbf{K}, \quad (5a)$$

$$\delta\mu_{\mathbf{K}} \div t_c[\hat{\mathcal{C}}_{\mathbf{K}} \circ (\mathbf{F}, \mathbf{K}, \dot{\mathbf{K}}, \omega)], \quad (5b)$$

165 “where the symbol “ \div ” indicates the conjugation induced by duality.”

166 3. Within the “classical doctrine”, the generalized forces \mathcal{Y}_u and \mathcal{Z} are not introduced, so that
 167 Equation (25) of [5] becomes

$$\mathbf{P} \div \text{Grad}\delta\chi, \quad \mathbf{Y}_u \div \mathbf{K}^{-1}\delta\mathbf{K}, \quad (6)$$

168 and the text two lines after Equation (25)

169 “; and \mathcal{Y}_u [...] $\delta\mathfrak{T}$.”

170 is no longer necessary. Moreover, the sentence three lines after Equation (25)

171 “The subscript [...], respectively.”

172 should be reformulated as (the emphasized text highlights the modifications of the original
 173 text)

174 “The subscript “u” in \mathbf{Y}_u indicates that *this force is* “unconstrained”, in the sense
 175 that, because of the presence of the Lagrange *multiplier* $\mu_{\mathbf{K}}$, *it is* associated with
 176 arbitrary (and, thus, “unconstrained”) variations $\delta\mathbf{K}$.”

177 Finally, Equation (26) becomes

$$\mathbf{f}, \boldsymbol{\tau} \div \delta\chi, \quad \mathbf{Z} \div \mathbf{K}^{-1}\delta\mathbf{K}, \quad (7)$$

178 while the sentence two lines after Equation (26)

179 “[...] from here on, \mathbf{Z} and \mathcal{Z} [...] respectively.”

180 should be reformulated as

181 “[...] from here on, \mathbf{Z} is said to be *external growth-conjugated stress-like force*.”

182 4. Remark 4 is no longer necessary.

183 5. In light of the comments above, the constrained expressions of the Principle of Virtual Work
 184 (PVW) reported in Equations (27) and (28) of [5] become

$$\begin{aligned} & \int_{\mathcal{B}} \mathbf{P} : \text{Grad}\delta\chi + \int_{\mathcal{B}} \mathbf{Y}_u : \mathbf{K}^{-1}\delta\mathbf{K} + \int_{\mathcal{B}} \mu_{\mathbf{K}}[\mathbf{K}^{-\text{T}} : \delta\mathbf{K}] + \int_{\mathcal{B}} t_c \delta\mu_{\mathbf{K}}[\hat{\mathcal{C}}_{\mathbf{K}} \circ (\mathbf{F}, \mathbf{K}, \dot{\mathbf{K}}, \omega)] \\ & = \int_{\mathcal{B}} \mathbf{f} \delta\chi + \int_{\partial_{\text{N}}^{\chi} \mathcal{B}} \boldsymbol{\tau} \delta\chi + \int_{\mathcal{B}} \mathbf{Z} : \mathbf{K}^{-1}\delta\mathbf{K}, \end{aligned} \quad (8a)$$

$$\begin{aligned} & \int_{\partial_{\text{N}}^{\chi} \mathcal{B}} \{\boldsymbol{\tau} - \mathbf{P}\mathbf{N}\} \delta\chi + \int_{\mathcal{B}} \{\text{Div}\mathbf{P} + \mathbf{f}\} \delta\chi + \int_{\mathcal{B}} \{\mathbf{Z} - \mu_{\mathbf{K}}\mathbf{I}^{\text{T}} - \mathbf{Y}_u\} : \mathbf{K}^{-1}\delta\mathbf{K} \\ & - \int_{\mathcal{B}} t_c \delta\mu_{\mathbf{K}}[\hat{\mathcal{C}}_{\mathbf{K}} \circ (\mathbf{F}, \mathbf{K}, \dot{\mathbf{K}}, \omega)] = 0. \end{aligned} \quad (8b)$$

185 Clearly, since neither the generalized forces dual to $\delta\mathfrak{T}$, i.e., \mathcal{Y}_u and \mathcal{Z} , nor the Lagrange
 186 multiplier $\mu_{\mathfrak{T}}$ are introduced in the present framework, the boundary value problem (BVP)

187 in Equations (29a)–(29h) of [5] reduces to

$$\text{Div} \mathbf{P} + \mathbf{f} = \mathbf{0}, \quad \text{in } \mathcal{B}, \quad (9a)$$

$$\chi = \chi_b, \quad \text{on } \partial_D^x \mathcal{B}, \quad (9b)$$

$$\mathbf{P} \mathbf{N} = \boldsymbol{\tau}, \quad \text{on } \partial_N^x \mathcal{B}, \quad (9c)$$

$$(\mathbf{Y}_u + \mu_{\mathbf{K}} \mathbf{I}^T) - \mathbf{Z} = \mathbf{0}, \quad \text{in } \mathcal{B}, \quad (9d)$$

$$\hat{\mathcal{C}}_{\mathbf{K}} \circ (\mathbf{F}, \mathbf{K}, \dot{\mathbf{K}}, \omega) = 0, \quad \text{in } \mathcal{B}. \quad (9e)$$

188 Note that the comments on Equations (29a) and (29d) of [5], which are identical to Equations (9a) and (9d), apply to the latter equations. Moreover, Equation (9e) is consistent with
 189 Equation (29g), while Equations (29e), (29f), and (29h) of [5] disappear from the present
 190 framework. We also remark that, in the current context, Equations (9a), (9d), and (9e) constitute a set of 13 scalar equations in the 13 scalar unknowns identified with the components
 191 of χ and \mathbf{K} , and with $\mu_{\mathbf{K}}$, respectively.
 192
 193

194 We emphasize that Equations (9a)–(9e) are identical to those of the original article [5] (see
 195 Equations (29a)–(29d) and (29g)), and, thus, the core message contained in them remains
 196 unchanged.

197 Analogously to the original article [5], the BVP (9a)–(9e) admits the equivalent formulation

$$\text{Div} \mathbf{P} + \mathbf{f} = \mathbf{0}, \quad \text{in } \mathcal{B}, \quad (10a)$$

$$\chi = \chi_b, \quad \text{on } \partial_D^x \mathcal{B}, \quad (10b)$$

$$\mathbf{P} \mathbf{N} = \boldsymbol{\tau}, \quad \text{on } \partial_N^x \mathcal{B}, \quad (10c)$$

$$\text{dev} \mathbf{Y}_u = \text{dev} \mathbf{Z}, \quad \text{in } \mathcal{B}, \quad (10d)$$

$$\mathbf{K}^{-T} : \dot{\mathbf{K}} = \hat{R}_{\gamma(\text{ph})} \circ (\mathbf{F}, \mathbf{K}, \omega), \quad \text{in } \mathcal{B}, \quad (10e)$$

198 in which Equation (9d) is replaced by its deviatoric part, and the constraint (9e) is written
 199 explicitly, while $\mu_{\mathbf{K}}$ is computed *a posteriori* as

$$\mu_{\mathbf{K}} = \frac{1}{3} \text{tr} \mathbf{Z} - \frac{1}{3} \text{tr} \mathbf{Y}_u, \quad \text{in } \mathcal{B}. \quad (11)$$

200 Again, it is important to remark that Equations (10a)–(10e) and (11) remain unchanged with
 201 respect to [5], and, indeed, correspond to Equations (34a)–(34e) and (33c) of [5], respectively.

202 6. In the present setting, the comments in the last two paragraphs of Section 5.1, and Equations
 203 (30a), (30b), and (31) do not come into play.

204 4 Review of Section 6 of [5]

205 The study of the dissipation inequality and of the constitutive laws as well as the considerations
 206 on the “*final form of the IBVP*” (initial and boundary value problem) (42a)–(42k), reported in
 207 Section 6 of [5], are unaffected by the present reformulation, with the exception of the preliminary
 208 discussion on the forces \mathcal{Y}_u and \mathcal{Z} , and the results presented in Equations (44b) and (45), which
 209 are no longer necessary. Moreover, the content of Subsection 6.4 of [5] remains unchanged.

210 5 Review of Section 7 of [5]

211 The content and the core message of Section 7 of [5] remain essentially unchanged within the present
 212 framework. However, since it holds that $\delta\mathcal{T} = 0$, the extension of the PVW expressed in Equation
 213 (8a) to the context of a theory of grade one in $J_{\mathbf{K}} := \det \mathbf{K}$, as is the case for the Cahn-Hilliard
 214 model, reads

$$\begin{aligned} & \int_{\mathcal{B}} \mathbf{P} : \text{Grad} \delta\chi + \int_{\mathcal{B}} q_u \frac{\delta J_{\mathbf{K}}}{J_{\mathbf{K}}} + \int_{\mathcal{B}} \mu_{\mathbf{K}} \frac{\delta J_{\mathbf{K}}}{J_{\mathbf{K}}} + \int_{\mathcal{B}} t_c \delta \mu_{\mathbf{K}} \left\{ \frac{\dot{J}_{\mathbf{K}}}{J_{\mathbf{K}}} + \left[\frac{1}{J_{\mathbf{K}}} \text{Div} \mathbf{v} - R_{\gamma(\text{ph})} \right] \right\} \\ & + \int_{\mathcal{B}} \mathbf{f} \frac{\text{Grad} \delta J_{\mathbf{K}}}{J_{\mathbf{K}}} = \int_{\mathcal{B}} \mathbf{f} \delta\chi + \int_{\partial_{\mathbb{N}}^{\chi} \mathcal{B}} \boldsymbol{\tau} \delta\chi + \int_{\mathcal{B}} z \frac{\delta J_{\mathbf{K}}}{J_{\mathbf{K}}}, \end{aligned} \quad (12)$$

215 with $R_{\gamma(\text{ph})} \equiv \hat{R}_{\gamma(\text{ph})} \circ (\mathbf{F}, \mathbf{K}, \omega)$, $\mathbf{K} = J_{\mathbf{K}}^{1/3} \mathbf{I}$, \mathbf{I} identity tensor, and $z := \frac{1}{3} \text{tr} \mathbf{Z}$. Moreover, $\mu_{\mathfrak{T}}$ does
 216 not appear in the present formulation.

217 6 Review of Section 8 of [5]

218 The comments summarized in Section 8 of [5] remain unchanged with respect to the original article,
 219 even though, in the present setting, the “*constrained version*” of the PVW [5] must be understood
 220 as in Equation (8a) above, i.e., with the virtual displacements $\delta \mathbf{K}$ taken at fixed time, i.e., with
 221 $\delta \mathfrak{T} = \delta \mathcal{T} = 0$. Moreover, Subsection 8.1 only requires to set $\delta \mathfrak{T} = \delta \mathcal{T} = 0$, and to disregard \mathfrak{T} , \mathcal{Y}_1
 222 and $\mu_{\mathfrak{T}}$, while Subsections 8.2 and 8.3 need no changes.

223 7 Review of Appendix A1 of [5]

224 Appendix A1 of [5], as it stands, suggests that the formulation that we have proposed therein is
 225 necessary for contextualizing the study of the non-holonomic and rheonomic constraint considered
 226 in our work to the framework developed in it. However, since this *is not* the case, we review
 227 here Appendix A1 accordingly, and we highlight below the sentences and equations that should be
 228 amended.

229 For our purposes, let us consider a mechanical system subjected to $m \in \mathbb{N}$ non-holonomic and
 230 rheonomic constraints², given by (see Equation (65) of [5])

$$\check{C}^i(q(t), \dot{q}(t), t) := \sum_{k=1}^n [a^i_k(q(t), t)] \dot{q}^k(t) + b^i(q(t), t) = 0, \quad i = 1, \dots, m. \quad (13)$$

231 In the jargon of [9], a system of this type is said to be “*acatastatic*” because of the presence of
 232 the terms $b^i(q(t), t)$, with $i = 1, \dots, m$. By adhering to the classical approach to the study of such
 233 systems (see e.g. [9, 4, 3, 2]), the virtual displacements $\delta q^1, \dots, \delta q^n$ that are *admissible* for the
 234 given constraints are, by definition, those that comply with the Pfaffian form of Equations (13) in
 235 the following way

$$\sum_{k=1}^n [a^i_k(q(t), t)] \delta q^k(t) + b^i(q(t), t) \delta \mathcal{T}(t) = \sum_{k=1}^n [a^i_k(q(t), t)] \delta q^k(t) = 0, \quad i = 1, \dots, m, \quad (14)$$

²See Lanczos [6].

236 i.e., with the variation $\delta\mathcal{T}(t) = 0$. In this sense, the virtual displacements are taken at fixed
 237 time, and Equation (14) replaces Equation (66) of [5] (see [4], page 14). Note that, since each
 238 $\check{C}^i(q(t), \dot{q}(t), t)$ is affine in the *true* generalized velocities, Equation (14), in fact, coincides with the
 239 set of Chetaev's conditions [3, 2, 7]

$$\sum_{k=1}^n \left[\frac{\partial \check{C}^i}{\partial \dot{q}^k}(q(t), \dot{q}(t), t) \right] \delta q^k(t) = \sum_{k=1}^n [a^i_k(q(t), t)] \delta q^k(t) = 0, \quad i = 1, \dots, m. \quad (15)$$

240 Moreover, since the virtual velocities of the system under consideration must satisfy Equations
 241 (14), or (15), with $\delta q^k(t)$ replaced by the corresponding virtual velocity $\nu^k(t)$, for $k = 1, \dots, n$, as
 242 pointed out in [9] (page 16), *the class of the true velocities does not coincide with the class of the*
 243 *virtual velocities*. Therefore, in this respect, the sentence of Appendix A1 of [5], eight lines after
 244 Equation (65), i.e.,

245 *“The relations [...] constraints.”*

246 is not consistent with the standard definition of virtual displacements or virtual velocities (see [9]),
 247 and it should be turned into

248 *“The relations obtained this way must be respected also by the virtual velocities of*
 249 *the considered mechanical system,” but as if the terms $b^1(q(t), t), \dots, b^m(q(t), t)$ were*
 250 *absent [4], “since, by definition, they [the virtual velocities] must be instantaneously in*
 251 *harmony with the imposed constraints.”*

252 Analogously, the text *“(be they virtual or real)”* [5] in the subsequent sentence should be turned
 253 into *“(in fact, the real ones)”*. Furthermore, since the virtual displacements are taken at $\delta\mathcal{T} = 0$,
 254 Equation (66) of [5] reduces to Equation (15) above, while Equation (67) becomes

$$\sum_{k=1}^n \mathcal{Q}_k(t) \delta q^k(t) + \sum_{i=1}^m \mu_i(t) \left\{ \sum_{k=1}^n [a^i_k(q(t), t)] \delta q^k(t) \right\} = 0. \quad (16)$$

255 Therefore, the PVW “sees” the constraints (13) *as if* the terms $b^1(q(t), t), \dots, b^m(q(t), t)$ were
 256 absent. Finally, Equations (68) and (69a)–(69c) of [5] become

$$\sum_{k=1}^n \left\{ \mathcal{Q}_k(t) + \sum_{i=1}^m \mu_i(t) [a^i_k(q(t), t)] \right\} \delta q^k(t) = 0, \quad (17a)$$

$$\mathcal{Q}_k(t) + \sum_{i=1}^m \mu_i(t) [a^i_k(q(t), t)] = 0, \quad k = 1, \dots, n. \quad (17b)$$

257 8 Conclusions

258 In this work, we have reviewed the main results of a previous article of ours [5] that were determined
 259 by regarding time as a fictitious Lagrangian parameter. In particular, we have reformulated some
 260 sentences, some equations, and some conclusions of [5] in light of an analysis of the constraint (2)
 261 that complies with the standard doctrine on non-holonomic and rheonomic constraints, and with
 262 the classical interpretation of the virtual displacements associated with this type of constraints [9].

263 In fact, we have shown that our main results are valid even though the virtual variations $\delta\mathfrak{T}$
264 and $\delta\mathcal{T}$ introduced in [5] are taken identically equal to zero, so that the constrained version of the
265 PVW is written in the form expressed in Equations (8a), or (8b), and (12). Moreover, we have
266 clarified some points of our previous work [5], and we have amended some statements of Appendix
267 A1 of [5] as well as some aspects of its formulation.

268 In summary, we would like to remark that having regarded time as a fictitious Lagrangian
269 parameter is not necessary for determining the crucial conclusions reported in [5], which we confirm
270 here. However, the approach presented in [5] could be useful for further research on growth and
271 on its connections with other biomechanical phenomena that are explicitly time dependent and
272 typically studied within the framework of Continuum Mechanics, as is the case e.g. for aging. For
273 such processes, indeed, a suitable adaptation of Nadile’s procedure [5] could lead to an insightful
274 interpretation of their physics.

275 Conflict of Interests

276 The Authors declare that they have no conflict of interests.

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286 References

- 287 [1] M. Favretti. Equivalence of dynamics for nonholonomic systems with transverse constraints.
288 *Journal of Dynamics and Differential Equations*, 10(4):511–536, 1998.
- 289 [2] M. R. Flannery. d’Alembert-lagrange analytical dynamics of nonholonomic systems. *Journal*
290 *of Mathematical Physics*, 52:032705–1–032705–29, 2011.
- 291 [3] M. R. Flannery. The elusive d’Alembert-lagrange dynamics of nonholonomic systems. *American*
292 *Journal of Physics*, 79(9):932–944, 2011.
- 293 [4] F. Gantmacher. *Lectures in analytical mechanics*. 1975.
- 294 [5] A. Grillo and S. Di Stefano. A formulation of volumetric growth as a mechanical problem
295 subjected to non-holonomic and rheonomic constraint. *Mathematics and Mechanics of Solids*,
296 2023.

- 297 [6] Cornelius Lanczos. *The Variational Principles of Mechanics*. Dover Publications, Inc. New
298 York, 1970.
- 299 [7] Jaume Llibre, Rafael Ramírez, and Natalia Sadovskaia. Nonlinear dynamics. *A new approach*
300 *to the vakonomic mechanics*, 78:2219–2247, 2014.
- 301 [8] A. Nadile. Sull’esistenza per i sistemi anolonomi soggetti a vincoli reonomi di un integrale
302 analogo a quello dell’energia. *Bollettino Unione Matematica Italiana*, 5:297–301, 1950.
- 303 [9] L.A. Pars. *A treatise on Analytical Dynamics*. Heinemann, London, 1965.