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Vacua of Marginal Deformations in Gauged Supergravity Models

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Declaration

I hereby declare that the contents and organization of this dissertation constitute, besides review material, my own and my collaborators' original work[1] and does not compromise in any way the rights of third parties, including those relating to the security of personal data.

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[1] In particular, the present thesis is mostly inspired by [1–3].

Abstract

After reviewing the main mathematical tools relevant to the formulation of a supergravity theory, both in its ungauged and gauged version, two specific examples are presented. In the first example, the $D=4$ $\mathcal{N}=3$ $\mathrm{SO}(3)\times\mathrm{SU}(3)$ gauged model inspired by the $\mathrm{AdS}_4\times\mathrm{N}^{0,1,0}$ compactification of M-theory is constructed. Two inequivalent $(\mathrm{SU}(1,1)/\mathrm{U}(1))^3$ truncations, obtained from singlets with respect to two different discrete groups, are discussed in detail. In the second example, a maximal $D=4$ $\mathcal{N}=8$ $\mathrm{SO}(1,1)\times\mathrm{SO}(6)\ltimes\mathbb{R}^{12}$ dyonically gauged supergravity is studied. An $\mathcal{N}=1$ truncation with respect to a $\mathbb{Z}_2^3$ discrete group is discussed. Both models share the property of having families of perturbatively stable vacua parameterized by supersymmetry breaking flat directions corresponding holographically to perturbatively stable non-supersymmetric conformal manifolds. In the latter example, a two-parameter family is uplifted to type IIB supergravity and pieces of evidence for non-perturbative stability of the conformal manifold are presented.

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Chapter 1

Brief Introduction and Motivation

Supergravity theories are the main topic of the present research. What is a Supergravity theory and why is it interesting? The answers to these questions are many and, in this Introduction, I will try to present my point of view on the subject.

As the name given by the scientific community to the theory suggests, the roots of a Supergravity theory reside in General Relativity. This latter is an essential ingredient in Theoretical Physics. Indeed, Gravity is one of the fundamental phenomena we experience every day and that we need to understand to have a complete scientific picture of our universe. Thanks to the efforts of the past century, today we know how to mathematically describe Gravity through General Relativity. In particular, the theory is developed in the framework of Differential Geometry and it aims to give, using the Einstein Equations, a classical description of Gravity. Gravity is encoded in the geometry of space-time. This latter is affected by the matter/energy present in it. This is where the limit of General Relativity becomes apparent. The theory of General Relativity classically describes very well the geometry of space-time but it has nothing to say about the nature of the matter deforming it. Even more elusive to General Relativity is the quantum nature of our world. Once this is understood, it is clear that there is a theoretical need to extend General Relativity to describe the other fundamental forces governing our world. In particular, those described by the well-known Standard Model. There are some crucial differences between the theoretical approach to General Relativity and the Standard Model. First of all, in the former, the space-time evolution for the metric tensor on a differential manifold is encoded in a set of partial non-linear differential equations while the latter is by definition a quantum field theory on Minkowski space-time where the metric tensor is fixed and given by the flat Minkowski metric. So, we see that the challenge here is not simply phenomenological but of theoretical nature too. One needs a framework able to support these very different mathematical descriptions.

At this point, it should be mentioned that the present research is theoretical rather than phenomenological so the interest is mainly on the mathematical assumptions and ingredients needed to formulate the relevant models and their implication on the physics that the theory can describe. It should be then understood that when adopting this point of view there is no limit to the path one can try to follow if not the mathematical consistency of the theory and those fundamental principles that are part of our way of thinking as a scientist. One of these principles, both present in classical and quantum physics, is the Lagrangian formulation of a system of particles. This framework is extensively used in classical mechanics and since the seminal work of Feynman, it found its applications in quantum mechanics and quantum field theory too. Concerning quantum field theory, the Lagrangian is one of the fundamental ingredients in the formulation of the action principle, the other being the space-time on which the Lagrangian is defined through the quantum fields present in the theory. In the case of the Standard Model, we have the fermionic and bosonic quantum fields describing the fundamental particles and interactions but Gravity through an action principle. Also in the case of Gravity, we can introduce an action principle, thanks to the Einstein-Hilbert Lagrangian. However, as well-known, the latter is a non-renormalizable theory. Meaning that it is consistent only at the classical level, the coupling constant being the square root of the Newton constant which in natural units has the dimension of a length. This means that, even when provided with the Einstein-Hilbert action and its Cartan formulation enabling us to couple the gravitational field to all fundamental particles and interactions, this route does not provide us with a consistent quantum theory. On the other hand, it is still a well-defined classical action principle that can be used to effectively describe some interactions between fermions, bosons, and Gravity. This is the starting point for formulating a Supergravity theory. What if we want to achieve a quantum description of gravity and we are not satisfied with an effective classical description?

One fruitful way of thinking comes from String Theory. Even if the model was introduced to describe what we know as Strong interaction it gave us a new way of describing Gravity. The simplest version of the theory involves a fundamental object in the shape of a one-dimensional string, in contrast to the zero-dimensional points we were used to, moving freely in a flat Minkowski space-time. Its free motion allows for vibrations which are described at the quantum level by creation and annihilation operators acting on a vacuum state. Surprisingly, some of the modes of the strings, the zero-energy ones, correspond to the known bosonic particles needed to describe all the fundamental forces, including Gravity too. However, this primitive model is in contrast with the fundamental principle that particles in Minkowski space-time must have non-negative squared masses. Indeed, a tachyon is present in the spectrum of the Bosonic String Theory signaling that the background around which the fields, metric included, were made to fluctuate, is not a true (i.e. stable) vacuum of the theory. Furthermore, the model is unsatisfactory from other perspectives. One of them is the fact that it does not describe half-integer spin particles, from the space-time point of view. Another one is the fact that even if the presence of interesting particles, such as the graviton is predicted, this simple approach to String Theory does not provide us with a dynamical picture. A crucial point is the interpretation of the coordinates describing the embedding of the string into space-time as quantum fields defined on the intrinsic two-dimensional surface that the string sweeps as time passes by. This introduces the distinction between the target space, the space-time where the string moves, and the world-sheet, the two-dimensional manifold intrinsically describing the motion of the string in space-time. This distinction is crucial since it gives us the correct framework in which to fix some issues of the bosonic string. Now, we are considering the motion of the string as originating from an action principle. This action principle allows for the introduction of new ingredients such as new fields defined on the world-sheet with a well-defined interpretation on the target space. Then, String Theory can be seen as a particular instance of a two-dimensional quantum field theory. Such a point of view is an important step forward toward a unified theory of fundamental forces. Indeed, we now have a quantistically consistent framework in which the graviton arises. Here, I do not want to focus on the details of String Theory. Instead, I want to mention one main ingredient that can be introduced in String Theory when regarded as a quantum field theory: Supersymmetry. This latter concept leads to Superstring Theory and solves at once two of the main theoretical problems of Bosonic String Theory. It introduces fermions and it allows the definition of a tachyon-free spectrum of particles. What is Supersymmetry and how it is introduced in the Einstein-Hilbert and String Theory action principle? What is the relation between the two?

Supersymmetry is a particular instance of another extremely important concept in physics: the Symmetry of a system. In particular, in a quantum field theory, we are used to dealing with continuous symmetries associated with a Lie Algebra, such as the global Poincaré symmetry or local Gauge symmetries. Particles are defined as irreducible representations of such symmetries. They are labeled by invariants describing physical properties. In this way, we can exploit the concept of symmetry to define the mass, the angular momentum, the spin, and in general the charges of a physical system. The key point here is that symmetry associated with a standard Lie Algebra defines irreducible representations with states of the same bosonic or fermionic nature. Indeed, the helicity of a particle is an invariant of the spinorial representations of the Lorentz group, the subgroup of the Poincaré one. So, each kind of particle must be introduced separately in a quantum field theory whose symmetries are of this kind. Supersymmetry is a much more unifying principle. Indeed, it can be regarded as an extension of the concept of Lie Algebras to the structure of Graded Lie Algebras. In particular, one can extend the Poincaré Lie Algebra of commutators to the SuperPoincaré Graded Lie Algebra of commutators and anti-commutators by adding to the usual translations, rotations and boosts generators new fermionic generators associated with supersymmetry. The fermionic nature of these symmetries is such that the irreducible representations exhibit states of different helicity so that a particle is now a collection of both bosons and fermions. Searching for a supersymmetric theory naturally constrains it to describe both bosons and fermions. When implementing this concept in String Theory we are led to Superstring Theory where, in the world-sheet approach, the bosonic fields describing the coordinates of the string are related by supersymmetry to new fermionic fields. The presence of these new ingredients in the quantum field theory defined on the intrinsic two-dimensional surface of the moving and vibrating string allows us to describe both bosonic and fermionic particles from the Target Space point of view. Requiring these latter to arrange in a supersymmetric way also from the Target Space point of view cuts out tachyonic modes from the string spectrum through the so-called GSO projection. In this way, we end up with a quantum consistent theory whose zero energy modes, including a graviton, but now supersymmetries require the presence of other unusual states. In particular, the gravitino, a vector and also a spinor from the Lorentz point of view, appears. Again, the details are more complicated and beyond the scope of this introduction. Let us instead mention how Supersymmetry is implemented on the Gravity side.

When dealing with Gravity, Supersymmetry has a different meaning than the one present in the SuperPoincaré case. Indeed, in General Relativity, translations are local and not only global symmetries. We are then dealing with an infinite dimensional algebra of symmetries as in the case of local gauge symmetries. One way of introducing supersymmetry in Gravity is to exploit the Cartan formulation where the gravitational field, encoded in the vielbein frame, naturally carries a representation of the Lorentz group. Then, in the same way as we associate with particles in a quantum field theory their supersymmetric partners we can associate with the vielbein its supersymmetric partner. It turns out that the properties of the latter, also named gravitino, are the same as the gravitino present in the spectrum of Superstring Theory. The difference with a supersymmetric quantum field theory is that now Supersymmetry is a local symmetry. This is what defines a Supergravity Theory. A Lagrangian theory of Gravity exhibiting local Supersymmetry in the sense just sketched. Then, the core of a Supergravity theory is a Lagrangian for the graviton and the gravitino fields, it is to say the Einstein-Hilbert Lagrangian coupled to the Rarita-Schwinger one, with a local symmetry that relates the two. Even if we already see some similarities from the particle content side, in principle a generic Supergravity theory is not related to Superstring theory. To explain what one means by this let me quickly introduce the case where the relation is well understood. The main bridge between the two is the quantum consistency of the Superstring Theory. Indeed, the requirement that all local symmetries of the latter are not anomalous translate into a set of differential equations describing the dynamics of the string modes. The low energy limit of these equations, in units of the string tension, can be recast as Euler equations derived from a specific Supergravity action. In this way, we have the striking correspondence between the infrared limit of type IIA/B Superstring Theory and the ten-dimensional type IIA/B Supergravity Theories. We now see the first relevant application of Supergravity. It is to say the effective description of the interactions involving the low energy modes of the superstring. This is extremely important since it allows us to explore some properties of the most prominent unifying theory which is the Superstring Theory. These latter are not the only possible Supergravity theories. In particular, they are formulated in a ten-dimensional space-time. More generically, Supersymmetry allows for an extension of a Gravity theory on a space-time up to eleven dimensions. Furthermore, depending on the dimensions of space-time we can allow for different Supersymmetric extensions of the global Poincaré Lie Algebra, and its local deformation, obtained by increasing the number of fermionic generators beyond the minimal extension. In this case, we talk about extended supersymmetry. The larger the extension, the more the symmetry and the constraint on the Lagrangian. The main one is that a state present in a supersymmetric particle must not exceed the spin of a graviton. In this sense, the eleven-dimensional Supergravity is special since in this case the Einstein-Hilbert Lagrangian admits a unique supersymmetric extension. Instead, in the main part of this work I will focus on a four-dimensional space-time. Why so? What is the relation between them and the supergravities originating from Superstring theories?

To understand these concepts one has to consider another main tool in theoretical physics that allows us to relate theories formulated in different space-time dimensions. It is to say the techniques of dimensional reduction and space-time (spontaneous) compactification. The technique of dimensional reduction was introduced by Kaluza and Klein in an attempt of unifying the electromagnetic field with Gravity. The main idea is that particles of different nature in a specific dimension can be interpreted as originating from fewer particles present in higher dimensions. In this way, Gravity coupled to a vector field and a scalar field in four dimensions can be interpreted as a particular instance of a pure Gravity theory in a five-dimensional space-time. In some sense, under reasonable assumptions, one can trade the number of dimensions by properly increasing the number of fields present in the theory. From the Action principle point of view, this means that we interpret the representation of the relevant symmetries in a dimension as originating from suitable combinations of symmetries present in a higher one. In particular, certain global and local internal symmetries in the lower dimensional theory naturally originate from space-time symmetries of the higher dimensional parent theory. In the previous examples, this would translate into breaking the representation of the spin-two particle in five dimensions in the sum of the spin-two particle, a spin-one, and a spin-zero particle in four. On top of this, we have to get rid of those dimensions we want to cut out by "integrating them out" to obtain an action principle defined in fewer dimensions. This latter step is less obvious than the former which is group-theoretically motivated. Indeed, to obtain some concrete Lagrangian to work with one has to assume a specific dependence of the fields on the extra dimensions. This can be done only once the nature of the part of space-time, also named the "internal space", describing the extra spatial dimensions, is specified. Once this is done we can study the fields that the latter can support. The most common choice is to assume the internal space to be compact. This assumption makes sense when trying to obtain an effective description of the original theory. Indeed, the natural scale introduced by the compact space, like the radius of a sphere, provides us with a natural energy scale for the dynamical processes. Again, with our simple example, we can imagine that the fifth dimension, the fourth spatial one, consists of a circle. This choice allows us to describe the internal dependence of the fields through Fourier expansions in units of the length of the circle. So that each four-dimensional field will show up together with an infinite tower of particles of the same nature but with mass increasing in discrete steps, it is to say the various Fourier modes or more generically the Kaluza-Klein modes. At this step, we are just describing the same model less intuitively so that there is a one-to-one correspondence between the configurations in four dimensions with all the Kaluza-Klein modes and the five-dimensional one. The next step for simplifying the theory consists in considering only the massless modes, which in our example correspond to the constant modes in the Fourier expansion. In this way, we have a four-dimensional effective description of the dynamics taking place at low energy in five dimensions. One can immediately understand that more complicated examples are those in which the dimension of the internal space increases. Nevertheless, one can try to explore the simple cases where an algebraic approach, as in the case of "coset spaces", can help us in studying the compactification of space-time. There are however some issues with naively applying this method to obtain a simpler theory in fewer dimensions from a complicated one defined on a higher dimensional space-time. The first one is the issue of spontaneous compactification. Indeed, consider our trivial example.

Let us assume that the Kaluza-Klein mechanism (and generalization thereof) provides the appropriate framework in which to classically unify gravity and electromagnetism in four dimensions by obtaining them from gravity on a five-dimensional space-time. Then it is reasonable to assume that the fifth dimension must be compact and possibly small since we do not experience it in our four-dimensional world. We can then proceed as described above to formulate the theory in four dimensions. We can go on and search for solutions of the effective model. Will they also describe solutions of the original five-dimensional theory? In general, this is not true. If we require this to be the case, we say that we search for a spontaneous compactification and a consistent truncation. In other words, we introduce the theoretically driven requirement that the form of the space-time that we choose to obtain a lower-dimensional theory must be consistent with the equations of motion of the higher-dimensional one. Furthermore, when removing a subset of degrees of freedom from the model we must be sure that they are not involved in the dynamics of the remaining ones. If this is the case we can proceed with the study of the lower dimensional theory while being sure that we are truly studying a subsector of the higher dimensional one. In this sense, the steps of compactification and then of consistent truncation of a higher dimensional theory to a lower dimensional one are a clever way to break down complicated equations of motion into simpler ones.

We are then interested in studying four-dimensional Supergravity theories since they provide an effective description of a subsector of the eleven or ten-dimensional ones as consistent truncations. More specifically, the construction of ten/eleven-dimensional backgrounds, in the higher-dimensional descriptions, would require the solution of often complicated partial differential equations. On the other hand, the same problem, for certain higher-dimensional backgrounds, in the four-dimensional framework reduces to the purely algebraic problem of extremizing a scalar potential. The landscape of supergravity theories is very rich. From a top-down point of view, each consistent choice of a seven/six-dimensional internal space of the eleven/ten-dimensional supergravities leads to effective descriptions captured by different four-dimensional models. Using a bottom-up approach, even when the field content of a four-dimensional supergravity is fixed by the relevant (super)symmetries, there is still some freedom in the kind of interactions that it can display. As we will see, in many cases we can try to deform a minimally interacting, or abelian, supergravity theory to a more interacting one through the so-called "gauging" procedure. Different gaugings can be related to different spontaneous compactifications. However, it must be pointed out that it is not always obvious how a Supergravity theory in four dimensions can be interpreted as a consistent truncation of the eleven or ten-dimensional ones and in many cases this is not the case. Even if many steps forward have been made, the latter is still an open problem. In this thesis, an example of a model exhibiting interesting configurations whose higher dimensional description, if present, is still not clear will be presented. On the other hand, in the past decades, using different techniques, many spontaneous compactifications of eleven and ten-dimensional Supergravities have been constructed. In this thesis, a particular example leading to a maximal Supergravity model exhibiting the so-called "S-fold" configurations will be presented.

During the past three decades a new point of view on Supergravity theories has been developed. In particular, a well-established interesting reason for studying Supergravity is the Gauge/Gravity correspondence. The latter consists of a conjectured duality between quantum gauge theories and quantum gravity theories. The main concept introduced by the conjecture is given by the identification between the Action principle of the two theories. In the ideal case, one would make the duality theoretically rigorous by proving the existence of a well-defined map between the fundamental ingredients, and the mathematical objects that can be defined with them, of the gravity model and of the gauge one. Clearly, this would require the knowledge and a full understanding of quantum gauge and quantum gravity theories. This is a very ambitious and challenging goal. However, much progress has been done by the scientific community in this direction. The most studied examples are those in which a (super)gravity configuration with an asymptotically locally anti-de Sitter space-time is dual to a conformal field theory. This example is known as the "AdS/CFT" correspondence. This specific case strongly relies on the algebraic notion of the conformal group/algebra and on two ways of geometrically interpreting it. Given a space with a well-defined metric field, the conformal group is given by the group of transformations preserving the metric field modulo a point-dependent rescaling. Then, a Conformal Field Theory on a given (pseudo)-Riemannian space can be defined as a quantum field theory exhibiting the conformal group of the given space as symmetry. This would also imply that we can use this symmetry to organize the degrees of freedom present in the model by exploiting the representation theory of the conformal group. On the other hand, one could try to find a space whose isometries correspond to conformal transformations. If this can be done, one can try to formulate a gravity theory in which the latter is a vacuum around which linear perturbative computations can be done. Then, because of their linear nature, the perturbations will carry a representation of the conformal group too. This can give a first insight into the possible duality between the degrees of freedom of the CFT and the perturbations around the vacuum of a gravity model. Even more promising, from the gauge/gravity conjecture point of view, is the scenario in which the gravity model under consideration originates from eleven/ten-dimensional supergravity. Indeed, even if not exploring the full duality, one can explore the AdS/CFT correspondence within a certain limit of String Theory. The main advantage of establishing a duality between two theories, or part of it, is that one has two ways of studying the same object. Working on the gravity side can shed light upon the quantum field theory side and vice versa.

As an example, an important question to ask when studying CFTs is whether they belong to a family of theories, known as a conformal manifold. The conformal manifold is spanned by exactly marginal deformations of the CFT. Without entering into technical details, exactly marginal deformations are contributions that can be added to the Lagrangian of a quantum field theory with no need for additional quantum corrections. Over the last decade, much insight has been gained into local properties of conformal manifolds of supersymmetric conformal field theories. On the other hand, no example is known of a non-supersymmetric conformal field theory in more than two dimensions featuring a conformal manifold. Indeed, they are widely believed not to exist. However, there are no "no-go theorems" that forbid non-supersymmetric conformal manifolds. As a result, the existence of non-supersymmetric conformal manifolds has been largely the subject of speculation.

Getting closer to the results presented in the next chapters, the two examples of four-dimensional gauged supergravity theories mentioned earlier in this introduction are relevant in making such speculations concrete. More precisely, some of the AdS vacua present in these models belong to families parameterized by continuous parameters corresponding to massless perturbations. From the AdS/CFT point of view the latter would correspond to exactly marginal deformations. The vacua under discussion preserve a different amount of supersymmetry, and in some cases, supersymmetry is completely broken. Then, the existence of such families of solutions on the gravity side would imply, given the AdS/CFT correspondence, the existence of a conformal manifold in which non-supersymmetric conformal field theories are continuously connected to supersymmetric ones. As already discussed, one could argue that working at the supergravity level allows us to make statements only valid at a "classical level". However, the examples related to the "S-fold" configurations involve four-dimensional supergravity theories whose ten-dimensional origin is clearer. As it will be shown, this allows us to provide evidence for the existence of a non-supersymmetric conformal manifold at a "quantum level".

Some technical gaps need to be filled before explicitly presenting such findings. The next chapter will be devoted to introducing more rigorously four-dimensional supergravity theories and their gaugings by focusing on the examples of interest.

Chapter 2

$D = 4$ Supergravity

As already motivated, it should be clear that the study of gauged supergravity models in lower dimensions provides an efficient framework in which to explore effectively the physics described by higher dimensional supergravity theories, such as $D = 10$ or $D = 11$ supergravity. The technique of consistent truncations allows us to define the lower dimensional models from higher dimensional ones. It is then relevant the choice of the background on which to reduce the theory since its properties such as its isometries will play an important role in the effective lower-dimensional description. The configuration of interest to our discussion will be the one described by $AdS_4 \times_w M_d$, $d = D - 4$ where $D = 10$ in the case of type II supergravities. The latter corresponds to an external 4-dimensional anti-de Sitter spacetime and an internal manifold M_d. In most cases M_d is chosen to be compact, however, configurations with non-compact internal direction are also of interest. We can relate to this kind of background by studying $D = 4$ supergravities. Finding vacua exhibiting the metric of AdS_4 in upliftable models automatically provides a solution for the higher dimensional model of the form just described. However, it is not completely satisfactory to limit the analysis to finding vacua of the model. It is indeed necessary to further study a given configuration to assess its stability. This can be done perturbatively by computing the mass spectrum of the oscillations around a given background. In some cases, one can go deeper in the analysis so to consider the non-perturbative properties of the configuration. Again, this is more interesting for those solutions admitting an uplift to type IIA/B supergravity since they can directly relate to non-perturbative superstring theory configurations. [1] We will explore the framework of Exceptional Field

[1] In particular, ten/eleven-dimensional supergravity theories include in their landscape of solutions the so-called p-branes which are extended black-objects with p-dimensional spatial extension. In ten dimensions, some of these supergravity solutions provide the low-energy description of microscopic objects belonging to the string spectrum known as D_p-branes. In supergravity theories, we can effectively study the interaction (usually referred to as

Theory [4, 5] to prove that some model of interest consistently originates from higher dimensional theories. As we will see, the same framework allows us to study the perturbative stability of interesting backgrounds [6]. When considering the non-maximal supergravity, when supersymmetry decreases, one can not directly exploit the latter framework. However, it is still possible to describe interesting classes of consistent truncations within the formalism of Generalized Geometry [7–9]. In this case, many examples of consistent truncations can be obtained from generalized G_S-structure manifolds with singlet intrinsic torsion. A complete understanding of the consistency of a given supergravity model is still missing.

Before moving to some interesting examples of upliftable models and of models whose higher dimensional origin is still unclear let us proceed with a short review of the general features of supergravity in four dimensions. First, the so-called "ungauged supergravity" and some details on its global symmetries are presented, then the procedure to promote some of the latter to a local gauge invariance of the model is discussed. This latter procedure, the "gauging procedure", is fundamental to building the model of interest in the present thesis.

2.1 Ungauged $D=4$ Supergravity

Following [10, 11], the simplest supergravity theory one can build, up to quadratic terms in the fermionic fields, has the following general form [2]:

$$\mathscr{L}_{\mathcal{N},D=4} = \mathscr{L}_{bosonic} + \mathscr{L}_{fermionic} + \mathscr{L}_{mixed}, \tag{2.1.1}$$

$$\frac{1}{\sqrt{|\mathrm{g}|}}\mathscr{L}_{bosonic} = -\frac{R}{2} + \frac{1}{2}\mathscr{G}_{st}(\phi)\partial_\mu\phi^s\partial^\mu\phi^t + \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}(\phi)F^{\Lambda}_{\mu\nu}F^{\Sigma\mu\nu} + \frac{1}{8\sqrt{|\mathrm{g}|}}\mathcal{R}_{\Lambda\Sigma}(\phi)\epsilon^{\mu\nu\rho\sigma}F^{\Lambda}_{\mu\nu}F^{\Sigma}_{\rho\sigma} \tag{2.1.2}$$

is the purely bosonic sector of the theory. R is the Ricci scalar for the metric $g_{\mu\nu}$ (here the mostly minus convention is adopted). As usual, $|\mathrm{g}| = \det(g_{\mu\nu})$. ϕ^s denotes the set of n_s scalar fields present in the theory. Their kinetic

"back-reaction") of a single brane with the background fields using the Dirac-Born-Infeld action on the world-volume of the former. These objects are non-perturbative string states in the following sense: in the perturbative limit (i.e. small string-coupling constant), the branes decouple from the low energy spectrum while their interaction becomes relevant at non-perturbative regimes (large string-coupling constant). We will be dealing with these objects and their supergravity description in section 6.2.

[2]Strictly $\mathcal{N}=1$ models can display a scalar potential too.

term describes a sigma model with Riemannian metric $\mathscr{G}_{st}(\phi)$. More on the scalar field and the space they parameterize will be added below. The vector sector is described by the Yang-Mills type and the θ-like term for the abelian curvatures $F^{\Lambda} = 2\partial_{[\mu} A^{\Lambda}_{\nu]}$ of the n_v vector fields describing an abelian $\mathrm{U}(1)^{n_v}$ gauge symmetry. Since fields are not charged under this latter we say that the theory describes an "Ungauged Supergravity". However, as we shall explicitly see in the next chapters, some non-abelian gaugings can be introduced consistently with supersymmetry. The latter, already at the ungauged level, allows for the couplings of the Yang-Mills contribution and of the θ-like part to depend on the scalar fields. This is expected from a top-down point of view on supergravities. Indeed, this is the case even in the simplest example of compactification by Kaluza and Klein.

The bosonic sector of the model is one of interest when searching for solutions defined by the vanishing of all fermionic fields. This latter is the most common configuration discussed in the literature. However, as we will see, the fermionic contributions are relevant when discussing supersymmetry and when considering perturbative phenomena around a solution of the theory. Let us write them down. We have[3]:

$$\mathscr{L}_{fermionic} = \epsilon^{\mu\nu\rho\sigma}\left(\overline{\psi}^{A}_{\mu}\gamma_{\nu}\mathscr{D}_{\rho}\psi_{A\sigma} - \overline{\psi}_{A\mu}\gamma_{\nu}\mathscr{D}_{\rho}\psi^{A}_{\sigma}\right) - \frac{i\sqrt{|\mathrm{g}|}}{2}\left(\overline{\lambda}^{\mathcal{I}}\gamma^{\mu}\mathscr{D}_{\mu}\lambda_{\mathcal{I}} + \overline{\lambda}_{\mathcal{I}}\gamma^{\mu}\mathscr{D}_{\mu}\lambda^{\mathcal{I}}\right) \tag{2.1.3}$$

The fields entering the latter kinetic terms, more precisely the Rarita-Schwinger and the Dirac one respectively, are the $\mathcal{N}$ gravitino fields $\psi_{A\mu}$ and the spin-$\frac{1}{2}$ field collectively denoted by $\lambda_{\mathcal{I}}$. The former is required in a theory whose supersymmetry is not only global but local too, in other words, a supergravity theory. Their number selects the $\mathcal{N}$-extended supersymmetry of the model. In particular, they all belong to the graviton multiplet together with 0, 1, 3, 6, 10, 16, 28 vectors, 0, 0, 1, 4, 11, 26, 56 spin-$\frac{1}{2}$ fields and 0, 0, 0, 2, 10, 30, 70 real scalar fields, when $\mathcal{N} = 1,2,3,4,5,6,8$. The rest of the vector fields belong to the vector multiplets, together with 1, 2, 3, 4 spin-$\frac{1}{2}$ fields and 0, 2, 6, 6 real scalar fields when $\mathcal{N} = 1,2,3,4$. When $\mathcal{N} > 4$, no vector multiplets are allowed for a theory describing only one metric such as a supergravity theory. When $\mathcal{N} = 1$, the remaining spin-$\frac{1}{2}$ fields and scalar fields can be arranged in matter multiplets. Each one of them provides 2 real scalar fields. We can add matter fields in the $\mathcal{N} = 2$ case too, and we refer to the multiplets containing spin-$\frac{1}{2}$ and scalar fields only as hypermultiplets. Each one of them contains 4 real scalar fields. When $\mathcal{N} > 2$, the scalar fields are required to parameterize a homogeneous symmetric manifold. Furthermore, no hypermultiplets are

[3]The conventions of [11] for spinors are implemented

allowed so that spin-$\frac{1}{2}$ fields sit only in the graviton multiplet or in the vector multiplet. The index $\mathcal{I}$ splits accordingly. In particular, λ_{AI} will denote the gauginos, present in the vector multiplets, while χ_{ABC} will denote the dilatinos, belonging to the graviton multiplet. In $\mathcal{N}=5$ models, an extra dilatino χ is present. In $\mathcal{N}=6$ models, six extra dilatinos χ_A are present.

Since it is not restrictive for the present discussion, let us focus on $\mathcal{N}>2$ theories [4]. Then, the scalar fields parameterize a coset manifold G/H with G semisimple and H maximally compact in G. They are conveniently described by a coset representative $L(\phi)\in G$. An isometry of the scalar manifold described by an element $g\in G$ with action of the generic form:

$$g:\ \phi^s \to \phi'^s = \phi'^s(\phi^r)\,, \tag{2.1.4}$$

is defined by the left action of the g element on the coset representative, modulo a compensating action of an element $h\in H$ from the right, more specifically:

$$g\cdot L(\phi) = L(\phi')\cdot h(\phi,g)\,, \tag{2.1.5}$$

where $h(\phi,g)\in H$ can in general depend on the point of the scalar manifold and on the isometry group element that acts on the latter. It is convenient to rewrite the Lie algebra $\mathfrak{g}$ of G as the direct sum of its maximal compact subalgebra $\mathfrak{H}$, generating H, and $\mathfrak{K}$ where the non-compact generators live:

$$\mathfrak{g} = \mathfrak{H}\oplus\mathfrak{K}. \tag{2.1.6}$$

This decomposition allows us to define relevant ingredients. In particular, as common in standard group theory, we can define from $L(\phi)$ a left-invariant 1-form $\Omega(\phi)$ with values in $\mathfrak{g}$

$$\Omega(\phi)\equiv L^{-1}dL(\phi) = \mathscr{Q}(\phi)+\mathscr{P}(\phi)\,. \tag{2.1.7}$$

The projections $\mathscr{Q}$, $\mathscr{P}$ of Ω belong to $\mathfrak{H}$ and $\mathfrak{K}$, respectively. By considering the exterior derivative of $\Omega(\phi)$ we can easily derive the Maurer-Cartan equations $d\Omega+\Omega\wedge\Omega=0$ from which we can obtain the following identities for $\mathscr{Q}$ and $\mathscr{P}$:

$$\begin{aligned}\mathcal{R}[\mathscr{Q}] &\equiv d\mathscr{Q}+\mathscr{Q}\wedge\mathscr{Q} = -\mathscr{P}\wedge\mathscr{P}\,,\\ \mathcal{D}\mathscr{P} &\equiv d\mathscr{P}+\mathscr{Q}\wedge\mathscr{P}+\mathscr{P}\wedge\mathscr{Q}=0\,.\end{aligned} \tag{2.1.8}$$

[4]In $\mathcal{N}=1$ models, the scalar manifold is required to be a Kähler manifold. In $\mathcal{N}=2$ models, it is the direct product of a Special Kähler and a Quaternionic Kähler manifold.

The above expressions allow us to interpret $\mathscr{Q}$ as an H-connection, $\mathcal{R}[\mathscr{Q}]$ being its $\mathfrak{H}$-valued curvature. $\mathcal{D}$ is then interpreted as an exterior H-covariant derivative under which $\mathscr{P}$ is charged.

The mixed terms presented later on (2.1.23), display couplings between fermions and scalars. Then, H-covariance must be extended to fermions so that the theory is not sensitive to the action of the local transformations h in (2.1.5). In other words, we want the supergravity theory to be locally H-invariant. Indeed, the action of $H \subset G$ (2.1.5) relates equivalent configurations of the scalar fields. In the kinetic term, this is implemented by the H-covariant derivative $\mathscr{D}$ which describes the action of $\mathscr{Q}$ on the various fermionic fields in a suitable representation. As usual, it also contains the Levi-Civita and spin-connection required to couple fermions to the metric field. Before giving the explicit form of the interaction terms involving fermionic fields it is useful to introduce some definitions relevant to making explicit a global G symmetry of the equations of motion of an extended supergravity theory. This global symmetry is already captured by the scalar manifold. Indeed, the global action of the group G is already well defined on the scalars by the left action of constant elements of G on $L(\phi)$. Even if this does not completely extend to a global symmetry of the theory it can still be extended to a symmetry relating physically equivalent configurations in an electric-magnetic duality fashion. In particular, once the G action is properly extended to the vectors and fermions it becomes a manifest global symmetry of the equations of motion of the model. Going on with [11], let us quickly review this feature. The first relevant ingredients are $G_{\Lambda\mu\nu}$, defined as

$$G_{\Lambda\mu\nu} = -\epsilon_{\mu\nu\rho\sigma}\frac{\delta\mathscr{L}_{\mathcal{N},D=4}}{\delta F^{\Lambda}_{\rho\sigma}}. \tag{2.1.9}$$

Together with their electric-magnetic duals $F^{\Lambda}_{\mu\nu}$[5], they carry a $2n_v$ dimensional representation $\mathscr{R}_v$ of G. We denote the $2n_v$ vector in this representation as $V^M = (V^\Lambda, V_\Lambda)$. Necessary conditions for the action of G to describe a symmetry of the equations of motion are

$$\mathscr{R}_v[g]^T\mathbb{C}\mathscr{R}_v[g] = \mathbb{C}, \tag{2.1.10}$$

[5] Note that in the case of a gauge theory with field-dependent couplings on a generic manifold such as the model in (2.1.1) one is led to the definition (2.1.9) for the electric-magnetic dual of F instead of its Hodge-dual. Indeed, in terms of the doublet $\mathcal{G}^M = (F^\Lambda, G_\Lambda)$, the Maxwell equations, in the case of vanishing fermionic fields, can be rewritten as $d\mathcal{G}^M = 0$. Furthermore, one can show that $\mathcal{G}$ is covariant under the global G-action which we interpret as the group of dualities relating different Lagrangians as discussed in the next section. The definition (2.1.9) reduces to the Hodge dual in a Yang-Mills theory with vanishing θ-term.

and

$$\mathcal{M}(\phi'=g(\phi))=\mathscr{R}_v^{-T}\mathcal{M}(\phi)\mathscr{R}_v^{-1}. \tag{2.1.11}$$

where $\mathbb{C}_{MN}$ is the standard $2n_v\times 2n_v$ symplectic matrix and

$$\mathcal{M}(\phi)=\begin{pmatrix} \mathcal{R}\mathcal{I}^{-1}\mathcal{R}+\mathcal{I} & -\mathcal{R}\mathcal{I}^{-1} \\ -\mathcal{I}^{-1}\mathcal{R} & \mathcal{I}^{-1} \end{pmatrix}(\phi). \tag{2.1.12}$$

The first property requires that $\mathscr{R}_v\in\mathrm{Sp}(2n_v,\mathbb{R})$ and the second one that the couplings, collected in the symplectic matrix $\mathcal{M}_{MN}(\phi)$ transform properly so to compensate the vector transformations. The scalar manifold structure, fixed by supersymmetry, ensures that a $2n_v$-dimensional symplectic representation of G is always realizable. Furthermore, the couplings captured by $\mathcal{M}$, constrained by supersymmetry too, in an extended supergravity theory in four dimensions are such that the latter property is always satisfied. In particular, defining an equivalent representation $\underline{\mathscr{R}}_v$ to $\mathscr{R}_v$ such that

$$\underline{\mathscr{R}}_v=\mathcal{S}^{-1}\mathscr{R}_v\mathcal{S},\quad \underline{\mathscr{R}}_v[h]\in\mathrm{SO}(2n_v)\quad h\in H, \tag{2.1.13}$$

and the hybrid coset representative $\mathbb{L}(\phi)=\mathscr{R}_v[L(\phi)]\mathcal{S}$, we have that the right action of H on $L(\phi)$ translates in an orthogonal right action on $\mathbb{L}(\phi)$. In terms of this latter, $\mathcal{M}(\phi)$ is computed as

$$\mathcal{M}(\phi)=\mathbb{C}\mathbb{L}(\phi)\mathbb{L}^T(\phi)\mathbb{C}. \tag{2.1.14}$$

In this basis, a vector is denoted by $V^{\underline{M}}=(V^{\underline{\Lambda}},V_{\underline{\Lambda}})$, where $V^{\underline{\Lambda}}$ further splits in $(V^{AB}=-V^{BA},V^I)$ describing the vector sector of the graviton multiplet and of the matter multiplet respectively [6].

The action of G must be extended to fermions too. In this case, as already mentioned the relevant transformations are the compensating ones. Indeed, fermions are coupled to bosons in such a way that they are required to transform in a complex representation of H. Because of this, it is convenient to consider a new complex basis for the representation of G defined by $\underline{\mathscr{R}}_v^c=\mathscr{A}\underline{\mathscr{R}}_v\mathscr{A}^\dagger$, where

$$\mathscr{A}=\frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{1} & i\mathbf{1} \\ \mathbf{1} & -i\mathbf{1} \end{pmatrix} \tag{2.1.15}$$

[6]In the $\mathcal{N}=6$ case, not relevant here, there is only the graviton multiplet and it contains two different kinds of vectors, those transforming under H as V^{AB} and a singlet. The singlet is described by the V^I sector.

is the $2n_v \times 2n_v$ Cayley matrix. In this new basis, the action of G is no longer symplectic but the action of H is block diagonal. Then, $\mathscr{Q}$ is represented by

$$\underline{\mathscr{R}}^c_v[\mathscr{Q}] = \begin{pmatrix} \mathscr{Q}^{\underline{\Lambda}}{}_{\underline{\Sigma}} & 0 \\ 0 & \mathscr{Q}_{\underline{\Lambda}}{}^{\underline{\Sigma}} = (\mathscr{Q}^{\underline{\Lambda}}{}_{\underline{\Sigma}})^* \end{pmatrix}, \tag{2.1.16}$$

where

$$\mathscr{Q}^{\underline{\Lambda}}{}_{\underline{\Sigma}} = \begin{pmatrix} \mathscr{Q}^{AB}{}_{CD} = 4\delta^{[A}_{[C}\mathscr{Q}^{B]}{}_{D]} & 0 \\ 0 & \mathscr{Q}^I{}_J \end{pmatrix}. \tag{2.1.17}$$

This is the relevant representation for the H-covariant derivative $\mathscr{D}$. Explicitly, the relevant expressions are

$$\begin{aligned} \mathscr{D}_\mu \psi_{A\nu} &= \nabla_\mu \psi_{A\nu} + \mathscr{Q}_{\mu A}{}^B \psi_{B\nu} & (2.1.18) \\ \mathscr{D}_\mu \chi_{ABC} &= \nabla_\mu \chi_{ABC} + 3\mathscr{Q}_{\mu[A}{}^D \chi_{|D|BC]} & (2.1.19) \\ \mathscr{D}_\mu \lambda_{AI} &= \nabla_\mu \lambda_{AI} + \mathscr{Q}_{\mu A}{}^B \lambda_{BI} + \mathscr{Q}_{\mu I}{}^J \lambda_{AJ}, & (2.1.20) \end{aligned}$$

where ∇_μ denotes the Lorentz covariant derivative. In the $\mathcal{N}=5$ model, χ is an H-singlet. In the $\mathcal{N}=6$ model, the expression for χ_A is the same as the one for the gauginos with the $\mathscr{Q}_{\mu I}{}^J$ component dropping out. On the other hand, $\mathscr{P}$ is represented by

$$\underline{\mathscr{R}}^c_v[\mathscr{P}] = \begin{pmatrix} 0 & \mathscr{P}^{\underline{\Lambda\Sigma}} \\ \mathscr{P}_{\underline{\Lambda\Sigma}} & 0 \end{pmatrix}. \tag{2.1.21}$$

Moreover, we can promote the hybrid coset representative to a complex matrix by defining

$$\mathbb{L}_c(\phi) = \mathbb{L}(\phi)\mathscr{A}^\dagger = \begin{pmatrix} f^\Lambda{}_{\underline{\Sigma}} & \overline{f}^{\Lambda\underline{\Sigma}} \\ h_{\Lambda\underline{\Sigma}} & \overline{h}_\Lambda{}^{\underline{\Sigma}} \end{pmatrix}. \tag{2.1.22}$$

In terms of this latter we have $\mathcal{M}(\phi) = \mathbb{C}\mathbb{L}_c(\phi)\mathbb{L}_c^\dagger(\phi)\mathbb{C}$ and

$$-\frac{1}{\sqrt{|\mathrm{g}|}}\mathscr{L}_{mixed} = \overline{\lambda}^{\mathcal{I}}\gamma^\mu\gamma^\nu\psi^A_\mu\partial_\nu\phi^s\mathscr{P}_{s\mathcal{I}A} - \frac{1}{2}F^{+\Lambda\mu\nu}\mathcal{I}_{\Lambda\Sigma}\overline{f}^{\Sigma\overline{\Gamma}}\mathcal{O}_{\overline{\Gamma}\mu\nu} + h.c., \tag{2.1.23}$$

where the dots refer to, in the $\mathcal{N}=5$ case, $\mathscr{P}_A = \frac{1}{24}\epsilon_{ABCDE}\mathscr{P}^{BCDE}$ and, in the $\mathcal{N}=6$ models, $\mathscr{P}_{AB} = \frac{1}{24}\epsilon_{ABCDEF}\mathscr{P}^{CDEF}$. Furthermore, $2F^+_{\Lambda\mu\nu} = F_{\Lambda\mu\nu} + \frac{1}{2}\sqrt{|\mathrm{g}|}\epsilon_{\mu\nu\rho\sigma}F^{\Lambda\rho\sigma}$, and the $\mathcal{O}_{\overline{\Gamma}\mu\nu}$ are fermion bilinears whose explicit expression is fixed by supersymmetry and it depends on the theory. Their relevant property is that they transform in the $\underline{\mathscr{R}}^c$ representation of H, the group of compensating transformations of the action of G.

The equations of motion of a $D=4$ $\mathcal{N}>2$ supergravity theory with the Lagrangian present here can be rewritten in terms of these ingredients to exhibit a manifest G invariance. This global symmetry of the equations of motion is one of the main features of supergravity theories in low dimensions. From a top-down point of view, in the case of a model originating from a spontaneous compactification, it is believed to capture some of the quantum dualities relating the different string theories and the conjectured M-theory. Supergravity theories can then give an effective description for strongly coupled string theories by exploiting a dual weakly coupled picture. Indeed, as we can easily see from the symplectic action of G on $\mathcal{G}^M = (F^\Lambda, G_\Lambda)$, the isometries of the scalar manifold act as an electric-magnetic duality when extended to all fields. Since in general this is a symmetry of the equations of motion and not of the action principle for a supergravity theory, the G action relates dual Lagrangians. There is some freedom in fixing what we call electric or magnetic. More precisely, the embedding of G in $\mathrm{Sp}(2n_v,\mathbb{R})$ depends on the so-called "symplectic frame". Meaning that at the level of the equations of motion we can always rotate $\mathcal{G}$ alone using a constant $\mathrm{Sp}(2n_v,\mathbb{R})$ element. Inequivalent frames will be those that can not be off-set by a local redefinition of the scalar fields (and consequently of fermions by the action of the relevant compensating transformation and vectors through $\mathscr{R}_v[G]$) or by a local block-diagonal $\mathrm{GL}(n_v,\mathbb{R}) \subset \mathrm{Sp}(2n_v,\mathbb{R})$ redefinition of the electric and magnetic fields separately. At the ungauged level, inequivalent symplectic frames describe equivalent equations of motion. When non-trivial interactions are introduced through the gauging procedure, different frames can yield inequivalent physical descriptions. As an example, the $\mathcal{N}>1$ ungauged models do not feature a scalar potential. This is not the case after the gauging is implemented. As presented in the next chapter, starting from different symplectic frames could result in introducing inequivalent scalar potentials. However, $\mathscr{R}_v[G]$ relates equivalent frames thus giving a tool for dualizing a supergravity theory. It is important to stress that the presence of G as a global symmetry is intrinsic in the supersymmetric nature of the model.

Now, we can be more explicit about supersymmetry by describing its action in the $\mathcal{N}>2$ models. Following [11], at lower order in the fermionic fields we have

$$
\begin{aligned}
\delta\phi^s \mathscr{P}_s^{ABCD} &= \Sigma^{ABCD} \\
\delta\phi^s \mathscr{P}_s^{IAB} &= \Sigma^{IAB} \\
\delta A_\mu^\Lambda &= \mathbb{L}^\Lambda_{c\underline{M}} \mathcal{O}_\mu^{\underline{M}} \\
\delta V_\mu^a &= i\bar{\epsilon}^A \gamma^a \psi_{\mu A} + i\bar{\epsilon}_A \gamma^a \psi_\mu^A \\
\delta\psi_{A\mu} &= \mathscr{D}_\mu \epsilon - \frac{1}{8} F^-_{\rho\sigma AB} \gamma^{\rho\sigma} \gamma_\mu \epsilon^B \\
\delta\chi_{ABC} &= i\partial_\mu \phi^s \mathscr{P}_{sABCD} \gamma^\mu \epsilon^D - \frac{3i}{4} F^-_{\mu\nu[AB} \gamma^{\mu\nu} \epsilon_{C]} \\
\delta\lambda_{IA} &= i\mathscr{P}_{sIAB} \partial_\mu \phi^s \gamma^\mu \epsilon^B - \frac{i}{4} F^-_{\mu\nu I} \gamma^{\mu\nu} \epsilon_A,
\end{aligned}
\tag{2.1.24}
$$

where $\Sigma^{\underline{\Lambda\Sigma}}$ depend linearly on the supersymmtry point-dependent fermionic parameter ϵ^A and the spin-$\frac{1}{2}$ fields. From the above expressions, we explicitly see the main property of supersymmetry, it is to say the exchange of bosons with fermions. Furthermore, we see that the supersymmetry rules define an infinite dimensional algebra extending the diffeomorphism symmetry of a classical gravity theory. Indeed, one can verify that commuting two supersymmetry transformations produces the action of a local diffeomorphism modulo the equations of motion and other local symmetries. It resembles the main property of a rigid supersymmetry algebra that the commutator of two supercharges produces a momentum operator modulo central charges. This is reasonable since one can make a rigid supersymmetry transformation by selecting a constant ϵ^A parameter. A more systematic approach to supersymmetry for a theory displaying a local diffeomorphism symmetry like gravity theories is given by the so-called "Rheonomic principle" [12–15]. This latter poses its basis on the algebraic formulation of a rigid supersymmetry algebra and it extends the concept to a local symmetry by exploiting the framework of differential geometry on "Supermanifolds". It is beyond our scope to review this subject and we refer to the bibliography.

For the present discussion, it is sufficient to think of supersymmetry as the local symmetry described by the above transformations. From the latter, we see another important property. Since the supersymmetry rules are written in terms of H-covariant quantities they inherit this property. In general, H splits as a direct product of the form $H = H_R \times H_{matter}$. H_R describes the action of the "R-symmetry" which is $\mathrm{U}(\mathcal{N})$ when $\mathcal{N} < 8$ and $\mathrm{SU}(8)$ in the $\mathcal{N} = 8$ maximal case. It acts as an automorphism of the supersymmetry transformations by rotating the supersymmetry parameters. H_{matter} is present due to those fields outside the graviton multiplet. In particular, $H_{matter} = \mathbf{1}$ when $\mathcal{N} > 4$ since in these models only the gravitational multiplet is allowed.

When searching for Lorentz preserving maximally symmetric bosonic solutions, i.e. vacua, of a supergravity model or for a more general bosonic solution of the equations of motion one tries to consistently solve the model with vanishing fermionic fields. However, a generic supersymmetry transformation can restore the presence of fermions. Those particular supersymmetries parameters, when present, that have the property of producing vanishing variations for all fields will describe residual supersymmetries of the specific background. They will select a subalgebra of the supersymmetry algebra that in many cases corresponds to a rigid supersymmetry algebra. As an example, for a Minkowski background, the residual supersymmetry will reproduce the super-Poincarè algebra. When the gauging procedure is implemented, AdS backgrounds usually appear. In these cases, the residual supersymmetry will reproduce the well-known super-conformal algebra. In the next chapters, this property will be exploited to make more precise statements on the AdS_4/CFT_3 correspondence. Let me conclude the discussion on supersymmetry by noting that I have omitted the transformations for the λ_I, χ and χ_F fields, specific to the $\mathcal{N}=3,5,6$ cases respectively. The former is the one of interest in the present work. At lower order in fermionic fields it reads

$$\delta\lambda_I = \frac{i}{2}\mathscr{P}_{sIAB}\partial_\mu\phi^s\gamma^\mu\epsilon_C\epsilon^{ABC}. \tag{2.1.25}$$

Now that all the main ingredients for an ungauged model have been presented, let us continue the discussion with the gauging procedure. Some interesting solutions, like black holes and other kinds of solitonic configurations, are already present in ungauged models. However ungauged supergravities are unsatisfactory from many points of view. As an example, as already discussed, $\mathcal{N} > 1$ models do not feature a scalar potential. We then encounter the phenomenological problem of moduli stabilization. It is to say that such models display several scalar fields whose background value is not fixed by dynamical considerations and that can lead to instabilities of the solution they belong to. Moreover, they appear in the interactions of other fields thus leading to non-predictive models. From a top-down point of view, ungauged supergravities usually originate from compactifications of higher-dimensional supergravities on a torus or on classes of Ricci-flat manifolds. When trying to obtain an effective description for a compactification on a different kind of manifold one is driven to gauged supergravity models. It is to say models featuring fields charged under gauge symmetries. Gauged models naturally display a scalar potential that allows fixing, through the equations of motion, the background values of scalar fields. The latter are interpreted, in upliftable models, as background values of higher dimensional fields. In particular, they can describe the geometrical properties of the internal manifold on which the higher dimensional theory is compactified. Then, the non-trivial scalar potential of gauged models offers the possibility of exploring various kinds of compactifications [7]. Furthermore, if those scalars appearing in the interactions of the Lagrangian are indeed dynamically fixed, one recovers the predictiveness of the model. The latter is relevant when studying supergravity for phenomenological purposes.

2.2 Gauging of the Global Symmetries

As reviewed in the previous section, a Supergravity model in general exhibits global symmetries due to the symmetric nature of the scalar manifold. It is natural to study how to promote them to local symmetries. Indeed, we know from many other examples that local symmetries enrich the structure of the theory offering new theoretical and phenomenological properties to explore. First of all, nontrivial interactions between gauge fields could be introduced. As we will see, these will induce a chain of modifications for the Lagrangian resulting in new interaction terms between fermions and scalars and in particular in the introduction of a scalar potential. Again, supersymmetry will be the main constraint and guide principle for the above procedure. As a trivial requirement, we can not introduce new vector fields. Indeed, the field content is already fixed in the ungauged model so the gauge group must be gauged through the vector fields already present in the theory. Then, one must find which subgroup G_g of G can be chosen consistently. As already introduced, G is a true symmetry of the equations of motion of the ungauged model but not of the Lagrangian. So it is clear that once the frame is fixed one has to choose a subgroup of the so-called "electric group". The latter is the subgroup of G whose elements are also a symmetry of the Lagrangian. Following [11], once the symplectic frame is fixed, the electric group G_e is the subgroup of G such that

$$\forall g \in G_e \Rightarrow \mathscr{R}_v[g] = \begin{pmatrix} A^{\Lambda}{}_{\Sigma} & 0 \\ C_{\Lambda\Sigma} & D_{\Lambda}{}^{\Sigma} = -A^{\Lambda}{}_{\Sigma}. \end{pmatrix} \tag{2.2.1}$$

Intuitively, such transformations do not change the set of vectors identified as electric by the frame. Then, it is clear that starting from a different frame, a different embedding of G in the symplectic group can result in a different G_e for the Lagrangian.

However, we can work in a more general set-up in which we do not fix the frame for the ungauged model. It is to say a G-covariant framework.

[7]See [11] for a review of the main known examples.

The latter is also referred to as the "embedding tensor formulation" of the gauging procedure in which a formal G-covariance of the field equations is made explicit by rewriting the gauged model in a G-covariant fashion through the "embedding tensor" Θ. The latter is a manifestly G-covariant tensor that captures all the information about the gauging. As common when searching for explicit invariance of a model under a particular symmetry, one has to introduce redundant degrees of freedom. The already present gauge fields A^{Λ}_{μ}, which can be considered "electric" and which are associated with the gauge generators X_{Λ}, are now doubled by adding their "magnetic" counterparts $A_{\Lambda\mu}$. They allow us to introduce the gauging of magnetic generators X^{Λ}. Furthermore one has to make room for new two-form fields $B_{\alpha} = (B_{\alpha\mu\nu})$, $\alpha = 1, \ldots, \dim(G)$, transforming in the adjoint representation G, the global symmetry group. As we will see, the conditions on the gauge algebra will ensure that once G_g is found, we can always rotate the model to an "electric frame", a frame in which G_g is a subgroup of a suitable G_e. In general, this last step can bring us to an inequivalent frame than the starting one meaning that to formulate the theory we must use a dual description of the starting Lagrangian.

The first ingredient in the G-covariant formulation of the gauging is the introduction of the "embedding tensor" Θ. It can be regarded as a map from the adjoint representation of the G algebra to $\mathscr{R}_{v^\star} = \mathscr{R}_v^{-T}$, the representation dual to $\mathscr{R}_v$. In particular, being t_α, $\alpha : 1,...,\dim(G)$, the generators of $\mathfrak{g}$, we have

$$X_M = \Theta_M{}^{\alpha}\, t_\alpha\,. \tag{2.2.2}$$

The X-tensor then selects which vectors can enter the gauge connection to be introduced for the standard minimal couplings in usual gauge covariant derivatives with coupling constant g. We have

$$\Omega_{g\,\mu} \equiv g\, A^M_\mu\, X_M\,. \tag{2.2.3}$$

Already at this level we explicitly see that this G-covariant formulation allows for magnetic vector fields to enter the gauge connection. When this is the case, and the frame in which the gauging is purely electric is inequivalent from the starting one, we have an example of a dyonic gauging. Later on, an explicit example in the maximal model will be presented. Moving on with the gauging, the outlined procedure is not straightforwardly consistent. Indeed, the action can be made locally G_g-invariant and $\mathcal{N}$-supersymmetric only if the embedding tensor satisfies the so called linear and quadratic constraints. The latter can be equivalently described by implementing the definition of the X-tensor tensor

$$X_{MN}{}^{P} \equiv \Theta_M{}^{\alpha}\, \mathscr{R}_{v^\star}[t_\alpha]_N{}^{P}\,.$$

The linear constraints have the following form:

$$X_{(MN}{}^{R}\mathbb{C}_{P)R}=0\,. \tag{2.2.4}$$

We have in general two quadratic constraints:

$$a):\ [X_M,X_N]+X_{MN}{}^{P}X_P=0\,, \tag{2.2.5}$$
$$b):\ \mathbb{C}^{MN}\Theta_M{}^{\alpha}\Theta_N{}^{\beta}=0\,. \tag{2.2.6}$$

The linear constraint translates in the gauge invariance of Θ and when it is satisfied ensures that the gauge fields are properly acted on by a global duality transformation of an element in G_g. It is to say, they must transform in the co-adjoint representation of the gauge group. This is the usual setup in all gauge theories. The quadratic constraints (2.2.6) are instead necessary to limit the number of vector fields that contribute to the gauging to n_v, more precisely there must be a number less or equal to n_v of linearly independent gauge generators. This constraint ensures that it is possible to introduce, using a global symplectic transformation, a particular symplectic frame such that the “magnetic” direction $\Theta^{\Lambda\alpha}$ of the embedding tensor Θ are vanishing. This frame is also referred to as an “electric” frame. The quadratic constraints are redundant in $\mathcal{N}\geq 3$ models. In the $8>\mathcal{N}>2$ case, upon the validity of the linear constraints, equation (2.2.6) can be obtained from (2.2.5), while in the maximal $\mathcal{N}=8$ theory, the two conditions are equivalent. In practice, these conditions tell us that a suitable gauge group is such that its dimension does not exceed the number of vector fields present in the theory and that its adjoint representation must be embedded in a suitable G_e. We see that this choice is purely algebraic for a given ungauged model.

Once the gauge algebra is properly chosen one proceeds by introducing the minimal couplings, and following standard procedure for gauge invariant theories one can promote partial derivatives to their covariant counterparts:

$$\partial_\mu\rightarrow\partial_\mu-\Omega_{g\,\mu}\,. \tag{2.2.7}$$

Analogously, one introduces the non-abelian field strengths:

$$\partial_\mu A^M_\nu-\partial_\nu A^M_\mu\ \rightarrow\ \partial_\mu A^M_\nu-\partial_\nu A^M_\mu+gX_{NP}{}^{M}\,A^N_{[\mu}A^P_{\nu]}\,. \tag{2.2.8}$$

The same applies to the previously defined Maurer-Cartan connection. In order to properly describe the scalar sectors of the gauged model, one promotes $\mathscr{P}$ and $\mathscr{Q}$ to their gauged version $\widehat{\mathscr{P}}$, $\widehat{\mathscr{Q}}$. The latter are defined through the gauged version of the Maurer-Cartan left-invariant 1-form in the following way:

$$\hat{\Omega}=\mathscr{R}_v[L^{-1}(d-\Omega_g)L]=\widehat{\mathscr{P}}+\widehat{\mathscr{Q}}\,. \tag{2.2.9}$$

Then, $\widehat{\mathscr{P}}$ encodes the gauged vielbein along the non-compact directions of the isometry group of the scalar manifold and $\widehat{\mathscr{Q}}$ described the gauged version of the H-connection which lives in the compact part of G. One proceeds by introducing the gauging in the various contributions to the Lagrangian, in particular the kinetic term for the scalar fields now reads

$$\frac{1}{\sqrt{|\mathrm{g}|}}\mathscr{L}_{scal.kin} = \frac{1}{2}\mathscr{G}_{rs}\mathscr{D}_\mu\phi^r\mathscr{D}^\mu\phi^s, \tag{2.2.10}$$

where $\mathscr{D}_\mu\phi^s = \partial_\mu\phi^s - gA_\mu^M\Theta_M^\alpha k_\alpha^s$ is the gauged covariant derivative in (2.2.7) made explicit for the scalar fields. In this expression, k_α^s are the Killing vectors describing the scalar manifold isometries along the t_α directions. The kinetic contribution for the vector fields must be modified too. It reads:

$$\frac{1}{\sqrt{|\mathrm{g}|}}\mathscr{L}_{v.kin} = \frac{1}{4}\mathcal{I}_{\Lambda\Sigma}\mathcal{H}^\Lambda_{\mu\nu}\mathcal{H}^{\Sigma\ \mu\nu} + \frac{1}{8\sqrt{|\mathrm{g}|}}\mathcal{R}_{\Lambda\Sigma}\epsilon^{\mu\nu\tau\gamma}\mathcal{H}^\Lambda_{\mu\nu}\mathcal{H}^\Sigma_{\tau\gamma}. \tag{2.2.11}$$

The main ingredients are now $\mathcal{H}^\Lambda$, corresponding to the electric directions of the improved field strength $\mathcal{H}^M = F^M + \frac{g}{2}\mathbb{C}^{MN}\Theta_M^\alpha B_\alpha$, it displays the symplectic non-abelian curvature $F^M = dA^M + \frac{g}{2}X_{NP}{}^M A^N \wedge A^P$. Even if its expression resembles a gauge covariant field-strength, generically it does not share this property with the electric F^Λ. This can be understood by noticing the property of the symplectic X tensor of not being anti-symmetric in the first two indices. We have in general, $X_{MN}{}^P = X_{[MN]}{}^P + X_{(MN)}{}^P$. Thus, X_{MN}^P can not be directly interpreted as a structure constant for the gauge algebra since it has a non-vanishing symmetric component and it does not satisfy a proper Jacobi identity. The definition of $\mathcal{H}^\Lambda$ requires also the $B_{\alpha\mu\nu}$ auxiliary fields. They must be present to promote F^M to the gauge covariant quantity $G^M = (\mathcal{H}^\Lambda, G_\Lambda)$[8]. The magnetic component of the latter, $G_\Lambda = -\epsilon_{\mu\nu\rho\sigma}\frac{\delta\mathscr{L}}{\delta\mathcal{H}_{\rho\sigma}}$, is the dual vector to $\mathcal{H}^\Lambda$[9]. Furthermore, in terms of these ingredients, new $\mathscr{L}_{top}$ topological terms and $\mathscr{L}_{CS}$ Chern-Simons interactions are required by gauge invariance. Their derivation is rather involved and we refer the reader to [11].

Concerning the fermionic kinetic terms, their expression is analogous to the ungauged model with the difference that now the H-connection $\mathscr{Q}$ is replaced by its gauged version $\widehat{\mathscr{Q}}$. The same recipes can be implemented to promote the remaining terms already present in the ungauged model to their gauged version. It is to say by replacing space-time derivatives, $\mathscr{P}$, $\mathscr{Q}$, and G^M with their gauged version. However, these modifications do not ensure that

[8] With a little abuse of notation we use G^M both for the ungauged and gauged symplectic vector.

[9] More precisely, the duality holds when the equations of motion derived from the variation of $B_{\alpha\mu\nu}$ are imposed. Once this is done, $\mathcal{H}_\Lambda$ is identified with G_Λ.

supersymmetry in the sense of the transformations (2.1.24) is preserved. In fact, it is not. The Lagrangian

$$\mathscr{L}^{(0)}_{\mathcal{N},D=4} = \mathscr{L}_{bosonic,gauged} + \mathscr{L}_{fermionic,gauged} + \mathscr{L}_{mixed,gauged} + \mathscr{L}_{top} + \mathscr{L}_{CS} \tag{2.2.12}$$

has a G_g local symmetry but it is not supersymmetric. However, this problem can be cured by introducing new terms in the Lagrangian and at the same time by introducing new contributions to the supersymmetry transformations using an expansion in the g coupling constant. Indeed, an explicit computation shows that linear terms in g survive the supersymmetry variation of $\mathscr{L}^{(0)}$. Then one can try to compensate them by adding terms of the same order to the Lagrangian and to the supersymmetry rules. In principle the procedure could take infinite steps. Luckily, it stops at second order in the coupling constant producing the following contributions

$$\begin{aligned} \frac{1}{\sqrt{|\mathrm{g}|}}\mathscr{L}^{(1)} &= g\left(2\overline{\psi}^A_\mu \gamma^{\mu\nu}\psi^B_\nu \mathbb{S}_{AB} + i\overline{\lambda}^{\mathcal{I}}_\mu \gamma^\mu \psi_{A\mu}\mathbb{N}_{\mathcal{I}}{}^A + \overline{\lambda}^{\mathcal{I}}\lambda^{\mathcal{J}}\mathbb{M}_{\mathcal{I}\mathcal{J}}\right) + h.c. \\ \frac{1}{\sqrt{|\mathrm{g}|}}\mathscr{L}^{(2)} &= -\frac{g^2}{\mathcal{N}}\left(\mathbb{N}_{\mathcal{I}}{}^A\mathbb{N}^{\mathcal{I}}{}_A - 12\mathbb{S}_{BC}\mathbb{S}^{BC}\right). \end{aligned} \tag{2.2.13}$$

The tensors $\mathbb{S}$, $\mathbb{N}$ and $\mathbb{M}$ entering the above expressions are called "fermionic shifts", and they do in general depend on the scalar fields. $\mathbb{N}_{\mathcal{I}}{}^A = (\mathbb{N}^{\mathcal{I}}{}_A)^\star$, $\mathbb{S}_{BC} = (\mathbb{S}^{BC})^\star$. We explicitly see that a scalar potential, $\sqrt{|\mathrm{g}|}V(\phi) = -\mathscr{L}^{(2)}$, naturally appears.

As already made explicit in the above expressions, supersymmetry relates the scalar potential to the fermionic shifts. Furthermore, it poses constraints between the latter. Let us be more explicit. The fermionic shifts are conveniently described in terms of the relevant object capturing the main properties of a gauged supergravity theory, the "T-tensor". It is the H-covariant, scalar dependent, tensor defined by

$$\mathbb{T}_{\underline{MN}}{}^{\underline{P}} = (\mathbb{L}_c^T)_{\underline{M}}^{\;Q}(\mathbb{L}_c^T)_{\underline{N}}^{\;R} X_{QR}{}^S (\mathbb{L}_c^{-T})_S^{\;\underline{P}}. \tag{2.2.14}$$

By definition, the T-tensor satisfies the same linear and quadratic constraints as the X-tensor, namely

$$\mathbb{T}_{(\underline{MN}}{}^{\underline{P}}\mathbb{C}_{\underline{Q})\underline{P}} \equiv \; -\mathbb{T}_{(\underline{MNQ})} = 0 \tag{2.2.15}$$

$$\mathbb{C}^{\underline{MN}}\mathbb{T}_{\underline{M}}\mathbb{T}_{\underline{N}} = 0 \tag{2.2.16}$$

$$\left[\mathbb{T}_{\underline{M}},\mathbb{T}_{\underline{N}}\right]_{\underline{R}}{}^{\underline{T}} + \mathbb{T}_{\underline{MN}}{}^{\underline{P}}\mathbb{T}_{\underline{PR}}{}^{\underline{T}} = 0. \tag{2.2.17}$$

The T-tensor, once expanded in H-irreducible representation, produces all the necessary ingredients for the definition of the shift tensors. A preliminary identification of the representations of interest is given by inspecting the couplings in (2.2.13). Indeed, as already discussed, we require the Lagrangian to be H-invariant. Further constraints are given by the so-called "gradient flow equations" and by the "potential Ward Identity". The latter are relation required by supersymmetry. The gradient flow equations take the following form

$$\begin{aligned} \mathscr{D}_s \mathbb{S}_{AB} &= \frac{1}{2} \mathscr{P}_{s\mathcal{I}(A} \mathbb{N}^{\mathcal{I}}{}_{B)} \\ \mathscr{D}_r \mathbb{N}_{\mathcal{I}}^A &= 2\mathscr{P}_{r\mathcal{I}B} \mathbb{S}^{BA} + 2\mathbb{M}_{\mathcal{I}\mathcal{J}} \mathscr{P}_r^{\mathcal{J}A} + \dots \,, \end{aligned} \tag{2.2.18}$$

and the potential Ward identity reads

$$\delta_B^A V(\phi) = g^2 \left(\mathbb{N}_{\mathcal{I}}{}^A \mathbb{N}^{\mathcal{I}}{}_B - 12 \mathbb{S}_{BC} \mathbb{S}^{AC} \right). \tag{2.2.19}$$

The same kind of relations can be derived from the T-tensor and its properties. Indeed, it can be shown that the latter follows from a particular instance of the quadratic constraint and the former can be obtained by differentiating the T-tensor. Direct comparison between these identities and the above equations allows us to correctly identify the $\mathbb{S}$, $\mathbb{N}$ and $\mathbb{M}$ tensors. As an example of the technique, an explicit derivation in the $\mathcal{N} = 3$ case is presented in next chapters. Once this is done, the computation automatically gives us the expression of the scalar potential. In terms of these ingredients the gauged Lagrangian is obtained by summing $\mathscr{L}^{(0)}$, $\mathscr{L}^{(1)}$, and $\mathscr{L}^{(2)}$. It is invariant under the gauged version of the ungauged supersymmetry transformations. In particular, the variations of fermionic fields are modified with the shift tensors in the following way

$$\begin{aligned} \delta\psi_{A\mu} &= \dots + ig \mathbb{S}_{AB} \gamma_\mu \epsilon^B \\ \delta\lambda_{\mathcal{I}} &= \dots + g \mathbb{N}_{\mathcal{I}}^A \epsilon_A . \end{aligned} \tag{2.2.20}$$

The dots refer to the contributions already present in the ungauged model but modified by the implementation of the gauging recipe.

At this point, we are ready to enter the details of the first example of interest. Namely, the explicit derivation of an $\mathcal{N} = 3$ $D = 4$ Gauged Supergravity. In passing, general mass formulae for the linear perturbations on a given background will be presented and used to derive interesting mass spectra. In general, the mass formulae will apply to other models too. This said, the second example we will analyze is part of a family of dyonic gaugings in the maximal $\mathcal{N} = 8$ $D = 4$ model. The latter models can be interpreted as originating from a spontaneous compactification of String Theory. We will show that this is the case by implementing recently discovered techniques exploiting the formalism of Exceptional Field Theory. We then take the opportunity to introduce this modern approach to the gauging of those maximal models and derive the mass formula this way.

Chapter 3

$D = 4$ $\mathcal{N} = 3$ $\mathrm{SO}(3) \times \mathrm{SU}(3)$ Gauged Supergravity

The present example consists of a supergravity model derived "from scratch", meaning that it is not directly obtained from a known consistent truncation of a spontaneously compactified supergravity theory on a higher dimensional spacetime. Nonetheless, I want to show that it is interesting on its own in that it displays a newly discovered feature of the vacua structure of gauged supergravity theories. Namely, within the present model, families of stable AdS_4 vacua parameterized by massless modes are present. These vacua preserve a different amount of supersymmetries depending on the values of the parameters. The other known example with such property will be discussed in the next chapters. The latter will also have the interesting feature of being a consistent truncation of type IIB supergravity. It can be conveniently presented in the framework of Exceptional Field Theory. On the other hand, the present chapter is a direct application of the topics introduced in the previous chapters.

Let me start by noticing that, even if there is no direct proof of the origin of the present model in terms of a consistent truncation, the theory is inspired by the spontaneous compactifications of $D = 11$ supergravity of the form

$$\mathrm{AdS}_4 \times \mathrm{N}^{0,1,0}, \tag{3.0.1}$$

where the compact factor corresponds to a tri-Sasakian manifold. In particular, it is an instance of an infinite family of the Sasakian spaces $\mathrm{N}^{\mathrm{p,q,r}}$ first studied in [16]. When p, q, r are set to $\{p, q, r\} = \{0, 1, 0\}$ the Sasakian structure enhances to a tri-Sasakian one[1]. The space $\mathrm{N}^{0,1,0}$ is the only known tri-Sasakian seven-

[1]A tri-Sasakian manifold consists of a manifold M whose cone $C(M) = \mathbb{R}_+ \times M$ admits an Hyper-Kähler structure. If this is the case, the cone is also a Calabi-Yau manifold. Indeed, all Hyper-Kähler manifolds are Calabi-Yau manifolds. Then M is also a Sasaki-Einstein

dimensional homogeneous manifold. Because of its tri-Sasakian nature, the background supports, in the full-dimensional picture, three Killing spinors. The first factor of the background (3.0.1) displays the bosonic conformal symmetry of a four-dimensional anti-de Sitter space. This latter, combined with the fermionic symmetries generated by the three Killing spinors, gives rise to the Osp(3|4) superalgebra which is the $\mathcal{N}=3$ superconformal algebra of the boundary theory dual to this configuration [16, 17]. This allows for the organization of the Kaluze-Klein spectrum in superconformal multiplets of the relevant Osp(3|4) superconformal algebra. Furthermore, the states transform nontrivially under the internal SU(3) group corresponding to the isometry group of the internal space. The Kaluza-Klein states will then be representations of

$$G_{iso} = \mathrm{OSp}(3|4) \times \mathrm{SU}(3) \ . \tag{3.0.2}$$

This was shown explicitly in [18] where general properties for the OSp(3|4) representations are presented and the relevant supermultiplets are listed. Since the background exhibits an AdS factor, it is natural to conjecture, in a proper regime, a duality between the Kaluza-Klein states and the primary operators of a suitable superconformal field theory. In [19], a proposal for the three-dimensional dual theory is given and the comparison between its primary operator and the supergravity spectrum is performed.

At the massless level, the supermultiplet structure is easily given. The massless graviton multiplet is part of the spectrum. Furthermore, the compactification on $\mathrm{N}^{0,1,0}$ gives rise to nine massless vector multiplets. This translates in the following fields present in the effective $\mathcal{N}=3$ lower dimensional language. The metric field $g_{\mu\nu}$ together with three gravitinos $\psi_{A\mu}$, three vectors A_μ^{AB} and a spinorial field $\chi_\bullet$ come from the massless vector multiplet. The vector fields A_μ^{AB} transforming in the adjoint of the R-symmetry group A_μ^{AB} can and do gauge the latter. The massless vector multiplets account for a vector A_μ, a triplet of spinors λ_A, and a spinor λ singlet under the R-symmetry group, a triplet of complex scalars. Eight of the nine vectors coming from the vector multiplets transform in the adjoint of SU(3) and contribute to the gauging of the latter. The remaining vector is a singlet under SU(3), its multiplet is interpreted as the Betti multiplet. The latter, as for the other eight vectors, are specific to the $\mathrm{N}^{0,1,0}$ compactification since they find their origin in the cohomology of the latter. The above model is what motivates us to study a particular instance of the gauged supergravity models of the previous chapters. We are interested in a $D=4$ model because of the AdS_4 factor in (3.0.2). Furthermore, we want to consider an $\mathcal{N}=3$ supersymmetry as suggested by the

manifold. Homogenous seven-dimensional Sasakian or tri-Sasakian manifolds are relevant in the study of spontaneous compactifications of eleven-dimensional supergravity of the form $AdS_4 \times M$ preserving at least $\mathcal{N}=2$ supersymmetry.

$\mathrm{N}^{0,1,0}$ compactification. Finally, we want to build a model with gauge group

$$G_g = \mathrm{SO}(3) \times \mathrm{SU}(3)\,, \tag{3.0.3}$$

gauged by a graviphoton and nine vector multiplets.

This is not sufficient in principle to consider the model as originating from a consistent truncation. Indeed, the analysis carried out in [9] seems to exclude this possibility. However, as we will describe in detail, we are going to consider a consistent subsector of the supergravity theory with a smaller scalar manifold than the one of the full model. In particular, the vacua we are going to study are all found in a $(\mathrm{SU}(1,1)/\mathrm{U}(1))^3$ truncation. This feature allows for the possibility of consistently embedding the solutions we shall discuss in higher dimensional supergravity. Indeed, the $(\mathrm{SU}(1,1)/\mathrm{U}(1))^3$ subsector could fit the analysis of [9]. However, this remains an open problem. At the level of our analysis, we can infer that the vacuum at the origin of the scalar manifold of the complete model captures the main properties of the background (3.0.1) such as the spectrum of massless modes and their packaging in superconformal multiplets. In the sequel, we will study the full model leaving aside the problem of consistently uplifting the vacua that it displays.

As already mentioned, the origin of the scalar manifold corresponds to an isolated $\mathcal{N}=3$ vacuum. We also unveil new interesting vacua exhibiting a different amount of preserved supercharges. In particular, the latter appear in two families displaying $\mathcal{N}=1$, $\mathcal{N}=2$ and $\mathcal{N}=3$ solutions each. Furthermore, $\mathcal{N}=0$ perturbatively stable vacua are also present within the same families. The amount of supersymmetry preserved by the solutions depends on the background value of three massless fields α_i parameterizing them. We can compute the superconformal multiplet associated with the supersymmetric ones and we also provide explicit RG flows connecting the two families to the isolated vacuum at the origin. It is to say, by fixing the constant values of α_i one can choose as IR fixed point of the flow any of the solutions within one of the two families. The UV fixed point is given by the isolated maximally symmetric point at the origin of the scalar manifold. An interesting feature of the α_i fields is that they parameterize a manifold with the compact topology T^3/K, $K \subset G_g$ being an isomorphic form of the discrete symmetric group S_4. Depending on the amount of preserved supersymmetries, the various vacua nicely fit the latter manifold. The $\mathcal{N}=3$ vacuum, different from the central one, is a point inside the manifold, the $\mathcal{N}=2$ vacua describe a line, the $\mathcal{N}=1$ describe a surface and the remaining points are filled by the non-supersymmetric vacua. In particular, this exotic family of connected AdS_4 vacua can be parameterized by three angles of a torus T^3 corresponding to the α_i fields subject to K-identifications. The scalar potential is constant along these angles so that, in the dual CFT_3 picture, they should correspond to exactly marginal deformations. It follows that, as we will also discuss in the other main example of gauged supergravity presented in the next chapters, the present analysis would holographically describe a perturbatively stable conformal manifold parametrized by supersymmetry breaking marginal deformations.

To be more precise, we find, within the present model, two of such examples corresponding to two inequivalent ways of truncating the full model to an $(\mathrm{SU}(1,1)/\mathrm{U}(1))^3$ scalar manifold. Briefly, concerning the $\mathcal{N}=3$ vacua, besides the one at the origin we find that the first one resides in an

$$\mathrm{SO}^{\mathrm{A}}_{\mathrm{diag}}(3) = \mathrm{diag}\,(\mathrm{SO}(3)\times\mathrm{SO}_{\mathrm{I}}(3)) \subset G_g$$

truncation, while a second one can be found in an

$$\mathrm{SO}^{\mathrm{B}}_{\mathrm{diag}}(3) = \mathrm{diag}\,(\mathrm{SO}(3)\times\mathrm{SO}_{\mathrm{II}}(3)) \subset G_g$$

truncation. In both cases the $\mathrm{SO}(3)\subset\mathrm{OSp}(3|4)$ factor corresponds to the R-symmetry of the $\mathcal{N}=3$ supersymmetric configurations and again in both cases we have $\mathrm{SO}_{\mathrm{I/II}}(3)\subset\mathrm{SU}(3)\subset G_g$. However, in the first cases, we have that $\mathrm{SO}_{\mathrm{I}}(3)$ is embedded in SU(3) such that the fundamental representation of the latter is mapped in the fundamental representation of the former:

$$\mathbf{3}_{\mathrm{SU}(3)} \xrightarrow{\mathrm{SO}_{\mathrm{I}}(3)} \mathbf{3}\,.$$

In the second truncation we have $\mathrm{SO}_{\mathrm{II}}(3)\sim\mathrm{SU}_{\mathrm{II}}(2)\subset\mathrm{SU}(3)$ under which the fundamental representation branches as

$$\mathbf{3}_{\mathrm{SU}(3)} \xrightarrow{\mathrm{SU}_{\mathrm{II}}(2)} \mathbf{2}\oplus\mathbf{1}.$$

As we will explicitly describe, each of the two non-central $\mathcal{N}=3$ vacua are part of a family of (non)supersymmetric vacua with the manifold structure T^3/K described above. Again they will be conveniently found in two different truncations analogous to the two $\mathrm{SO}^{\mathrm{A/B}}_{\mathrm{diag}}(3)$ ones.

3.1 Building the Model

Before entering the details of the vacua structure we will start with specializing the general features of a supergravity model in $D=4$ to $\mathcal{N}=3$ supergravity. We will then explicitly illustrate the gauging procedure for the gauge group of interest. Furthermore, it will provide explicit examples of the derivation of the fermion-shift tensors, the mass matrices, and the scalar potential from the H-irreducible components of the T-tensor.

3.1.1 The Ungauged Model

As anticipated, the ungauged model of this kind features, besides the supergravity multiplet, a number n of vector multiplets[2]. As well-known, for an $\mathcal{N}=3$ model in $D=4$, the graviton multiplet is made up of the metric field $g_{\mu\nu}$ together with the three gravitinos $\psi_{A\mu}$ necessary to make supersymmetry local, with $A=1,\dots,3$, an R-symmetry triplet of vector potentials (graviphotons) A^{AB}_{μ}, and one R-symmetry singlet fermion, the dilatino, $\chi_{ABC}=\chi_{\bullet}\epsilon_{ABC}$. In the $\mathcal{N}=3$ case vector multiplets are allowed. We introduce n of them and we use the label $I=1,\dots,n$ to count them. They provide $1\times n$ vector fields $A_{I\mu}$, 4×9 gauginos λ_{IA}, λ_I, and $6\times n$ real scalar fields that are conveniently arranged in $3\times n$ complex variables ϕ_{IAB}. Alltogether the theory exhibits $n_v=3+n$ vector potentials.

The complex nature of the scalar sector was first described in [20, 21] where it is shown that the scalar manifold of an $\mathcal{N}=3$ $D=4$ supergravity must be of the form:

$$\mathcal{M}_{scalar}=\frac{\mathrm{SU}(3,n)}{\mathrm{SU}(3)\times\mathrm{SU}(n)\times\mathrm{U}(1)}\,. \tag{3.1.1}$$

The latter is a non-compact manifold. Furthermore, the natural metric that follows from the standard coset geometry makes it a Kähler manifold, as expected for a supersymmetric gravity theory. The matter content of the model is then fixed by a convenient choice of the number "n" of vector multiplets. We will fix later on $n=9$. However, many preliminary considerations are true for generic n. As already introduced in previous chapters, the isotropy group of the scalar manifold is described by the R-symmetry factor, $H_R=\mathrm{U}(3)$ in our case, and by a matter contribution $H_{\mathrm{matter}}=\mathrm{SU}(n)$ which acts exclusively on the fields originating from the vector multiplets. At the level of the equation of motion, the global symmetry of the model is the isometry group of the scalar manifold. The action of the latter on the scalar manifold is extended to a symplectic linear action on the electric field strengths and their redundant magnetic counterparts.

It is useful to describe the electric-magnetic G-action in the following complex representation:

$$\mathscr{R}_{\eta}=(\mathbf{3}+\mathbf{n})\oplus\overline{(\mathbf{3}+\mathbf{n})}\,. \tag{3.1.2}$$

The complex $(\mathbf{3}+\mathbf{n})$ representation has the following branch when decomposed with respect to the isotropy group H:

$$(\mathbf{3}+\mathbf{n})\to(\mathbf{3},\mathbf{1})_{-1}\oplus(\mathbf{1},\mathbf{n})_{\frac{3}{n}}\,. \tag{3.1.3}$$

[2] Even if we are interested in the $n=9$ case, many statements hold with generic n

Then, we can introduce the natural complex basis for the vector space of the $\mathscr{R}_\eta$ representation such that G is represented by block-diagonal matrices, the two blocks corresponding to the two contributions in (3.1.2). In particular, we have that an element of the $\mathscr{R}_\eta$ vector space is of the form[3]

$$V^{\underline{M}} = (V^{\underline{\Lambda}}, V_{\underline{\Lambda}}) \ , \ \ V^{\underline{\Lambda}} = (V^{AB}, V_I) \ , \ \ V_{\underline{\Lambda}} = (V^{\underline{\Lambda}})^* \, . \tag{3.1.4}$$

As the notation suggests, V^{AB} corresponds to the $(\mathbf{3},\mathbf{1})_{-1}$ part of the branching (3.1.3) and analogously V_I describes the $(\mathbf{1},\mathbf{n})_{+\frac{3}{n}}$ contribution of the same equation. In the basis just introduced we can represent an SU$(3,n)$ element $\mathrm{T} \in G$ in the following way:

$$\mathrm{T} \in G \ \rightarrow \ \mathscr{R}_\eta[\mathrm{T}]_{\underline{M}}{}^{\underline{N}} \equiv \begin{pmatrix} T & \mathbf{0} \\ \mathbf{0} & T^* \end{pmatrix} . \tag{3.1.5}$$

In the above expression $T = (T_{\underline{\Lambda}}{}^{\underline{\Sigma}})$ is an element of fundamental representation of G and it satisfies $T^\dagger \eta T = \eta$, where the standard SU$(3,n)$-invariant metric is $\eta = \mathrm{diag}(+1,+1,+1,-1,\ldots,-1)$. The H-branching allows to further split T into the following blocks:

$$T = \begin{pmatrix} T_{AB}{}^{CD} & T_{AB\,J} \\ T^{I\,CD} & T^I{}_J \end{pmatrix} . \tag{3.1.6}$$

It is important to notice that $\mathscr{R}_\eta[\mathrm{T}]_{\underline{M}}{}^{\underline{N}}$, with $\mathrm{T} \in G$, is not an Sp$(2n_v,\mathbb{R})$ element. That is to say, $\mathscr{R}_\eta$ is not a symplectic representation of G.[4] This apparent obstruction can be easily overcome by a proper change of basis. In particular, one can rotate all elements in a symplectic representation where all the G elements are described by Sp$(2n_v,\mathbb{R})$ symplectic matrices in the following way:

$$V^M = (\mathscr{A}^\dagger \mathscr{O})^M{}_{\underline{N}} V^{\underline{N}} . \tag{3.1.8}$$

[3] Only in this chapter we use underlined indices for the representation $\mathscr{R}_\eta$.

[4] The $\mathscr{R}_\eta[T]$ representation has however a pseudo-symplectic action defined by the $2n_v \times 2n_v$ matrix $\mathbb{C}_\eta$ whose expression in a convenient notation is:

$$\mathbb{C}_\eta \equiv \begin{pmatrix} \mathbf{0} & \eta \\ -\eta & \mathbf{0} \end{pmatrix} . \tag{3.1.7}$$

By implementing the identity $T^\dagger \eta T = \eta$, it can be checked that $\mathscr{R}_\eta[T]^t \, \mathbb{C}_\eta \mathscr{R}_\eta[T] = \mathbb{C}_\eta$.

The vector V^M in the proper symplectic basis is now a real element with electric and magnetic components $V^M = (V^\Lambda, V_\Lambda)$. Here we give the expression of $\mathscr{O}$:

$$\mathscr{O} = \begin{array}{c c} & \begin{array}{cccc} 3 & n & 3 & n \end{array} \\ \begin{array}{c} 3 \\ n \\ 3 \\ n \end{array} & \left(\begin{array}{cc||cc} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \hline\hline \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{array}\right) \end{array} .$$

The role of the latter matrix is to bring the $\mathscr{R}_\eta$ representation to the $\underline{\mathscr{R}}_v^c$ representation. The further action of the Cayley matrix $\mathscr{A}$ has the $\mathscr{R}_v$ representation as image. In particular, it can be regarded as a map from G to the symplectic group $\mathrm{Sp}(2n_v,\mathbb{R})$:

$$\mathscr{R}_v : \mathrm{G} \longrightarrow \mathrm{Sp}(2(3+n),\mathbb{R}) \Leftrightarrow \forall T \in \mathrm{G} : \mathscr{R}[T]^T \cdot \mathbb{C} \cdot \mathscr{R}[T] = \mathbb{C} . \tag{3.1.9}$$

In the chosen notation, the symplectic invariant-metric has the standard expression

$$\mathbb{C} \equiv \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} .$$

The $\mathscr{R}_v$ basis is the correct one to use to define the Lagrangian of the model, as discussed in the previous chapter. In particular this is the basis naturally describing the electric field strengths $F^\Lambda_{\mu\nu} = \partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu$ and their magnetic counterparts $G_{\Lambda\mu\nu}$. It is defined as

$$G_{\Lambda\mu\nu} = -\epsilon_{\mu\nu\rho\sigma} \frac{\delta \mathscr{L}}{\delta F^\Lambda_{\rho\sigma}} . \tag{3.1.10}$$

They are part of the $\mathscr{R}_v$ vector

$$G^M_{\mu\nu} \equiv \begin{pmatrix} F^\Lambda_{\mu\nu} \\ G_{\Lambda\mu\nu} \end{pmatrix} , \tag{3.1.11}$$

An $\mathrm{SU}(3,n)$ group element T will act through its $\mathrm{Sp}(2n_v,\mathbb{R})$ representation. In general, it will mix the electric and the magnetic components of G^M in the following way:

$$G^M_{\mu\nu} \rightarrow G'^M_{\mu\nu} = \mathscr{R}[T]^{-1}{}_N{}^M \, G^N_{\mu\nu} . \tag{3.1.12}$$

The symplectic element G^M can be locally interpreted as the abelian field strength of the symplectic electric-magnetic potential $A^M = (A^\Lambda, A_\Lambda)$ through

	A^M_μ	$\psi_{A\mu}$	$\chi_\bullet$	λ_{IA}	λ_I	$F^{AB}_{\mu\nu}$	$F^I_{\mu\nu}$
G	$(\mathbf{3}+\mathbf{n})+\overline{(\mathbf{3}+\mathbf{n})}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
H	$(\mathbf{1},\mathbf{1})_0$	$(\mathbf{3},\mathbf{1})_{+\frac{1}{2}}$	$(\mathbf{1},\mathbf{1})_{+\frac{3}{2}}$	$(\mathbf{3},\mathbf{n})_{\frac{n+6}{2n}}$	$(\mathbf{1},\mathbf{n})_{\frac{3(n+2)}{2n}}$	$(\mathbf{3},\mathbf{1})_{-1}$	$(\mathbf{1},\overline{\mathbf{n}})_{-\frac{3}{n}}$

Table 3.1 The G and H representations of vector and fermionic fields. The gauginos λ_{IA} and the λ_I ones have opposite chirality.

the relation $G^M = dA^M$. As far as vectors and fermions are concerned we list in Table 3.1 their representations with respect to the duality group G and the isotropy group $H \sim H_R \times H_{\text{matter}}$.

According to the general theory of $\mathcal{N}>2$ supergravity theories, the scalar fields parameterize a coset manifold. In our case the $\phi = (\phi^s)$ fields are interpreted as coordinates of an $L(\phi) \in \mathrm{SU}(3,n)$ element which is acted on by means of the left action of an isometry $T \in \mathrm{SU}(3,n)$ modulo a compensating transformation H acting on the right. This action in turn determines the transformation of the scalar fields after an isometry transformation is performed. To explicitly describe the coset element $L(\phi)$ it is convenient to work at the level of the Lie algebra $\mathfrak{g}$ of G. In the case at hand we have the decomposition of $\mathfrak{g} = \mathfrak{su}(3,n)$ in its compact and non-compact subspaces given by $\mathfrak{g} = \mathfrak{H} \oplus \mathfrak{K}$ where $\mathfrak{H} = \mathfrak{u}(3) \oplus \mathfrak{su}(n)$ is the Lie algebra of the isotropy group H and K can be interpreted as the generators corresponding to the scalar fields ϕ^s. When this is done we have

$$L(\phi) \in e^{\mathfrak{K}}. \tag{3.1.13}$$

It follows that we are using a parametrization in which H acts covariantly on the scalar fields. Indeed, the coset structure implies that $[\mathfrak{H}, \mathfrak{K}] \subset \mathfrak{K}$ so that $\mathfrak{K}$ and the scalar fields live in an H representation. In particular, for the scalar fields ϕ^s, the relevant representation is the $(\overline{\mathbf{3}},\mathbf{n})_k \oplus (\mathbf{3},\overline{\mathbf{n}})_{-k}$ one. The index s then splits as

$$\phi = (\phi_{AB\,J}, \phi^{AB\,J})\,, \quad \phi^{AB\,J} = (\phi_{AB\,J})^*\,. \tag{3.1.14}$$

Accordingly, the expression of the Maurer-Cartan form (2.1.7) in the $\mathscr{R}_v$ representation is:

$$\Omega = \begin{pmatrix} \mathscr{Q}_{AB}{}^{CD} & \mathscr{P}_{AB\,J} \\ \mathscr{P}^{I\,CD} & \mathscr{Q}^I{}_J \end{pmatrix}. \tag{3.1.15}$$

As for the scalar fields we have $\mathscr{P}_{AB\,I} = (\mathscr{P}^{I\,AB})^*$. For convenience we shall also introduce $\mathscr{P}^{AB\,I} = (\mathscr{P}_{AB\,I})^*$ and $\mathscr{P}_{I\,AB} = (\mathscr{P}^{I\,AB})^*$. As usual, the Riemannian coset metric can be computed as

$$\mathscr{G}_{st}(\phi)\, d\phi^s \otimes d\phi^t = \mathscr{P}_s^{AB\,I} \mathscr{P}_{AB\,I|t}\, d\phi^s \otimes d\phi^t\,. \tag{3.1.16}$$

3.1.2 The Gauged Model

Now that we have presented the main ingredients of the ungauged model under discussion and the relevant properties of the latter under the action of the electric-magnetic duality group $\mathrm{G} = \mathrm{SU}(3, n)$ we can move on to implement the gauging procedure previously introduced. We recall here that one of the main reasons to gauge an $\mathcal{N} > 2$ $D = 4$ supergravity theory is to introduce a scalar potential without spoiling the supersymmetry invariance of the model. This is crucial when searching for vacua different from the Minkowski one that all ungauged model features. The general case for electric gaugings has been originally studied in [21]. Here we refine the discussion by implementing the duality covariant procedure based on the embedding tensor [22–25], see [10, 11] for reviews, and presented in previous chapters.

In our case, the gauged scalar kinetic term reads as

$$\frac{1}{\sqrt{|\mathrm{g}|}}\mathscr{L}_{scal.kin} = \frac{1}{2}\mathscr{G}_{rs}\mathscr{D}_\mu\phi^r\,\mathscr{D}^\mu\phi^s = \frac{1}{2}\mathrm{Tr}(\widehat{\mathscr{P}}_\mu\cdot\widehat{\mathscr{P}}^\mu) = \widehat{\mathscr{P}}_\mu^{ABI}\,\widehat{\mathscr{P}}^\mu_{ABI}\,, \quad (3.1.17)$$

while, as already discussed, the Yukawa terms have the general form:

$$\frac{1}{\sqrt{|\mathrm{g}|}}\mathcal{L}_{\text{Yukawa}} = g\left(2\overline{\psi}^A_\mu\gamma^{\mu\nu}\,\psi^B_\nu\,\mathbb{S}_{AB} + \mathrm{i}\,\overline{\lambda}^{\mathcal{I}}\gamma^\mu\psi_{A\mu}\,\mathbb{N}_{\mathcal{I}}{}^A + \overline{\lambda}^{\mathcal{I}}\lambda^{\mathcal{I}}\,\mathbb{M}_{\mathcal{I}\mathcal{J}}\right) + \text{h.c.}. \quad (3.1.18)$$

where $\lambda_{\mathcal{I}}$ in this specific case collectively denotes the gauginos and the dilatino spin-1/2 field with the same chirality

$$\lambda_{\mathcal{I}} \equiv \{\lambda_{IA}, \lambda^I, \chi_\bullet\}\,.$$

We will denote the opposite chirality spinors as $\lambda^{\mathcal{I}}$. In $\mathcal{N} = 3$ supergravities, the fermionic shifts and mass matrices $\mathbb{S}_{AB} = \mathbb{S}_{BA}$, $\mathbb{N}_{\mathcal{I}}{}^A$ and $\mathbb{M}_{\mathcal{I}\mathcal{J}}$ can be expressed in terms of their H-covariant components

$$\mathbb{S}_{AB} = \mathbb{S}_{BA}\ ,\ \mathbb{N}^{IA}{}_B\ ,\ \mathbb{N}^{AI}\ ,\ \mathbb{N}_A\ ,\ \mathbb{M}_{IA}{}^J\ ,\ \mathbb{M}^{\bullet,IA}\ ,\ \mathbb{M}^{\bullet,I}\ ,\ \mathbb{M}^{IA,JB}\,, \quad (3.1.19)$$

We also introduce the complex conjugate of the latter objects $\mathbb{S}^{AB} \equiv (\mathbb{S}_{AB})^*$, $\mathbb{N}^{\mathcal{I}}{}_A \equiv (\mathbb{N}_{\mathcal{I}}{}^A)^*$, $\mathbb{M}^{\mathcal{I}\mathcal{J}} \equiv (\mathbb{M}_{\mathcal{I}\mathcal{J}})^*$. A small difference with respect to the general discussion is that in this case, we use the complex basis $\mathscr{R}_\eta$ so that the T-tensor

$$\mathbb{T}_{\underline{MN}}{}^{\underline{P}} \equiv (\widetilde{L}^{-1})_{\underline{M}}{}^Q\left(\widetilde{L}^{-1}\mathscr{R}[X_Q]\,\widetilde{L}\right)_{\underline{N}}{}^{\underline{P}}. \quad (3.1.20)$$

is obtained by dressing $X_{MN}{}^P$ with $\widetilde{L}^{-1}$, where the left index of $\widetilde{L}^M{}_{\underline{N}}$ refers to the real symplectic basis and the right one refers to the complex basis symplectic respect to $\mathbb{C}_\eta$ in which the action of G is block diagonal. This does not introduce subtleties, indeed both the linear and the quadratic constraints (2.2.15) (2.2.17) are necessary for the consistency of the gauging procedure to hold. The linear constraint restricts the representation of the T-tensor, generally transforming in the $\mathscr{R}_v \times \mathrm{Adj}(G)$, to the

$$(0,1,0,\dots,0,1) \oplus (1,0,\dots,0,1,0) \tag{3.1.21}$$

components, as the Dynkin labels of G-representations, one being the complex conjugate of the other. This translates into having as only non-vanishing components the following:

$$\mathbb{T}_{\underline{\Lambda\Sigma}}{}^{\underline{\Gamma}} = \mathbb{T}_{[\underline{\Lambda\Sigma}]}{}^{\underline{\Gamma}}\,,\quad \mathbb{T}^{\underline{\Lambda\Sigma}}{}_{\underline{\Gamma}} = \left(\mathbb{T}_{\underline{\Lambda\Sigma}}{}^{\underline{\Gamma}}\right)^* = \mathbb{T}^{[\underline{\Lambda\Sigma}]}{}_{\underline{\Gamma}}\,.$$

Recall that in this chapter we use underlined indices $\underline{\Lambda} : 1,...,3+n$ for the electric and magnetic components of an element in the basis (3.1.5) which is complex and symplectic with respect to $\mathbb{C}_\eta$. We are able to identify the fermionic shift and mass matrices (3.1.19) as $H = \mathrm{S}[\mathrm{U}(3)\times\mathrm{U}(n)]$-irreducible tensors extracted from the non-vanishing T-tensor components[5] in the following way:

$$\begin{aligned}
\mathbb{S}_{AB} &= -\frac{1}{2}\epsilon_{(A|CD}\mathbb{T}^{CD}{}_{B)}\,,\\
\mathbb{N}^{B} &= \mathbb{T}^{EB}{}_{E}\,,\\
\mathbb{N}_{CI} &= \epsilon_{ABC}\mathbb{T}^{AB}{}_{I}\,,\\
\mathbb{N}^{IA}{}_{B} &= -2\,\mathbb{T}^{IA}{}_{B} + \mathbb{T}^{IC}{}_{C}\delta^{A}_{B}\,.
\end{aligned} \tag{3.1.22}$$

As already anticipated, the gradient flow equations [26, 11] can be found by inspection of the H-components of the identity:

$$\mathcal{D}\mathbb{T}_{MN}{}^{P} = -\mathscr{R}_v[\mathscr{P}]_M{}^{Q}\mathbb{T}_{QN}{}^{P} + [\mathbb{T},\mathscr{R}_v[\mathscr{P}]]_N{}^{P}\,, \tag{3.1.23}$$

where $\mathcal{D}$ stands for the H-covariant derivative which acts on $\mathbb{T}$ accordingly to its H-representation. Indeed, one can easily obtain the above equation by considering the T-tensor definition and the splitting of the Maurer-Cartan form in (2.1.7) once translated in the suitable representation. As far as the quadratic constraints (2.2.17) are concerned, they imply that the potential of the model can be computed from the so-called "potential Ward identity" (2.2.19). In our case, the above identity takes the form specific to $\mathcal{N}=3$:

$$\mathbb{N}_A\mathbb{N}^B + \mathbb{N}_{AI}\mathbb{N}^{BI} + \mathbb{N}_{IC}{}^{B}\mathbb{N}^{IC}{}_{A} - 12\,\mathbb{S}_{AC}\mathbb{S}^{BC} = \delta^B_A V\,. \tag{3.1.24}$$

[5]More detailed identities and their derivation can be found in Appendices B and C.

Here, we have rescaled the embedding tensor and consequently its components entering the definition of the shift and mass matrices to absorb g, the coupling constant of the model. This will be a useful notation since for the case under discussion we will deal with a gauge group G_g allowing for two coupling constants. They will be captured by independent components of the embedding tensor. As discussed in the general case, the above equation and the full quadratic constraints are necessary conditions for the gauging procedure to be consistent with the original $\mathcal{N}=3$ supersymmetry of the ungauged model. In Appendix A we present how to obtain the potential Ward identity starting from the quadratic constraint of the case under discussion. Now, all the ingredients necessary to study the model are given. It is left to choose a suitable gauge group G_g and to explicitly compute all the relevant tensors. However, before entering the details of a specific gauging let us briefly present general mass formulae for the fermionic and bosonic fields. We will use them to study the perturbative stability of the solutions (vacua) presented below. Properly modified they will hold for a generic gauging and generic $\mathcal{N}$.

3.2 General Mass Formulae

The formulae presented in this section will be relevant for the computation of the spectrum of perturbation around the relevant AdS_4 vacua. The latter are by construction purely bosonic configurations and the equation of motion drastically simplify when expanded at quadratic order in the bosonic and fermionic fields. A further simplification is given by the scalar field configuration. Later on, we will provide examples of configuration with coordinate-dependent scalars. However, here we shall focus on constant scalar configurations $\phi_0=(\phi_0^s)$ so that for a solution of the model we have

$$\left.\frac{\partial V}{\partial \phi^s}\right|_{\phi=\phi_0}=0\,. \tag{3.2.1}$$

As usual, we can relate the extremum $V_0=V(\phi_0)$ of the scalar function V to the cosmological constant Λ. In our notation, we have the exact match between the two: $\Lambda=V_0$. We are going to study the case in which the background metric describes an AdS geometry with a negative cosmological constant. The natural scale of the relevant AdS space will be related to the cosmological constant through the relation $L=\sqrt{-\frac{3}{V_0}}$.

3.2.1 Scalar Masses

We can obtain the mass formula for the scalar fields by expanding to quadratic order the scalar sector of the theory

$$\frac{1}{\sqrt{|g|}}\mathscr{L}_{scal}=\frac{1}{2}\mathscr{G}_{rs}\,\mathscr{D}_{\mu}\phi^{r}\,\mathscr{D}^{\mu}\phi^{s}-V(\phi)\,, \tag{3.2.2}$$

around the background value ϕ_0. The other fields do not contribute to the computations. By implementing the procedure just outlined, we can derive the linearized scalar equation of motion which correspond to the Klein-Gordon equation with square-mass matrix given by:

$$M^{(scal)}{}_{r}{}^{t}=\mathscr{G}^{ts}\,\frac{\partial^{2}V}{\partial\phi^{s}\partial\phi^{r}}\bigg|_{\phi=\phi_0}\,. \tag{3.2.3}$$

We then compute the scalar spectrum by evaluating the latter matrix and by extracting its eigenvalues which will correspond to the square of the masses of the scalar perturbations.

3.2.2 Vector Mass Matrix

The relevant sector of the theory for the vector masses comes from the coupling of the latter with the scalars, indeed all fermions vanish in our backgrounds. In this case, we derive the Euler-Lagrange equation

$$\epsilon^{\mu\nu\rho\sigma}\,\mathscr{D}_{\nu}G^{M}_{\rho\sigma}=2\,\mathbb{C}^{MN}\,\frac{\delta\mathscr{L}_{matter}}{\delta A^{M}_{\mu}}\,. \tag{3.2.4}$$

The latter encodes, in terms of the duality covariant vector G^M, the dynamical Maxwell-like equations of motion for the physical fields F^{Λ} and their Bianchi identities. We will now make use of the so-called "twisted self-duality condition" in the case of a bosonic background

$$^{*}G=-\mathbb{C}\cdot\mathcal{M}\cdot G \tag{3.2.5}$$

which follows from the definition (2.1.9) of G_{Λ}, and we expand equation (3.2.4) at linear order in the gauge fields. The result is of the form

$$\mathcal{M}_{MN}{}^{*}\mathscr{D}^{*}G^{N}\;=g^{2}\Theta^{\alpha}_{M}\,k^{r}_{\alpha}\,\mathscr{G}_{rs}\,k^{s}_{\beta}\,\Theta^{\beta}_{N}A^{N}\,. \tag{3.2.6}$$

so that the squared masses of the vector fields can be computed from the matrix

$$M^{(vector)P}{}_M = g^2 \mathscr{R}_v[L^{-1}]_Q{}^P \mathscr{R}_v[L^{-1}]_Q{}^N \mathscr{K}_{NM} = -g^2 \left(\mathcal{M}^{-1} \cdot \mathscr{K}\right){}^P{}_M \tag{3.2.7}$$

with

$$\mathscr{K}_{MN} \equiv \Theta_M^\alpha\, k_\alpha^r\, \mathscr{G}_{rs}\, k_\beta^s\, \Theta_N^\beta \Big|_{\phi=\phi_0} . \tag{3.2.8}$$

Again, by extracting the eigenvalues of the latter quantity we obtain the mass spectrum for the spin-one fields. One can already predict that half of the spectrum will correspond to massless modes. However, they are not to be considered physical degrees of freedom. Indeed, the duality covariant formulation of gauged supergravity is redundant since we have introduced the "magnetic" counterparts of the physical "electric" field strengths. In other words, by performing a global G transformation, at the level of the equation of motion, we can rotate the model in a pure electric frame. In the latter configuration, $\Theta^{\Lambda\alpha} = 0$, and the quadratic constraints imply that the matrix has a Kernel with dimension at least equal to n_v. Furthermore, we can derive an expression for the vector masses in terms of the T-tensor. Indeed, we note that

$$\det\left(M^{(vector)}\right) \propto \det\left(\mathscr{R}_v[L^{-1}] \cdot \mathscr{K} \cdot \mathscr{R}_v[L^{-T}]\right) ,$$

so that the mass spectrum can be equivalently computed by extracting the eigenvalues of the following object

$$\mathbb{M}^{(vector)}_{PN} = \frac{g^2}{4} \mathrm{Tr}\left(\mathbb{T}_P \cdot \mathbb{T}_N + \mathbb{T}_P \cdot (\mathbb{T}_N)^\dagger\right) . \tag{3.2.9}$$

Where we have used that

$$\mathscr{K}_{MN} = \frac{1}{2} \mathrm{Tr}\left(K_M K_N\right) , \tag{3.2.10}$$

with

$$K_M \equiv \frac{1}{2}\left(\mathscr{R}_v[L]^{-1} \cdot X_M \cdot \mathscr{R}_v[L] + (\mathscr{R}_v[L]^{-1} \cdot X_M \cdot \mathscr{R}_v[L])^\dagger\right)$$

3.2.3 Fermionic masses

The relevant terms for the fermionic masses are the Yukawa terms (3.1.18). Here, we explicitly see the role that the fermionic shifts and mass matrices $\mathbb{S}_{AB}$, $\mathbb{N}_{\mathcal{I}}^A$ and $\mathbb{M}_{\mathcal{I}\mathcal{J}}$ play in the dynamics of the model. In particular, $\mathbb{M}_{\mathcal{I}\mathcal{J}}$ and $\mathbb{S}_{AB}$ are relevant for the $\lambda^{\mathcal{I}}$ masses and the gravitinos masses respectively. The $\mathbb{N}_{\mathcal{I}}^A$ shifts will be relevant for the definition of goldstinos, the supersymmetric analogous of the goldstone bosons.

Gravitinos masses and Supersymmetry breaking

Since we are working with a bosonic configuration we have as background values for the fermionic fields

$$\langle \psi_\mu^A \rangle = \langle \lambda_{\mathcal{I}} \rangle = 0\,.$$

Among the supersymmetries $\epsilon_A Q^A$ we can find those that satisfy:

$$\begin{aligned} \langle \delta\psi_{A\mu} \rangle &= \mathscr{D}_\mu \epsilon_A + i\mathbb{S}_{AB}|_{\phi=\phi_0}\gamma_\mu \epsilon^B = 0 && (3.2.11)\\ \langle \delta\lambda_{\mathcal{I}} \rangle &= \mathbb{N}_{\mathcal{I}}{}^A|_{\phi=\phi_0}\epsilon_A = 0 && (3.2.12) \end{aligned}$$

These are necessary conditions for the supersymmetry $\epsilon_A Q^A$ to be a symmetry of the bosonic background. Indeed, equations (3.2.11)(3.2.12) describe a supersymmetry variation of the fermionic fields on the bosonic solution. They must hold true if we want to preserve the background. If this is the case, the Killing spinor equation (3.2.11) can have ϵ_a, $a:1,...,\mathcal{N}^{'} \leq \mathcal{N}$ independent solutions. One can easily derive from the latter the following integrability condition constraining the gravitinos shift matrix:

$$\mathbb{S}_{aA}\mathbb{S}^{bA}|_{\phi=\phi_0} = -\frac{V_0}{12}\delta_a^b\,. \qquad (3.2.13)$$

On the other hand, consistency of this assumption with the second equation (3.2.12) tells us that the ϵ^a directions must be in the Kernel of the $\lambda^{\mathcal{I}}$ shift matrices so that in a proper R-symmetry basis we have

$$\mathbb{N}_{\mathcal{I}}^a = 0\,.$$

A closer look at the Yukawa terms unveils that the gravitinos shift matrix plays also the role of their mass matrix. This makes sense since, as in the case of a bosonic partial symmetry breaking, a massless or massive mode corresponding to a field gauging a symmetry relates to a preserved or spontaneously broken generator respectively. In the case $\mathcal{N}^{'}=3$ all gravitinos are to be considered massless and their mass matrix

$$\mathbb{S}\mathbb{S}^\star|_{\phi=\phi_0}$$

is, in a proper basis, diagonal with non-vanishing entries $m_\psi^2 = -\frac{V_0}{12}$. As expected in the case of fully preserved supersymmetry on an AdS background. Otherwise, the modes corresponding to preserved supersymmetry directions will have the latter squared mass and the masses of the ones signaling a broken supersymmetry can again be extracted from the $\mathbb{S}\mathbb{S}^\star|_{\phi=\phi_0}$ eigenvalues.

Spin-$\frac{1}{2}$ masses

As explained above, when some supersymmetry generators are not preserved by the vacuum under consideration the $\mathbb{N}$ shifts will not vanish. We can use them to define the "goldstinos"

$$\eta_A = \lambda_{\mathcal{I}} \mathbb{N}^{\mathcal{I}}_A \,.$$

The latter can be used to decouple the gravitinos from the other fermionic fields through the redefinition

$$\psi^A_\mu \to \psi^A_\mu + \frac{i}{12} \sum_C \left(\frac{\mathbb{S}}{\mathbb{S}\mathbb{S}^\star + \frac{V_0}{12} \mathbf{1}_{3\times 3}} \right)^{AC} \gamma_\mu \eta_C \,. \tag{3.2.14}$$

Here, the summed R-symmetry index refers to the broken supersymmetry directions. The quantity $\mathbb{S}\mathbb{S}^\star + \frac{V_0}{12}\mathbf{1}_{3\times 3}$ restricted to the latter is non-degenerate. By implementing the latter equation and by considering, at the level of the equations of motion, first-order contributions in $\lambda_{\mathcal{I}}$ one obtains the Dirac equation

$$i\gamma^\mu \mathscr{D}_\mu \lambda_{\mathcal{I}} = \left(2\mathbb{M}_{\mathcal{I}\mathcal{J}} - \frac{1}{3} \sum_{AB} \left(\frac{\mathbb{S}}{\mathbb{S}\mathbb{S}^\star + \frac{V_0}{12} \mathbf{1}_{3\times 3}} \right)_{AB} \mathbb{N}^A{}_{\mathcal{I}} \mathbb{N}^B{}_{\mathcal{J}} \right) \lambda^{\mathcal{J}} \equiv \mathrm{M}_{\mathcal{I}\mathcal{J}} \lambda^{\mathcal{J}} \tag{3.2.15}$$

From the latter, we easily obtain the masses for the $\lambda_{\mathcal{I}}$ fields by computing the eigenvalues of the squared mass matrix $\mathrm{MM}^{\dagger\mathcal{J}}_{\mathcal{I}}$.

3.3 The Model with Gauge Group SO(3) × SU(3)

Let us now specialize our analysis to the relevant case in which the gauge group of the model is chosen to be $G_g = \mathrm{SO}(3) \times \mathrm{SU}(3)$. As already argued, the second factor corresponds to the isometry of the central AdS_4 $\mathcal{N} = 3$ vacuum which we believe to effectively describe the background (3.0.1) with internal space $\mathrm{N}^{0,1,0}$ while the first factor corresponds to the R-symmetry of the relevant superconformal symmetry preserved by the configuration. However, the model is interesting even when not considering the relation with higher dimensional models. Indeed, we find, in the case of the gauge group under discussion, new families of (non)-supersymmetric vacua. For the sake of clarity, the analysis is carried out in a pure electric gauging so that the only non-vanishing entries of the embedding tensor are the $\Theta_\Lambda{}^\alpha$ ones. For the case at hand, a dyonic formulation is physically equivalent and it does not add new features. The choice $G_g = \mathrm{SO}(3) \times \mathrm{SU}(3)$ is fixed by setting $n = 9$ and it is consistent with the quadratic constraints (2.2.5). In particular, the 24-dimensional representation $\mathscr{R}_v$ contains two copies of the 11-dimensional adjoint representation of G_g. Indeed, we have the splitting

$$\mathbf{12} \xrightarrow{\mathrm{SO}(3)\times\mathrm{SU}(3)} (\mathbf{3},\mathbf{1})\oplus(\mathbf{1},\mathbf{8})\oplus(\mathbf{1},\mathbf{1})\,. \tag{3.3.1}$$

of the G fundamental representation with respect to $\mathrm{SO}(3)\times\mathrm{SU}(3)$. In the above formula, the singlet corresponds to the vector field of the Betti multiplet. There are no scalar fields in the graviton multiplet and they parameterize the coset manifold

$$\mathcal{M}_{scalar} = \frac{\mathrm{SU}(3,9)}{\mathrm{S}[\mathrm{U}(3)\times\mathrm{U}(9)]}\,, \tag{3.3.2}$$

Out of the G isometries generators t_α we select the G_g gauged ones through the embedding tensor which in turn defines the X-tensor (2.2.2). It is convenient to directly present the expression of X_M. We use the notation $\widehat{t}_\ell$, $\ell=1,2,3$, and $\widehat{t}_m$, $m=1,\ldots,8$ for the generators of SO(3) and SU(3), respectively. Their explicit form is given in appendix C.1. We then associate a different coupling constant g_1, g_2 to each of the two G_g factors by defining

$$X_\ell = g_1\,\widehat{t}_\ell\,,\ \ X_m = g_2\,\widehat{t}_m\,. \tag{3.3.3}$$

The latter, and their complex conjugate, correspond to the non-vanishing entries of the 24-dimensional vector X_M. We have,

$$\{X_\Lambda\} = \{X_\ell, X_m, X_{\Lambda=12} = \mathbf{0}\}\,.$$

Furthermore,

$$X^\Lambda = \mathbf{0}$$

because of our choice of working in an electric frame in which the A_Λ magnetic vectors are not involved in the gauging. The explicit form of the generators $\widehat{t}_\ell$ and $\widehat{t}_m$ in the $\mathscr{R}_\eta$ representation (3.1.5) is given by

$$\mathscr{R}_\eta[\widehat{t}] = \begin{pmatrix} \mathrm{adj}(\widehat{t}) & 0_{3\times 9} & 0_{3\times 3} & 0_{3\times 9} \\ 0_{9\times 3} & 0_{9\times 9} & 0_{9\times 3} & 0_{9\times 9} \\ 0_{3\times 9} & 0_{3\times 9} & \mathrm{adj}(\widehat{t})^* & 0_{3\times 9} \\ 0_{9\times 3} & 0_{9\times 9} & 0_{9\times 3} & 0_{9\times 9} \end{pmatrix} \quad \widehat{t}\in\mathfrak{so}(3) \tag{3.3.4}$$

$$\mathscr{R}_\eta[\hat{t}] = \begin{pmatrix} 0_{3\times3} & 0_{3\times8} & 0 & 0_{3\times3} & 0_{3\times8} & 0 \\ 0_{8\times3} & \mathrm{adj}(\hat{t}) & 0 & 0_{8\times3} & 0_{8\times8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0_{3\times3} & 0_{3\times8} & 0 & 0_{3\times3} & 0_{3\times8} & 0 \\ 0_{8\times3} & 0_{8\times8} & 0 & 0_{8\times3} & \mathrm{adj}(\hat{t}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \hat{t} \in \mathfrak{su}(3)\,. \tag{3.3.5}$$

The electric vector fields naturally belong to the

$$A_\mu^\Lambda \;:\quad (\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8}+\mathbf{1})\,, \tag{3.3.6}$$

$G_g = \mathrm{SO}(3) \times \mathrm{SU}(3)$ representation. As far as the scalar fields are concerned, we implement the H-covariant parametrization of the coset manifold (3.3.2) in which the action of G_g is simpler. In particular, the scalar fields belong to the representation

$$\phi^s \;:\quad (\mathbf{3},\mathbf{8})\,[\phi^{\ell,m}] \oplus (\mathbf{3},\mathbf{1})\,[\phi^\ell] + c.c.$$

as they correspond to the $\mathfrak{K}$ directions in (2.1.6).

3.4 Consistent Truncations and Two Classes of Vacua

As in many gauged supergravity theories, working with the full scalar manifold is often a very complicated task. In our particular example, we have 54 real scalars and they will appear in the scalar potential computed from (3.1.24) in a nonlinear manner. It is then not obvious how to extremize $V(\phi)$ and thus to have a complete picture of the vacua of the model. However, one can try to analyze a sufficiently small subsector of the theory exhibiting a residual symmetry. It is to say, one can simplify the model to a consistent truncation by restricting the fields to the singlets of the action of a suitable subgroup of the symmetries of the theory. The consistency relies on the fact that interactions between singlets can only give rise to other singlets. In particular, to perform such truncation, one can choose a subgroup of G preserving the X_M tensor. This latter condition is sufficient for the equation of motion to be preserved. As we explain below, we can restrict to two different consistent truncations in which the scalar manifold reduces to

$$\mathcal{M}_{scalar,trunc} \sim \left[\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right]^3$$

and it is parameterized by three complex scalar fields. This will lead us to a simpler scalar potential that can be analytically extremized. Interestingly, two new classes of AdS vacua are found and we study them in detail. The first class of solutions is found in the "Type (i)" truncation under the discrete group

$$\begin{aligned}\text{Type (i):}\ \ &\mathbf{g}_1=\exp(\pi(\widehat{J}_1+\widehat{\lambda}_1))\in \mathrm{SU}(2)_D\subset G_g\subset G\\ &\mathbf{g}_2=\exp(\pi(\widehat{J}_2+\widehat{\lambda}_2))\in \mathrm{SU}(2)_D\subset G_g\subset G\ ,.\end{aligned}\tag{3.4.1}$$

while the second one belongs to the "type (ii)" truncation identified by

Type (ii):

$$\begin{aligned}&\mathbf{g}_1=\mathrm{diag}(1,-1,-1,1,1,1,-1,-1,-1,-1,1,1)=\exp(\pi(-\widehat{J}_1+2\widehat{\lambda}_2))\in \mathrm{SO}(3)_D\subset \mathrm{Adj}(G_g)\subset G\\ &\mathbf{g}_2=\mathrm{diag}(-1,1,-1,-1,-1,1,1,1,-1,-1,1,1)=\exp(\pi(-\widehat{J}_2+2\widehat{\lambda}_5))\in \mathrm{SO}(3)_D\subset \mathrm{Adj}(G_g)\subset G\\ &\mathbf{g}_3=\mathrm{diag}(1,-1,-1,-1,1,-1,1,-1,1,-1,-1,-1)\not\subset \mathrm{Adj}(G_g)\subset G\,,\end{aligned}\tag{3.4.2}$$

where $\widehat{J}_i$ and $\widehat{\lambda}_I$ denotes the T-representation (3.1.5) of the gauge generators $J_i\in\mathfrak{so}(3)$, $i\lambda_I/2\in\mathfrak{su}(3)$ presented in Appendix C.1. The $\mathrm{SU}(2)_D$ and $\mathrm{SO}(3)_D$ groups correspond to a suitable diagonal combination of the R-symmetry algebra and an $\mathfrak{so}(3)$ subalgebra of the remaining part of the gauge symmetry. The above truncations give rise to two different scalar manifolds of the form $\left[\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right]^3$ corresponding to two inequivalent ways of embedding $\mathfrak{s}u(1,1)^3$ in $\mathfrak{K}$, the non-compact directions of the scalar manifold. Let us be more explicit by considering an explicit basis for the representation of an element $\mathbf{k}\in\mathfrak{K}$ as a particular element of the fundamental representation of $\mathfrak{su}(3,9)$. We have

$$\mathbf{k}=\left(\begin{array}{c|c}\mathbf{0}_{3\times3} & \mathbf{X}_{3\times9}\\ \hline \mathbf{X}^\dagger_{9\times3} & \mathbf{0}_{9\times9}\end{array}\right),\qquad \mathbf{X}\in\mathrm{Mat}_{3\times9}(\mathbb{C})\tag{3.4.3}$$

The $\mathbf{X}$s corresponding to the two inequivalent truncations are given by

$$\text{Type (i):}\qquad \mathbf{X}=\begin{pmatrix} z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & z_3 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}\tag{3.4.4}$$

$$\text{Type (ii):}\qquad \mathbf{X}=\begin{pmatrix} 0 & z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & z_2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & z_3 & 0 & 0\end{pmatrix}\tag{3.4.5}$$

and the embedding of the manifold $\mathcal{M}_{scal,trunc}$ parameterized by (z_1, z_2, z_3) is naturally given by

$$\left[\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}\right]^3 \hookrightarrow \mathcal{M}_S : \ (z_1, z_2, z_3) \mapsto \exp(\mathbf{k}) \tag{3.4.6}$$

No other scalar fields are allowed when truncating the model by means of the discrete groups (3.4.1) or (3.4.2). The properties of the latter groups as subgroups of G will play a role in the physical features of the vacua found in the corresponding truncations, so let us briefly present the relevant points. In the type (i) case, one can check that the group elements (3.4.1) close a quaternionic group. Indeed, one has the natural isomorphism ($\mathbf{g}_1 \to i, \ \mathbf{g}_2 \to j$), where i and j are the usual imaginary units of quaternions. On the other hand, in the type (ii) case, we have a discrete group generated by three commuting elements of order two. Then, the group (3.4.2) is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. One must not be confused by the fact that in the type (ii) case the $\mathbf{g}_3$ element is not inside the gauge group. Indeed, it is sufficient for the latter to act as a symmetry of the X-tensor. This is the case, so we are sure that (3.4.5) defines a consistent truncation. Another important difference between the two groups is given by their embedding in G. In particular, in the type (i) and type (ii) cases there are elements of a diagonal combination between the $\mathrm{SO}(3)_R$ R-symmetry and the groups

$$\mathrm{SU}(3) \supset \begin{cases} \mathrm{SU}(2) \text{ generated by } \{\lambda_1, \lambda_2, \lambda_3\} \text{ (i)} \\ \mathrm{SO}(3) \text{ generated by } \{\lambda_2, \lambda_5, \lambda_7\} \text{ (ii)} \end{cases} \tag{3.4.7}$$

respectively. In the first case, we see that the action of the $\mathrm{SO}(3)_R$ rotation is combined with rotations of SU(2) corresponding to a spinorial representation of SO(3). Then, the elements $\mathbf{g}_1$ and $\mathbf{g}_2$ of the type (i) can be interpreted as two non-commuting rotations by $\pi/2$ and they are sufficient to generate the whole quaternionic group. In the second case the rotations $\mathbf{g}_1$ and $\mathbf{g}_2$ correspond instead to a vector representation of SO(3) and they commute. As we will see, the two different ways of embedding the three-dimensional rotation group in SU(3) as in (3.4.7) reflect in different physical properties of the truncations (i) and (ii). As an example, we will present two analogous families of solutions with very similar features. However, their spectrum of perturbation will organize in different combinations of (super)conformal multiplets exhibiting different R-symmetry representations, in the supersymmetric case. The $\mathcal{N} = 3$ non-central vacuum corresponding to the (i) case gives rise to spinorial representations of the R-symmetry while the spectrum of the analogous type (ii) vacuum will involve vector representations only. As already discussed in detail, once we have identified the relevant truncation we can easily define the corresponding coset representative and use it to dress the X-tensor. In this way, we obtain the T-tensor and from the latter we extract the fermionic shifts. The power of working with a consistent truncation is that we can algebraically restrict to the scalar singlets so that the extrema of the scalar potential

$$V|_{\text{singlet}} = \frac{1}{3}\text{Tr}\left(\mathbb{N}_A\mathbb{N}^A + \mathbb{N}_{AI}\mathbb{N}^{AI} + \mathbb{N}_{IC}{}^A\mathbb{N}^{IC}{}_A - 12\mathbb{S}_{AC}\mathbb{S}^{AC}\right)\Big|_{\text{singlet}} . \tag{3.4.8}$$

will correspond to actual extrema of the full potential in (3.1.24). For the actual computation, we choose a parametrization of the scalar fields (z_1, z_2, z_3) in terms of a radial and an angular coordinate. In particular we set

$$z_j = r_j \exp(i\alpha_j),\ j = 1,2,3, \quad \text{where } r_j \in \mathbb{R}_{\geq 0} \text{ and } \alpha_j \in [0,2\pi) . \tag{3.4.9}$$

The geometry of the manifold $\mathcal{M}_{scal,trunc}$ is described by the following coset metric

$$ds^2 = \sum_i^3 \left(2dr_i{}^2 + \frac{1}{2}\sinh^2(2r_i)\, d\alpha_i{}^2\right) . \tag{3.4.10}$$

The scalar potential obtained from the Type (i) truncation is given by

$$\begin{aligned} V(r_i,\alpha_i) \;=\;& g_1^2(-3 - 2\cosh(2r_3) - \cosh(2r_1)(2 + \cosh(2r_2) + \cosh(2r_3)) - \\ & \cosh(2r_2)(2 + \cosh(2r_3))) + g_2^2(3 + \cosh(2r_2)(-2 + \cosh(2r_3)) \\ & -2\cosh(2r_3) + \cosh(2r_1)(-2 + \cosh(2r_2) + \cosh(2r_3))) \end{aligned} \tag{3.4.11}$$

and the analogous expression for the Type (ii) case is obtained as

$$\begin{aligned} V(r_i,\alpha_i) \;=\;& g_1^2(-3 - 2\cosh(2r_3) - \cosh(2r_1)(2 + \cosh(2r_2) + \cosh(2r_3)) - \\ & \cosh(2r_2)(2 + \cosh(2r_3))) + \frac{g_2^2}{4}(3 + \cosh(2r_2)(-2 + \cosh(2r_3)) \\ & -2\cosh(2r_3) + \cosh(2r_1)(-2 + \cosh(2r_2) + \cosh(2r_3))) \end{aligned} \tag{3.4.12}$$

As one can easily check, they just differ in the second term. In particular, one can obtain the first one from the second one by substituting $g_2 \to 2g_2$. This latter fact can be traced back to the two different embeddings in (3.4.7). Note that the above expressions do not depend on α_i. Since these angular variables are not Goldstone bosons, they correspond to genuine flat directions.

Now, by virtue of the Gradient-Flow equations, the potential in (3.4.11,3.4.12) can be re-interpreted in terms of a "superpotential"; such a superpotential, $\mathcal{W}$, is strictly dependent on the eigenvalues of the fermionic shift $\mathbb{S}_{AB}$, which are given by

$$\text{Type (i):}\quad \mathbb{S}_{AB} = \delta_{AB}\left(g_1\prod_{j=1}^{3}\cosh(r_j) - g_2 e^{i(-\alpha_B+\alpha_C+\alpha_D)}\prod_{j=1}^{3}\sinh(r_j)\right)\,, \tag{3.4.13}$$

$$\text{Type (ii):}\quad \mathbb{S}_{AB} = \delta_{AB}\left(g_1\prod_{j=1}^{3}\cosh(r_j) - \frac{g_2}{2} e^{i(-\alpha_B+\alpha_C+\alpha_D)}\prod_{j=1}^{3}\sinh(r_j)\right)\,, \tag{3.4.14}$$

with $\alpha_B \neq \alpha_C \neq \alpha_D$. In both type (i) and (ii) truncations, we can construct the superpotential $\mathcal{W}(r_i,\alpha_i)$ in terms of the modulus of any of the diagonal entries of $\mathbb{S}_{AB}$ (e.g. $\mathbb{S}_{11}$):

$$\mathcal{W}(r_i,\alpha_i) = 2\,|\mathbb{S}_{AA}|\,. \tag{3.4.15}$$

The scalar potential is defined through the "superpotential equation"

$$V(r_j) = 2\mathscr{G}^{rs}\frac{\partial}{\partial\phi^r}\mathcal{W}(r_j,\alpha_j)\frac{\partial}{\partial\phi^s}\mathcal{W}(r_j,\alpha_j) - 3\,\mathcal{W}(r_j,\alpha_j)^2\,, \tag{3.4.16}$$

which holds both for Type (i) and Type (ii) vacuum. Notice that the dependence on α_i drops out in the expression of the potential. For this reason, we can define an α_i-independent superpotential as follows:

$$\mathcal{W}_0(r_i) \equiv \mathcal{W}(r_i,\alpha_i=0)\,, \tag{3.4.17}$$

in terms of which the potential reads:

$$V(r_k) = \sum_{i=1}^{3}\left(\frac{\partial}{\partial r_i}\mathcal{W}_0\right)^2 - 3\,\mathcal{W}_0^2\,. \tag{3.4.18}$$

We shall use this function to derive the domain wall solution in section 3.6. We find a scalar potential with three flat directions (i.e. not Goldstone bosons) when restricted to the truncations defined above. In the dual CFT, these flat directions are natural candidates for exactly marginal deformations. In fact the three angles will parametrize two 3-tori $(T^3_{(i)}, T^3_{(ii)})$ of vacua, to be discussed below. Although the potential at these extrema does not depend on α_i, the amount of preserved supersymmetry does, thus realizing a phenomenon of spontaneous supersymmetry breaking through marginal deformations. To our knowledge, these manifolds of vacua of the $\mathcal{N}=3$ model under consideration, preserving different amounts of supersymmetry, have not been discussed in the literature so far. Let us discuss them in detail.

3.4.1 The Two Families of Vacua

Inspection of the gradient of the potential shows that one can consistently set[6] $r_1 = r_2 = r_3 = r$. This allows us to write a more compact formula for the scalar potential to be extremized

$$\text{Type (i):} \qquad V(r,\alpha_1,\alpha_2,\alpha_3) = V(r) = -12\left[g_1^2\cosh^4(r) - g_2^2\sinh^4(r)\right], \tag{3.4.19}$$

$$\text{Type (ii):} \qquad V(r,\alpha_1,\alpha_2,\alpha_3) = V(r) = -12\left[g_1^2\cosh^4(r) - \frac{g_2^2}{4}\sinh^4(r)\right]. \tag{3.4.20}$$

The extremality condition $\frac{\partial V}{\partial r} = 0$ determines the following three distinct values $r = r_{\text{vac}}$ for r at the extrema:

$$\text{Type (i):} \qquad r_{\text{vac}} = \frac{1}{2}\log\left(\frac{g_2+g_1}{g_2-g_|}\right) \qquad \Longrightarrow T^3_{(i)} \text{ of extrema: } \exists\, g_2 > g_1\,, \tag{3.4.21}$$

$$\text{Type (ii):} \qquad r_{\text{vac}} = \frac{1}{2}\log\left(\frac{g_2+2g_1}{g_2-2g_1}\right) \qquad \Longrightarrow T^3_{(ii)} \text{ of extrema: } \exists\, g_2 > 2g_1\,, \tag{3.4.22}$$

$$\text{Origin:} \qquad r_{\text{vac}} = 0 \qquad \Longrightarrow \text{isolated extremum: } \forall\, g_1, g_2\,. \tag{3.4.23}$$

We see that we have one isolated vacuum that exists for all values of the couplings g_1, g_2. It is located at the origin $\mathcal{O}$ of the scalar manifold as expected. Aside from it, there are two types of non-trivial vacuum manifolds: both of them are three-tori T^3 parameterized by $(\alpha_1,\alpha_2,\alpha_3)$, though embedded differently into the scalar manifold $\mathcal{M}_{scalar}$. The Type (i) and type (ii) T^3-vacua only exist for $g_2 > g_1$ and $g_2 > 2g_1$, respectively. The corresponding values of the

[6] The other vacua of the truncations have $r_2 = r_3 = 0$ (when $g_1 = 0$) or $r_1 = r_2$ and $r_3 = 0$ (modulo permutations of the radii). They correspond to supersymmetric Minkowski vacua or non-supersymmetric and perturbatively unstable AdS vacua respectively. The first case corresponds to a model with ungauged graviphotons. Here we shall focus on perturbatively stable AdS vacua.

scalar potential V (i.e. the cosmological constants at the extrema) are:

$$\text{Type (i):} \qquad \Lambda = V|_{r_{\text{vac}}} = -12\frac{g_1^2 g_2^2}{g_2^2 - g_1^2}\,, \tag{3.4.24}$$

$$\text{Type (ii):} \qquad \Lambda = V|_{r_{\text{vac}}} = -12\frac{g_1^2\, g_2^2}{g_2^2 - 4g_1^2}\,, \tag{3.4.25}$$

$$\text{Origin:} \qquad V|_{r_{\text{vac}}} = -12\, g_1^2\,. \tag{3.4.26}$$

Thus all vacua have a negative constant scalar curvature, as expected for AdS_4 spacetime geometries. We still need to introduce one more refinement since the discussion above was slightly imprecise. The points of the tori T^3 of Type (i) or (ii) are not all gauge inequivalent. There is a discrete subgroup $\Gamma \subset G_g$ of the gauge group that identifies them. It acts on the (z_1, z_2, z_3) coordinates introduced in (3.4.6) in terms of a 3-dimensional irreducible representation

$$\begin{array}{llll} \text{Inversions:} & \boxed{\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} & \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ \text{Permutations:} & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \boxed{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}} & \boxed{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}} \end{array} \tag{3.4.27}$$

The first line represents inversions of all possible pairs of the z-coordinates (shifts of their α-phases by π), while the second line acts by permutations. These matrices generate the discrete group

$$\Gamma \simeq S_4 \simeq S_3 \ltimes K_4 \simeq T_{24}^h \simeq \mathrm{O}_{24} \tag{3.4.28}$$

where S_n is the symmetric group of n objects, $K_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ is the Kleinian four-group, $T_{24}^h \subset \mathrm{O}(3)$ is the full tetrahedral group (including inversions) and finally $\mathrm{O}_{24} \subset \mathrm{SO}(3)$ is the rotational (orientation preserving) octahedral group. The discrete group $\Gamma \simeq S_4$ can be presented by 3 generators and relations among them. A possible choice of these generators (in the 3-dimensional irrep) consists of the 3 boxed matrices in (3.4.27). So the conclusion of this analysis is that the vacuum manifold $\mathcal{M}_{\text{vac}}$ depends on the couplings g_1, g_2 and takes the form

$$\mathcal{M}_{\rm vac}=\begin{cases} g_2\le g_1: & \mathcal{O}\\ g_1<g_2\le 2g_1: & \mathcal{O}\cup T^3_{(i)}/S_4\\ g_2>2g_1: & \mathcal{O}\cup T^3_{(i)}/S_4\cup T^3_{(ii)}/S_4\end{cases} \tag{3.4.29}$$

We may interpret the appearance of new vacua for the above ranges of the coupling constants in terms of the occurrence of phase transitions. As will be discussed in the sequel, according to the specific phases, different RG-flows between the above vacua can exist.

Next, we will characterize interesting submanifolds of the vacuum manifold according to supersymmetry or gauge symmetry-breaking patterns. To analyze supersymmetry breaking it is sufficient to study the kernel (or equivalently image) of the generalized fermionic shift tensor $\mathbb{N}_{\mathcal{I}}{}^{A}$ of spin$-\frac{1}{2}$ fields. The index $\mathcal{I}$ runs over all spin$-\frac{1}{2}$ fields in the theory. In the case of an $\mathcal{N}=3$ supergravity in $d=4$ dimensions under consideration it means $\mathcal{I}=1,\dots,37$ in the following order: $\mathcal{I}\in\{$1 dilatino, 9×1 gaugino R–symmetry singlets, 9×3 gaugino R–symmetry triplets$\}$. Then the number of unbroken supersymmetries in a given vacuum is determined as

$$\mathcal{N}_{\rm vac}=\dim\left(\mathrm{Ker}\mathbb{N}_{\mathcal{I}}{}^{A}\Big|_{\rm vac}\right) \tag{3.4.30}$$

In light of the potential Ward identity (3.1.24), the number of preserved supersymmetries is equal to the number of eigenvalues $\mathbb{S}_{AA}$ of the diagonal matrix $\mathbb{S}_{AB}$ (see (3.4.13) and (3.4.14)) satisfying

$$|\mathbb{S}_{AA}|=\frac{1}{2L}=\sqrt{-V_0/12}, \tag{3.4.31}$$

where $L=\sqrt{-3/V_0}$ is the AdS radius. Both for type i) and ii), the above condition is met (modulo permutations in angles α_i) for one, two, and three eigenvalues when:

$$\begin{aligned}&\mathcal{N}=1\quad \alpha_1=\alpha_2+\alpha_3\,,\\ &\mathcal{N}=2\quad \alpha_1=\alpha_2\,,\ \alpha_3=0\,,\\ &\mathcal{N}=3\quad \alpha_1=\alpha_2=\alpha_3=0\,.\end{aligned} \tag{3.4.32}$$

All other points break supersymmetry completely. In Figure 3.1 we graphically illustrate the structure of both type i) and ii) vacua, parametrized by α_1, $\alpha_2\,\alpha_3$, where the identifications implemented by the group $\varGamma$ are taken into account. The inversions in this group amount to shifting two angles by $\pm\pi$, leaving the

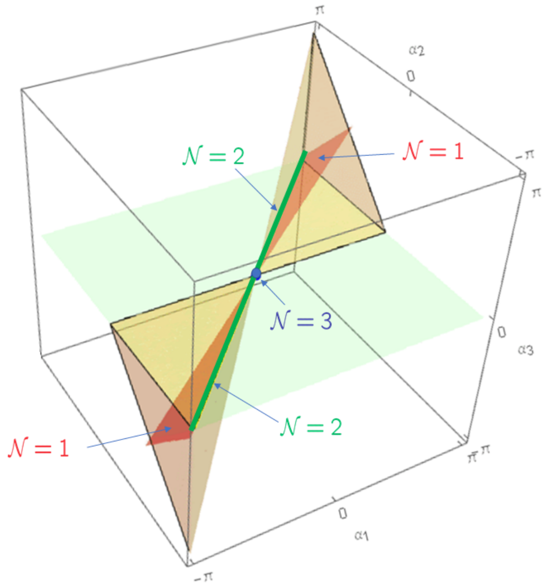

Figure 3.1 Representation of one of the two manifolds of vacua parametrized by $\alpha_1, \alpha_2, \alpha_3$. There is a residual identification (3.4.34) among the points on the plane $\alpha_3 = 0$. The vertices $(-\pi, -\pi, -\pi)$ and (π, π, π) are also identified.

third unaltered. We can fix these symmetries, as well as the permutations in Γ, by restricting the values of the angles to the following domains:

$$\begin{aligned} &D_1 : -\pi \leq \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq 0\,, \\ &D_2 : 0 \leq \alpha_3 \leq \alpha_2 \leq \alpha_1 \leq \pi\,. \end{aligned} \tag{3.4.33}$$

which are represented in Figure 3.1 by the colored tetrahedra. There is still an identification to be considered among the points in the shaded region of the graph. It identifies the two triangular faces of the tetrahedra at $\alpha_3 = 0$ and acts as follows:

$$(\alpha_1, \alpha_2) \in D_1 \;\sim\; (\alpha_2 + \pi, \alpha_1 + \pi) \in D_2\,. \tag{3.4.34}$$

Hence we can describe the independent $\mathcal{N} = 2$ vacua ($\alpha_1 = \alpha_2$, $\alpha_3 = 0$) by the segment belonging to D_1 only. Let us now describe the gauge group breaking patterns in various vacua. To determine the subgroup $H_0 \subset G_g$ of the gauge group that remains unbroken in the vacuum, one solves for the centralizer $h_0 \in \mathrm{Lie}(H_0) \subset \mathfrak{su}(3,9)$ of the coset generator $\mathbf{k}$ in (3.4.6) evaluated at the given vacuum

$$\left[\mathbf{k}\big|_{\mathrm{vac}}, h_0\right] = 0 \tag{3.4.35}$$

Equipped with this knowledge let us classify the submanifolds of $\mathcal{M}_{\rm vac}$ based on the residual gauge symmetry. We systematize the discussion starting from most generic submanifolds with the least residual gauge symmetry, going to more restricted submanifolds with a bigger gauge symmetry according to the following chain of subgroups

$$
\mathbf{1} \subset \cdots \subseteq H_0^{(k)} \subseteq \cdots \subseteq H_0^{(1)} \subset G_g \tag{3.4.36}
$$

Below we give the list of special submanifolds of $\mathcal{M}_{\rm vac}$, together with their properties, i.e. topology, preserved supersymmetry and residual gauge symmetry [7]

Type (i): $g_2 > g_1$

$$
\boxed{\begin{array}{c}(\alpha_1,\alpha_2,\alpha_3)\ \text{generic}\\ \mathcal{M}_{\rm vac}=T^3/S_4\\ \mathcal{N}=0\\ H_0=\mathrm{U}(1)\end{array}}
\begin{array}{c}\supset\\[2em] \subseteq\end{array}
\boxed{\begin{array}{c}(\alpha_2+\alpha_3,\alpha_2,\alpha_3)\\ \mathcal{M}_{\rm vac}=T^2/K_4\\ \mathcal{N}=1;\ (\alpha_2,\alpha_3\neq 0)\\ H_0=\mathrm{U}(1)\end{array}}
\begin{array}{c}\supset\\[2em] \subset\end{array}
\boxed{\begin{array}{c}(\alpha_2,\alpha_2,0)\\ \mathcal{M}_{\rm vac}=S^1/\mathbb{Z}_2\\ \mathcal{N}=2;\ (\alpha_2\neq 0)\\ H_0=\mathrm{U}(1)_D\times\mathrm{U}(1)\end{array}}
\begin{array}{c}\not\supset\\[2em] \subset\end{array}
$$

$$
\boxed{\begin{array}{c}(\alpha_1,\alpha_1,\alpha_1)\\ \mathcal{M}_{\rm vac}=S^1/\mathbb{Z}_2\\ \begin{cases}\alpha_1\neq 0:\ \mathcal{N}=0\\ \alpha_1=0:\ \mathcal{N}=3\end{cases}\\ H_0=\mathrm{SU}(2)_D\times\mathrm{U}(1)\end{array}}
\xrightarrow{r\to 0}
\boxed{\begin{array}{c}\mathcal{M}_{\rm vac}=\mathrm{pt}=\mathcal{O}\\ \mathcal{N}=3\\ H_0=G_g\end{array}} \tag{3.4.37}
$$

[7]In the following diagrams, the upper inclusion sign captures the relation between various submanifolds, while the lower one represents relations among unbroken gauge groups H_0. The inclusion between gauge groups is regular, but this is not always the case for the vacuum manifolds. For instance the two circles (with antipodal identification) are disjoint up to one point that they share. The first circle is $\mathcal{N}=2$, the second one is $\mathcal{N}=0$ and the single common point is $\mathcal{N}=3$.

Type (ii): $g_2 > 2g_1$

$$
\begin{array}{ccccccc}
\boxed{\begin{array}{c}(\alpha_1,\alpha_2,\alpha_3)\ \text{generic}\\ \mathcal{M}_{\text{vac}} = T^3/S_4\\ \mathcal{N}=0\\ H_0=\mathbf{1}\end{array}} & \begin{array}{c}\supset\\ \\ \subseteq\end{array} & \boxed{\begin{array}{c}(\alpha_2+\alpha_3,\alpha_2,\alpha_3)\\ \mathcal{M}_{\text{vac}} = T^2/K_4\\ \mathcal{N}=1;\ (\alpha_2,\alpha_3\neq 0)\\ H_0=\mathbf{1}\end{array}} & \begin{array}{c}\supset\\ \\ \subset\end{array} & \boxed{\begin{array}{c}(\alpha_2,\alpha_2,0)\\ \mathcal{M}_{\text{vac}} = S^1/\mathbb{Z}_2\\ \mathcal{N}=2;\ (\alpha_2\neq 0)\\ H_0=\mathrm{U}(1)_D\end{array}} & \begin{array}{c}\not\supset\\ \\ \subset\end{array} \\
\boxed{\begin{array}{c}(\alpha_1,\alpha_1,\alpha_1)\\ \mathcal{M}_{\text{vac}} = S^1/\mathbb{Z}_2\\ \begin{cases}\alpha_1\neq 0:\ \mathcal{N}=0\\ \alpha_1=0:\ \mathcal{N}=3\end{cases}\\ H_0=\mathrm{SO}(3)_D\end{array}} & \xrightarrow{r\to 0} & \boxed{\begin{array}{c}\mathcal{M}_{\text{vac}} = \text{pt} = \mathcal{O}\\ \mathcal{N}=3\\ H_0=G_g\end{array}} & & &
\end{array}
\tag{3.4.38}
$$

As we commented in (3.4.7), Type (i) vacua are associated with the embedding $\mathrm{SU}(2)\subset\mathrm{SU}(3)$ which has a U(1) commutant. Namely, one takes the diagonal combination of this SU(2) subgroup with the SO(3) factor in the gauge group (taking also into account the U(1) commutant) to arrive at (see (3.4.36))

$$H_0^{(1)} = \mathrm{SU}(2)_D\times\mathrm{U}(1) \tag{3.4.39}$$

This is the residual gauge symmetry of the $S^1/\mathbb{Z}_2$ vacua (first box on second line of (3.4.37)). The gauge groups of all other vacua in the Type (i) chains are subgroups of this one. Similarly, the only other non-abelian subgroup of SU(3) is SO(3). The embedding $\mathrm{SO}(3)\subset\mathrm{SU}(3)$ has no commutant, so in this case, one arrives at

$$H_0^{(1)} = \mathrm{SO}(3)_D \tag{3.4.40}$$

which is the residual gauge group of highest rank for Type (ii) vacua in (3.4.38). Moreover, let us remark that the singlets with respect to these maximal subgroups $H_0^{(1)}$ are the unique ones given in (3.4.4) and (3.4.5) (with the appropriate specification of phases shown in (3.4.37) and (3.4.38)). To argue this, as a first step, it is useful to remind the branching rules of the adjoint representation $\mathbf{8}$ of SU(3) with respect to its only two non-abelian subgroups

SU(2) and SO(3)

$$\mathbf{8}\Big|_{\mathrm{SU}(3)} \to (\mathbf{1}\oplus 2\times\mathbf{2}\oplus\mathbf{3})\Big|_{\mathrm{SU}(2)} \tag{3.4.41}$$

$$\mathbf{8}\Big|_{\mathrm{SU}(3)} \to (\mathbf{3}\oplus\mathbf{5})\Big|_{\mathrm{SO}(3)} \tag{3.4.42}$$

Recall that the scalar fields parameterizing the scalar manifold $\mathcal{M}_{scalar}$ transform in the $(\mathbf{3},\mathbf{8}+\mathbf{1})$ representation under $G_g = \mathrm{SO}(3)\times\mathrm{SU}(3)$. So combining the above decomposition with the adjoint representation $\mathbf{3}$ of SO(3) and restricting to the diagonal subgroups results in

$$(\mathbf{3},2\times\mathbf{1}\oplus 2\times\mathbf{2}\oplus\mathbf{3})\Big|_{\mathrm{SO}(3)\times\mathrm{SU}(2)} \to (\mathbf{1}\oplus 2\times\mathbf{2}\oplus 3\times\mathbf{3}\oplus 2\times\mathbf{4}\oplus\mathbf{5})\Big|_{\mathrm{SU}(2)_{\mathrm{D}}} \tag{3.4.43}$$

$$(\mathbf{3},\mathbf{1}\oplus\mathbf{3}\oplus\mathbf{5})\Big|_{\mathrm{SO}(3)\times\mathrm{SO}(3)} \to (\mathbf{1}\oplus 3\times\mathbf{3}\oplus 2\times\mathbf{5}\oplus\mathbf{7})\Big|_{\mathrm{SO}(3)_{\mathrm{D}}} \tag{3.4.44}$$

We see that in both cases there is a unique singlet as we claimed.

Having analyzed the residual supersymmetry of our distinguished subset of vacua, we move on to calculating mass spectra in each of these vacua. We use the general formulae for mass matrices for fields of all spins given in previous chapters. Then we apply these techniques and compute the spectra in all supersymmetric points and show that they organize into $\mathrm{OSp}(\mathcal{N}|4)$ supermultiplets, for $\mathcal{N}=1,2,3$ [8]. As usual in the AdS/CFT correspondence literature, this result suggests the duality between our backgrounds and three-dimensional (super)conformal field theories in which operators dual to the states described by the (super)conformal multiplets are present. The multiplets also provide the relevant conformal data for the dual operators. In this framework, the domain wall solutions presented in the next sections will correspond to RG flows between different conformal field theories triggered by relevant operators. As discussed in detail, the massless parameter α_i will, in the dual picture, classify families of RG flows between conformal field theories exhibiting different amounts of superconformal symmetries.

[8] We computed mass spectra also for AdS_4 vacua that break supersymmetry completely to SO(3,2).

3.5 Organizing supergravity fields into OSp($\mathcal{N}|4$) supermultiplets

Here we will show results for vacua that preserve $\mathcal{N} = 1,2,3$ supersymmetry. There are however $\mathcal{N} = 0$ vacua, which completely break supersymmetry and the mass spectrum of supergravity excitations around these vacua has been computed as well. However, it is not particularly illuminating and for this reason, it will not be presented explicitly. Besides the importance of the computation given the gauge/gravity correspondence, the matching of the spectrum with (super)conformal multiplets is a consistency check for our computation. Indeed, on general grounds, one expects this behavior for linearized perturbations around an AdS_4 background whose isometry in four dimensions plays the role of the conformal algebra in one dimension less.

3.5.1 General comments on OSp($\mathcal{N}|4$) supermultiplets

To describe supermultiplets we will follow the notation of [27]. The particular case of OSp($\mathcal{N}|4$) supermultiplets relevant in this chapter was also studied earlier in [28]. We briefly summarize just the necessary conventions and definitions of [27] useful in our special case. For details, the reader is kindly asked to consult the original paper. Supermultiplets of OSp($\mathcal{N}|4$) will be classified by Dynkin labels of its maximal compact subgroup $\mathrm{SO}(\mathcal{N})_R \times \mathrm{SO}(3)_J \times \mathrm{SO}(2)_\Delta$. The first factor represents the R–symmetry, the second the (Wick rotated) Lorentz transformations in three dimensions and finally the last factor is generated by the dilatation operator D. At the level of algebras, we use for the first two factors the isomorphism $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$, whenever available (always for the spin part and the R-symmetry if $\mathcal{N} = 3$). In such a situation, R and J are understood as $\mathfrak{su}(2)$ weights[9]. For $\mathcal{N} = 2$, the $\mathrm{SO}(2) \simeq \mathrm{U}(1)$ R-charge takes values in real numbers, $R \in \mathbb{R}$. Finally, if $\mathcal{N} = 1$, there is no R-symmetry, and states are labeled just by spin and scaling dimension. Then a supermultiplet will be denoted by its lowest weight state

$$X[J]_\Delta^{(R)}, \quad \text{where } X = L, A_1, A_2, B_1, B_2 \tag{3.5.1}$$

from which the complete supermultiplet is constructed by raising operators. As explained above R is the R–symmetry charge, J the spin and Δ the scaling dimension. The letter X specifies the type of the supermultiplet: L stands for a long supermultiplet, A for a short supermultiplet at the threshold (i.e. its scaling dimension Δ_A can be continuously approached from above), while B represents

[9] We use the half-integer convention for J, as it indicates the spin of the particles, and the integer one for the Dynkin label R. In [27], the integer convention is used both for J and R.

an isolated short multiplet (i.e. its scaling dimension $\Delta_B < \Delta_A$ is separated by a gap). From supergravity computations at the classical level [10] one obtains not directly the scaling dimensions, but rather masses of the particles (here we refer to the uncorrected mass; the AdS_4 mass is then obtained by combining this uncorrected mass with curvature contributions). It is thus useful to build a dictionary between the uncorrected masses and the scaling dimensions Δ or equivalently energies E_0, depending on whether we are using a gauge theory or gravity language. For particles of various spin it takes the form

spin	$\Delta \equiv E_0$
0	$\frac{1}{2}\left(3 \pm \sqrt{9+4m^2}\right)$
1	$\frac{1}{2}\left(3 \pm \sqrt{1+4m^2}\right)$
$\frac{1}{2}$, $\frac{3}{2}$	$\frac{1}{2}(3+2\lvert m\rvert)$

(3.5.2)

3.5.2 $\mathcal{N} = 3$ vacua

OSp(3|4) supermultiplets

The R-symmetry Lie algebra is $\mathfrak{so}(3) \simeq \mathfrak{su}(2)$. To label the states we will use the Dynkin label (R) of $\mathfrak{su}(2)$. The remaining labels of states in a supermultiplet are the spin and the scaling dimension. In Appendix E.1 we list only those OSp(3|4) supermultiplets that will be relevant to the present discussion (in the tables the R-symmetry representation is denoted by its dimension, i.e. **2** for the fundamental):

$\mathcal{N} = 3$ vacuum preserving $H_0 = G_g$

The mass spectrum in this isolated $\mathcal{N} = 3$ maximally symmetric vacuum is summarized in Table F.1. A quick consistency check employs the Goldstone theorem. There are 11 unbroken gauge generators and no broken ones in this vacuum. Therefore we expect no massive vector fields and $11+1$ massless ones. The additional vector comes from a completely decoupled massless vector supermultiplet – the Betti multiplet. This supermultiplet will be present in all the following spectra. Later, it will be included without further comments. The supergravity excitations can be assembled into the following supermultiplets of

[10] For $\mathcal{N} = 3$ the scaling dimension and hence the mass is a function of quantized quantities only – the spin and the R-charge. It cannot receive any corrections and is thus exact.

OSp(3|4)

$$\mathrm{Spec} = \underbrace{A_1[0]^{(0)}_{\frac{3}{2}}}_{\substack{\text{massless graviton}\\ \text{multiplet}}} \oplus \underbrace{9 \times B_1[0]^{(2)}_{1}}_{\substack{\text{massless vector}\\ \text{multiplets}}}, \tag{3.5.3}$$

as can be easily checked by comparing Table F.1 with the field content of the supermultiplets, which was summarized in the previous section.

$\mathcal{N} = 3$ vacuum preserving $H_0 = \mathrm{SU}(2)_D \times \mathrm{U}(1) \subset G_g$

The spectrum at the single $\mathcal{N} = 3$ supersymmetric point (lying on $S^1/\mathbb{Z}_2$ manifold of vacua, spanned by $\alpha_1 = \alpha_2 = \alpha_3$, invariant under the same subgroup H_0 of the gauge group) is shown in Table F.2. Inspection of the supermultiplet tables presented in Appendix E.1 leads to the conclusion that the spectrum given in Table F.2 is organized into the following supermultiplets

$$\mathrm{Spec} = \underbrace{A_1[\tfrac{1}{2}]^{(0)}_{\frac{3}{2}}}_{\substack{\text{massless graviton}\\ \text{multiplet}}} \oplus \underbrace{B_1[0]^{(4)}_{2} \oplus 2 \times B_1[0]^{(3)}_{\frac{3}{2}}}_{\text{massive vector multiplets}} \oplus \underbrace{2 \times B_1[0]^{(2)}_{1}}_{\substack{\text{massless vector}\\ \text{multiplets}}}. \tag{3.5.4}$$

A consistency check is provided by the Goldstone theorem. The gauge symmetry breaking pattern in this vacuum tells that there are 7 broken generators and 4 unbroken ones. Hence the number of massive vector fields is 7 and that of the massless ones is $4+1$, in agreement with the above tables.

$\mathcal{N} = 3$ vacuum preserving $H_0 = \mathrm{SO}(3)_D \subset G_g$

As in the previous case, the vacuum manifold that is invariant under the subgroup $H_0 = \mathrm{SO}(3)_D$ is $S^1/\mathbb{Z}_2$, spanned by $\alpha_1 = \alpha_2 = \alpha_3$. Again, there exists a single supersymmetric point on this circle of vacua which preserves $\mathcal{N} = 3$ supersymmetry. The spectrum at this special vacuum consists of states listed in Table F.3. Comparison with the supermultiplet tables results in a unique grouping of the states in Table F.3 into OSp(3|4) supermultiplets

$$\mathrm{Spec} = \underbrace{A_1[\tfrac{1}{2}]^{(0)}_{\frac{3}{2}}}_{\substack{\text{massless graviton}\\ \text{multiplet}}} \oplus \underbrace{B_1[0]^{(6)}_{3} \oplus B_1[0]^{(4)}_{2}}_{\text{massive vector multiplets}} \oplus \underbrace{B_1[0]^{(2)}_{1}}_{\substack{\text{massless vector}\\ \text{multiplet}}}. \tag{3.5.5}$$

Goldstone theorem serves as a check of consistency. There are 3 unbroken and 8 broken gauge generators in this vacuum and hence 8 massive and $3+1$ massless vector fields. Looking at the tables we see that this is true.

3.5.3 $\mathcal{N}=2$ vacua

OSp(2|4) **supermultiplets**

We have a $\mathrm{SO}(2)\simeq \mathrm{U}(1)$ R-symmetry and thus states of the OSp(2|4) supermultiplets are labeled by the U(1) R-charge $R\in\mathbb{R}$, spin and scaling dimension. There are two independent supercharges with R-charge $(+1)$ and (-1), respectively. We know that the $\mathcal{N}=2$ vacua spontaneously breaks the R–symmetry to $\mathrm{U}(1)\leftarrow\mathrm{SO}(3)$, hence we can infer the R–charges content from the breaking pattern of R–symmetry representations present in the corresponding $\mathcal{N}=3$ vacua. In Appendix E.2, more details on the OSp(2|4) supermultiplet are given, they are also relevant for another main example to be discussed in the next chapters. Taking all these comments into account, we find a unique way to organize the spectra in OSp(2|4) supermultiplets. In Appendix E.2 we list the relevant ones.

$\mathcal{N}=2$ **vacuum preserving** $H_0=\mathrm{U}(1)_D\times\mathrm{U}(1)\subset\mathrm{SU}(2)\times\mathrm{U}(1)\subset G_g$

The gauge symmetry-breaking pattern in this vacuum takes the form $G_g=\mathrm{SO}(3)\times\mathrm{SU}(3)\to H_0=\mathrm{U}(1)_D\times\mathrm{U}(1)$. According to Goldstone theorem the 12 vector fields split into 9 massive ones and $2+1$ massless ones (two gauging H_0 and one belonging to the Betti multiplet). The supergravity mass spectrum displayed in Table F.4 can be arranged into the following supermultiplets

$$\begin{aligned}\mathrm{Spec}=\;&\underbrace{A_1\overline{A}_1[1]_3^{(0)}}_{\substack{\text{massless graviton}\\ \text{multiplet}}}\oplus\underbrace{L\overline{L}[\tfrac{1}{2}]^{(0)}_{|m_G^{(1)}|+\frac{1}{2}}}_{\substack{\text{long massive}\\ \text{gravitino multiplet}}}\oplus\underbrace{L\overline{L}[0]_2^{(0)}}_{\substack{\text{long massive}\\ \text{vector multiplet}}}\oplus 2\times\underbrace{A_2\overline{A}_2[0]_1^{(0)}}_{\substack{\text{massless vector}\\ \text{multiplets}}}\\ &\oplus\left(\underbrace{2\times L\overline{A}_2[0]^{(\frac{1}{2})}_{\frac{3}{2}}}_{\substack{\text{short massive}\\ \text{vector multiplets}}}\underbrace{\oplus 2\times L\overline{B}_1[0]_1^{(1)}\oplus 2\times L\overline{B}_1[0]^{(\frac{3}{2})}_{\frac{3}{2}}\oplus L\overline{B}_1[0]_2^{(2)}}_{\frac{1}{2}-\text{hypermultiplets}}\right)\oplus\ (R\to -R)\end{aligned} \tag{3.5.6}$$

Where

$$m_G^{(1)}=\frac{\sqrt{g_1^4+g_2^4-2g_1^2g_2^2\cos 2\alpha_2}}{g_2^2-g_1^2} \tag{3.5.7}$$

is the mass of the single massive gravitino.

$\mathcal{N}=2$ vacuum preserving $H_0 = \mathrm{U}(1)_D \subset \mathrm{SO}(3)_D \subset G_g$

The gauge symmetry $G_g = \mathrm{SO}(3)\times\mathrm{SU}(3)$ is partially spontaneously broken to $H_0 = \mathrm{U}(1)_D$. We conclude that out of the 12 vector fields 10 become massive, while $1+1$ (one belonging to the Betti multiplet) remain massless. This agrees with the supergravity mass spectrum shown in Table F.5, which can be organized in a supermultiplet structure given below

$$\begin{aligned}\mathrm{Spec} = & \underbrace{A_1\overline{A}_1[1]_3^{(0)}}_{\text{massless graviton multiplet}} \oplus \underbrace{L\overline{L}[\tfrac{1}{2}]^{(0)}_{|m_G^{(2)}|+\frac{1}{2}}}_{\text{long massive gravitino multiplet}} \oplus \underbrace{L\overline{L}[0]_2^{(0)}}_{\text{long massive vector multiplet}} \oplus \underbrace{A_2\overline{A}_2[0]_1^{(0)}}_{\text{massless vector multiplet}} \oplus \underbrace{L\overline{L}[0]_3^{(0)}}_{\text{long massive vector multiplet}} \\ & \oplus \left(\underbrace{L\overline{L}[0]_3^{(1)}}_{\text{long massive vector multiplet}} \oplus \underbrace{L\overline{A}_2[0]_3^{(2)}}_{\text{short massive vector multiplet}} \oplus \underbrace{L\overline{B}_1[0]_3^{(3)} \oplus L\overline{B}_1[0]_2^{(2)} \oplus L\overline{B}_1[0]_1^{(1)}}_{\frac{1}{2}-\text{hypermultiplets}} \right) \oplus (R \to -R)\end{aligned} \tag{3.5.8}$$

Where

$$m_G^{(2)} = \frac{\sqrt{16g_1^4 + g_2^4 - 8g_1^2 g_2^2 \cos 2\alpha_2}}{g_2^2 - g_1^2} \tag{3.5.9}$$

is the mass of the single massive gravitino.

3.5.4 $\mathcal{N}=1$ vacua

OSp(1|4) **supermultiplets**

Since the R-symmetry is trivial, states of irreducible representations of OSp(1|4) are labeled just by spin and scaling dimension. In Appendix E.3 we list only six supermultiplets that will be needed, four long and two short ones.

$\mathcal{N}=1$ vacuum preserving $H_0 = \mathrm{U}(1) \subset \mathrm{U}(1)_\mathrm{D} \times \mathrm{U}(1) \subset \mathrm{SU}(2)_\mathrm{D} \times \mathrm{U}(1) \subset G_g$

The gauge symmetry $\mathcal{G} = \mathrm{SO}(3)\times\mathrm{SU}(3)$ in this vacuum is partially spontaneously broken to $H_0 = \mathrm{U}(1)$. Thus there are $\dim(G_g) - \dim H_0 = 10$ broken generators and the Goldstone theorem implies in this situation that the total 12 vector fields split into 10 massive and $1+1$ massless ones (one in the Betti multiplet). Indeed, the above reasoning complemented by the computation of the mass spectrum within supergravity, reported in Table F.6 leads to a unique

$\mathcal{N}=1$ supermultiplet spectrum in AdS_4 (i.e. OSp(1|4))

$$\begin{aligned}\text{Spec} = &\underbrace{A_1[\tfrac{3}{2}]_{\frac{5}{2}}}_{\substack{\text{massless}\\ \text{graviton}\\ \text{multiplet}}} \oplus \underbrace{L[1]_{\Delta^{(1)}_{G1}} \oplus L[1]_{\Delta^{(1)}_{G2}}}_{\substack{\text{massive gravitino}\\ \text{multiplets}}} \oplus \underbrace{2\times A_1[\tfrac{1}{2}]_{\frac{3}{2}}}_{\substack{\text{massless vector}\\ \text{multiplets}}} \oplus \underbrace{4\times L[\tfrac{1}{2}]_2 \oplus L[\tfrac{1}{2}]_{\Delta^{(1)}_{V1}} \oplus L[\tfrac{1}{2}]_{\Delta^{(1)}_{V2}}}_{\text{massive vector multiplets}}\\ &\underbrace{\oplus L'[0]_3 \oplus 2\times L'[0]_2 \oplus 8\times L'[0]_{\frac{3}{2}} \oplus L'[0]_{\Delta^{(1)}_{H1}} \oplus L'[0]_{\Delta^{(1)}_{H2}} \oplus 6\times L'[0]_1}_{\text{matter multiplets}}\end{aligned} \tag{3.5.10}$$

When comparing the supermultiplet spectrum (3.5.10) to the mass spectrum of supergravity presented in Table F.6, the Higgs phenomenon has to be taken into account. Namely, the longitudinal modes of massive vectors (gravitini) are massless scalars (spin-$\frac{1}{2}$ fermions). The scaling dimensions (energies) appearing in Table F.6 are expressed in terms of the parameters of the supergravity theory as follows

$$\Delta^{(1)}_{G1} = \Delta^{(1)}_{H1} = 1 + \frac{\sqrt{g_1^4 + g_2^4 - 2g_1^2 g_2^2 \cos(2\alpha_2)}}{g_2^2 - g_1^2} \tag{3.5.11}$$

$$\Delta^{(1)}_{G2} = \Delta^{(1)}_{H2} = 1 + \frac{\sqrt{g_1^4 + g_2^4 - 2g_1^2 g_2^2 \cos(2\alpha_3)}}{g_2^2 - g_1^2} \tag{3.5.12}$$

$$\Delta^{(1)}_{V1} = 1 + \frac{\sqrt{\beta^{(1)}_1 - 4\sqrt{\beta^{(1)}_2}}}{2(g_2^2 - g_1^2)} \tag{3.5.13}$$

$$\Delta^{(1)}_{V2} = 1 + \frac{\sqrt{\beta^{(1)}_1 + 4\sqrt{\beta^{(1)}_2}}}{2(g_2^2 - g_1^2)} \tag{3.5.14}$$

$$\beta^{(1)}_1 = 5g_1^4 + 5g_2^4 - 2g_1^2 g_2^2 (4\cos(2\alpha_2) + 4\cos(2\alpha_3) - 3) \tag{3.5.15}$$

$$\begin{aligned}\beta^{(1)}_2 = &\, g_1^8 + 2g_1^6 g_2^2 + 10 g_1^4 g_2^4 + 2g_1^2 g_2^6 + g_2^8 +\\ & 8g_1^4 g_2^4 \cos(2(\alpha_2+\alpha_3)) + 2g_1^2 g_2^2 \left(g_1^2 + g_2^2\right)^2 (\cos(2(\alpha_2-\alpha_3)) - 2\cos(2\alpha_2) - 2\cos(2\alpha_3))\end{aligned} \tag{3.5.16}$$

$\mathcal{N}=1$ vacuum preserving $H_0 = \{\mathbf{1}\} \subset \mathrm{U}(1)_\mathrm{D} \subset \mathrm{SO}(3)_\mathrm{D} \subset G_g$

In this vacuum, we observe a complete spontaneous symmetry breaking $G_g = \mathrm{SO}(3)\times\mathrm{SU}(3) \to H_0 = \{\mathbf{1}\}$. Hence Goldstone theorem dictates that there are 11 massive vector fields and just a single massless vector in the Betti multiplet. The mass spectrum of supergravity fields summarized in Table F.7 is organized into the following OSp(1|4) supermultiplets

$$\mathrm{Spec} = \underbrace{A_1[\tfrac{3}{2}]_{\frac{5}{2}}}_{\substack{\text{massless}\\\text{graviton}\\\text{multiplet}}} \oplus \underbrace{L[1]_{\Delta^{(2)}_{G1}} \oplus L[1]_{\Delta^{(2)}_{G2}}}_{\substack{\text{massive gravitino}\\\text{multiplets}}} \oplus \underbrace{A_1[\tfrac{1}{2}]_{\frac{3}{2}}}_{\substack{\text{massless}\\\text{vector}\\\text{multiplets}}} \oplus \underbrace{5\times L[\tfrac{1}{2}]_{\frac{7}{2}} \oplus L[\tfrac{1}{2}]_{\Delta^{(2)}_{V1}} \oplus L[\tfrac{1}{2}]_{\Delta^{(2)}_{V2}}}_{\text{massive vector multiplets}}$$
$$\underbrace{\oplus\, 3\times L'[0]_4 \oplus 8\times L'[0]_3 \oplus 2\times L'[0]_2 \oplus L'[0]_{\Delta^{(2)}_{H1}} \oplus L'[0]_{\Delta^{(2)}_{H2}} \oplus 3\times L'[0]_1}_{\text{matter multiplets}} \tag{3.5.17}$$

The values of scaling dimensions determining the supergravity mass spectrum presented in Table F.7 take the form of the ones in the corresponding type (i) vacua with the replacement $g_1 \to 2g_1$.

3.6 Domain wall solutions

In the previous section, we studied the (super)conformal multiplet arrangement of the fields on the new AdS_4 vacua. In this section, we will show that the latter can be interpreted as fixed points of RG-flows triggered by relevant operators which pertain to the CFT dual to the central vacuum. To do this, we consider a $(3+1)$-dimensional bulk space-time, parametrized by the coordinates $x^\mu = \left(x^i, y\right)$, and use the standard domain-wall (DW) ansatz for the metric, which has the usual form

$$ds^2 = e^{2A(y)}\, ds^2_{1,2} - dy^2 = e^{2A(y)}\, dx^i \eta_{ij}\, dx^j - dy^2 \quad , \quad \eta_{ij} = (+,-,-) \quad , \tag{3.6.1}$$

$$\phi^r = \phi^r\,(y) \quad , \quad i,j = 0,1,2 \quad , \tag{3.6.2}$$

where $ds^2_{1,2}$ defines the flat Minkowski metric in three dimensions, $A(y)$ is the scale factor, y is the coordinate transverse to the wall, and all scalar fields $\phi(y)$ depend only on the transverse coordinate y [11].From the AdS/CFT point of view, the domain wall ansatz corresponds to an RG flow between the UV and IR fixed points described by the asymptotic regions $y \to \pm\infty$. Let us be more explicit by considering the consistent truncation described in Section 3.4, generated by the three complex scalar fields z_1, z_2, z_3. We recall that solutions of the truncated theory are solutions of the complete theory and that all fields in the DW solution are functions of the transverse coordinate y only. From the coset metric (3.4.10) and the ansatz in (3.6.1) one can obtain, after consistently

[11] From now on, we will omit the y-dependence of the scalar fields and the scale factor in the DW metric.

setting all fermions and vector fields to zero, the effective Lagrangian density [12]

$$\mathscr{L} = -e^{3A}\sum_i^3\left[3A'' + 6A'^2 + \left(r_i'\right)^2 + \frac{1}{4}\sinh\left(2r_i\right)^2\left(\alpha_i'\right)^2 + V(r_i,\alpha_i)\right] , \tag{3.6.3}$$

where the potential for Type (i) and Type (ii) models was given in (3.4.11) and (3.4.12), respectively.

We leave the details of the DW solutions in appendix D. Here we focus on the main properties and their possible interpretation in the dual picture. In particular, we search for configurations in which the radii r_i are equal to the same field r. Then the phases α_i do not depend on y. Therefore, the constant values of the phases α_i select the critical point at the end of the flow (IR fixed point) as in Table (3.4.37) (or (3.4.38) for Type (ii) vacuum), the starting point being the central $\mathcal{N}=3$ vacuum (UV fixed point). The "shape" of the domain wall is implicitly governed by the field $r(y)$ through the warping function $A(y(r))$. For the sake of simplicity let us consider the Type (i) consistent truncation (3.4.4) (Type (ii) consistent truncation gives the same results after substituting $g_1 \to 2g_1$), which provides the vacuum at the origin and the one described by (3.4.21). In this case, we obtain the DW solution, whose explicit expression is given in eq. (D.1.15) Appendix D. It is useful to perform the following change of coordinates to study the behavior near the fixed points of the flow:

$$x^i \mapsto \left(g_1^2 - \varepsilon g_2^2\right) x^i \quad , \quad r = r(y) \quad , \quad \varepsilon = \begin{cases} 0 & r \to 0 \\ 1 & r \to r^\star \end{cases} \tag{3.6.4}$$

where $r(y)$ is the solution for r in the DW background. It is enough to know the expression for the inverse relation $y(r)$ given by (D.1.16). Then the DW metric becomes

$$ds^2 = \frac{1}{4}\left(\frac{(g_1\text{csch}(r) - g_2\text{sech}(r))^2}{g_1^4}dx^i dx_i - \frac{\text{csch}^2(r)\text{sech}^4(r)}{(g_1 - g_2\tanh(r))^2}dr^2\right) . \tag{3.6.5}$$

Now, we consider the limit $r \to 0$ to obtain

$$ds^2 \sim ds^2_{\text{UV}} = \frac{1}{4r^2 g_1^2}\left(-dr^2 + dx^i dx_i\right) \tag{3.6.6}$$

[12]Here, primes denote derivatives respect to the y direction

which is the metric for an AdS_4 space with radius

$$R^2 = -\frac{3}{\Lambda} = \frac{1}{4{g_1}^2} , \tag{3.6.7}$$

in agreement with the value of Λ at $r=0$ in (3.4.24). This expression provides directly the asymptotic behavior of r near the conformal boundary. Indeed, in this particular case, the metric is in the usual Poincaré coordinates with radial direction z. Hence, we have $r \sim z$ and $\Delta_r = 1$. On the other side, expanding ds^2 near $r \to r^\star$ we get

$$ds^2 \sim ds^2_{\text{IR}} = R^2 \left(u^2 dx^j dx_j - \frac{du^2}{u^2} \right) \tag{3.6.8}$$

where $u = (r - r^\star)$ and

$$R^2 = -\frac{3}{V(r^\star)} = \frac{{g_2}^2 - {g_1}^2}{4{g_1}^2{g_2}^2} , \tag{3.6.9}$$

as expected from (3.4.24). The relation with Poincaré coordinates is given by $u = \frac{1}{z}$. So that $(r - r^\star) \sim z^{-1}$ and $\Delta_u = -1$.

The interpretation as an RG-flow is the following. When we switch on the r source (the combination $\delta r_1 + \delta r_2 + \delta r_3$) at the origin of the scalar manifold we introduce a relevant deformation, indeed the scaling dimension of the operator coupled to r will be $\Delta_{\mathcal{O}_r}|_0 = 2$. This triggers an RG-flow that eventually ends at $r = r^\star$ where the operator becomes irrelevant, indeed $\Delta_{\mathcal{O}_r}|_{r^\star} = 4$. We are flowing from the $\mathcal{N} = 3$ $SCFT_3$ dual to the AdS_4 background at $r = 0$ (the UV region) to a CFT_3 dual to the AdS_4 background at $r = r^\star$ (in the IR region). In general, the IR three-dimensional dual theory will not be superconformal. For particular values of α_i the IR critical point will correspond to a $SCFT_3$ with different amounts of supersymmetries, in agreement with the classification given in 3.4.37.

As a check for our interpretation we compute the scalar spectrum of the truncation near $r = 0$ and $r = r^*$ and we obtain the masses $(-2,-2,-2,-2,-2,-2)$ and $(4,-2,-2,0,0,0)$ respectively. The latter correspond to the combinations $(\delta r_1 + \delta r_2 + \delta r_3, \delta r_2 - \delta r_1, \delta r_3 - \delta r_1, \delta\alpha_1, \delta\alpha_2, \delta\alpha_3)$. Another relevant check of the interpretation of $r = 0$ as the UV critical point and $r = r^\star$ as the IR one is provided by the holographic c-theorem [29, 30]. Following these works we compute

$$a(y) = A'(y)^{-2} , \tag{3.6.10}$$

where

$$A'(y) = -2g_2 \sinh^3(r(y)) + 2g_1 \cosh^3(r(y)) . \tag{3.6.11}$$

It follows that $a(y(r))$ is monotonically decreasing as a function of $r \in [0, r^\star]$, consistently with the holographic c-theorem $a_{UV} \geq a_{IR}$.

This analysis concludes our detailed discussion of the first interesting example of supergravity vacua deformed by supersymmetry breaking marginal deformations. In particular, within this $\mathcal{N}=3$ model, the backgrounds exhibit perturbative stability as explicitly checked with standard mass formulae. However, we are not yet aware of the possibility of embedding this model inside an upliftable one. By this, we mean that we do not know if the present theory can arise as a consistent truncation of a spontaneous compactification of higher dimensional supergravity. Among other implications, this means that we have very little to say about Kaluza-Klein spectrometry for the solutions and even less about the non-perturbative stability of the conformal manifolds discussed above. To fill this gap, we will present another explicit example in next chapters. In this new case, families of vacua similar to the one found in the $\mathcal{N}=3$ model will be presented in a maximal $\mathcal{N}=8$ model. This latter framework will allow us to discuss the relation between supergravity theories in four dimensions and their cousins in ten dimensions (by focusing on the type IIB case). By implementing the relatively new techniques of Exceptional Field theory a detailed discussion on Kaluza-Klein spectrometry and the stability of the solutions can be carried out. Before entering the details of the second example, let us quickly introduce the new framework in which their properties can be properly studied.

Chapter 4

Overview on a Class of Upliftable $\mathcal{N}=8$ Models

The duality covariant formulation of gauged supergravities [22–24], discussed in previous chapters has provided a valuable tool for discovering new superstring/M-theory compactifications and their duality connections. A consistent truncation of the low-lying modes of superstring/M-theory, in certain compactifications, is captured by an effective extended supergravity theory whose Lagrangian typically exhibits characteristic minimal couplings, associated with a gauge group G_g, Yukawa terms, and a scalar potential. All these features of the effective low-energy description depend on general characteristics of the higher-dimensional background, such as the geometry of the internal manifold M_{int} and various kinds of fluxes that are present in the solution. As already introduced in the $\mathcal{N}=3$ case, all these features are present in a gauged supergravity model and they can be all encoded in a single object called the embedding tensor. This tensor is formally covariant with respect to the on-shell global symmetry group G of the corresponding ungauged theory. Although the presence of minimal couplings typically breaks G, formal G-invariance of the field equations and the Bianchi identities are preserved, provided the embedding tensor is transformed together with all the other fields.

As far as $D=4$ $\mathcal{N}=8$ supergravities are concerned, the on-shell global symmetry group is of exceptional type $G=\mathrm{E}_{7(7)}$. In these cases a direct relation between certain gauged models and superstring/M-theory can be established within the framework of Exceptional Field Theory (ExFT) [31, 4, 32, 33]. The latter provides a manifestly $\mathrm{E}_{7(7)}$-covariant description of 11–dimensional and Type–II supergravities and shows how to embed certain gauged supergravities within the higher-dimensional ones, as consistent truncations, through a generalized Scherk–Schwarz ansatz [5]. Recently, this framework has also proven to be very useful in performing Kaluza–Klein spectrometry for those compactifications fitting into the generalized Scherk–Schwarz ansatz [34, 6]. As a key simplification, the construction only relies on the scalar harmonics, corresponding to the maximally symmetric point of the lower dimensional supergravity. Let us then present the general feature of the $\mathcal{N}=8$ supergravity theories and discuss how to describe some classes of the latter in the ExFT framework.

4.1 $D=4$ $\mathcal{N}=8$ Supergravity

As a particular example of the general framework discussed in chapter 2, ungauged $\mathcal{N}=8$ supergravity in four dimensions only describes a gravitational multiplet consisting of the graviton, 8 gravitini, 28 vector fields, 56 spin-1/2 fields and 70 scalars spanning the scalar manifold $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ [35]. The on-shell global symmetry group of the ungauged model is $\mathrm{E}_{7(7)}$ which acts as an electric-magnetic duality group on the 28 vector field strengths and their magnetic duals. This duality action is defined by the symplectic 56-dimensional representation of $\mathrm{E}_{7(7)}$. The gravitini transform in the fundamental representation of SU(8), the R-symmetry group. In this case $\mathfrak{H}=\mathfrak{su}(8)$ and $\mathfrak{K}$ describes the 70-dimesnional representation of $\mathfrak{H}$. The $\mathscr{R}_v^c$ representation is such that

$$\mathscr{R}_v^c[\mathscr{Q}] = \begin{pmatrix} \mathscr{Q}^{\underline{AB}}{}_{\underline{CD}} = 4\delta^{[\underline{A}}_{[\underline{C}}\mathscr{Q}^{\underline{B}]}{}_{\underline{D}]} & 0 \\ 0 & \mathscr{Q}_{\underline{AB}}{}^{\underline{CD}} = -\mathscr{Q}^{\underline{AB}}{}_{\underline{CD}} \end{pmatrix}, \qquad (4.1.1)$$

and

$$\mathscr{R}_v^c[\mathscr{P}] = \begin{pmatrix} 0 & \mathscr{P}^{\underline{ABCD}} \\ \mathscr{P}_{\underline{ABCD}} = \frac{1}{24}\epsilon_{\underline{ABCDEFGH}}\mathscr{P}^{\underline{EFGH}} & 0 \end{pmatrix}, \qquad (4.1.2)$$

where $\mathscr{Q}^{\underline{AB}}{}_{\underline{CD}}$, $\underline{A}=1,...,8$, belong to the **28** two-fold antisymmetric representation of $\mathfrak{su}(8)$ and $\mathscr{P}_{\underline{ABCD}}$ belong to the **70** four-fold antisymmetric representation of the same algebra. However, we will make extensive use of the $\mathrm{SL}(8,\mathbb{R})$ frame in which the gaugings of interest are conveniently described. In this symplectic frame the off-shell global symmetry group is $\mathrm{SL}(8,\mathbb{R}) \subset \mathrm{E}_{7(7)}$. If $A,B=1,\ldots,8$ label the fundamental 8-dimensional representation of this group, the 28 electric vector fields $A_\mu^{[AB]}$ and their magnetic counterparts $A_{[AB]\mu}$ are labeled by the antisymmetric couple $[AB]$. These fields are conveniently described by a symplectic 56-component vector A_μ^M,

$M = 1, \ldots, 56$, of the form: $A^M_\mu = (A^{[AB]}_\mu, A_{[AB]\mu})$[1]. The generators of $\mathrm{E}_{7(7)}$ consist of the $\mathrm{SL}(8,\mathbb{R})$ generators $t^A{}_B$ and generators $t^{ABCD} = t^{[ABCD]}$ (or their duals $t_{ABCD} = \frac{1}{24}\epsilon_{ABCDEFGH} t^{EFGH}$ in the representation **70** of the same group. The relation between these generators and those of $\mathfrak{H}$ and $\mathfrak{K}$, associated to $\mathscr{Q}$ and $\mathscr{P}$ respectively, is obtained by branching SU(8) and $\mathrm{SL}(8,\mathbb{R})$ and their representation with respect to the common SO(8) subgroup. In particular the generators of $\mathfrak{H}$ correspond to the **28** plus the $\mathbf{35}_s$ representations of SO(8). It is to say the compact generators of $\mathrm{SL}(8,\mathbb{R})$ plus the anti-selfdual combination $\frac{1}{2}\left(t^{ABCD} - t_{ABCD}\right)$. Instead, the non-compact generators of $\mathfrak{K}$ are described by the $\mathbf{35}_v$ plus the $\mathbf{35}_c$ representation of SO(8). It is to say the non-compact generators of $\mathrm{SL}(8,\mathbb{R})$ plus the selfdual combination $\frac{1}{2}\left(t^{ABCD} + t_{ABCD}\right)$. After this quick review of the ingredients in the ungauged case let us move to the main features of the gauging procedure in the maximal case.

4.1.1 Generalities on the Gauged Model

In the symplectic-covariant formulation of the gauging procedure, the gauge algebra is described by a 56-component symplectic vector of generators X_M, $M = 1, \ldots, 56$, each represented by a matrix $(X_M)_N{}^P$ in the symplectic 56-dimensional representation of the $\mathrm{E}_{7(7)}$ generators:

$$X_{MN}{}^P \mathbb{C}_{QP} = X_{MQ}{}^P \mathbb{C}_{NP}\,, \tag{4.1.3}$$

where $\mathbb{C}_{NP}$ is the antisymmetric 56×56 symplectic invariant matrix and the X-tensor $X_{MN}{}^P \equiv \Theta^\alpha_M \mathscr{R}_v[t_\alpha]_N{}^P$ is expressed in terms of the embedding tensor as usual. It is a formal $\mathrm{E}_{7(7)}$-tensor encoding all information about the embedding of the gauge algebra within the global symmetry one. All the additional terms, required by the gauging procedure, in the Lagrangian (Yukawa terms and scalar potential) and in the supersymmetry transformation laws are expressed in terms of $X_{MN}{}^P$ or equivalently by the T-tensor. By standard algebraic arguments it can be shown that the linear constraint (2.2.4) restricts the X-tensor to the **912** representation[2] of $\mathrm{E}_{7(7)}$. Once this constraint is imposed, the quadratic constraints (2.2.5) are equivalent. The former and the latter are necessary for the gauged Lagrangian to be supersymmetric. In the maximal case the gauge

[1]For the $\mathrm{SL}(8,\mathbb{R}) \subset \mathrm{E}_{7(7)}$ indices we use the notation that contraction over an antisymmetric couple $[AB]$ should be multiplied times a factor 1/2: $V_M W^M = \frac{1}{2}(V_{[AB]} W^{[AB]} + V^{[AB]} W_{[AB]})$.

[2]This is true when excluding gauging associated to an extra on-shell scaling symmetry, the "trombone symmetry", which is not described by $\mathrm{E}_{7(7)}$ transformations.

connection reads:

$$\Omega_{g\,\mu} \equiv g\,A_\mu^M\,X_M = \frac{g}{2}\left(A_\mu^{[AB]}\,X_{[AB]} + A_{[AB]\,\mu}\,X^{[AB]}\right), \tag{4.1.4}$$

where g is the gauge coupling and the right-hand side is specific to the $\mathrm{SL}(8,\mathbb{R})$ frame. Besides $A_{[AB]\,\mu}$, also a set of antisymmetric 2-forms $B_{a\,\mu\nu}$, $a=1,\ldots,133$, transforming in the adjoint representation of $\mathrm{E}_{7(7)}$, has to be introduced. This is a redundant description of the field content which is required when we gauge a group G_g using vector fields that are not electric in the symplectic frame of the original ungauged Lagrangian. This will be the case for the gauging of the main example we will discuss. The fermionic shifts are encoded in the T-tensor which, as described in (2.2.14), is an H-covariant (SU(8)-covariant in our case) tensor constructed out of the X-tensor. The latter is in the **912** of $\mathrm{E}_{7(7)}$. When branched respect to H gives the **36** and the **420** representations of SU(8) plus their conjugates. The former is the one describing the gravitini shift matrix $\mathbb{S}_{\underline{AB}}$ and the latter describes the dilatini shift matrix $\mathbb{N}_{\underline{ABC}}{}^{\underline{D}}$. Then the potential ward identity gives as potential

$$V(\phi) = g^2\left(\frac{1}{48}\mathbb{N}_{\underline{ABC}}{}^{\underline{D}}\mathbb{N}^{\underline{ABC}}{}_{\underline{D}} - \frac{3}{2}\mathbb{S}_{\underline{AB}}\mathbb{S}^{\underline{AB}}\right) \tag{4.1.5}$$

It can also be recast in the G-covariant expression [36][37]:

$$V(\phi) = \frac{g^2}{672}\mathcal{M}^{MN}\left(\mathcal{M}^{PQ}\mathcal{M}_{RS}\,X_{MP}{}^{R}\,X_{NQ}{}^{S} + 7\,\mathrm{Tr}(X_M\,X_N)\right). \tag{4.1.6}$$

The symplectic, symmetric matrix $\mathcal{M}_{MN}(\phi)$ (2.1.14) is defined in terms of the coset representative $\mathbb{L}^M{}_{\underline{N}} \in \frac{\mathrm{E}_{7(7)}}{\mathrm{SU}(8)}$ of the scalar manifold in the representation **56** of $\mathrm{E}_{7(7)}$. $\underline{M},\underline{N}=1,\ldots,56$ denote the SU(8) indices labeling the $\mathbf{28}+\overline{\mathbf{28}}$ representation. In (4.1.6) $\mathcal{M}^{MN}$ describes the inverse matrix of $\mathcal{M}_{MN}$. On a specific vacuum, mass matrices can be computed by specializing the general mass formulae of Section 3.2, explicit expressions can be found in [11]. However, we will be interested in computing the supergravity spectrum as a subsector of the full Kaluza-Klein spectrum of the supergravity vacua. Indeed, the examples presented below are derived in the contest of models upliftable to type IIB supergravity. Let us proceed by presenting the class of gaugings we want to focus on in the embedding tensor formalism. As we will see, by describing the latter in the framework of ExFT, we can explicitly derive their uplift and their full Kaluza-Klein spectrum.

4.1.2 Gaugings in the $\mathrm{SL}(8,\mathbb{R})$ frame

The fundamental of $\mathrm{E}_{7(7)}$ splits in the representation $\mathbf{28}'$ describing the electric vector fields, plus its dual describing the magnetic ones, of $\mathrm{SL}(8,\mathbb{R})$ when branched respect to the latter while the $\mathbf{133}$ splits in the adjoint representation plus the $\mathbf{70}$. We then have that the embedding tensor representation, the $\mathbf{912}$, splits in sum of the $\mathbf{36}'$ and the $\mathbf{420}$ of $\mathrm{SL}(8,\mathbb{R})$ plus their duals. In this frame the electric group is $G_e=\mathrm{SL}(8,\mathbb{R})$. This implies that if we want to describe a gauging proper to this frame only electric vector fields can be selected by the embedding tensor. This means that only the $\mathbf{28}\times\mathbf{63}$ part of the $\mathbf{56}\times\mathbf{133}$ (when branched respect to G_e) describing $\Theta_{[AB]D}{}^{C}$ is not vanishing. In particular, in the case of an electric gauging only the $\mathbf{36}$ component of the embedding tensor survives. The latter is given by

$$\Theta_{[AB]D}{}^{C}=\delta^{C}_{[A}\eta^{(p,q,r)}_{B]D}, \tag{4.1.7}$$

where $\eta^{(p,q,r)}$ is the diagonal matrix with p "+1"s, q "-1"s and $r=8-p-q$ "0"s on the diagonal [22][24][38]. The above embedding tensor correspond to a $G_g=\mathrm{CSO}(p,q,r)=\mathrm{SO}(p,q)\ltimes\mathbb{R}^{r(p+q)}$. However, the G-duality covariant formulation allows for more general gaugings than the electric ones. Because of the algebraic argument given above, it is clear that as soon as magnetic components of the embedding tensor are switched on we end up in a proper dyonic gauging. This means that even if we work in the $\mathrm{SL}(8,\mathbb{R})$ frame, when using magnetic vectors, we are describing a gauge group G_g which is gauged by electric vectors of a frame inequivalent to the $\mathrm{SL}(8,\mathbb{R})$ one. Following this idea, we can use the full power of the G-covariant formulation described in previous sections and we can consider an embedding tensor, generalising the previous one, with entries $\Theta_{[AB]D}{}^{C}$ and $\Theta^{[AB]}{}_{D}{}^{C}$ describing the $\mathbf{36}$ and $\mathbf{36}'$ components respectively. In particular, the former has the same expression as (4.1.7) while the latter has the analogous form [39]

$$\Theta^{[AB]}{}_{D}{}^{C}=\delta^{[A}_{D}\xi^{B]C}_{(p',q',r')}. \tag{4.1.8}$$

In this case the quadratic constraints (2.2.5) (or equivalently (2.2.6)) are not trivially satisfied as in the purely electric case and we have

$$\eta^{(p,q,r)}\xi_{(p',q',r')}\otimes\mathbf{1}_{8\times8}-\mathbf{1}_{8\times8}\otimes\eta^{(p,q,r)}\xi_{(p',q',r')}=0\ . \tag{4.1.9}$$

The possible gaugings of this form can be divided into two cases. It is to say the case $r=0$ and $r\neq0$. In the former case $\eta^{(p,q,0)}$ is invertible and the quadratic constraint are solved by $\xi_{(p',q',r')}=\xi_{(p,q,0)}=c\eta^{-1}_{(p,q,0)}$. They describe the gauge group $G_g = \mathrm{SO}(p,q)$ which is now embedded dyonically in G. In other words, when $c \neq 0$ both the electric and magnetic vector fields of the $\mathrm{SL}(8,\mathbb{R})$ frame are involved in the gauging of the $\mathfrak{so}(p,q)$ generators. When $c=0$ one recovers the electric version of the gauging. These gaugings are referred to as $\mathrm{SO}(p,q)_\omega$ models [40], being ω a compact parameter, expressed in terms of c, parameterizing an SO(2) rotation whose symplectic action relates, at the classical level, the original frame in which G_g is gauged dyonically and the frame in which G_g is gauged by electric vectors only. The case $r \neq 0$ is solved, without loss of generality, by considering $\xi_{(p',q',r')}$ with $p'+q' \leq r$. In particular, the non-vanishing entries of the latter are confined to the vanishing entries of $\eta_{(p',q',r')}$. Then, G_g is of the form

$$G_g = \Big(\mathrm{SO}(p,q) \times \mathrm{SO}(p',q')\Big) \ltimes \exp\Big(N_1^{(8-r)(r-p'-q')} \oplus N_2^{(8-r')(r-p'-q')} \oplus N_3^{(8-r)(8-r')}\Big). \tag{4.1.10}$$

In the above expression, the first SO factor and N_1 are gauged by electric vectors only, the second SO factor and N_2 are gauged by magnetic vectors only while N_3 is gauged by a combination of the two. N_1, N_2 and N_3 are nilpotent algebras. This concludes, from a $D=4$ point of view, the general discussion on the class of gaugings we will consider. Next, we present the main example corresponding to the specific values $p=6$, $q=0$, $p'=1$ and $q'=1$. This latter case and others within the class of models presented above have a direct higher dimensional interpretation. This feature is best explored in the context of Exceptional Field Theory. Even if we will not be exhaustive on the subject, we are going to review the features of the latter framework relevant to our discussion. The reader is referred to the bibliography for the interesting details of ExFT. We will merely use the theory for its powerful techniques. In particular for describing the uplift of supergravity models and for computing the Kaluza-Klein spectrum of the vacua of interest.

4.2 The Gauged $[\mathrm{SO}(6)\times\mathrm{SO}(1,1)]\ltimes\mathbb{R}^{12}$ model

In this section we consider the gauged model in which the gauge group has the form [39, 41–43]:

$$G_g = [\mathrm{SO}(6)\times\mathrm{SO}(1,1)]\ltimes\mathbb{R}^{12}\,. \tag{4.2.1}$$

In the SL(8,$\mathbb{R}$)-symplectic frame the embedding tensor $X_{MN}{}^{P}$ of the gauging reads:

$$\begin{aligned} X_{[AB],[CD]}{}^{[EF]} &= -X_{[AB]}{}^{[EF]}{}_{[CD]} = 8\,\delta^{[E}_{[A}\theta_{B][C}\delta^{F]}_{D]}\,, \\ X^{[AB]}{}_{[CD]}{}^{[EF]} &= -X^{[AB][EF]}{}_{[CD]} = 8\,\delta^{[A}_{[C}\xi^{B][E}\delta^{F]}_{D]}\,, \end{aligned} \tag{4.2.2}$$

where a possible choice for the θ and ξ up to change of SL(8,$\mathbb{R}$) basis is

$$\theta_{AB} = \mathrm{diag}(1,1,1,1,1,0,0,1)\;,\;\; \xi^{AB} = \mathrm{diag}(0,0,0,0,0,1,-1,0)\,. \tag{4.2.3}$$

As expected for the dyonic case, the "magnetic" vectors $A_{[AB]\mu}$ are involved in the gauge connection. In our discussion about this model, we shall follow, unless stated otherwise, the notations of [43]. Following [43], the vacua we are interested in can all be described within a $\mathbb{Z}_2^3$-invariant sector [44][45] which describes an $\mathcal{N}=1$ supergravity coupled to seven chiral multiplets with complex scalars $z_i = -\chi_i + i\,e^{-\varphi_i}$, $i:1,\dots,7$. The coset representative, in a suitable basis of the $\mathrm{E}_{7(7)}$ generators, is chosen to be:

$$\mathbb{L} = \exp\left(\sum_{i=1}^{7}\chi_i e_i\right)\cdot\exp\left(\sum_{i=1}^{7}\varphi_i h_i\right) \in \left[\frac{\mathrm{SL}(2)}{\mathrm{SO}(2)}\right]^7 \subset \frac{\mathrm{E}_{7(7)}}{\mathrm{SU}(8)}\,, \tag{4.2.4}$$

where the generators h_i, e_i satisfy the relations $[h_i,e_j]=\delta_{ij}\,e_j$, $[e_i,(e_j)^t]=2\delta_{ij}\,h_i$. They are related to the generators g_{χ_i}, g_{φ_i} in [43] as follows: $h_i = g_{\varphi_i}/4$, $e_i = -12\,g_{\chi_i}$. We have set, without loss of generality, the gauging parameter c to $c=1$. Indeed, the latter is an on/off parameter. The model features anti-de Sitter vacua with supersymmetry $\mathcal{N}=0$, 1, 2 and 4. We shall focus below on the $\mathcal{N}=2$ class of vacua, compute the (bosonic) Kaluza-Klein spectrum on them and eventually provide their uplift to $D=10$.

4.2.1 The $\mathcal{N}=2$ Vacua and their Spectra

We shall focus our discussion on the $\mathcal{N}=2$ vacua, defined by the following expectation values for the scalars z_i:

$$z_1 = -\bar{z}_3 = -\chi + \frac{i}{\sqrt{2}}\,, \qquad z_2 = z_4 = z_6 = i\,, \qquad z_5 = z_7 = \frac{1}{\sqrt{2}}(1+i)\,. \tag{4.2.5}$$

The vector and fermionic fields are all set to zero. The metric describes a geometry of AdS with cosmological constant

$$\Lambda = -3g^2 \tag{4.2.6}$$

This family of vacua is parameterized by a continuous parameter χ. From a low energy $D=4$ supergravity perspective, this parameter takes values in $\mathbb{R}$. It describes an $\mathrm{SU}(2)\times\mathrm{U}(1)$-invariant vacuum only when $\chi=0$. In general, for $\chi\neq 0$, the vacuum posses a $\mathrm{U}(1)^2$ residual gauge symmetry. As we shall see, this picture is strongly modified when considering the whole KK spectra of these backgrounds, or, equivalently, the corresponding $D=10$ solution. This analysis will show that χ is in fact periodic. At the level of supergravity dynamics we can already see that the spectrum depends on χ. In particular the values of the latter discriminate between $\mathrm{U}(1)^2$ and $\mathrm{SU}(2)\times\mathrm{U}(1)$ symmetric vacua. A compact way of presenting the spectra of the latter is to make explicit the superconformal symmetry. It is to say to arrange the spectra modes in superconformal multiplets. In the $\mathrm{U}(1)^2$ case, since there must be a massless gravity multiplet which contains two gravitini with[3] $m^2=1$ and one massless vector, only one massless vector multiplet must be considered with the other vectors being massive. Furthermore, they must come in pairs with opposite R–charges in order to fit into $\mathfrak{u}(1)_R$ representations. The remaining fields live in pairs of matter multiplets. The spectrum is organized into the following $\mathrm{OSp}(2|4)$ supermultiplets[4]

$$\begin{aligned}&A_1\overline{A}_1[1]_2^{(0)}\oplus L\overline{A}_1[\tfrac{1}{2}]_{\frac{5}{2}}^{(1)}\oplus A_1\overline{L}[\tfrac{1}{2}]_{\frac{5}{2}}^{(-1)}\oplus 4\times L\overline{L}[\tfrac{1}{2}]^{0}_{\frac{1}{2}+\sqrt{2+\chi^2}}\oplus A_2\overline{A}_2[0]_1^{(0)}\\&\oplus L\overline{B}_1[0]_2^{(2)}\oplus B_1\overline{L}[0]_2^{(-2)}\oplus 2\times L\overline{L}[0]^{(0)}_{\frac{1}{2}+\frac{1}{2}\sqrt{1+16\chi^2}}\oplus 2\times L\overline{L}[0]^{(0)}_{\frac{1}{2}+\frac{1}{2}\sqrt{17}}\,.\end{aligned}\tag{4.2.7}$$

In the $\mathrm{SU}(2)\times\mathrm{U}(1)$ symmetric vacuum identified by $\chi=0$, some of the long multiplets in (4.2.7) reach the unitarity bound and the following branching rule applies

$$L\overline{L}[0]^{(0)}_{\frac{1}{2}+\frac{1}{2}\sqrt{1+16\chi^2}}\ \xrightarrow{\chi\to 0}\ A_2\overline{A}_2[0]_1^{(0)}\oplus L\overline{B}_1[0]_2^{(2)}\oplus B_1\overline{L}[0]_2^{(-2)}\,. \tag{4.2.8}$$

The resulting shortened multiplets join their copies in (4.2.7) to combine into an $\mathrm{SU}(2)$ vector. In particular, two massive vectors become massless and join into the gauge vectors of the enhanced $\mathrm{SU}(2)$ symmetry. Nothing more can be said about the symmetry of the latter vacua and the properties of χ. As in the $\mathcal{N}=3$ example in order to have more insight into these solutions a higher dimensional point of view is useful if not necessary. This is when the present example becomes more interesting. Indeed, the model originates from

[3] Here all masses are normalized in units of $1/L=\sqrt{-V_0/3}=g$.

[4] We refer to appendix E.2 for notation and details on these multiplets.

the gauging of the maximal model which, at the ungauged level originates from an effective description of string theory. As already anticipated, the dyonic gauging just described can be uplifted to type IIA/B models. The explicit uplift is provided by a generalized Shecrk-Schwarz reduction of $E_{7(7)}$ ExFT. Let us present these ingredients.

4.3 ExFT Framework

ExFT [46][31][4][32][47][48] is a reformulation of 10-/11-dimensional supergravity in fully covariant framework respect to the exceptional groups $E_{d(d)}$, $d = 6,7,8,9$. In particular, it allows us to explicitly describe the action of the latter duality symmetry from a higher dimensional point of view on the supergravity fields. As we will briefly review in a specific case, this can be exploited in order to formulate a wide class of consistent truncation of the higher dimensional supergravities to lower dimensional ones.

We discuss the case $d = 7$. Indeed, this is the framework we can use to describe the S-fold configurations presented in next chapters. $E_{7(7)}$-ExFT is defined on an extended spacetime spanned by the four-dimensional coordinates x^μ, $\mu = 0,1,2,3$, and 56 internal ones $Y^{\mathtt{M}}$, $\mathtt{M} = 1,\dots,56$, in the representation **56** of $E_{7(7)}$. So that the bosonic field content of the model

$$\left\{\widehat{g}_{\mu\nu}, \widehat{\mathcal{M}}_{\mathtt{MN}}, \widehat{A}_\mu{}^{\mathtt{M}}, \widehat{B}_{\mu\nu\mathtt{a}}, \widehat{B}_{\mu\nu\mathtt{M}}\right\}, \quad \begin{aligned} &\mu = 0,\dots,3\,, \\ &\mathtt{M} = 1,\dots,56\,, \\ &\mathtt{a} = 1,\dots,133\,, \end{aligned} \tag{4.3.1}$$

will in general depend on both x^μ and $Y^{\mathtt{M}}$. In particular $\widehat{g}_{\mu\nu}$ and $\widehat{\mathcal{M}}_{\mathtt{MN}}$ describe an internal and external metric respectively, with the latter parametrizing the coset space $E_{7(7)}/SU(8)$. The fields, $\mathcal{A}_\mu{}^{\mathtt{M}}$, are vectors from a four-dimensional point of view and they also transform in the **56** of the group $E_{7(7)}$. The remaining fields are 2-forms, from the internal spacetime point of view and they transform in the adjoint and fundamental representation of $E_{7(7)}$ respectively. In analogy to the local diffeomorphism invariance of supergravity theories, ExFT exhibits local invariance under the action of generalized diffeomorphisms. The latter extends the former symmetry to the generalized $(4+56)$ dimensional spacetime. The action of this generalized local symmetry is conveniently described, in a $E_{7(7)}$ covariant fashion, in terms of generalized Lie derivatives as explicitly defined in [4]. The so-called "section constraints" give a necessary requirement in order to actually realize these transformations as a symmetry of the ExFT model. In particular one has that the action of the operators

$t_{\mathtt{a}}^{\mathtt{MN}}\partial_{\mathtt{M}}\partial_{\mathtt{N}}$, $t_{\mathtt{a}}^{\mathtt{MN}}\partial_{\mathtt{M}}\otimes\partial_{\mathtt{N}}$ and $\mathbb{C}^{\mathtt{MN}}\partial_{\mathtt{M}}\otimes\partial_{\mathtt{N}}$ must vanish on all fields. In the latter expressions, $t_{\mathtt{a}}^{\mathtt{MN}}$ stands for the generators describing the $\mathrm{E}_{7(7)}$ action in the fundamental representation. The first relevant solution to the above constraints, also called "section constraints", is given by restricting the $Y^{\mathtt{M}}$ dependence of all fields to the seven ones with the maximum grading with respect to the GL(1) factor of the maximal $\mathrm{GL}(7)\subset\mathrm{E}_{7(7)}$. When this condition is imposed one correctly embeds $D=11$ supergravity in $\mathrm{E}_{7(7)}$ ExFT. Another relevant solution to the section constraints is given by restricting the external dependence to the six coordinates with maximum grading respect to the GL(1) factor of the maximal $\mathrm{GL}(6)\times\mathrm{SL}(2,\mathbb{R})\subset\mathrm{E}_{7(7)}$. This latter condition allows to embed type IIB supergravity in $\mathrm{E}_{7(7)}$ ExFT [5].

The dynamics of the bosonic fields is derived from the action principle

$$\mathcal{S}_{\mathrm{ExFT}}=\int d^4x d^{56}Y\sqrt{|\hat{g}|}\left(\hat{R}+\frac{1}{48}\hat{g}^{\mu\nu}\widehat{\mathscr{D}}_{\mu}\widehat{\mathcal{M}}^{\mathtt{MN}}\widehat{\mathscr{D}}_{\nu}\widehat{\mathcal{M}}_{\mathtt{MN}}-\frac{1}{8}\widehat{\mathcal{M}}_{\mathtt{MN}}\widehat{F}^{\mathtt{M}}_{\mu\nu}\widehat{F}^{\mathtt{N}\mu\nu}-V(\widehat{\mathcal{M}},\hat{g})+\ldots\right). \tag{4.3.2}$$

Here the dots stand for topological terms. In addition to this, the first-order twisted self-duality constraint

$$\widehat{F}^{\mathtt{M}}_{\mu\nu}=-\frac{1}{2}\sqrt{|\hat{g}|}\epsilon_{\mu\nu\rho\sigma}\mathbb{C}^{\mathtt{MN}}\widehat{\mathcal{M}}_{\mathtt{NP}}\widehat{F}^{\mathtt{P}}_{\mu\nu} \tag{4.3.3}$$

must be imposed. The Lagrangian and the self-duality constraint in the above expressions are reminiscent of the bosonic part of Lagrangian (2.2.12) and condition (3.2.5) for a $D=4$ $\mathcal{N}=8$ supergravity model. Indeed, the ExFT Lagrangian is inspired by the latter, however the ExFT theory is formulated on a $(4+56)$ dimensional spacetime so that all the ingredients in (4.3.2) depend on all the extended coordinates and they are a generalization of the usual fields one defines for a supergravity theory as explained in [4]. Nonetheless, the resemblance becomes explicit when the section constraints are imposed so that the ExFT equations of motion become equivalent to those of $D=11$ or type IIB supergravity [31][4]. The bosonic action (4.3.2) admits a supersymmetric generalization where supersymmetry can be considered an extension of generalized diffeomorphisms and can be interpreted as local symmetries on a suitable superspace [49][50].

The bosonic formulation is sufficient for our discussion since we are interested in (supersymmetric) bosonic solutions of type IIB supergravity. This is a crucial step in which ExFT becomes technically relevant. Indeed, the solutions we will

[5] Later on, a slightly modified version of this solution will be considered. While the present one is useful for describing electric gaugings, the one presented in [41] is relevant in the case of dyonic gaugings.

discuss are first obtained as solutions of a dynonically gauged supergravity, of the kind described in section 4.1.2, and then uplifted to the higher dimensional theory. This is possible thanks to the property of the latter models of being a consistent truncation of a spontaneous compactification of type IIB supergravity, as shown in [41]. The proof is based on the ExFT formulation of type IIB supergravity. The latter is a natural framework for studying Kaluza-Klein reductions and the full spectrum of the Kaluza-Klein modes [5][34][6]. In particular, once the section constraints are imposed, it naturally describes the ten-dimensional theory with a $4+6$ decomposition of the spacetime coordinates. Furthermore, in the ExFT formulation, the higher-dimensional models are expressed in a gauged supergravity fashion allowing for a direct comparison between the lower-dimensional effective descriptions. Let us quickly describe the latter techniques and how to apply them to the model of interest.

4.3.1 Generalized Sherck-Schwarz Reductions in ExFT

The main lesson to learn from the standard Scherck-Schwarz reduction is that one can generalize the ordinary dimensional reduction giving rise to toroidal compactifications to more interesting internal manifolds by exploiting the symmetries of the original higher dimensional theory [51]. In the first formulation of the technique it was shown that in a theory invariant under local diffeomorphisms, such as (super)gravity theory, one can perform different consistent truncations depending on the subset of the original local symmetries one can effectively describe in the resulting reduced model. The simplest reduction is obtained by dropping the internal coordinates dependence on the local diffeomorphisms resulting in the standard toroidal compactification.

However one can try a more general choice for the local parameter of the symmetries of the form

$$\lambda^\mu(x^\mu, y^i) = \lambda^\mu(x^\mu),\ \lambda^i(x^\mu, y^m) = u^{-1}(y^m)^i{}_j \lambda^j(x^\mu) \tag{4.3.4}$$

where, on top of the external coordinates x^μ dependence, an internal y^m, $m = 1, \dots, n$ dependence is allowed for the internal directions. While the toroidal reduction produces a theory with a $\mathrm{U}(1)^n$ local invariance associated with constant translations along the original internal direction, the above choice allows for a non abelian gauge symmetry for the effective model. Indeed, consistency of (4.3.4) under the usual commutator of two local symmetries results in the condition

$$2\, u^{-1 i}{}_j u^{-1 k}{}_l \partial_{[i} u^p{}_{k]} = f_{jl}{}^p \tag{4.3.5}$$

where ${f_{jl}}^p$ are structure constants for a suitable Lie algebra so that the internal dependence in the above expression drops out. The latter algebra will coincide with the resulting gauge algebra. In this way one can see that, when applying this procedure to supergravity theories, one can recover lower dimensional ungauged supergravities as toroidal compactifications of higher dimensional ones while some lower dimensional gauged models can be described by the above Scherck-Schwarz ansatz. If this is the case, once a proper electric frame is chosen, the structure constant can be related to the X-tensor. This is natural since they are both defining the gauge algebra of the model. The resulting model can be seen as originating from a compactification on an internal manifold which admits as symmetry the action of the gauge algebra under consideration such as in the case (but not exclusively) of group manifolds of dimension n. In particular, the matrix $u(y)$ can be chosen so as to describe the action of the gauge symmetry on the original internal space by means of the differential operator[6] $L_i = {u^{-1}}^j{}_i \partial_i$. From the ansatz (4.3.4), the ansatz for the fields present in the theory follows from their representation respect to the internal direction of the local symmetry. As an example, the Kaluza-Klein vectors A^i_μ and scalars originating from the metric decomposition in the external and internal components will have the ansatz

$$A(x,y)^i_\mu = {u(y)^{-1}}^i{}_j A(x)^j_\mu \tag{4.3.6}$$

since they are vectors under the action of internal symmetries. As another example, the external part of the metric $g(x,y)_{\mu\nu}$ will simply drop the internal dependence since it is a scalar under the action internal symmetries.

One can apply the same logic to ExFT. In this case the local symmetry is given by generalized diffeomorphisms capturing both the usual local symmetries of a gravity theory in four dimensions and local symmetries along the external $Y^{\mathtt{M}}$ directions. By using the fully $\mathrm{E}_{7(7)}$ covariant formulation of ExFT one can generalize the procedure of the standard Scherk-Shwarz reduction to more general ansatz [5]. Furthermore, because of its strong relation to $\mathcal{N}=8$ $D=4$ gauged supergravity, it provides an explicit description of those gaugings originating from such generalized reduction. In this case, the relevant choice of internal parameters is given by [7]

$$\Lambda(x,Y)^{\mathtt{M}} = \rho^{-1}(Y) U(Y)^{-1\mathtt{M}}_{N} \Lambda^N(x) \tag{4.3.7}$$

[6]More precisely, in (4.3.4) one should distinguish between the left and right index of u^{-1} since they refer to a local symmetry of the original model and to a global symmetry of the reduced model respectively. Here, instead both indices refer to the complete higher dimensional model. As matrices, in both cases they have the same entries.

[7]Here we distinguish between the $\mathtt{M}$ better suited for the quantities acted upon by generalized diffeomorphisms and the global N $\mathrm{E}_{7(7)}$ indices.

where U and ρ, defining the so-called "twist matrix", are elements of $E_{7(7)}$ and $\mathbb{R}^+$ respectively. Condition (4.3.7) is analogous to equation (4.3.5). The analogy can be understood in the following way. The standard case exploits the symmetry related to the tangent bundle of a differential manifold of dimension n with has in general $GL(n)$ as structure group. On the other hand, generalized diffeomorphisms can be described in the framework of "Generalized Geometry"[52][53] as objects living in a generalized tangent bundle with $E_{7(7)} \times \mathbb{R}^+$ as structure group. The latter will describe the global symmetries of the reduced equation of motion. In particular, $E_{7(7)}$ is associated with the global duality group encoded in the isometries of the scalar manifold $E_{7(7)}/SU(8)$ while the $\mathbb{R}^+$ factor is associated to the so-called "trombone" scaling symmetry displayed by the equation of motion of the ungauged maximal supergravity model [54].

Consistency of the generalized diffeomorphisms imposes

$$\left[U(Y)_N^{-1\mathtt{M}} U(Y)_P^{-1\mathtt{Q}} \partial_{\mathtt{M}} U(Y)_{\mathtt{Q}}{}^R\right]_{\mathbf{912}} = \frac{1}{7}\rho(Y)\,\Theta_N{}^\alpha t_{\alpha P}{}^R = \frac{1}{7}\rho(Y) X_{NP}{}^R \quad (4.3.8)$$

which relates the U matrix to the embedding tensor in the **912** of $E_{7(7)}$ describing the gaugings of $E_{7(7)}$ electric generators in the effective model captured by the generalized Scherck-Schwarz reduction. By the same token one has to require

$$\partial_{\mathtt{M}} U(Y)_N^{-1\mathtt{M}} - 3\rho^{-1}\partial_{\mathtt{M}}\rho\, U(Y)_N^{-1\mathtt{M}} = 2\rho\,\theta_N \quad (4.3.9)$$

providing the description of a component of the embedding tensor in the **56** of $E_{7(7)}$. The latter describes gaugings of the trombone symmetry. However, as shown in the latter studies, models with a gauged trombone symmetry do not have a Lagrangian description [55][56]. The ansatz for describing effectively local four-dimensional diffeomorphisms and global $E_{7(7)}$ duality symmetries translates in the following ansatz for the bosonic fields of ExFT

$$\begin{aligned}
\widehat{g}(x,Y)_{\mu\nu} &= \rho^{-2}(Y) g_{\mu\nu}(x) \\
\widehat{A}(x,Y)_\mu^{\mathtt{M}} &= \rho(Y)^{-1} A(x)_\mu^N U(Y)_N^{-1\mathtt{M}} \\
\widehat{B}(x,Y)_{\mu\nu\mathtt{a}} &= \rho(Y)^{-2} U(Y)_{\mathtt{a}}{}^\alpha B(x)_{\mu\nu\alpha} \\
\widehat{B}_{\mu\nu\mathtt{M}} &= -2\rho(Y)^{-2} U(Y)_N^{-1\mathtt{P}} \partial_{\mathtt{M}} U(Y)_{\mathtt{P}}{}^R B_{\mu\nu\alpha} t_R^{\alpha N} \\
\widehat{\mathcal{M}}_{\mathtt{MN}} &= U(Y)_{\mathtt{M}}{}^P U(Y)_{\mathtt{N}}{}^Q \mathcal{M}_{PQ},
\end{aligned} \quad (4.3.10)$$

where $U(Y)_{\mathtt{a}}{}^{\alpha}$ denotes the U matrix in the adjoint representation of $\mathrm{E}_{7(7)}$.

With the above ansatz, the ExFT equations of motion reduce to the ones of $\mathcal{N}=8$ $D=4$ supergravity with gauging described by the embedding tensor (4.3.8) and (4.3.9). This also means that, given the equivalence between ExFT and higher dimensional supergravities, once the latter equations can be solved for a given embedding tensor of the $D=4$ theory, the Generalized Sherck-Schwarz ansatz provides the uplift of the lower dimensional model to $D=10/11$ supergravity models. In [5], by using these techniques, the $\mathrm{CSO}(p,q,r)$ gaugings described in previous sections were proven to describe consistent truncations of $D=11$ supergravity on seven-dimensional manifolds with topology $H^{p,q}\times\mathbb{R}^r$. Let us review how the formalism is applied to the dyonic gaugings with embedding tensor (4.1.7)(4.1.8).

4.3.2 The Dyonic Gaugings in the ExFT Framework

As already discussed, $\mathrm{E}_{7(7)}$ is a symmetry of the equations of motion of $D=4$ gauged supergravity but in general it is not an off-shell symmetry of the model. In the case of the $\mathrm{SL}(8,\mathbb{R})$ frame, the actual off-shell symmetry of the model is $G_e=\mathrm{SL}(8,\mathbb{R})$. It is then reasonable, as shown in [5], to restrict the U matrix to an element of $\mathrm{SL}(8)$ in order to make the connection with the $\mathrm{CSO}(p,q,r)$ and dyonic gaugings formulated in the $\mathrm{SL}(8,\mathbb{R})$ frame. The 56 internal coordinates are branched accordingly to $Y^{\mathtt{M}}=(Y^{[\mathtt{AB}]},Y_{[\mathtt{AB}]})$, transforming in the $\mathbf{28}$ and $\mathbf{28}'$ of $\mathrm{SL}(8,\mathbb{R})$ respectively. This is formally the same splitting of the fundamental representation of $\mathrm{E}_{7(7)}$ as the global symmetry group of the lower dimensional equations of motion. The U matrix, as an element of $\mathrm{SL}(8,\mathbb{R})$ diagonally embedded in $\mathrm{E}_{7(7)}$, reads

$$U_{\mathtt{M}}{}^{N}(Y)=\begin{pmatrix} U_{[\mathtt{AB}]}{}^{[CD]}(Y) & 0 \\ 0 & U^{[\mathtt{EF}]}{}_{[GH]}(Y)=U^{-1}_{[GH]}{}^{[\mathtt{EF}]}(Y)\end{pmatrix}, \tag{4.3.11}$$

where the 28×28 matrix $U_{[\mathtt{AB}]}{}^{[CD]}(Y)$ is expressed in terms of the 8×8 one $U_{\mathtt{A}}{}^{B}(Y)$, which describes the same $\mathrm{SL}(8,\mathbb{R})$ element in the representation $\mathbf{8}$, as follows:

$$U_{[\mathtt{AB}]}{}^{[CD]}=2\,U_{[\mathtt{A}}{}^{[C}\,U_{\mathtt{B}]}{}^{D]}\,. \tag{4.3.12}$$

In the case of the electric $\mathrm{CSO}(p,q,r)$ gaugings, following [5], one can consider the following explicit form for the entries of the $U=U_e$ matrix and the scale factor $\rho=\rho_e$

$$(U_e^{p,q,r})_{\mathtt{A}}{}^{B} = \begin{pmatrix} \beta^r U_{p,q} & 0 \\ 0 & \beta^{-(p+q)}\mathbf{1}_{r\times r} \end{pmatrix}, \; \rho_e = (1-v)^{\frac{1}{4}} \tag{4.3.13}$$

where the $(p+q)\times(p+q)$ matrix $U_{p,q}$ and β are given by

$$U_{p,q}^{-1} = \begin{pmatrix} (1-v)^{\frac{p+q-1}{p+q}} & \eta_{\bar{i}\bar{j}} y^{\bar{j}} (1-v)^{\frac{p+q-2}{2p+2q}} K(u,v) \\ \eta_{\bar{i}\bar{j}} y^{\bar{j}} (1-v)^{\frac{p+q-2}{2p+2q}} & (1-v)^{-\frac{1}{p+q}} \left(\delta^{\bar{i}\bar{j}} + \eta_{\bar{i}\bar{k}}\eta_{\bar{j}\bar{l}} y^{\bar{k}} y^{\bar{l}} K(u,v)\right) \end{pmatrix} \tag{4.3.14}$$

and

$$\beta = (1-v)^{-\frac{1}{(p+q)8}}, \; u = y^{\bar{i}} y^{\bar{i}}, \; v = \eta_{\bar{i}\bar{j}} y^{\bar{i}} y^{\bar{j}}. \tag{4.3.15}$$

In the above formulae, $y^{\bar{i}}$ with $\bar{i} = 1,\ldots,p+q-1$ denote a subset of the seven physical coordinates chosen among the $Y^{\mathtt{M}}$ in agreement with the section constraints. The matrix $\eta_{\bar{i}\bar{j}}$ denotes the invariant SO$(p-1,q)$ metric with $p-1$ "+"s and q "-"s. The function $K(u,v)$ solves the differential equation

$$2(1-v)(u\partial_u K + v\partial_v K) + (u - (1+q-p)(1-v)) + 1 = 0. \tag{4.3.16}$$

As proven in [5], once the ansatz (4.3.13) is implemented and provided that $p+q+r=8$ and $D=4$ (as in our case of study), the conditions (4.3.8)(4.3.9) reduce to the above equation. In particular, the U_e matrix given above correctly reproduces, through the condition (4.3.8), the CSO(p,q,r) embedding tensor while the left-hand side of (4.3.9) identically vanishes, meaning that the trombone symmetry is not a local on-shell symmetry of the resulting model.

Building on the above results, in [41] the cases of dyonic gauge groups of the form (4.1.10) with $2 \le p+q \le 6$ have been considered[8]. The main idea in generalizing the electric results is to combine two of the U_e matrices of the electric case so that the complete U can depend on a more general set of coordinates, still compatible with the section constraints, involving the dual representation of the usual electric ones. Let us be more explicit by noticing that U_e depends only on a subset of the 7 internal physical coordinates leaving

[8] It should be clear that when we speak of electric gaugings p and q refer to the SO(p,q) factor of the gauge group while they refer to the electric SO factor of (4.1.10) in the dyonic case.

room for a more general dependence. As shown in (4.1.10), one can define the following U matrix and scale factor ρ

$$U_d(y) = U^{(1)}(y_e)U^{(2)}_m(y_m) \tag{4.3.17}$$
$$\rho_d(y) = \rho^{(1)}(y_e)\rho^{(2)}(y_m), \tag{4.3.18}$$

where $U^{(1/2)}$ are of the form of the CSO(p,q,r) case as explained below. In the above expressions the internal 7/6 physical type IIA/B y coordinates are split into "electric" and "magnetic". The former are chosen from the $Y^{[\mathtt{AB}]}$ and the latter from the $Y_{[\mathtt{AB}]}$. Their type IIA/B origin can be understood by considering the following branches of the fundamental representation of $\mathrm{E}_{7(7)}$ respect to the maximal subgroups $\mathrm{SO}(1,1)\times\mathrm{GL}(6)$ and $\mathrm{SL}(2,\mathbb{R})\times\mathrm{GL}(6)$ respectively

$$\mathbf{56} \to \mathbf{6}'_{-4}+\mathbf{1}_{-3}+\mathbf{6}_{-2}+\mathbf{15}_{-1}+\mathbf{6}_{+4}+\mathbf{1}_{+3}+\mathbf{6}'_{+2}+\mathbf{15}'_{+1}\ (IIA)$$
$$\mathbf{56} \to (\mathbf{6}',\mathbf{1})_{-4}+(\mathbf{6},\mathbf{2})_{-2}+(\mathbf{20},\mathbf{1})_{0}+(\mathbf{6},\mathbf{1})_{+4}+(\mathbf{6}',\mathbf{2})_{+2}\ (IIB).$$

In the type IIA case, the internal physical coordinates can be chosen to describe the $\mathbf{6}'_{-4}+\mathbf{1}_{-3}$ representation while in the type IIB case they can be associated to the $(\mathbf{6}',\mathbf{1})_{-4}$ representation. Which of the 7/6 coordinates is considered electric or magnetic depends on how $\mathrm{SO}(1,1)\times\mathrm{GL}(6)$ or $\mathrm{SL}(2,\mathbb{R})\times\mathrm{GL}(6)$ is embedded in $\mathrm{SL}(8,\mathbb{R})$ which in turn defines the $28+28$ splitting of the 56 generalized coordinates $Y^{\mathtt{M}}$. Once the choice is fixed, one can set

$$U^{(1)}(y_e) = U_e^{p,q,p'+q'}(y^{\bar{i}} = y_e^{\bar{j}}),\ \bar{j} = 1,\dots,p+q-1, \tag{4.3.19}$$

and

$$U^{(2)}(y_m)_{\mathtt{A}}{}^{B} = \begin{pmatrix} \beta^{(p'+q')}(y^{\bar{i}} = y_m^{\bar{a}})\mathbf{1}_{r'\times r'} & 0 \\ 0 & \beta^{-r'}U^{-T}_{p',q'}(y^{\bar{i}} = y_m^{\bar{a}}) \end{pmatrix},\ \bar{a} = 1,\dots,p'+q'-1. \tag{4.3.20}$$

The expression for the magnetic factor is essentially the same as the electric one with entries (4.3.11)(4.3.13) but with the role of the $Y^{[AB]}$and $Y_{[AB]}$ directions inverted. Moreover, the blocks in (4.3.13) are swapped so that, modulo scale factors, $U^{(1)}$ and $U^{(2)}$ act non-trivially on the electric and the magnetic directions respectively. They commute as $\mathrm{SL}(8,\mathbb{R})$ elements and they are mutually compatible in the sense of [41]. This property is analogous to the one for the matrices $\xi_{(p',q',r')}$ and $\eta_{(p,q,r)}$ given in section 4.1.2 of being complementary one to each other. The scale factor ρ is given by

$$\rho^{(1)} = \rho_e(y^{\bar{i}} = y_e^{\bar{j}}),\ \rho^{(2)} = \rho_e(y^{\bar{i}} = y_m^{\bar{a}}). \tag{4.3.21}$$

In [41], the embeddings for the cases $2 \leq p+q \leq 6$ and $p'+q' = 8-p-q$ are considered explicitly[9]. In particular, $p+q = 2,4,6$ is consistent with an ExFT description of type IIB supergravity. Instead, the other cases are consistent with the ExFT description of type IIA supergravity and the set of $(y_e^{\bar{i}}, y_m^{\bar{a}})$ coordinates, belonging to the $\mathbf{6}'_{-4}$ representation in (4.3.19), can be extended with an extra coordinate corresponding to the $\mathbf{1}_{-3}$ representation appearing in the same branching. The U_d matrix and the scale factor ρ_d do not depend on the latter, however, it is relevant to describe the model in terms of $D = 11$ supergravity. Following again [41], it is proven that U_d and ρ_d given above satisfy the conditions (4.3.8)(4.3.9). In particular, the resulting dyonic X tensor is given by the sum of the two embedding tensors identified by $U^{(1)}$ and $U^{(2)}$ respectively. It is to say an embedding tensor describing an electric CSO(p,q,r) gauging combined with a magnetic CSO(p',q',r') gauging. This allows us to interpret some of the $D = 4$ maximal models with gauge group given in (4.1.10) as originating from a consistent truncation of the generalized Scherk-Schwarz reduction described in this section. The other way around, the above construction provides an uplift to type IIA/B supergravity theories of the $D = 4$ $\mathcal{N} = 8$ gauged models with gauge group (4.1.10) and $2 \leq p+q = r' \leq 6$. In the next chapter, we will see how to apply this result to study interesting solutions in the $p = 6$, $q = 0$, $p' = 2$ case. We will also use the main example to review the ExFT technique for computing Kaluza-Klein spectra of upliftable solutions of the lower dimensional model.

[9] In this section we are using a different SL$(8,\mathbb{R})$ basis than the one in section 4.2 which is the same used in next chapters. It is clear that it is a matter of conventions

Chapter 5

Uplifting the $\mathcal{N}=2$ Solutions

To summarize the previous chapter and what we are going to discuss, we have introduced gaugings in four dimensions, which involve, in a standard symplectic frame, magnetic components of the embedding tensor (*dyonic gaugings*) [39, 57, 40]. While some of these models are constructed by gauging a same simple gauge group of the form $\mathrm{SO}(p,q)$ in different frames, others involve non-semisimple gauge groups and have the general form $[\mathrm{SO}(p,q)\times\mathrm{SO}(p',q')]\ltimes N$, with N being a subgroup generated by nilpotent generators. The dyonic nature of the latter gaugings (i.e. the non-vanishing magnetic components of the embedding tensor) is encoded in a deformation parameter c which if non-vanishing can always be set to a fixed value by field redefinitions, e.g. $c=1$. All these gaugings generalize their electric simple and semi-simple counterparts [58, 59] (the non-semisimple gaugings, for $c=0$, reduce to the electric $\mathrm{CSO}(p,q,r)$ gaugings). We have discussed how the non-semisimple dyonic gaugings can be embedded in Type II supergravity in the framework of ExFT. The latter not only provides explicit uplift formulae for the models allowing to explore interesting properties of the higher dimensional configurations. For example, the dyonic ISO(7)-model was shown to be a consistent truncation of massive Type IIA supergravity [60] on a background of the form $\mathrm{AdS}_4\times S^6$ [61–63, 45, 64, 65]. The general embedding of the models featuring non-semisimple dyonic gaugings within Type II supergravities was derived, employing the ExFT framework, in [41].

Here we are interested in describing the application of the latter results to the four-dimensional maximal supergravity with dyonic gauging

$$G_g=[\mathrm{SO}(6)\times\mathrm{SO}(1,1)]\ltimes\mathbb{R}^{12}\,, \tag{5.0.1}$$

which features AdS_4 vacua with $\mathcal{N}=0$, 1, 2 and 4 supersymmetries [66, 41–43]. Some of these were lifted to Type IIB S-folds of Janus solutions, which have a spacetime geometry of the form $\mathrm{AdS}_4\times S^1\times S^5$, with S^5 being a deformed five-sphere. These backgrounds are characterized by a monodromy $\mathfrak{M}_{S^1}$ around the non-contractible S^1 with radius $\frac{T}{2\pi}$, with $\mathfrak{M}_{S^1}$ a hyperbolic element of the $\mathrm{SL}(2,\mathbb{Z})_{\mathrm{IIB}}$ duality group. In other words, these solutions feature different local geometric descriptions patched together by a non-perturbative Type IIB S-duality transformation. They can also be constructed as suitable quotients of Janus-like solutions in Type IIB [67, 68]. The $\mathcal{N}=4$ vacuum with SO(4) residual gauge symmetry was found in [66] and uplifted to Type IIB theory in [41]. The $\mathcal{N}=0,1$ vacua were discovered in [42]. The $\mathcal{N}=0$ vacuum with symmetry SU(4) and the $\mathcal{N}=1$ one with symmetry SU(3) were uplifted, in the same work, to ten-dimensional S-folds of type IIB. In [43], a new family of $\mathcal{N}=2$ $\mathrm{U}(1)^2$ symmetric vacua was found. The vacua of this family are labeled by a continuous, non-compact parameter χ.[1] At $\chi=0$ the residual gauge symmetry is enhanced to $\mathrm{SU}(2)\times\mathrm{U}(1)$ and the type IIB uplift at this particular value was found in the same work. The corresponding S-fold solutions are conjectured to be holographically dual to interface super-Yang Mills theories in $D=3$. Interesting examples are given in [71], where a class of S-fold $\mathcal{N}=4$ $\mathrm{AdS}_4\times K_6$ solutions with compact K_6 internal manifold is given. Following the authors, these solutions can be obtained as quotients of known non-compact ones, with the quotient defined by an $\mathrm{SL}(2,\mathbb{Z})_{\mathrm{IIB}}$ action on the latter. Furthermore, by translating this procedure on the corresponding $\mathcal{N}=4$ CFT_3 Janus-type theories [72][73], they were able to find strong candidates for their SCFT_3 duals.

By employing the ExFT methods, we perform a Kaluza-Klein analysis on the $\mathrm{U}(1)^2$-symmetric $\mathcal{N}=2$ family of vacua found in [43]. We perform their uplift to Type IIB S-fold solutions of the whole 1-parameter $\mathcal{N}=2$ family. In particular, we give χ a geometrical interpretation as a 10-dimensional metric modulus. We find that the dependence on χ of the type IIB solution can be interpreted as a global twist in the internal geometry, and, in particular, involving a squashed S^3 submanifold of the deformed S^5, which is fibered over S^1. This fibration involves a non-trivial twist of the points of S^3, as we move around S^1, which depends on χ. As we shall prove, the $D=10$ S-fold solutions corresponding to the $\chi\neq 0$ vacua are locally related to the one associated with the $\chi=0$ vacuum by the above reparameterization, although globally different. In particular, χ only enters through the dependence of the fields on the point of the squashed S^3 and thus does not affect the axion-dilaton field. We are also able to relate χ with a complex structure modulus associated with the internal submanifold $S^3\times S^1$. Indeed, writing S^3 as a Hopf fibration of a circle over S^2 and combining the circular fiber with the external S^1 into a

[1] In fact this family of vacua will feature at least two moduli fields, as the conformal manifold ought to be complex. The supergravity moduli fields are expected to be a subset of the four scalar massless modes found in [43]. Recently a 2-parameter extension of the $\mathcal{N}=2$ vacua studied here was constructed in [69, 70].

2-torus T^2, the manifold $S^3\times S^1$ can be written as a toroidal fibration over S^2. We show that χ defines the real part of the modular parameter of the toroidal fiber T^2 and, due to the invariance of the complex structure of the torus under a Dehn twist, χ has period $\frac{2\pi}{T}$. All these global properties of the $D=10$ background, associated with the χ parameter, cannot be seen from the four-dimensional supergravity perspective, but are apparent from the analysis of the Kaluza-Klein spectrum of these vacua, which we perform. At the special values $\chi=\frac{\pi m}{T}$, $m\in\mathbb{Z}$, two vectors in the full KK spectrum, but outside the supergravity truncation, become massless, thus enhancing $\mathrm{U}(1)^2$ to $\mathrm{SU}(2)\times\mathrm{U}(1)'$. This corresponds to a "space invaders" scenario [74, 75].

As far as the dual 3-dimensional theory is concerned, we can still rely on the constructions put forward in [76], building on [71]. One of these possibilities involves the strong coupling regime of the T[U(N)] theory by Gaiotto-Witten [77] in which the $\mathrm{U}(N)\times\mathrm{U}(N)$ global symmetry is gauged by a $\mathrm{U}(N)$ $\mathcal{N}=2$ vector multiplet, to preserve $\mathcal{N}=2$ supersymmetry in the IR limit. The parameter χ would parameterize a further exactly marginal deformation of this $\mathcal{N}=2$ model, thus defining a direction in the conformal manifold of the dual theory. Our analysis, unveiling the compact nature of χ, sheds some light on the global properties of the conformal manifold.

5.1 Embedding the Model in ExFT

In this section, we shall specialize the discussion of the previous chapter on the framework of $\mathrm{E}_{7(7)}$-exceptional field theory (ExFT) [4] to uplift the one-parameter $\mathcal{N}=2$ family of vacua to $D=10$ backgrounds of Type IIB supergravity. The $D=4$ supergravity vacua under discussion have non-vanishing gravitational and scalar fields. To perform the uplift of the vacua, we only need the fields $\widehat{g}_{\mu\nu}(x,Y)$ and the generalized metric $\widehat{\mathcal{M}}_{\mathtt{MN}}(x,Y)$ of the theory, the vector and tensor fields being consistently set to zero. These fields are related to their counterparts $g_{\mu\nu}(x)$, $\mathcal{M}_{MN}(\phi(x))$ of the four-dimensional supergravity as described in section 4.3.1, using the generalized Scherk-Schwarz ansatz [5]:

$$
\begin{aligned}
\widehat{g}_{\mu\nu}(x,Y) &= \rho(Y)^{-2}\,g_{\mu\nu}(x)\,,\\
\widehat{\mathcal{M}}_{\mathtt{MN}}(x,Y) &= U_{\mathtt{M}}{}^{K}(Y)\,U_{\mathtt{N}}{}^{L}(Y)\,\mathcal{M}_{KL}(\phi(x))\,.
\end{aligned}
\tag{5.1.1}
$$

The relationship between $U_M{}^N(Y)$ and the constant embedding tensor $X_{MN}{}^P$ (4.2.2) in the four-dimensional theory is:

$$
U^{-1}{}_{M}{}^{\mathtt{R}}U^{-1}{}_{N}{}^{\mathtt{Q}}\partial_{\mathtt{R}}U_{\mathtt{Q}}{}^{P}\Big|_{\mathbf{912}} = \frac{\rho}{7}\,X_{MN}{}^{P}\,,
\tag{5.1.2}
$$

with the scalar function $\rho = \rho(Y)$ from (5.1.1). Equivalently, this condition is expressed as [78]

$$\mathcal{L}_{\mathcal{U}_M}\mathcal{U}_N = X_{MN}{}^P\,\mathcal{U}_P\,, \tag{5.1.3}$$

via the action of generalized diffeomorphisms, where the $\mathcal{U}_M$ denote the generalized (56-dimensional) vectors

$$(\mathcal{U}_M)^{\mathtt{N}} = \rho^{-1}(U^{-1})_M{}^{\mathtt{N}}\,, \tag{5.1.4}$$

given by the columns of the inverse twist matrix. In our case, the twist matrix $U_{\mathtt{M}}{}^N(Y)$ is an element of the $\mathrm{SL}(8,\mathbb{R})$ subgroup of $\mathrm{E}_{7(7)}$ and in a suitable $\mathrm{SL}(8,\mathbb{R})$ basis it is diagonally embedded as in (4.3.11). To embed Type IIB supergravity in ExFT we need the branching of the relevant $\mathrm{E}_{7(7)}$ representations with respect to the subgroup $\mathrm{SL}(6,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})_{\mathrm{IIB}}\times\mathrm{SO}(1,1)$, where $\mathrm{SL}(2,\mathbb{R})_{\mathrm{IIB}}$ is the global symmetry group of the Type-IIB:

$$\mathbf{56} \to (\mathbf{6}',\mathbf{1})_{-2} + (\mathbf{6},\mathbf{2})_{-1} + (\mathbf{20},\mathbf{1})_0 + (\mathbf{6}',\mathbf{2})_{+1} + (\mathbf{6},\mathbf{1})_{+2}\,, \tag{5.1.5}$$

the subscript being the $\mathrm{SO}(1,1)$-grading. Correspondingly $Y^{\mathtt{M}}$ splits as follows:

$$Y^M \to y^m\,,\; y_{\widehat{\alpha}m}\,,\; y_{mnp}\,,\; y^{\widehat{\alpha}m}\,,\; y_m\,, \tag{5.1.6}$$

where $m,n,p = 1,\dots,6$ and $\widehat{\alpha} = 1,2$ labels the components of an $\mathrm{SL}(2,\mathbb{R})_{\mathrm{IIB}}$ doublet. Restricting the ExFT fields to the y^m coordinates only, the section constraints are satisfied and the field equations of ExFT reduce to those of Type IIB supergravity. To identify the above components of $Y^{\mathtt{M}}$ with the components of the same vector in the basis $Y^{[AB]}$, $Y_{[AB]}$ it is necessary to further split the $\mathrm{SL}(6,\mathbb{R})$ representations with respect to its $\mathrm{SL}(5,\mathbb{R})\times\mathrm{SO}(1,1)$ subgroup, so that $y^i = y^i_e$, $i = 1,\dots,5$, are identified with $Y^{[i8]}$ while $y^6 = y^6_m$, to be denoted by $\tilde{y}$, is identified with $Y_{[67]}$, and we can write $(y^m) = (y^i,\tilde{y})$ so that the internal coordinates are split in "electric" and "magnetic" directions respectively[2]. We refer to [41] and [42, 43] for the detailed correspondence between the quantities in the decomposition (5.1.6) and the components $Y^{[\mathtt{AB}]}$, $Y_{[\mathtt{AB}]}$.[3] The explicit

[2]Below, the subscript 'e' will drop out from the electric coordinates.

[3]As opposed to the notations used in [43], here we label by an upper (or lower) index m a vector transforming in the $\mathbf{6}'$ (or $\mathbf{6}$) representation of $\mathrm{SL}(6,\mathbb{R})$.

form of (the inverse of) $U_{\mathtt{A}}{}^{B}(y^m)$ is given by [41].

$$\sqrt{\frac{\rho_e}{\rho_m}}(U^{-1})_A{}^{\mathtt{B}} = \begin{array}{cc} & \begin{array}{cc} 5 & 3 \end{array} \\ \begin{array}{c} 5 \\ 3 \end{array} & \left(\begin{array}{c|ccc} \delta^{ij}+\eta_{ik}\eta_{jl}y^k y^l K_e(u,v) & 0 & 0 & \rho_e^2\eta_{ij}y^j \\ \hline 0 & 1 & \rho_m^{-2}K_m(\tilde{u},\tilde{v})\,\tilde{y} & 0 \\ 0 & \rho_m^{-2}\,\tilde{y} & \rho_m^{-4}(1+K_m(\tilde{u},\tilde{v})\tilde{u}) & 0 \\ \rho_e^2\eta_{ij}y^j K_e(u,v) & 0 & 0 & \rho_e^4 \end{array}\right) \end{array} .$$

The latter expression is the same as the one given in (4.3.18) and (4.3.19) (4.3.20) with $p=6$, $q=0$, $p'=1$, $q'=1$ but in a slightly different SL(8,$\mathbb{R}$) basis. To express the components of the matrix $\widehat{\mathcal{M}}_{\mathtt{MN}}(x,Y)$ in terms of $D=10$ fields we further need the decomposition of the **133** of $\mathrm{E}_{7(7)}$, which branches as follows

$$\mathbf{133}\to(\mathbf{1},\mathbf{2})_{+3}+(\mathbf{15}',\mathbf{1})_{+2}+(\mathbf{15},\mathbf{2})_{+1}+(\mathbf{35}+\mathbf{1},\mathbf{1})_0+(\mathbf{1},\mathbf{3})_0+(\mathbf{15}',\mathbf{2})_{-1}+(\mathbf{15},\mathbf{1})_{-2}+(\mathbf{1},\mathbf{2})_{-3}\,, \tag{5.1.7}$$

with the $\mathrm{E}_{7(7)}$ generators splitting accordingly into

$$\left\{t^{\widehat{\alpha}}\,,\ t^{mnpq}\,,\ t^{\widehat{\alpha}mn}\,,\ t^m{}_n\,,\ t^{\widehat{\alpha}}{}_{\widehat{\beta}}\,,\ t_{\widehat{\alpha}mn}\,,\ t_{mnpq}\,,\ t_{\widehat{\alpha}}\right\}. \tag{5.1.8}$$

Next we write $\widehat{\mathcal{M}}_{\mathtt{MN}}(x,Y)$ in (5.1.1) as

$$\widehat{\mathcal{M}}(x,Y)=\mathcal{V}_{\mathrm{IIB}}(x,y)\cdot\mathcal{V}_{\mathrm{IIB}}(x,Y)^T\,,$$

where $\mathcal{V}_{\mathrm{IIB}}(x,y)$ is a representative of the coset $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ in the solvable gauge which is appropriate to the Type IIB theory [79, 80]:

$$\mathcal{V}_{\mathrm{IIB}}(x,y)=e^{t^{\widehat{\alpha}}B_{\widehat{\alpha}}}\cdot e^{\frac{1}{24}t^{mnpq}C_{mnpq}}\cdot e^{\frac{1}{2}t^{\widehat{\alpha}mn}B_{\widehat{\alpha}mn}}\cdot\mathcal{V}_2\cdot\mathcal{V}_6\,, \tag{5.1.9}$$

where $B_{\widehat{\alpha}}$ are the scalars dual in $D=4$ to $B_{\widehat{\alpha}\mu\nu}$, C_{mnpq} are the internal components of the 4-form, $B_{\widehat{\alpha}mn}$ are the internal components of the 2-forms, $\mathcal{V}_6$ is the representative of GL(6,$\mathbb{R}$)/SO(6) and $\mathcal{V}_2$ that of $\mathrm{SL}(2,\mathbb{R})_{\mathrm{IIB}}/\mathrm{SO}(2)$, depending on the $D=10$ axion C_0 and dilaton ϕ fields. In our notations the doublet of ten dimensional 2-forms $B^{\widehat{\alpha}}_{(2)}$ is defined in terms of the NS-NS and R-R fields $B_{(2)}, C_{(2)}$ as follows: $B^{\widehat{\alpha}}_{(2)}=\epsilon^{\widehat{\alpha}\widehat{\beta}}B_{\widehat{\beta}\,(2)}=(B_{(2)},C_{(2)})$.[4]. After having computed the matrix $\widehat{\mathcal{M}}(x,Y)$ on the $\mathcal{N}=2$ vacua, the internal metric $G_{mn}(y)$, the internal components of the 2-forms $B^{\widehat{\alpha}}_{mn}=\epsilon^{\widehat{\alpha}\widehat{\beta}}B_{\widehat{\beta}mn}$, and the internal

[4] In our conventions $\epsilon_{12}=\epsilon^{12}=+1$

components of the 4-form C_{mnpq} in the $D=10$ solution, can be computed as follows [41–43]:

$$\begin{aligned}
G^{mn} &= G^{\frac{1}{2}}\widehat{\mathcal{M}}^{mn} \\
B_{mn}^{\widehat{\alpha}} &= G^{\frac{1}{2}}G_{mp}\epsilon^{\widehat{\alpha}\widehat{\beta}}\widehat{\mathcal{M}}^{p}{}_{\beta n} \\
C_{mnpq}-\frac{3}{2}\epsilon_{\widehat{\alpha}\widehat{\beta}}B_{m[n}^{\widehat{\alpha}}B_{pq]}^{\widehat{\beta}} &= -G^{\frac{1}{2}}G_{mr}\widehat{\mathcal{M}}^{r}{}_{npq} \\
m_{\widehat{\alpha}\widehat{\beta}} &= \frac{1}{6}G\left(\widehat{\mathcal{M}}^{mn}\widehat{\mathcal{M}}_{\widehat{\alpha}m\,\widehat{\beta}m}+\widehat{\mathcal{M}}^{m}{}_{\widehat{\alpha}n}\widehat{\mathcal{M}}^{n}_{\widehat{\beta}m}\right)
\end{aligned} \tag{5.1.10}$$

where $G\equiv\det(G_{mn})$. The matrix $m_{\widehat{\alpha}\widehat{\beta}}$ is an element of $\mathrm{SL}(2,\mathbb{R})_{\mathrm{IIB}}/\mathrm{SO}(2)$ and is defined as:

$$m_{\widehat{\alpha}\widehat{\beta}}\equiv(\mathcal{V}_2\cdot\mathcal{V}_2^t)_{\widehat{\alpha}\widehat{\beta}}=\frac{1}{\mathrm{Im}(\tau)}\begin{pmatrix}|\tau|^2 & -\mathrm{Re}(\tau)\\ -\mathrm{Re}(\tau) & 1\end{pmatrix}, \tag{5.1.11}$$

where $\tau\equiv C_0+i\,e^{-\phi}$. In the next section, we shall perform the Kaluza-Klein analysis on the $\mathcal{N}=2$ vacua and in section 5.3, using the above formulas, we shall give the corresponding class of one-parameter $D=10$ solutions.

5.2 The $\mathcal{N}=2$ Kaluza-Klein Spectrum from ExFT

The ExFT formulation of supergravity not only provides a powerful tool for uplifting lower-dimensional solutions but also for computing the Kaluza-Klein spectra around the resulting higher-dimensional backgrounds. The formalism has been set up in [34, 6] and here we briefly review the relevant formulas. As a general structure, the Kaluza-Klein fluctuations around such a background are expressed as a product of the modes of the consistent truncation (4.3.10) captured by the U matrix, with a complete basis of functions on the compactification manifold. In the case at hand, the basis of functions $\{\mathcal{Y}^{\varsigma}\}$ can be chosen to be a tensor product of the scalar harmonics on the round S^5 with a standard Fourier expansion on S^1. More precisely, we can use the following basis for harmonics

$$\mathcal{Y}^{\varsigma}=\left\{\mathcal{Y}^{\sigma}\otimes\mathcal{Y}^{(n)}\right\}, \tag{5.2.1}$$

where

$$\mathcal{Y}^{\sigma}=\{\mathcal{Y}^{a},\mathcal{Y}^{a_1a_2},\ldots,\mathcal{Y}^{a_1\ldots a_n},\ldots\}\,,\qquad a_i=1,\ldots,6, \tag{5.2.2}$$

are the sphere harmonics on S^5 constructed as traceless symmetric products $\mathcal{Y}^{a_1\dots a_n}=\mathcal{Y}^{((a_1}\dots\mathcal{Y}^{a_n))}$, in terms of the fundamental harmonics, $\mathcal{Y}^a$, on S^5, which satisfy $\mathcal{Y}^a\mathcal{Y}^a=1$, and

$$\mathcal{Y}^{(n)}=\exp\left(\frac{2\pi i n}{T}\eta\right), \tag{5.2.3}$$

are the S^1 harmonics with periodicity $\eta=\eta+T$ of the S^1 coordinate η. The harmonics are related to the twist matrices from (4.3.10) by a linear action of generalized diffeomorphisms

$$\mathcal{L}_{\mathcal{U}_M}\mathcal{Y}^{\varsigma}=-\mathcal{T}_M{}^{\varsigma}{}_{\varphi}\,\mathcal{Y}^{\varphi}, \tag{5.2.4}$$

with gauge parameters (5.1.4), see [34, 6] for details. For the following, we simply note that this relation defines a set of constant matrices $(\mathcal{T}_M)^{\varsigma}{}_{\varphi}$, satisfying the algebra

$$[\mathcal{T}_M,\mathcal{T}_N]=X_{MN}{}^{P}\,\mathcal{T}_P, \tag{5.2.5}$$

which realizes the embedding tensor $X_{MN}{}^{P}$ as structure constants. For the specific twist matrix $U_{\mathtt{M}}^{\underline{N}}(Y)$ defined above, the matrices $(\mathcal{T}_M)^{\varsigma}{}_{\varphi}$ acting on the harmonics (5.2.1) have the following non-zero entries

$$\mathcal{T}_{AB}{}^{c_1\dots c_n}{}_{d_1\dots,d_n}=2n\,t_{[A}{}^{((c_1}t_{B]((d_1}\delta^{c_2}_{d_2}\dots\delta^{c_n))}_{d_n))}, \qquad \mathcal{T}^{67,(n)}{}_{(m)}=\frac{2\pi i n}{T}\,\delta^{(n)}_{(m)}, \tag{5.2.6}$$

where the matrix

$$t_A{}^{c}=\begin{cases}\delta_{A,c} & A\le 5\\ \delta_{A-2,c} & A=8\\ 0 & A=6,7\end{cases}, \tag{5.2.7}$$

takes care of the embedding of the harmonics into the basis used to define $U_{\mathtt{M}}^{\underline{N}}(Y)$. The fluctuation Ansatz of the ExFT fields (4.3.1) around an AdS$_4$ vacuum extends the Ansatz for the consistent truncation (4.3.10) and is given by [34, 6]

$$\begin{aligned}
\widehat{g}_{\mu\nu}(x,y)&=\rho^{-2}\left(g_{\mu\nu}(x)+\sum_{\varsigma}\mathcal{Y}^{\varsigma}\,h_{\mu\nu,\varsigma}(x)\right),\\
\widehat{A}_{\mu}{}^{\mathtt{M}}(x,y)&=\rho^{-1}\,(\mathring{U}^{-1})_{\underline{N}}{}^{\mathtt{M}}\sum_{\varsigma}\mathcal{Y}^{\varsigma}\,A_{\mu}{}^{\underline{N},\varsigma}(x),\\
\widehat{\mathcal{M}}_{\mathtt{MN}}(x,y)&=\mathring{U}_{\mathtt{M}}{}^{\underline{P}}\mathring{U}_{\mathtt{N}}{}^{\underline{Q}}\left(\delta_{\underline{PQ}}+\pi_{\underline{PQ},I}\sum_{\varsigma}\mathcal{Y}^{\varsigma}j_{I,\varsigma}(x)\right),
\end{aligned} \tag{5.2.8}$$

where the Kaluza-Klein fluctuations for the metric, vector fields and scalars are labeled by $h_{\mu\nu,\varsigma}(x)$, $A_{\mu}{}^{M,\varsigma}$, and $j_{I,\varsigma}\in\mathfrak{e}_{7(7)}\ominus\mathfrak{su}(8)$, respectively. The twist matrix $\mathring{U}_{\mathtt{M}}{}^{\underline{N}}$ appearing in (5.2.8) is obtained from the twist matrix from (4.3.10) upon dressing with the scalar hybrid coset representative of the four-dimensional supergravity, $\mathbb{L}_{M}{}^{\underline{N}} \in \mathrm{E}_{7(7)}/\mathrm{SU}(8)$, evaluated at the scalar configuration specifying the $\mathcal{N}=2$ vacuum as

$$\mathring{U}_{\mathtt{M}}{}^{N}(y) = U_{\mathtt{M}}{}^{P}(y)\,\mathbb{L}_{P}{}^{\underline{N}}\,. \tag{5.2.9}$$

The scalar fluctuations in (5.2.8) moreover appear under projection $\pi_{\underline{MN},I}$, with $I=1,\ldots,70$, over the non-compact $\mathrm{E}_{7(7)}$-generators resulting from the expansion of the group element $\widehat{\mathcal{M}}_{\mathtt{MN}}$ on the 70-dimensional coset space $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$. The normalization of $\pi_{\underline{MN},I}$ is not relevant since it drops out of the mass matrix when normalized relative to the scalar kinetic term. Evaluating the ExFT field equations from [4] with the fluctuation Ansatz (5.2.8) induces the mass matrices for the bosonic Kaluza-Klein spectrum which are expressed in terms of the T-tensor of the resulting gauged supergravity model $T_{\underline{MN}}{}^{\underline{P}}$ corresponding to (4.2.2), and the dressed matrices $\mathcal{T}$ from (5.2.4), (5.2.6),

$$\mathcal{T}_{\underline{M}} = (\mathbb{L}^{-1})_{\underline{M}}{}^{N}\,\mathcal{T}_{N}\,. \tag{5.2.10}$$

The mass matrices are obtained by linearizing the ExFT field equations with the fluctuation ansatz (5.2.8) [34, 6], and we give them in compact form as

$$\begin{aligned}
\mathbb{M}^{(\text{spin-2})}_{\varsigma\varphi} &= -(\mathcal{T}_{\underline{M}}\mathcal{T}_{\underline{M}})_{\varsigma\varphi}\,,\\
\mathbb{M}^{(\text{vector})}_{\underline{M}\varsigma,\underline{N}\varphi} &= (\varPi\varPi^{T})_{\underline{M}\varsigma,\underline{N}\varphi}\,,\\
\mathbb{M}^{(\text{scalar})}_{I\varsigma,J\varphi} &= \mathcal{M}^{(0)}_{IJ}\,\delta_{\varsigma\varphi} + \delta_{IJ}\,\mathbb{M}^{(\text{spin-2})}_{\varsigma\varphi} + \mathcal{N}_{IJ}{}^{\underline{M}}\,\mathcal{T}_{\underline{M},\varsigma\varphi} - \tfrac{1}{6}(\varPi^{T}\varPi)_{I\varsigma,J\varphi}\,.
\end{aligned} \tag{5.2.11}$$

The tensors appearing in these expressions are given by

$$\begin{aligned}
\varPi_{\underline{M}\varsigma,I\varphi} &= \delta_{\varsigma\varphi}\,T_{\underline{MN}}{}^{\underline{P}}\,\pi_{\underline{NP},I} - 12\,\pi_{\underline{MN},I}\,\mathcal{T}_{\underline{N}\,\varphi\varsigma}\,,\\
\mathcal{N}_{IJ}{}^{\underline{M}} &= -4\left(T_{\underline{MN}}{}^{\underline{P}} + 12\,T_{\underline{NP}}{}^{\underline{M}}\right)\pi_{\underline{NQ}}{}^{[I}\,\pi_{\underline{PQ}}{}^{J]}\,,\\
\mathcal{M}^{(0)}_{IJ} &= \tfrac{1}{7}\left(7\,T_{\underline{MR}}{}^{\underline{S}}T_{\underline{NS}}{}^{\underline{R}} + T_{\underline{MR}}{}^{\underline{S}}T_{\underline{NR}}{}^{\underline{S}} + T_{\underline{RM}}{}^{\underline{S}}T_{\underline{RN}}{}^{\underline{S}} + T_{\underline{RS}}{}^{\underline{M}}T_{\underline{RS}}{}^{\underline{N}}\right)\pi_{\underline{MQ},I}\,\pi_{\underline{NQ},J}\\
&\quad + \tfrac{2}{7}\left(T_{\underline{MC}}{}^{\underline{R}}T_{\underline{NQ}}{}^{\underline{R}} - T_{\underline{MR}}{}^{\underline{C}}T_{\underline{NR}}{}^{\underline{Q}} - T_{\underline{RM}}{}^{\underline{C}}T_{\underline{RN}}{}^{\underline{Q}}\right)\pi_{\underline{MN},I}\,\pi_{\underline{CQ},J}\,.
\end{aligned} \tag{5.2.12}$$

In particular, the matrix $\mathcal{M}^{(0)}_{IJ}$ is the mass matrix derived from the scalar potential of $D=4$ supergravity, describing the masses of the 70 scalars at the lowest Kaluza-Klein level. The corresponding mass formulas for the fermionic sector have been worked out in [81].

5.2.1 The Kaluza-Klein spectrum around the $\mathcal{N}=2$ backgrounds

Before applying the ExFT technology to the family of $\mathcal{N}=2$ vacua of interest, let us first work out to which extent the structure of the spectrum is constrained from the representation structure of the underlying supergroup OSp($\mathcal{N}|4$). The generic supermultiplet of this group is of long type

$$L\overline{L}[J]_{\Delta}^{(R)}\,, \qquad J=0,\tfrac{1}{2},1\,, \tag{5.2.13}$$

with J referring to the Lorentz spin of the highest weight state (HWS), such that its different values in (5.2.13) correspond to the long vector, gravitino, and graviton multiplets, respectively. Labels Δ, and R refer to the conformal dimensions and the $\mathrm{U}(1)_R$ R-symmetry charge of the HWS, respectively. Unitarity implies a lower bound for the conformal dimension

$$\Delta \;\geq\; 1+|R|+J\,. \tag{5.2.14}$$

When the bound is saturated, the long multiplet decomposes into shortened multiplets[5]. The presence of an S^1 factor in our backgrounds implies that all masses continuously depend on the inverse circle radius. Only at generic values of the radius, the spectrum thus necessarily assembles into long multiplets (5.2.13). At specific values of the inverse radius (and in particular for the zero modes on the circle) some of the long multiplets fall to the unitarity bound (5.2.14) and decompose into shortened multiplets. To make the results explicit, let us recall the character/partition function of the long multiplets (5.2.13), given by

$$\begin{aligned}
Z_{L\overline{L}[0]_{\Delta}^{(R)}} &= Z_0[\Delta,R] \equiv t^{\Delta}u^{R}\Big(1-\sqrt{t}\,\tfrac{\sqrt{z}}{u}\Big)\Big(1-\sqrt{t}\,\tfrac{1}{\sqrt{z}u}\Big)\Big(1-\sqrt{t}\sqrt{z}\,u\Big)\Big(1-\sqrt{t}\,\tfrac{u}{\sqrt{z}}\Big)\,,\\
Z_{L\overline{L}[\frac{1}{2}]_{\Delta}^{(R)}} &= Z_{\frac{1}{2}}[\Delta,R] \equiv -t^{\Delta}u^{R}\Big(\sqrt{z}+\tfrac{1}{\sqrt{z}}\Big)\Big(1-\sqrt{t}\,\tfrac{\sqrt{z}}{u}\Big)\Big(1-\sqrt{t}\,\tfrac{1}{\sqrt{z}u}\Big)\Big(1-\sqrt{t}\sqrt{z}\,u\Big)\Big(1-\sqrt{t}\,\tfrac{u}{\sqrt{z}}\Big)\,,\\
Z_{L\overline{L}[1]_{\Delta}^{(R)}} &= Z_1[\Delta,R] \equiv t^{\Delta}u^{R}\Big(z+1+\tfrac{1}{z}\Big)\Big(1-\sqrt{t}\,\tfrac{\sqrt{z}}{u}\Big)\Big(1-\sqrt{t}\,\tfrac{1}{\sqrt{z}u}\Big)\Big(1-\sqrt{t}\sqrt{z}\,u\Big)\Big(1-\sqrt{t}\,\tfrac{u}{\sqrt{z}}\Big)\,.
\end{aligned} \tag{5.2.15}$$

c.f. appendix E.2. Here, exponents of t, u, and z count the conformal dimension, R-charge, and Lorentz spin, respectively. Following the above discussion, the partition function for the full Kaluza-Klein spectrum can thus be written in the form

$$Z_{\mathrm{KK}} = \nu_0\, Z_0[0,0] + \nu_{1/2}\, Z_{\frac{1}{2}}[0,0] + \nu_1\, Z_1[0,0]\,, \tag{5.2.16}$$

[5]In section 3.5 the notation and details on these multiplets and the shortening patterns are discussed.

with the characters ν_0, $\nu_{1/2}$, ν_1, carrying the HWS of the long multiplets. Except for the masses, the remaining quantum numbers of the spectrum can be inferred from the fluctuation ansatz (5.2.8), upon multiplying the fields of $\mathcal{N}=8$ supergravity with the tower of scalar harmonics. To this end, let us note that the U(2) symmetry, preserved at the $\chi=0$ vacuum, is embedded into the SO(6) part of the gauge group, such that the gravitini decompose as

$$\begin{aligned} \mathbf{8}_s &\longrightarrow 2\times[0]_{+1}+2\times[0]_{-1}+2\times[\tfrac{1}{2}]_0\,, \\ \text{i.e.}\quad \nu_8 &= 2\,u+\tfrac{2}{u}+2\sqrt{x}+\tfrac{2}{\sqrt{x}}\,, \end{aligned} \tag{5.2.17}$$

where x counts the $\mathrm{U}(1)\subset\mathrm{SU}(2)$ charges. From this, the U(2) representation content of the full $\mathcal{N}=8$ supergravity multiplet can be deduced as

$$\begin{aligned} \text{graviton}: &\quad \mathbf{28}:\ \mathbf{1}\,, \\ \text{gravitini}: &\quad \mathbf{28}:\ \mathbf{8}_s\,, \\ \text{vectors}: &\quad \mathbf{28}:\ \mathbf{8}_s\wedge\mathbf{8}_s\,, \\ \text{spin-}\tfrac{1}{2}\text{ fermions}: &\quad \mathbf{56}:\ \mathbf{8}_s\wedge\mathbf{8}_s\wedge\mathbf{8}_s\,, \\ \text{scalars}: &\quad \mathbf{70}:\ \mathbf{8}_s\wedge\mathbf{8}_s\wedge\mathbf{8}_s\wedge\mathbf{8}_s\,. \end{aligned} \tag{5.2.18}$$

The S^5 sphere harmonics in turn decompose as

$$\begin{aligned} \mathbf{6} &\longrightarrow 2\times[0]_0+[\tfrac{1}{2}]_{+1}+[\tfrac{1}{2}]_{-1}\,, \\ \text{i.e.}\quad \nu_6 &= 2+\sqrt{x}\,u+\tfrac{u}{\sqrt{x}}+\tfrac{1}{\sqrt{x}u}+\tfrac{\sqrt{x}}{u}\,, \end{aligned} \tag{5.2.19}$$

under U(2). The full Kaluza-Klein spectrum then is obtained by multiplying (5.2.18) with the symmetric tower of S^5 harmonics (5.2.19) and the tower of S^1 harmonics, the latter amounting to a standard Fourier expansion. Comparing the result to the general form (5.2.16), we may read off the characters ν_J except for the conformal dimensions, i.e. setting $t=1$, and find

$$\begin{aligned} \nu_0\Big|_{t=1} &= \frac{(1-q^2)\left(x+3+\frac{1}{x}\right)}{(1-q)^2\left(1-q\,\frac{\sqrt{x}}{u}\right)\left(1-q\,\frac{1}{\sqrt{x}u}\right)\left(1-q\sqrt{x}\,u\right)\left(1-q\,\frac{u}{\sqrt{x}}\right)}\,\frac{1+s}{1-s}\,, \\ \nu_{1/2}\Big|_{t=1} &= \frac{2\,(1-q^2)\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)}{(1-q)^2\left(1-q\,\frac{\sqrt{x}}{u}\right)\left(1-q\,\frac{1}{\sqrt{x}u}\right)\left(1-q\sqrt{x}\,u\right)\left(1-q\,\frac{u}{\sqrt{x}}\right)}\,\frac{1+s}{1-s}\,, \\ \nu_1\Big|_{t=1} &= \frac{1-q^2}{(1-q)^2\left(1-q\,\frac{\sqrt{x}}{u}\right)\left(1-q\,\frac{1}{\sqrt{x}u}\right)\left(1-q\sqrt{x}\,u\right)\left(1-q\,\frac{u}{\sqrt{x}}\right)}\,\frac{1+s}{1-s}\,. \end{aligned} \tag{5.2.20}$$

Here, exponents of q, s, count levels for the S^5 and the S^1 harmonics, respectively. The S^1 factor $\frac{1+s}{1-s}$ simply encodes the fact that at S^1–level $n>0$ the harmonics (Fourier modes) are complex. Representation theory alone thus determines the Kaluza-Klein spectrum to be of the form (5.2.16), (5.2.20). The last and central information which completes this spectrum is the assignment of conformal dimensions/masses to all the states. It is at this step, that the ExFT technology described in the previous subsection becomes relevant. After evaluating the mass matrices (5.2.11) for the spin-2, the vector and the scalar fields, respectively, we can extract a general formula for the conformal dimensions $\varDelta$ of the HWS of the supermultiplets, counted by (5.2.20), as

$$\begin{gathered}\varDelta=\tfrac{1}{2}+\sqrt{\tfrac{17}{4}+\tfrac{1}{2}R^2-J(J+1)-2k(k+1)+\ell(\ell+4)+4\left(\tfrac{\pi n}{T}-j\chi\right)^2}\\ \text{for a HWS of type }\; q^\ell\, s^n\, u^R\, x^j\, z^J\;\text{ and SU(2) spin } k\,.\end{gathered}\tag{5.2.21}$$

The conformal dimensions inside the multiplets then follow from the multiplet structure (5.2.15). Combining (5.2.16), (5.2.20), and (5.2.21) thus produces the full Kaluza-Klein spectrum. We stress again, that for some multiplets, the conformal dimensions determined from (5.2.21) may saturate the unitarity bound (5.2.14), such that the corresponding long multiplets appearing in the expansion (5.2.16) split into shortened multiplets. The mass formula (5.2.21) explicitly shows that a non-vanishing $\chi\neq 0$ breaks SU(2) by terms proportional to the U(1) $\subset$ SU(2) charge. Moreover, it exhibits an interesting interplay between the χ-dependence and the S^1–level n: all masses receive correction terms proportional to

$$\left(\tfrac{\pi n}{T}-j\chi\right)^2\,.\tag{5.2.22}$$

In particular, this allows us to deduce that the full mass spectrum is mapped onto itself under shifts $\chi\to\chi+\frac{2\pi}{T}$. Indeed, upon switching on χ, the SU(2) representations at a given S^1–level n break up into their U(1) constituents which then at $\chi=\frac{2\pi}{T}$ recombine (over various levels) into a copy of the original SU(2) representations. More precisely, a state of SU(2) spin k at level n and generic value of the deformation parameter χ breaks up into the $2k+1$ states of U(1) charge

$$j\in\{-k,-k+1,\dots,k\}\,,\tag{5.2.23}$$

with conformal dimensions $\varDelta_\chi$ given by (5.2.21), thus deformed by contributions in (5.2.22). For $\chi=\frac{2\pi}{T}$ on the other hand, every level $\tilde{n}$ in the range

$$\tilde{n}\in\{|n-k|,|n-k+1|,\dots,n+k\}\,,\tag{5.2.24}$$

carries a state of conformal dimension $\varDelta_0$ which recombine into a spin k representation of a (newly enhanced) SU(2) symmetry. As an illustration, Figure 5.1 depicts the spectrum of spin-2 masses at fixed S^5–level $\ell=3$. It

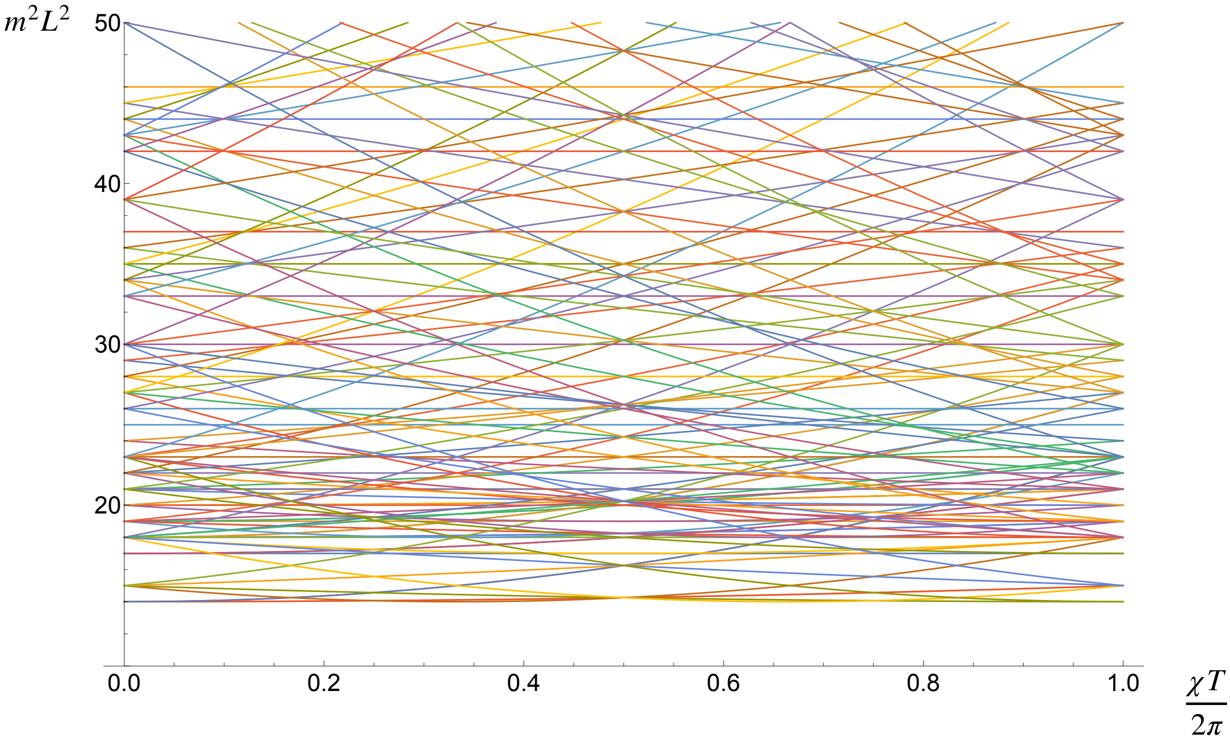

Figure 5.1 Spin-2 masses at level $\ell=3$, as a function of χ. L denotes the AdS$_4$ radius.

shows the breaking and recombining of the spin-2 states as a function of the deformation parameter χ running from 0 to $\frac{2\pi}{T}$. The spectra at the two endpoints $\chi=0$ and $\chi=\frac{2\pi}{T}$ are identical.

It is also instructive to illustrate this pattern at the lowest S^5–level $\ell=0$. At this level, the spectrum combines into supermultiplets

$$\begin{aligned} &4\times L\overline{L}[0]^{(0)}_{\frac{1}{2}+\sqrt{\frac{17}{4}+\frac{4\pi^2n^2}{T^2}}} \oplus\; 2\times L\overline{L}[0]^{(0)}_{\frac{1}{2}+\sqrt{\frac{1}{4}+\left(\frac{2\pi n}{T}\pm 2\chi\right)^2}} \oplus\; 2\times L\overline{L}[0]^{(0)}_{\frac{1}{2}+\sqrt{\frac{1}{4}+\frac{4\pi^2n^2}{T^2}}} \\ &\quad\oplus\; 4\times L\overline{L}[\tfrac{1}{2}]^{(0)}_{\frac{1}{2}+\sqrt{2+\left(\frac{2\pi n}{T}\pm\chi\right)^2}} \oplus\; 2\times L\overline{L}[1]^{(0)}_{\frac{1}{2}+\sqrt{\frac{9}{4}+\frac{4\pi^2n^2}{T^2}}}\,, \end{aligned} \tag{5.2.25}$$

for S^1–level $n>0$, accompanied by (4.2.7) at level $n=0$. At $\chi=0$, some of the conformal dimensions in (5.2.25) degenerate with the corresponding supermultiplets joining into irreducible SU(2) representations of spin [1] and $[\frac{1}{2}]$, respectively. At level 0, this moreover induces the multiplet shortening (4.2.8) with the two arising massless vector multiplets $A_2\overline{A}_2[0]^{(0)}_1$ manifesting the symmetry enhancement $\mathrm{U}(1)\to\mathrm{SU}(2)$, as discussed in the previous section. In contrast, at $\chi=\frac{2\pi}{T}$, those additional massless vector multiplets arise from multiplet shortening of the second multiplet in (5.2.25) at level $n=2$. This is an explicit realization of a (bosonic version of the) space invader scenario encountered in other compactifications [74], in which massive fields from higher Kaluza-Klein levels turn into massless gauge fields. However, the structure of the spectrum (5.2.25) shows an even more remarkable structure at the intermediate value $\chi=\frac{\pi}{T}$. At this value, multiplet shortening of the second multiplet in (5.2.25) now at level $n=1$ gives rise to two additional massless vector multiplets which reveal another SU(2) symmetry enhancement at this point. In contrast with the symmetry enhancement at $\chi=\frac{2\pi}{T}$, the full Kaluza-Klein spectrum at this intermediate point is different from the one at $\chi=0$. A closer look at the χ-dependence (5.2.22) of the masses shows that under $\chi\to\chi+\frac{\pi}{T}$, the spectrum of states of integer SU(2) spin maps into itself whereas the states of half-integer SU(2) spin acquire different masses. This is also visible in Figure 5.1 with the degeneracies due to the symmetry enhancement to an inequivalent spectrum at the intermediate point $\chi=\frac{\pi}{T}$. It is worth pointing out that the truncation of the level 0 spectrum (4.2.7) to integer SU(2) spin amounts to truncating the four-dimensional $\mathcal{N}=8$ supergravity to a half-maximal $\mathcal{N}=4$ theory. In section 5.3, we will discuss the higher-dimensional origin responsible for these patterns. First, let us point out the relevant symmetries of the Kaluza-Klein spectrum. Inspection of the Kaluza-Klein spectrum shows the following two symmetries:

$$\chi\to\chi+\frac{2\pi}{T}\ ,\quad n\to n+2j\,, \tag{5.2.26}$$

$$\chi\to-\chi\ ,\quad j\to-j\,. \tag{5.2.27}$$

The above symmetries combine into a reflection symmetry of the spectrum in the $\chi=\pi/T$ vertical line:

$$\chi\to\frac{2\pi}{T}-\chi\ ,\quad n\to n-2j\ ,\quad j\to-j\,, \tag{5.2.28}$$

which is perceptible in Figure 5.1[6]. Later in Section 5.3.1, we will give a characterization of the symmetries (5.2.26) and (5.2.27) in terms of the geometric properties of an elliptic fibration within the internal manifold. In this construction χ will be identified with the real part of the complex structure modulus of a torus fibered over S^2. The symmetry (5.2.26) will then be interpreted as

[6]More explicitly, one can imagine following a line corresponding to a state with non-vanishing U(1) charge, it is to say with decreasing/increasing energy with respect to χ, until it reaches the middle point $\frac{\chi T}{2\pi}=\frac{1}{2}$. Because of the SU(2) degeneracy occurring at this point, one will cross the line corresponding to a state with opposite U(1) charge, it is to say with increasing/decreasing energy, until the endpoint $\frac{\chi T}{2\pi}=1$. The symmetry is trivially realized for those states having vanishing U(1) charge $j=0$. Indeed, they correspond to horizontal lines since their energy does not depend on χ (5.2.22).

the Dehn twist, see Subsection 5.3.2 on the fiber, which can be reabsorbed in a globally well-defined reparametrization of the deformed S^3, while (5.2.27) as the effect of a parity transformation on the same fiber, see Subsection 5.3.5.

5.2.2 Multiplet shortening

As discussed above, at $\chi=0$, the symmetry enhances according to $\mathrm{U}(1)^2 \to \mathrm{U}(2)$. At the same time, at these values, the conformal dimensions (5.2.21) of several supermultiplets hit the unitarity bound (5.2.14) and the generic long multiplets split up into shortened multiplets according to the patterns reviewed in appendix E.2. Explicitly, combining the saturation of the unitarity bound

$$\Delta = 1+|R|+J\,, \tag{5.2.29}$$

with the formula (5.2.21) translates into the condition

$$8+2\ell(\ell+4) = (|r|+2J)(|r|+2J+2)+4k(k+1)\,. \tag{5.2.30}$$

Combining this with the bounds derived from the specific characters (5.2.20), we conclude that multiplet shortening appears for the multiplets whose HWS charges satisfy

$$|R|=\ell\,,\quad k=1+\tfrac{1}{2}\ell-J\,. \tag{5.2.31}$$

This reveals six series of long multiplets which sit on the unitarity bound and each decomposes into semi-short multiplets according to (E.2.3)

$$\begin{gathered}
[\tfrac{\ell}{2}]\otimes L\overline{L}[1]^{(\pm\ell)}_{\ell+2} \quad\longrightarrow\quad [\tfrac{\ell}{2}]\otimes\begin{cases} L\overline{A}_1[1]^{(\ell)}_{\ell+2}+L\overline{A}_1[\tfrac{1}{2}]^{(\ell+1)}_{\ell+5/2} \\[2ex] A_1\overline{L}[1]^{(-\ell)}_{\ell+2}+A_1\overline{L}[\tfrac{1}{2}]^{(-\ell-1)}_{\ell+5/2}\end{cases}\,, \\[4ex]
[\tfrac{\ell+1}{2}]\otimes L\overline{L}[\tfrac{1}{2}]^{(\pm\ell)}_{\ell+\frac{3}{2}} \quad\longrightarrow\quad [\tfrac{\ell+1}{2}]\otimes\begin{cases} L\overline{A}_1[\tfrac{1}{2}]^{(\ell)}_{\ell+\frac{3}{2}}+L\overline{A}_2[0]^{(\ell+1)}_{\ell+2} \\[2ex] A_1\overline{L}[\tfrac{1}{2}]^{(-\ell)}_{\ell+\frac{3}{2}}+A_2\overline{L}[0]^{(-\ell-1)}_{\ell+2}\end{cases}\,, \\[4ex]
[\tfrac{\ell+2}{2}]\otimes L\overline{L}[0]^{(\pm\ell)}_{\ell+1} \quad\longrightarrow\quad [\tfrac{\ell+2}{2}]\otimes\begin{cases} L\overline{A}_2[0]^{(\ell)}_{\ell+1}+L\overline{B}_1[0]^{(\ell+2)}_{\ell+2} \\[2ex] A_2\overline{L}[0]^{(-\ell)}_{\ell+1}+B_1\overline{L}[0]^{(-\ell-2)}_{\ell+2}\end{cases}\,,
\end{gathered} \tag{5.2.32}$$

at level $\ell>0$. Similar multiplet shortening occurs at $\chi=\frac{\pi}{T}$. More remarkably, multiplet shortening happens at every value of χ that is a rational multiple of

$\frac{2\pi}{T}$. More precisely, at

$$\chi = \frac{p}{q}\frac{2\pi}{T}\,, \qquad p, q \in \mathbb{N}\,, \tag{5.2.33}$$

shortening occurs for the multiplets whose HWS have U(1) charge

$$j = \frac{q\,n}{2\,p} \in \tfrac{1}{2}\mathbb{N}\,. \tag{5.2.34}$$

These multiplets appear at S^1 levels n that are integer multiples of p, i.e. $n = mp$ with $m \in \mathbb{N}$. We stress, however, that the resulting shortened multiplets are not necessarily protected, as they can potentially recombine again into the original long multiplets. It remains an open question to what extent they can be recovered in the dual conformal field theory.

5.3 The Type IIB Uplift of the 1-Parameter $\mathcal{N}=2$ Vacua

Just as in the $\chi = 0$ case, the $D = 10$ dimensional solution corresponding to the 1-parameter family of $\mathcal{N}=2$ vacua has the geometry of $\mathrm{AdS}_4 \times S^5 \times S^1$, where S^5 denotes here a deformed five-sphere. As we shall see this family is locally related to the $\chi = 0$ solution by a coordinate transformation involving the coordinates of S^1 and a squashed S^3 within S^5.

5.3.1 Geometry of the Internal Space

We locally parameterize S^5 by coordinates θ, φ, α, β, γ and S^1 by the coordinate η, with the following ranges

$$0 \le \eta < T\,,\ 0 \le \theta \le \frac{\pi}{2}\,,\ 0 \le \varphi < 2\pi\,,\ 0 \le \alpha \le 2\pi\,,\ 0 \le \beta \le \pi\,,\ 0 \le \gamma + \frac{\pi}{2} < 4\pi\,. \tag{5.3.1}$$

The coordinates θ, φ parametrize an S^2 within S^5, while α, β, γ parametrize an S^3 within the same manifold. We begin by describing the internal geometry for the $\chi = 0$ solution, before explaining how it is modified when $\chi \neq 0$. To understand the effect of $\chi \neq 0$, it is sufficient to focus on S^3 and S^1. In the full solution, discussed in section 5.3.3, the S^3 is fibred over S^2 in such a way that, for $\chi = 0$, only an $\mathrm{SU}(2) \times \mathrm{U}(1)' \subset \mathrm{SU}(2) \times \mathrm{SU}(2)'$ isometry of S^3 remains. Hence it is convenient to describe S^3 using the isomorphism $\mathrm{SU}(2) \simeq S^3$, given by

$$g(\alpha,\beta,\gamma)\equiv\begin{pmatrix} w_1 & w_2 \\ -\overline{w}_2 & \overline{w}_1\end{pmatrix}=\begin{pmatrix} e^{\frac{1}{2}i(\alpha+\gamma+\pi/2)}\cos\left(\frac{\beta}{2}\right) & e^{\frac{1}{2}i(-\alpha+\gamma+\pi/2)}\sin\left(\frac{\beta}{2}\right) \\ -e^{-\frac{1}{2}i(-\alpha+\gamma+\pi/2)}\sin\left(\frac{\beta}{2}\right) & e^{-\frac{1}{2}i(\alpha+\gamma+\pi/2)}\cos\left(\frac{\beta}{2}\right)\end{pmatrix}, \tag{5.3.2}$$

where w_1, w_2 satisfy $|w_1|^2+|w_2|^2=1$ and define the embedding of S^3 in $\mathbb{C}^2$. The map from (5.3.1) to the y^m coordinates in (5.1.6) is

$$\frac{y^1}{\cos\theta}=\mathrm{Re}(w_1)\,,\qquad \frac{y^4}{\cos\theta}=\mathrm{Im}(w_1)\,,\qquad \frac{y^5}{\cos\theta}=\mathrm{Re}(w_2)\,, \tag{5.3.3}$$

and

$$y^2=\cos\phi\sin\theta\,,\qquad y^3=\sin\phi\sin\theta\,,\qquad \tilde{y}=\sinh\eta\,. \tag{5.3.4}$$

For $\chi=0$, we can express the squashed S^3 metric in terms of the left-invariant 1-forms σ^i, $i=1,\ldots,3$, defined as

$$g^{-1}dg=\sum_{i=1}^{3}\sigma^i(i\boldsymbol{\sigma}^i)\,, \tag{5.3.5}$$

where $\boldsymbol{\sigma}^i$ are the three Pauli matrices. The σ^i satisfy the Maurer-Cartan equations

$$d\sigma^i-\epsilon^{ijk}\,\sigma^j\wedge\sigma^k=0\,, \tag{5.3.6}$$

with $\epsilon^{ijk}=\pm1$ the structure constants of SU(2). Evaluating the Maurer-Cartan forms in terms of the coordinates (5.3.1), we find

$$\begin{aligned}&\sigma^1=\frac{1}{2}(\mathrm{d}\gamma\cos(\alpha)\sin(\beta)-\mathrm{d}\beta\sin(\alpha))\,,\qquad \sigma^2=\frac{1}{2}(\mathrm{d}\beta\cos(\alpha)+\mathrm{d}\gamma\sin(\alpha)\sin(\beta))\,,\\ &\sigma^3=\frac{1}{2}(\mathrm{d}\alpha+\mathrm{d}\gamma\cos(\beta))\,.\end{aligned} \tag{5.3.7}$$

The dependence of the internal metric and the other fields, in the $\chi=0$ solution, on the point in S^3 is expressed in terms of σ^i and thus the solution features an SU(2) symmetry group acting from the left on $g(\alpha,\beta,\gamma)$ and thus leaving σ^i invariant. Due to the squashing of the S^3 geometry, only a $\mathrm{U}(1)'$ subgroup of the $\mathrm{SU}(2)'$ group acting on $g(\alpha,\beta,\gamma)$ from the right is a symmetry of the $\chi=0$ solution. In fact the group $\mathrm{U}(1)'$ coincides with the $\mathcal{N}=2$ R-symmetry group, previously denoted by $\mathrm{U}(1)_R$. For $\chi\neq0$ the solution features a fibration of S^3 over S^1 in which a point of S^1 is associated with an S^3 parametrized by coordinates (α',β',γ') which define the following SU(2)-element

$$g(\alpha',\beta',\gamma')=\widehat{g}(\alpha,\beta,\gamma,\eta)\equiv h(\eta)\cdot g\,(\alpha,\beta,\gamma)\,, \tag{5.3.8}$$

where

$$h(\eta) \equiv \begin{pmatrix} \cos(\eta\chi) & \sin(\eta\chi) \\ -\sin(\eta\chi) & \cos(\eta\chi) \end{pmatrix} \in \mathrm{SU}(2)\,. \tag{5.3.9}$$

The relation (5.3.8) defines the transition function on the S^3 fiber when changing chart on S^1 and introduces a monodromy on the same fiber as $\eta \to \eta + T$, represented by the left action of the element $h(T) = h(\eta)^{-1} h(\eta + T)$ in SU(2). Then, the total space of the 4-dimensional fiber-bundle, with fiber S^3 and base S^1, is given by the quotient space $S^3 \times [0, T]/\sim$ where the identification $\sim$ is defined as follows

$$[g\,(\alpha,\beta,\gamma)\,,\,\eta=0] \sim [h(T)\cdot g\,(\alpha,\beta,\gamma)\,,\,\eta=T]\,. \tag{5.3.10}$$

The presence of this monodromy further breaks the SU(2) isometry, that the squashed S^3 has for $\chi = 0$, to the subgroup of SU(2) commuting with $h(T)$. In general, this subgroup is given by U(1), while the isometry U(1)$'$ coming from the right-action remains of course unbroken by $h(T)$. The values $\chi = \frac{2\pi}{kT}$ are particularly interesting since then the element $h(T)$ generates the cyclic group $\mathbb{Z}_k$. For $k = 1$, the quotient is trivial so that four-dimensional manifold, like the $\chi = 0$ case, is a direct product of $S^3 \times S^1$ with isometry group SU(2). This will be further clarified in section 5.3.2, where we show that for $k = 1$, i.e. $\chi = \frac{2\pi}{T}$, the solution is equivalent to the $\chi = 0$ one. For $k = 2$, namely $\chi = \frac{\pi}{T}$, the twist commutes with all of SU(2) since the $\mathbb{Z}_2$ group it generates is the center of SU(2). Thus for $k = 1, 2$ the U(1) isometry is enhanced to SU(2). This explains the symmetry enhancement, for those special values of χ, observed in section 5.2 by inspection of the Kaluza-Klein spectrum. Concerning the geometric description of the internal space, we observe that *locally* $h(\eta)$ in (5.3.8) can be absorbed into a coordinate transformation:

$$\{\alpha,\beta,\gamma,\eta\} \to \{\alpha'(\alpha,\beta,\gamma,\eta), \beta'(\alpha,\beta,\gamma,\eta), \gamma'(\alpha,\beta,\gamma,\eta), \eta' = \eta\}\,, \tag{5.3.11}$$

where $\alpha'(\alpha,\beta,\gamma,\eta)$, $\beta'(\alpha,\beta,\gamma,\eta)$, $\gamma'(\alpha,\beta,\gamma,\eta)$ are defined by the solution to the matrix equation

$$g(\alpha',\beta',\gamma') = \widehat{g}(\alpha,\beta,\gamma,\eta)\,. \tag{5.3.12}$$

Therefore, we can express the $\chi \neq 0$ solution by computing the new left-invariant 1-forms $\widehat{\sigma}^i$ associated with $\widehat{g}(\alpha,\beta,\gamma,\eta)$ or, equivalently, $g(\alpha',\beta',\gamma')$, as

$$\widehat{g}^{-1} d\widehat{g} = \sum_{i=1}^{3} \widehat{\sigma}^i (i\,\boldsymbol{\sigma}^i)\,. \tag{5.3.13}$$

We find

$$\begin{aligned}
\widehat{\sigma}^1 &\equiv \sigma^1 + \chi\,(-\cos(\alpha)\cos(\beta)\cos(\gamma) + \sin(\alpha)\sin(\gamma))\mathrm{d}\eta\,,\\
\widehat{\sigma}^2 &\equiv \sigma^2 - \chi\,(\sin(\alpha)\cos(\beta)\cos(\gamma) + \cos(\alpha)\sin(\gamma))\mathrm{d}\eta\,,\\
\widehat{\sigma}^3 &\equiv \sigma^3 + \chi\,\cos(\gamma)\sin(\beta)\mathrm{d}\eta\,.
\end{aligned} \tag{5.3.14}$$

As we will show, the $D=10$ background for $\chi \neq 0$ can be obtained from the $\chi = 0$ solution given in [43] through the replacement

$$\sigma^i \to \widehat{\sigma}^i\,. \tag{5.3.15}$$

However, it is important to emphasise that the local coordinate redefinition (5.3.11) is not globally well-defined and therefore does not define a diffeomorphism, except for the case $\chi = \frac{2\pi}{T}$, as shown clearly in section 5.3.2. Hence, χ amounts to a physical modulus of the $D=10$ solution with periodicity $\frac{2\pi}{T}$.

5.3.2 χ as a Complex Structure Modulus

The parameter χ can also be interpreted as a complex structure modulus on $M_4 \sim S^3 \times S^1$, which gives another perspective on its geometric role and most clearly elucidates its periodicity $\chi \in [0, \frac{2\pi}{T})$. For this, it is best to view S^3 as the Hopf fibration, such that the Hopf fiber and S^1 combine into an elliptic fibration over S^2. As we will now show, χ forms part of the complex structure modulus of the T^2 fiber. We begin by considering a different, yet equivalent, parameterization of S^3 with coordinates $(\Phi,\, \xi,\, \psi)$[7], defined by

$$g(\alpha,\beta,\gamma) = g\left(0, \frac{\pi}{2}, \pi\right)\cdot g(\Phi,\xi,\psi)\,. \tag{5.3.16}$$

A point in the four-dimensional total space that we are considering is now given by

$$p = (\widehat{g}(\Phi,\xi,\psi,\eta),\eta)\,, \tag{5.3.17}$$

with

$$\widehat{g}(\Phi,\xi,\psi,\eta) \equiv h(\eta)\cdot g\left(0,\frac{\pi}{2},\pi\right)\cdot g(\Phi,\xi,\psi) \in \mathrm{SU}(2)\,. \tag{5.3.18}$$

The projection map $\pi : M_4 \to S^2$ is essentially given by the usual Hopf map

$$\pi : (\zeta_1,\zeta_2) \mapsto \mathbf{r} = \left(\mathrm{Re}(2\zeta_1\overline{\zeta_2}),\, \mathrm{Im}(2\zeta_1\overline{\zeta_2}),\, |\zeta_1|^2 - |\zeta_2|^2\right)\,, \tag{5.3.19}$$

[7] Their ranges are the same as the $(\alpha,\, \beta,\, \gamma)$ ones.

with

$$\begin{pmatrix} \zeta_1 & \zeta_2 \\ -\bar{\zeta}_2 & \bar{\zeta}_1 \end{pmatrix} = g\left(0, \frac{\pi}{2}, \pi\right)^{-1} \cdot \widehat{g}(\Phi, \xi, \psi, \eta)\,. \tag{5.3.20}$$

It is straightforward to check that $\mathbf{r}$ defined by (5.3.19) satisfies $\mathbf{r} \in S^2 \subset \mathbb{R}^3$ since $\mathbf{r} \cdot \mathbf{r} = 1$ and that ψ and η are projected out in (5.3.19). Thus, ψ and η provide local coordinates on the T^2 fiber. We can now read off the complex structure on the elliptic fiber, for example by studying the connection 1-forms on M_4. These are given by the right-invariant 1-forms

$$\begin{aligned} \omega_\psi &= d\psi - 2\chi d\eta + \cos\xi\, d\Phi\,, \\ \omega_\eta &= d\eta\,. \end{aligned} \tag{5.3.21}$$

Thus, the local holomorphic coordinate on the elliptic fiber is given by

$$\varrho = \psi + \widehat{\tau}\,\eta\,, \tag{5.3.22}$$

with $\widehat{\tau} = i - 2\chi$ defining the complex structure and the periodicity of ϱ given by

$$\varrho \sim \varrho + 4\pi \sim \varrho + \widehat{\tau}\, T\,. \tag{5.3.23}$$

Moreover, the $\widehat{\sigma}^i$ now read

$$\begin{aligned} \widehat{\sigma}^1 &= \frac{1}{2}(\sin(\xi)\cos(\Phi)(d\psi - 2\chi d\eta) - d\xi \sin(\Phi))\,, \\ \widehat{\sigma}^2 &= \frac{1}{2}(\sin(\xi)\sin(\Phi)(d\psi - 2\chi d\eta) + d\xi \cos(\Phi))\,, \\ \widehat{\sigma}^3 &= \frac{1}{2}(\cos(\xi)(d\psi - 2\chi d\eta) + d\Phi)\,, \end{aligned} \tag{5.3.24}$$

or in terms of the complex coordinate ϱ

$$\begin{aligned} \widehat{\sigma}^1 &= \frac{1}{4}(\sin(\xi)\cos(\Phi)(d\varrho + d\bar{\varrho}) - 2d\xi \sin(\Phi))\,, \\ \widehat{\sigma}^2 &= \frac{1}{4}(\sin(\xi)\sin(\Phi)(d\varrho + d\bar{\varrho}) + 2d\xi \cos(\Phi))\,, \\ \widehat{\sigma}^3 &= \frac{1}{4}(\cos(\xi)(d\varrho + d\bar{\varrho}) + d\Phi)\,, \end{aligned} \tag{5.3.25}$$

where there is no explicit dependence on χ. Thus, it is clear that χ only affects the complex structure of the T^2 fiber. The complex structure $\widehat{\tau} = i - 2\chi$ now makes the periodicity of χ clear. First, recall that ψ has periodicity 4π whereas η has periodicity T. Let us thus rescale $\psi \to \psi' = \frac{\psi}{4\pi}$ and $\eta = \eta' = \frac{\eta}{T}$ which have standard periodicities

$$\psi' \sim \psi' + 1\,, \qquad \eta' \sim \eta' + 1\,. \tag{5.3.26}$$

The local holomorphic coordinate, u, is given in terms of these by

$$\varrho = 4\pi \left(\psi' + \tau\,\eta'\right)\,, \tag{5.3.27}$$

with the complex structure

$$\tau = \frac{i}{4\pi} - \frac{\chi T}{2\pi}\,. \tag{5.3.28}$$

It is now clear that $\chi \to \chi + \frac{2\pi}{T}$ just corresponds to a Dehn twist, $\tau \to \tau - 1$, and can be reabsorbed by a globally well-defined reparameterization. Thus, χ has periodicity $\frac{2\pi}{T}$.

5.3.3 The Metric

The spacetime metric has the following form

$$ds^2 = (2l)^{-1}\left(ds^2_{\mathrm{AdS}_4} + ds^2_6\right)\,, \tag{5.3.29}$$

where

$$l \equiv (6 - 2\cos(2\theta))^{-\frac{1}{4}}\,. \tag{5.3.30}$$

The internal metric ds^2_6 has the following form

$$ds^2_6 = ds^2_{S^2} + ds^2_{S^3\times S^1}\,, \tag{5.3.31}$$

where

$$ds^2_{S^2} = d\theta^2 + \sin^2(\theta)\,d\varphi^2\,,\ ds^2_{S^3\times S^1} = \cos^2(\theta)\left(\widehat{\sigma}_2^2 + 8\,\varDelta^4\,(\widehat{\sigma}_1^2 + \widehat{\sigma}_3^2)\right) + d\eta^2\,, \tag{5.3.32}$$

and, for a fixed θ, $S^3 \times S^1$ denotes the twisted product described in the previous section. Note that the squashing of the S^3, arising from the different factors multiplying the $\widehat{\sigma}^i$, breaks the $\mathrm{SU}(2)\times\mathrm{SU}(2)'$ symmetry of the round S^3 to $\mathrm{SU}(2)\times\mathrm{U}(1)'$, with the $\mathrm{U}(1)'$ rotating $\widehat{\sigma}^1$ with $\widehat{\sigma}^3$. As discussed in section 5.3.1, when $\chi \neq 0$, the SU(2) is also broken to U(1). Finally, the symmetries of S^2 are broken by the dependence on θ and φ of the solution.

5.3.4 The 2-Forms, the 4-Form, the Dilaton and the Axion

As mentioned earlier, the expressions of the 2-forms and the 4-forms are the same as in the $\chi=0$ case, given in [43], aside from the replacement $\sigma^i \to \widehat{\sigma}^i$ as in (5.3.15). Thus, in the notation of [43], we can write,

$$B^{\widehat{\alpha}}_{(2)} = A(\eta)^{\widehat{\alpha}}{}_{\widehat{\beta}}\,\mathfrak{b}^{\widehat{\beta}}_{(2)}\,, \tag{5.3.33}$$

where

$$A(\eta)^{\widehat{\alpha}}{}_{\widehat{\beta}} \equiv \begin{pmatrix} \cosh(\eta) & \sinh(\eta) \\ \sinh(\eta) & \cosh(\eta) \end{pmatrix}\,, \tag{5.3.34}$$

is an $\mathrm{SL}(2,\mathbb{R})_{\mathrm{IIB}}$ twist and

$$\begin{aligned} \mathfrak{b}^1_{(2)} &= \frac{1}{\sqrt{2}}\cos(\theta)\left[\left(\cos(\phi)\,d\theta + \frac{1}{2}\sin(2\theta)\,d(\cos(\phi))\right)\wedge\widehat{\sigma}_2 + \cos(\phi)\frac{4\sin(2\theta)}{6-2\cos(2\theta)}\widehat{\sigma}_1\wedge\widehat{\sigma}_3\right]\,, \\ \mathfrak{b}^2_{(2)} &= -\frac{1}{\sqrt{2}}\cos(\theta)\left[\left(\sin(\phi)\,d\theta + \frac{1}{2}\sin(2\theta)\,d(\sin(\phi))\right)\wedge\widehat{\sigma}_2 + \sin(\phi)\frac{4\sin(2\theta)}{6-2\cos(2\theta)}\widehat{\sigma}_1\wedge\widehat{\sigma}_3\right]\,. \end{aligned} \tag{5.3.35}$$

The self-dual 5-form field strength reads:

$$\begin{aligned} \widetilde{F}_5 \equiv dC_{(4)} + \frac{1}{2}\epsilon_{\widehat{\alpha}\widehat{\beta}}B^{\widehat{\alpha}}_{(2)}\wedge H^{\widehat{\beta}}_{(3)} &= (1+\star)4\Delta^4\sin(\theta)\cos^3(\theta)\,[3\,d\theta\wedge d\phi\wedge\widehat{\sigma}_1\wedge\widehat{\sigma}_2\wedge\widehat{\sigma}_3 \\ &\quad -d\eta\wedge\left(\cos(2\theta)\,d\theta - \frac{1}{2}\sin(2\theta)\sin(2\phi)\,d\phi\right)\wedge\widehat{\sigma}_1\wedge\widehat{\sigma}_2\wedge\widehat{\sigma}_3\Big]\,, \end{aligned} \tag{5.3.36}$$

where $H^{\widehat{\beta}}_{(3)} = dB^{\widehat{\alpha}}_{(2)}$. Finally, the axion and the dilaton fields are encoded in the matrix $m_{\widehat{\alpha}\widehat{\beta}}$ in (5.1.11) which, in our solution, reads

$$m_{\widehat{\alpha}\widehat{\beta}} = \left(A^{-1}(\eta)\right)^{\widehat{\sigma}}{}_{\widehat{\alpha}}\left(A^{-1}(\eta)\right)^{\widehat{\gamma}}{}_{\widehat{\beta}}\,\mathfrak{m}_{\widehat{\sigma}\widehat{\gamma}}\,, \tag{5.3.37}$$

where

$$\mathfrak{m}_{\widehat{\sigma}\widehat{\gamma}} = 2\,\Delta^2\begin{pmatrix} \sin^2(\theta)\cos^2(\phi)+1 & -\frac{1}{2}\sin^2(\theta)\sin(2\phi) \\ -\frac{1}{2}\sin^2(\theta)\sin(2\phi) & \sin^2(\theta)\sin^2(\phi)+1 \end{pmatrix}\,. \tag{5.3.38}$$

Note that the axion-dilaton system is the same as in the $\chi=0$ solution. This is because the extra dependence on η when $\chi\neq 0$ is entirely induced by the matrix $h(\eta)$ in (5.3.8), and only affects those fields which depend on the point in S^3. As mentioned earlier, the explicit dependence of the axion and dilaton on the coordinates θ, φ mean that the isometries of S^2 are not a symmetry of the whole solution, while $\mathrm{U}(1)^2=\mathrm{U}(1)\times\mathrm{U}(1)'$ is. This is true for all values of χ. For the special values of χ

$$\chi=\frac{m\pi}{T}\,,\qquad m\in\mathbb{Z}\,, \tag{5.3.39}$$

the twist in the local product of $S^3\times S^1$ is either trivial (m even) or $\mathbb{Z}_2$ (m odd), as discussed at the end of subsection 5.3.1, and the symmetry of the solution is enhanced to $\mathrm{U}(2)=\mathrm{SU}(2)\times\mathrm{U}(1)'$. Moreover, when m is even, the solution is equivalent to $\chi=0$, so we should identify χ as a periodic modulus $\chi\sim\chi+\frac{2\pi}{T}$. The dependence on η through the $\mathrm{SL}(2,\mathbb{R})_{\mathrm{IIB}}$-twist matrix $A(\eta)^{\widehat{\alpha}}{}_{\widehat{\beta}}$ of the two-forms and the axion-dilaton system is the same as in the $\chi=0$ case, so we can apply to this family of solutions the same discussion about the corresponding $\mathrm{SL}(2,\mathbb{R})_{\mathrm{IIB}}$-monodromy matrix $\mathfrak{M}_{S^1}$ made in [43]. As $\eta\to\eta+T$ the twist matrix A induces a monodromy

$$\mathfrak{M}_{S^1}\equiv A^{-1}(\eta)\cdot A(\eta+T)=\begin{pmatrix}\cosh(T) & \sinh(T)\\ \sinh(T) & \cosh(T)\end{pmatrix}. \tag{5.3.40}$$

By generalizing the twist matrix and suitably choosing the value of T [43], one can construct backgrounds in which the monodromy has the form $\mathfrak{M}_{S^1}=-\mathcal{S}\,\mathcal{T}^k\in\mathrm{SL}(2,\mathbb{Z})_{\mathrm{IIB}}$, thus defining a family of S-fold solutions of Type IIB theory.

5.3.5 The χ-twist in the Kaluza-Klein spectrum

As we have seen in section 5.3.1, a non-vanishing value of χ induces an extra dependence on η of those fields which, in the $\chi=0$ solution, were non-trivial functions of the point of S^3, due to the fibration of the latter over S^1. We can use this feature to determine the χ-dependence on the full Kaluza-Klein tower of states. To do so, it is easiest to consider the background underformed, i.e. as for $\chi=0$, and instead modify the Kaluza-Klein states' dependence on η. Thus, fields transforming in the SU(2) representation $[k]$, now acquire an η-dependence through the $[k]$-representation of the SU(2)-element $h(\eta)$ given in (5.3.9). The corresponding twist matrix has eigenvalues

$$e^{2ij\chi\eta}\,,\qquad \text{with } j=-k,\,-k+1,\ldots,k-1,\,k\,. \tag{5.3.41}$$

As an example, consider the three vector fields A^i_μ, which, for $\chi=0$, gauge the SU(2) isometry group. For $\chi=0$, these transform as the right-invariant

Killing vectors K_i, defining the infinitesimal left-translations on $g(\alpha,\beta,\gamma)$. Indeed, these vectors, on a group manifold, are defined as:

$$g^{-1}\cdot t_i\cdot g = K_i{}^{\ell}\sigma^s{}_{\ell}\,t_s\,, \tag{5.3.42}$$

where t_i are SU(2) generators, with $i=1,2,3$, and we have written the left-invariant 1-forms σ^i as $\sigma^i=\sigma^i{}_{\ell}\,dx^{\ell}$, $x^i\equiv(\alpha,\beta,\gamma)$. When $\chi\neq 0$, the vector fields are modified. Transforming g by the twist:

$$g(\alpha,\beta,\gamma)\to\widehat{g}(\alpha,\beta,\gamma,\eta)=h(\eta)\cdot g(\alpha,\beta,\gamma)\,, \tag{5.3.43}$$

where $h(\eta)$ is the 2×2 twist matrix given in (5.3.9), we find

$$\widehat{g}^{-1}\cdot t_i\cdot\widehat{g}=g^{-1}\cdot h^{-1}\cdot t_i\cdot h\cdot g=h_i{}^{\ell}\,(K_{\ell}{}^{k}\sigma^s{}_k\,t_s)=\widehat{K}_i{}^{\ell}\sigma^s{}_{\ell}\,t_s\,. \tag{5.3.44}$$

Here $h_i{}^j$ denotes the adjoint action of h:

$$h^{-1}\cdot t_i\cdot h\equiv h_i{}^j t_j\,. \tag{5.3.45}$$

Therefore, as expected, the Killing vectors, and therefore A^i_{μ}, transform in the $k=1$ representation acted on by the 3×3 matrix $h^i{}_{\ell}(\eta)$. The twisted vectors $\widehat{A}_{(0)}{}^i_{\mu}$, at KK level $n=0$ on S^1, therefore now have a η-dependence due to the twist

$$\widehat{A}_{(0)}{}^i_{\mu}(x,\eta)=h^i{}_{\ell}(\eta)\,A^{\ell}_{\mu}(x)\,. \tag{5.3.46}$$

This additional η dependence makes two of the vectors massive. As a result, SU(2) is broken at level $n=0$. Similarly, the corresponding vectors at level n on S^1 have an η-dependence of the form

$$\widehat{A}_{(n)}{}^i_{\mu}(x,\eta)=h^i{}_{\ell}(\eta)\,A^{\ell}_{\mu}(x)e^{\frac{2i\pi n\eta}{T}}\,,\qquad \widehat{A}_{(n)}{}^i_{\mu}(x,\eta)^*=h^i{}_{\ell}(\eta)\,A^{\ell}_{\mu}(x)e^{-\frac{2i\pi n\eta}{T}}\,. \tag{5.3.47}$$

We can now see that when $\chi=p\pi/T$, with $p\in\mathbb{Z}$, the SU(2) symmetry is restored. Two of the eigenvalues of $\partial/\partial\eta$ on these vectors are now

$$\pm\left(2i\chi-\frac{2i\pi n}{T}\right)=\pm\left(\frac{2i\pi p}{T}-\frac{2i\pi n}{T}\right), \tag{5.3.48}$$

which vanish for $n=p$. These correspond to the two gauge vectors at level $n>0$ which become massless for these values of χ and enhance the U(1), seen at S^1 KK level $n=0$, back to SU(2). This is a bosonic version of the *space invaders* scenario [74, 75], where higher Kaluza-Klein modes become massless. In general, on a field $\Phi^{[k]}_{(n)}$, in S^1 KK level n and in the $[k]$-representation of

SU(2), the operator $\partial/\partial\eta$ will have eigenvalues

$$\pm 2i\left(j\chi - \frac{\pi n}{T}\right),\ j = -k, -k+1, \ldots, k-1, k, \tag{5.3.49}$$

where j can easily be identified with the $\mathrm{U}(1) \subset \mathrm{SU}(2)$ charge. The same conclusion can be reached by thinking of χ as part of the complex structure modulus of an elliptic fibration over S^2, as in section 5.3.2. Now a field obtains an additional η-dependence by the replacement of the Hopf fibre coordinate $\psi \to \psi - 2\chi\eta$. As above, the ψ-dependence is determined by the field's $\mathrm{U}(1) \subset \mathrm{SU}(2)$ charge, so that the field's eigenvalues under $\partial/\partial\eta$ are again given by (5.3.49). This explains the dependence on χ of the KK spectrum, as noted in section 5.2, see eq. (5.2.22).

We can summarize our geometrical understanding of the symmetries (5.2.26), (5.2.27) of the Kaluza-Klein spectrum as follows. The former amounts to a Dehn twist of the toroidal fiber over S^2 which can be undone by a globally well-defined reparameterization of the fiber. In particular for $\chi = 2\pi/T$ the elliptic fibration is globally $S^3 \times S^1$ where S^3 denotes the deformed three-sphere with isometry $\mathrm{SU}(2) \times \mathrm{U}(1)$, and thus the $\mathrm{U}(1)^2$ symmetry is enhanced to $\mathrm{SU}(2) \times \mathrm{U}(1)$. As far as (5.2.27) is concerned, the transformation $\chi \to -\chi$ corresponds to a transformation $\tau \to -\overline{\tau}$ of the complex structure modulus of the toroidal fiber. This amounts in turn to a reflection in the imaginary axis of the torus, seen as a complex manifold, since it implies $u \to -\overline{u}$ as we also transform $\psi \to -\psi$. It is not an invariance of the complex manifold itself, since it changes its orientation, but rather a parity transformation with respect to which the higher dimensional theory is invariant.[8] Note that a change $\psi \to -\psi$ amounts, in the Kaluza-Klein modes, to changing the sign of the corresponding j quantum number, as in (5.2.27). We have finally seen how the ExFT technology for uplifting lower dimensional solutions and computing Kaluza-Klein spectra can help us in understanding the interesting properties of supergravity vacua. In our case, the framework was necessary for explaining the role of the χ parameter in the higher dimensional picture. This analysis is a step forward with respect to the one in the $\mathcal{N}=3$ example of previous chapters. In that case, the parameters α_i $i: 1,2,3$ play a similar role as χ. However, it is not clear if the uplift of the vacua of the $\mathcal{N}=3$ model is possible. At the present level, we can note some similarities such as the presence of free parameters governing the symmetries of different loci in the families of solutions. In this chapter, we have presented a parameter χ selecting bosonic symmetries. However, as in the $\mathcal{N}=3$ model, the one-dimensional manifold parameterized by χ sits in a bigger manifold classifying vacua exhibiting different unbroken supersymmetries if not completely broken. So, as we will discuss below, the

[8] For a discussion of parity symmetry in extended supergravities see [11].

present upliftable model share with the $\mathcal{N} = 3$ one the property of exhibiting a manifold of moduli parameterizing stable AdS_4 vacua with a continuum of non-supersymmetric points.

Chapter 6

More general manifolds of (non)-Supersymmetric Vacua

In the present chapter, we are going to review other known families of S-fold configurations generalizing the previous example. They mainly arise from uplifts of "S-foldable" vacua of the $\mathcal{N} = 8$ and of five-dimensional models. By this, we mean that the lower dimensional solutions under discussion, once uplifted to type IIB supergravity can be twisted with the procedure outlined in [41] following the general considerations of [82]. As above (5.3.40), the folding can take place since the uplift of the solutions gives rise to an $M_5 \times \mathbb{R}$ internal topology. The non-compact internal direction can be compactified. Indeed, its dependence can be factorized in a $\mathrm{SL}(2,\mathbb{R})_{IIB}$ twist implying that the solution exhibits a translational isometry along the latter. Then a monodromy analogous to (5.3.40) is introduced and can be extended to an S-fold configuration. Some of these backgrounds are of interest to us since they generalize the $\mathcal{N} = 2$ solution presented in previous chapters to a family of supersymmetric and non-supersymmetric configurations sharing some properties with the vacua found in the $\mathcal{N} = 3$ model.

We will focus on an interesting subfamily connecting the $\mathcal{N} = 4$ vacuum of [66] and the $\mathcal{N} = 2$ discussed here through a continuum of $\mathcal{N} = 0$ stable solutions. We will also present some interesting observations on their uplift and their holographic interpretation. While in the $\mathcal{N} = 3$ case our analysis was limited to the study of RG-flows due to the lack of knowledge on the possible higher dimensional origin of the model, in the present case we have a direct connection with type IIB supergravity and one can build on well-established holographic results. As already introduced, the main example of an S-fold solution was given in [41]. In the latter study, the $\mathcal{N} = 4$ solution of [66] belonging the $\mathcal{N}=8$ $[\mathrm{SO}(6)\times\mathrm{SO}(1,1)]\ltimes\mathbb{R}^{12}$ gauged model[1] is uplifted to type IIB supergravity. The interpretation as an S-fold is based on the results of [83][82]. Indeed, as explained in [41] one can consider the four-dimensional model as originating from two consecutive compactifications. The first step from ten to five dimensions is captured by the "electric" part of the U-matrix (4.3.18) while the second and crucial step from five to four dimensions is described by a Scherk-Schwarz reduction essentially encoded in the magnetic $\mathrm{SL}(2,\mathbb{R})$-valued part of the same twist matrix. This is where the analysis of [83][82] becomes explicit. Indeed, the latter step exploits the dependence of the axion-dilaton field on the remaining external direction to generate a four-dimensional solution exhibiting an $\mathrm{SL}(2,\mathbb{R})$-monodromy, once the external direction is compactified to a circle.

This point of view is useful in understanding the holographic interpretations of the S-folds. Indeed, the five-dimensional picture suggests a strong relation with a compactified version of the Janus configurations [84, 67, 85, 72, 68, 86, 73, 77, 87, 71]. Building on these results, in [42] new S-fold solutions are presented. In particular, the authors analyze an $\mathcal{N}=2$ sector of the dyonic model under discussion identified by an $\mathrm{U}(1)^2$ truncation. Within this truncation, an $\mathcal{N}=1$ SU(3) invariant vacuum and an $\mathcal{N}=0$ SU(4) invariant vacuum are found. Thanks to the ExFT formalism these vacua are uplifted to type IIB configurations with topology $AdS_4\times M_5\times\mathbb{R}$ were $M_5\simeq\mathbb{CP}_2\rtimes S^1$ and $M_5=S^5$ for the $\mathcal{N}=1$ and the $\mathcal{N}=0$ solution respectively. Once again, following [83][82][41], these configurations can be S-folded by exploiting the axion-dilaton dependence on the external $\mathbb{R}$ direction giving rise to a background of the form $AdS_4\times M_5\times S^1$ with an $\mathrm{SL}(2,\mathbb{R})$ monodromy around the S^1 factor. The analysis has been refined in [43] by truncating the theory to the $\mathcal{N}=1$ sector which was already shown to be fruitful in the study of other gaugings [88][45]. It turns out that the $\mathcal{N}=4,1,0$ solutions fit in the latter $\mathbb{Z}_2^3$ invariant sector. In the very same analysis the $\mathcal{N}=2$ family of vacua introduced in previous chapters was found. Furthermore, the authors found that the $\mathcal{N}=0$ solution, from a four-dimensional point of view, actually comes in a family of three arbitrary parameters. The values of the latter, in the spirit of the parameter χ of the $\mathcal{N}=2$ vacua, discriminate between SU(4), $\mathrm{SU}(3)\times\mathrm{U}(1)$, $\mathrm{SU}(2)\times\mathrm{U}(1)^2$ and SU(2) invariant subfamilies. Analogously, the $\mathcal{N}=1$ vacuum sits in a two-parameter family of vacua with possible SU(3), $\mathrm{SU}(2)\times\mathrm{U}(1)^2$ and SU(3) residual symmetry.

Following the result presented in previous chapters for the uplift of the χ family, in [89] the full family of $\mathcal{N}=1$ solutions has been uplifted to type IIB. By exploiting the technique introduced in the $\mathcal{N}=2$ case, and presented above, the three parameters are shown to be compact in the full ten-dimensional

[1]Other vacua of the same model were also found in [39].

picture. Furthermore, in the same study, a Domain Wall solution connecting the $\mathcal{N} = 1$ SU(3) solution with the $\mathcal{N} = 2$ SU(2) × U(1) one is presented, showing that these families of vacua are not isolated. As shown in [69], solutions with different supersymmetry are not only related by RG flows but they can be part of a common family. This was first noted in [76], where it is argued that the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ solutions should be part, in the CFT dual picture, of the same conformal manifold. This argument is confirmed, by using the five-dimensional approach, by the authors of [69]. In the latter study, a one-parameter family of solutions containing both the $\mathcal{N} = 4$ SO(4) symmetric solution and the $\mathcal{N} = 2$ SU(2) × U(1) solution, once a further Scherk-Schwarz reduction to four dimensions is performed, is presented. Exploring the same approach, in [70] this family was extended to a two-parameter family, one of them being the χ parameter of the previous section. The latter two-dimensional conformal manifold preserves in general $\mathcal{N} = 2$ supersymmetry. As explained in the same work, the solutions are obtained by combining the coset representative describing the $\mathcal{N} = 2$ solution of [69] with the one describing the χ family of previous chapters. This interesting idea for constructing new families of vacua has been systematically explored in [90] where it is shown that given a G_0 symmetric S-fold solution one can always deform it with χ-like flat directions associated with the generators of the cartan of G_0. Thus, it is natural that the $\mathcal{N} = 0$ SU(4) symmetric and that the $\mathcal{N} = 1$ SU(3) symmetric solutions of [42] come in a three and two-parameter families of vacua respectively. By the same token, one expects the $\mathcal{N} = 4$ SO(4) solution to sit in a family of two χ-like parameters, thus different from the one found in [70]. This is exactly what the authors of [90] found. The main feature of this new family is that, like the families found in the $\mathcal{N} = 3$ model of previous chapters, it contains both non-supersymmetric and supersymmetric vacua. In particular, the family interpolates between the $\mathcal{N} = 4$ vacuum and the symmetric $\mathcal{N} = 2$ vacua through a two-dimensional manifold of $\mathcal{N} = 0$ $\mathrm{U}(1)^2$ symmetric solutions.

However, as we understood in the previous chapter, to fully understand the nature of the marginal deformations parameterizing the latter families a higher dimensional description of the S-fold Backgrounds is necessary. An analysis based on the computation Kaluza-Klein spectrum for the supersymmetric family described in [70] is carried out in [91]. The χ parameter is shown to be compact not only in the particular section corresponding to the $\mathcal{N} = 2$ solutions of previous chapters but along the whole family. More elusive is the nature of the second parameter and we are not going to discuss it in the present chapter. Instead, we will focus on the higher dimensional interpretation of the two χ-like flat directions of [90]. As we will see, the interpretation of the presence of the χ parameter as a local coordinate redefinition can be transposed in this example and it will give us interesting insights into the perturbative and non-perturbative stability of the dual non-supersymmetric conformal manifold.

6.1 Uplift of an interesting two-parameter family

As already introduced in Chapter 1 the S-fold configurations under discussion relate to the study of conformal manifolds, i.e. the manifold spanned by marginal operators whose β-functions vanish exactly to all orders. Much progress has been made in the study of such objects and their properties in some particular cases [92–96]. For example, it is not uncommon for four-dimensional $\mathcal{N}=1$ and three-dimensional $\mathcal{N}=2$ CFTs to possess conformal manifolds, whose dimensions can be deduced from the symmetry of the CFTs, without the need to compute β-functions or even having a Lagrangian description. However, it is widely believed that non-supersymmetric CFTs in more than two dimensions do not exhibit a conformal manifold. The main reason is that it is unclear how the precise cancellations in the β-functions will be achieved without supersymmetry. However, there are no "no-go theorems" that forbid non-supersymmetric conformal manifolds. As a result, the existence of non-supersymmetric conformal manifolds has been largely the subject of speculation, with only a few systematic analyses performed recently [97–100].

The AdS/CFT correspondence [84, 101, 102] between anti-de Sitter (AdS) solutions of string theory and CFTs provides a powerful tool to address this question, at least in the "large-N limit" where the rank of the gauge group of the CFT is taken to be large. The correspondence maps the conformal manifold of a CFT to a continuous family, known as the "moduli space", of AdS solutions of string theory. As yet, no continuous family of non-supersymmetric AdS solutions of string theory has been constructed, with the possible exception of [103]. Indeed, non-supersymmetric AdS solutions of string theory are conjectured to be unstable [104], with only a handful of isolated potentially stable non-supersymmetric AdS vacua known [65].

In this section we are going to argue that the $\mathcal{N}=8$ gauged model with an uplift to type IIB supergravity provides holographic evidence for a three-dimensional non-supersymmetric conformal manifold. We do this by constructing a 2-parameter non-supersymmetric deformation of an $\mathcal{N}=4$ supersymmetric AdS_4 vacuum describing a non-geometric solution of Type IIB superstring theory. We will prove that the entire 2-parameter family is perturbatively stable in IIB supergravity, and show that it does not suffer from various non-perturbative instabilities. We note that just as for the supersymmetric deformations considered in the χ-family of previous chapters, the non-supersymmetric deformations we study here can also locally be absorbed by coordinate redefinitions, which are, however, not globally well-defined. This implies that any local diffeomorphism-invariant quantities, such as those controlling higher-derivative corrections of string theory, are independent of the deformations. This provides hope that our conformal manifold may also exist beyond the large-N limit of the CFT.

We construct our non-supersymmetric 2-parameter family of AdS_4 vacua of IIB string theory by uplifting the corresponding family of AdS_4 vacua of four-dimensional $[\mathrm{SO}(6)\times\mathrm{SO}(1,1)]\ltimes\mathbb{R}^{12}$ supergravity [90] using the truncation Ansatz of [41]. Our family of AdS_4 vacua depends on two "axionic" parameters χ_1, χ_2 [90]. For generic values of $\chi_{1,2}$, the AdS_4 vacua are non-supersymmetric and preserve a $\mathrm{U}(1)^2$ symmetry. Three patterns of (super) symmetry enhancement occur at special loci of the (χ_1,χ_2) parameter space. For $\chi_1=\pm\chi_2$, there is an $\mathcal{N}=2$ supersymmetry enhancement whereas a $\mathrm{U}(1)^2$ symmetry is still preserved. For $\chi_1=0$ or $\chi_2=0$, the vacua are non-supersymmetric but the residual symmetry gets enhanced to $\mathrm{SU}(2)\times\mathrm{U}(1)$. Lastly, an $\mathcal{N}=4$ and $\mathrm{SO}(4)$ symmetric AdS_4 vacuum appears at the special point $\chi_1=\chi_2=0$. As a result, $\chi_{1,2}$ parameterise non-supersymmetric deformations of the $\mathcal{N}=4$ AdS_4 S-fold vacuum of IIB string theory [41]. The ten-dimensional geometry we obtain is a non-supersymmetric "S-fold" of the form $\mathrm{AdS}_4\times S^1_\eta\times S^5$, where $S^5=\mathcal{I}\times S^2_1\times S^2_2$ and $\mathcal{I}$ is an interval with angular coordinate $\alpha\in[0,\frac{\pi}{2}]$. As in the other S-fold examples, the 10-dimensional solution has an $\mathrm{SL}2,\mathbb{Z}$ S-duality monodromy of IIB string theory as we move around the S^1_η circle. The corresponding dual CFT is known as a J-fold CFT obtained by compactifying $\mathcal{N}=4$ super Yang-Mills theory on a circle with an $\mathrm{SL}2,\mathbb{Z}$ twist [71]. Holography has recently proven powerful in studying supersymmetric AdS_4 vacua of these types and their supersymmetric deformations [41, 105, 69, 90, 106, 89, 70, 91, 107]. More concretely, the S-fold solution can be constructed out of the following 10-dimensional solution of classical Type IIB supergravity:

$$\begin{aligned} ds^2_{10} = \Delta^{-1}\Big[&\tfrac{1}{2}\,ds^2_{\mathrm{AdS}_4}+d\eta^2+d\alpha^2 \\ &+\frac{\cos^2\alpha}{2+\cos(2\alpha)}d\Omega_1+\frac{\sin^2\alpha}{2-\cos(2\alpha)}d\Omega_2\Big]\,, \end{aligned} \tag{6.1.1}$$

where χ_i-twisted two-spheres Ω_i have metrics

$$d\Omega_i = d\theta_i^2+\sin^2\theta_i\,d\varphi_i'^2 \quad \text{with} \quad d\varphi_i' = d\varphi_i+\chi_i\,d\eta\ , \tag{6.1.2}$$

and the non-singular warping factor is

$$\Delta^{-4} = 4-\cos^2(2\alpha)\ . \tag{6.1.3}$$

The two-form potential B_2 and C_2 take the form

$$\begin{aligned} B_2 &= -2\sqrt{2}\,e^{-\eta}\,\frac{\cos^3\alpha}{2+\cos(2\alpha)}\,\mathrm{vol}_{\Omega_1}\,, \\ C_2 &= -2\sqrt{2}\,e^{\eta}\,\frac{\sin^3\alpha}{2-\cos(2\alpha)}\,\mathrm{vol}_{\Omega_2}\,, \end{aligned} \tag{6.1.4}$$

whereas the dilaton $g_s = e^{\Phi}$ and the axion C_0 read

$$e^{\Phi} = \sqrt{2}\,e^{-2\eta}\,\frac{2-\cos(2\alpha)}{\sqrt{7-\cos(4\alpha)}}\,, \qquad \text{and} \qquad C_0 = 0\,. \tag{6.1.5}$$

The four-form potential C_4, yielding a self-dual field strength $\widetilde{F}_5 = dC_4 + \frac{1}{2}\,(B_2 \wedge dC_2 - C_2 \wedge dB_2)$, reads

$$\begin{aligned} C_4 = {} & \tfrac{3}{2}\,\omega_3 \wedge \left(d\eta + \tfrac{2}{3}\sin(2\alpha)\,d\alpha\right) \\ & - \tfrac{1}{2}\,f(\alpha)\,d\alpha \wedge (A_1 \wedge \mathrm{vol}_{\Omega_2} + \mathrm{vol}_{\Omega_1} \wedge A_2)\,, \end{aligned} \tag{6.1.6}$$

where $d\omega_3 = \mathrm{vol}_{\mathrm{AdS}_4}$ with AdS radius $L_{\mathrm{AdS}_4} = 1$. The function $f(\alpha)$ in (6.1.6) is given by

$$f(\alpha) = \sin^2(2\alpha)\,\frac{\cos(4\alpha) - 55}{\left(7-\cos(4\alpha)\right)^2}\,, \tag{6.1.7}$$

where we have introduced one-forms $A_i = -\cos\theta_i\,d\varphi_i'$ so that $dA_i = \mathrm{vol}_{\Omega_i}$. Note that the function $f(\alpha)$ in (6.1.7) vanishes at $\alpha = 0,\, \frac{\pi}{2}$, where each of the S^2 shrinks to zero size in a smooth way so that the compact space is topologically $S^1_\eta \times S^5$.

The S-fold solution, characterised by an $\mathrm{SL}(2,\mathbb{Z})$ monodromy along S^1_η, can then be obtained from the above solution through a suitable $\mathrm{SL}(2,\mathbb{R})$-transformation together with an appropriate choice of the period T, according to the prescription given in [41, 71]. In this way the monodromy can be chosen, for instance, to be a hyperbolic element of the form

$$J_k = \begin{pmatrix} k & 1 \\ -1 & 0 \end{pmatrix}\,, \qquad k > 2\,. \tag{6.1.8}$$

This choice requires the S^1_η radius to be

$$T = \log\left(k + \sqrt{k^2-4}\right) - \log 2\,. \tag{6.1.9}$$

Moreover, k can be chosen such that the supergravity approximation remains valid throughout the S-fold solution, because the dilaton and derivatives of the axio-dilaton remain small throughout [71].

We shall refrain from further discussing these aspects of the solution, since they do not affect our present analysis, which focuses on the 2-parameter deformation of the background and is independent of the duality twist. The $\chi_{1,2}$ deformations only appear in the background via the combination (6.1.2) and thus can locally be absorbed by the coordinate redefinition

$$\varphi_i' = \varphi_i + \chi_i \eta \,. \tag{6.1.10}$$

However, due to the periodicity of $\eta \to \eta + T$, this is only a well-defined coordinate transformation when $\chi_i = \frac{2\pi k_i}{T}$ for $k_i \in \mathbb{Z}$. This suggests that the deformation parameters are periodic with period $\frac{2\pi}{T}$. However, there is a subtlety because of how the spinors are defined on the S^1_η. In fact, by looking at the spinors, as we will demonstrate later in (6.2.3) through the Kaluza-Klein spectrum, we see that the correct periodicity is in fact $\chi_i \in \left[0, \frac{4\pi}{T}\right)$. This means that the non-supersymmetric conformal manifold is compact and has topology $T^2/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ corresponds to the interchange $\chi_1 \leftrightarrow \chi_2$.

An alternative description of the parameters χ_i comes from their oxidation to the five-dimensional supergravity obtained by reducing IIB string theory on S^5. As noted in [89, 90] (see also [107, 106]) the χ_i define non-trivial one-form deformations or Wilson loops, for the vector fields along S^1_η. For the $\mathcal{N}=4$ S-fold, this corresponds to turning on Wilson loops for the $\mathfrak{su}(2)\times\mathfrak{su}(2)$-valued gauge fields breaking the symmetry down to its Cartan subgroup. It is instructive to compare the deformation of the $\mathcal{N}=4$ S-fold solution analysed here, with the deformation, discussed in [103], of the maximally supersymmetric $\mathrm{AdS}_5\times S^5$ Type IIB background, which generalises the Lunin-Maldacena construction [108]. The holographic dual to this solution is conjectured to be a non-supersymmetric marginal deformation of $\mathcal{N}=4$ four-dimensional SYM theory. However, [109] suggested that conformal symmetry of this dual theory is absent, while [110, 111] hint at the existence of a tachyonic instability in the corresponding superstring background. In [103], the deformation parameters γ_I, $I=1,2,3$, were the effect of shift transformations in the $\mathrm{O}(3,3)$ group acting on the three angular directions associated with translational isometries [112] along internal angular coordinates. These shift transformations were, however, preceded and followed by T-dualities of the kind $R_I \to 1/R_I$ along all the three directions. Just as S^5 in the $\mathrm{AdS}_5\times S^5$ background, the internal manifold $\mathcal{I}\times S^2_1\times S^2_2\times S^1_\eta$ of the $\mathcal{N}=4$ S-fold solution features three angular coordinates $\xi^I = \varphi_1, \varphi_2, \eta$ each associated with a translational symmetry of the internal metric, although, in the latter case, a constant translation along η is not a symmetry of the whole solution due to the $\mathrm{SL}(2,\mathbb{R})_{\mathrm{IIB}}$-twist. As opposed to the construction of [103], the χ_i-deformation discussed here only results from a shift transformation in $\mathrm{GL}(3,\mathbb{R}) \subset \mathrm{O}(3,3)$, with no T-dualities. This is effected by the $\mathrm{GL}(3,\mathbb{R})$ matrix

$$A = \begin{pmatrix} 1 & 0 & \chi_1 \\ 0 & 1 & \chi_2 \\ 0 & 0 & 1 \end{pmatrix}, \tag{6.1.11}$$

which acts linearly on the I-component of all the fields. The components $g = (g_{IJ})$ of the internal metric along the angular directions ξ^I, for instance, transforms as follows:

$$g \to A^t g A. \tag{6.1.12}$$

Our χ_i deformations thus change the metric on the $S^5 \times S^1_\eta$ compactification, while leaving the fibration structure unchanged. This is analogous to complex structure deformations of $T^2 \sim S^1 \times S^1$, which can also locally be absorbed by diffeomorphisms that are, however, not globally well-defined. Indeed, our χ_i appear like the real part of complex structure deformations of the $\varphi_i \times S^1_\eta$ tori. A more precise definition is in terms of the mapping torus of S^5 [90]: the χ_i deformations imply that as we move around S^1_η, we end up in a different point on S^5. If $\chi_i \to \chi_i + 2\pi k_i/T$, $k_i \in \mathbb{Z}$, the deformation is in $\mathrm{GL}(3,\mathbb{Z})$ and the internal geometry is not affected. Invariance of the full spectrum, however, due to the presence of states with half-integer j_1, j_2, extends the periodicity of χ_i to $4\pi/T$, as will be discussed below. Via the AdS/CFT correspondence, our family of non-supersymmetric AdS_4 vacua of IIB string theory suggests that the dual "J-fold" CFT_3 should belong to a non-supersymmetric conformal manifold. However, this is not the case if the non-supersymmetric AdS_4 vacua are unstable, as conjectured in [104].

6.2 Perturbative and non-Perturbative (in)Stability

In general, instabilities could arise due to some scalar fluctuation in the Kaluza-Klein spectrum violating the Breitenlohner-Freedman bound, or via a non-perturbative phenomenon. Let us now address these concerns.

First, we can prove that the Kaluza-Klein spectrum has no tachyons, i.e. the AdS_4 vacua are perturbatively stable. To do this, we use the technology developed in [34, 6] to compute the full Kaluza-Klein spectrum around the family of non-supersymmetric AdS_4 vacua we consider here. It is easiest to express the Kaluza-Klein spectrum as a deformation of the spectrum of the $\mathcal{N} = 4$ vacuum [2]. The conformal dimension of the highest weight state of each

[2] The full $\mathcal{N} = 4$ spectrum was computed in [113] and we discuss it in appendix G.

supermultiplet is given by

$$\Delta = \frac{3}{2} + \frac{1}{2}\sqrt{9 + 2\ell(\ell+4) + 4\sum_{i=1,2}\ell_i(\ell_i+1) + 2\left(\frac{2n\pi}{T}\right)^2}\,, \tag{6.2.1}$$

where ℓ denotes the S^5 Kaluza-Klein level, n the S^1 level and ℓ_1, ℓ_2 the SO(4) spin of the highest weight state (in this case, the graviton). These $\mathcal{N}=4$ supermultiplets are counted by the generating function for their highest weight states:

$$\nu = \frac{1}{(1-q^2)(1-qu)(1-qv)}\frac{1+s}{1-s}\,, \tag{6.2.2}$$

where the exponent of q and s determine the Kaluza-Klein levels on the S^5, ℓ, and S^1, n, while the exponents of u and v count the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ spins, ℓ_1 and ℓ_2. The effect of the $\chi_{1,2}$ deformations is to shift the conformal dimension of each field by replacing

$$\frac{2n\pi}{T} \longrightarrow \frac{2n\pi}{T} + (j_1+j_2)\chi_+ + (j_1-j_2)\chi_-\,, \tag{6.2.3}$$

in (6.2.1), where j_1, j_2 are the charges of the field under the $\mathrm{U}(1)\times\mathrm{U}(1)$ Cartan of SO(4) and we defined $\chi_\pm = \frac{1}{2}(\chi_1 \pm \chi_2)$. Note from (6.2.2) that, while j_1, j_2 are half-integers, $j_1 \pm j_2$ are always integers. Thus, we manifestly see that the full background has periodicity $\chi_\pm \to \chi_\pm + \frac{2\pi}{T}$, upon which the Kaluza-Klein spectrum gets mapped back to itself with an appropriate reshuffling of the fields amongst the S^1 level with $n \to n-(j_1 \pm j_2)$, just like in [3]. Notice that χ_1, χ_2 separately have period $4\pi/T$, which can only be seen from the spinors on the AdS_4 background which have half-integers charges under the $\mathrm{U}(1)\times\mathrm{U}(1)$ Cartan. Even more importantly, we can see that the masses for all the fields are bounded from below by the masses of the fields of the four-dimensional supergravity at the $\mathcal{N}=4$ vacuum, i.e. where $\ell=\ell_1=\ell_2=n=\chi_i=0$. This in particular implies that all scalars have masses above the Breitenlohner-Freedman bound for any value of χ_i. Thus, the non-supersymmetric vacua are perturbatively stable. One may also wonder whether the AdS_4 vacua are secretly supersymmetric in 10 dimensions, with some gravitinos amongst the higher Kaluza-Klein modes becoming light, akin to the “space invaders” scenario [74, 75, 3]. However, from (6.2.1), (6.2.3), we can easily see that such gravitinos can only restore supersymmetry when the combination $\frac{2n\pi}{T} + j_1\chi_1 + j_2\chi_2 = 0$. This can only occur when either $n=0$ and $\chi_1 = \pm\chi_2$, corresponding to supersymmetry enhancement that already occurs in the four-dimensional supergravity [90], or $\chi_\pm = \frac{2\pi k_\pm}{T}$, for $k_\pm \in \mathbb{Z}$ when some gravitinos at S^1 level $n = -(j_1+j_2)k_+ - (j_1-j_2)k_-$ become massless. This latter condition is precisely when the background is mapped back to itself, so that for $0 < \chi_\pm < \frac{2\pi}{T}$, $\chi_1 \neq \pm\chi_2$, the AdS_4 vacua are not supersymmetric in the full Type IIB string theory.

Next we investigate the non-perturbative stability of the non-supersymmetric AdS_4 vacua. Since the AdS_4 vacua arise as near-horizon limits of certain brane configurations, one may worry that for the non-supersymmetric vacua the corresponding brane configurations become unstable [114]. We search for signs of such instabilities by considering single probe Dp-branes (and single probe NS5-branes) with rigid embeddings in our AdS_4 vacua. In particular, we check whether the branes are unstable due to a greater repulsive force of the fluxes coming from the WZ term than the attractive (i.e. towards the interior of the AdS spacetime) gravitational force due to the DBI term. Indeed, [104] conjectures that there should always be some branes that are unstable in this way, see also [115]. However, we find that single probe Dp-branes and NS5-branes without worldvolume flux remain stable when placed in the non-supersymmetric backgrounds (6.1.1)–(6.1.7). The stability of these probe branes can be understood in the following way. Firstly, note that we can perform the diffeomorphism (6.1.10) to remove the χ_i deformation from the metric. However, now the coordinates respect the deformed periodicities

$$\begin{aligned} &\varphi_i' \to \varphi_i' + 2\pi\,, \\ &\eta \to \eta + T\,, \quad \varphi_i' \to \varphi_i' + \chi_i T\,. \end{aligned} \tag{6.2.4}$$

As a result, the only well-defined embeddings of branes wrapping η must also wrap φ_i'. In particular, let us denote by $\xi \sim \xi + T$ the relevant wrapped worldvolume coordinate on the brane. Then, the only well-defined embeddings are given by

$$\eta(\xi) = q\,\xi\,, \qquad \varphi_i'(\xi) = \left(p_i \frac{2\pi}{qT} + \chi_i\right)\xi\,, \tag{6.2.5}$$

with $p_i \in \mathbb{Z}$. We see that as χ_i is turned on, a brane wrapping S^1_η must also wrap increasing amounts of φ_i', so that the DBI part of the action increases. At the same time, for p-branes, with $p \neq 5$, the WZ coupling is insensitive to wrapping along φ_i', unless the brane is completely internal. Therefore, these branes either become more stable as χ_i are turned on or they are completely internal branes, which cannot trigger non-perturbative instabilities in the usual way. Finally, an explicit computation for NS5- and D5-branes shows that they also remain stable as χ_i are turned on in the backgrounds (6.1.1) – (6.1.7). Finally, non-supersymmetric vacua may also decay due to bubbles of nothing [116], which requires a circle or sphere [117] to collapse. However, our internal space $S^5 \times S^1_\eta$ is topologically protected from such a collapse: the S^5 cannot collapse as it is supported by flux, whereas the S^1_η cannot collapse since the spinors do not have anti-periodic boundary conditions on it [116], but instead general periodicities along S^1_η, provided $(\chi_1, \chi_2) \neq (\frac{2\pi}{T}, 0), (0, \frac{2\pi}{T})$. This means that a straightforward bubble of nothing cannot occur. Still, our vacua could decay semi-classically via more complicated bubbles of nothing containing defects, e.g. a D3-brane in S^5 similar to [118, 119] or an O7-plane in S^1 [120]. However, because the volume form of the compactification is independent of the χ_i deformations, our non-supersymmetric AdS_4 vacua are likely to be stable against the instanton decay constructed in [119], which is delocalised on the compactification space. On the other hand, constructing the localised instanton solutions is extremely technically challenging. Moreover, the mechanism of [119] treats a shrinking dilaton as equivalent to a shrinking S^1. Aside from the validity of this equivalence, a similar shrinking dilaton would be problematic for our S-fold vacua, where the dilaton is not well-defined due to the quantized $\mathrm{SL}(2,\mathbb{Z})$ monodromy along S^1_η.

So far, we have proven that our AdS_4 vacua are perturbatively stable and have provided evidence that they may also be stable against semi-classical decay. Still, one may worry that while our AdS_4 geometries are solutions of IIB supergravity, the higher-derivative corrections of IIB string theory will spoil our solutions. In the dual CFT, this would imply that some $\frac{1}{N}$ corrections lift the conformal manifold. However, even if not possible globally, the deformations χ_i can always be locally absorbed by the coordinate redefinition (6.1.10). Therefore, all local diffeomorphism-invariant quantities are independent of the χ_i. In particular, this means that each term of the higher-derivative corrections of string theory, involving powers of the curvature tensor or the fluxes, are also independent of $\chi_{1,2}$. Thus, our non-supersymmetric AdS_4 vacua are equally valid solutions of IIB string theory as the $\mathcal{N}=4$ vacuum. Moreover, the χ_i deformations actually correspond to parity-odd (pseudo) scalars in the maximal supergravity [90], so the potential $1/N$ tadpole destabilisation of [121] cannot take place for our backgrounds. There could still be some string corrections, e.g. from branes wrapping the compactification, which are sensitive to χ_i and which could thus spoil our solutions. For example, Dp-instantons could wrap some $(p+1)$-cycle of the compactification, and depend on χ_i. However, our solutions are also protected against such instanton corrections, since the compactification $S^5 \times S^1_\eta$ only has non-trivial 1-, 5- and 6-cycles. Therefore, we can only have D5-instantons wrapped on the full $S^5 \times S^1_\eta$. But since the volume form is independent of χ_i, these instantons give no corrections to our solutions. Nonetheless, one could expect some other extended state to do so, corresponding to some $\frac{1}{N}$ correction in the dual CFT.

6.3 Holographic Considerations

According to the proposal put forward in [71], the SCFT dual to the $\mathcal{N}=4$ background emerges as the effective IR description of a $3d$ $\mathrm{T}[\mathrm{U}(N)]$ theory

[77] in which the diagonal subgroup of the $\mathrm{U}(N)\times\mathrm{U}(N)$ flavour group has been gauged using an $\mathcal{N}=4$ vector multiplet. In addition, a Chern-Simons term at level k must be introduced which is defined by the $J_k=-\mathcal{S}\mathcal{T}^k\in\mathrm{SL}(2,\mathbb{Z})_{\mathrm{IIB}}$ monodromy along the S^1_η. The effective $\mathcal{N}=4$ superpotential [122] $W_{\text{eff}}=(2\pi/k)\,\mathrm{Tr}(\mu_H\,\mu_C)$ is marginal in the IR and, in [70], a shift $W_{\text{eff}}\rightarrow W_{\text{eff}}+\lambda\,\mathrm{Tr}(\mu_H\,\mu_C)$ with $\lambda\in\mathbb{C}$ was proposed as an exactly marginal deformation preserving $\mathcal{N}=2$. The scalar superconformal primary operators μ_H and μ_C are respectively described by the moment maps of the Higgs and Coulomb branch of $\mathrm{T}[\mathrm{U}(N)]$. Each of the μ_H and μ_C fields is a component of a vector in the adjoint representation of the corresponding $\mathrm{SU}(2)$ subgroup of the $\mathrm{SO}(4)$ R-symmetry group (to be denoted by $\mathrm{SU}(2)_H$ and $\mathrm{SU}(2)_C$, respectively). We can therefore associate with μ_H the quantum numbers $j_1=1, j_2=0$ and with μ_C the values $j_1=0, j_2=1$, having identified j_1, j_2 with the eigenvalues of the Cartan generators of $\mathrm{SU}(2)_H$ and $\mathrm{SU}(2)_C$, respectively. Note that χ_1 (χ_2) only breaks $\mathrm{SU}(2)_H$ ($\mathrm{SU}(2)_C$) to its $\mathrm{U}(1)_H$ ($\mathrm{U}(1)_C$) subgroup. The combination $(\chi_1-\chi_2)/2$ of these two parameters, for $\chi_1=-\chi_2$, should already be encoded in the λ parameter of the $\mathcal{N}=2$ exactly marginal deformation proposed in [70]. We suggest that the orthogonal combination $(\chi_1+\chi_2)/2$, be encoded in the conjectured exactly marginal deformation of the $3d$ Lagrangian:

$$\partial_\alpha\mathcal{O}\,\partial^\alpha\overline{\mathcal{O}}\ , \tag{6.3.1}$$

where $\mathcal{O}\equiv\mathrm{Tr}(\mu_H\,\overline{\mu}_C)$ is an operator with $j_1=1, j_2=-1$ and ∂_α denote the partial derivatives with respect to the (real) scalar fields. As opposed to $\mathrm{Tr}(\mu_H\,\mu_C)$, the above term does not originate from a holomorphic deformation of the superpotential and thus would break all supersymmetries. The exact marginality of the operator (6.3.1) is here conjectured in light of the holographic evidence we put forward. Note that the resulting $\mathcal{N}=0$ theory would be parity symmetric in both the Higgs and the Coulomb sector: By performing, for instance, a parity transformation in the Coulomb sector which changes sign to the complex structure of the hyper-Kähler manifold (described as a complex Kähler space), $\mu_C\rightarrow\overline{\mu}_C$, and $\mathcal{O}$ would be exchanged with the exactly marginal operator $\mathrm{Tr}(\mu_H\,\mu_C)$ in the superpotential proposed in [70]. The same transformation would correspond in the bulk to a parity $\varphi_2\rightarrow-\varphi_2$ in S^2_2 and, correspondingly, to $\chi_2\rightarrow-\chi_2$. It is therefore the simultaneous presence of the deformations $\mathcal{O}$, $\overline{\mathcal{O}}$ and $\mathrm{Tr}(\mu_H\,\mu_C)$ in the Lagrangian which breaks supersymmetry. Finally, our computation of the Kaluza-Klein spectrum (6.2.1), (6.2.3) reveals the $\frac{4\pi}{T}$ periodicity of the exactly marginal deformations parameterised by χ_i, and it also gives the anomalous dimensions of all operators of the CFT dual to supergravity modes along the non-supersymmetric conformal manifold.

Chapter 7

Conclusions

Let us proceed with the conclusive remarks by quickly summarizing the contents of the present work. First of all, it should be now clear that the main innovations and contributions presented here belong to the supergravity field of study. However, the subjects of previous chapters are deeply related, as I have tried to explain, to other main theoretical frameworks such as String Theory and Conformal Field Theory. Nevertheless, I have tried to be concise on those aspects not strictly related to the analysis fruit of the last few years of research.

In Chapter 2 a detailed description of a $D = 4$ Supergravity theory is presented. In the review, we adopt the point of view of [11]. We first present the relevant properties of an Ungauged model, the minimal version of a supergravity model, such as the field content of the theory, the local and global symmetries it exhibits, the structure of the couplings, the on-shell properties of the model and so on. In particular, we put our focus on the representations of the various fields with respect to the on-shell global symmetry group G and with respect to its compact subgroup H. Indeed, the latter are the main characters in the gauging procedure and the understanding of many supergravity models as consistent truncations of $D = 10/11$ supergravities. We have explained how to properly choose a subset of the isometry group of the scalar manifold as a good candidate for a local symmetry of the model and how the gauging procedure helps us in introducing interesting dynamics for the scalar fields. The main technical ingredient is the G-covariant formulation in terms of the tensor $\mathbb{T}$. The latter "knows" everything about the resulting gauged supergravity. Indeed, the scalar potential and the couplings appearing in the Lagrangian can all be expressed in terms of H-tensors properly identified from the T-tensor. We have shown how the consistency of the model boils down to purely algebraic constraints, the so-called "linear and quadratic constraints", in $\mathbb{T}$.

In Chapter 3 the formalism is specialized to the $\mathcal{N} = 3$ supersymmetric extended case where the global symmetry group is given by $G = \mathrm{SU}(3, n)$. The latter being the isometry group of the scalar manifold G/H of the model, with isotropy group $H = \mathrm{S}[\mathrm{U}(3) \times \mathrm{U}(\mathrm{n})]$. We focus on the $n = 9$ case and we argue how this could be interesting given the compactification of $D = 11$ supergravity on a tri-Sasakian manifold. Indeed, the model feature, in a purely electric frame, according to the linear and quadratic constraints a gauge group of the form $G_g = \mathrm{SO}(3) \times \mathrm{SU}(3)$. We identify the T-tensor and its relevant H-invariant blocks necessary to define the shift tensors and mass matrices. Of interest is the property of the gauge group of being a direct product of two semi-simple factors allowing us to introduce two coupling constants g_1 and g_2. In particular, we build the scalar potential of the model which depends on the 54 scalar fields and we restrict ourselves to two simpler truncations identified by three complex scalars $z_i = r_i\, e^{i\,\alpha_i}$ each. Within the latter subsectors, we can find all the solutions with constant scalars, vanishing fermions and vectors. In particular, both truncations share the central $\mathcal{N} = 3$ $\mathrm{SO}(3) \times \mathrm{SU}(3)$ symmetric vacuum. Each of the two truncations contains a family of AdS_4 vacua which can be arranged in a space topologically equivalent to a three-torus quotiented by the action of a discrete group $\Gamma \sim S_4$. Interestingly, the latter families are present only for specific ranges of the coupling constants. The first family exists when $g_1 \leq g_2 \leq 2g_1$ while for $g_2 > 2g_1$ they are both present. For each family one finds a single $\mathcal{N} = 3$ vacuum, a one-parameter subfamily of $\mathcal{N} = 2$ vacua, a two-parameter family of $\mathcal{N} = 1$ vacua. They describe a point, a line, and a surface inside T^3/Γ respectively, and the rest of the space is filled with non-supersymmetric points. The subgroups of SO(3) preserving the latter loci inside the torus correspond to the preserved R-symmetry of the solutions[1]. The angular coordinates of T^3/Γ exactly correspond to the scalar fields α_i which are flat directions of the scalar potential. We then expect the latter to holographically describe marginal deformations of the dual CFT at the boundary of AdS_4. In particular, since all the presented solutions are perturbatively stable we expect the latter three-dimensional conformal manifold to be perturbatively stable, even when the non-supersymmetric points are considered. We are also able to find Domain Wall solutions connecting the isolated maximally symmetric vacuum at the origin of the scalar manifold to any of the solutions presented in 3.4.1. Again, the constant values of the scalar fields α_i select the supersymmetry preserved far away from the "Wall". In the dual picture, they would correspond to RG-flows from the UV theory dual to the $\mathrm{SO}(3) \times \mathrm{SU}(3)$ symmetric vacuum and the IR conformal field theories dual to the T^3/Γ family. As we refrain below, this property is reminiscent of the picture arising in the $\mathcal{N} = 8$ models presented in previous chapters. However, in the $\mathcal{N} = 3$ a higher dimensional description of the configurations is still missing. An important question to address in the future is whether the loci of vacua

[1]We understand that the analysis of [123] is related to the same model but it does not cover this new results.

described here can be uplifted to $D = 10/11$-dimensional supergravities. As an example, one could try to embed a suitable truncation of the model studied here (containing the 3-tori of vacua) into maximal supergravity, and then use exceptional field theory techniques [46, 4, 5] to uplift them to string or M-theory. Another possibility is that the truncation of the model describing the new vacua studied here does not fit in a maximal supergravity. In this case, one should work with less supersymmetric consistent truncations possibly implementing the analysis of [7]. In particular one could try to obtain a subsector of our model capturing the new solutions and the central one as a compactification of string or M-theory using a suitable G_S-structure manifold. There is also the possibility that no consistent truncation can describe our solutions. If the uplift of the whole new family of vacua is possible, assessing perturbative stability of the corresponding $\mathcal{N} = 0$ backgrounds would in principle require the computation of the corresponding Kaluza-Klein spectrum to check if the $D = 4$ scalar-modes have all squared masses exceeding the BF bound.

Moving to the second main case of study, in Chapter 4 we apply the general formalism for the description of the ungauged maximal model in four dimensions, it is to say the $\mathcal{N} = 8$ $D = 4$ supergravity exhibiting a global $G = \mathrm{E}_{7(7)}$ symmetry. We present the general theory of the gauging in the $\mathrm{SL}(8,\mathbb{R})$ frame. In particular, the relevant component of the embedding tensor necessary to promote an electric $\mathrm{CSO}(p,q,r)$ subgroup of G and its dyonic counterpart are presented. We then focus on the case $G_g = [\mathrm{SO}(1,1) \times \mathrm{SO}(6)] \ltimes \mathbb{R}^{12}$ dyonic gauging. Again, the full scalar potential is a complicated function of the 70 scalar fields of the model and we restrict to the analysis of a $[\mathrm{SL}(2)/\mathrm{SO}(2)]^7$ subsector of the scalar manifold consistently identified as the one parametrized by singlets respect to a discrete $(\mathbb{Z}_2)^3$ symmetry of the theory. We review the known solutions which in this case have a direct interpretation in terms of $D = 10$ type IIB supergravity. Indeed, thanks to the ExFT framework and the generalized Sherk-Schwarz ansatz one can in general obtain the U-matrix for the $\mathrm{CSO}(p,q,r)$ gaugings and the dyonic gaugings with gauge group (4.1.10).

In chapter 5 we focus on a particular family of $\mathcal{N} = 2$ AdS_4 vacua present in the $G_g = \mathrm{SO}(6) \times \mathrm{SO}(1,1)] \ltimes \mathbb{R}^{12}$ case originally found in [43]. Interestingly we can provide an uplift of the latter family to type IIB supergravity. We can do so thanks to the general results of [41] which we specialize to the case at hand. We find that the one-parameter family of $\mathcal{N} = 2$ uplift to a so-called S-fold configuration in ten dimensions with the geometry locally described by $\mathrm{AdS}_4 \times S^5 \times S^1$. Its global features are determined by the fact that going around the S^1 circle causes the solution to transform in an $\mathrm{SL}(2,\mathbb{Z})$ dual configuration which can be interpreted as a patching transformation between two local descriptions of the same background. Indeed, $\mathrm{SL}(2,\mathbb{Z})$ is a well-known type IIB global symmetry. The uplift gives us a new insight into the physical properties of the vacuum. Indeed, as far as the four-dimensional analysis is concerned, the flat direction χ parameterizing the family of solutions appears as a non-compact variable which when vanishing corresponds to an $\mathcal{N}=2$ SU(2) $\times$ U(1)$_R$ symmetric solution. For other constant values of χ, the solution is $\mathcal{N}=2$ U(1) $\times$ U(1)$_R$ symmetric. However, the ten-dimensional picture unveils new interesting features. Indeed, we show, both by studying the full Kaluza-Klein spectrum of the background and by giving it a nice geometrical interpretation in terms of a complex structure of the $S^3 \times S^1$ part of the internal space, that χ is periodic with period given by $\frac{2\pi}{T}$, the latter being the period of the S^1 factor in the internal geometry. In particular, the Kaluza-Klein spectrum undergoes under a non-trivial permutation of its modes as $\chi \to \chi + \frac{2\pi}{T}$. Furthermore, we find that $\chi \sim 0$ are not the only special points of the one-parameter family. The values $\chi \sim \frac{\pi}{T}$ give explicit examples of the "space-invaders" scenario where vector in the Kaluza-Klein tower become massless and are responsible for a symmetry enhancement of the solution to SU(2) $\times$ U(1)$_R$ [74]. These features have a clear interpretation from the geometrical point of view too. In particular, we show that χ can be interpreted locally as a coordinates redefinition applied to the $\chi = 0$ configuration. However, the latter change of coordinates is only locally well-defined and it can not be extended globally unless $\chi = \frac{2\pi}{T}$. Otherwise, the global geometry of the space is found to be a fibration of $S^3 \subset S^5$ over S^1 with monodromy

$$h(T) = \begin{pmatrix} \cos(\chi T) & \sin(\chi T) \\ -\sin(\chi T) & \cos(\chi T) \end{pmatrix} \tag{7.0.1}$$

around the S^1 cycle. The latter fibration is topologically equivalent to the trivial one so that we can still rely on the $S^5 \times S^1$ harmonics to compute the Kaluza-Klein spectrum for generic values of χ. However, they differ in the structure of the metric tensor defined on the latter space. First of all, the line element in the generic case does not exhibit the SU(2) $\times$ U(1) isometry of the $\chi = 0$ case. Indeed, $h(T)$ breaks the SU(2) isometry to its commutant with $h(T)$ which, for generic values of χ, is given by U(1). However, when $\chi = \frac{\pi}{kT}$, with $k \in \mathbb{Z}$, $h(T) \subset \mathbb{Z}_k$. This explains the special features observed when $\chi = \frac{\pi}{T}$, since now $h(T) = -\mathbf{1}$, preserves all of SU(2) and leaves invariant all SU(2) integer-spin states. When $\chi \to \frac{2\pi}{T}$ the shift can be now reabsorbed in a globally well-defined change of coordinates. This can be seen in a particularly clear way by writing the background as a T^2-fibration over S^2, which is further warped over S^2. As we showed, the parameter χ then appears as a complex structure modulus of the T^2-fibre and a shift $\chi \to \chi + \frac{2\pi}{T}$ corresponds to a Dehn twist of the T^2.

As far as the Kaluza-Klein spectrum of the $\mathcal{N}=2$ background for generic values of χ is concerned we find that all perturbative modes can be arranged in $\mathcal{N}=2$ $D=4$ long superconformal multiplets. In general, the conformal dimension of Kaluza-Klein modes depends on the radius of the internal T which can, at the $D=10$ supergravity level of analysis, assume different values compatible with the S-folding procedure, see below 5.3.40. Then only for specific values of the latter, the conformal dimensions will be exactly the ones saturating the relevant unitarity bounds of the long multiplets, causing them to split into short ones. This simple observation turns out to be very useful. Indeed, we can infer the superconformal multiplet structure of the perturbative modes just by looking at the Kaluza-Klein tower of the gravitational field. This technique, applied to the known $\mathcal{N}=4$ AdS_4 vacuum of the same $[\mathrm{SO}(6)\times\mathrm{SO}(1,1)]\ltimes\mathbb{R}^{12}$ gauged supergravity [43] is reviewed in appendix G. Given these results, some issues are left to investigate. As an example, our interpretation of the χ twist as the complex structure of the $S^3\times S^1$ once described with an elliptic fibration opens the possibility of adding an extra modulus deforming the solution. The latter would be different from the non-compact modulus studied in [91] whose geometric role is still unclear. Another open problem is given by the fact that other $\mathcal{N}=2$ solutions could be obtained once a suitable quotient is performed on the one studied here. Indeed, a detailed inspection of the Kaluza-Klein spectrum suggests that at the intermediate point $\chi=\frac{\pi}{T}$ one could consistently truncate the model to states corresponding to the vector representation of the SU(2) isometry group. One can go further in this argument by noticing that one may replace the internal S^3 geometry with $S^3/\mathbb{Z}_k$, $k\in\mathbb{Z}^+$. Since the quotient does not break the $\mathrm{U}(1)_R$ R-symmetry, the resulting AdS_4 vacua would still be $\mathcal{N}=2$ supersymmetric. However, the $\mathbb{Z}_k$ quotient, for $k\geq 3$, breaks the isometries of the background, for all χ, to $\mathrm{U}(1)\times\mathrm{U}(1)'$, while the $\mathbb{Z}_2$ quotient preserves the $\mathrm{SU}(2)\times\mathrm{U}(1)'$ isometry at $\chi=0$. The periodicity of the modulus χ is also affected by the quotient. For a $\mathbb{Z}_k$ quotient, it is now given by $\chi\sim\chi+\frac{2\pi}{kT}$. This can be seen by noticing that for this value of χ, the monodromy matrix $h(T)\in\mathbb{Z}_k$ now acts trivially on $S^3/\mathbb{Z}_k$. The same conclusion can be reached by looking at the complex structure of the T^2 fibration over S^2 as in (5.3.2), where the Hopf fiber of $S^3/\mathbb{Z}_k$ and the S^1 parametrized by η make up the T^2 fiber. Since the Hopf fibre now has periodicity $\frac{4\pi}{k}$, the shift $\chi\to\chi+\frac{2\pi}{kT}$ corresponds to a Dehn twist. Furthermore, the $\mathbb{Z}_k$ quotient projects out various states, thus reducing the KK spectrum and isometries. In particular, at Kaluza-Klein level 0, only the states corresponding to 4-dimensional $\mathcal{N}=4$ supergravity survive the projection. This includes the modulus χ. For $k=2$, the vacua seem to admit a consistent truncation with 6 vector multiplets, while for $k\geq 3$, the vacua seem to admit a consistent truncation with 4 vector multiplets. These truncations can, in principle, be constructed by performing the $\mathbb{Z}_k$ quotient on the twist matrices (5.2.9) and assembling the invariant objects into a half-maximal structure [124].

Other generalizations of the $\mathcal{N}=2$ family of solutions parameterized by the compact χ modulus are discussed in Chapter 6. After reviewing the recent results present in the literature we focus on the particular case of a family described by two χ-like parameters, (χ_1, χ_2). With similar techniques to the one implemented in the previous case, we can prove χ's compactness both from the geometric and the Kaluza-Klein spectrum point of view. The topology of the internal manifold is still given by $S^5 \times S^1$, however in this case it is useful to interpret S^5 as $S^2_1 \times S^2_1 \times \mathcal{I}$ and χ_i is interpreted as a modulus of the S^2_i part of the geometry. This configuration corresponds to a marginal deformation of the $\mathcal{N}=4$ S-fold background so that in a unique family we encompass both the latter solution and the $\mathcal{N}=2$ χ family suggesting that the SCFT should admit at least a two-dimensional conformal manifold. Interestingly, non-supersymmetric cases are included for generic values of the moduli χ_1 and χ_2. We analyze in detail the perturbations on such configurations and we find that they are indeed perturbatively stable. Moreover, we also give arguments suggesting that this two-parameter family of S-fold backgrounds may also be stable against non-perturbative phenomena. In particular, we exclude brane-jet instabilities. We argue that these configurations should be stable against "bubble of nothing" solutions [116]. Furthermore, the backgrounds for generic values of the moduli, hence the non-supersymmetric ones too, are as valid as the $\mathcal{N}=4$ solution when considering higher derivative corrections to go beyond the supergravity approximation. Many of these perturbative and non-perturbative considerations rely on the remarkable property that χ_1 and χ_2 can be locally reabsorbed by a local diffeomorphism which however is not globally well-defined. This example teaches us that non-supersymmetric vacua connected to stable supersymmetric ones by continuous parameters are expected to be perturbatively stable. We can then borrow this argument and we can apply it to the $\mathcal{N}=3$ model of previous chapters to anticipate perturbative stability of the T^3/Γ vacua if uplifted to higher dimensional supergravity. Non-perturbative statements will then require a more detailed analysis.

The presence of non-supersymmetric (non)-perturbatively stable points on the (χ_1, χ_2) manifold is a crucial feature to investigate given the conjectured relation between the S-fold solutions and certain CFT in three dimensions. In particular, the five-dimensional point of view on this type of backgrounds, already discussed in previous chapters, suggests a deep relationship with the so-called Janus configurations. Type IIB Janus solutions were first discussed in [67]. Their formulation is based on the AdS_4 slicing of AdS_5 which allows us to describe the AdS_5 conformal boundary as divided into two regions separated by an interface located at the point where the conformal factor of the boundary metric diverges. When working in the Poincaré patch for AdS_4 this gives an $\mathbb{R}^{1,3}$ conformal boundary sliced in two by an $\mathbb{R}^{1,2}$ surface. This gives a natural set-up for dual theories to Janus solution as interface theories in $D=4$ $\mathcal{N}=4$

SYM. The solution studied in [67] is a non-supersymmetric stable deformation of the famous $AdS_5 \times S^5$ solution where the dilaton has a non-trivial profile giving two different boundary values on the two parts of the conformal boundary. It is to say, the boundary configuration exhibits piece-wise constant dilaton, with a discontinuity through an interface, where the two pieces of the boundary join. The name “interface” is because no extra degrees of freedom are present on the discontinuity. This is discussed in [85] where a proposal for the CFT dual to the Janus configuration is given and tested. It is essentially the $D = 4$ SYM theory with varying gauge coupling across an interface which preserves SO(2,3) conformal symmetry and breaks supersymmetry. In [68] a generalization of the latter solution has been constructed providing an $\mathcal{N} = 1$ supersymmetric Janus configuration on the gravity side. Its dual is believed to be the $\mathcal{N} = 1$ interface SYM found again in [85]. Their construction is based on the results of the classification of supersymmetric interface gauge theories in [72]. The symmetry of the solutions would be $\mathrm{SO}(2,3) \times \mathrm{SU}(3) \times \mathrm{U}(1) \times \mathrm{SL}(2,\mathbb{R})_{\mathrm{IIB}}$ corresponding to the internal manifold topology $\mathbb{R} \times AdS_4 \times S^5$ with a squashed S^5 given by a fibration of S^1 over $\mathbb{CP}_2$. The $\mathrm{U}(1) \times \mathrm{SL}(2,\mathbb{R})_{\mathrm{IIB}}$ factor acts as a duality of the solutions. An analysis of the asymptotic behavior near the conformal boundary of the external metric shows that the type IIB two-form $B_{\mu\nu}$ provides a delta-like source for operators on the interface. This is a main difference with the original Janus solution where the SO(6) internal symmetry forbids the presence of a non-vanishing B field. This is consistent with the analysis of [72][85]. In [72], a classification of supersymmetric Janus solutions on the CFT side is carried on. This classification would correspond to a classification of supersymmetric Janus configurations in Supergravity. A main difference with the non-supersymmetric case is that the $\mathcal{N} = 4$ SYM fields on the two sides of the interface provide delta-like sources with support on the interface. All the possible interface gauge invariant renormalizable operators are discussed and added to the canonical $\mathcal{N} = 4$ SYM theory. In this way, the inequivalent supersymmetric configurations with maximal internal symmetry are classified. Notable mentions to our discussions are given by the $\mathcal{N} = 4$ $\mathrm{SU}(2) \times \mathrm{SU}(2)$-symmetric model and the $\mathcal{N} = 2$ $\mathrm{U}(1) \times \mathrm{SU}(2)$-symmetric one. The classification can be refined by considering deformations further breaking the flavour symmetry. In this way an $\mathcal{N} = 2$ model $\mathrm{U}(1) \times \mathrm{U}(1)$-symmetric is found. Already at this level, we see the strong similarities between the latter case and the χ deformation studied in the present work. The $\mathcal{N} = 1$ case of [85] with SU(3) internal symmetry is also recovered. In [73], the $\mathcal{N} = 4$ supersymmetric theory of [72] is generalized so to include a varying θ-angle. This gives a well-defined $\mathrm{SL}(2,\mathbb{R})_{\mathrm{IIB}}$ action on the Janus configurations on the field theory side which was used in [68] to generalize the original Janus configuration of [67]. In particular, they find that these new Janus solutions correspond to a pure dilatonic configuration acted upon with an $\mathrm{SL}(2,\mathbb{R})_{\mathrm{IIB}}$.

Then, as in [72] these supersymmetric theories have a superconformal limit corresponding to a generic Yang-Mills coupling jumping on the interface. The tools for this construction are put forward in [86] and further studied in [77]. They are the basis for a more general understanding of interfaces in $\mathcal{N}=4$ SYM. In the latter, the action of S-duality on these theories is discussed in detail and paired to mirror symmetry in three-dimensional gauge theories. In particular, the so-called "$T(G)$ theories" are constructed. As shown in [71], the latter theories are relevant in the dual formulation of the S-fold supergravity configurations. Indeed, the authors present strong evidence that the theories dual to the $\mathcal{N}=4$ S-fold configurations of [41] are given by what they refer to as J_n CFTs. They argue that the UV behavior of the latter can be described by the three-dimensional interface $T[\mathrm{U}(N)]$ which exhibits global $\mathrm{U}(N)\times\mathrm{U}(N)$ symmetry with the addition of a level n Chern-Simons term. Furthermore, the latter model is coupled to a four-dimensional gauge theory with two $\mathrm{U}(N)$ vector multiplets on each side of the interface. The novelty with respect to the original description of the $T[\mathrm{U}(N)]$ models is that the latter gauge fields are used to gauge a $\mathrm{U}(N)$ subgroup of the $\mathrm{U}(N)\times\mathrm{U}(N)$ global symmetry on the interface. By flowing in the IR one then obtains a $\mathrm{U}(N)$ gauge theory with a level n supersymmetric Chern-Simons term coupled to the $T[\mathrm{U}(N)]$ model. When considering the large N-limit of the latter theories, one obtains a good candidate for the SCFT's dual to the $\mathcal{N}=4$ S-fold configuration, as the authors of [71] show by explicitly comparing the on-shell action of the type IIB configuration and the three-sphere partition function of the J_n theory. Once this correspondence is established it can be exploited to understand the nature of the theories dual to marginal deformations of the $\mathcal{N}=4$ S-fold background and the other S-fold configurations.

As far as the two-parameter family of S-folds is concerned, we provided the first holographic evidence for the existence of a non-supersymmetric conformal manifold. Nonetheless, the fate of this family of non-supersymmetric AdS_4 vacua deserves further investigation. The brane-web whose near-horizon limit corresponds to the AdS_4 vacua could still suffer from some other instability mechanism. For example, it could feature some tachyon in its fluctuation spectrum, see e.g. [125, 126] for recent discussions. However, because we do not know the brane-web that would give rise to the AdS_4 vacua, it is currently unclear which probe branes to use for this computation. Still, the existence of a continuous limit to the $\chi_i=0$ supersymmetric case could help in taming such potential instabilities. Also, some non-perturbative string corrections could lift the moduli space. Finally, the CFT_3 interpretation of the χ_i deformations deserves further exploration. As already mentioned, the vacua of the $\mathcal{N}=3$ gauged supergravity presented in previous chapters and their possible uplifts to $D=10/11$ supergravity deserve further investigation. The ten or eleven-dimensional backgrounds, if found, would then provide further holographic evidence in favor of the existence of non-supersymmetric conformal manifolds.

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Appendix A

Ward Identity

A particular case of eq. (2.2.17) is the following one

$$\mathbb{T}_{\underline{\Lambda}\underline{R}}{}^{\underline{\Pi}}\mathbb{T}^{\underline{\Sigma}}{}_{\underline{\Pi}}{}^{\underline{\Delta}} - \mathbb{T}^{\underline{\Sigma}}{}_{\underline{R}}{}^{\underline{\Pi}}\mathbb{T}_{\underline{\Lambda}\underline{\Pi}}{}^{\underline{\Delta}} + \mathbb{T}_{\underline{\Lambda}}{}^{\underline{\Sigma}}{}_{\underline{\Pi}}\mathbb{T}^{\underline{\Pi}}{}_{\underline{R}}{}^{\underline{\Delta}} = 0 \tag{A.0.1}$$

Terms like $\mathbb{T}_{M}{}^{\underline{\Lambda}\underline{\Sigma}}$ and $\mathbb{T}_{M\underline{\Lambda}\underline{\Sigma}}$ do not appear because $\mathscr{R}$ is block-diagonal. We further restrict (A.0.1) to

$$\begin{aligned}
\mathbb{T}^{AD\underline{\Pi}}\mathbb{T}_{B\underline{\Pi}C} - \mathbb{T}_{B}{}^{D\underline{\Pi}}\mathbb{T}^{A}{}_{\underline{\Pi}C} + \mathbb{T}^{A}{}_{B\underline{\Pi}}\mathbb{T}^{\underline{\Pi}D}{}_{C} &= \\
\mathbb{T}^{AD}{}_{E}\mathbb{T}_{B}{}^{E}{}_{C} - \mathbb{T}_{B}{}^{D}{}_{E}\mathbb{T}^{AE}{}_{C} + \mathbb{T}^{A}{}_{B}{}^{E}\mathbb{T}_{E}{}^{D}{}_{C} & \\
+\mathbb{T}^{AD}{}_{I}\mathbb{T}_{B}{}^{I}{}_{C} - \mathbb{T}_{B}{}^{D}{}_{I}\mathbb{T}^{AI}{}_{C} + \mathbb{T}^{A}{}_{B}{}^{I}\mathbb{T}_{I}{}^{D}{}_{C} &= 0
\end{aligned} \tag{A.0.2}$$

Now we recall

$$\mathbb{T}_{\underline{\Lambda}A}{}^{B} = -\mathbb{T}_{\underline{\Lambda}}{}^{B}{}_{A} \quad \text{and} \quad \mathbb{T}_{\underline{\Lambda}A}{}^{I} = \mathbb{T}_{\underline{\Lambda}}{}^{I}{}_{A} \tag{A.0.3}$$

to obtain

$$\begin{aligned}
Q^{AD}{}_{BC} \equiv -\mathbb{T}^{AD}{}_{E}\mathbb{T}_{BC}{}^{E} + \mathbb{T}_{BE}{}^{D}\mathbb{T}^{AE}{}_{C} + \mathbb{T}^{AE}{}_{B}\mathbb{T}_{EC}{}^{D} & \\
+\mathbb{T}^{AD}{}_{I}\mathbb{T}_{BC}{}^{I} - \mathbb{T}_{BI}{}^{D}\mathbb{T}^{AI}{}_{C} - \mathbb{T}^{AI}{}_{B}\mathbb{T}_{IC}{}^{D} &= 0
\end{aligned} \tag{A.0.4}$$

The following decomposition holds true[1]

$$\begin{aligned}
\mathbb{T}^{AD}{}_{E} &= \frac{1}{2}(\epsilon^{ADB}S_{BE} + \delta_{E}^{[A}N^{D]}) \\
\mathbb{T}^{AD}{}_{I} &= \epsilon^{BAD}T_{BI}
\end{aligned} \tag{A.0.5}$$

[1] $S_{AB} = S_{BA}$

In terms of S_{AB} and N^A, we compute[2]

$$
\begin{aligned}
-\epsilon_{\overline{A}AD}\epsilon^{\overline{B}BC}T^{AD}{}_{E}T_{BC}{}^{E} &= -(SS^*)_{\overline{A}}{}^{\overline{B}} - \frac{1}{2}(\epsilon_{\overline{A}ED}N^D S^{\overline{B}E} + \epsilon^{\overline{B}ED}N_D S_{\overline{A}E}) \\
&\quad -\frac{1}{4}(\delta_{\overline{A}}^{\overline{B}}\mathbb{N}_A\mathbb{N}^A - \mathbb{N}_{\overline{A}}\mathbb{N}^{\overline{B}}) \\
\epsilon_{\overline{A}AD}\epsilon^{\overline{B}BC}\mathbb{T}_{BE}{}^{D}\mathbb{T}^{AE}{}_{C} &= \frac{1}{4}(\delta_{\overline{A}}^{\overline{B}}\mathrm{Tr}\{SS^*\} - (SS^*)_{\overline{A}}{}^{\overline{B}} + \frac{1}{2}(\epsilon_{\overline{A}ED}N^D S^{\overline{B}E} + \epsilon^{\overline{B}ED}N_D S_{\overline{A}E}) \\
&\quad +\frac{1}{4}\delta_{\overline{A}}^{\overline{B}}N_A N^A - \frac{3}{4}N_{\overline{A}}N^{\overline{B}}) \\
\epsilon_{\overline{A}AD}\epsilon^{\overline{B}BC}\mathbb{T}^{AD}{}_{I}\mathbb{T}_{BC}{}^{I} &= 4T_{\overline{A}I}T^{\overline{B}I} \\
\epsilon_{\overline{A}AD}\epsilon^{\overline{B}BC}\mathbb{T}^{AE}{}_{B}\mathbb{T}_{EC}{}^{D} &= \epsilon_{\overline{A}AD}\epsilon^{\overline{B}BC}\mathbb{T}_{BE}{}^{D}\mathbb{T}^{AE}{}_{C} \\
-\epsilon_{\overline{A}AD}\epsilon^{\overline{B}BC}\mathbb{T}_{BI}{}^{D}\mathbb{T}^{AI}{}_{C} &= -\epsilon_{\overline{A}AD}\epsilon^{\overline{B}BC}\mathbb{T}^{AI}{}_{B}\mathbb{T}_{IC}{}^{D}
\end{aligned} \tag{A.0.6}
$$

It is easy to verify the last two equations. Indeed, eq.(2.2.15) implies

$$
\mathbb{T}_{\underline{\Lambda\Sigma}}{}^{\Delta} = -\mathbb{T}_{\underline{\Sigma\Lambda}}{}^{\Delta} \tag{A.0.7}
$$

We get rid of terms of the form $S\cdot N$ thanks to

$$
\begin{aligned}
Q^{AD}{}_{BD} = 0 \Leftrightarrow \epsilon^{AE\overline{C}}S_{\overline{C}B}N_E &= -\frac{1}{2}\delta_B^A N_C N^C + \frac{1}{2}N^A N_B \\
&\quad + 4\,\delta_B^A T_{CI}T^{CI} - 4\,T_{BI}T^{AI} \\
&\quad - 4\,\mathbb{T}_{IB}{}^{D}\mathbb{T}^{IA}{}_{D} + 4\,\mathbb{T}^{IA}{}_{B}\mathbb{T}_{ID}{}^{D}
\end{aligned} \tag{A.0.8}
$$

Finally, we obtain[3]

$$
\epsilon_{\overline{A}AD}\epsilon^{\overline{B}BC}Q^{AD}{}_{BC} - \frac{32}{3}Q^{AD}{}_{AD}\delta_{\overline{A}}^{\overline{B}} = 0 \Leftrightarrow \mathbb{N}_A\mathbb{N}^B + \mathbb{N}_{AI}\mathbb{N}^{BI} + \mathbb{N}_{IC}{}^{B}\mathbb{N}^{IC}{}_{A} - 12\,\mathbb{S}_{AC}\mathbb{S}^{BC} = \delta_{\overline{A}}^{\overline{B}}V \tag{A.0.9}
$$

[2] $S^{AB} = (S_{AB})^*$

[3] Actually, to get (3.1.22) we must redefine $S_{AB} = -2\mathbb{S}_{AB}$, $N^D = 2\mathbb{N}^D$, $T_{CI} = \frac{1}{2}\mathbb{N}_{CI}$.

Appendix B

Fermion Shift Tensors and Mass Matrices from $\mathbb{T}$-tensor

We present a systematic way to identify the interesting components of the $\mathbb{T}$ tensor involved in the definitions of fermionic shifts and mass matrices.

B.1 Fermionic shifts

In order to identify fermionic shifts inside $\mathbb{T}$ we consider what their H representation should correspond to. This task is easy since we know that they enter fermionic supersymmetry transformations with parameter $\epsilon_A \in (\mathbf{3},\mathbf{1})_{+\frac{1}{2}}$. Indeed, we have

$$
\begin{array}{rccl}
(\mathbf{3},\mathbf{1})_{+\frac{1}{2}} & \langle\delta\psi_{A\mu}\rangle = \langle\nabla_\mu\epsilon_A + \mathrm{i}\mathbb{S}_{AB}\gamma_\mu\epsilon^B\rangle & \Rightarrow \mathbb{S}_{AB} \in (\mathbf{6},\mathbf{1})_{+1} \\
(\mathbf{1},\mathbf{1})_{+\frac{3}{2}} & \langle\delta\chi\rangle = \langle\mathbb{N}^D\epsilon_D\rangle & \Rightarrow \mathbb{N}^D \in (\overline{\mathbf{3}},\mathbf{1})_{+1} \\
(\mathbf{3},\mathbf{n})_{\frac{n+6}{2n}} & \langle\delta\lambda_{IA}\rangle = \langle{\mathbb{N}_{IA}}^B\epsilon_B\rangle & \Rightarrow {\mathbb{N}_{IA}}^B \in (\mathbf{8}+\mathbf{1},\mathbf{n})_{+\frac{3}{n}} \\
(\mathbf{1},\mathbf{n})_{\frac{3n+6}{2n}} & \langle\delta\lambda_I\rangle = \langle\mathbb{N}_{IA}\epsilon^A\rangle & \Rightarrow \mathbb{N}_{IA} \in (\mathbf{3},\mathbf{n})_{\frac{2n+3}{n}}
\end{array}
\tag{B.1.1}
$$

We see that the wanted components of $\mathbb{T}$, possibly projected with $\mathcal{G} \subset H$-invariant tensors, must have one or two R-symmetry indices and no more than one matter index. The independent choices, obtained from ${\mathbb{T}^{AB}}_C$, ${\mathbb{T}^{IA}}_{\ \ B}$,

$\mathbb{T}^{AB}{}_I$ up to complex conjugation, are

$$\begin{aligned} &\epsilon_{AB(D}\mathbb{T}^{AB}{}_{C)} \in (\mathbf{6},\mathbf{1})_{+1} && \epsilon_{AB[D}\mathbb{T}^{AB}{}_{C]} \Leftrightarrow \mathbb{T}^{EB}{}_B \in (\overline{\mathbf{3}},\mathbf{1})_{+1} \\ &\mathbb{T}_{IA}{}^B \in (\mathbf{8}+\mathbf{1},\mathbf{n})_{+\frac{3}{n}} && \epsilon_{CAB}\mathbb{T}^{AB}{}_I \in (\mathbf{3},\mathbf{n})_{\frac{2n+3}{n}} \end{aligned} \tag{B.1.2}$$

These are exactly the needed representation in the definition of fermionic shifts.

B.2 Fermionic Mass Matrices

Now we move to $\mathbb{M}_{\mathcal{I}\mathcal{J}}$. We play the same game as before. In this case we discover their representations from the possible $\overline{\lambda}^{\mathcal{I}}\mathbb{M}_{\mathcal{I}\mathcal{J}}\lambda^{\mathcal{J}} \in (\mathbf{1},\mathbf{1})_0$ interactions[1] which are of the following form

$$\overline{\chi}_\bullet\mathbb{M}^{\bullet\bullet}\chi_\bullet \quad \Rightarrow \mathbb{M}^{\bullet\bullet} \in (\mathbf{1},\mathbf{1})_{-3} \tag{B.2.1}$$

$$\overline{\chi}_\bullet\mathbb{M}_I\lambda^I \quad \Rightarrow \mathbb{M}_I \in (\mathbf{1},\mathbf{n})_{\frac{3}{n}} \tag{B.2.2}$$

$$\overline{\chi}_\bullet\mathbb{M}^{AI}\lambda_{IA} \quad \Rightarrow \mathbb{M}^{IA} \in (\overline{\mathbf{3}},\overline{\mathbf{n}})_{-\frac{2n+3}{n}} \tag{B.2.3}$$

$$\overline{\lambda}^I\mathbb{M}_{IJ}\lambda^J \quad \Rightarrow \mathbb{M}_{IJ} \in \left(\mathbf{1},\frac{\mathbf{1}}{\mathbf{2}}\mathbf{n}(\mathbf{n}+\mathbf{1})\right)_{\frac{3(n+2)}{n}} \tag{B.2.4}$$

$$\overline{\lambda}^I\mathbb{M}_I{}^{AJ}\lambda_{AJ} \quad \Rightarrow \mathbb{M}_I{}^{AJ} \in (\overline{\mathbf{3}},\mathbf{n}\times\overline{\mathbf{n}})_{+1} \tag{B.2.5}$$

$$\overline{\lambda}_{AI}\mathbb{M}^{AI|BJ}\lambda_{BJ} \quad \Rightarrow \mathbb{M}^{AI|BJ} \in \left(\mathbf{3},\frac{\mathbf{1}}{\mathbf{2}}\overline{\mathbf{n}}(\overline{\mathbf{n}}-\mathbf{1})\right)_{-\frac{n+6}{n}} \tag{B.2.6}$$

One can get easily convinced that the only $\mathbb{T}$ components matching these representations are

$$\mathbb{T}^{IJ}{}_J \in (\mathbf{1},\overline{\mathbf{n}})_{-\frac{3}{n}} \quad \mathbb{T}^{AJ}{}_I \in (\overline{\mathbf{3}},\mathbf{n}\times\overline{\mathbf{n}})_{+1} \quad \epsilon^{ABC}\mathbb{T}^{IJ}{}_C \in \left(\mathbf{3},\frac{\mathbf{1}}{\mathbf{2}}\overline{\mathbf{n}}(\overline{\mathbf{n}}-\mathbf{1})\right)_{-\frac{n+6}{n}} \quad \epsilon_{CAB}\mathbb{T}^{AB}{}_I \in (\mathbf{3},\mathbf{n})_{\frac{2n+3}{n}}$$

These are the only ones entering gradient flow equations. Then, $\mathbb{M}_{IJ}$ and $\mathbb{M}_{\bullet\bullet}$ are consistently vanishing.

The precise relations between the mass matrices and the corresponding components of the $\mathbb{T}$-tensor are given in Appendix C.

[1] One could find the needed components for $\mathbb{N}^A_{\mathcal{I}}$ and $\mathbb{S}_{AB}$ looking for a gravitino-gravitino and gravitino-fermions mass terms.

Appendix C

The Gradient Flow equations

We consider here the different projections of eq. (3.1.23) into H-covariant components:

$$\begin{aligned}
\mathcal{D}\mathbb{N}^{A} &= \frac{1}{2}\mathscr{R}[\mathscr{P}]^{E}{}_{I}\mathbb{N}^{IA}{}_{E}+\frac{1}{2}\epsilon^{EAC}\mathbb{N}_{CI}\mathscr{R}[\mathscr{P}]^{I}{}_{E}+\frac{1}{2}\mathscr{R}[\mathscr{P}]^{A}{}_{I}\mathbb{N}^{IE}{}_{E}\\
\mathcal{D}\mathbb{N}_{CI} &= 2\epsilon_{ABC}\mathscr{R}[\mathscr{P}]^{B}{}_{J}\mathbb{T}^{AJ}{}_{I}-2\mathbb{S}_{CD}\mathscr{R}[\mathscr{P}]^{D}{}_{I}+\epsilon_{CDB}\mathbb{N}^{B}\mathscr{R}[\mathscr{P}]^{D}{}_{I}\\
\mathcal{D}\mathbb{N}^{IA}{}_{B} &= -2\mathscr{R}[\mathscr{P}]^{I}{}_{C}\epsilon^{CAD}\mathbb{S}_{DB}+\mathscr{R}[\mathscr{P}]^{I}{}_{B}\mathbb{N}^{A}\\
&\quad+\mathscr{R}[\mathscr{P}]^{J}{}_{D}\left(-2\delta^{D}_{B}\mathbb{T}^{IA}{}_{J}+\mathbb{T}^{ID}{}_{J}\delta^{A}_{B}\right)+\mathscr{R}[\mathscr{P}]^{D}{}_{J}\left(2\delta^{A}_{D}\mathbb{T}^{IJ}{}_{B}-\mathbb{T}^{IJ}{}_{D}\delta^{A}_{B}\right)\\
\mathcal{D}\mathbb{S}_{BE} &= -\frac{1}{2}\epsilon_{AD(B}\mathbb{N}^{ID}{}_{E)}\mathscr{R}[\mathscr{P}]^{A}{}_{I}-\frac{1}{2}\mathscr{R}[\mathscr{P}]^{I}_{(\ E}\mathbb{N}_{B)I}
\end{aligned}\tag{C.0.1}$$

On the other hand, using the general form of the gradient flow equations required by the supersymmetry of the gauged Lagrangian, see [11], specialized to the $\mathcal{N}=3$ models, we find:

$$\begin{aligned}
\mathcal{D}\mathbb{N}^{A} &= \mathscr{R}[\mathscr{P}]^{E}{}_{I}\mathbb{T}^{AI}{}_{E}+\mathbb{T}^{AE}{}_{I}\mathscr{R}[\mathscr{P}]^{I}{}_{E}-2\mathscr{R}[\mathscr{P}]^{A}{}_{I}\mathbb{M}^{I}{}_{\bullet}-2\mathscr{R}[\mathscr{P}]^{I}{}_{F}\mathbb{M}_{\bullet IE}\epsilon^{EAF}\\
\mathcal{D}\mathbb{N}_{CI} &= -2\mathbb{S}_{CD}\mathscr{R}[\mathscr{P}]^{D}{}_{I}-2\mathbb{M}_{IJ}\mathscr{R}[\mathscr{P}]^{J}{}_{C}-2\mathbb{M}_{I}{}^{JA}\mathscr{R}[\mathscr{P}]^{B}{}_{J}\epsilon_{ACB}\\
\mathcal{D}\mathbb{N}^{IA}{}_{B} &= -2\mathscr{R}[\mathscr{P}]^{I}{}_{C}\epsilon^{CAD}\mathbb{S}_{DB}+\mathscr{R}[\mathscr{P}]^{J}{}_{E}\mathbb{T}^{IE}{}_{J}\delta^{A}_{B}+\mathscr{R}[\mathscr{P}]^{E}{}_{J}\mathbb{T}^{IJ}{}_{E}\delta^{A}_{B}\\
&\quad-2\mathscr{R}[\mathscr{P}]^{J}{}_{B}\mathbb{M}^{IA}{}_{J}-2\mathbb{M}^{IA|JC}\mathscr{R}[\mathscr{P}]^{D}{}_{J}\epsilon_{CBD}\\
\mathcal{D}\mathbb{S}_{BE} &= -\frac{1}{2}\epsilon_{AD(B}\mathbb{N}^{ID}{}_{E)}\mathscr{R}[\mathscr{P}]^{A}{}_{I}-\frac{1}{2}\mathscr{R}[\mathscr{P}]^{I}_{(\ E}\mathbb{N}_{B)I}
\end{aligned}\tag{C.0.2}$$

Direct comparison between (C.0.1) and (C.0.2) suggest the following identifications

$$\mathbb{T}^{IE}{}_{E}=2\mathbb{M}^{I}{}_{\bullet},\quad \mathbb{T}^{IA}{}_{J}+\frac{1}{2}\delta^{I}_{J}\mathbb{N}^{A}=\mathbb{M}^{AI}{}_{J},\quad \mathbb{T}^{IJ}{}_{A}=-\frac{1}{2}\epsilon_{ABC}\mathbb{M}^{IB|JC},\quad \mathbb{T}^{AB}{}_{I}=2\epsilon^{ABC}\mathbb{M}_{\bullet IC}\tag{C.0.3}$$

and

$$\mathbb{M}_{IJ} = 0\,. \tag{C.0.4}$$

The latter condition is consistent with the discussion of Appendix B, where it is also shown that the mass matrix $\mathbb{M}_{\bullet\bullet}$, which does not enter the above gradient flow equations, is in fact vanishing.

C.1 Gauge Generators

The $\mathrm{SO}(3)\times\mathrm{SU}(3)$ generators $\widehat{t}_\ell$, $\widehat{t}_m$ in the fundamental representations of the respective groups read:

$$\widehat{t}_{\ell=1} = J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \;;\; \widehat{t}_{\ell=2} = J_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \;;\; \widehat{t}_{\ell=2} = J_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\widehat{t}_{m=3+I} = \frac{i}{2}\,\lambda_I\,, \quad I = 1,\dots,8\,,$$

where

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;;\; \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;;\; \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \;;\; \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \;;\; \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \;;\; \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{C.1.1}$$

Appendix D

Solving for the DW solutions

When computed on the DW metric (3.6.1), the components of the Ricci tensor read

$$R_{ij} = e^{2A}\left[3(A')^2 + A''\right]\eta_{ij}\,, \tag{D.0.1}$$

$$R_{yy} = -3\left[(A')^2 + A''\right], \tag{D.0.2}$$

where the $'$ denotes the derivative with respect to the transverse coordinate y and the Ricci scalar is

$$R = 6\left[2(A')^2 + A''\right]. \tag{D.0.3}$$

The Euler-Lagrange equations of motion for (3.6.3) are

$$e^{3A}\left[2r_i'' + 6A'r_i' - \frac{1}{2}\sinh(4r_i)\,\alpha_i'^2 - \partial_{r_i}V(r_i,\alpha_i)\right] = 0\,, \tag{D.0.4}$$

$$e^{3A}\sinh(2r_i)\left[4\cosh(2r_i)\,r_i'\alpha_i' + \sinh(2r_i)\left(3A'\alpha_i' + \alpha_i''\right)\right] = 0\,, \tag{D.0.5}$$

while Einstein equations read

$$e^{2A}\left[A'' + 3A'^2 + V(r_i,\alpha_i)\right] = 0\,, \tag{D.0.6}$$

$$3A'' + 3A'^2 + V(r_i,\alpha_i) + \sum_i^3\left(2r_i'^2 + \frac{1}{2}\sinh(2r_i)^2\,\alpha_i'^2\right) = 0\,. \tag{D.0.7}$$

The critical points of the potentials (3.4.11) and (3.4.12), that we choose as end–points of the RG-flow, consist of the origin $\mathcal{O}$ and other vacua at fixed radii

$$\text{Type (i):} \qquad r_1 = r_2 = r_3 = r_{\text{vac}} = \frac{1}{2}\log\left(\frac{|g_2|+|g_1|}{|g_2|-|g_1|}\right) \quad , \quad |g_2| > |g_1| \; , \tag{D.0.8}$$

$$\text{Type (ii):} \qquad r_1 = r_2 = r_3 = r_{\text{vac}} = \frac{1}{2}\log\left(\frac{|g_2|+2|g_1|}{|g_2|-2|g_1|}\right) \quad , \quad |g_2| > 2|g_1| \; . \tag{D.0.9}$$

When imposing that the moduli of all z_i are equal, (D.0.4) leads to the conclusion that α_i have to be constant. In fact, setting all r_i to the same value r, and combining the three equations in (D.0.4), one obtains

$$e^{3A}\sinh(4r)\left(\alpha_1'^2 - \alpha_2'^2\right) = 0 \; , \tag{D.0.10}$$

$$e^{3A}\sinh(4r)\left(\alpha_1'^2 - \alpha_3'^2\right) = 0 \; , \tag{D.0.11}$$

$$e^{3A}\sinh(4r)\left(\alpha_2'^2 - \alpha_3'^2\right) = 0 \; . \tag{D.0.12}$$

D.1 The solution

Setting all α_i to constant values along the flow, the equations reduce to the EOM for the field r and the Einstein equations, which read

$$r'' + 3A'r' - \frac{1}{6}\partial_r V(r) = 0 \; , \tag{D.1.1}$$

$$A'' + 3\left(A'\right)^2 + V(r) = 0 \; , \tag{D.1.2}$$

$$3\left[A'' + \left(A'\right)^2 + 2\left(r'\right)^2\right] + V(r) = 0. \tag{D.1.3}$$

$V(r,\alpha_i)$ being the potential given in (3.4.19) for Type (i) solution or (3.4.20) for Type (ii). The last two equations can be combined into the following constraint

$$3\left(A'\right)^2 - 3\left(r'\right)^2 + V(r) = 0 \; . \tag{D.1.4}$$

Now, this system of equations can be obtained from an effective action of the form

$$\begin{aligned} \mathscr{L}_{\text{eff}} &= e^{3A}\left[3\left(A'\right)^2 - 3\left(r'\right)^2 - V(r)\right] = \\ &= \frac{1}{2}G_{ij}\Phi'^i\Phi'^j - \mathscr{V}(\Phi) \; , \end{aligned} \tag{D.1.5}$$

with $\Phi^i = (A, r)$, $\mathscr{V}(\Phi) = e^{3A} V(r)$ and $G_{ij} = 6\, e^{3A}\, \mathrm{diag}(1, -1)$.
The Hamiltonian corresponding to the above Lagrangian is defined via the Legendre transform

$$H = \Pi_i \Phi'^i - \mathscr{L}_{\mathrm{eff}} = \frac{1}{2} G^{ij} \Pi_i \Pi_j + \mathscr{V}(\Phi) \,, \tag{D.1.6}$$

where

$$\Pi_i = \frac{\delta \mathscr{L}_{\mathrm{eff}}}{\delta \Phi'^i} = G_{ij} \Phi'^j \tag{D.1.7}$$

are the usual canonical momenta. Then we can recast the second-order field equations in the form of first order ones by considering the Hamilton-Jacobi problem, namely by writing

$$\Pi_i = \frac{\delta W(\Phi)}{\delta \Phi^i} \,, \tag{D.1.8}$$

where $W(\Phi)$ is the Hamilton's characteristic function, solution to the Hamilton-Jacobi equation:

$$H = \frac{1}{2} G^{ij} \partial_i W \partial_j W + \mathscr{V}(\Phi) \,. \tag{D.1.9}$$

The characteristic function $W(\Phi)$ can be expressed in terms of a α_i-independent "superpotential" $\mathcal{W}_0$, defined in (3.4.17), as follows

$$W(A, r) = 2\, e^{3A}\, \mathcal{W}_0(r) \,, \tag{D.1.10}$$

Note that this "superpotential" also describes non-supersymmetric flows. Again this is related to the fact that α_i, which connect supersymmetric vacua to non-supersymmetric ones, are constant along the flow. In terms of the superpotential $\mathcal{W}_0(r)$, the scalar potential is defined through the "superpotential equation"

$$V(r) = \frac{1}{3} \left(\partial_r \mathcal{W}_0(r)\right)^2 - 3 \mathcal{W}_0(r)^2 \,, \tag{D.1.11}$$

which holds both for Type (i) and Type (ii) vacuum. Now, from (D.1.7) and (D.1.8) we obtain

$$\Phi'^i = G^{ij} \frac{\partial W}{\partial \Phi^j} \,, \tag{D.1.12}$$

so that the general form of the first order equations is

$$\begin{aligned}
\text{Type (i)} \quad : \quad A'(y) &= \mathcal{W}_0(r) = 2\Big|\left[|g_1|\cosh^3(r) - |g_2|\sinh^3(r)\right]\Big| \,, \\
r'(y) &= -\sinh(2r)\left[|g_1|\cosh(r) - |g_2|\sinh(r)\right] \,,
\end{aligned} \tag{D.1.13}$$

$$\begin{aligned}
\text{Type (ii)} \quad : \quad A'(y) &= \mathcal{W}_0(r) = \Big|2\,|g_1|\cosh^3(r) - |g_2|\sinh^3(r)\Big| \,, \\
r'(y) &= -\frac{1}{2}\sinh(2r)\left[2\,|g_1|\cosh(r) - |g_2|\sinh(r)\right] .
\end{aligned} \tag{D.1.14}$$

These equations can be easily integrated to give

$$\text{Type (i)} \quad : \quad A(y) = c_1 + \ln\left[\frac{|g_1|\cosh(r) - |g_2|\sinh(r)}{\sinh(2r)}\right] \,, \tag{D.1.15}$$

$$\begin{aligned}
y = c_2 - \frac{1}{2|g_1||g_2|}\Bigg(& 2|g_1|\arctan\left[\tanh\left(\frac{r}{2}\right)\right] + |g_2|\ln\left[\tanh\left(\frac{r}{2}\right)\right] + \\
&+ 2\sqrt{|g_2|^2 - |g_1|^2}\,\tanh^{(-1)}\left[\frac{|g_2| - |g_1|\tanh(\frac{r}{2})}{\sqrt{|g_2|^2 - |g_1|^2}}\right]\Bigg) \,,
\end{aligned} \tag{D.1.16}$$

$$\text{Type (ii)} \quad : \quad A(y) = c_1 + \ln\left[\frac{2\,|g_1|\cosh(r) - |g_2|\sinh(r)}{\sinh(2r)}\right] \,, \tag{D.1.17}$$

$$\begin{aligned}
y = c_2 - \frac{1}{2|g_1||g_2|}\Bigg(& 4|g_1|\arctan\left[\tanh\left(\frac{r}{2}\right)\right] + |g_2|\ln\left[\tanh\left(\frac{r}{2}\right)\right] + \\
&+ 2\sqrt{|g_2|^2 - 4|g_1|^2}\,\tanh^{(-1)}\left[\frac{|g_2| - 2|g_1|\tanh(\frac{r}{2})}{\sqrt{|g_2|^2 - 4|g_1|^2}}\right]\Bigg) .
\end{aligned}$$

c_1 and c_2 are integration constants that can be set to zero by performing a shift in the x^i coordinates.

Appendix E

Relevant Supermultiplets

E.1 $\mathcal{N} = 3$

$A_1[\frac{1}{2}]^{(0)}_{\frac{3}{2}}$: massless graviton multiplet			
spin	Δ	$\mathfrak{su}(2)_R$ irrep	m^2
$\frac{1}{2}$	$\frac{3}{2}$	$\mathbf{1}$	0
1	2	$\mathbf{3}$	0
$\frac{3}{2}$	$\frac{5}{2}$	$\mathbf{3}$	1
2	3	$\mathbf{1}$	0

$B_1[0]^{(2)}_1$: massless vector multiplet			
spin	Δ	$\mathfrak{su}(2)_R$ irrep	m^2
0	1	$\mathbf{3}$	-2
	2	$\mathbf{3}$	-2
$\frac{1}{2}$	$\frac{3}{2}$	$\mathbf{1}$	0
	$\frac{3}{2}$	$\mathbf{3}$	0
1	2	$\mathbf{1}$	0

$B_1[0]^{(3)}_{\frac{3}{2}}$: masssive vector multiplet			
spin	Δ	$\mathfrak{su}(2)_R$ irrep	m^2
0	$\frac{3}{2}$	**4**	$-\frac{9}{4}$
	$\frac{5}{2}$	**4**	$-\frac{5}{4}$
	$\frac{5}{2}$	**2**	$-\frac{5}{4}$
$\frac{1}{2}$	2	**2**	$\frac{1}{4}$
	2	**4**	$\frac{1}{4}$
	3	**2**	$\frac{9}{4}$
1	$\frac{5}{2}$	**2**	$\frac{3}{4}$

$B_1[0]^{(4)}_{2}$: masssive vector multiplet			
spin	Δ	$\mathfrak{su}(2)_R$ irrep	m^2
0	2	**5**	-2
	3	**1**	0
	3	**3**	0
	3	**5**	0
	4	**1**	4
$\frac{1}{2}$	$\frac{5}{2}$	**3**	1
	$\frac{5}{2}$	**5**	1
	$\frac{7}{2}$	**1**	4
	$\frac{7}{2}$	**3**	4
1	3	**3**	2

$B_1[0]^{(6)}_{3}$: masssive vector multiplet			
spin	Δ	$\mathfrak{su}(2)_R$ irrep	m^2
0	3	**7**	0
	4	**3**	4
	4	**5**	4
	4	**7**	4
	5	**3**	10
$\frac{1}{2}$	$\frac{7}{2}$	**5**	4
	$\frac{7}{2}$	**7**	4
	$\frac{9}{2}$	**3**	9
	$\frac{9}{2}$	**5**	9
1	4	**5**	6

E.2 $\mathcal{N} = 2$

In short, the OSp(2|4) multiplet will be classified by Dynkin labels of the maximal compact subgroup $\mathrm{U}(1) \times \mathrm{SO}(3)_J \times \mathrm{U}(1)_\Delta$. The first factor represents the R–symmetry, whose charges we label by real $R \in \mathbb{R}$. In accordance with the two independent supersymmetries Q and $\overline{Q}$, a generic Osp(2|4) multiplet will be of the form

$$X\overline{Y}[J]_{\Delta}^{(R)}, \tag{E.2.1}$$

where $X \in \{L, A_1, A_2, B_1\}$ and $\overline{Y} \in \{\overline{L}, \overline{A}_1, \overline{A}_2, \overline{B}_1\}$ refer to the long and the different shortened structures with respect to Q and $\overline{Q}$, respectively. The parameters R, J, and Δ refer to the R–charge, Lorentz spin, and conformal dimension of the highest weight state of the multiplet (E.2.1), respectively.[1] Unitarity implies the lower bound for the conformal dimension

$$\Delta \ \geq \ 1+|R|+J\,. \tag{E.2.2}$$

For $\Delta > 1+|R|+J$, the multiplet is of the long type $L\overline{L}[J]_{\Delta}^{(R)}$ and is given by the tensor product of its HWS with the representation generated by the action of the 4 supercharges on a scalar vacuum, c.f. section 4.2 of [27]. Therefore, its character factors are as in (5.2.15). Evaluating the product and organizing the fields according to their Lorentz spins, yields the explicit field content which we summarize in Tables E.1, E.2, for $J=0,\frac{1}{2},1$. When the unitarity bound is saturated, the multiplets are shortened. More precisely, for $R>0$ and $\Delta = 1+R+J$, the right factor $\overline{L}$ in (E.2.1) breaks according to

$$\begin{aligned}
\overline{L}[1]_{\Delta}^{(R)} &\longrightarrow \overline{A}_1[1]_{\Delta}^{(R)} + \overline{A}_1[\tfrac{1}{2}]_{\Delta+\frac{1}{2}}^{(R+1)}\,,\\
\overline{L}[\tfrac{1}{2}]_{\Delta}^{(R)} &\longrightarrow \overline{A}_1[\tfrac{1}{2}]_{\Delta}^{(R)} + \overline{A}_2[0]_{\Delta+\frac{1}{2}}^{(R+1)}\,,\\
\overline{L}[0]_{\Delta}^{(R)} &\longrightarrow \overline{A}_2[0]_{\Delta}^{(R)} + \overline{B}_1[\tfrac{1}{2}]_{\Delta+1}^{(R+2)}\,,
\end{aligned} \tag{E.2.3}$$

whereas for $R<0$ it is the left factor L in (E.2.1) which breaks accordingly. At $R=0$, further shortening occurs, and massless multiplets show up. We list the relevant shortened multiplets in Tables E.3 and E.4, where we restrict to the long-short case $L\overline{A}_1$, etc.. The short-long multiplets are obtained from the former upon replacing R with $-R$ (we can refer to this operation as "conjugation"). In particular, a half–hypermultiplet combined with its conjugate forms a complete hypermultiplet.

[1] In contrast to the notation used in [27], our J is half-integer, referring to the spin, not the Dynkin label.

$L\overline{L}[0]_{\Delta}^{(R)}$: long vector multiplet			
spin	Δ	R	m^2
0	Δ	R	$(\Delta)(\Delta-3)$
	$\Delta+1$	$R-2$	$(\Delta+1)(\Delta-2)$
	$\Delta+1$	$R+2$	$(\Delta+1)(\Delta-2)$
	$\Delta+1$	R	$(\Delta+1)(\Delta-2)$
	$\Delta+2$	R	$(\Delta+2)(\Delta-1)$
$\frac{1}{2}$	$\Delta+1/2$	$R-1$	$(\Delta-1)^2$
	$\Delta+1/2$	$R+1$	$(\Delta-1)^2$
	$\Delta+3/2$	$R-1$	Δ^2
	$\Delta+3/2$	$R+1$	Δ^2
1	$\Delta+1$	R	$(\Delta)(\Delta-1)$

$L\overline{L}[\frac{1}{2}]_{\Delta}^{(R)}$: long gravitino multiplet			
spin	Δ	R	m^2
0	$\Delta+1/2$	$R-1$	$(\Delta+1/2)(\Delta-5/2)$
	$\Delta+1/2$	$R+1$	$(\Delta+1/2)(\Delta-5/2)$
	$\Delta+3/2$	$R-1$	$(\Delta+1/2)(\Delta-5/2)$
	$\Delta+3/2$	$R+1$	$(\Delta+3/2)(\Delta-3/2)$
$\frac{1}{2}$	Δ	R	$(\Delta-3/2)^2$
	$\Delta+1$	$R-2$	$(\Delta-1/2)^2$
	$\Delta+1$	$R+2$	$(\Delta-1/2)^2$
	$\Delta+1$	R	$(\Delta-1/2)^2$
	$\Delta+1$	R	$(\Delta-1/2)^2$
	$\Delta+2$	R	$(\Delta+1/2)^2$
1	$\Delta+1/2$	$R-1$	$(\Delta-1/2)(\Delta-3/2)$
	$\Delta+1/2$	$R+1$	$(\Delta-1/2)(\Delta-3/2)$
	$\Delta+3/2$	$R-1$	$(\Delta+1/2)(\Delta-1/2)$
	$\Delta+3/2$	$R+1$	$(\Delta+1/2)(\Delta-1/2)$
3/2	$\Delta+1$	R	$(\Delta-1/2)^2$

Table E.1 Long $\mathcal{N}=2$ multiplets $L\overline{L}[0]_{\Delta}^{(R)}$ and $L\overline{L}[\frac{1}{2}]_{\Delta}^{(R)}$.

$L\overline{A}_2[0]^{(R)}_{R+1}$: short masssive vector multiplet			
spin	Δ	R	m^2
0	$R+1$	R	$(R+1)(R-2)$
	$R+2$	$R-2$	$(R+2)(R-1)$
	$R+2$	R	$(R+2)(R-1)$
$\frac{3}{2}$	$R+\frac{3}{2}$	$R-1$	R^2
	$R+\frac{3}{2}$	$R+1$	R^2
	$R+\frac{5}{2}$	$R-1$	$(R+1)^2$
1	$R+2$	R	$R(R+1)$

$L\overline{L}[1]^{(R)}_{\Delta}$: long graviton multiplet			
spin	Δ	R	m^2
0	Δ	R	$\Delta(\Delta-3)$
$\frac{1}{2}$	$\Delta+1/2$	$R-1$	$(\Delta-1)^2$
	$\Delta+1/2$	$R+1$	$(\Delta-1)^2$
	$\Delta+3/2$	$R-1$	Δ^2
	$\Delta+3/2$	$R+1$	Δ^2
1	Δ	R	$(\Delta-1)(\Delta-2)$
	$\Delta+1$	R	$\Delta(\Delta-1)$
	$\Delta+1$	R	$\Delta(\Delta-1)$
	$\Delta+1$	$R-2$	$\Delta(\Delta-1)$
	$\Delta+1$	$R+2$	$\Delta(\Delta-1)$
	$\Delta+2$	R	$\Delta(\Delta+1)$
$\frac{3}{2}$	$\Delta+1/2$	$R-1$	$(\Delta-1)^2$
	$\Delta+1/2$	$R+1$	$(\Delta-1)^2$
	$\Delta+3/2$	$R-1$	Δ^2
	$\Delta+3/2$	$R+1$	Δ^2
2	$\Delta+1$	R	$(\Delta+1)(\Delta-2)$

Table E.2 Short $L\overline{A}_2[0]^{(R)}_{R+1}$ and long $L\overline{L}[1]^{(R)}_{\Delta}$ $\mathcal{N} = 2$ multiplets.

$L\overline{A}_1[\frac{1}{2}]^{(R)}_{R+\frac{3}{2}}$: short masssive gravitino multiplet			
spin	Δ	R	m^2
0	$R+2$	$R-1$	$(R+2)(R-1)$
$\frac{1}{2}$	$R+3/2$	R	R^2
	$R+5/2$	$R-2$	$(R+1)^2$
	$R+5/2$	R	$(R+1)^2$
1	$R+2$	$R-1$	$R(R+1)$
	$R+2$	$R+1$	$R(R+1)$
	$R+3$	$R-1$	(R+2)(R+1)
3/2	$R+5/2$	R	$(R+1)^2$

$L\overline{B}_1[0]^{(R)}_R$: half-hypermultiplet			
spin	Δ	R	m^2
0	R	R	$R(R-3)$
	$R+1$	$R-2$	$(R+1)(R-2)$
1/2	$R+1/2$	$R-1$	$(R-1)^2$

Table E.3 Shortened $\mathcal{N}=2$ multiplets.

$A_1\overline{A}_1[1]^{(0)}_2$: massless graviton multiplet			
spin	Δ	R	m^2
1	2	0	0
$\frac{3}{2}$	5/2	-1	1
	5/2	$+1$	1
2	3	0	0

$A_2\overline{A}_2[0]^{(0)}_1$: massless vector multiplet			
spin	Δ	R	m^2
0	1	0	-2
	2	0	-2
$\frac{1}{2}$	3/2	-1	0
	3/2	$+1$	0
1	2	0	0

Table E.4 Massless $\mathcal{N}=2$ multiplets.

E.3 $\mathcal{N} = 1$

Here we list the relevant $\mathcal{N} = 1$ multiplets.

$A_1[3]_{\frac{5}{2}}$: massless gravity multiplet		
spin	Δ	m^2
$\frac{3}{2}$	$\frac{5}{2}$	1
2	3	0

$L[2]_{\Delta>2}$: massive gravitino multiplet		
spin	Δ	m^2
$\frac{1}{2}$	$\Delta+\frac{1}{2}$	$(\Delta-1)^2$
1	Δ	$(\Delta-1)(\Delta-2)$
	$\Delta+1$	$\Delta(\Delta-1)$
$\frac{3}{2}$	$\Delta+\frac{1}{2}$	$(\Delta-1)^2$

$A_1[1]_{\frac{3}{2}}$: massless vector multiplet		
spin	Δ	m^2
$\frac{1}{2}$	$\frac{3}{2}$	0
1	2	0

$L[1]_{\Delta>\frac{3}{2}}$: massive vector multiplet		
spin	Δ	m^2
0	$\Delta+\frac{1}{2}$	$(\Delta+\frac{1}{2})(\Delta-\frac{5}{2})$
$\frac{1}{2}$	Δ	$(\Delta-\frac{3}{2})^2$
	$\Delta+1$	$(\Delta-\frac{1}{2})^2$
1	$\Delta+\frac{1}{2}$	$(\Delta-\frac{1}{2})(\Delta-\frac{3}{2})$

$L[0]_{\Delta>1}$: matter multiplet		
spin	Δ	m^2
0	Δ	$\Delta(\Delta-3)$
	$\Delta+1$	$(\Delta+1)(\Delta-2)$
$\frac{1}{2}$	$\Delta+\frac{1}{2}$	$(\Delta-1)^2$

$L'[0]_{\Delta>\frac{1}{2}}$: matter multiplet		
spin	Δ	m^2
0	Δ	$\Delta(\Delta-3)$
	$\Delta+1$	$(\Delta+1)(\Delta-2)$
$\frac{1}{2}$	$\Delta+\frac{1}{2}$	$(\Delta-1)^2$

Appendix F

Supergravity spectra in the $\mathcal{N} = 3$ model

F.1 $\mathcal{N} = 3$

spin	m^2	$\Delta \equiv E_0$	multiplicity
0	-2	$\{1,2\}$	54=27+27
$\frac{1}{2}$	0	$\frac{3}{2}$	37
1	0	2	12
$\frac{3}{2}$	1	$\frac{5}{2}$	3
2	0	3	1

Table F.1 Mass spectrum of the $\mathcal{N} = 3$ vacuum preserving the full gauge group $\mathrm{SO}(3) \times \mathrm{SU}(3)$

spin	m^2	$\Delta \equiv E_0$	multiplicity
0	4	4	1
	0	3	16
	$-\frac{5}{4}$	$\frac{5}{2}$	12
	-2	1,2	17
	$-\frac{9}{4}$	$\frac{3}{2}$	8
$\frac{1}{2}$	4	$\frac{7}{2}$	4
	$\frac{9}{4}$	3	4
	1	$\frac{5}{2}$	8
	$\frac{1}{4}$	2	12
	0	$\frac{3}{2}$	9
1	2	3	3
	$\frac{3}{4}$	$\frac{5}{2}$	4
	0	2	5
$\frac{3}{2}$	1	$\frac{5}{2}$	3
2	0	3	1

Table F.2 Mass spectrum of the single $\mathcal{N}=3$ vacuum invariant under the subgroup $SU(2)\times U(1)$ of the gauge group.

spin	m^2	$\Delta \equiv E_0$	multiplicity
0	10	5	3
	4	4	16
	0	3	24
	-2	1,2	11
$\frac{1}{2}$	9	$\frac{9}{2}$	8
	4	$\frac{7}{2}$	16
	1	$\frac{5}{2}$	8
	0	$\frac{3}{2}$	5
1	6	4	5
	2	3	3
	0	4	4
$\frac{3}{2}$	1	$\frac{5}{2}$	3
2	0	3	1

Table F.3 Mass spectrum of the $\mathcal{N} = 3$ vacuum invariant under the subgroup $\mathrm{SO}(3)_D$ of the gauge group.

F.2 $\mathcal{N}=2$

spin	m^2	$\Delta \equiv E_0$	multiplicity
0	$-\frac{9}{4}$	$\frac{3}{2}$	8
	-2	$\{1,2\}$	$15=6+9$
	$-\frac{5}{4}$	$\frac{5}{2}$	12
	0	3	14
	4	4	1
	$(R+2)(R-1)$	$R+2$	2
	$R(R+3)$	$R+3$	2
$\frac{1}{2}$	0	$\frac{3}{2}$	9
	$\frac{1}{4}$	2	12
	$\frac{9}{4}$	3	4
	1	$\frac{5}{2}$	4
	4	$\frac{7}{2}$	2
	R^2	$R+\frac{3}{2}$	1
	$(R+1)^2$	$R+\frac{5}{2}$	4
	$(R+2)^2$	$R+\frac{7}{2}$	1
1	0	2	3
	$\frac{3}{4}$	$\frac{5}{2}$	4
	2	3	1
	$R(R+1)$	$R+2$	2
	$(R+1)(R+2)$	$R+3$	2
$\frac{3}{2}$	1	$\frac{5}{2}$	2
	$(R-1)^2$	$R+\frac{5}{2}$	1
2	0	3	1

Table F.4 Mass spectrum of the $\mathcal{N}=2$ vacuum invariant under a $\mathrm{U}(1)_D \times \mathrm{U}(1)$ subgroup of the gauge group.

spin	m^2	$\Delta \equiv E_0$	multiplicity
0	-2	$\{1,2\}$	$9=4+5$
	0	3	22
	4	4	16
	10	5	3
	$(R+2)(R-1)$	$R+2$	2
	$R(R+3)$	$R+3$	2
$\frac{1}{2}$	0	$\frac{3}{2}$	5
	1	$\frac{5}{2}$	4
	4	$\frac{7}{2}$	14
	9	$\frac{9}{2}$	8
	R^2	$R+\frac{3}{2}$	1
	$(R+1)^2$	$R+\frac{5}{2}$	4
	$(R+2)^2$	$R+\frac{7}{2}$	1
1	0	2	2
	2	3	1
	6	4	5
	$R(R+1)$	$R+2$	2
	$(R+1)(R+2)$	$R+3$	2
$\frac{3}{2}$	1	$\frac{5}{2}$	2
	$(R-1)^2$	$R+\frac{5}{2}$	1
2	0	3	1

Table F.5 Mass spectrum of the $\mathcal{N}=2$ vacuum invariant under a $\mathrm{U}(1)_D$ subgroup of the gauge group.

F.3 $\mathcal{N}=1$

spin	m^2	$\Delta \equiv E_0$	multiplicity
0	$-\frac{9}{4}$	$\frac{3}{2}$	8
	-2	$\{1,2\}$	$14=12+2$
	$-\frac{5}{4}$	$\frac{5}{2}$	12
	0	3	13
	4	4	1
	$(\Delta_{V1}+\frac{1}{2})(\Delta_{V1}-\frac{5}{2})$	$\Delta_{V1}+\frac{1}{2}$	1
	$(\Delta_{V2}+\frac{1}{2})(\Delta_{V2}-\frac{5}{2})$	$\Delta_{V2}+\frac{1}{2}$	1
	$\Delta_{H1}(\Delta_{H1}-3)$	Δ_{H1}	1
	$(\Delta_{H1}+1)(\Delta_{H1}-2)$	$\Delta_{H1}+1$	1
	$\Delta_{H2}(\Delta_{H2}-3)$	Δ_{H2}	1
	$(\Delta_{H2}+1)(\Delta_{H2}-2)$	$\Delta_{H2}+1$	1
$\frac{1}{2}$	0	$\frac{3}{2}$	10
	$\frac{1}{4}$	2	12
	1	$\frac{5}{2}$	2
	$\frac{9}{4}$	3	4
	4	$\frac{7}{2}$	1
	$(\Delta_{G1}-1)^2$	$\Delta_{G1}+\frac{1}{2}$	2
	$(\Delta_{G2}-1)^2$	$\Delta_{G2}+\frac{1}{2}$	2
	$(\Delta_{V1}-\frac{3}{2})^2$	Δ_{V1}	1
	$(\Delta_{V1}-\frac{1}{2})^2$	$\Delta_{V1}+1$	1
	$(\Delta_{V2}-\frac{3}{2})^2$	Δ_{V2}	1
	$(\Delta_{V2}-\frac{1}{2})^2$	$\Delta_{V2}+1$	1
1	0	2	2
	$\frac{3}{4}$	$\frac{5}{2}$	4
	$(\Delta_{G1}-1)(\Delta_{G1}-2)$	Δ_{G1}	1
	$\Delta_{G1}(\Delta_{G1}-1)$	$\Delta_{G1}+1$	1
	$(\Delta_{G2}-1)(\Delta_{G2}-2)$	Δ_{G2}	1
	$\Delta_{G2}(\Delta_{G2}-1)$	$\Delta_{G2}+1$	1
	$(\Delta_{V1}-\frac{1}{2})(\Delta_{V1}-\frac{3}{2})$	$\Delta_{V1+\frac{1}{2}}$	1
	$(\Delta_{V2}-\frac{1}{2})(\Delta_{V2}-\frac{3}{2})$	$\Delta_{V2+\frac{1}{2}}$	1
$\frac{3}{2}$	1	$\frac{5}{2}$	1
	$(\Delta_{G1}-1)^2$	$\Delta_{G1}+\frac{1}{2}$	1
	$(\Delta_{G2}-1)^2$	$\Delta_{G2}+\frac{1}{2}$	1
2	0	3	1

Table F.6 Mass spectrum of the $\mathcal{N}=1$ vacuum invariant under a U(1) subgroup of the gauge group.

spin	m^2	$\varDelta \equiv E_0$	multiplicity
0	-2	$\{1,2\}$	$6+2=8$
	0	3	21
	4	4	16
	10	5	3
	$(\varDelta_{V1}+\frac{1}{2})(\varDelta_{V1}-\frac{5}{2})$	$\varDelta_{V1}+\frac{1}{2}$	1
	$(\varDelta_{V2}+\frac{1}{2})(\varDelta_{V2}-\frac{5}{2})$	$\varDelta_{V2}+\frac{1}{2}$	1
	$\varDelta_{H1}(\varDelta_{H1}-3)$	$\varDelta_{H1}$	1
	$(\varDelta_{H1}+1)(\varDelta_{H1}-2)$	$\varDelta_{H1}+1$	1
	$\varDelta_{H2}(\varDelta_{H2}-3)$	$\varDelta_{H2}$	1
	$(\varDelta_{H2}+1)(\varDelta_{H2}-2)$	$\varDelta_{H2}+1$	1
$\frac{1}{2}$	0	$\frac{3}{2}$	6
	1	$\frac{5}{2}$	2
	4	$\frac{7}{2}$	13
	9	$\frac{9}{2}$	8
	$(\varDelta_{G1}-1)^2$	$\varDelta_{G1}+\frac{1}{2}$	2
	$(\varDelta_{G2}-1)^2$	$\varDelta_{G2}+\frac{1}{2}$	2
	$(\varDelta_{V1}-\frac{3}{2})^2$	$\varDelta_{V1}$	1
	$(\varDelta_{V1}-\frac{1}{2})^2$	$\varDelta_{V1}+1$	1
	$(\varDelta_{V2}-\frac{3}{2})^2$	$\varDelta_{V2}$	1
	$(\varDelta_{V2}-\frac{1}{2})^2$	$\varDelta_{V2}+1$	1
1	0	2	1
	6	4	5
	$(\varDelta_{G1}-1)(\varDelta_{G1}-2)$	$\varDelta_{G1}$	1
	$\varDelta_{G1}(\varDelta_{G1}-1)$	$\varDelta_{G1}+1$	1
	$(\varDelta_{G2}-1)(\varDelta_{G2}-2)$	$\varDelta_{G2}$	1
	$\varDelta_{G2}(\varDelta_{G2}-1)$	$\varDelta_{G2}+1$	1
	$(\varDelta_{V1}-\frac{1}{2})(\varDelta_{V1}-\frac{3}{2})$	$\varDelta_{V1+\frac{1}{2}}$	1
	$(\varDelta_{V2}-\frac{1}{2})(\varDelta_{V2}-\frac{3}{2})$	$\varDelta_{V2+\frac{1}{2}}$	1
$\frac{3}{2}$	1	$\frac{5}{2}$	1
	$(\varDelta_{G1}-1)^2$	$\varDelta_{G1}+\frac{1}{2}$	1
	$(\varDelta_{G2}-1)^2$	$\varDelta_{G2}+\frac{1}{2}$	1
2	0	3	1

Table F.7 Mass spectrum of the $\mathcal{N}=1$ vacuum with completely broken gauge symmetry.

Appendix G

Kaluza-Klein spectrum of the $\mathcal{N} = 4$ Vacuum

The $\mathcal{N} = 4$ AdS_4 vacuum, first presented in [66], is defined by the following expectation values of the z_i:

$$z_1 = z_2 = z_3 = i\,, \qquad z_4 = z_5 = z_6 = -\bar{z}_7 = \frac{1}{\sqrt{2}}(1+i)\,. \tag{G.0.1}$$

The vacuum has $\mathrm{SU}(2)\times\mathrm{SU}(2)$ symmetry, corresponding to the superconformal R-symmetry.

As argued in section 5.2.1, the fact that the background contains a S^1, whose radius can be varied, implies that the Kaluza-Klein states with non-zero modes on the S^1 must fit into long supermultiplets. Moreover, by decomposing the $\mathcal{N} = 8$ multiplet into long $\mathcal{N} = 4$ multiplets, we deduce that, for generic values of the S^1 radius, the entire Kaluza-Klein spectrum organises itself into long *graviton* multiplets. Therefore, the full Kalzua-Klein spectrum of the $\mathcal{N} = 4$ vacuum can be determined from just its spin-2 spectrum, which has been worked out in [113].

Indeed, a direct computation using the tools of [34, 6] and reviewed in section 5.2 confirms that all Kaluza-Klein modes can be organised into long graviton multiplets. These are counted by the character for the highest-weight states, i.e. the gravitons,

$$\nu_1 = \frac{1}{(1-q^2)\,(1-q\,u)\,(1-q\,v)}\frac{1+s}{1-s}\,. \tag{G.0.2}$$

Here exponents of q, s count levels for the S^5 and S^1 harmonics, respectively, while exponents of u, v count the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ spins.

We find that the conformal dimension, Δ, of the highest weight state of the supermultiplets, as counted by (G.0.2), is given by

$$\Delta = \frac{3}{2} + \frac{1}{2}\sqrt{9 + 2\ell(\ell+4) + 4\ell_1(\ell_1+1) + 4\ell_2(\ell_2+1) + \frac{2n^2\pi^2}{T^2}}\,, \tag{G.0.3}$$

for a HWS of type $q^\ell\, s^n\, u^{\ell_1}\, v^{\ell_2}$. This precisely matches the spin-2 Kaluza-Klein masses computed in [113].